ROBUST HETERODIMENSIONAL CYCLES OF CO-INDEX TWO VIA SPLIT BLENDING MACHINES

PABLO G. BARRIENTOS, LORENZO J. DÍAZ, YURI KI, CRISTINA LIZANA, AND SEBASTIÁN A. PÉREZ

ABSTRACT. We consider diffeomorphisms f with heterodimensional cycles of co-index two, associated with saddles P and Q having unstable indices ℓ and $\ell+2$, respectively. In a partially hyperbolic setting, where a two-dimensional center direction and strong invariant manifolds are defined, we introduce the class of *non-escaping cycles*, where the strong stable manifold of P and the strong unstable manifold of Q are involved in the cycle. This configuration guarantees the existence of orbits that remain in a neighbourhood of the cycle.

We show that such diffeomorphisms f can be C^1 approximated by diffeomorphisms exhibiting simultaneously C^1 robust heterodimensional cycles of co-indices one and two, encompassing all possible combinations among hyperbolic sets of unstable indices ℓ, $\ell+1$, and $\ell+2$.

The proof relies on the construction of *split blending machines*. This tool extends Asaoka's blending machines to a partially hyperbolic setting, providing a mechanisms to generate and control robust intersections within a two-dimensional central bundle.

We also present simple dynamical settings where such cycles occur, namely skew product dynamics with surface fiber maps. Non-escaping cycles also appear in contexts such as Derived from Anosov diffeomorphisms and matrix cocycles on $\mathrm{GL}(3,\mathbb{R})$.

To Jacob Palis, in memoriam

1. INTRODUCTION

1.1. **Framework.** Homoclinic tangencies and heterodimensional cycles are conjectured to be the two primary sources of nonhyperbolic dynamics, see [32]. These bifurcations give rise to cascades of new bifurcations and, in many cases, to robustly non-hyperbolic dynamics. Here we focus on *heterodimensional cycles*, a configuration in which the invariant manifolds of two hyperbolic basic sets with different u-*indices* (dimension of the unstable bundle) intersect cyclically. Due to a deficiency in dimension, one of these intersections cannot be transverse. A threshold for the “lack of transversality" is given by the *co-index* of the cycle, defined as the absolute value of the difference between the

Date: November 18, 2025.

2020 *Mathematics Subject Classification.* Primary: 37C20. Secondary: 37C29, 37D20, 37D30.

Key words and phrases. Blender, Blending machine, Heterodimensional cycle, Partial hyperbolicity, Skew-product.

P. Barrientos was supported by PID2020-113052GB-I00 funded by MCIN, PQ 305352/2020-2 (CNPq), and JCNE E-26/201.305/2022 (FAPERJ), Brazil. L. J. Díaz is partially supported by CAPES-Finance Code 001, CNPq 310069/2020-3, CNPq Projeto Universal 404943/2023-3, INCT-FAPERJ E-26/200866/2018, and CNE-FAPERJ E26/204046/2024, Brazil. C. Lizana was partially supported by CNPq projects 406750/2021-1 and 404943/2023-3, Brazil. S. A. Pérez was partially supported by FONDECYT Regular 1250697, Chile. The authors thank the hospitality of PUC-Rio (Brazil), UFBA (Brazil), PUCV (Chile), and CMUP (Portugal).

u-indices of the sets. Heterodimensional cycles can only occur in dimensions three or higher. In dimension three, heterodimensional cycles always have a co-index one.

Recall that a basic set Λ of diffeomorphism f has a unique and well-defined continuation Λ_g for every diffeomorphism g sufficiently close to f. A heterodimensional cycle between two basic sets Λ and Γ of a diffeomorphism f is said to be C^r *robust* if there exists a C^r neighbourhood $\mathscr{U}$ of f such that, for every $g \in \mathscr{U}$, the continuations Λ_g and Γ_g also form a heterodimensional cycle. By the Kupka-Smale genericity theorem, every robust cycle involves some *nontrivial* hyperbolic set (i.e., a set containing non-periodic orbits). The presence of such cycles prevents hyperbolicity; thus, robust heterodimensional cycles give rise to open sets of non-hyperbolic diffeomorphisms.

Following [14, Preface], examples of open sets of diffeomorphisms consisting entirely of non-hyperbolic maps are roughly divided into two categories: critical and non-critical. Critical behavior is associated with the occurrence of homoclinic tangencies, whereas non-critical behavior is related to partial hyperbolicity, with typical bifurcations involving heterodimensional cycles. The first category includes constructions such as those in [28, 29], which deal with the persistence of homoclinic tangencies. The examples in [1, 37] belong to the second category. In these constructions, the invariant manifolds of two hyperbolic sets with different u-indices intersect cyclically, forming heterodimensional cycles, although this terminology was not used in the original works. Indeed these cycles are robust.

A natural question in dynamics is to determine when the unfolding of a cycle leads to robust ones. In the case of heterodimensional cycles, the co-index measures the dimension gap that must be bridged to produce “transverse-like" intersections, those capable of generating cyclic and robust interactions between the invariant manifolds of the sets in the cycle. To overcome this difficulty, additional geometric structures are required. This issue was resolved for co-index one cycles in [11] (in the C^1 setting) and [25] (for higher C^r regularity), where the key ingredient is the construction of blenders. See Section 1.2 for further discussion.

Within the terminology above, the dynamics that we consider fall into the non-critical category. We study co-index two heterodimensional cycles within a partially hyperbolic setting that ensures the saddles in the cycle have well-defined strong stable and unstable manifolds. In this context, the generation of new cycles requires addressing two independent difficulties: bridging the two-dimensional gap to achieve “transverse-like properties" and overcoming the “loss of recurrence" along the central directions of the cycle. The latter is managed by assuming that the strong invariant manifolds are “involved" in the cycle. We call such cycles *non-escaping,* see Definition 2.1 and the heuristic discussion in Section 2.1.1. We prove that these cycles simultaneously give rise to C^1 robust heterodimensional cycles of both co-index one and two.

To state our main result, note that a co-index two cycle associated with saddles of u-indices ℓ and $(\ell+2)$ may generate hyperbolic sets of u-indices ℓ, $(\ell+1)$, and $(\ell+2)$, which can in turn be involved in new cycles. To capture the full complexity of the arising cycles, we introduce the following definition:

Definition 1.1 (Cycle of type (ℓ, m))**.** A heterodimensional cycle is a *cycle of type* (ℓ, m), with $\ell < m$, if it is associated with hyperbolic sets of u-indices ℓ and m.

Theorem A. *Let f be a C^1 diffeomorphism having a non-escaping cycle of type $(\ell,\ell+2)$ associated to a pair of saddles. Then there is g arbitrarily C^1 close to f having robust cycles of type $(\ell,\ell+2)$, $(\ell,\ell+1)$, and $(\ell+1,\ell+2)$.*

Remark 1.2. Theorem A applies to a broader class of cycles which, following the terminology of [25], we call cycles with a *contour of double type I*, defined in Section 3.2 involving the notion of a contour, see Section 2.1. Informally, such cycles behave as if they carried two independent type I contours of a cycle of co-index one (in the sense of [25]), one associated with the strong stable manifold of the saddle of u-index ℓ, and the other with the strong unstable manifold of the saddle of u-index $\ell+2$.

Viewing non-escaping cycles as cycles with a double type I contour, the coexistence of a pair of robust co-index one cycles, obtainable without destroying the original cycle, follows naturally from the approach in [25]. The genuinely new and technically demanding phenomenon lies in the coexistence of robust co-index two cycles.

Question 1. It remains an open problem whether there is a transitive set (for instance, a homoclinic class) that contains all the hyperbolic sets involved in the three types of cycles in the theorem. A further question is whether such a set could also contain the continuations of the saddles in the cycle. These questions are closely related to the problem of stabilization of cycles; see [13, 25]. A more precise version of this question should involve contours. Let us observe that there are co-index one cycles (precisely those having a contour of type I) that cannot be stabilizable, see [25, Theorem C].

Our aim is to present the geometric ideas underlying robust cycles of co-index two, setting aside the regularity issues involved in the perturbations. The approach in [25] provides a natural starting point for addressing higher regularity, which remains a delicate and technically demanding aspect of the theory. In line with the study of co-index one cycles, our strategy is to first introduce the geometric framework (as in [11]).

Before turning to the setting of high co-index cycles, we revisit the co-index one case. There are two (non-disjoint classes) of co-index one cycles: twisted and non-twisted (following the terminology of [13]), or type I and type II (as in [25]). In the case of twisted or type I cycles, the bifurcating diffeomorphism may exhibit very rich dynamics, see Theorem 5.10. As shown in [25, Theorems 4 and 5], the saddles in the cycle may be accumulated by other hyperbolic sets, which, however, fail to be homoclinically or heteroclinically related to the saddles in the original cycle. This leads to a form of latent underlying dynamics[1], whose study is essential for understanding the global dynamics of the cycle. In the non-escaping setting, a similar form of latent dynamics arises.

Our interest in co-index two cycles arises from a class of skew-products with fiber dynamics on the two-sphere, a setting reminiscent of [22]; see Section 2.2. This class leads us to introduce non-escaping cycles and to place them within a broader context.

[1]This is somewhat reminiscent of the so-called "ghost dynamics" in saddle-node cycles, described via transition maps in [31]; see also [19, Section 11.3]. The key difference is that ghost dynamics are virtual, suppressed dynamics that only appear upon unfolding the cycle, whereas latent dynamics are actual and present in the system.

Within nonhyperbolic dynamics, non-escaping cycles already appear in the class of diffeomorphisms constructed by Abraham-Smale [1] and Shub [36], as well as in Manning's versions of them in [26]; see Section 11.1.

Although our primary focus is on diffeomorphisms, an additional motivation arises from the study of matrix cocycles in $\mathrm{GL}(3,\mathbb{R})$, which are directly related to skew-products with fiber maps defined on the two-sphere. We note that heterodimensional cycles of co-index one have already appeared in the study of the boundary of hyperbolicity for matrix cocycles, see [3, Theorem 4.1], even though the terminology of cycles is not explicitly used there (see also the discussion in [18, Section 1.2]). In the case of matrix cocycles in $\mathrm{GL}(3,\mathbb{R})$, we observe that the corresponding skew-product dynamics often exhibit non-escaping cycles; see Section 11.2 for further discussion.

1.2. **Bridging the dimension deficit using blenders.** Blenders were introduced in [10] as a tool for constructing robustly transitive diffeomorphisms. Since then, various versions have been developed to serve different purposes. Rather than surveying all these variants, we focus on a key dynamical feature of them: a *blender* is a hyperbolic set with a special geometric superposition property that makes certain non-transverse intersections between invariant manifolds robust under perturbations, behaving as transverse ones. A bit more precisely, let Λ be a blender, and denote by d^{s} and d^{u} the dimensions of its stable and unstable bundles, respectively. Then Λ is called a cu*-blender* if its stable manifold "behaves" as if it had dimension $d^{\mathrm{s}}+j^{\mathrm{s}}$, for some $j^{\mathrm{s}}\geqslant 1$. Similarly, Λ is a cs-blender if its unstable manifold behaves as if it had dimension $d^{\mathrm{u}}+j^{\mathrm{u}}$, for some $j^{\mathrm{u}}\geqslant 1$. For details see Definition 6.11. We call the number j^{s} *(stable) dimension jump of the* cu*-blender,* similarly for j^{u}. For an informal presentation of blenders, see [9]. The robust cycles obtained in [11, 25] are associated to co-index one cycles and involve a saddle and a blender which provides a one-dimensional jump. To study cycles of higher co-index, a more refined analysis of blenders is required.

The first examples of robust cycles were provided in [1] in dimension four and later extended to dimension three in [37]. These configurations are specific and not associated with bifurcations. In [8, Section 4.2], these constructions are revisited by embedding Plykin attractors on two disks in a three-dimensional manifold as normally hyperbolic basic sets of u-index two. The resulting sets have a two-dimensional stable manifold, while their stable bundle is one-dimensional. Hence they can be regarded as cu-blenders and used as a dynamical plug to generate robust cycles of co-index one. To complete the construction it is enough to relate the normally hyperbolic Plykin set with a saddle of u-index one. We refer to this type of blender as a cu-blender of Plykin type.

A crucial difference between the constructions in [1, 37, 8] and those in [11, 25] lies in the mechanism by which robust cycles are obtained. The former relies on rigid, semi-global configurations in which blenders are present in the initial system. In contrast, the latter requires only the existence of a co-index one cycle associated to a pair of saddles, a flexible configuration that appears in many contexts. The key point is that in [1, 37, 8], blenders are assumed and used as tools, whereas in [11, 25], blenders are obtained by unfolding the cycles and thereafter used as plugs to produce robust cycles.

We now turn to robust cycles of co-index $k \geqslant 2$. One approach, following [8], starts with a hyperbolic attractor of unstable dimension $k \geqslant 2$ and embeds it as a saddle-type basic set by adding a normally hyperbolic unstable direction. The resulting set is a cu-blender with jump $j^{\mathrm{s}} = k$, which serves as a dynamical plug to create robust cycles of co-index k. This, however, is an ad hoc construction.

A different, more semi-global approach, proposed in [5], perturbs the product of a diffeomorphism with a horseshoe and the identity on a k-dimensional manifold. This configuration also produces robust cycles of co-index k, involving a blender horseshoe with dimension jump k and a saddle, but it is not intrinsically tied to cycles and requires starting from an already rich dynamics.

In the previous constructions, the dimension deficit required for transverse-like intersections is resolved by using a blender with a dimension jump equal to the full deficit. Our alternative approach, is to split the deficit k into two jumps j^{s} and j^{u}, corresponding to unstable and stable blenders, with $j^{\mathrm{s}} + j^{u} = k$. This strategy was implemented in [2]. Our construction is inspired by the ideas developed there, although substantial modifications and new ingredients are required to overcome the difficulties specific to our setting. This approach allows us to consider cycles involving just a pair of saddles.

The previous discussion leads to the question below, where the term "simple" is intentionally vague, referring to cycles involving saddles and specific conditions on the intersections of their invariant manifolds. This excludes cases where, by assumption, the saddles belong to "big" hyperbolic sets (as in the first two cases discussed above). For the second item in the question, have in mind the constructions in [34], where the cycles are not associated to blenders but to sets with large Hausdorff dimension. Also note that the question below can be posed in any C^r topology.

Question 2. Consider any $k \in \mathbb{N}$.

(1) Does there exist a "simple" class of (heterodimensional) cycles of co-index k that generates robust cycles of co-index k?
(2) Are such cycles associated with blenders? If so, what are the corresponding dimension jumps of these blenders?

In the case $k = 1$, this question was positively answered in [11, 25] obtaining a blender with dimension jump one. We introduce the class of non-escaping cycles (of co-index two) and provide a positive answer to Question 2 for them. To fill the two-dimensional deficit in the indices of the sets, we get the simultaneous existence of a cs-blender Λ_1 and a cu-blender Λ_2 of u-indices ℓ and $\ell + 2$, respectively, and both with dimension jump equal to one. The blender property implies that the sets $W^{\mathrm{u}}(\Lambda_1)$ and $W^{\mathrm{s}}(\Lambda_2)$ ($W^{\mathrm{s}}(\cdot)$ and $W^{\mathrm{u}}(\cdot)$ denote the stable and unstable manifolds) behave as manifolds of dimensions $\ell+1$ and $(n-\ell-2)+1$, respectively (here n is the dimension of the ambience). Hence the sum of the "dimensions" is n and can provide intersections that behave as transverse ones and hence may be done robust.

Note that in our construction the blenders have a dimension jump equal to one. This leads to a second question, which refines item (2) of Question 2.

Question 3. Given $j \geqslant 2$, does there exist a "simple" class of heterodimensional cycles that generates blenders exhibiting a dimension jump equal to j? Are these blenders involved in robust cycles?

A recent approach [6] provides robust homoclinic tangencies from robust heterodimensional cycles of the lifted dynamics in the Grassmannian manifold, in a partially hyperbolic setting with center dimension at least two. This strategy was further refined in [2] to produce C^2 robust homoclinic tangencies of high co-dimension in a non-dominated setting, though not directly related to explicit bifurcations. In the same work, Asaoka also revisits the theory of blenders, introducing the notion of a *blending machine*, a mechanism for generating robust intersections of dynamically defined Cantor sets in higher dimensions. We adapt this concept to the dominated setting by introducing the *split blending machine*, a variant designed for partially hyperbolic systems with two- or higher-dimensional center direction, reflecting the co-index two cycles considered here (see Section 6.2).

2. Setting, toy model, and sketch of the proof

In Section 2.1, we state precisely the setting where Theorem A applies. In Section 2.2, we present a toy non-escaping cycle. In Section 2.3, we sketch the main ingredients of our constructions.

Throughout the paper, we consider a diffeomorphism f defined on a compact boundaryless Riemannian manifold M of dimension $n \geqslant 4$, having two saddles P and Q forming a heterodimensional cycle of co-index two (in what follows we omit the heterodimensional), where the u-index of P is smaller than that of Q.

2.1. **Non-escaping cycles of co-index two.**

2.1.1. *Heuristics.* The dynamics associated with a cycle depend on several factors which are considered in the the definition of non-escaping ones (see Remark 2.2): (i) domination or partial hyperbolicity in the cycle (if any); (ii) the so-called central multipliers of the saddles; (iii) the geometry of the intersections $W^{\mathrm{s}}(P)\cap W^{\mathrm{u}}(Q)$ and $W^{\mathrm{u}}(P)\cap W^{\mathrm{s}}(Q)$; and (iv) the transitions along the cycle between their saddles. These aspects are not entirely independent.

Concerning (iii), the way the invariant manifolds of the saddles in the cycle intersect plays a fundamental role. A key problem in analyzing the unfolding of a cycle is the generation of "recurrences" (used here as a deliberately vague term) around the cycle. The simplest precursor of this type of question appear in surface diffeomorphisms with heteroclinic tangencies whose limit sets and non-wandering sets are different, see [30, Figure 4.4]. In such cases, the relative position of the tangencies is crucial in determining the recurrent behaviour. In the specific case of heterodimensional cycles, recurrences always occur in the co-index one case, a key fact in establishing their robustness. However, for higher co-indices, there may exist "escaping directions" that obstruct the recurrence mechanism. The rough idea is that, in the unfolding of a cycle, orbits tend to follow the available center-strong stable direction to enter the cycle, and the unstable directions to exit it. In non-escaping cycles, these directions are intrinsically part of the cycle structure, which guarantees recurrence. We now formalize this heuristic.

2.1.2. *Contours of a cycle.* For simplicity, we assume that the saddles P and Q are fixed points. The f-invariant *maximal contour of the cycle* is the compact set

$$\Theta \stackrel{\text{def}}{=} \{P\} \cup \{Q\} \cup \Big(W^{\mathrm{s}}(P) \cap W^{\mathrm{u}}(Q)\Big) \cup \Big(W^{\mathrm{u}}(P) \cap W^{\mathrm{s}}(Q)\Big). \tag{2.1}$$

After unfolding the cycle, one aims to control the orbits that remain within a small neighbourhood of Θ (or some prescribed part of it, since in many cases the set Θ may be very big). Hence one may consider compact f-invariant subsets Υ of Θ that contain the saddles P and Q, along heteroclinic points in both sets $W^{\mathrm{s}}(P) \cap W^{\mathrm{u}}(Q)$ and $W^{\mathrm{u}}(P) \cap W^{\mathrm{s}}(Q)$. We refer to such a set as a *contour of the cycle*.

To be more precise, we suppose there exists a contour Υ having a partially hyperbolic splitting adapted to the co-index two

$$T_\Upsilon M = E^{\mathrm{sss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{uuu}}, \tag{2.2}$$

where E^{sss} and E^{uuu} are uniformly contracting and expanding, respectively. Adapted means that E^{c} is a central direction of dimension two such that the stable and unstable bundles of the saddles of the cycle satisfy

$$\begin{aligned} &E^{\mathrm{s}}(P) = E^{\mathrm{sss}}(P) \oplus E^{\mathrm{c}}(P), && E^{\mathrm{u}}(P) = E^{\mathrm{uuu}}(P),\\ &E^{\mathrm{s}}(Q) = E^{\mathrm{sss}}(Q), && E^{\mathrm{u}}(Q) = E^{\mathrm{c}}(Q) \oplus E^{\mathrm{uuu}}(Q). \end{aligned} \tag{2.3}$$

The bundle E^{c} may split in a dominated way or not. We consider the first case, assuming that $E^{\mathrm{c}} = E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2$, where E^{c}_1 is more contracting than E^{c}_2. These bundles play asymmetric roles, depending on the saddle of the cycle as explained below. Putting together the previous bundles we get a dominated splitting over Υ,

$$T_\Upsilon M = E^{\mathrm{sss}} \oplus E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2 \oplus E^{\mathrm{uuu}}. \tag{2.4}$$

The results in [16, 20] show that, for co-index one cycles, the resulting dynamics primarily depends on the central direction, which is one-dimensional and further exploited in [11, 25] to get robust cycles. Co-index two cycles, however, require a more careful analysis. Note from (2.3) that the saddle P admits several stable subbundles:

$$E^{\mathrm{sss}}(P) \subset E^{\mathrm{ss}}(P) \subset E^{\mathrm{s}}(P), \quad \text{with} \quad E^{\mathrm{ss}}(P) \stackrel{\text{def}}{=} E^{\mathrm{sss}}(P) \oplus E^{\mathrm{c}}_1(P).$$

We refer to these as the *strongest stable*, *strong stable*, and *stable* bundles, respectively. Similarly, for the saddle Q there are expanding bundles

$$E^{\mathrm{uuu}}(Q) \subset E^{\mathrm{uu}}(Q) \subset E^{\mathrm{u}}(Q), \quad \text{with} \quad E^{\mathrm{uu}}(Q) \stackrel{\text{def}}{=} E^{\mathrm{c}}_2(Q) \oplus E^{\mathrm{uuu}}(Q),$$

Accordingly, one can define the invariant manifolds tangent to them [24]:

$$W^{\mathrm{sss}}(P) \subset W^{\mathrm{ss}}(P) \subset W^{\mathrm{s}}(P) \qquad \text{and} \qquad W^{\mathrm{uuu}}(Q) \subset W^{\mathrm{uu}}(Q) \subset W^{\mathrm{u}}(Q). \tag{2.5}$$

The orbits generated in the unfolding of the cycle split in segments as follows: first, iterations close to Q; then a transition from Q to P following some heteroclinic point in $\Upsilon \cap W^{\mathrm{s}}(P) \cap W^{\mathrm{u}}(Q)$; next, iterations close to P; and finally a transition from P to Q following some heteroclinic point in $\Upsilon \cap W^{\mathrm{u}}(P) \cap W^{\mathrm{s}}(Q)$. Typically, these orbits exit from Q along E^{c}_2 and approach to P along E^{c}_1. Therefore, both directions must be involved in the cycle, otherwise, points may leave the neighbourhood of the cycle along E^{c}_2 and fail to return to the cycle along E^{c}_1. This observation is done more precise using $W^{\mathrm{ss}}(P)$ and $W^{\mathrm{uu}}(Q)$: the interaction of the central directions is guaranteed if $W^{\mathrm{ss}}(P)$ and $W^{\mathrm{uu}}(Q)$ both intersect the contour Υ. This leads to the definition of non-escaping cycles.

2.1.3. *Non-escaping contours and cycles.* We introduce conditions (NE1)–(NE2) for a contour Υ of the cycle associated to P and Q to be *non-escaping.* Recall the definition of the maximal contour Θ in (2.1).

(NE1) There are points $X_1, X_2, Z_1, Z_2 \in W^s(P) \pitchfork W^u(Q)$ with pairwise disjoint orbits and $Y \in W^{\mathrm{u}}(P) \cap W^{\mathrm{s}}(Q)$ such that

$$\Upsilon \stackrel{\text{def}}{=} \{P\} \cup \{Q\} \cup \bigcup_{i\in\mathbb{Z}} f^i(\{X_1, X_2, Z_1, Z_2, Y\}) \subset \Theta \tag{2.6}$$

has a dominated splitting

$$T_\Upsilon M = E^{\mathrm{sss}} \oplus E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2 \oplus E^{\mathrm{uuu}}, \qquad E^{\mathrm{c}} \stackrel{\text{def}}{=} E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2, \tag{2.7}$$

with four non-trivial bundles satisfying (2.3) and such that $\dim E^{\mathrm{c}}_1 = \dim E^{\mathrm{c}}_2 = 1$.

The eigenvalues of Df at P and Q corresponding to E^{c}_1 and E^{c}_2 are the *central eigenvalues* in (ii) in Section 2.1.1. By hypothesis, they are real and different in modulus. There is no assumption on their signals.

