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Normalization of partial wave CP asymmetries in three-body decays of heavy hadrons

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Abstract: CP violation in hadronic multi-body decays has been extensively studied, and the experimental breakthrough in the heayy baryon sector was made recently. Partial-Wave CP Asymmetries (PWCPAs) in multi-body decays of heavy hadrons, which although provide us with more interference information, suffer from the normalization problem, as is pointed out in this paper. We propose a novel solution to this problem. We introduce a set of extra factors to rescale the PWCPAs to the proper sizes. Instead of determining the set of factors according to the normalization requirement, we demand that all the PWCPAs have the same statistical errors. In this way, we obtain a set of quasi-normalized PWCPAs, in the sense that they are close to the ideal normalized ones. As an application, we perform an analysis of PWCPAs in the decay channel $B^\pm \to \pi^+\pi^-\pi^\pm$. We focus on the phase space region where the invariant mass of the $\pi^+\pi^-$ pair varies around the vector resonance $\rho^0(1450)$. Based on the data of the LHCb collaboration, the interference patterns among the resonances $\rho^0(1450)$, $f_2(1270)$, and $f_0(1500)$ and their contributions to the quasi-normalized PWCPAs are analyzed. The analysis indicates that the quasi-normalized PWCPAs can avoid potential misleading or distorted results comparing with some other alternatively defined ones.

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Contents

1 Introduction

CP violation (CPV) plays a crucial role in the Standard Model (SM) and cosmology. In the SM, CPV originates from the complex elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, which are associated with quark-level transition amplitudes [1, 2]. The experimental observations of CPV have been firmly established in the decays of K, B, and D mesons [3–10] to date. Recently, the first observation of CPV in baryon decays was reported [11], providing valuable insights into the matter-antimatter asymmetry puzzle in the Universe.

CPV occurs via different mechanisms and can be expressed by different CPV observables in the decay processes of hadrons. Typically, CP Asymmetries (CPAs) are induced by the interference of decay amplitudes with different weak phases. The most extensively studied type of CPV is the direct CPV which is characterized by the difference between the branching ratios of a pair of CP-conjugate decay processes. However, the interference terms containing CPV sometimes do not show up in this type of CPV. For example, some of the interference terms become zero after the differential decay widths are integrated over the whole phase space, rather, they show up non-trivially in the angular distributions, such as the triple product asymmetries of the momenta and/or spins of the involved particles [12–22]. Consequently, it is necessary to study the angular distributions in the case of direct CPV more extensively.

There are also some other typical examples of CPAs corresponding to the angular distributions. The first one corresponds to the Lee-Yang decay parameters in the baryon decay $\Lambda \to p\pi^-$ [23], where the CPV observable is conventionally defined as [24]

$$A_{CP,\alpha}(\Lambda \to p\pi^-) \equiv \frac{\alpha - \overline{\alpha}}{\alpha + \overline{\alpha}}. \tag{1.1}$$

The second example is the partial-wave CPAs (PWCPAs), which are introduced in multi-body decays of heavy hadrons [25]. Take a three-body decay as an example. One first perform the Legendre expansion to the decay amplitude squared (see Eq. (2.1) below), then the PWCPAs are conventionally defined as

$$A_{CP,l}^{\rm conv} = \frac{w_l - \bar{w}_l}{w_l + \bar{w}_l}, \tag{1.2}$$

where w_l and $\overline{w}_l$ are the weights in the Legendre expansion of the decay amplitude squared for a pair of CP-conjugate processes, respectively.

The definition of CPAs in the above examples are clearly not normalized, since neither the decay parameters such as α nor the weights in the Legendre expansions w_l are positive definite. The allowed values of $A_{CP,\alpha}$ and $A_{CP,l}^{\rm conv}$ are in principle in the range $(-\infty, +\infty)$. The normalization problem for CPAs corresponding to the decay parameters of the hyperon decays is not serious since the CPAs in this case were measured to be small [26, 27]. Furthermore, this problem can be overcome by redefining the CPA observable as $A_{CP,\alpha} = \frac{1}{2}(\alpha + \overline{\alpha})$, since α and $\bar{\alpha}$ are normalized by definition. However, the PWCPAs defined in Eq. (1.2) cannot be solved in this way because w_l and $\overline{w}_l$ are not normalized. There are also alternative definitions for these CPAs. For example, the PWCPAs can also be defined as [28]

$$\mathring{A}_{CP,l} = \frac{w_l - \overline{w}_l}{w_0 + \overline{w}_0}. \tag{1.3}$$

However, such definitions cannot solve the normalization problem either. One of the motivation of the present paper is to solve such a normalization problem.

This paper is organized as follows. In Sec. 2, we present an alternative definition of PWCPAs and discuss their normalization. In Sec. 3, we apply our framework to the decays $B^{\pm} \to \pi^{\pm}\pi^{+}\pi^{-}$. We briefly give the summary in Sec. 4.

2 The partial wave CP asymmetries and their normalization

For a three-body decay process $M \to M_1 M_2 M_3$, the decay amplitude, $\mathcal{M} = \mathcal{M}(s_{12}, s_{23})$, generally depends on the invariant masses of M_1M_2 and M_2M_3, denoted by s_{12} and s_{23}, respectively. The invariant mass squared s_{12} can be expressed as $s_{12} = \Delta_{12}\cos\theta + \Sigma_{12}$, where θ is the helicity angle which is defined as the angle between the momenta of M_1 and M in the centre-of-mass frame of the M_1M_2 system, $\Delta_{12} = (s_{12,\rm max} - s_{12,\rm min})/2$, and $\Sigma_{12} = (s_{12,\rm max} + s_{12,\rm min})/2$, with $s_{12,\rm max}$ and $s_{12,\rm min}$ respectively being the kinematically allowed maximum and minimum values of s_{12}. The kinematical parameter s_{23} is irrelevant to our discussion here, so we simply ignore it in the following discussion.

The decay amplitude squared can be expanded by the Legendre polynomials as

$$|\mathcal{M}|^2 = \sum_l w_l P_l(c_\theta), \tag{2.1}$$

where P_l is the l-th Legendre polynomial, w_l is, as is introduced in the previous section, the corresponding weight in the expansion. We define the PWCPAs as

$$A_{CP,l} = \eta_l \mathring{A}_{CP,l} = \eta_l \frac{w_l - \overline{w}_l}{w_0 + \overline{w}_0} = \frac{\eta_l (2l+1)\int \left[|\mathcal{M}|^2 - |\overline{\mathcal{M}}|^2\right] P_l(c_\theta) \tilde{d}c_\theta}{\int \left[|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2\right] \tilde{d}c_\theta}, \tag{2.2}$$

where $\overline{\mathcal{M}}$ is the decay amplitude for the CP-conjugate process, $\tilde{d}c_\theta \equiv \Delta_{12} dc_\theta$, and η_l can be viewed as a set of normalization factors. The motivation for introducing the factors η_l is to ensure that the PWCPAs are comparable with each other, as well as with other CPA observables, so that one can compare these CPA observables in a reasonable way, and determine which CPAs are expected to be observed more easily in experiments. The guaidlines for determining these factors η_l will be presented in the following.

In order to obtain a proper form of the factors η_l, let us first introduce a set of experiment-friendly CPA observables,

$$\hat{A}_{CP,l} \equiv \frac{\int \left[|\mathcal{M}|^2 - |\overline{\mathcal{M}}|^2\right] \mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta}{\int \left[|\mathcal{M}|^2 + |\overline{\mathcal{M}}|^2\right] \tilde{d}c_\theta} = \frac{\sum_i (N_i - \overline{N}_i)\mathrm{sgn}_{l,i}}{N + \overline{N}}, \tag{2.3}$$

where in the second equality we have divided the interval of c_θ into small bins (denoted by i in the subscript), and express $\hat{A}_{CP,l}$ by N_i and $\overline{N}_i$, the event yields of the CP-conjugate processes in each bin, respectively, and N and $\overline{N}$, the corresponding total ones. The sign functions in Eq. (2.3) are defined as $\mathrm{sgn}(x) = \pm 1$, if $x \gtrless 0$, and $\mathrm{sgn}_{l,i}$ is the abbreviation for $\mathrm{sgn}(P_l(c_{\theta_i}))$. It can be easily seen that 1) $\hat{A}_{CP,l}$'s satisfy the normalization condition $-1 \leq \hat{A}_{CP,l} \leq +1$, 2) for $l = 0$, $\hat{A}_{CP,l}$ is just the direct CPA, and 3) the statistical errors of $\hat{A}_{CP,l}$'s are approximately the same, i.e., $\sigma_{\hat{A}_{CP,l}} \approx \hat{\sigma}$, where the $\hat{\sigma}$ is the statistical error which takes the familiar form $1/\sqrt{N + \overline{N}}$ in the absence of the background. In principle, both sets of observables $\{A_{CP,l}\}$ and $\{\hat{A}_{CP,l}\}$ are equivalent. Moreover, the latter can be expressed in terms of the former as

$$\hat{A}_{CP,l} = \sum_k \Omega_{lk} A_{CP,k}, \tag{2.4}$$

where Ω is the matrix connecting these two sets of CPA observables, the elements of which can be easily shown to take the form $\Omega_{lk} = \frac{\omega_{lk}}{\eta_k}$, where

$$\omega_{lk} = \frac{1}{2}\int_{-1}^{+1} \mathrm{sgn}\left(P_l(c_\theta)\right) P_k(c_\theta) dc_\theta. \tag{2.5}$$

The matrix elements ω_{lk} have the following properties: 1) they take non-zero values only when l and k are both even or odd; 2) the diagonal elements are much larger than the off-diagonal ones in the same rows or columns.

As a first approximation, we demand that the diagonal elements of Ω equal 1, so we have

$$\hat{A}_{CP,l} = A_{CP,l} + \sum_{k \neq l} \Omega_{lk} A_{CP,k}. \tag{2.6}$$

The advantages for this choice is clear, the elements of Ω_{lk} are quite smaller than 1 for $k \neq l$, hence one can see transparently that $\hat{A}_{CP,l}$ will get the most important contribution from $A_{CP,l}$. Since $\hat{A}_{CP,l}$ are clearly normalized, we conclude that $A_{CP,l}$ are now *roughly* normalized. The choice of the diagonal elements of Ω implies that the factors η_l are chosen as

$$\tilde{\eta}_l = \omega_{ll} = \frac{1}{2}\int_{-1}^{+1} \mathrm{sgn}\left(P_l(c_\theta)\right) P_l(c_\theta) dc_\theta = \frac{1}{2}\int_{-1}^{+1} |P_l(c_\theta)|\, dc_\theta, \tag{2.7}$$

where the tilde above η_l indicates that $\tilde{\eta}_l$ should be viewed as the first approximation for the factors η_l.