Consider the invariant manifolds $W^{\mathrm{sss}}(P) \subset W^{\mathrm{ss}}(P) \subset W^{\mathrm{s}}(P)$ and $W^{\mathrm{uuu}}(Q) \subset W^{\mathrm{uu}}(Q) \subset W^{\mathrm{u}}(Q)$ in (2.5). Each of them has co-dimension one in the next one. Thus, the sets

$$W^{\mathrm{s}}(P) \setminus W^{\mathrm{ss}}(P), \quad W^{\mathrm{ss}}(P) \setminus W^{\mathrm{sss}}(P), \quad W^{\mathrm{u}}(Q) \setminus W^{\mathrm{uu}}(Q), \quad W^{\mathrm{uu}}(Q) \setminus W^{\mathrm{uuu}}(Q)$$

each have two connected components, which we denote by $W^{\mathrm{s}}_j(P)$, $W^{\mathrm{ss}}_j(P)$, $W^{\mathrm{u}}_j(Q)$, and $W^{\mathrm{uu}}_j(Q)$, for $j \in \ell, r$.

(NE2) The heteroclinic points X_1, X_2, Z_1, Z_2 in the contour Υ such that

$$X_1, X_2 \in W^{\mathrm{uu}}(Q) \cap (W^{\mathrm{s}}(P) \setminus W^{\mathrm{ss}}(P)) \quad \text{and} \quad Z_1, Z_2 \in W^{\mathrm{ss}}(P) \cap (W^{\mathrm{u}}(Q) \setminus W^{\mathrm{uu}}(Q))$$

satisfy one of the following properties:

1. either (a) X_1, X_2 lie in different components of $W^{\mathrm{uu}}(Q) \setminus W^{\mathrm{uuu}}(Q)$, or (b) they lie in the same component of $W^{\mathrm{uu}}(Q) \setminus W^{\mathrm{uuu}}(Q)$ but in different components of $W^{\mathrm{s}}(P) \setminus W^{\mathrm{ss}}(P)$;
2. either (a) Z_1, Z_2 lie in different components of $W^{\mathrm{ss}}(P) \setminus W^{\mathrm{sss}}(P)$, or (b) they lie in the same component of $W^{\mathrm{ss}}(P) \setminus W^{\mathrm{sss}}(P)$ but in different components of $W^{\mathrm{u}}(Q) \setminus W^{\mathrm{uu}}(Q)$.

Definition 2.1 (Non-escaping contours and cycles)**.** A contour Υ of a cycle of co-index two associated to saddles P and Q is called *non-escaping* if it satisfies conditions (NE1) and (NE2). A co-index two cycle is *non-escaping* if it has some non-escaping contour.

Remark 2.2. Condition (NE2) directly addresses item (iii) in Section 2.1.1 and implicitly involves items (i) and (ii). The transverse intersection $W^{\mathrm{s}}(P) \pitchfork W^{\mathrm{u}}(Q)$ captures both $W^{\mathrm{ss}}(P)$ and $W^{\mathrm{uu}}(Q)$. Regarding the transitions in item (iv), the forward transitions from Q to P along the orbits of X_j avoid the strong stable manifold of P. Similarly, the backward transitions from P to Q along the orbits of Z_j avoid the strong unstable manifold of Q. These conditions hold open and densely in the setting under consideration.

2.2. **Toy non-escaping cycles.** We introduce a class of symbolic skew-products, whose fiber maps are defined on $\mathbb{S}^2$, having non-escaping cycles. This construction is motivated by [22]. For simplicity, and with cocycles in $\mathrm{GL}(3,\mathbb{R})$ in mind, we present the construction with fiber maps on $\mathbb{S}^2$, although it can be adapted to any surface. These constructions can be translated to the differentiable setting by replacing the symbolic part with a horseshoe.

Consider C^1 diffeomorphisms $f_0, f_1 \colon \mathbb{S}^2 \to \mathbb{S}^2$ defined on the two sphere satisfying conditions (a)-(e) below depicted in Figure 1.

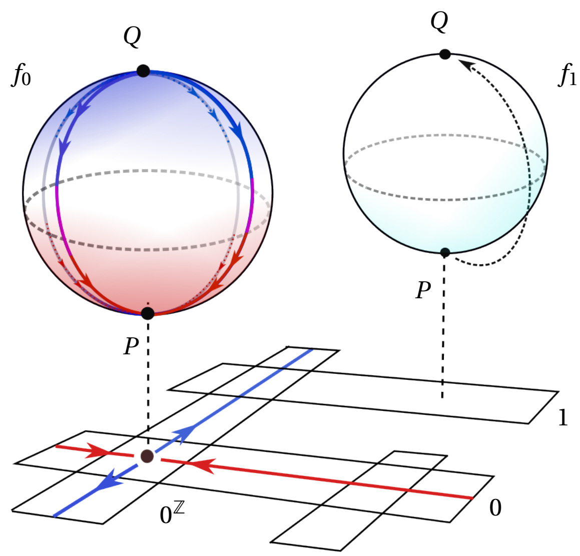

FIGURE 1. Toy non-escaping cycle

(a) f_0 has two fixed points, a global repeller Q and global attractor P,
(b) $Df_0(P)$ and $Df_0(Q)$ have real eigenvalues with different modulus.

Let E^{ss}_P and E^{cs}_P be the eigenspaces of $Df_0(P)$ where E^{ss}_P is associated to the most contracting eigenvalue. Similarly, we let E^{cu}_Q and E^{uu}_Q the eigenspaces of $Df_0(Q)$, where E^{uu}_Q is associated to the most expanding eigenvalue. Hence, there are one-dimensional strong stable and unstable foliations, denoted by $\mathscr{F}^{\mathrm{ss}}$ and $\mathscr{F}^{\mathrm{uu}}$, defined on $W^{\mathrm{s}}(P, f_0)$ and $W^{\mathrm{u}}(Q, f_0)$, respectively. In particular, the one-dimensional strong stable manifold $W^{\mathrm{ss}}(P, f_0) \subset \mathbb{S}^2$ and $W^{\mathrm{uu}}(Q, f_0) \subset \mathbb{S}^2$ are defined. Note that every point in

$$W^{\mathrm{s}}(P, f_0) \cap W^{\mathrm{u}}(Q, f_0) = \mathbb{S}^2 \setminus \{P, Q\}$$

has strong stable and unstable leaves. Note also that $W^{\mathrm{ss}}(P, f_0) \setminus \{P\}$ has two connected components $W^{\mathrm{ss}}_\ell(P, f_0)$ and $W^{\mathrm{ss}}_r(P, f_0)$. Similarly, we can write

$$W^{\mathrm{uu}}(Q, f_0) \setminus \{Q\} = W^{\mathrm{uu}}_\ell(Q, f_0) \cup W^{\mathrm{uu}}_r(Q, f_0).$$

(c) For $j = 1, 2$, there are points

$$X_j \in W^{\mathrm{uu}}(Q, f_0) \pitchfork \mathscr{F}^{\mathrm{ss}}(X_j, f_0) \quad \text{and} \quad Z_j \in W^{\mathrm{ss}}(P, f_0) \pitchfork \mathscr{F}^{\mathrm{uu}}(Z_j, f_0),$$

such that

$$X_1 \in W^{\mathrm{ss}}_{\ell}(P,f_0)\setminus W^{\mathrm{uu}}(Q,f_0), \qquad X_2 \in W^{\mathrm{ss}}_{r}(P,f_0)\setminus W^{\mathrm{uu}}(Q,f_0),$$
$$Z_1 \in W^{\mathrm{uu}}_{\ell}(Q,f_0)\setminus W^{\mathrm{ss}}(P,f_0), \qquad Z_2 \in W^{\mathrm{uu}}_{r}(Q,f_0)\setminus W^{\mathrm{ss}}(P,f_0).$$

(d) $f_1(P)=Q$.

We finally assume the transversality condition

(e) $Df_1(E^{\mathrm{ss}}_P)\oplus E^{\mathrm{uu}}_Q = T_Q\mathbb{S}^2$.

Let $\Sigma_2=\{0,1\}^{\mathbb{Z}}$. Given $\underline{\mathrm{i}}\in\Sigma_2$, we write $\underline{\mathrm{i}}=\underline{\mathrm{i}}^-.\underline{\mathrm{i}}^+=(\dots i_{-2}i_{-1}.i_0i_1\dots)$, where "." marks the 0th position, $\underline{\mathrm{i}}^-=(i_j)_{j<0}$, and $\underline{\mathrm{i}}^+=(i_j)_{j\geq 0}$. Consider the *shift map,*

$$\sigma\colon \Sigma_2\to\Sigma_2, \quad \sigma(\underline{\mathrm{i}})=\underline{\mathrm{i}}', \quad \text{where} \quad i'_j=i_{j+1} \quad \text{for every} \quad j\in\mathbb{Z}.$$

We consider the transition matrix

$$\mathfrak{M}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},$$

the subspace $\Sigma_{\mathfrak{M}}$ of Σ_2 formed by sequences without two consecutive symbols equal to 1, and the restriction $\sigma_{\mathfrak{M}}$ of σ to $\Sigma_{\mathfrak{M}}$. The skew-product map $F=F_{\mathfrak{M},\mathscr{F}}$ associated to $\sigma_{\mathfrak{M}}$ and the family $\mathscr{F}=\{f_0,f_1\}$ is defined by

$$F\colon \Sigma_{\mathfrak{M}}\times\mathbb{S}^2\to\Sigma_{\mathfrak{M}}\times\mathbb{S}^2, \quad F(\underline{\mathrm{i}},x)=(\sigma_{\mathfrak{M}}(\underline{\mathrm{i}}),f_{i_0}(x)), \quad \text{where} \quad \underline{\mathrm{i}}=(\dots i_{-1}.i_0i_1\dots). \tag{2.8}$$

This map can be thought as a partially hyperbolic one, where the subshift provides a hyperbolic part with (arbitrarily) strong contraction and expansion and the fiber maps provide the central dynamics. In this way, the points

$$\mathbf{Q}=(0^{\mathbb{Z}},Q) \qquad \text{and} \qquad \mathbf{P}=(0^{\mathbb{Z}},P) \tag{2.9}$$

are "saddles" of F with different "types of hyperbolicity" (their indices differ by two). We see that F exhibits a non-escaping cycle. Although we refrain from giving a translation of this notion for skew-products. We outline the key steps.

We first see that $\mathbf{P}$ and $\mathbf{Q}$ form "heterodimensional" cycle. For that, fix any

$$X\in\mathbb{S}^2\setminus\{Q,P\}=W^{\mathrm{u}}(Q,f_0)\cap W^{\mathrm{s}}(P,f_0)$$

and consider

$$\mathbf{X}\overset{\text{def}}{=}(0^{\mathbb{Z}},X) \qquad \text{and} \qquad \mathbf{Y}\overset{\text{def}}{=}(0^{-\mathbb{N}}.10^{\mathbb{N}},P). \tag{2.10}$$

It follows that

$$\mathbf{X}\in W^{\mathrm{u}}(\mathbf{Q},F)\cap W^{\mathrm{s}}(\mathbf{P},F) \quad \text{and} \quad \mathbf{Y}\in W^{\mathrm{s}}(\mathbf{Q},F)\cap W^{\mathrm{u}}(\mathbf{P},F), \tag{2.11}$$

obtaining the claimed cycle.

The definition of a non-escaping cycle involves a dominated structure and invariant manifolds. We adapt them to the skew-product setting. We first discuss the strongest and strong invariant sets analogous to the ones in (2.5). We focus on the point $\mathbf{P}$, the analysis for $\mathbf{Q}$ is similar. We let

$$W^{\mathrm{sss}}(\mathbf{P},F)\overset{\text{def}}{=}W^{\mathrm{s}}(0^{\mathbb{Z}},\sigma_{\mathfrak{M}})\times\{P\}, \qquad W^{\mathrm{ss}}(\mathbf{P},F)\overset{\text{def}}{=}W^{\mathrm{s}}(0^{\mathbb{Z}},\sigma_{\mathfrak{M}})\times W^{\mathrm{ss}}(P,f_0).$$

Hence

$$W^{\mathrm{ss}}(\mathbf{P},F)\setminus W^{\mathrm{sss}}(\mathbf{P},F)=W^{\mathrm{ss}}_{\ell}(\mathbf{P},F)\cup W^{\mathrm{ss}}_{r}(\mathbf{P},F)$$

where
$$W^{\mathrm{ss}}_j(\mathbf{P},F) \stackrel{\mathrm{def}}{=} W^{\mathrm{s}}(0^{\mathbb{Z}},\sigma_{\mathfrak{M}}) \times W^{\mathrm{ss}}_j(P,f_0), \qquad j=\ell,r.$$
The sets $W^{\mathrm{uuu}}(\mathbf{Q},F)$, $W^{\mathrm{uu}}(\mathbf{Q},F)$, $W^{\mathrm{uu}}_j(\mathbf{Q},F)$, $W^{\mathrm{uu}}_j(Q,f_0)$ are analogously defined[2]. This completes the discussion about invariant sets.

Given the points X_1,X_2 and Z_1,Z_1 in condition (c), we define

$$\mathbf{X}_1 \stackrel{\mathrm{def}}{=} (0^{\mathbb{Z}},X_1), \quad \mathbf{X}_2 \stackrel{\mathrm{def}}{=} (0^{\mathbb{Z}},X_2), \quad \mathbf{Z}_1 \stackrel{\mathrm{def}}{=} (0^{\mathbb{Z}},Z_1), \quad \mathbf{Z}_2 \stackrel{\mathrm{def}}{=} (0^{\mathbb{Z}},Z_2). \tag{2.12}$$

We consider the contour Υ as in (2.6) with elements $\mathbf{Q},\mathbf{P},\mathbf{X}_1,\mathbf{X}_2,\mathbf{Z}_1,\mathbf{Z}_2$ and $\mathbf{Y}$.

Transversality conditions (c) and (e) imply that it satisfies (NE1). Condition (NE2) is guaranteed by the choices of the points X_1,X_2,Z_1,Z_2 and equation (2.12).

The previous construction is summarised as follows:

Proposition 2.3. *The map F has a non-escaping cycle associated with $\mathbf{P}$ and $\mathbf{Q}$.*

2.3. **Organization and Ingredients of the proof.** To prove our results, we work on three levels: quotient two-dimensional dynamics (corresponding to the the center direction), skew-products, and diffeomorphisms. As in [11], the strategy is hierarchical, but here we place the results under a more conceptual framework. For the first two levels, we introduce preblending and split blending machines, while the extension to diffeomorphisms follows directly, since the considered skew-products reproduce the semi-local dynamics of a diffeomorphism and Asaoka's machinery in [2] applies. We also follow the approach in [25], considering a contour of the cycle and the dynamics in neighbourhoods of it.

In Section 3, we introduce a class of perturbations in a contour of the cycle leading to what we call simple contours. In a neighbourhood of them, the transverse dynamic to the two-dimensional center direction is affine and preserves the strongest contracting and expanding directions, allowing us to consider the quotient dynamics. The role of the quotient dynamics here is reminiscent of the quadratic limit dynamics arising in the renormalization of homoclinic bifurcations, see [33]. Perturbations of the diffeomorphisms that preserve the transverse dynamics are called adapted. The dynamics around the contour is then formulated in terms of skew-products.

In Section 4, we introduce a dictionary relating the quotient dynamics of the adapted perturbations to the corresponding skew-product. These relations allow us to identify strong homoclinic intersections of points with neutral derivatives, a configuration that leads to robust co-index one cycles (see [11]). A key property is that these cycles are obtained without destroying the initial co-index two cycle.

Using the quotient dynamics, in Section 5, for each central direction, we construct periodic points with neutral derivative in that direction. On the one hand, this leads to the occurrence of strong homoclinic intersections (which on its turn leads to robust cycles of co-index one). On the other hand, this allows to construct the rectangles that will serve as the base for building first preblending machines and thereafter split blending machines (see Section 6), which are the germ of the co-index two cycles and extend the blending machines in [2] to contexts with splittings with more than two bundles. These

[2]Note that in the symbolic case these sets are not connected. The corresponding translation to diffeomorphisms provides connected sets.

structures are robust and provide an abstract framework capturing the mechanism behind blenders constructed using dynamical rectangles and covering properties. In Section 6, we revisit and adapt tools in [2] for this purpose. Our approach is more flexible and considers non-hyperbolic covering properties (covering of some projections, see Remark 7.12) which may enable for further potential applications.

In Section 7, we consider one-step skew-products, whose base and fiber dynamics are given by an affine horseshoe and a family of two-dimensional local diffeomorphisms. These maps arise naturally in our context. We also introduce preblending machines, which induce split blending machines within the skew-product framework.

In Section 9, following [2], we introduce adapted transitions between preblending machines and show that they give rise to a pair of split blending machines (one machine is central attracting and the other central repelling) having a robust cycle. The proof of Theorem A is completed in Section 10.

Finally, in Section 11, we present two classes of dynamics that exhibit non-escaping cycles: robustly transitive diffeomorphisms with two-dimensional center and dynamics related to matrix cocycles in $\mathrm{GL}(3,\mathbb{R})$.

Caveat 2.4. Throughout, all perturbations considered are C^1 and arbitrarily small.

3. Simple cycles, skew-products, and quotient dynamics

Throughout this section, $f\colon M\to M$ denotes a diffeomorphism with a non-escaping contour Υ as in Section 2.1.2. We analyse the dynamics of f near Υ, for that we introduce the semi-local dynamics in Section 3.1. In Section 3.3, we prove that f admits C^1 perturbations preserving the contour, whose dynamics near the contour is affine. Following [11, Section 3.1], we refer to such contours as *simple*. Section 3.4 introduces the associated two-dimensional quotient dynamics.

3.1. **Dynamics around the cycle contour.**

Recall that the contour Υ has elements elements Q,P,X_1,X_2,Z_1,Z_2,Y see equation (2.6) and the splitting in (2.7). For notational simplicity, we write $A=X_1$, $B=X_2$, $D=Z_1$, $E=Z_2$, and $Y=C$.

3.1.1. *Local dynamics and itineraries.* Consider the set

$$\mathscr{C}\stackrel{\text{def}}{=}\{\mathtt{q},\mathtt{p},\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{e}\}, \tag{3.1}$$

and, for $\mathtt{x}\in\mathscr{C}$, neighbourhoods $U_{\mathtt{x}}$ of X and numbers $n_{\mathtt{x}}\in\mathbb{N}$, called *transition times,* with $n_{\mathtt{p}}=n_{\mathtt{q}}=1$.

Define the *local strong stable manifold* $W^{\mathrm{ss}}_{\mathrm{loc}}(P)$ *of* P as the connected component of $W^{\mathrm{ss}}(P,f)\cap U_{\mathtt{p}}$ that contains P. Similarly, the *local strong unstable manifold* $W^{\mathrm{uu}}_{\mathrm{loc}}(Q)$ *of* Q is the connected component of $W^{\mathrm{uu}}(Q,f)\cap U_{\mathtt{q}}$ that contains Q.

Remark 3.1 (Neighbourhoods and transition times)**.** After shrinking these neighbourhoods, replacing the heteroclinic points by some iterate in their orbits, and increasing the transition times, recalling conditions (NE1)-(NE2) in Section 2.1.3, we can assume that (see Figure 2)

(1) $f(U_{\mathtt{q}})\cap U_{\mathtt{p}}=\emptyset=f(U_{\mathtt{p}})\cap U_{\mathtt{q}}$,

(2) $\bigcup_{\mathtt{x}\in\{\mathtt{a},\mathtt{b},\mathtt{d},\mathtt{e}\}} U_{\mathtt{x}} \subset U_{\mathtt{q}}$ and $U_{\mathtt{c}} \subset U_{\mathtt{p}}$,
(3) $f(\bigcup_{\mathtt{x}\in\{\mathtt{a},\mathtt{b},\mathtt{d},\mathtt{e}\}} U_{\mathtt{x}}) \cap U_{\mathtt{q}} = \emptyset$ and $f(U_{\mathtt{c}}) \cap U_{\mathtt{p}} = \emptyset$,
(4) $\bigcup_{\mathtt{x}\in\{\mathtt{a},\mathtt{b},\mathtt{d},\mathtt{e}\}} f^{n_{\mathtt{x}}}(U_{\mathtt{x}}) \subset U_{\mathtt{p}}$ and $f^{n_{\mathtt{c}}}(U_{\mathtt{c}}) \subset U_{\mathtt{q}}$,
(5) $\big(f^{n_{\mathtt{a}}}(U_{\mathtt{a}}) \cup f^{n_{\mathtt{b}}}(U_{\mathtt{b}})\big) \cap W^{\mathrm{ss}}_{\mathrm{loc}}(P) = \emptyset$ and $(U_{\mathtt{d}} \cup U_{\mathtt{e}}) \cap W^{\mathrm{uu}}_{\mathrm{loc}}(Q) = \emptyset$, and
(6) the family of sets below is pairwise disjoint,

$$U_{\mathtt{p}}, U_{\mathtt{q}}, \Big\{ f^i(U_{\mathtt{j}}) \colon i \in \{1, \dots, n_{\mathtt{j}} - 1\}, \mathtt{j} \in \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{e}\} \Big\}.$$

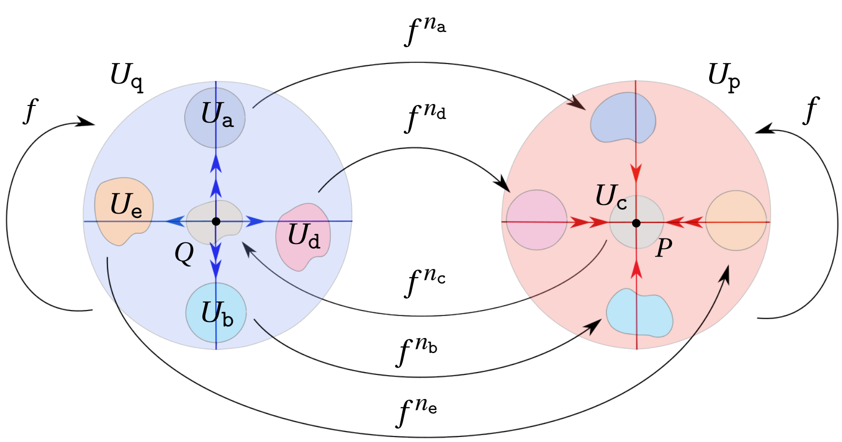

FIGURE 2. Neighbourhoods and transition times

We consider the maximal invariant set of f in a neighbourhood U_Υ of the contour Υ defined by

$$\Lambda_f(U_\Upsilon) \stackrel{\text{def}}{=} \bigcap_{i\in\mathbb{Z}} f^i(U_\Upsilon), \qquad U_\Upsilon \stackrel{\text{def}}{=} U_{\mathtt{q}} \cup U_{\mathtt{p}} \cup \left(\bigcup_{\mathtt{j}\in\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{e}\}} \left(\bigcup_{i=1}^{n_{\mathtt{j}}-1} f^i(U_{\mathtt{j}}) \right) \right). \tag{3.2}$$

3.1.2. *Symbolic spaces, shifts, and skew-products.* To study the dynamics in $\Lambda_f(U_\Upsilon)$, we consider the set $\mathscr{C}$ in (3.1) as an alphabet, the symbolic space $\Sigma_{\mathscr{C}} \stackrel{\text{def}}{=} \mathscr{C}^{\mathbb{Z}}$, and the family of local diffeomorphisms, called *local maps*,

$$\mathfrak{F} = \mathfrak{F}_{\mathscr{C}} \stackrel{\text{def}}{=} \Big\{ f_{\mathtt{x}} \stackrel{\text{def}}{=} f^{n_{\mathtt{x}}}|_{U_{\mathtt{x}}} \colon \mathtt{x} \in \mathscr{C} \Big\}. \tag{3.3}$$

For these maps, we have properties similar to the ones in Remark 3.1.

Notation 3.2. Similarly to the notation in Section 2.2, given $\underline{\mathtt{i}} \in \Sigma_{\mathscr{C}}$, we write $\underline{\mathtt{i}} = \underline{\mathtt{i}}^-.\underline{\mathtt{i}}^+$, where "." indicates the 0th position, $\underline{\mathtt{i}}^- = (i_j)_{j<0}$, and $\underline{\mathtt{i}}^+ = (i_j)_{j\geq 0}$. We endow $\Sigma_{\mathscr{C}}$ with the standard metric that makes it a compact space whose topology coincides with the one generated by cylinder sets.

We let $\Sigma^*_{\mathscr{C}} \stackrel{\text{def}}{=} \bigcup_{k\geq 1} \mathscr{C}^k$. An element $\mathtt{i} = i_0 \dots i_n \in \Sigma^*_{\mathscr{C}}$ is called a *word*. We denote by $|\mathtt{i}|$ the *length* of $\mathtt{i}$ (the number $n+1$ of letters). A word $\mathtt{j}$ of the form $\mathtt{j} = i_0 \dots i_k$ with $k \leq n$ is called a *prefix of* $\mathtt{i}$.

Consider the matrix T of possible transitions between the sets $U_{\mathtt{j}}$ by the maps in $\mathfrak{F}_{\mathscr{C}}$:

$$T \stackrel{\text{def}}{=} \big(t_{\mathtt{x},\mathtt{y}}\big), \quad \text{where} \quad t_{\mathtt{x},\mathtt{y}} = \begin{cases} 1 & \text{if } f_{\mathtt{x}}(U_{\mathtt{x}}) \cap U_{\mathtt{y}} \neq \emptyset, \\ 0 & \text{if } f_{\mathtt{x}}(U_{\mathtt{x}}) \cap U_{\mathtt{y}} = \emptyset. \end{cases} \tag{3.4}$$

Associated to T, consider the shift map $\sigma_T\colon \Sigma_T \to \Sigma_T$ and define Σ_T^* as the subset $\Sigma_{\mathscr{C}}^*$ formed by the words that can be extended to a sequence in Σ_T. The words in Σ_T^* are called *admissible*.

Notation 3.3. We use the following notation for the compositions of maps in $\mathfrak{F}$:

$$f_{\mathtt{i}} = f_{i_0\cdots i_n} \stackrel{\text{def}}{=} f_{i_n} \circ \cdots \circ f_{i_0}, \quad \text{where} \quad \mathtt{i} = i_0 \dots i_n \in \Sigma_T^*, \tag{3.5}$$

where the map $f_{\mathtt{i}}$ is defined on its maximal domain (which may be an empty set)

$$U_{\mathtt{i}} = U_{\mathtt{i},\mathfrak{F}} \stackrel{\text{def}}{=} \{X \in U_{i_0} \colon f_{i_0\cdots i_j}(X) \in U_{i_{j+1}},\, j = 0, \dots, n-1\}. \tag{3.6}$$

Given $\mathtt{i} \in \Sigma_T^*$, the $\mathtt{i}$-orbit of a point $X \in U_{\mathtt{i}}$ is defined by

$$\mathscr{O}_{\mathtt{i}}(X) \stackrel{\text{def}}{=} \{f_{\mathtt{j}}(X) \colon \mathtt{j} \text{ is a prefix of } \mathtt{i}\}. \tag{3.7}$$

We define similarly orbits of sets.

We finally define the skew-product

$$F = F_{T,\mathfrak{F}} \colon \Lambda_{T,\mathfrak{F}} \to \Lambda_{T,\mathfrak{F}}, \quad F(\underline{i}, X) = (\sigma_T(\underline{i}), f_{i_0}(X)), \tag{3.8}$$

where

$$\Lambda_{T,\mathfrak{F}} \stackrel{\text{def}}{=} \Sigma_T \times U_{\mathscr{C}} \qquad \text{with} \qquad U_{\mathscr{C}} \stackrel{\text{def}}{=} \bigcup_{\mathtt{i} \in \mathscr{C}} U_{\mathtt{i}}. \tag{3.9}$$

3.2. **Double type I contours.** The definition of type I contours of co-index one in [25] involves the so-called Shilnikov cross-form coordinates from [35] and invokes [21, Lemma 6] to obtain formulas for the composition of local maps. Here, we adapt this idea to express them in terms of local coordinates.

Definition 3.4 (Double type I contour)**.** Let f be a diffeomorphism having a non-escaping cycle associated with saddles P and Q of u-indices ℓ and $\ell+2$ with contour Υ as in Section 3.1. The contour Υ is of *double type I* if there are $\bar{A}, \bar{D} \in \Upsilon$ and local coordinates at $U_{\mathtt{p}}$ and $U_{\mathtt{q}}$ such that, for some $\rho > 1$,

(1) The local invariant manifolds of P and Q in (2.5) are of the form

$$\begin{aligned}
W^{\mathrm{u}}_{\mathrm{loc}}(P) &= \{0^{n-\ell}\} \times [-\rho,\rho]^{\ell}, & W^{\mathrm{s}}_{\mathrm{loc}}(Q) &= [-\rho,\rho]^{n-\ell-2} \times \{0^{\ell+2}\},\\
W^{\mathrm{s}}_{\mathrm{loc}}(P) &= [-\rho,\rho]^{n-\ell} \times \{0^{\ell}\}, & W^{\mathrm{u}}_{\mathrm{loc}}(Q) &= \{0^{n-\ell-2}\} \times [-\rho,\rho]^{\ell+2},\\
W^{\mathrm{ss}}_{\mathrm{loc}}(P) &= [-\rho,\rho]^{n-\ell-1} \times \{0^{\ell+1}\}, & W^{\mathrm{uu}}_{\mathrm{loc}}(Q) &= \{0^{n-\ell-1}\} \times [-\rho,\rho]^{\ell+1},\\
W^{\mathrm{sss}}_{\mathrm{loc}}(P) &= [-\rho,\rho]^{n-\ell-2} \times \{0^{\ell+2}\}, & W^{\mathrm{uuu}}_{\mathrm{loc}}(Q) &= \{0^{n-\ell}\} \times [-\rho,\rho]^{\ell},
\end{aligned}$$

(2) the points $\bar{A}, \bar{D} \in U_{\mathtt{q}}$ and $f^{n_{\mathtt{a}}}(\bar{A}), f^{n_{\mathtt{d}}}(\bar{D}) \in U_{\mathtt{p}}$ are of the form

$$\begin{aligned}
&\text{(i)} & \bar{A} &= (0^{\mathrm{sss}}, 0, a_2, a^{\mathrm{uuu}}), & a_2 &> 0, & f^{n_{\mathtt{a}}}(\bar{A}) &= (\bar{a}^{\mathrm{sss}}, \bar{a}_1, \bar{a}_2, 0^{\mathrm{uuu}}), & \bar{a}_2 &> 0,\\
&\text{(ii)} & \bar{D} &= (0^{\mathrm{sss}}, d_1, d_2, d^{\mathrm{uuu}}), & d_1 &> 0, & f^{n_{\mathtt{a}}}(\bar{D}) &= (\bar{d}^{\mathrm{sss}}, \bar{d}_1, 0, 0^{\mathrm{uuu}}), & \bar{d}_1 &> 0;
\end{aligned}$$

(3) the derivative of the transition map $f^{n_{\mathtt{c}}}$ is such that

$$\text{(i)} \quad Df^{n_{\mathtt{c}}}(C)(0^{\mathrm{sss}},1,0,0^{\mathrm{uuu}}) \cdot (0^{\mathrm{sss}},1,0,0^{\mathrm{uuu}}) > 0,$$

$$\text{(ii)} \quad Df^{n_{\mathtt{c}}}(C)(0^{\mathrm{sss}},0,1,0^{\mathrm{uuu}}) \cdot (0^{\mathrm{sss}},0,1,0^{\mathrm{uuu}}) > 0.$$

To Υ we associate the integer $\iota(\Upsilon) = \ell$ where ℓ is the u-index of P.

Lemma 3.5. *Given any neighbourhood U of a non-escaping contour of a diffeomorphism f, there exists an arbitrarily small perturbation g of f with a non-escaping cycle associated to the saddles of the initial one having a double type I contour contained in U.*

Proof. Recall the heteroclinic points A, B, C, D of a non-escaping cycle as in Definition 2.1.

We can select local charts η_P and η_Q satisfying (1). These points fall into the cases (2)(i) (A or B) or (2)(ii) (D or E), but their coordinate signs are not a priori yet determined. We now show how to control them.

In these coordinates, consider the central involutions

$$\iota_1(x^{\mathrm{s}}, x_1, x_2, x^{\mathrm{u}}) \mapsto (x^{\mathrm{s}}, -x_1, x_2, x^{\mathrm{u}}), \qquad \iota_2(x^{\mathrm{s}}, x_1, x_2, x^{\mathrm{u}}) \mapsto (x^{\mathrm{s}}, x_1, -x_2, x^{\mathrm{u}}).$$

The conditions in (1) remain valid after replacing η_P with $\eta_P \circ \iota_i$ and η_Q with $\eta_Q \circ \iota_i$.

The analysis of conditions (2)(i), (3)(i) (involving A, B and the center coordinate x_2) and conditions (2)(ii), (3)(ii) (involving D, E and the coordinate x_1) are independent, and can be solved using the involutions ι_2 and ι_1, respectively. Hence we focus only on case (i). In the chosen coordinates, write

$$X = (0^{\mathrm{sss}}, 0, x_2, x^{\mathrm{uuu}}), \qquad f^{n_{\mathsf{x}}}(X) = (\bar{x}^{\mathrm{sss}}, \bar{x}_1, \bar{x}_2, 0^{\mathrm{uuu}}), \quad X \in \{A, B\}.$$

The involutions ι_2 allows us to assume that for instance $a_2 > 0$, and that $Df^{n_{\mathsf{c}}}$ satisfies (3)(i). There are four possible cases:

(1) $\bar{a}_2 > 0$.
(2) $\bar{a}_2 < 0$, $b_2 > 0$, $\bar{b}_2 > 0$.
(3) $\bar{a}_2 < 0$, $b_2 < 0$, $\bar{b}_2 > 0$.
(4) $\bar{a}_2 < 0$, $b_2 < 0$, $\bar{b}_2 < 0$.

In case (1), take $\bar{A} = A$. In case (2), take $\bar{A} = B$. In case (4), replace η_P and η_Q by $\eta_P \circ \iota_2$ and $\eta_Q \circ \iota_2$ and then take $\bar{A} = B$ (note that this does not modify (3)(i)). It remains to consider case (3), where we need to consider a new contour and perhaps doing a perturbation: it is enough to consider double round points (are depicted in Figure 3) in a small neighbourhood of A having, in the local coordinates, a return of the form

$$f_{\mathsf{b}} f_{\mathsf{q}}^m f_{\mathsf{c}} f_{\mathsf{p}}^n f_{\mathsf{a}},$$

where m, n are chosen such that admissible domain $U_{\mathsf{ap}^n\mathsf{cq}^m\mathsf{b}} \neq \emptyset$ (hence $U_{\mathsf{ap}^n\mathsf{cq}^m} \neq \emptyset$), which may involve local perturbations f_{p} and f_{q}. This allows to take a new heteroclinic point $\bar{A} = A'$ having a return $n_{\mathsf{a}'} = \mathsf{ap}^n\mathsf{cq}^m\mathsf{b}$ satisfying (2)(i).

Note that all previous conditions do not change after introducing the involution ι_1 in the coordinates. This implies that the analysis for the points D, E can be performed independently. The proof is now complete. □

In view of Lemma 3.5, in what follows we will deal with double type I contours, considering only the points $\bar{A} = A$ and $\bar{D} = D$ and the corresponding transitions. Then, we remove from the alphabet $\mathscr{C}$ the symbols b and e, as well as, the corresponding transition maps. We still denote this alphabet by $\mathscr{C}$ and the family of local maps by $\mathfrak{F}$.

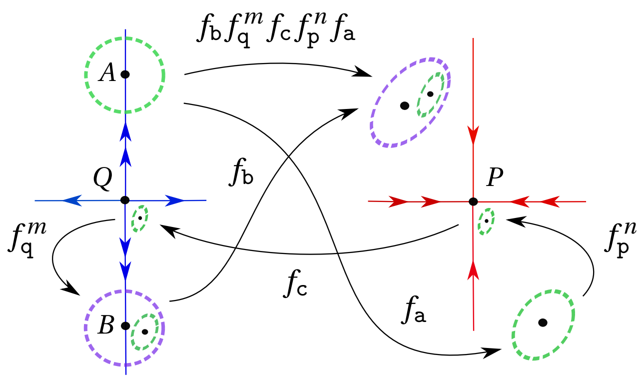

FIGURE 3. Case (3)

3.3. **Simple contours.** The arguments in this section are standard in the C^1 topology and follow [11, Section 3.1], hence the details are omitted. Our first result is a translation of [11, Proposition 3.5]. The only difference is that here we consider two transverse heteroclinic points in $W^{\mathrm{s}}(P) \pitchfork W^{\mathrm{u}}(Q)$ whose orbits can be analysed independently and also deal with two central directions (indeed, this is not an issue since in [11, Proposition 3.5] this question is also addressed).

Lemma 3.6 (Simple double type I contours)**.** *Let Υ be a double type I contour of a cycle of f whose elements are Q, P, A, C, D. After replacing the heteroclinic points A, C, D for points in their orbits, shrinking the neighbourhoods $U_{\mathtt{x}}$, $\mathtt{x} \in \mathscr{C}$, and increasing the numbers $n_{\mathtt{x}}$, $\mathtt{x} \in \mathscr{C} \setminus \{\mathtt{p}, \mathtt{q}\}$, there is a perturbation g of f having the same double type I contour Υ whose local maps $g_{\mathtt{x}}$, $\mathtt{x} \in \mathscr{C}$, preserve the splitting $E^{\mathrm{sss}} \oplus E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2 \oplus E^{\mathrm{uuu}}$ in (2.4) and such that, in the local coordinates in $U_{\mathtt{p}}$ and $U_{\mathtt{q}}$:*

(1) *$g_{\mathtt{p}}$ and $g_{\mathtt{q}}$ are linear in $U_{\mathtt{p}}$ and $U_{\mathtt{q}}$;*
(2) *for $\mathtt{x} = \mathtt{a}, \mathtt{c}, \mathtt{d}$, the maps $g_{\mathtt{x}} \colon U_{\mathtt{x}} \to g_{\mathtt{x}}(U_{\mathtt{x}})$ are affine and their derivatives restricted to $E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2$ are of the form*

$$Dg_{\mathtt{a}}(A) = \begin{pmatrix} \epsilon_{\mathtt{a}} & 0 \\ 0 & \pm 1 \end{pmatrix}, \qquad Dg_{\mathtt{d}}(D) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \epsilon_{\mathtt{d}} \end{pmatrix}, \qquad Dg_{\mathtt{c}}(C) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $\epsilon_{\mathtt{a}}, \epsilon_{\mathtt{d}}, a_{11}, a_{22} \neq 0$.

Following [11, Definition 3.4], we call a cycle as in Lemma 3.6 *simple double type I*. To avoid heavy notation, in what follows we call these cycles just simple.

It follows the sketch of the proof of the lemma. The first step is to linearize (after a perturbation) in neighbourhoods of Q and P. By shrinking the neighbourhoods $U_{\mathtt{q}}$ and $U_{\mathtt{p}}$, we may assume that the maps $f_{\mathtt{q}}$ and $f_{\mathtt{p}}$ are both linear.

The second step uses domination. In the neighbourhoods $U_{\mathtt{q}}$ and $U_{\mathtt{p}}$, we consider the invariant splittings given by the derivatives. We replace the point A by a point of the form $A' = f^{-i}(A)$ and redefine $n_{\mathtt{a}}$ as $n_{\mathtt{a}'} \stackrel{\text{def}}{=} i + n_{\mathtt{a}} + j$, for suitably large i and j. After a new perturbation, we can assume that the linearizing splitting in $U_{\mathtt{q}}$ is mapped into the linearizing splitting in $U_{\mathtt{p}}$. Adjusting the values of i and j, and taking into account that

- Df^i expands E^{c}_2 in $U_{\mathtt{q}}$,
- $Df^{n_{\mathtt{a}}}$ has derivative along E^{c}_2 bounded above and below, and
- Df^j contracts E^{c}_2 in $U_{\mathtt{p}}$,

we can choose i and j such that the total derivative along E^{c}_2 has modulus close to one. As the number of iterations can be done arbitrarily large, a final small perturbation yields derivative of modulus one. From this point on, we replace A by A' and $n_{\mathtt{a}}$ by $n_{\mathtt{a}'}$. The argument for the other points proceeds in a similar way. The fact that in the derivative of $Dg_{\mathtt{c}}(0,0)$ the numbers $a_{11}, a_{22} \neq 0$ follows from the transversality conditions. This completes our sketch.

Remark 3.7. In the case of diffeomorphisms, the matrix corresponding to $Dg_{\mathtt{c}}$ can be chosen diagonal. Aiming applications to skew-products, we stated the previous, a bit more general, version. Note that in the skew-product, the fiber component of the orbit of the heteroclinic point going from P to Q is finite. Therefore, the previous argument can not applied.

3.4. **Quotient dynamics.** Recall the alphabet $\mathscr{C}$ in (3.1) and the local maps $f_{\mathtt{x}}$ in (3.3). In what follows, we assume the contour Υ of f is simple cycle. For each $\mathtt{x} \in \mathscr{C}$, consider the quotient of the neighbourhood $\mathtt{U}_{\mathtt{x}}$ of $U_{\mathtt{x}}$ by the $E^{\mathrm{ss}} \oplus E^{\mathrm{uu}}$, that we identify with subsets of $\mathbb{R}^2$. The quotient points are denoted by $\mathtt{Q}, \mathtt{P}, \mathtt{A}, \mathtt{C}, \mathtt{D}$. We denote by $\varphi_{\mathtt{x}}$ the quotient maps induced by $f_{\mathtt{x}}$ and consider the quotient family of $\mathfrak{F}$,

$$\mathscr{F} = \{\varphi_{\mathtt{x}} \colon \mathtt{x} \in \mathscr{C}\}, \qquad \varphi_{\mathtt{x}} \colon \mathtt{U}_{\mathtt{x}} \to \mathbb{R}^2. \tag{3.10}$$

By Lemma 3.6, the maps in $\mathscr{F}$ have the following form:

(QD1) *Local dynamics at* $\mathtt{P}$ *and* $\mathtt{Q}$. The points $\mathtt{Q}$ and $\mathtt{P}$ are identified with $(0,0)$ and $\varphi_{\mathtt{q}} \colon \mathtt{U}_{\mathtt{q}} \to \mathbb{R}^2$ and $\varphi_{\mathtt{p}} \colon \mathtt{U}_{\mathtt{p}} \to \mathbb{R}^2$, are linear maps

$$\varphi_{\mathtt{q}}(x,y) = (\beta_1\, x, \beta_2\, y) \qquad \text{and} \qquad \varphi_{\mathtt{p}}(x,y) = (\alpha_1\, x, \alpha_2\, y), \tag{3.11}$$

where $0 < |\alpha_1| < |\alpha_2| < 1 < |\beta_1| < |\beta_2|$.

(QD2) *Transverse heteroclinic transitions.* We can assume that

$$\mathtt{A} = (0,1) \in \mathtt{U}_{\mathtt{a}} \subset \mathtt{U}_{\mathtt{q}} \qquad \text{and} \qquad \varphi_{\mathtt{a}}(\mathtt{A}) \stackrel{\text{def}}{=} \mathtt{A}' = (0,1) \in \mathtt{U}_{\mathtt{p}}, \tag{3.12}$$

and that $\varphi_{\mathtt{a}} \colon \mathtt{U}_{\mathtt{a}} \to \mathtt{U}_{\mathtt{p}}$ is an affine map of the form

$$\varphi_{\mathtt{a}}(x, 1+y) = (\epsilon_{\mathtt{a}} x, 1 \pm y), \quad \text{where} \quad \epsilon_{\mathtt{a}} \neq 0. \tag{3.13}$$

We let $\mathtt{D}' = \varphi_{\mathtt{d}}^{-1}(\mathtt{D}) = (1,0)$ and $\mathtt{U}'_{\mathtt{d}} = \varphi_{\mathtt{d}}^{-1}(\mathtt{U}_{\mathtt{d}})$. We assume similar conditions for the transition $\varphi_{\mathtt{d}}^{-1} \colon \mathtt{U}'_{\mathtt{d}} \to \mathtt{U}_{\mathtt{p}}$.

(QD3) *Nontransverse heteroclinic transitions.* In the local charts, the point $\mathtt{C}$ is identified with $(0,0)$ and the cycle map $\varphi_{\mathtt{c}} \colon \mathtt{U}_{\mathtt{c}} \to \mathtt{U}_{\mathtt{q}}$ is given by

$$\varphi_{\mathtt{c}}(x,y) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad a_{11}, a_{22} > 0. \tag{3.14}$$

Remark 3.8. The family $\mathfrak{F} = \{f_{\mathtt{x}} \colon \mathtt{x} \in \mathscr{C}\}$ and its quotient family $\mathscr{F} = \{\varphi_{\mathtt{x}} \colon \mathtt{x} \in \mathscr{C}\}$ have the same transition matrix T defined in (3.4).

4. SKEW-PRODUCTS AND QUOTIENT DYNAMICS

Throughout this section, f denotes a diffeomorphism with a simple contour as in Section 3.3. We begin by defining a special type of perturbation of f and its associated skew-product (see Section 4.1). In Section 4.2, we derive properties of the dynamics of diffeomorphisms from those of the skew-product. Using these relations and strong homoclinic points, in Section 4.3, we state sufficient conditions for the occurrence of robust cycles of co-index one.

4.1. **Adapted perturbations.** Recall the family $\mathfrak{F}$ in (3.3), the point C in Section 3.1, and the alphabet $\mathscr{C}$ in (3.1).

Definition 4.1. An *adapted perturbation of f with respect to $\mathfrak{F}$* is a C^1 perturbation g of f such that:

- g and f coincide on the set $\Upsilon \setminus \bigcup_{i\in\mathbb{Z}} f^i(C)$;
- g coincides with f outside a neighbourhood of Υ;
- g preserves the foliations $\mathscr{F}^{\mathrm{sss}}$ and $\mathscr{F}^{\mathrm{uuu}}$ in a neighbourhood of Υ; and
- for every $\mathtt{x} \in \mathscr{C}$, the quotient map $\psi_{\mathtt{x}}$ of $g_{\mathtt{x}}$ (defined in the obvious way) is a C^1 perturbation of $\varphi_{\mathtt{x}}$.

Moreover, when g coincides with f in the whole set Υ, we say that *preserves the contour*, and hence the cycle which is also of double type I (but not simple, since affinity in the central direction may be lost).

For an adapted perturbation g of f with respect to $\mathfrak{F}$ (in what follows we will omit this dependence) we consider the family $\mathfrak{G} = \{g_{\mathtt{x}} \colon \mathtt{x} \in \mathscr{C}\}$, where $g_{\mathtt{x}}$ is defined in the obvious way, the sets $\Lambda_{T,\mathfrak{G}}$, and the skew-product $F_{T,\mathfrak{G}}$ defined as in (3.9) and (3.8). We also consider the maximal g-invariant set in U_Υ,

$$\Lambda_g(U_\Upsilon) \stackrel{\text{def}}{=} \bigcap_{i\in\mathbb{Z}} g^i(U_\Upsilon). \tag{4.1}$$

Since g preserves the splitting, it allows us to consider the corresponding quotient family

$$\mathscr{G} = \mathscr{G}_g = \{\psi_{\mathtt{x}} \colon \mathtt{x} \in \mathscr{C}\}, \quad \text{with} \quad \psi_{\mathtt{x}} \colon \mathtt{U}_{\mathtt{x},\mathscr{G}} \to \mathbb{R}^2. \tag{4.2}$$

By construction, the quotient maps satisfy

$$\psi_{\mathtt{x}}(\mathtt{X}) = \varphi_{\mathtt{x}}(\mathtt{X}), \qquad \text{for every} \quad \mathtt{x} \in \mathscr{C}.$$

For simplicity, and with a slight abuse of notation, we say that the family $\mathscr{G}$ *is an adapted perturbation of the family $\mathscr{F}$ in* (3.10). Also, again with an abuse of notation, we denote the sets $\mathtt{U}_{\mathtt{x},\mathscr{G}}$ in (4.2) corresponding to $\mathscr{G}$ by $\mathtt{U}_{\mathtt{x}}$.

Consider the matrix T in (3.4), similarly as in (3.9) and (3.8), we consider the set

$$\Gamma_{T,\mathscr{G}} \stackrel{\text{def}}{=} \{(\underline{\mathtt{i}},\mathtt{X}) \in \Sigma_T \times \mathtt{U}_{\mathscr{C}} \colon \mathtt{X} \in \mathtt{U}_{i_0}\}, \qquad \text{where} \qquad \mathtt{U}_{\mathscr{C}} \stackrel{\text{def}}{=} \bigcup_{\mathtt{i}\in\mathscr{C}} \mathtt{U}_{\mathtt{i}}, \tag{4.3}$$

and the skew-product

$$\Psi = \Psi_{T,\mathscr{G}} \colon \Gamma_{T,\mathscr{G}} \to \Sigma_T \times \mathtt{U}_{\mathscr{C}}, \quad \Psi(\underline{\mathtt{i}},\mathtt{X}) = (\sigma_T(\underline{\mathtt{i}}), \psi_{i_0}(\mathtt{X})). \tag{4.4}$$

The fiber dynamics of this skew-product corresponds to the model families in [11, Section 3.2][3]. We will now focus on its study.