Interestingly, besides the fact that $A_{CP,l}$'s are roughly normalized, one can also see that they share roughly the same statistical errors with $\hat{A}_{CP,l}$. This loose correlation inspires us to pursue more practical and reasonable estimations of η_l. To this end, we assume that the Legendre expansion is truncated at some finite number L, so that the matrix Ω and ω become dimension $L+1$ ones, which will be denoted as Ω_L and ω_L, respectively. It can be shown that the matrices Ω_L and ω_L are reversible. This allows us to express $A_{CP,l}$ inversely in terms of $\hat{A}_{CP,k}$:

$$A_{CP,l} = \sum_{k=0}^{L} (\Omega_L^{-1})_{lk} \hat{A}_{CP,k} = \eta_l \sum_{k=0}^{L} (\omega_L^{-1})_{lk} \hat{A}_{CP,k}, \tag{2.8}$$

where Ω_L^{-1} and ω_L^{-1} are the inverse matrices of Ω_L and ω_L, respectively, and the relation $(\Omega_L^{-1})_{lk} = \eta_l(\omega_L^{-1})_{lk}$ has been used. The errors of $A_{CP,l}$ can be expressed as

$$\sigma_l^2 = \left[\Omega_L^{-1} \hat{\rho} \left(\Omega_L^{-1}\right)^T\right]_{ll} \hat{\sigma}^2 = \eta_l^2 \left[\omega_L^{-1} \hat{\rho} \left(\omega_L^{-1}\right)^T\right]_{ll} \hat{\sigma}^2, \tag{2.9}$$

where $\hat{\rho}_{kk'}$ are the correlation coefficients between $\hat{A}_{CP,k}$ and $\hat{A}_{CP,k'}$, which, as will be shown below, can be well estimated as

$$\hat{\rho}_{kk'} \approx \frac{1}{2} \int_{-1}^{+1} \mathrm{sgn}(P_k(c_\theta))\mathrm{sgn}(P_{k'}(c_\theta)) dc_\theta. \tag{2.10}$$

Instead of the requirement of normalization, it would be practically more useful to choose the η_l's in such a way that *the statistical error of $A_{CP,l}$ also equals to $\hat{\sigma}$, i.e., $\sigma_l = \hat{\sigma}$*. This means that η_l should be chosen as

$$\eta_l^{(L)} = \frac{1}{\sqrt{\left[\omega_L^{-1}\hat{\rho}(\omega_L^{-1})^T\right]_{ll}}} = \frac{1}{\sqrt{\sum_{k,k'=0}^{L}(\omega_L^{-1})_{lk}\omega_L^{-1})_{lk'}\hat{\rho}_{kk'}}}, \quad l = 0, 1, \cdots, L, \tag{2.11}$$

as is indicated by Eq. (2.9), where the subscript "L" indicates that the results are truncation-dependent. This is a more accurate estimation for the factors η_l comparing with Eq. (2.7), in the sense that the statistical error of $A_{CP,l}$ is closer to that of $\hat{A}_{CP,l}$.

In the following we present the reasons for the expression of the correlation coefficients $\hat{\rho}_{kk'}$ in Eq. (2.10). The correlation matrix is defined according to

$$\hat{\sigma}^2 \hat{\rho}_{lk} = \sum_i \left(\frac{\partial \hat{A}_{CP,k}}{\partial N_i} \frac{\partial \hat{A}_{CP,l}}{\partial N_i} \sigma_{N_i}^2 + \frac{\partial \hat{A}_{CP,l}}{\partial \overline{N}_i} \frac{\partial \hat{A}_{CP,k}}{\partial \overline{N}_i} \sigma_{\overline{N}_i}^2 \right).$$

Hence $\hat{\rho}_{lk}$ takes the form

$$\hat{\rho}_{lk} = \frac{1}{(N+\overline{N})^2} \sum_i \Bigg\{ \frac{\sigma_{N_i}^2 + \sigma_{\overline{N}_i}^2}{\hat{\sigma}^2} \left[\mathrm{sgn}_{l,i}\mathrm{sgn}_{k,i} + \frac{\sum_j (N_j - \overline{N}_j)\mathrm{sgn}_{l,j} \sum_{j'}(N_{j'} - \overline{N}_{j'})\mathrm{sgn}_{k,j'}}{(N+\overline{N})^2} \right]$$
$$- \frac{\sigma_{N_i}^2 - \sigma_{\overline{N}_i}^2}{\hat{\sigma}^2} \left[\frac{\mathrm{sgn}_{l,i} \sum_j (N_j - \overline{N}_j)\mathrm{sgn}_{k,j} + \mathrm{sgn}_{k,i} \sum_j (N_j - \overline{N}_j)\mathrm{sgn}_{l,j}}{N+\overline{N}} \right] \Bigg\}.$$

In the limit of CP symmetry, we obtain [1]

$$\hat{\rho}_{lk} \approx \frac{1}{(N+\overline{N})^2}\sum_i \left(\frac{\sigma^2_{N_i}+\sigma^2_{\overline{N}_i}}{\hat{\sigma}^2}\right)\text{sgn}_{l,i}\text{sgn}_{k,i} = \frac{\sum_i \left(N_i+\overline{N}_i\right)\text{sgn}_{l,i}\text{sgn}_{k,i}}{N+\overline{N}}.$$

This is clearly a very good approximation if the CPAs are not quite large. Not only that, at the end of this section we will show that there is no need for the assumption of CP symmetry at all. At first sight, $\hat{\rho}$ depends on the event distributions, or the decay amplitudes, which is a disadvantage since it means that ρ is not universal. However, as one will see that this dependence is quite mild and can be neglected safely. First of all, one can see that the diagonal elements of $\hat{\rho}$ equal 1, hence are independent of the decay amplitudes and are universal. Secondly, although the off-diagonal elements depend on the decay amplitudes, they are much smaller than the diagonal ones, which means that we can safely assume plain distributions of the event yields for estimation, so that a universal estimation of the matrix elements of $\hat{\rho}$ are obtained in Eq. (2.10). Based on the same argument, it is expected that the factors $\eta_l^{(L)}$ obtained in Eq. (2.11) should be close to the uncorrelated limit, i.e., $\hat{\rho}_{kk'} = \delta_{kk'}$.

Taking $L=4$ as an example. One first calculates the numerical values of the matrix ω and $\hat{\rho}$ respectively according to Eqs. (2.5) and (2.10), which read

$$\omega = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & -0.1250 & 0 \\ -0.1547 & 0 & 0.3849 & 0 & -0.0642 \\ 0 & -0.1 & 0 & 0.3250 & 0 \\ -0.0400 & 0 & -0.0768 & 0 & 0.2866 \end{pmatrix},$$

and

$$\hat{\rho} = \begin{pmatrix} 1 & 0 & -0.1547 & 0 & -0.0423 \\ 0 & 1 & 0 & -0.5492 & 0 \\ -0.1547 & 0 & 1 & 0 & -0.2475 \\ 0 & -0.5492 & 0 & 1 & 0 \\ -0.0423 & 0 & -0.2475 & 0 & 1 \end{pmatrix},$$

respectively [2]. Then substituting these results into Eq. (2.11), one obtains

$$\eta_0^{(4)} = 1,\ \eta_1^{(4)} = 0.5418,\ \eta_2^{(4)} = 0.3847,\ \eta_3^{(4)} = 0.3312,\ \eta_4^{(4)} = 0.2829. \tag{2.12}$$

For comparison, the factors η_l for the uncorrelated limit read

$$\eta_0^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 1,\ \eta_1^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.4308,\ \eta_2^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.3540,\ \eta_3^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.2942,\ \eta_4^{(4)}\Big|_{\hat{\rho}=\mathbb{1}} = 0.2674, \tag{2.13}$$

[1] Note that this equation also works when the contributions of the background are included in the statistical errors. Typically, the background may rescale the statistical errors, for example, the error $\hat{\sigma}$ would be rescaled to $\lambda/\sqrt{N+\overline{N}}$, where λ is the rescale factor. Meanwhile, the errors σ_{N_i} and $\sigma_{\overline{N}_i}$ will also be rescaled to $\lambda\sqrt{N_i}$ and $\lambda\sqrt{\overline{N}_i}$. Consequently, because of the cancellation of the rescale factors, one obtain $\frac{\sigma^2_{N_i}+\sigma^2_{\overline{N}_i}}{\hat{\sigma}^2} = (N_i+\overline{N}_i)(N+\overline{N})$.

[2] One can see from $\hat{\rho}$ that the largest correlation occurs between $\hat{A}_{CP,1}$ and $\hat{A}_{CP,3}$.

	$L \backslash l$	0	1	2	3	4	5	6	7	8
$\eta_l^{(L)}$	0	1								
	1	1	0.5							
	2	1	0.5	0.3896						
	3	1	0.5418	0.3896	0.3312					
	4	1	0.5418	0.3847	0.3312	0.2829				
	5	1	0.5552	0.3847	0.3332	0.2829	0.2615			
	6	1	0.5552	0.3899	0.3332	0.2807	0.2615	0.2345		
	7	1	0.5656	0.3899	0.3312	0.2807	0.2630	0.2345	0.2205	
	8	1	0.5656	0.3967	0.3312	0.2793	0.2630	0.2322	0.2205	0.2064
$\eta_l^{(L)}\Big\vert_{\hat{\rho}=\mathbb{1}}$	0	1								
	1	1	0.5							
	2	1	0.5	0.3804						
	3	1	0.4308	0.3804	0.2942					
	4	1	0.4308	0.3540	0.2942	0.2674				
	5	1	0.4383	0.3540	0.2787	0.2674	0.2441			
	6	1	0.4383	0.3200	0.2787	0.2423	0.2441	0.2131		
	7	1	0.4256	0.3200	0.2668	0.2423	0.2232	0.2131	0.1998	
	8	1	0.4256	0.3252	0.2668	0.2378	0.2232	0.1986	0.1998	0.1915
$\tilde{\eta}_l$		1	0.5	0.3849	0.3250	0.2866	0.2592	0.2385	0.2220	0.2086

Table 1: The numerical values of $\eta_l^{(L)}$, $\eta_l^{(L)}\Big\vert_{\hat{\rho}=\mathbb{1}}$ (for truncation $L \leq 8$), and $\tilde{\eta}_l$ (for $l \leq 8$).

and the first few $\tilde{\eta}_l$'s based on the rough estimation in Eq. (2.7) take the following values:

$$\tilde{\eta}_0 = 1, \quad \tilde{\eta}_1 = 0.5, \quad \tilde{\eta}_2 = 0.3849, \quad \tilde{\eta}_3 = 0.3250, \quad \tilde{\eta}_4 = 0.2866. \tag{2.14}$$

The comparison of Eqs. (2.12), (2.13), and (2.14) shows that 1) the contributions to the factors η_l from the correlation between the statistical errors of different CPA observables $A_{CP,l}$ are small, 2) the rough estimation of the η_l based on Eq. (2.7), i.e., $\tilde{\eta}_l$, works pretty well. The results of $\eta_l^{(4)}$ in Eq. (2.12), together with the numerical values of $\tilde{\eta}_l$ in Eq. (2.14), will be adopted in the application in the next section. More numerical values of $\eta_l^{(L)}$, $\eta_l^{(L)}\Big\vert_{\hat{\rho}=\mathbb{1}}$, and $\tilde{\eta}$ are presented in the Table 1. As one can clearly see from this table, the truncation-dependence of $\eta_l^{(L)}$ and $\eta_l^{(L)}\Big\vert_{\hat{\rho}=\mathbb{1}}$ is small.