4.2. **A dictionary between quotient and global dynamics.** In what follows, g stands for an adapted perturbation of f as in the previous section. We derive relations between the dynamics of g and those of its associated skew-product Ψ, whose fiber maps are denoted by $\psi_{\mathtt{j}}$, with $\mathtt{j} \in \Sigma_T^*$.

Remark 4.2. A periodic point $(\underline{\mathtt{i}}, \mathtt{X})$ of Ψ is *hyperbolic* if $\underline{\mathtt{i}} = \mathtt{i}^{\mathbb{Z}}$ for some word $\mathtt{i} \in \Sigma_T^*$, $\psi_{\mathtt{i}}(\mathtt{X}) = \mathtt{X}$, and all eigenvalues of $D\psi_{\mathtt{i}}(\mathtt{X})$ are different from ± 1. Note that, due to domination, these eigenvalues are real and have distinct moduli. This point may be attracting, a saddle, or repelling. We denote by $W^*_{\mathrm{loc}}(\mathtt{A}, \psi_{\mathtt{i}})$, $* = \mathrm{s}, \mathrm{u}$, the corresponding local invariant manifolds of $\mathtt{A}$. If $\mathtt{A}$ is attracting, then $W^{\mathrm{s}}_{\mathrm{loc}}(\mathtt{A}, \psi_{\mathtt{i}})$ is a disk and $W^{\mathrm{u}}_{\mathrm{loc}}(\mathtt{A}, \psi_{\mathtt{i}}) = \{\mathtt{A}\}$, and from the previous comments, $W^{\mathrm{ss}}_{\mathrm{loc}}(\mathtt{A}, \psi_{\mathtt{i}})$ is a well defined curve. The analogous description holds for saddle and repelling points.

We can also define heteroclinic and homoclinic intersections for hyperbolic periodic points. Specific examples of these relations were given in Section 2.2. The remark bellow follows as in [17, Section 2.1] and hence details are omitted.

Remark 4.3 (Hetero and homoclinic relations)**.** Let $(\mathtt{i}^{\mathbb{Z}}, \mathtt{A})$ and $(\mathtt{j}^{\mathbb{Z}}, \mathtt{B})$ be hyperbolic periodic points of Ψ such that there exists an admissible word $\mathtt{r}$ with

$$\mathtt{A} \in \mathtt{U}_{\mathtt{r}} \qquad \text{and} \qquad \psi_{\mathtt{r}}(\mathtt{A}) \in W^{\mathrm{s}}_{\mathrm{loc}}(\mathtt{B}, \psi_{\mathtt{j}}). \tag{4.5}$$

Then, exactly as in (2.10) and (2.11),

$$(\mathtt{i}^{-\mathbb{N}}.\mathtt{r}\mathtt{j}^{\mathbb{N}}, \mathtt{A}) \in W^{\mathrm{u}}\big((\mathtt{i}^{\mathbb{Z}}, \mathtt{A}), \Psi\big) \cap W^{\mathrm{s}}\big((\mathtt{j}^{\mathbb{Z}}, \mathtt{B}), \Psi\big). \tag{4.6}$$

This allows us to define homoclinic relations (again, no transversality is required, transversality-like properties follow from the dominated splitting of the central bundle). Note (4.6) provides intersections in the set $\Gamma_{T,\mathscr{G}}$ (see equation (4.3)).

When the points have different u-indices this leads to heterodimensional cycles, which may have co-index one or two.

Given g as above, the following two remarks establish a dictionary between its quotient, its skew-product, and its dynamics.

Remark 4.4. To each point in $(\underline{\mathtt{i}}, \mathtt{X}) \in \Gamma_{T,\mathscr{G}}$ corresponds a point in $X = X_{(\underline{\mathtt{i}}, \mathtt{X})} \in \Lambda_g(U_\Upsilon)$ with the "same itinerary" (we refrain to formalize this). If $(\underline{\mathtt{i}}, \mathtt{X})$ is periodic and hyperbolic, the same holds for X, where its u-index is the sum of $\dim E^{\mathrm{uuu}}$ and the u-index of $\mathtt{X}$. Intersections between the invariant sets of the skew-product translate to the same intersection for g, where the intersection points are contained in $\Lambda_g(U_\Upsilon)$.

Remark 4.5 (Strongest and strong manifolds)**.** Let $A = A_{(\mathtt{i}^{\mathbb{Z}}, \mathtt{A})}$ be a periodic point of g such that $(\mathtt{i}^{\mathbb{Z}}, \mathtt{A})$ is central repelling for Ψ. Then each connected component of $W^{\mathrm{uu}}_{\mathrm{loc}}(A, g) \setminus W^{\mathrm{uuu}}_{\mathrm{loc}}(A, g)$ projects (by $\mathscr{F}^{\mathrm{uuu}}$) into one component of $W^{\mathrm{uu}}_{\mathrm{loc}}(\mathtt{A}, \psi_{\mathtt{i}}) \setminus \{\mathtt{A}\}$. Similarly, if $(\mathtt{i}^{\mathbb{Z}}, \mathtt{A})$ is central repelling then each component of $W^{\mathrm{ss}}_{\mathrm{loc}}(A, g) \setminus W^{\mathrm{sss}}_{\mathrm{loc}}(A, g)$ projects (by $\mathscr{F}^{\mathrm{sss}}$) into one component of $W^{\mathrm{ss}}_{\mathrm{loc}}(\mathtt{A}, \psi_{\mathtt{i}}) \setminus \{\mathtt{A}\}$.

[3]We chose to follow the skew-product approach, which is somewhat more abstract and general.

4.3. **Strong homoclinic intersections and robust cycles of co-index one.** We recall the definition of *strong homoclinic points* from [11, Definition 3.2]. Given a nonhyperbolic periodic point S with a splitting $E^{ss}\oplus E^{c}\oplus E^{uu}$, where E^{ss} and E^{uu} are uniformly contracting and expanding and E^{c} is one-dimensional, we can consider the strong manifolds $W_{E^{ss}}(S)$ and $W_{E^{uu}}(S)$ of S tangent to E^{ss} and E^{uu} (we use this notation to prevent overlap with the notation in (2.5), this interference wil appear below). A point $X\in W_{E^{ss}}(S)\cap W_{E^{uu}}(S)$ such that $T_XW_{E^{ss}}(S)\oplus T_XW_{E^{uu}}(S)$ is called a *quasi-transverse strong homoclinic point of S*. The relevance of this configuration comes from [11, Theorem 2.4]:

Lemma 4.6 (Robust co-index one cycles from strong homoclinic intersection)**.** *Let h be a diffeomorphism with a quasi-transverse strong homoclinic intersection X associated with a nonhyperbolic periodic point S. Then, for every neighbourhood V of the closure of the orbit of X, there exists a C^1 perturbation of h with C^1 robust heterodimensional cycles of co-index one and type $(\ell,\ell+1)$ whose contours are contained in V, where $\ell=$* $\dim E^{uu}(S)$.

We now explain how strong homoclinic points arise in our context and are detected using the skew-product. First, recall the dominated splitting $E^{sss}\oplus E^{c}_1\oplus E^{c}_2\oplus E^{uuu}$ from (2.4), together with the strong invariant manifolds $W^{sss}(S)$, $W^{uuu}(S)$, in (2.5). Recall also the definition of $\iota(\Upsilon)$ in Definition 3.4.

Remark 4.7. There are the following cases for a nonhyperbolic periodic point S:

(1) If $E^{c}_2(S)$ is nonhyperbolic (thus $E^{c}_1(S)$ is contracting), then $E^{ss}=E^{sss}\oplus E^{c}_1$, $E^{c}=E^{c}_2$, $E^{uu}=E^{uuu}$ and thus $W_{E^{uu}}(S)=W^{uuu}(S)$. Hence if S has a quasi-transverse strong homoclinic point it generates robust cycles of type $(\iota(\Upsilon),\iota(\Upsilon)+1)$.
(2) If $E^{c}_1(S)$ is nonhyperbolic (thus $E^{c}_2(S)$ is expanding), then $E^{ss}=E^{sss}$ and thus $W_{E^{ss}}(S)=W^{sss}(S)$, $E^{c}=E^{c}_1$, and $E^{uu}=E^{c}_2\oplus E^{uuu}$. Hence if S has a quasi-transverse strong homoclinic point it generates robust cycles of type $(\iota(\Upsilon)+1,\iota(\Upsilon)+2)$.

We now focus on points $A=A_{(\mathtt{i}^{\mathbb{Z}},\mathtt{A})}$ in case (1) above, case (2) is analogous.

Lemma 4.8. *Let $(\mathtt{i}^{\mathbb{Z}},\mathtt{A})$ be a periodic point of Ψ such that the eigenvalues of $D\psi_{\mathtt{i}}(\mathtt{A})$ have modulus one and less than one (hence $W^{ss}_{loc}(\mathtt{A},\psi_{\mathtt{i}})$ is well defined and one-dimensional) and there is an admissible word $\mathtt{r}$ such that*

$$\psi_{\mathtt{r}}(\mathtt{A})\in W^{ss}_{loc}(\mathtt{A},\psi_{\mathtt{i}}),\quad \mathtt{r}\ \textit{is not a prefix of}\ \mathtt{i}.$$

Then there is a perturbation h of g having a robust heterodimensional cycle of type $(\iota(\Upsilon),\iota(\Upsilon)+1)$ *associated with sets contained in U_Υ.*

Proof. Let $A=A_{(\mathtt{i}^{\mathbb{Z}},\mathtt{A})}$ and note that it is a nonhyperbolic periodic point of g. Let

$$X=X_{(\mathtt{i}^{-\mathbb{N}}.\mathtt{r}\mathtt{i}^{\mathbb{N}},\mathtt{A})}\in\Lambda_g(U_\Upsilon).$$

Recalling that $W^{uuu}(A,g)$ projects into points, adapting Remarks 4.3 and 4.4, and using Remark 4.7, we get that X is a quasi-transverse strong homoclinic point of A. Lemma 4.6 implies the assertion after noting that the final perturbation is local around the quasi-transverse point and hence can be done preserving the co-index two cycle of g. □

5. Periodic points with neutral directions and consequences

Using the quotient dynamics from the previous sections, we construct adapted perturbations that preserve the contour and generate periodic points with neutral derivative in the central direction E^{c}_1 (close to ± 1) within a neighbourhood of the contour. An analogous family with neutral derivative in the E^{c}_2 direction also arises; see Proposition 5.1. The proof of this proposition begins in Section 5.1 and is completed in Section 5.2.

For simplicity, we introduce a compact notation for words

$$\mathtt{i}_{(m,n)} \stackrel{\text{def}}{=} \mathtt{a}\mathtt{p}^m \mathtt{c}\mathtt{q}^n \qquad \text{and} \qquad \mathtt{j}_{(r,t)} \stackrel{\text{def}}{=} \mathtt{p}^r \mathtt{c}\mathtt{q}^t \mathtt{d}. \tag{5.1}$$

Proposition 5.1 (Neutral periodic points and strong homoclinic intersections)**.** *Let f be diffeomorphism with a simple contour and consider its quotient family. Then for every $\ell_0 \in \mathbb{N}$, there exists an arbitrarily small adapted perturbation g of f, which preserves the contour, and whose associated quotient family $\mathscr{G}$ satisfies the following properties:*

For $k = 1, \dots, 4$ there are natural numbers $m_k, n_k, r_k, t_k > \ell_0$ (increasing with k), words $\mathtt{i}_k = \mathtt{i}_{(m_k,n_k)}, \mathtt{j}_k = \mathtt{j}_{(r_k,t_k)} \in \Sigma^*_T$, and points $\mathtt{A}_{\mathtt{i}_k}, \mathtt{D}'_{\mathtt{j}_k}$ such that, for $k = 1,2,3$

$$\mathtt{A}_{\mathtt{i}_k} = (a_{\mathtt{i}_k}, 1) \to \mathtt{A} = (0,1) \in \mathtt{U}_\mathtt{q}, \quad \textit{and} \quad \mathtt{D}'_{\mathtt{j}_k} = (1, d_{\mathtt{j}_k}) \to \mathtt{D} = (1,0) \in \mathtt{U}_\mathtt{p},$$

when $m_k, n_k, r_k, t_k \to \infty$, and

$$\begin{aligned} &\psi_{\mathtt{i}_k}(\mathtt{A}_{\mathtt{i}_k}) = \mathtt{A}_{\mathtt{i}_k} \quad \textit{with} \quad D\psi_{\mathtt{i}_k}(\mathtt{A}_{\mathtt{i}_k}) = \begin{pmatrix} \theta_{\mathtt{i}_k} & 0 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \theta_{\mathtt{i}_k} \to 0, \\ &\psi_{\mathtt{j}_k}(\mathtt{D}'_{\mathtt{j}_k}) = \mathtt{D}_{\mathtt{j}_k} \quad \textit{with} \quad D'\psi_{\mathtt{j}_k}(\mathtt{D}'_{\mathtt{j}_k}) = \begin{pmatrix} 1 & 0 \\ 0 & \kappa_{\mathtt{j}_k} \end{pmatrix} \quad \textit{and} \quad \kappa_{\mathtt{j}_k} \to \infty. \end{aligned} \tag{5.2}$$

Moreover,

$$\psi_{\mathtt{i}_4}(\mathtt{A}_{\mathtt{i}_1}) \in W^{\mathrm{ss}}_{\mathrm{loc}}(\mathtt{A}_{\mathtt{i}_1}, \psi_{\mathtt{i}_1}), \quad \textit{and} \quad (\psi_{\mathtt{j}_4})^{-1}(\mathtt{D}'_{\mathtt{j}_1}) \in W^{\mathrm{uu}}_{\mathrm{loc}}(\mathtt{D}'_{\mathtt{j}_1}, \psi_{\mathtt{j}_1}). \tag{5.3}$$

Observe that the conditions in (5.2) allow us to consider the strong manifolds in (5.3).

This proposition has two key consequences. First, in Section 5.4, using the points $A_{\mathtt{i}_1}$ and $D'_{\mathtt{j}_1}$ together with their strong homoclinic intersections obtained in (5.3), we prove Theorem 5.10, which establishes the existence of the pair of robust co-index one cycles in Theorem A. Second, in Section 5.5, we derive a family of preliminary rectangles from the points $A_{\mathtt{i}_2}, A_{\mathtt{i}_3}, D'_{\mathtt{j}_2}, D'_{\mathtt{j}_3}$. These rectangles will later serve as the basis for the preblending machines built in Section 10.1.1.

Remark 5.2. There are the following estimates for $\theta_{\mathtt{i}_k}$ and $\kappa_{\mathtt{j}_k}$ in (5.2)[4]

$$|\theta_{\mathtt{i}_k}| \approx |\alpha_1^{m_k} \beta_1^{n_k}|, \qquad |\kappa_{\mathtt{j}_k}| \approx |\alpha_2^{r_k} \beta_2^{t_k}|.$$

The estimate for $|\theta_{\mathtt{i}_k}|$ follow from (5.7) and (5.8) in the proof of the proposition. The estimates for $|\kappa_{\mathtt{j}_k}|$ follows symmetrically. We also have,

$$|a_{\mathtt{i}_k}| \approx \left(\frac{|\beta_1|}{|\beta_2|} \right)^{n_k}, \qquad |d_{\mathtt{j}_k}| \approx \left(\frac{|\alpha_1|}{|\alpha_2|} \right)^{r_k}.$$

The estimate for $a_{\mathtt{i}_k}$ follows from (5.9). The estimates for $|d_{\mathtt{j}_k}|$ follows symmetrically.

[4] The notation $a(n) \approx b(n)$ means that two quantities are of the same order as $n \to \infty$, that is, $\lim_{n\to\infty} \frac{a(n)}{b(n)}$ exists and is different from 0.

5.1. **Periodic points with neutral directions.** The main step in the proof of Proposition 5.1 is the next lemma:

Lemma 5.3. *For every $\ell_0 \in \mathbb{N}$, there exists an arbitrarily small adapted perturbation g of f such that there are admissible words $\mathtt{i} = \mathtt{i}(m,n)$, with $|\mathtt{i}| > \ell_0$, and points $\mathtt{A}_\mathtt{i} = (a_\mathtt{i}, 1) \to \mathtt{A} = (0,1)$ when $|\mathtt{i}| \to \infty$, with $\psi_\mathtt{i}(\mathtt{A}_\mathtt{i}) = \mathtt{A}_\mathtt{i}$. Moreover, the derivative satisfies*

$$D\psi_\mathtt{i}(\mathtt{A}_\mathtt{i}) = \begin{pmatrix} \delta^1_\mathtt{i} & \delta^2_\mathtt{i} \\ \delta^3_\mathtt{i} & 1 \end{pmatrix}, \quad \text{where} \quad \delta^j_\mathtt{i} \to 0 \quad \text{as} \quad |\mathtt{i}| \to \infty. \tag{5.4}$$

The proof of this lemma relies on the iterations of the point $\mathtt{A}$ near $\mathtt{P}$ and $\mathtt{Q}$ under $\varphi_\mathtt{p}$ and $\varphi_\mathtt{q}$, see equation (3.11). The construction involves a sequence of adapted perturbations g of f, standard in the C^1 topology (see [12, Section 4.1.2] for details). We denote the quotient maps of these perturbations by $\mathscr{G}$, keeping the same notation after each perturbation.

Before going into details, we sketch a heuristic proof of the lemma. Here, the point $\mathtt{A}$ serves as the "base". To find fixed points of $\psi_{\mathtt{ap}^m\mathtt{cq}^n}$ close to $\mathtt{A}$ we consider large $m, n \in \mathbb{N}$, and returns of boxes centered at $\mathtt{A}$:

$$\Delta_{\mathtt{A},\varrho} = \mathtt{A} + \Delta_\varrho \subset \mathtt{U}_\mathtt{q}, \quad \Delta_\varrho = [-\varrho, \varrho]^2. \tag{5.5}$$

After adjusting the eigenvalues of $\psi_\mathtt{p}$ and $\psi_\mathtt{q}$, an adapted perturbation yields $\psi_{\mathtt{ap}^m\mathtt{cq}^n}(\bar{\mathtt{A}}) = \bar{\mathtt{A}}$ for some $\bar{\mathtt{A}}$ close to $\mathtt{A}$. We will also control the derivatives.

To perform the perturbations, observe first that a direct calculation from equations (3.11), (3.13),(3.14), gives

$$\varphi_{\mathtt{ap}^m\mathtt{cq}^n}(0,1) = \big(a_{12}\alpha_2^m\beta_1^n, a_{22}\alpha_2^m\beta_2^n\big).$$

We first select arbitrarily large m and the associated n defined by

$$|\alpha_2^m\beta_2^{n-1}| < 1 \qquad \text{and} \qquad |\beta_2| \geqslant |\alpha_2^m\beta_2^n| \geqslant 1.$$

This choice implies that if $m \to \infty$ then $n \to \infty$ and

$$|\alpha_2^m\beta_1^n| = \left|\alpha_2^m\beta_2^n\frac{\beta_1^n}{\beta_2^n}\right| \leqslant |\beta_2|\left|\left(\frac{\beta_1}{\beta_2}\right)^n\right| \to 0 \quad \text{as} \quad m, n \to \infty. \tag{5.6}$$

The idea now is to replace α_2 and β_2 by close values $\bar\alpha_2$ and $\bar\beta_2$ (these values converge to α_2 and β_2 when $m, n \to \infty$) such that

$$a_{22}\bar\alpha_2^m\bar\beta_2^n = 1.$$

This choice implies that

$$a_{12}\bar\alpha_2^m\beta_1^n = \frac{a_{12}}{a_{22}}\left(\frac{\beta_1}{\bar\beta_2}\right)^n \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $\mathtt{A}$ is "almost" a fixed point of the perturbation $\psi_{\mathtt{ap}^m\mathtt{cq}^n}$. The final step is to see that the image of the square $\Delta_{\mathtt{A},\varrho}$ by $\psi_{\mathtt{ap}^m\mathtt{cq}^n}$ intersects $\Delta_{\mathtt{A},\varrho}$ in a Markovian way, providing the fixed point close to $\mathtt{A}$. We now go into the details.

We begin with some simple facts. The first result is the following remark:

Claim 5.4. *Given $\alpha, \beta, \gamma \in \mathbb{R}$, $\varepsilon > 0$, and $n_0 \in \mathbb{N}$ there are $\bar\alpha$, $\bar\beta$ with $\bar\alpha \in (\alpha - \varepsilon, \alpha + \varepsilon)$ and $\bar\beta \in (\beta - \varepsilon, \beta + \varepsilon)$ and natural numbers $m_1, n_1 > n_0$ such that*

$$\bar\alpha^{m_1}\bar\beta^{n_1} = |\gamma|.$$

Moreover, if either $\alpha<0$ or $\beta<0$ we may have $\bar\alpha^{m_1}\bar\beta^{n_1}=\gamma$.

Proof. It is enough to slightly modify α,β to get large numbers n_1,m_1 with

$$n_1\frac{\log\bar\beta}{\log\bar\alpha}=-m_1+\frac{\log|\gamma|}{\log\bar\alpha}.$$

The second part of the claim follows considering odd natural numbers. □

Claim 5.5. *Assume configuration $(+,+)$. Given $\varepsilon,\delta>0$ and $n_0\in\mathbb{N}$ there are $\bar\alpha_2\in(\alpha_2-\delta,\alpha_2+\delta)$, $\bar\beta_2\in(\beta_2-\delta,\beta_2+\delta)$, and $m_1,n_1\geqslant n_0$ such that*

$$\text{(i)}\qquad |\bar\alpha_2^{m_1}\beta_1^{n_1}|<\varepsilon/100\qquad\textit{and}\qquad\text{(ii)}\qquad a_{22}\bar\alpha_2^{m_1}\bar\beta_2^{n_1}=1.$$

Proof. Let $\alpha=\alpha_2$, $\beta=\beta_2$, $\gamma=1/a_{22}>0$ and take $\bar\alpha=\bar\alpha_2$, $\bar\beta=\bar\beta_2$ and m_1,n_1 as in Claim 5.4 (associated to γ), hence

$$a_{22}\bar\alpha_2^{m_1}\bar\beta_2^{n_1}=1.$$

This implies (ii). Finally, noting that $|\beta_1|<|\bar\beta_2|$, we get

$$\bar\alpha_2^{m_1}\beta_1^{n_1}=\bar\alpha_2^{m_1}\bar\beta_2^{n_1}\left(\frac{\beta_1}{\bar\beta_2}\right)^{n_1}=\frac{1}{a_{22}}\left(\frac{\beta_1}{\bar\beta_2}\right)^{n_1}\to 0,\quad\text{as}\quad n_1\to\infty,$$

that implies item (i). □

We now consider an adapted perturbation g of f such that $\psi_{\mathtt{j}}=\varphi_{\mathtt{j}}$, for $\mathtt{j}\neq\mathtt{q},\mathtt{p}$, and such that and $\psi_{\mathtt{q}}$ and $\psi_{\mathtt{p}}$ are (in the local coordinates) of the form $\psi_{\mathtt{q}}(x,y)=(\beta_1x,\bar\beta_2y)$ and $\psi_{\mathtt{p}}(x,y)=(\alpha_1x,\bar\alpha_2y)$, where $\bar\alpha_2$ and $\bar\beta_2$ are as in Claim 5.5.

We now study the images of the boxes $\Delta\stackrel{\text{def}}{=}\Delta_{\mathtt{A},\varrho}$ in (5.5) by the maps $\psi_{\mathtt{ap}^{m_1}\mathtt{cq}^{n_1}}$. In the following calculations, and without loss of generality, we will assume that ϵ_a in (3.13) are positive. Note that

$$\begin{aligned}
\psi_{\mathtt{a}}(\Delta)&=[-\epsilon_a\varrho,\epsilon_a\varrho]\times[1-\varrho,1+\varrho]\subset\mathtt{U}_{\mathtt{p}}\\
\psi_{\mathtt{ap}^m}(\Delta)&=[-\epsilon_a\varrho\alpha_1^m,\epsilon_a\varrho\alpha_1^m]\times[\bar\alpha_2^m-\varrho\bar\alpha_2^m,\bar\alpha_2^m+\varrho\bar\alpha_2^m]\subset\mathtt{U}_{\mathtt{p}},\\
\psi_{\mathtt{ap}^m\mathtt{c}}(\Delta)&=\{(a_{12}\bar\alpha_2^m+a_{11}x+a_{12}y,a_{22}\bar\alpha_2^m+a_{21}x+a_{22}y)\colon|x|<\epsilon_a\varrho\alpha_1^m,\ |y|<\varrho\bar\alpha_2^m\}.
\end{aligned}$$

Consider the projections $p_1(x,y)=x$ and $p_2(x,y)=y$ on $\mathtt{U}_{\mathtt{q}}$. We have

$$\begin{aligned}
p_1\big(\psi_{\mathtt{ap}^m\mathtt{c}}(\Delta)\big)&=\{a_{12}\bar\alpha_2^m+a_{11}x+a_{12}y\colon|x|<\epsilon_a\varrho\alpha_1^m,\quad|y|<\varrho\bar\alpha_2^m\},\\
p_2\big(\psi_{\mathtt{ap}^m\mathtt{c}}(\Delta)\big)&=\{a_{22}\bar\alpha_2^m+a_{21}x+a_{22}y\colon|x|<\epsilon_a\varrho\alpha_1^m,\quad|y|<\varrho\bar\alpha_2^m\}
\end{aligned}$$

and

$$\psi_{\mathtt{ap}^m\mathtt{c}}(\Delta)\subset\Delta_m,\quad\text{where}\quad\Delta_m\stackrel{\text{def}}{=}p_1\big(\psi_{\mathtt{ap}^m\mathtt{c}}(\Delta)\big)\times p_2\big(\psi_{\mathtt{ap}^m\mathtt{c}}(\Delta)\big)\subset U_{\mathtt{q}}.$$

Denote by $\ell_i(m)$ the length of $p_i\big(\psi_{\mathtt{ap}^m\mathtt{c}}(\Delta)\big)$, $i=1,2$.

We now study the boxes $\psi_{\mathtt{q}}^{n_1}(\Delta_{m_1})$ for the values $\bar\alpha_2,\bar\beta_2,m_1,n_1$ in Claim 5.5. Note that $n_1=n_1(m_1)\to\infty$ as $m_1\to\infty$. We see that the lengths of the basis of these boxes go to zero as $m_1\to\infty$ (Claim 5.6) while the heights have uniform size (Claim 5.7).

Claim 5.6. $\beta_1^{n_1}\ell_1(m_1)\to 0$ *as* $m_1\to\infty$.

Proof. As $n_1 \to \infty$ as $m_1 \to \infty$, it is enough to check that for every $|x| < \epsilon_a \varrho \alpha_1^{m_1}$ and $|y| < \varrho \bar{\alpha}_2^{m_1}$ it holds

$$\beta_1^{n_1}(a_{21}\bar{\alpha}_2^{m_1} + a_{11}x + a_{12}y) \to 0, \quad \text{as} \quad m_1 \to \infty.$$

Using that $|\alpha_1| < |\bar{\alpha}_2|$ we get $|\alpha_1^{m_1}\beta_1^{n_1}| < |\bar{\alpha}_2^{m_1}\beta_1^{n_1}|$ and by Claim 5.5-(i) the later term can be chosen arbitrarily small if m_1 (and hence n_1) is large enough. Noting that

$$|a_{12}\bar{\alpha}_2^{m_1} + a_{11}x + a_{12}y| \leqslant |a_{12}\bar{\alpha}_2^{m_1}| + |a_{11}\epsilon_a\varrho\alpha_1^{m_1}| + |a_{12}\varrho\bar{\alpha}_2^{m_1}|,$$

the claim follows from the previous observations. □

Claim 5.7. $|\bar{\beta}_2^{n_1}|\ell_2(m_1) \to 2\varrho$ *as* $m_1 \to \infty$.

Proof. Note that the supremum and the infimum of $a_{21}x + a_{22}y$ restricted to the values $|x| < \epsilon_a\varrho|\alpha_1|^{m_1}$ and $|y| < \varrho|\bar{\alpha}_2|^{m_1}$ are respectively

$$|a_{21}\epsilon_a\varrho\alpha_1^{m_1}| + |a_{22}\varrho\bar{\alpha}_2^{m_1}| \qquad \text{and} \qquad -|a_{21}\epsilon_a\varrho\alpha_1^{m_1}| - |a_{22}\varrho\bar{\alpha}_2^{m_1}|.$$

Using

$$\sup(I+J) = \sup(I) + \sup(J) \qquad \text{and} \qquad \inf(I+J) = \inf(I) + \inf(J)$$

and that $|\bar{\alpha}_2^{m_1}\bar{\beta}_2^{n_1}|$ is bounded, it follows

$$|\bar{\beta}_2^{n_1}|\sup\{a_{21}x + a_{22}y\} - |\bar{\beta}_2^{n_1}|\inf\{a_{21}x + a_{22}y\} = 2|\bar{\beta}_2^{n_1}|\big(|a_{21}\epsilon_a\varrho\alpha_1^{m_1}| + |a_{22}\varrho\bar{\alpha}_2^{m_1}|\big).$$

As $\bar{\beta}_2^{n_1}\alpha_1^{m_1} \to 0$ and $\bar{\beta}_2^{n_1}\bar{\alpha}_2^{m_1} = a_{22}^{-1}$ the claim follows. □

5.1.1. *End of the proof of Lemma 5.3.* We got an itinerary $\mathtt{i}_{(m_1,n_1)}$ of $\mathtt{A} = (0,1)$ such that

$$\psi_{\mathtt{i}}(0,1) = (a_{12}\bar{\alpha}_2^{m_1}\beta_1^{n_1}, 1), \quad \text{where} \quad a_{12}\bar{\alpha}_2^{m_1}\beta_1^{n_1} = \frac{a_{12}}{a_{22}}\left(\frac{\beta_1}{\bar{\beta}_2}\right)^{n_1}.$$

Note that the later term goes to 0 as $n_1 \to \infty$. To get the assertion on the derivative (see (5.4)), note that

$$D\psi_{\mathtt{i}}(\mathtt{A}) = \begin{pmatrix} a_{11}\epsilon_a\beta_1^{n_1}\alpha_1^{m_1} & \pm a_{12}\beta_1^{n_1}\bar{\alpha}_2^{m_1} \\ a_{21}\epsilon_a\bar{\beta}_2^{n_1}\alpha_1^{m_1} & 1 \end{pmatrix}, \tag{5.7}$$

where

$$|a_{11}\epsilon_a\beta_1^{n_1}\alpha_1^{m_1}|, |a_{12}\beta_1^{n_1}\bar{\alpha}_2^{m_1}|, |a_{21}\epsilon_a\bar{\beta}_2^{n_1}\alpha_1^{m_1}| \to 0 \quad \text{as} \quad m_1, n_1 \to \infty. \tag{5.8}$$

Thus, the eigenvalues of $D\psi_{\mathtt{i}}(\mathtt{A})$ are close to 0 and 1. By domination, the eigenspace associated to 1 is almost vertical. Therefore, after a new perturbation we can assume that the vertical direction is invariant. We now modify the maps along the orbit of $\mathtt{A}$ to see that $\psi_{\mathtt{i}}(\Delta)$ intersects Δ in a Markovian way (see Figure 4). This provides a fixed point with derivative close to one. As the number of iterates is arbitrarily large a new perturbation provides a point $\mathtt{A}_{\mathtt{i}}$ of the form $\mathtt{A}_{\mathtt{i}} = (a_{\mathtt{i}}, 1)$ with $\psi_{\mathtt{i}}(\mathtt{A}_{\mathtt{i}}) = \mathtt{A}_{\mathtt{i}}$ and multiplier equal to 1. Note that the other eigenvalue is arbitrarily small (hence of contracting type). We observe that by construction

$$|a_{\mathtt{i}}| \approx \left(\frac{|\beta_1|}{|\bar{\beta}_2|}\right)^{n_1}. \tag{5.9}$$

The proof of the lemma is now complete. □

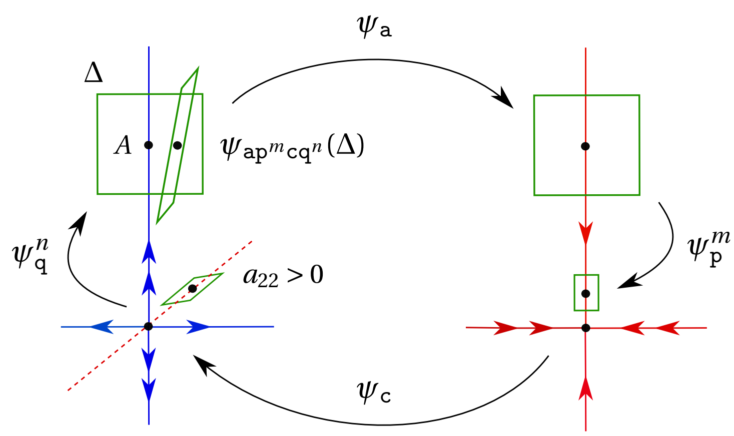

FIGURE 4. Itineraries $\mathtt{i} = \mathtt{ap}^m\mathtt{cq}^n$

5.2. **End of the proof of Proposition 5.1.** Suppose that $\mathtt{i}_1$ and $A_{\mathtt{i}_1}$ are defined. We now explain how to get large m_2, n_2 such that, taking $\mathtt{i}_2 = \mathtt{i}_{(m_2,n_2)}$, there is a fixed point $\mathtt{A}_{\mathtt{i}_2}$ of $\psi_{\mathtt{i}_2}$ satisfying (5.2). We observe that the orbit of $\mathtt{A}_{\mathtt{i}_1}$ is “separated" from both $\mathtt{Q}$ and $\mathtt{P}$, and both, the cycle and transitions, are preserved. As the cycle is not destroyed, we can repeat the construction choosing $n_2 \gg n_1$ and $m_2 \gg m_1$. This implies that a new perturbation can be made “deeper" than the first one (done to get $\mathtt{A}_{\mathtt{i}_1}$), meaning there are points in the orbit of $\mathtt{A}_{\mathtt{i}_2}$ much closer to $\mathtt{Q}$ and $\mathtt{P}$. Consequently, a new itinerary with a fixed point can be obtained without altering the orbit of $\mathtt{A}_{\mathtt{i}_1}$. See Figure 4. This can be repeated to get a new itinerary $\mathtt{i}_3$ and a point $\mathtt{A}_{\mathtt{i}_3}$.

Remark 5.8. By construction, for large m_k, n_k, $k = 1,2,3$, (see Figure 6)

$$W^{\mathrm{ss}}(\mathtt{A}_{\mathtt{i}_k}, \psi_{\mathtt{i}_k}) \pitchfork W^{\mathrm{uu}}(\mathtt{Q}, \psi_{\mathtt{q}}) \neq \emptyset, \quad \mathtt{i}_k = \mathtt{i}_{(m_k,n_k)}.$$

To get the relations in (5.3) for a word $\mathtt{i}_4$, we proceed similarly to the previous argument. This time, we increase the number m_4 so that we are closer to $\mathtt{P}$ than the periodic orbits of $\mathtt{A}_{\mathtt{i}_1}, \mathtt{A}_{\mathtt{i}_2}, \mathtt{A}_{\mathtt{i}_3}$. This allows us to perform a deeper perturbation without modifying the orbits of $\mathtt{A}_{\mathtt{i}_1}, \mathtt{A}_{\mathtt{i}_2}, \mathtt{A}_{\mathtt{i}_3}$ in a such a way, that, for an appropriate and sufficiently large n_4, m_4, we have $\psi_{\mathtt{i}_4}(\mathtt{A}_{\mathtt{i}_1}) \in W^{\mathrm{ss}}_{\mathrm{loc}}(\mathtt{A}_{\mathtt{i}_1}, \psi_{\mathtt{i}_1})$, here we use Remark 5.8.

Finally note that, by construction, to the words $\mathtt{i}_2$ and $\mathtt{i}_3$ there are associated fixed points $\mathtt{A}_{\mathtt{i}_2}$ and $\mathtt{A}_{\mathtt{i}_3}$ satisfying (5.2). This completes the proof for the points $\mathtt{A}_{\mathtt{i}_k}$.

To get the points $\mathtt{D}'_{\mathtt{j}_k}$, with a neutral direction and strongly expanding eigenvalues, we proceed as above, but now considering the transition $\varphi_{\mathtt{d}}^{-1}$. This construction can be done preserving the points $\mathtt{A}_{\mathtt{i}_k}$ and their homoclinic intersections. Note that the perturbations in the construction of these points are done in the two disjoint cuspidal regions depicted in Figure 5, thus they can be performed independently. This ends the proof of the proposition. □

5.3. **Transverse heteroclinic intesections.** The next remark will be used to get the transverse intersection between the invariant manifolds of sets in cycles of co-index two.

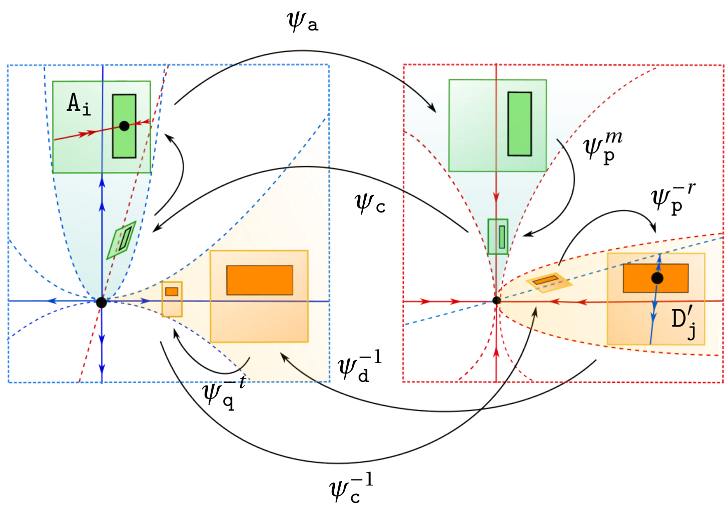

FIGURE 5. Cuspidal regions for compatible perturbations

Remark 5.9. By construction, for large m_k, n_k, r_k, t_k and $k = 1,2,3$,

$$W^{ss}(\mathtt{A}_{\mathtt{i}_k}, \psi_{\mathtt{i}_k}) \pitchfork W^{uu}(\mathtt{Q}, \psi_{\mathtt{q}}) \neq \emptyset, \quad W^{uu}(\mathtt{D}'_{\mathtt{j}_k}, \psi_{\mathtt{j}_k}) \pitchfork W^{ss}(\mathtt{P}, \psi_{\mathtt{p}}) \neq \emptyset,$$

see Figure 6. Noting that, for large n, m,

$$\text{if} \quad |\beta_2^n \alpha_2^m| \approx 1 \quad \text{then} \quad |\beta_1^n \alpha_1^m| \approx 0.$$

we obtain that

$$\psi_{\mathtt{q}}^n \circ \psi_{\mathtt{c}} \circ \psi_{\mathtt{p}}^m \left(W^{uu}(\mathtt{D}'_{\mathtt{j}_k}, \psi_{\mathtt{j}_k}) \right) \pitchfork W^{ss}(\mathtt{A}_{\mathtt{i}_k}, \psi_{\mathtt{i}_k}) \neq \emptyset. \tag{5.10}$$

Thus, if $\mathtt{A}_{\mathtt{i}_k}$ and $\mathtt{D}_{\mathtt{j}_k}$ are hyperbolic points, then

$$W^s\left((\mathtt{i}_k^{\mathbb{Z}}, \mathtt{A}_{\mathtt{i}_k}), \Psi\right) \pitchfork W^u\left((\mathtt{j}_k^{\mathbb{Z}}, \mathtt{D}'_{\mathtt{j}_k}), \Psi\right) \neq \emptyset.$$

As a consequence of Remark 4.4, the periodic points $A_{(\mathtt{i}_k^{\mathbb{Z}}, \mathtt{A}_{\mathtt{i}_k})}$ and $D'_{(\mathtt{j}_k^{\mathbb{Z}}, \mathtt{D}'_{\mathtt{i}_k})}$ of the diffeomorphism g associated to Ψ are hyperbolic and satisfy

$$W^s(A_{(\mathtt{i}_k^{\mathbb{Z}}, \mathtt{A}_{\mathtt{i}_k})}, g) \pitchfork W^u(D'_{(\mathtt{j}_k^{\mathbb{Z}}, \mathtt{D}'_{\mathtt{i}_k})}, g) \neq \emptyset.$$

5.4. **Co-index one robust cycles.** We now derive the occurrence of robust cycles from Proposition 5.1.

Theorem 5.10. *Let f be a diffeomorphism with a simple contour Υ. Then, for every neighbourhood U_Υ of Υ, there exists a diffeomorphism h arbitrarily C^1 close to f having simultaneously a contour of double type I contained in U_Υ and a pair of C^1 robust heterodimensional cycles of co-index one, of types $(\iota(\Upsilon), \iota(\Upsilon)+1)$ and $(\iota(\Upsilon)+1, \iota(\Upsilon)+2)$, associated to hyperbolic sets contained in U_Υ.*

Proof. Consider the points $\mathtt{A}_{\mathtt{i}_1}$ and $\mathtt{D}'_{\mathtt{j}_1}$ and the words $\mathtt{i}_4$ and $\mathtt{j}_4$ given by Proposition 5.1 such that $\psi_{\mathtt{i}_4}(\mathtt{A}_{\mathtt{i}_1}) \in W^{ss}_{loc}(\mathtt{A}_{\mathtt{i}_1}, \psi_{\mathtt{i}_1})$ and $(\psi_{\mathtt{i}_4})^{-1}(\mathtt{D}'_{\mathtt{j}_1}) \in W^{uu}_{loc}(\mathtt{D}'_{\mathtt{j}_1}, \psi_{\mathtt{j}_1})$. Using the first

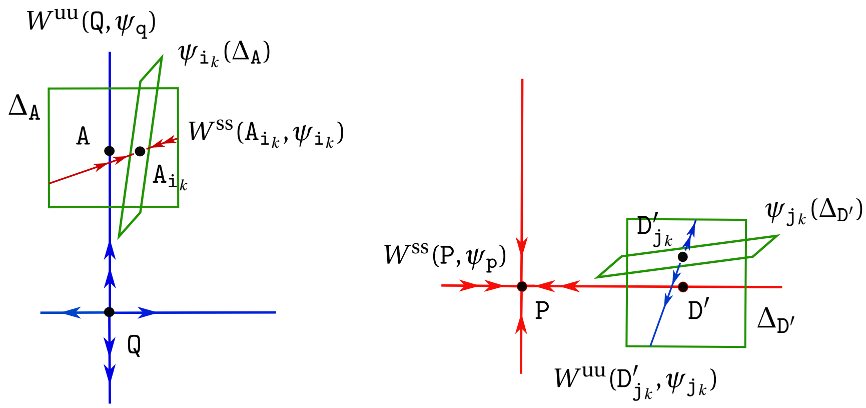

FIGURE 6. Transverse intersections $W^{ss}(\mathtt{A}_{\mathtt{i}_k},\psi_{\mathtt{i}_k}) \pitchfork W^{uu}(\mathtt{Q},\psi_{\mathtt{q}}) \neq \emptyset$ and $W^{uu}(\mathtt{D}'_{\mathtt{j}_k},\psi_{\mathtt{j}_{k,}}) \pitchfork W^{ss}(\mathtt{P},\psi_{\mathtt{p}}) \neq \emptyset$

condition, Lemma 4.8 provides the cycle of type $(\iota(\Upsilon),\iota(\Upsilon)+1)$. The second condition and the lemma provides the cycle of type $(\iota(\Upsilon)+1,\iota(\Upsilon)+2)$. $\square$

5.5. **Selection of rectangles.** We use the points $A_{\mathtt{i}_2}, A_{\mathtt{i}_3}, D'_{\mathtt{j}_2}$, and $D'_{\mathtt{j}_3}$ as base points of a family of rectangles that will be used in the construction of preblending machines. We assume that α_j,β_i in (3.11) are both positive. We assume that the numbers m_k,n_k,r_k,t_k in the words $\mathtt{i}_k=\mathtt{i}_{(m_k,n_k)}$ and $\mathtt{j}_k=\mathtt{j}_{(r_k,t_k)}$ are arbitrarily large. Recall that $\mathtt{A}_{\mathtt{i}_k}=(a_{\mathtt{i}_k},1)$, where $a_{\mathtt{i}_k}\approx\left(\frac{\beta_1}{\beta_2}\right)^{n_k}>0$ (recall Remark 5.2).

We denote by $|J|$ the lenght of an interval J.

Claim 5.11. *Given $\varepsilon>0$ there is $\vartheta_0>0$ such that for every $\vartheta\in(0,\vartheta_0)$ there is an ε adapted perturbation of $\mathscr{G}$ (whose quotient maps will be denoted as those of $\mathscr{G}$) and closed intervals*

$$\mathtt{V}(\vartheta)\overset{\text{def}}{=}[1-\vartheta,1+\vartheta],\quad \mathtt{H}_{\mathtt{i}_2},\quad \mathtt{H}_{\mathtt{i}_3},\quad \textit{and}\quad \mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}$$

satisfying the following for $k=2,3$:

(1) $a_{\mathtt{i}_k}\in\operatorname{int}(\mathtt{H}_{\mathtt{i}_k})$ and $\mathtt{H}_{\mathtt{i}_2}\cup\mathtt{H}_{\mathtt{i}_3}\subset\operatorname{int}(\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3})$,
(2) $|\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}|\approx|\mathtt{H}_{\mathtt{i}_2}|$ and $|\mathtt{H}_{\mathtt{i}_k}|\approx|\alpha_1^{m_k}\beta_1^{n_k}|<\left(\frac{\beta_1}{\beta_2}\right)^{n_k}$,
(3) the restriction of $\psi_{\mathtt{i}_k}$ to the rectangle $\mathtt{H}_{\mathtt{i}_k}\times\mathtt{V}(\vartheta)\subset\mathtt{U}_{\mathtt{q}}$ preserves the horizontal and vertical lines, it acts as the identity in the vertical direction, and
(4) $\psi_{\mathtt{i}_k}(\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}\times\mathtt{V}(\vartheta))=\mathtt{H}_{\mathtt{i}_k}\times\mathtt{V}(\vartheta)$.

Proof. Recall that $\mathtt{i}_2=\mathtt{i}_{(m_2,n_2)}$ and $\mathtt{i}_3=\mathtt{i}_{(m_3,n_3)}$, the proof follows taking the m_k,n_k sufficiently large. Also recall $m_2<m_3$ and $n_2<n_3$. We start with a pair of auxiliary intervals $\widehat{\mathtt{H}}_{\mathtt{i}_k}$ of size $\left(\frac{\beta_1}{\beta_2}\right)^{n_k}$ containing $a_{\mathtt{i}_k}$. After performing an small adapted perturbation of $\mathscr{G}$, we may assume that the restriction of $\psi_{\mathtt{i}_k}$ to $\mathtt{H}_{\mathtt{i}_k}\times\mathtt{V}(\vartheta)$ preserves both horizontal and vertical directions, and acts as the identity along vertical lines. The set $\mathtt{H}_{\mathtt{i}_k}$ in the claim will be a subset of $\widehat{\mathtt{H}}_{\mathtt{i}_k}$, and thus it satisfies condition (3). To get the intervals $\mathtt{H}_{\mathtt{i}_k}$ and $\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}$, note that the contraction of $\psi_{\mathtt{i}_k}$ in the horizontal direction is of order of $\alpha_1^{m_k}\beta_1^{n_k}$.