At the end of this section, we want to point out that the factors η_l in fact reflect the bound of the ratio w_l/w_0. Hence our mathematical conclusion is that $|w_l/w_0|$ is bounded roughly by $1/\eta_l$, with η_l (at least approximately) taking the forms in Eq. (2.11), or even more simply, in Eq. (2.7). This means that the introduction and the determination of the factors η_l can be done without the introduction of PWCPAs. To see this, we first introduce the PWCPAs alternatively as $\tilde{A}_{CP,l} = (A_l - \overline{A}_l)/2$, where $A_l = \eta_l w_l/w_0$ and $\overline{A}_l = \eta_l \overline{w}_l/\overline{w}_0$

[29] [3]. Let us focus on A_l. One can introduce a set of experiment-friendly observables $\hat{A}_l \equiv \frac{\int[|\mathcal{M}|^2]\mathrm{sgn}(P_l(c_\theta))\tilde{d}c_\theta}{\int[|\mathcal{M}|^2]\tilde{d}c_\theta} = \frac{\sum_i N_i \mathrm{sgn}_{l,i}}{N}$, and perform almost exactly the same procedure, in this way one can obtain exactly the same expressions for $\tilde{\eta}_l$ and $\eta_l^{(L)}$. Moreover, the same procedure clearly also applies to $\overline{A}_l$, so that both $\tilde{\eta}_l$ and $\eta_l^{(L)}$ should be the same for the pair of CP-conjugate processes, regardless how large the CPAs are. Clearly, neither $\tilde{\eta}_l$ nor $\eta_l^{(L)}$ can ensure that A_l or $\overline{A}_l$, as well as $\tilde{A}_{CP,l}$ or $A_{CP,l}$, are normalized, but with our choices of $\tilde{\eta}_l$ or $\eta_l^{(L)}$, they must be not far away from normalization. Consequently, we will call them quasi-normalized ones. For example, with our choices of η_l in Eqs. (2.7) or (2.11), $A_{CP,l}$ will be called as quasi-normalized PWCPAs.

3 Application to $B^\pm \to \pi^\pm\pi^+\pi^-$ decays

As an application, we present the analysis of the PWCPAs for the decays $B^\pm \to \pi^\pm\pi^+\pi^-$ in this section. We perform this analysis based on the data of the LHCb collaboration in Ref. [30], focusing on the phase space where the invariant mass of the $\pi^+\pi^-$ pair $m_{\pi\pi,\mathrm{low}}$ is around the vicinity of the $\rho^0(1450)$ resonance. There are several resonances that may contribute to this decay, including $f_0(1370)$, $f_0(1500)$, $f_2(1270)$, $f_2(1430)$, $f_2'(1525)$, etc.. After various fitting scenarios were attempted, we found that the best fit to the LHCb data of Fig. 12 in Ref. [30] for $m_{\pi\pi,\mathrm{low}}$ ranging between 1.262 and 1.676 GeV can be obtained with the resonances $\rho^0(1450)$, $f_0(1500)$, and $f_2(1270)$ with the quantum numbers J^{PC} being 1^{--}, 0^{++}, and 2^{++}, respectively. This means that the Legendre expansion of the decay amplitude squared should be truncated at $l = 4$, which reads

$$|\mathcal{M}^\pm|^2 = \sum_{l=0}^{4} w_l^\pm P_l(c_\theta), \tag{3.1}$$

where "$\pm$" in the superscript indicates the electric charge of the B meson. The weights of the expansion $w_l^\pm$ are found to take the forms

$$\begin{aligned} w_0^\pm &= \left(|\mathcal{M}_{f_0}^\pm|^2 + \frac{1}{5}|\mathcal{M}_{f_2}^\pm|^2 + \frac{1}{3}|\mathcal{M}_{\rho^0}^\pm|^2\right), \quad w_1^\pm = \left(2\mathrm{Re}[\mathcal{M}_{\rho^0}^\pm \mathcal{M}_{f_0}^{\pm\,*}] + \frac{4}{5}\mathrm{Re}[\mathcal{M}_{\rho^0}^\pm \mathcal{M}_{f_2}^{\pm\,*}]\right), \\ w_2^\pm &= \left(\frac{2}{3}|\mathcal{M}_{\rho^0}^\pm|^2 + \frac{2}{7}|\mathcal{M}_{f_2}^\pm|^2 + 2\mathrm{Re}[\mathcal{M}_{f_0}^\pm \mathcal{M}_{f_2}^{\pm\,*}]\right), \quad w_3^\pm = \frac{6}{5}\mathrm{Re}[\mathcal{M}_{\rho^0}^\pm \mathcal{M}_{f_2}^{\pm\,*}], \\ w_4^\pm &= \frac{18}{35}|\mathcal{M}_{f_2}^\pm|^2, \end{aligned} \tag{3.2}$$