Therefore the horizontal size of $\psi_{\mathtt{i}_k}(\widehat{\mathtt{H}}_{\mathtt{i}_k}\times V(\vartheta))$ is of order of

$$\alpha_1^{m_k}\beta_1^{n_k}\left(\frac{\beta_1}{\beta_2}\right)^{n_k}.$$

The previous number can be take arbitrarily close to 0 taking m_k, n_k large enough. This allows us to chose an interval $\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}\subset\widehat{\mathtt{H}}_{\mathtt{i}_2}\cap\widehat{\mathtt{H}}_{\mathtt{i}_3}$ and intervals $\mathtt{H}_{\mathtt{i}_2},\mathtt{H}_{\mathtt{i}_3}\subset\mathrm{int}(\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3})$ such that $a_{\mathtt{i}_k}\in\mathrm{int}(\mathtt{H}_{\mathtt{i}_k})$, with $|\mathtt{H}_{\mathtt{i}_k}|\approx|\alpha_1^{m_k}\beta_1^{n_k}|\leqslant|\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}|\approx|\alpha_1^{m_2}\beta_1^{n_2}|$, and

$$\psi_{\mathtt{i}_k}(\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}\times\mathtt{V}(\vartheta))=\mathtt{H}_{\mathtt{i}_k}\times\mathtt{V}(\vartheta).$$

Hence (1), (2), and (4) hold. $\square$

We let $\widehat{\mathtt{R}}_{\mathtt{i}_k}\stackrel{\text{def}}{=}\mathtt{H}_{\mathtt{i}_k}\times\mathtt{V}(\vartheta)$ and consider the canonical projections

$$P_{\mathtt{r}},Q_{\mathtt{r}}\colon\mathtt{U}_{\mathtt{r}}\to\mathbb{R},\quad P_{\mathtt{r}}(x,y)=x,\quad Q_{\mathtt{r}}(x,y)=y,\quad \mathtt{r}=\mathtt{q},\mathtt{p}.$$

Let $\mathscr{G}$ be a perturbation given in the Claim 5.11. After a new (standard) perturbation one gets:

Claim 5.12. *Given $\varepsilon>0$ there is $\vartheta_0>0$ such that for every $\vartheta\in(0,\vartheta_0)$, there is an ε perturbation of $\mathscr{G}$ (whose quotient maps will be denoted as in $\mathscr{G}$) and rectangles*

$$\mathtt{R}_{\mathtt{i}_k}\stackrel{\text{def}}{=}\mathtt{H}_{\mathtt{i}_k}\times\mathtt{V}_k(\vartheta),\quad \textit{with}\quad \mathtt{V}_k(\vartheta)\quad\textit{close to}\quad\mathtt{V}(\vartheta),\quad k=2,3 \tag{5.11}$$

containing $\mathtt{A}_k$ in their interiors such that

1. *$\mathtt{V}(\vartheta)\subset\mathrm{int}(Q_{\mathtt{q}}(\mathtt{R}_{\mathtt{i}_2})\cup Q_{\mathtt{q}}(\mathtt{R}_{\mathtt{i}_3}))$,*
2. *the restriction of $\psi_{\mathtt{i}_k}$ to $\mathtt{R}_{\mathtt{i}_k}$ is affine with derivative $\left(\begin{smallmatrix}\rho_{\mathtt{i}_k} & 0\\ 0 & \theta_{\mathtt{i}_k}\end{smallmatrix}\right)$, where $|\rho_{\mathtt{i}_k}|$ is of order of $\alpha_1^{m_k}\beta_1^{n_k}$ (hence arbitrarily small) and $|\theta_{\mathtt{i}_k}|\in(0.99,1)$,*
3. *$\mathtt{R}_{\mathtt{i}_k}\subset\psi_{\mathtt{i}_k}^{-1}(\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}\times\mathrm{int}(\mathtt{V}(\vartheta))$, where $\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}$ is the interval in Claim 5.11 containing $\mathtt{H}_{\mathtt{i}_2}\cup\mathtt{H}_{\mathtt{i}_3}$ in its interior.*

Arguing as above, replacing the points $\mathtt{A}_{\mathtt{i}_k}$ by $\mathtt{D}'_{\mathtt{j}_k}$ and having again in mind the independence of the adapted perturbations to get the periodic points, it follows:

Claim 5.13. *Given $\varepsilon>0$ there exist $\delta_0>0$ such that for every $\delta\in(0,\delta_0)$ there are an ε adapted perturbation $\mathscr{G}$ of $\mathscr{F}$ satisfying Claim 5.12 and rectangles*

$$\mathtt{R}_{\mathtt{j}_k}\stackrel{\text{def}}{=}\mathtt{H}_k(\delta)\times\mathtt{V}_{\mathtt{j}_k},\quad\textit{with}\quad\mathtt{H}_k(\vartheta)\quad\textit{close to}\quad\mathtt{H}(\delta)\stackrel{\text{def}}{=}[1-\delta,1+\delta],\quad k=2,3. \tag{5.12}$$

containing $\mathtt{D}_k$ in their interiors such that

1. *$\mathtt{H}(\delta)\subset\mathrm{int}(P_{\mathtt{p}}(\mathtt{R}_{\mathtt{j}_2})\cup P_{\mathtt{p}}(\mathtt{R}_{\mathtt{j}_3}))$,*
2. *the restriction of $\psi_{\mathtt{j}_k}$ to $R_{\mathtt{j}_k}$ is affine with derivative $\left(\begin{smallmatrix}\lambda_{\mathtt{j}_k} & 0\\ 0 & \eta_{\mathtt{j}_k}\end{smallmatrix}\right)$ where $|\lambda_{\mathtt{j}_k}|\in(1,1.01)$ and $|\eta_{\mathtt{j}_k}|$ is of order of $\alpha_2^{t_k}\beta_2^{r_k}$ (hence arbitrarily large),*
3. *$\psi_{\mathtt{j}_k}(\mathtt{R}_{\mathtt{j}_k})\subset\mathrm{int}(\mathtt{H}(\delta))\times\mathtt{V}_{\mathtt{j}_2,\mathtt{j}_3}$, where $\mathtt{V}_{\mathtt{j}_2,\mathtt{j}_3}$ is an interval whose interior contains $\mathtt{V}_{\mathtt{j}_2}\cup\mathtt{V}_{\mathtt{j}_3}$.*

6. SPLIT BLENDING MACHINES

We introduce *split blending machines*, extending the concept of a blending machine from [2, Section 2.4], with the key distinction of an additional "center direction". Accordingly, we revisit and adapt the tools developed in [2]. Remark 6.9 provides a comparison of these two concepts.

These machines provide an abstract setting that captures the mechanism behind blenders constructed using the covering property (see Remark 7.12). The setting involves local dynamics, invariant splittings, and systems of rectangles. Although the presence of splittings suggests partial hyperbolicity, no hyperbolicity is assumed. The system of rectangles displays intersection properties similar to those found in blenders, placing split blending machines as nonhyperbolic counterparts to blenders, see Section 6.3. Before, we provide some preliminaries in Section 6.1, followed by the definition and main properties in Section 6.2.

6.1. Splittings, cones, and rectangles.

For $n \geqslant 1$, denote the box norm in $\mathbb{R}^n$ by

$$\|x\| = \|(x_1, \dots, x_n)\| \stackrel{\text{def}}{=} \max\{|x_1|, \dots, |x_n|\}, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

We identify the tangent space $T_x\mathbb{R}^n$ with $\mathbb{R}^n$.

Given manifolds M_1 and M_2, an open set U of M_1, and a C^1 map $f\colon U \to M_2$, we denote by Df the tangent map of f between the tangent bundles TU and TM_2. For notational simplicity, when no confusion can arise, given $x \in U$ and $v \in T_xU$, we simply write $Df(v)$ instead $D_xf(v)$.

In what follows, fix an open Riemannian manifold N and positive integers p and q.

Definition 6.1 (Pairs, splittings, rectangles, and cones)**.** Given C^1 maps $P\colon N \to \mathbb{R}^p$ and $Q\colon N \to \mathbb{R}^q$, consider the map $\mathfrak{S} = (P, Q)$ defined by

$$\mathfrak{S}\colon N \to \mathbb{R}^p \times \mathbb{R}^q, \quad \mathfrak{S}(x) \stackrel{\text{def}}{=} \big(P(x), Q(x)\big).$$

We say that $\mathfrak{S}$ is a (p,q)*-pair* of N. Associated to $\mathfrak{S} = (P, Q)$, we consider the (q, p)-pair $\mathfrak{S}^t \stackrel{\text{def}}{=} (Q, P)$.

We say that $\mathfrak{S}$ is a (p,q)*-splitting* of N if it is a diffeomorphism onto its image. In particular, the dimension of N is $p+q$.

A subset R of N is called a $\mathfrak{S}$*-rectangle* if the (p,q)-pair $\mathfrak{S} = (P,Q)$ is a diffeomorphism between R and $P(R) \times Q(R)$. For $\theta > 0$ and $x \in N$, we define the $(\mathfrak{S},\theta)$*-cone* at x by

$$C_\theta(x, \mathfrak{S}) \stackrel{\text{def}}{=} \big\{v \in T_xN\colon \|DQ(v)\| \leqslant \theta \|DP(v)\|\big\}. \tag{6.1}$$

Throughout the paper, we will use the terms "splitting" and "splitting pair" interchangeably. In particular, we will refer to $\mathfrak{S}$ as a splitting pair.

Remark 6.2. We always assume that the (p,q)-pairs $\mathfrak{S} = (P, Q)$ are *non-degenerate,* meaning that both kernels, $\operatorname{Ker} D_xP$ and $\operatorname{Ker} D_xQ$, are not equal to T_xN for every $x \in N$.

Definition 6.3 (Invariance, dilatation, and domination)**.** Let $\mathfrak{S}_\tau = (P_\tau, Q_\tau)$ be a (p,q)-pair of an open manifold N_τ, with $\tau = 1,2$. Consider an open set U of N_1, a C^1 embedding $f\colon U \to N_2$, a subset $R \subset U$, and constants $\mu, \gamma, \theta > 0$.

(1) The map f satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta)$-*cone invariance condition* on R if there exists $0<\theta'<\theta$ such that for every $x\in R$ it holds
$$Df^{-1}\big(C_\theta(f(x),\mathfrak{S}_2)\big)\subset C_{\theta'}(x,\mathfrak{S}_1) \quad\text{and}\quad Df\big(C_\theta(x,\mathfrak{S}_1^t)\big)\subset C_{\theta'}(f(x),\mathfrak{S}_2^t).$$
(2) The map f satisfies $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-*dilatation condition on* R if the following holds
$$\|D(P_1\circ f^{-1})(w)\|>\mu\|DP_2(w)\| \quad \text{for every } w\in C_\theta\big(f(x),\mathfrak{S}_2\big)\setminus\{0\} \text{ and } x\in R,$$
$$\|D(Q_2\circ f)(v)\|>\gamma\|DQ_1(v)\| \quad \text{for every } v\in C_\theta(x,\mathfrak{S}_1^t)\setminus\{0\} \text{ and } x\in R.$$
(3) The map f satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-*cone condition* on R if it satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta)$-cone invariance and the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-dilatation conditions on R.
(4) The map f is $(\mathfrak{S}_1,\mathfrak{S}_2,\mu,\gamma)$-*dominated* on R if it satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-cone condition on R for every $\theta>0$.

When $N=N_1=N_2$ and $\mathfrak{S}=\mathfrak{S}_1=\mathfrak{S}_2$, we refer to these conditions as $(\mathfrak{S},\theta)$-cone invariance, $(\mathfrak{S},\theta,\mu,\gamma)$-cone, and $(\mathfrak{S},\theta,\mu,\gamma)$-dilatation and $(\mathfrak{S},\mu,\gamma)$-domination.

Using the notation and terminology in Definition 6.3, we have the following:

Lemma 6.4. *Suppose that f is $(\mathfrak{S}_1,\mathfrak{S}_2,\mu,\gamma)$-dominated on R. Then for every $x\in R$ and every $v,w\in T_xU$ such that $v\notin \operatorname{Ker} D_x(P_2\circ f)$ and $w\notin \operatorname{Ker} D_xQ_1$ it holds*
$$\|D(P_2\circ f)(v)\|<\mu^{-1}\,\|DP_1(v)\| \quad\textit{and}\quad \|D(Q_2\circ f)(w)\|>\gamma\,\|DQ_1(w)\|. \tag{6.2}$$
When $\mu\gamma>1$, the inequalities in (6.2) imply that f is $(\mathfrak{S}_1,\mathfrak{S}_2,\mu,\gamma)$-dominated on R.

Proof. We prove the first inequality in (6.2), the other one follows analogously. By hypothesis, the map f satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-cone condition on R for every $\theta>0$. In particular, f satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-dilatation condition on R, therefore,
$$\|DP_1(v)\|>\mu\|D(P_2\circ f)(v)\| \quad \text{for every } x\in R \text{ and } v\in Df^{-1}\big(C_\theta(f(x),\mathfrak{S}_2)\big)\setminus\{0\}. \tag{6.3}$$
Since Df^{-1} is an isomorphism and
$$\bigcup_{\theta>0} C_\theta\big(f(x),\mathfrak{S}_2\big)\setminus\{0\}=T_{f(x)}f(U)\setminus \operatorname{Ker} D_{f(x)}P_2 \tag{6.4}$$
condition (6.3) holds for every $v\in T_xU\setminus Df^{-1}(\operatorname{Ker} D_{f(x)}P_2)$, implying the first inequality and proving the first part of the lemma.

Assuming now that $\mu\gamma>1$ and equation (6.2) holds, we prove that f satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-cone condition on R for every $\theta>0$. To obtain the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta)$-cone invariance condition on R take
$$\theta'\overset{\text{def}}{=}\theta\,(\mu\gamma)^{-1}<\theta. \tag{6.5}$$
To get $Df\big(C_\theta(x,\mathfrak{S}_1^t)\big)\subset C_{\theta'}\big(f(x),\mathfrak{S}_2^t\big)$, note that
$$\|DP_1(v)\|\leqslant\theta\,\|DQ_1(v)\|, \tag{6.6}$$
for every $x\in R$ and every $v\in C_\theta(x,\mathfrak{S}_1^t)$. Hence,
$$\begin{aligned}\|D(P_2\circ f)(v)\| &\overset{(6.2)}{<}\mu^{-1}\|DP_1(v)\|\overset{(6.6)}{\leqslant}\mu^{-1}\theta\,\|DQ_1(v)\|\\ &\overset{(6.2)}{<}\mu^{-1}\gamma^{-1}\theta\,\|D(Q_2\circ f)(v)\|\overset{(6.5)}{=}\theta'\,\|D(Q_2\circ f)(v)\|.\end{aligned}$$

Thus, $Df(v) \in C_{\theta'}\big(f(x), \mathfrak{S}_2^t\big)$, proving the inclusion. To get $Df^{-1}(C_\theta\big(f(x),\mathfrak{S}_2)\big) \subset C_{\theta'}(x,\mathfrak{S}_1)$, observe that

$$\|DQ_2(w)\| \leqslant \theta\,\|DP_2(w)\| \text{ for every } w \in C_\theta\big(f(x),\mathfrak{S}_2\big). \tag{6.7}$$

Take any $w \in C_\theta\big(f(x),\mathfrak{S}_2\big)$ and consider $v = Df^{-1}(w) \in T_xN_1$. Then

$$\begin{aligned}\|DQ_1(v)\| &\overset{(6.2)}{\leqslant} \gamma^{-1}\|D(Q_2\circ f)(v)\| = \gamma^{-1}\|DQ_2(w)\| \overset{(6.7)}{\leqslant} \gamma^{-1}\theta\,\|DP_2(w)\|\\ &= \gamma^{-1}\theta\,\|D(P_2\circ f)(v)\| \overset{(6.2)}{\leqslant} \gamma^{-1}\mu^{-1}\theta\,\|DP_1(v)\| \overset{(6.5)}{=} \theta'\,\|DP_1(v)\|.\end{aligned}$$

This implies that $Df^{-1}(w) \in C_{\theta'}(x,\mathfrak{S}_1)\setminus\{0\}$. Thus, the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta)$-cone invariance condition on R holds for f.

It remains to prove the dilatation condition in the cones. For $w \in C_\theta\big(f(x),\mathfrak{S}_2\big)\setminus\{0\}$, letting $v = Df^{-1}(w)$, equation (6.4) implies that $\|D(P_2\circ f)(v)\| = \|DP_2(w)\| \neq 0$. Thus, $v \in T_xU \setminus \operatorname{Ker} D_x(P_2\circ f)$ and equation (6.2) implies

$$\|D(P_1\circ f^{-1})(w)\| > \mu\|DP_2(w)\| \quad \text{for every } w \in C_\theta(f(x),\mathfrak{S}_2)\setminus\{0\}.$$

Similarly, if $v \in C_\theta(x,\mathfrak{S}_1^t) \subset T_xU\setminus \operatorname{Ker} D_xQ_1$, then $\|DQ_1(v)\| \neq 0$ and from (6.2) follows that

$$\|D(Q_2\circ f)(v)\| > \gamma\|DQ_1(v)\| \quad \text{for every } v \in C_\theta(x,\mathfrak{S}_1^t)\setminus\{0\}.$$

Therefore, f satisfies the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-cone condition on R for every $\theta>0$, ending the proof. ☐

Remark 6.5. In general, $(\mathfrak{S}_1,\mathfrak{S}_2,\mu,\gamma)$-domination of f does not hold for small C^1 perturbations of f. In that sense, $(\mathfrak{S}_1,\mathfrak{S}_2,\mu,\gamma)$-domination is not a robust property. However, fixed any $\theta>0$, the $(\mathfrak{S}_1,\mathfrak{S}_2,\theta,\mu,\gamma)$-cone condition on a compact set is a C^1 robust property. This sort of robustness is enough in our setting.

The next two lemmas will play a key role in Section 8. The first one is a version of [2, Corollary 2.7] in our setting, it deals with diameters of images of rectangles by compositions of pairs and maps.

Given a set $S \subset \mathbb{R}^n$, we denote by $\operatorname{diam}(S)$ its diameter (with respect to the metric induced by the box norm).

Lemma 6.6. *Let N_τ be open manifolds and $\mathfrak{S}_\tau = (P_\tau, Q_\tau)$ a (p,q)-splitting of N_τ, where $\tau = 1,2$. Let U be an open subset of N_1, $f\colon U \to N_2$ a C^1 embedding, and R a $(P_1, Q_2\circ f)$-rectangle in U.*

Consider C^1 maps $P\colon \mathbb{R}^p \to \mathbb{R}^\ell$ and $Q\colon \mathbb{R}^q \to \mathbb{R}^m$, where $1 \leqslant \ell < p$ and $1 \leqslant m < q$, and the (ℓ,m)-pairs $\mathfrak{X}_\tau = (P\circ P_\tau, Q\circ Q_\tau)$ of N_τ with $\tau = 1,2$.

Suppose that f satisfies the $(\mathfrak{X}_1,\mathfrak{X}_2,\theta,\mu,\gamma)$-cone condition on R. Then,

$$\begin{aligned}\operatorname{diam}\big(Q\circ Q_1(R)\big) &\leqslant \theta\operatorname{diam}\big(P\circ P_1(R)\big) + \gamma^{-1}\operatorname{diam}\big(Q\circ Q_2(f(R))\big),\\ \operatorname{diam}\big(P\circ P_2(f(R))\big) &\leqslant \theta\operatorname{diam}\big(Q\circ Q_2(f(R))\big) + \mu^{-1}\operatorname{diam}\big(P\circ P_1(R)\big).\end{aligned} \tag{6.8}$$

Proof. Fix any pair of points $x, x' \in R$ and the point $\big(P_1(x), Q_2(f(x'))\big) \in P_1(R)\times Q_2\big(f(R)\big)$. Since R is a $(P_1, Q_2\circ f)$-rectangle, there exists $x_* \in R$ such that

$$P_1(x_*) = P_1(x) \quad \text{and} \quad Q_2\big(f(x_*)\big) = Q_2\big(f(x')\big).$$

Define the following submanifolds

$$R^- \stackrel{\text{def}}{=} \{y \in R : Q_2(f(y)) = Q_2(f(x_*))\} \quad \text{and} \quad R^+ \stackrel{\text{def}}{=} \{y \in R : P_1(y) = P_1(x_*)\}.$$

Note that $\{x', x_*\} \subset R^-$ and $\{x, x_*\} \subset R^+$. In particular,

$$Q \circ Q_2\big(f(y)\big) = Q \circ Q_2\big(f(x_*)\big) \quad \text{for every } y \in R^-. \tag{6.9}$$

Given any vector $v \in T_y R^-$, equation (6.9) implies that $D(Q \circ Q_2 \circ f)(v) = 0$. Thus $Df(v) \in C_\theta\big(f(y), \mathfrak{X}_2\big)$. Since f satisfies the $(\mathfrak{X}_1, \mathfrak{X}_2, \theta)$-cone invariance condition on R, it holds

$$v \in Df^{-1}\big(C_\theta(f(y), \mathfrak{X}_2)\big) \subset C_\theta(y, \mathfrak{X}_1).$$

Therefore, by the definition of $(\mathfrak{X}_1, \theta)$-cone (recall equation (6.1)), we have

$$\|D(Q \circ Q_1)(v)\| \leqslant \theta \, \|D(P \circ P_1)(v)\| \quad \text{for every } y \in R^- \text{ and } v \in T_y R^-.$$

We claim that this implies that

$$\operatorname{diam}\big(Q \circ Q_1(R^-)\big) \leqslant \theta \operatorname{diam}\big(P \circ P_1(R^-)\big). \tag{6.10}$$

In fact, take any regular curve $t \in [0,1] \mapsto \rho(t) \in R^-$ and consider $\alpha(t) \stackrel{\text{def}}{=} Q \circ Q_1\big(\rho(t)\big)$ and $\beta(t) \stackrel{\text{def}}{=} P \circ P_1\big(\rho(t)\big)$. The inequality in (6.10) is obtained from the following relation between the lengths of α and β:

$$\int_0^1 \|\alpha'(t)\| \, dt = \int_0^1 \|D(Q \circ Q_1)(\rho'(t))\| \, dt \leqslant \theta \int_0^1 \|D(P \circ P_1)(\rho'(t))\| \, dt = \theta \int_0^1 \|\beta'(t)\| \, dt.$$

Similarly, for every $y' \in R^+$ and every vector $v' \in T_{y'} R^+$ it holds $D(P \circ P_1)(v') = 0$, and hence $v' \in C_\theta(y', \mathfrak{X}_1^t)$. The $(\mathfrak{X}_1, \mathfrak{X}_2, \theta, \mu, \gamma)$-dilatation condition implies that

$$\|D(Q \circ Q_1)(v')\| \leqslant \gamma^{-1} \, \|D(Q \circ Q_2 \circ f)(v')\|.$$

As above, this implies that

$$\operatorname{diam}\big(Q \circ Q_1(R^+)\big) \leqslant \gamma^{-1} \operatorname{diam}\big(Q \circ Q_2(f(R^+))\big). \tag{6.11}$$

Since $\{x', x_*\} \subset R^-$ and $\{x, x_*\} \subset R^+$, it follows that

$$\begin{aligned}
\|(Q \circ Q_1)(x') - (Q \circ Q_1)(x)\| &\leqslant \|Q \circ Q_1(x') - Q \circ Q_1(x_*)\| + \|Q \circ Q_1(x_*) - Q \circ Q_1(x)\| \\
&\leqslant \operatorname{diam}\big(Q \circ Q_1(R^-)\big) + \operatorname{diam}\big(Q \circ Q_1(R^+)\big) \\
&\leqslant \theta \operatorname{diam}\big(P \circ P_1(R^-)\big) + \gamma^{-1} \operatorname{diam}\big(Q \circ Q_2(f(R^+))\big) \\
&\leqslant \theta \operatorname{diam}\big(P \circ P_1(R)\big) + \gamma^{-1} \operatorname{diam}\big(Q \circ Q_2(f(R))\big).
\end{aligned}$$

where the third inequality follows from (6.10) and (6.11) and the last inequality follows from $R^-, R^+ \subset R$. As the points $x, x' \in R$ are arbitrary, the first inequality in (6.8) follows.

The second inequality in (6.8) follows similarly. This ends the proof of the lemma. □

The following result is a version of Lemma 2.5 in [2] in our setting, it deals with dynamically defined rectangles under compositions of embeddings and pairs.

Lemma 6.7 (Lemma 2.5 in [2] with $N = 3$)**.** *Consider open manifolds N_i, open subsets U_i of N_i, and (p,q)-splittings $\mathfrak{L}_i = (P_i, Q_i)$ of N_i, where $i = 1, \ldots, 4$. Consider, for $i = 1,2,3$, embeddings $g_i \colon U_i \to N_{i+1}$ and compact $(P_i, Q_{i+1} \circ g_i)$-rectangles R_i in U_i such that*

$$P_{j+1}(g_j(R_j)) \subset \operatorname{int}(P_{j+1}(R_{j+1})) \quad \text{and} \quad Q_{j+1}(R_{j+1}) \subset \operatorname{int}(Q_{j+1}(g_j(R_j)), \quad j = 1, 2.$$

Let

$$G_3 \stackrel{\text{def}}{=} g_3 \circ g_2 \circ g_1, \quad G_2 \stackrel{\text{def}}{=} g_2 \circ g_1, \quad G_1 \stackrel{\text{def}}{=} g_1$$

be defined on the maximal open subset of U_1 *where the composition is well defined. Suppose that there is* $0 < \theta \leqslant 1$ *such that* g_i *satisfies the* $(\mathfrak{L}_i, \mathfrak{L}_{i+1}, \theta)$*-cone invariance condition on* R_i *for* $i = 1, 2, 3$*. Then the set*

$$R_* \stackrel{\text{def}}{=} R_1 \cap G_1^{-1}(R_2) \cap G_2^{-1}(R_3)$$

is a $(P_1, Q_4 \circ G_3)$*-rectangle such that*

$$P_1(R_*) = P_1(R_1) \quad \textit{and} \quad Q_4 \circ G_3(R_*) = Q_4 \circ g_3(R_3).$$

In particular, the set R_* *is non-empty.*

6.2. **Split blending machines.** We now extend the definition of a blending machine from [2, Section 2.4] by adding a central direction. To this end, we introduce conditions (BM3) and (BM4) in Definition 6.8. For further discussion, see Remark 6.9.

Given two sets A and B, the notation $A \Subset B$ means that the closure of A is contained in the interior of B.

Definition 6.8 (Split blending machine)**.** Let $f : U \to N$ be a C^1 map defined on an open subset U of a manifold N. Let d_s, d_c, and d_u be positive integers and set $d_{cs} \stackrel{\text{def}}{=} d_s + d_c$. Consider (see Figure 7)

- a family $\mathscr{R} = \{R_i\}_{i \in I}$ of compact subsets of U, where I is a finite set with at least two elements,
- a (d_{cs}, d_u)-splitting $\mathfrak{S} = (P_{cs}, Q_u)$ of N,
- a (d_s, d_c)-splitting $\mathfrak{C} = (P_s, P_c)$ of $\mathbb{R}^{d_{cs}}$,
- a $\mathfrak{S}$-rectangle $Z \subset U$, and
- a real number $0 < \theta < 1$.

The tuple $\mathfrak{B}_f = (\mathscr{R}, Z, \mathfrak{S}, \mathfrak{C}, \theta)$ is a *split blending machine for* f if the following holds:

(BM1) for every $i \in I$, the restriction of f to some neighbourhood of R_i is a C^1 embedding satisfying the $(\mathfrak{S}, \theta)$-cone invariance condition,

(BM2) for every $i \in I$, the set R_i is a $(P_{cs}, Q_u \circ f)$-rectangle such that

$$Q_u(R_i) \Subset Q_u(Z), \quad (Q_u \circ f)(R_i) = Q_u(Z), \quad \text{and} \quad (P_{cs} \circ f)(R_i) \Subset P_{cs}(Z),$$

(BM3) for every $i \in I$, the set $P_{cs}(R_i)$ is a $\mathfrak{C}$-rectangle such that $P_s \circ P_{cs}(Z) \Subset P_s \circ P_{cs}(R_i)$,

(BM4) $\{\text{int}\big((P_c \circ P_{cs})(R_i)\big)\}_{i \in I}$ is a covering of $\overline{P_c \circ P_{cs}(Z)}$ having a Lebesgue number Δ such that

$$\theta \, \|DP_c\| \operatorname{diam} Q_u(Z) < \Delta.$$

We call $\mathfrak{S}$ the *stable center splitting*, $\mathfrak{C}$ the *center splitting*, P_c the *central projection*, Z the *superposition domain*, θ the *cone width*, and Δ the *Lebesgue number* of the split blending machine. We say that the split blending machine $\mathfrak{B}_f$ has dimensions (d_s, d_c, d_u).

Remark 6.9 (Split blending machines versus blending machines)**.**

(i) As $d_s, d_c, d_u \geqslant 1$ then $\dim(N) \geqslant 3$.

R_2

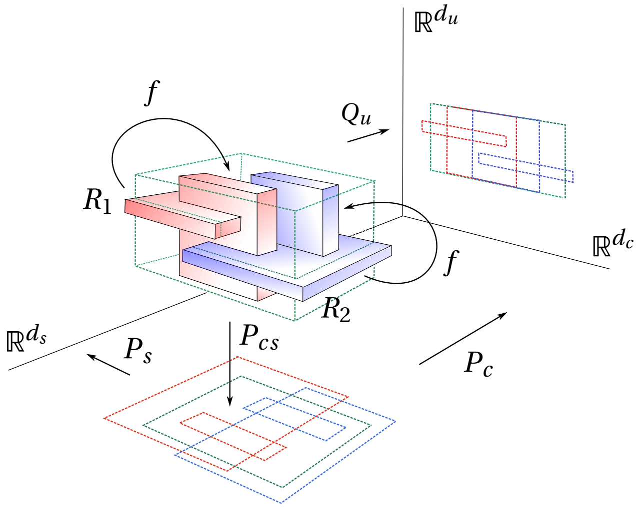

FIGURE 7. Split blending machine

(ii) The definition of a blending machine in [2] is recovered from Definition 6.8 taking $d_{cs} = d_c$ and $\mathfrak{C} = (P_s, P_c)$, with P_s the trivial projection onto $\{\bar{0}\}$ and P_c the identity map in $\mathbb{R}^{d_{cs}}$.

(iii) Blending machines in [2] are defined in dimension at least two. In particular, any horseshoe of a surface diffeomorphism satisfies the definition of blending machine. To avoid this case, for that we assume that $d_s \geqslant 1$ and hence $d_c < d_{cs}$.

(iv) Condition $\theta < 1$ is implicitly used in [2]. Here this condition is explicitly used in several proofs along the paper.

Next remark states the robustness of split blending machines, see [2, Remark 2.9].

Remark 6.10 (Robustness of split blending machines)**.** Let $\mathfrak{B}_f = (\mathscr{R}, Z, \mathfrak{S}, \mathfrak{C}, \theta)$ be a split blending machine, where $\mathscr{R} = \{R_i\}_{i \in I}$. For every $i \in I$, consider a neighbourhood U_i of R_i and a compact subset $K_i \subset \mathrm{int}(R_i)$. Then there is a C^1 neighbourhood $\mathscr{U}$ of f such that every $g \in \mathscr{U}$ has a split blending machine $\mathfrak{B}_g = (\mathscr{R}', Z, \mathfrak{S}, \mathfrak{C}, \theta)$, where $\mathscr{R}' = \{R'_i\}_{i \in I}$ is a family of sets with $K_i \subset \mathrm{int}(R'_i) \subset U_i$ for every $i \in I$.

In what follows, if no confusion arises, we will refer to split blending machines simply as blending machines.

6.3. **Superposition properties of split blending machines.** Blending machines were introduced to generate cu-blenders (see [2, Proposition 2.10]). In Proposition 6.12, we show a similar for split blending machines. To this end, we first recall the notion of a cu-*blender* and introduce some terminology.

Definition 6.11 (cu-blender, Definition 3.1 in [7])**.** Let Γ be a hyperbolic compact set of a diffeomorphism f with a Df-invariant dominated splitting $E^s \oplus E^c \oplus E^{uu}$ into three non-trivial bundles, where E^s and $E^u = E^c \oplus E^{uu}$ are the stable and unstable bundles of Γ. Let $\mathscr{D}$ be an open set of disks[5] of dimension $d_{uu} = \dim(E^{uu})$. The pair $(\Gamma, \mathscr{D})$ is a cu-*blender* if there is a C^1 neighbourhood $\mathscr{U}$ of f such that

$$D \cap W^s_{\mathrm{loc}}(\Gamma_g) \neq \emptyset \quad \text{for every } g \in \mathscr{U} \text{ and every } D \in \mathscr{D}, \tag{6.12}$$

where Γ_g is the continuation of Γ for g. The number $d_c = \dim(E^c) > 0$ is the *central dimension* and the set $\mathscr{D}$ is the *superposition region* of the cu-blender. With a slight abuse of notation (when the role of $\mathscr{D}$ is omitted), we also call the set Γ a blender.

Let X be a topological space, Y an open set of X, K a compact subset of Y, and $f\colon Y \to X$ a continuous map. The *maximal forward invariant set* $\Lambda^s(K,f)$ of f in K is

$$\Lambda^s(K,f) \stackrel{\text{def}}{=} \bigcap_{n\geqslant 0} f^{-n}(K). \tag{6.13}$$

For invertible maps f, we also define

$$\Lambda^u(K,f) \stackrel{\text{def}}{=} \Lambda^s(K,f^{-1}) \quad \text{and} \quad \Lambda(K,f) \stackrel{\text{def}}{=} \Lambda^s(K,f) \cap \Lambda^u(K,f).$$

Given a split blending machine $\mathfrak{B}_f = (\mathscr{R}, Z, \mathfrak{S}, \mathfrak{C}, \theta)$ with $\mathscr{R} = \{R_i\}_{i\in I}$, its *maximal forward invariant set* is the set given by

$$\Lambda^s(\mathfrak{B}_f) \stackrel{\text{def}}{=} \Lambda^s\big(\bigcup_{i\in I} R_i, f\big). \tag{6.14}$$

The next result is a variation of [2, Proposition 2.10] for split blending machines, where the intersection property in (6.12) is reformulated: the family of disks is replaced by a family of graphs, and the local stable set of the blender by the forward invariant set of the split blending machine. This proposition also gives a sufficient condition for a split blending machine to generate a blender.

Proposition 6.12. *Let $f\colon U \to N$ be a C^1 embedding defined on an open subset U of N and $\mathfrak{B}_f = (\mathscr{R}, Z, \mathfrak{S}, \mathfrak{C}, \theta)$ a split blending machine for f with dimensions (d_s, d_c, d_u). Let $\mathfrak{S} = (P_{cs}, Q_u)$ and $\mathfrak{C} = (P_s, P_c)$. Consider the set $\mathscr{S}$ of C^1 maps $\sigma\colon Q_u(Z) \to \mathrm{int}\big(P_{cs}(Z)\big)$ with $\|D\sigma\| < \theta$. Then, for every $\sigma \in \mathscr{S}$, it holds*

$$\mathrm{graph}(\sigma) \cap \Lambda^s(\mathfrak{B}_f) \neq \emptyset \qquad \textit{where} \qquad \mathrm{graph}(\sigma) \stackrel{\text{def}}{=} \{x \in Z\colon P_{cs}(x) = \sigma\big(Q_u(x)\big)\}.$$

Moreover, assume that

- *$\mathscr{D}$ is a set of C^1 embedded compact disks of dimension d_u such that for every $D \in \mathscr{D}$ there is $\sigma \in \mathscr{S}$ with $\mathrm{graph}(\sigma) \subset \mathrm{int}(D)$, and*
- *Γ is a hyperbolic set of f such that*
  - (a) *there is a neighbourhood K of $\bigcup R_i$, $R_i \in \mathscr{R}$, with $\Lambda(K,f) \subset \Gamma$, and*
  - (b) *Γ has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^{uu}$, where E^s and $E^u = E^c \oplus E^{uu}$ are the stable and unstable bundles of Γ and $\dim(E^s) = d_s$, $\dim(E^c) = d_c$, and $\dim(E^{uu}) = d_u$.*

Then $(\Gamma, \mathscr{D})$ is a cu-blender with central dimension d_c.

[5] Here the elements of the set $\mathscr{D}$ are images of embedded disks and a suitable metric is considered in the space of these disks, see [7, Section 3.1] for details.

The proof of this proposition follows the ideas in [2, Proposition 2.10] adapting the argument of graph transform in [5]. We proceed to the details.

Take any $\sigma \in \mathscr{S}$ and note that $\operatorname{graph}(\sigma) \subset Z$. Then

$$\begin{aligned} \operatorname{diam}\big((P_c \circ P_{cs})(\operatorname{graph}(\sigma))\big) &\leqslant \|DP_c\| \operatorname{diam}\big(P_{cs}(\operatorname{graph}(\sigma))\big) = \\ &= \|DP_c\| \operatorname{diam}\big(\sigma(Q_u(\operatorname{graph}(\sigma)))\big) \\ &\leqslant \|DP_c\| \, \|D\sigma\| \operatorname{diam}\big(Q_u(\operatorname{graph}(\sigma))\big) \\ &< \|DP_c\| \, \theta \operatorname{diam}\big(Q_u(Z)\big) \overset{\text{(BM4)}}{<} \Delta, \end{aligned} \tag{6.15}$$

where the first equality follows from the definition of $\operatorname{graph}(\sigma)$ and Δ is the Lebesgue number of the split blending machine $\mathfrak{B}_f$, see Definition 6.8.

We need the following lemma used instead of the Lebesgue number of the covering $\{\operatorname{int}\big(P_{cs}(R_i)\big)\}_{i\in I}$ of $\overline{P_{cs}(Z)}$ in [2].

Lemma 6.13. *Consider any set $K \subset \overline{P_{cs}(Z)}$ with $\operatorname{diam}\big(P_c(K)\big) < \Delta$. Then there is $i \in I$ such that $K \subset \operatorname{int}\big(P_{cs}(R_i)\big)$.*

Proof. Note that $P_c(K) \subset P_c\big(\overline{(P_{cs})(Z)}\big) \subset \overline{(P_c \circ P_{cs})(Z)}$ and (recall (BM4)) $\operatorname{diam}\big(P_c(K)\big) < \Delta$. As Δ is a Lebesgue number of the covering $\{\operatorname{int}(P_c \circ P_{cs})(R_i)\}_{i\in I}$ of $\overline{(P_c \circ P_{cs})(Z)}$, there exists $i \in I$ such that

$$P_c(K) \subset \operatorname{int}\big(P_c \circ P_{cs}(R_i)\big). \tag{6.16}$$

Moreover,

$$P_s(K) \subset \overline{(P_s \circ P_{cs})(Z)} \overset{\text{(BM3)}}{\subset} \operatorname{int}(P_s \circ P_{cs}(R_i)). \tag{6.17}$$

As $P_{cs}(R_i)$ is a $\mathfrak{C}$-rectangle (BM3) and $\mathfrak{C} = (P_s, P_c)$, these inclusions imply that

$$\begin{aligned} K &\subset \mathfrak{C}^{-1}\big(P_s(K) \times P_c(K)\big) \overset{(6.17),(6.16)}{\subset} \mathfrak{C}^{-1}\big(\operatorname{int}(P_s \circ P_{cs})(R_i) \times \operatorname{int}(P_c \circ P_{cs})(R_i)\big) \\ &= \operatorname{int}\big(\mathfrak{C}^{-1}\big((P_s \circ P_{cs})(R_i) \times (P_c \circ P_{cs})(R_i)\big)\big) \\ &= \operatorname{int}\big(\mathfrak{C}^{-1}(\mathfrak{C}(P_{cs}(R_i))\big) = \operatorname{int}(P_{cs}(R_i)), \end{aligned}$$

proving the lemma. □

Equation (6.15) allows us to apply Lemma 6.13 to the set $P_{cs}\big(\operatorname{graph}(\sigma)\big) \subset P_{cs}(Z)$ to get $i_0 \in I$ such that $P_{cs}\big(\operatorname{graph}(\sigma)\big) \subset \operatorname{int}\big(P_{cs}(R_{i_0})\big)$. For each $t \in Q_u(Z)$ consider the set

$$D(t) \overset{\text{def}}{=} \{x \in R_{i_0} \colon Q_u \circ f(x) = t\}.$$

Lemma 6.14. *The intersection* $\operatorname{graph}(\sigma) \cap D(t)$ *is a singleton for every* $t \in Q_u(Z)$.

Proof. Fix $t \in Q_u(Z)$. As R_{i_0} is a $(P_{cs}, Q_u \circ f)$-rectangle (BM2) we can consider the C^1 map $g\colon P_{cs}(R_{i_0}) \to Q_u(R_{i_0})$ defined by $g = Q_u \circ (P_{cs}|_{D(t)})^{-1}$.

Claim 6.15. $\|Dg\| < \theta$.

We are postponing the proof of the claim for now to conclude the proof of the lemma.

Recall that $\mathfrak{S} = (P_{cs}, Q_u)$ and observe that $\mathfrak{S}(D(t))$ is the graph of g,

$$\mathfrak{S}(D(t)) = \{(s, g(s)) : s \in P_{cs}(R_{i_0})\}. \tag{6.18}$$

From $P_{cs}\big(\mathrm{graph}(\sigma)\big) \subset \mathrm{int}\big(P_{cs}(R_{i_0})\big)$ and the definition of g, we define the map

$$G\colon P_{cs}(R_{i_0}) \times Q_u(R_{i_0}) \to P_{cs}(R_{i_0}) \times Q_u(R_{i_0}), \quad G(s,t) \stackrel{\text{def}}{=} (\sigma(t), g(s)).$$

Since $\|Dg\|$ and $\|D\sigma\|$ are less than $\theta < 1$, the map G is a uniform contraction. Hence, by the compactness of its domain, G has a unique fixed point.

Note that $x \in \mathrm{graph}(\sigma) \cap D(t)$ if and only if $\big(P_{cs}(x), Q_u(x)\big)$ can be simultaneously written as $(\sigma(t), t)$ and $(s, g(s))$, for some s and t. This implies that $\big(P_{cs}(x), Q_u(x)\big)$ is a fixed point of G. As G has a unique fixed point, this intersection is a singleton, proving the lemma. It remains to prove the claim.

Proof of Claim 6.15. Consider $x \in D(t)$ and any vector $v \in T_xD(t)$. Since $Q_u \circ f(x) = t$, it follows that $D(Q_u \circ f)(v) = 0$ and thus $Df(v) \in C_\theta\big(f(x), \mathfrak{S}\big)$. As f satisfies the $(\mathfrak{S},\theta)$-cone invariance condition on R_{i_0}, we get $v \in Df^{-1}\big(C_\theta(f(x),\mathfrak{S})\big) \subset C_\theta(x,\mathfrak{S})$. Therefore, by the definition of $C_\theta(x,\mathfrak{S})$ in (6.1), it holds that

$$\|DQ_u(v)\| \leqslant \theta \|DP_{cs}(v)\| \quad \text{for every } x \in D(t) \text{ and } v \in T_x D(t).$$

This inequality and $g = Q_u \circ (P_{cs}|_{D(t)})^{-1}$ imply that $\|Dg\| < \theta$, proving the claim. □

The proof of the lemma is now complete. □

We are now ready to complete the proof of Proposition 6.12.

Proof of Proposition 6.12. Fix any $\sigma_0 \in \mathscr{S}$. Lemma 6.14 allows us to define a C^1 map

$$\sigma_1\colon Q_u(Z) \to P_{cs}(Z) \quad \text{such that} \quad f\big(\mathrm{graph}(\sigma_0) \cap R_{i_0}\big) = \mathrm{graph}(\sigma_1).$$

Using that $P_{cs}\big(f(R_{i_0})\big) \Subset P_{cs}(Z)$, (BM2), and that f satisfies the $(\mathfrak{S},\theta)$-cone invariance condition on R_{i_0}, (BM1), one obtain that $\sigma_1 \in \mathscr{S}$. Iterating this process, one gets sequences $\{i_n\}_n$ in I and $\{\sigma_n\}_n$ in $\mathscr{S}$ such that

$$f\big(\mathrm{graph}(\sigma_n) \cap R_{i_n}\big) = \mathrm{graph}(\sigma_{n+1}) \quad \text{for every } n \geqslant 0.$$

By construction,

$$\emptyset \neq \bigcap_{n\geqslant 0} f^{-n}\big(\mathrm{graph}(\sigma_n)\big) \subset \bigcap_{n\geqslant 0} f^{-n}(R_{i_n}) \cap \mathrm{graph}(\sigma_0) \subset \Lambda^s(\mathfrak{B}_f) \cap \mathrm{graph}(\sigma_0),$$

which proves the first part of the proposition.

To prove the second part of the proposition, it suffices to observe that, by definition of Γ, it holds $\Lambda^s(\mathfrak{B}_f) \subset W^s_{\mathrm{loc}}(\Gamma)$, ending the proof. □

7. SPLIT BLENDING MACHINES FOR SKEW-PRODUCTS

In the spirit of Sections 3-5, we consider skew-products whose base and fiber dynamics are given by an affine horseshoe map and a family of local diffeomorphisms, see Section 7.1. These systems arise from diffeomorphisms with simple contours and their adapted perturbations. In Section 7.2, we define *preblending machines*, which serve as germs of split blending machines. Section 7.3 contains Theorem 7.14, showing how preblending machines yield split blending machines for skew-products.

7.1. **Skew-products over affine horseshoes.**

Definition 7.1 (Locally affine horseshoe maps)**.** Fix integers $k, d_s, d_u \geqslant 1$ and $\nu \in (0,1)$. Let

$$\mathsf{D} \stackrel{\text{def}}{=} H \times V, \quad \text{where } H \stackrel{\text{def}}{=} [0,1]^{d_s} \text{ and } V \stackrel{\text{def}}{=} [0,1]^{d_u}. \tag{7.1}$$

A *locally affine* (k, d_s, d_u, ν)*-horseshoe map* (for short a (k, d_s, d_u, ν)*- or a* (k,ν)*-horseshoe map* if the roles of d_s and d_u are not relevant) is a diffeomorphism $A\colon \mathbb{R}^{d_s+d_u} \to \mathbb{R}^{d_s+d_u}$ such that there are families:

- $(V_i)_{i=1}^k$ of pairwise disjoint compact disks of dimension d_u contained in $\mathrm{int}(V)$,
- $(H_i)_{i=1}^k$ of pairwise disjoint of compact disks of dimension d_s contained in $\mathrm{int}(H)$,
- $(A_i^s)_{i=1}^k$ and $(A_i^u)_{i=1}^k$ of affine maps $A_i^s\colon \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$ and $A_i^u\colon \mathbb{R}^{d_s} \to \mathbb{R}^{d_s}$, with

$$\max\big\{\|DA_i^s\|, \|(DA_i^u)^{-1}\|, \, i \in \{1,\dots,k\}\big\} < \nu \tag{7.2}$$

such that for every $i = 1,\dots,k$ it holds:

(a) $A(H \times V_i) = H_i \times V$,

(b) the restriction $A|_{\mathsf{H}_i}$ of A to $\mathsf{H}_i \stackrel{\text{def}}{=} H \times V_i$ coincides with $A_i \stackrel{\text{def}}{=} A_i^s \times A_i^u$.

The number ν is called a *stretching* of A.

Remark 7.2 (Families of legs)**.** We let $\Sigma_k = \{1,\dots,k\}^{\mathbb{Z}}$ and Σ_k^* the family of finite words. Given $\mathtt{i} \in \Sigma_k^*$, using the notation for composition of maps in (3.5), define the sets

$$H_{\mathtt{i}} \stackrel{\text{def}}{=} A_{\mathtt{i}}^s(H) \quad \text{and} \quad V_{\mathtt{i}} \stackrel{\text{def}}{=} (A_{\mathtt{i}}^u)^{-1}(V) \tag{7.3}$$

and

$$\mathsf{H}_{\mathtt{i}} \stackrel{\text{def}}{=} H \times V_{\mathtt{i}} \quad \text{and} \quad \mathsf{V}_{\mathtt{i}} \stackrel{\text{def}}{=} H_{\mathtt{i}} \times V = A_{\mathtt{i}}(\mathsf{H}_{\mathtt{i}}). \tag{7.4}$$

Note that for $\mathtt{i} = i_0 \dots i_n$ it holds

$$A^j(\mathsf{H}_{\mathtt{i}}) \subset \mathsf{H}_{i_j} \quad \text{for every } j = 0,1,\dots,n-1.$$

Furthermore, A^j is a (k^j, d_s, d_u, ν^j)-horseshoe map whose restriction to $\mathsf{H}_{\mathtt{i}}$, $A^j|_{\mathsf{H}_{\mathtt{i}}}$, with $|\mathtt{i}| = j$, coincides with $A_{\mathtt{i}} = A_{\mathtt{i}}^s \times A_{\mathtt{i}}^u$. We refer to the family $(\mathsf{H}_{\mathtt{i}})_{\mathtt{i}\in\Sigma_k^*}$ of D as the *legs* of the horseshoe map A.

Remark 7.3 (Inverse horseshoe maps)**.** Given a (k, d_s, d_u, ν)-horseshoe map A, its inverse $B = A^{-1}$ is the (k, d_u, d_s, ν)-horseshoe map obtained considering the maps $(B_i^s)_{i=1}^k$ and $(B_i^u)_{i=1}^k$ given by $B_i^s = (A_i^u)^{-1}$ and $B_i^u = (A_i^s)^{-1}$.

Consider a manifold M, pairwise disjoint open sets $(U_i)_{i=1}^k$ of M, and a family of local diffeomorphisms $\mathscr{F} = \{\phi_i \colon U_i \to M, \, i = 1,\dots,k\}$. Consider the compositions $\phi_{\mathtt{i}}$, $\mathtt{i} \in \Sigma_k^*$, and their domains $U_{\mathtt{i}}$ as in (3.6). We also consider the corresponding transition matrix T as in (3.4) and the shift space Σ_T and the set Σ_T^* of admissible words.

Definition 7.4 (Skew-product over an affine horseshoe)**.** Let A be a horseshoe map as in Definition 7.1 and $\mathscr{F} = \{\phi_i \colon U_i \to M, \, i = 1,\dots,k\}$ a familiy of local C^1 diffeomorphisms of M. We define the skew-product with base map A and fiber maps $\mathscr{F}$ by

$$\Phi \stackrel{\text{def}}{=} A \ltimes \mathscr{F} \colon \bigcup_{i=1}^k (H_i \times U_i) \to \mathbb{R}^{d_s+d_u} \times M, \qquad \Phi|_{\mathsf{H}_i \times U_i} \stackrel{\text{def}}{=} A_i \times \phi_i. \tag{7.5}$$

Remark 7.5. Note that $\Phi^j|_{\mathsf{H}_{\mathtt{i}} \times U_{\mathtt{i}}} = A_{\mathtt{i}} \times \phi_{\mathtt{i}}$ for every $\mathtt{i} \in \Sigma_k^*$ with $|\mathtt{i}| = j$.