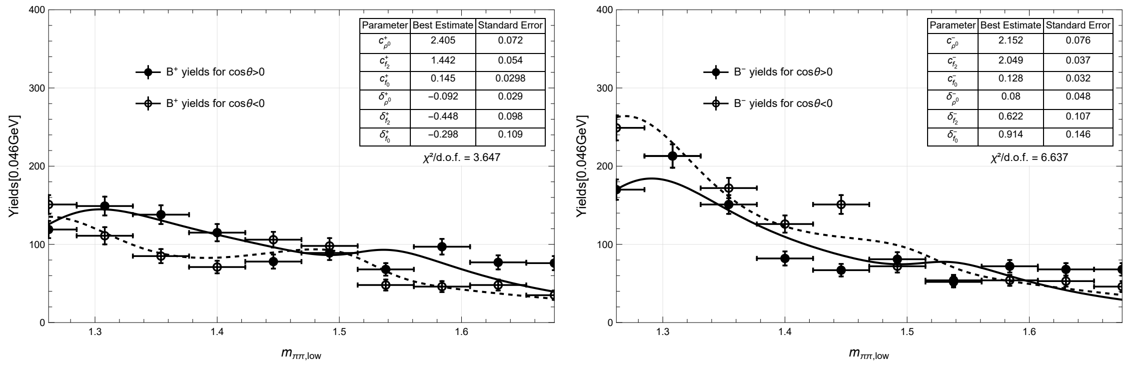
where $\mathcal{M}_R^\pm$ represent the decay amplitudes of the cascade decays $B^\pm \to R(\to \pi^+\pi^-)\pi^\pm$, with $R = f_0$, ρ^0, and f_2 representing the resonances $f_0(1500)$, $\rho^0(1450)$ and $f_2(1270)$,

[3]Clearly, $\tilde{A}_{CP,l}$ and $A_{CP,l}$ are equivalent. However, the physical meaning of A_l and $\overline{A}_l$ for even l seems not transparent. This is the reason why we did not perform the analysis with $\tilde{A}_{CP,l}$, as well as A_l and $\overline{A}_l$, in the main text.

respectively. The decay amplitudes are parameterized as

$$\mathcal{M}_R^{\pm} = c_R^{\pm} F_R^{BW} e^{i\delta_R^{\pm}} P_l(c_\theta), \tag{3.3}$$

where F_R^{BW} is the the Breit-Wigner line-shape of resonance R, $c_R^{\pm}$ and $\delta_R^{\pm}$ are the corresponding amplitude and the relative phase, respectively.



(a) (b)

Figure 1: The fitting results of the parameters (shown in the upper right corner of each figure) from the event yields of $B^+ \to \pi^+\pi^-\pi^+$ (a) and $B^- \to \pi^+\pi^-\pi^-$ (b) for $\cos\theta > 0$ and $\cos\theta < 0$.

The fitting results are presented in Fig. 1. With the values of the parameters $c_R^{\pm}$ and $\delta_R^{\pm}$, we are able to make predictions for the quasi-normalized PWCPAs, which are defined in Eq. (2.2), and the results with η_l taking the values $\eta_l^{(4)}$ are presented in Fig. 2. The comparison of PWCPAs with different definitions are presented in Fig. 3.

One simple fact about the quasi-normalized PWCPAs from Fig. 2 is that all the four PWCPAs, $A_{CP,l}$ (l=1, 2, 3, 4), range roughly around [-0.20, 0.20]. Note that this is a quite strong conclusion and depends crucially on the choice of the factors η_l. The four diagrams in Fig. 3 provide strong evidences that the choices of the factors η_l in Eqs. (2.7) and (2.11) lead to quite reasonable relative sizes of all the four quasi-normalized PWCPAs in Fig. 2. These evidences, as can be seen from Fig. 3, include 1) $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$ and $A_{CP,l}|_{\eta_l=\tilde{\eta}_l}$ are very close to $\hat{A}_{CP,l}$; 2) $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$ and $A_{CP,l}|_{\eta_l=\tilde{\eta}_l}$ are very close to each other; and 3) $\mathring{A}_{CP,l}$ are quite different form $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$, $A_{CP,l}|_{\eta_l=\tilde{\eta}_l}$, and $\hat{A}_{CP,l}$.

From Fig. 3(a) one can also see that the largest difference between $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$ ($A_{CP,l}|_{\eta_l=\tilde{\eta}_l}$) and $\hat{A}_l$ occurs when $l = 1$. This is understandable. According to Eq. (2.6), we have $\hat{A}_{CP,1} = A_{CP1} + (\omega_{13}/\tilde{\eta}_3)A_{CP,3} = A_{CP1} + (-0.1250/0.3250)A_{CP,3} \approx A_{CP1} - 0.385A_{CP,3}$. Since $A_{CP,3}$ takes the value of about 0.2, the deviation of $\hat{A}_{CP,1}$ from $A_{CP,1}$ is as large as about 0.08, which is just the difference between $\hat{A}_{CP,1}$ and $A_{CP,1}$ in Fig. 3(a).

For completeness, we also present the conventionally defined PWCPAs, $A_{CP,l}^{\text{conv}}$, in Fig. 4. Clearly, $A_{CP,l}^{\text{conv}}$'s are not normalized. Even worse, they are not even bounded sometime, as can been clearly seen from $A_{CP,1}^{\text{conv}}$ and $A_{CP,3}^{\text{conv}}$ in this figure. Simply taking the absolute values of w_l and $\overline{w}_l$ in the denominator of the definition of $A_{CP,1}^{\rm conv}$ does not solve this problem. It only forces $A_{CP,l}^{\rm conv}$ to be equal to ± 1 when $|A_{CP,l}^{\rm conv}| > 1$. This is also clearly illustrated in Fig. 4, by the solid and dashed lines which are -1 in the main part of the $m_{\pi\pi,\rm low}$ range. The comparison between Figs. 2 and 4 shows the large differences between the quasi-normalized and the conventionally defined PWCPAs. These differences indicate clearly that it is necessary to study the PWCPAs with the quasi-normalized ones in order to avoid misleading or distorted results.

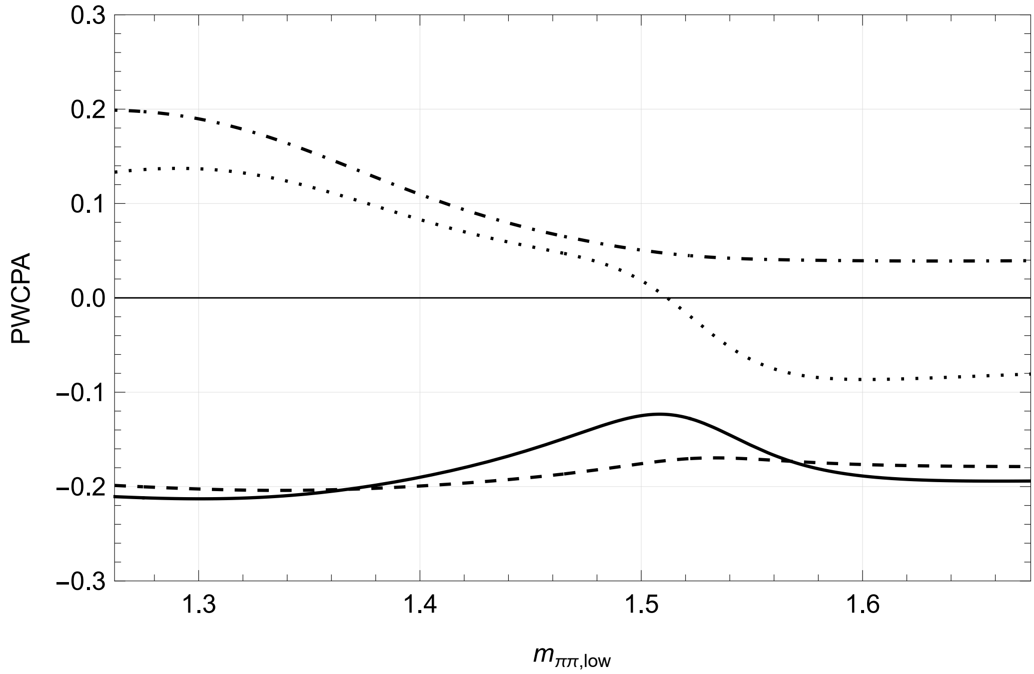
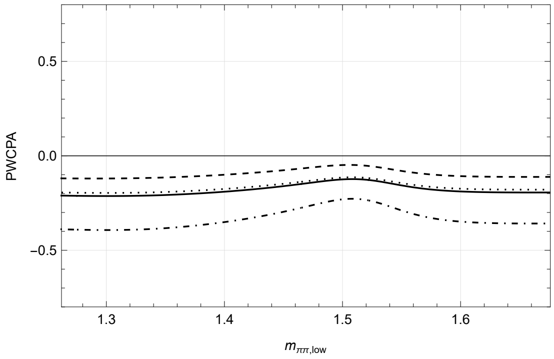
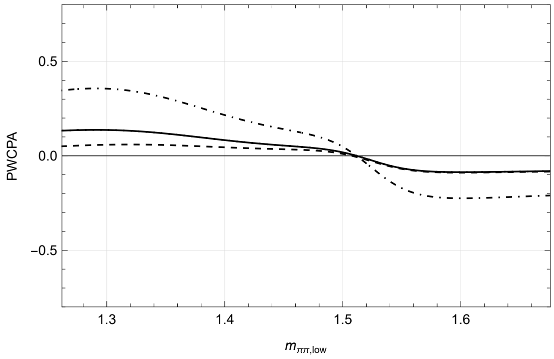


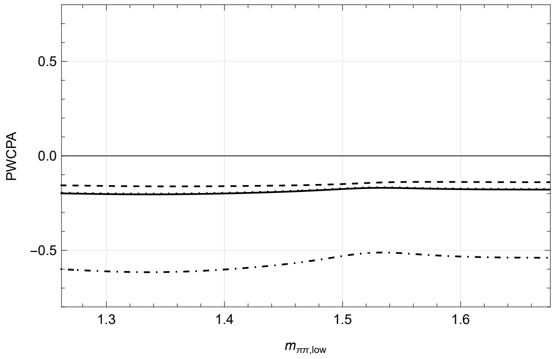
Figure 2: Predictions of the quasi-normalized PWCPAs $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$ for $B^\pm \to \pi^+\pi^-\pi^\pm$ as a function of the invariant mass $m_{\pi\pi,\rm low}$. The solid, dotted, dashed, and dash-dotted lines correspond to $l = 1$, 2, 3, and 4, respectively.



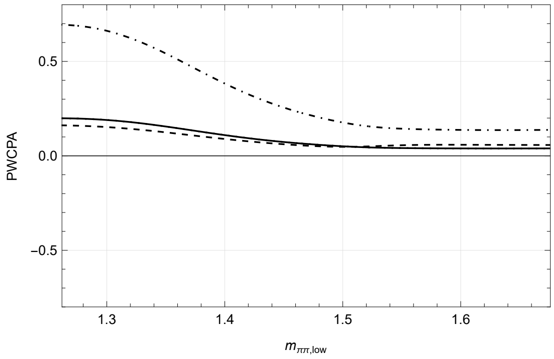
(a)



(b)



(c)



(d)

Figure 3: Comparisons of various definitions of PWCPAs, where the solid, dotted, dashed , and dot-dashed lines correspond to $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$, $A_{CP,l}|_{\eta_l=\tilde{\eta}_l}$, $\hat{A}_{CP,l}$, and $\mathring{A}_{CP,l}$, respectively. (a), (b), (c), and (d) correspond to $l=1$, 2, 3, and 4, respectively. Note that $A_{CP,l}|_{\eta_l=\eta_l^{(4)}}$ and $A_{CP,l}|_{\eta_l=\tilde{\eta}_l}$ almost coincide with each other.

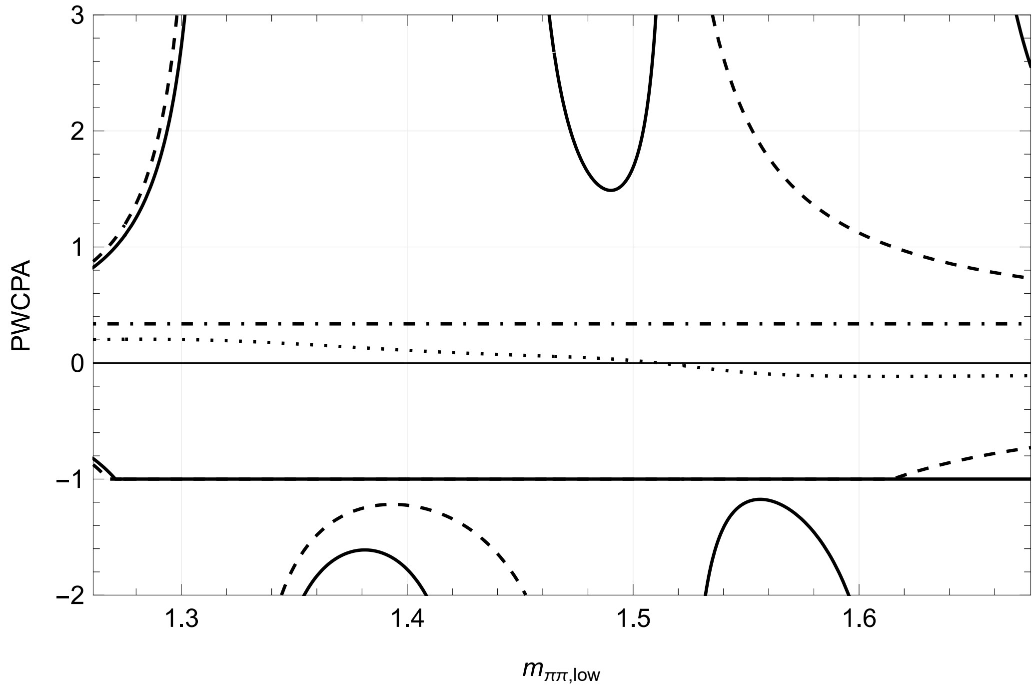


Figure 4: The conventionally defined PWCPAs, $A_{CP,l}^{\text{conv}}$, as a function of $m_{\pi\pi,\text{low}}$, where the solid, dotted, dashed and dash-dotted lines correspond to $l = 1$, 2, 3, and 4, respectively. We also present the results for another alternative PWCPAs, which are defined as $\tilde{A}_{CP,l}^{\text{conv}} \equiv \frac{w_l - \bar{w}_l}{|w_l| + |\bar{w}_l|}$, with l corresponding to the same line types as those for $A_{CP,l}^{\text{conv}}$. For $l = 1$ and 3, $\tilde{A}_{CP,l}^{\text{conv}}$ are exactly -1 in the main part of the region of $m_{\pi\pi,\text{low}}$; while for $l = 2$ and 4, $\tilde{A}_{CP,l}^{\text{conv}}$ exactly coincides with $A_{CP,l}^{\text{conv}}$.

4 Summary

In this paper, we pointed out the problem of the normalization of the PWCPAs in multi-body decay processes of heavy hadrons. To solve this problem, we redefined the PWCPAs according to Eq. (2.2), which are called as the quasi-normalized ones in this paper. To obtain the proper forms of the normalization factors η_l, the experiment-friendly CPV observables $\hat{A}_{CP,l}$'s, which are clearly normalized and have a proper form of the statistical errors, were introduced. These properties of $\hat{A}_{CP,l}$, together with the correlation between $\hat{A}_{CP,l}$ and $A_{CP,l}$, allowed us to derive reasonable estimations of the factors η_l in Eqs. (2.7) and (2.11), where the former was obtained based on rough estimation, while the latter was obtained by requiring that the statistical errors of $A_{CP,l}$ and $\hat{A}_{CP,l}$ be approximately the same. The numerical values of the factors η_l were also presented in Table 1 for future reference. Interestingly and reasonably, the numerical values of η_l obtained based on Eqs. (2.7) and (2.11) are quite close to each other.

As an illustration, we presented an analysis of the quasi-normalized PWCPAs in Eq. (2.2) for the decay processes $B^\pm \to \pi^+\pi^-\pi^\pm$. With the amplitudes extracted from the data of the LHCb collaboration in Ref. [30], we were able to obtain the quasi-normalized PWCPAs in the region around the intermediate resonance $\rho^0(1450)$. The results illustrated in Figs. 2, 3, and 4 show clearly that the quasi-normalized PWCPAs indeed can overcome the deficiencies of some other alternatively defined PWCPAs.

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