7.1.1. *Adapted stretching and partial hyperbolicity.* We now introduce mild assumptions on a skew-product $\Phi = A \ltimes \mathscr{F}$ and on the pair $(\mathfrak{X}_1, \mathfrak{X}_2)$ to guarantee that the locally affine horseshoe map $A^{|\mathtt{i}|}$ is $(\mathfrak{X}_1, \mathfrak{X}_2)$-adapted to $\phi_{\mathtt{i}}$ for every $\mathtt{i} \in \Sigma_k^*$.

Definition 7.6 (Adapted stretching)**.** Let ϕ be a $(\mathfrak{X}_1, \mathfrak{X}_2, \mu, \gamma)$-dominated C^1 local diffeomorphism on a subset R of N. A (k, ν)-horseshoe map A (or its stretching ν) is $(\mathfrak{X}_1, \mathfrak{X}_2)$-*adapted to* ϕ *on* R if $\nu < \mu^{-1} < \gamma < \nu^{-1}$. If $\mathfrak{X} = \mathfrak{X}_1 = \mathfrak{X}_2$, we say it is $\mathfrak{X}$-adapted.

Lemma 7.7. *Consider a family* $\mathscr{F} = \{\phi_1, \dots, \phi_k\}$ *of* (λ, ρ)*-Lipschitz local diffeomorphisms of* M *(see (7.8)). Let* $\mathfrak{X}_\tau = (P_\tau, Q_\tau)$*,* $\tau = 1, 2$*, be a splitting of an open set* $N_\tau \subset M$ *such that there is a constants* $C_\tau \geqslant 1$ *with*

$$\frac{1}{C_\tau}\,\|v\| \leqslant \|D\mathfrak{X}_\tau(v)\| \leqslant C_\tau\,\|v\| \quad \textit{for every } x \in N_\tau \textit{ and } v \in T_x N_\tau. \tag{7.6}$$

Suppose that $\phi_{\mathtt{i}}$*,* $\mathtt{i} \in \Sigma_k^*$*, is* $(\mathfrak{X}_1, \mathfrak{X}_2, \mu, \gamma)$*-dominated on a subset* R *of* N_1*. Then*

$$\frac{\lambda^{|\mathtt{i}|}}{C} \leqslant \mu^{-1} \quad \textit{and} \quad \gamma \leqslant C\,\rho^{|\mathtt{i}|} \quad \textit{where } C \overset{\text{def}}{=} C_1 C_2.$$

Furthermore, let A *be a* (k, ν)*-horseshoe map such that*

$$\nu^{|\mathtt{i}|} < \frac{\lambda^{|\mathtt{i}|}}{C} \quad \textit{and} \quad C\,\rho^{|\mathtt{i}|} < \nu^{-|\mathtt{i}|} \tag{7.7}$$

then $A^{|\mathtt{i}|}$ *is* $(\mathfrak{X}_1, \mathfrak{X}_2)$*-adapted to* $\phi_{\mathtt{i}}$ *on* R*.*

Proof. The (λ, ρ)-Lipschitz of the family $\mathscr{F}$ means

$$\lambda^{|\mathtt{i}|} < m(D_x\phi_{\mathtt{i}}) \leqslant \|D_x\phi_{\mathtt{i}}\| < \rho^{|\mathtt{i}|} \quad \text{for every } x \in M \text{ and } \mathtt{i} \in \Sigma_k^*. \tag{7.8}$$

Fix $x \in R$ and (recall (6.1)) consider any vector in $v \in C_1(x, \mathfrak{X}_1^t) \setminus \mathrm{Ker}(D_x Q_1)$ (note that this vector always exist). Then $\|DP_1(v)\| \leqslant \|DQ_1(v)\|$ and thus

$$\|D\mathfrak{X}_1(v)\| = \|DQ_1(v)\|. \tag{7.9}$$

Since $\phi_{\mathtt{i}}$ is $(\mathfrak{X}_1\mathfrak{X}_2, \mu, \gamma)$-dominated on R, we get

$$\begin{aligned} \gamma\,\|DQ_1(v)\| &\overset{(6.2)}{\leqslant} \|D(Q_2\circ\phi_{\mathtt{i}})(v)\| \leqslant \|D(\mathfrak{X}_2\circ\phi_{\mathtt{i}})(v)\| \overset{(7.6)}{\leqslant} C_2\,\|D\phi_{\mathtt{i}}(v)\| \\ &\overset{(7.8)}{\leqslant} C_2\,\rho^{|\mathtt{i}|}\,\|v\| \overset{(7.6)}{\leqslant} C_2 C_1\,\rho^{|\mathtt{i}|}\,\|D\mathfrak{X}_1(v)\| \overset{(7.9)}{=} C\,\rho^{|\mathtt{i}|}\|DQ_1(v)\|. \end{aligned}$$

As $\|DQ_1(v)\| \neq 0$ we get $\gamma \leqslant C\rho^{|\mathtt{i}|}$ and hence the second inequality in the lemma.

Similarly, take any $w \in C_1(\phi_{\mathtt{i}}(x), \mathfrak{X}_2) \setminus \mathrm{Ker}(D_{\phi_{\mathtt{i}}(x)}P_2)$. Hence, $\|DQ_2(w)\| \leqslant \|DP_2(w)\|$ and $\|D\mathfrak{X}_2(w)\| = \|DP_2(w)\|$. As $\phi_{\mathtt{i}}$ satisfies $(\mathfrak{X}_1, \mathfrak{X}_2, \mu, \gamma)$-dominated condition on R, by definition, it satisfies the $(\mathfrak{X}_1, \mathfrak{X}_2, 1)$-cone invariance condition on R. Thus, $v = D\phi_{\mathtt{i}}^{-1}(w) \in C_{\theta'}(x, \mathfrak{X}_1)$ for some $0 < \theta' < 1$. This implies that $\|DQ_1(v)\| \leqslant \|DP_1(v)\|$ and hence $\|D\mathfrak{X}_1(v)\| = \|DP_1(v)\|$. As above, Lemma 6.4 and equations (7.6) and (7.8) imply that

$$\begin{aligned} \mu^{-1}\,\|DP_1(v)\| &\geqslant \|D(P_2\circ\phi_{\mathtt{i}})(v)\| = \|DP_2(w)\| = \|D\mathfrak{X}_2(w)\| \overset{(7.6)}{\geqslant} \frac{1}{C_2}\,\|w\| \\ &\geqslant \frac{1}{C_2}\,\|D\phi_{\mathtt{i}}(v)\| \overset{(7.8)}{\geqslant} \frac{\lambda^{|\mathtt{i}|}}{C_2}\,\|v\| \overset{(7.6)}{\geqslant} \frac{\lambda^{|\mathtt{i}|}}{C_1 C_2}\,\|D\mathfrak{X}_1(v)\| = \frac{\lambda^{|\mathtt{i}|}}{C}\,\|DP_1(v)\|. \end{aligned}$$

As $\|DP_1(v)\| \neq 0$, this implies that $C\mu^{-1} \geqslant \lambda^{|\mathtt{i}|}$, which concludes the first part of the lemma. The second part of the lemma follows immediately, concluding the proof. $\square$

Definition 7.8. Given a (k,ν)-horseshoe map A and a finite family $\mathscr{F}$ of (λ,ρ)-Lipschitz local diffeomorphisms, the map $\Phi = A \ltimes \mathscr{F}$ in (7.5) is (ν,λ,ρ)-*partially hyperbolic* if

$$\nu < \lambda < \rho < \nu^{-1}. \tag{7.10}$$

Remark 7.9. Consider a family $\mathscr{F}$, a pair $(\mathfrak{X}_1,\mathfrak{X}_2)$, and a constant C as in Lemma 7.7. Let A be a (k,ν)-horseshoe such that the skew-product $\Phi = A \ltimes \mathscr{F}$ is (ν,λ,ρ)-partially hyperbolic and consider $\mathtt{i} \in \Sigma_k^*$ such that

$$|\mathtt{i}| > \max\left\{\frac{2\log C}{\log(\lambda\nu^{-1})}, \frac{2\log C}{-\log(\rho\nu)}\right\}.$$

Since this inequality implies (7.7) in Lemma 7.7, the horseshoe map $A^{|\mathtt{i}|}$ is $(\mathfrak{X}_1,\mathfrak{X}_2)$-adapted to $\phi_{\mathtt{i}}$ on R. Furthermore, if we have $C=1$ (as in the case of canonical projections), then it suffices to take $|\mathtt{i}| \geqslant 1$. In particular, if $\Phi = A \ltimes \mathscr{F}$ is partially hyperbolic, then A is adapted to any preblending machine $\mathfrak{D}_{\mathscr{F}} = (\mathscr{R}, B, \mathfrak{X})$ for $\mathscr{F}$ (see Definition 7.10) with splitting $\mathfrak{X} = (P,Q)$ given by canonical projections P and Q.

7.2. **Preblending machines.**

We now introduce preblending machines and state their robustness, see Remark 7.11.

Definition 7.10 (Preblending machine)**.** Let $\mathscr{F} = \{\phi_i : U_i \to M,\ i = 1,\dots,k\}$ be a family of local diffeomorphisms of a manifold M, an open subset N of M, and a positive integer $d_c < d = \dim(M)$. Consider

- a family $\mathscr{R} = \{R_{\mathtt{i}}\}_{\mathtt{i}\in I}$ of (nonempty) compact subsets of N, where $R_{\mathtt{i}} \subset U_{\mathtt{i}}$ and I is a finite subset of Σ_k^* with at least two elements,
- a $(d_c, d-d_c)$-splitting $\mathfrak{X} = (P,Q)$ of N, and
- a $\mathfrak{X}$-rectangle $B \subset N$.

The three-tuple $\mathfrak{D}_{\mathscr{F}} = (\mathscr{R}, B, \mathfrak{X})$ is a *preblending machine* for $\mathscr{F}$ if:

(PB1) for every $\mathtt{i} \in I$ there are $\mu_{\mathtt{i}}, \gamma_{\mathtt{i}}$, with $0 < \mu_{\mathtt{i}}^{-1} < \gamma_{\mathtt{i}}$, such that the map $\phi_{\mathtt{i}}$ is a C^1 embedding from an open neighbourhood of $R_{\mathtt{i}}$ into N which is $(\mathfrak{X}, \mu_{\mathtt{i}}, \gamma_{\mathtt{i}})$-dominated on $R_{\mathtt{i}}$,

(PB2) for every $\mathtt{i} \in I$, the set $R_{\mathtt{i}}$ is a $(P, Q\circ\phi_{\mathtt{i}})$-rectangle such that

$$Q(R_{\mathtt{i}}) \Subset Q(B), \quad (Q\circ\phi_{\mathtt{i}})(R_{\mathtt{i}}) = Q(B), \quad \text{and} \quad (P\circ\phi_{\mathtt{i}})(R_{\mathtt{i}}) \Subset P(B),$$

(PB3) the family $\{\mathrm{int}(P(R_{\mathtt{i}}))\}_{\mathtt{i}\in I}$ covers $\overline{P(B)}$.

We call B the *preblending region*, I the *symbols*, d_c *the central dimension*, and $(\mu_{\mathtt{i}})_{\mathtt{i}\in I}$, $(\gamma_{\mathtt{i}})_{\mathtt{i}\in I}$ the *dominated constants* of $\mathfrak{D}_{\mathscr{F}}$. A *Lebesgue number* of $\mathfrak{D}_{\mathscr{F}}$ is any Lebesgue number of the covering $\{\mathrm{int}(P(R_{\mathtt{i}}))\}_{\mathtt{i}\in I}$. The preblending machine is *expanding* if $\mu_{\mathtt{i}} > 1$ and $\gamma_{\mathtt{i}} > 1$ for every $\mathtt{i} \in I$.

Remark 7.11 (Robustness of preblending machines)**.** Let $\mathfrak{D}_{\mathscr{F}} = (\mathscr{R}, B, \mathfrak{X})$ be a preblending machine for the family $\mathscr{F} = \{\phi_i : U_i \to M, i = 1,\dots,k\}$, where $\mathscr{R} = \{R_{\mathtt{i}}\}_{\mathtt{i}\in I}$. For each $\mathtt{i} \in I$, consider a small neighbourhood $V_{\mathtt{i}}$ of $R_{\mathtt{i}}$, $\overline{V}_{\mathtt{i}} \subset U_{\mathtt{i}}$, and a compact subset $K_{\mathtt{i}} \subset \mathrm{int}(R_{\mathtt{i}})$. Then, for every $i = 1,2,\dots,k$, there exist a C^1 neighbourhood $\mathscr{U}_i$ of ϕ_i such that every family $\mathscr{G} = \{\varphi_i : U_i : M, i = 1,\dots,k\}$ with $\varphi_i \in \mathscr{U}_i$, has a preblending machine $\mathfrak{D}_{\mathscr{G}} = (\mathscr{R}', B, \mathfrak{X})$ where $\mathscr{R}' = \{R'_{\mathtt{i}}\}_{\mathtt{i}\in I}$ is a family of sets with $K_{\mathtt{i}} \subset R'_{\mathtt{i}} \subset V_{\mathtt{i}}$, $\mathtt{i} \in I$, where $V_{\mathtt{i}}$ is contained in $U_{\mathscr{G},\mathtt{i}}$ (the set $U_{\mathtt{i}}$ corresponding to the family $\mathscr{G}$).

Remark 7.12. Considering $d_c = d$, $P = \mathrm{id}$, and Q as the trivial projection, Definition 7.10 can be restated as follows:

(PB'1) For every $\mathtt{i} \in I$, the map $\phi_{\mathtt{i}}$ is a C^1 embedding on a neighborhood U of $R_{\mathtt{i}}$ satisfying $\|D\phi_{\mathtt{i}}(v)\| \leqslant \mu_{\mathtt{i}}^{-1}\|v\|$ for every $v \in T_x U$.

(PB'2) For every $\mathtt{i} \in I$, it holds that $\phi_{\mathtt{i}}(R_{\mathtt{i}}) \Subset B$.

(PB'3) $\overline{B} \subset \bigcup_{\mathtt{i}\in I} \mathrm{int}(R_{\mathtt{i}})$.

If the family $\mathscr{F}$ is invertible, the conditions above recover the definition of a blending region for $\mathscr{F}^{-1} \stackrel{\text{def}}{=} \{\phi_1^{-1}, \dots, \phi_k^{-1}\}$ without assuming contraction for $\mathscr{F}^{-1}$ (i.e., expansion for $\mathscr{F}$). Contracting blending regions were introduced in [27, 5] (see also [4, Definition 6.1]). Later, in [6, Definition 5.2], the authors considered blending regions that contract in some directions and expand in others, while still providing a cover as in (PB'3). Note that Definition 7.10 requires neither contraction nor expansion. Furthermore, the covering condition (PB3) is of a different nature, as it only requires covering the projection of the set B. These differing features are depicted in Figure 8.

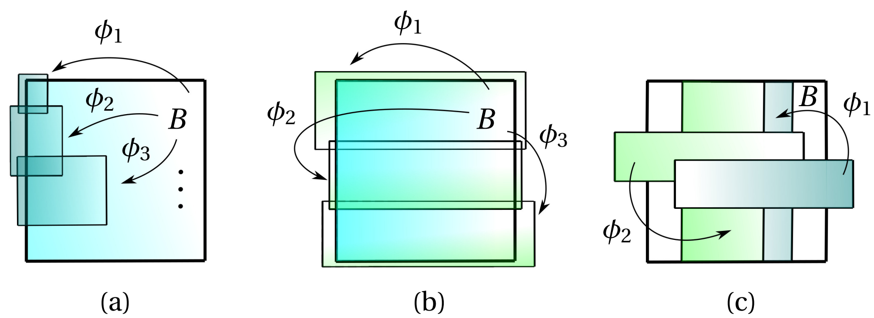

FIGURE 8. (a) Contracting blending region. (b) Non-contractive blending region. (c) Preblender region

7.3. **Preblending machines leading to split blending machines.** Theorem 7.14 states how preblending machines associated to the fiber maps of skew-product over horseshoes lead to split blending machines of tower maps.

We start by defining the latter element. Given a skew-product map $\Phi = A \ltimes \mathscr{F}$, where $\mathscr{F} = \{\phi_i \colon U_i \to M, i = 1, \dots, k\}$, as in Definition 7.4, consider the tupla $\mathfrak{T} = (\Phi, \mathscr{R}, I)$, where I is a finite subset of Σ_k^*, $\mathscr{R} = \{R_{\mathtt{i}}\}_{\mathtt{i}\in I}$ is a family of (nonempty) compact subsets of $\mathsf{D} \times M$ (recall (7.1)) with $R_{\mathtt{i}} \subset \mathsf{H}_{\mathtt{i}} \times U_{\mathtt{i}}$ for every $\mathtt{i} \in I$.

Definition 7.13 (Skew-product tower map)**.** The *skew-product tower map* $F = F_{\mathfrak{T}}$ *associated to* $\mathfrak{T} = (\Phi, \mathscr{R}, I)$ is defined by

$$F \colon \bigcup_{\mathtt{i}\in I} R_{\mathtt{i}} \to \mathsf{D} \times M, \qquad F|_{R_{\mathtt{i}}} \stackrel{\text{def}}{=} \Phi^{|\mathtt{i}|}|_{R_{\mathtt{i}}} = A_{\mathtt{i}} \times \phi_{\mathtt{i}}, \tag{7.11}$$

By Remark 7.5, this map is well defined.

Theorem 7.14 (Split blending machines for tower maps)**.** *Let* $\mathscr{F} = \{\phi_1, \dots, \phi_k\}$ *be a family of* C^1 *local diffeomorphisms of a manifold* M *and* A *a* (k, d_s, d_u, ν)*-horseshoe map with domain* $\mathsf{D} = [0,1]^{d_s} \times [0,1]^{d_u} = \mathsf{H} \times \mathsf{V}$ *with elements as in Definition 7.1. Consider*

- *a preblending machine $\mathfrak{D}_{\mathscr{F}} = (\mathscr{R}, B, \mathfrak{X})$ with central dimension $d_c \leqslant d = \dim(M)$ for $\mathscr{F}$, where $\mathscr{R} = \{R_{\mathtt{i}}\}_{\mathtt{i}\in I}$, $I \subset \Sigma_k^*$, and $\mathfrak{X} = (P, Q)$,*
- *the skew-product tower map F associated to $(A \ltimes \mathscr{F}, \mathscr{R}, I)$, with*

$$\mathscr{R} = \{\mathbf{R}_{\mathtt{i}}\}_{\mathtt{i}\in I}, \qquad \mathbf{R}_{\mathtt{i}} \stackrel{\text{def}}{=} \mathsf{H}_{\mathtt{i}} \times R_{\mathtt{i}}, \quad \text{where } \mathsf{H}_{\mathtt{i}} \text{ as in (7.4)}.$$

Assume that the stretching of A is adapted to $\mathfrak{D}_{\mathscr{F}}$. Then there is $\theta_0 > 0$ such that for every $0 < \theta \leqslant \theta_0$ the tuple $\mathfrak{B}_{\mathfrak{D}_{\mathscr{F}},F} = (\mathscr{R}, Z, \mathfrak{S}, \mathfrak{C}, \theta)$ is a split blending machine for F with dimension $(d_s, d_c, d - d_c + d_u)$, where

(1) *$Z = \mathsf{D} \times B$,*
(2) *$\mathfrak{S} = (P_{cs}, Q_u)$ is the $(d_s + d_c, d - d_c + d_u)$-splitting of $\mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times M$ given by*

$$\begin{aligned} P_{cs}(x^s, x^u, x) &\stackrel{\text{def}}{=} \big(\Pi_s(x^s, x^u), P(x)\big) = \big(x^s, P(x)\big) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_c} \quad \text{and} \\ Q_u(x^s, x^u, x) &\stackrel{\text{def}}{=} \big(Q(x), \Pi_u(x^s, x^u)\big) = \big(Q(x), x^u\big) \in \mathbb{R}^{d-d_c} \times \mathbb{R}^{d_u}, \end{aligned} \tag{7.12}$$

where $(x^s, x^u, x) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times M$ and

(3) *$\mathfrak{C} = (\pi_s, \pi_c)$ is the (d_s, d_c)-splitting of $\mathbb{R}^{d_{cs}}$, $d_{cs} \stackrel{\text{def}}{=} d_c + d_s$, given by the standard projections from $\mathbb{R}^{d_{cs}} = \mathbb{R}^{d_s} \times \mathbb{R}^{d_c}$ into $\mathbb{R}^{d_s}$ and $\mathbb{R}^{d_c}$ respectively.*

We say that the split blending machine $\mathfrak{B}_{\mathfrak{D}_{\mathscr{F}},F}$ is associated to $\mathfrak{D}_{\mathscr{F}}$ and F.

Proof. We begin with a technical lemma.

Lemma 7.15. *Consider (p, q)-splittings $\mathfrak{X}_\tau = (P_\tau, Q_\tau)$ of open manifolds N_τ, $\tau = 1, 2$, an open set U of N_1, a subset $R \subset U$, and a C^1 embedding $\phi\colon U \to N_2$. Assume that:*

- *ϕ is $(\mathfrak{X}_1, \mathfrak{X}_2, \mu, \gamma)$-dominated on R,*
- *there exists a (k, d_s, d_u, ν)-horseshoe map A with legs $\{\mathsf{H}_{\mathtt{i}}\}_{\mathtt{i}\in\Sigma_k^*}$,*
- *for some $j \geqslant 1$, the (k^j, d_s, d_u, ν^j)-horseshoe map A^j is $(\mathfrak{X}_1, \mathfrak{X}_2)$-adapted to ϕ on R.*

Let Π_s and Π_u be the canonical projections from $\mathbb{R}^{d_s} \times \mathbb{R}^{d_u}$ into $\mathbb{R}^{d_s}$ and $\mathbb{R}^{d_u}$, respectively. Define the maps

$$\begin{aligned} P_{cs,\tau}(x^s, x^u, x) &\stackrel{\text{def}}{=} \big(\Pi_s(x^s, x^u), P_\tau(x)\big) = \big(x^s, P_\tau(x)\big) \in \mathbb{R}^{d_s} \times \mathbb{R}^{p} \quad \text{and} \\ Q_{u,\tau}(x^s, x^u, x) &\stackrel{\text{def}}{=} \big(Q_\tau(x), \Pi_u(x^s, x^u)\big) = \big(Q_\tau(x), x^u\big) \in \mathbb{R}^{q} \times \mathbb{R}^{d_u}, \end{aligned}$$

where $(x^s, x^u, x) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times N_\tau$ with $\tau = 1, 2$.

Then for every $\mathtt{i} \in \Sigma_k^*$ with $|\mathtt{i}| = j$, the map $A_{\mathtt{i}} \times \phi$ is $(\mathfrak{S}_1, \mathfrak{S}_2, \mu, \gamma)$-dominated on $R_{\mathtt{i}} \stackrel{\text{def}}{=} \mathsf{H}_{\mathtt{i}} \times R$, where $\mathfrak{S}_\tau \stackrel{\text{def}}{=} (P_{cs,\tau}, Q_{u,\tau})$ is a $(d_s + p, q + d_u)$-splitting of $\mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times N_\tau$, $\tau = 1, 2$.

Proof. Fix $\mathtt{i} \in \Sigma_k^*$ with $|\mathtt{i}| = j$. Write $\bar{x} = (x^s, x^u, x) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times N_1$ and $\mathbf{v} = (v^s, v^u, v) \in \mathbb{R}^{d_s} \times \mathbb{R}^{d_u} \times T_x N_1$. Since ϕ is $(\mathfrak{X}_1, \mathfrak{X}_2, \mu, \gamma)$-dominated on R, Lemma 6.4 implies that

$$\|D(P_2 \circ \phi)(v)\| \leqslant \mu^{-1} \|DP_1(v)\| \quad \text{and} \quad \|D(Q_2 \circ \phi)(v)\| \geqslant \gamma \|DQ_1(v)\|. \tag{7.13}$$

Using these inequalities and $\nu^{|\mathtt{i}|} < \mu^{-1} < \gamma < \nu^{-|\mathtt{i}|}$, recalling Definition 7.6, and setting $f = A_{\mathtt{i}} \times \phi$, we get

$$\begin{aligned} \|D(P_{cs,2} \circ f)(\mathbf{v})\| &= \big\|\big(\nu^{|\mathtt{i}|} v^s, D(P_2 \circ \phi)(v)\big)\big\| \stackrel{(7.13)}{\leqslant} \big\|\big(\nu^{|\mathtt{i}|} v^s, \mu^{-1} DP_1(v)\big)\big\| \\ &\leqslant \mu^{-1} \big\|\big(v^s, DP_1(v)\big)\big\| = \mu^{-1} \|DP_{cs,1}(\mathbf{v})\| \end{aligned} \tag{7.14}$$

and

$$\|D(Q_{u,2}\circ f)(\mathbf{v})\| = \|\big(D(Q_2\circ\phi)(v), \nu^{-|\mathtt{i}|}v^u\big)\| \overset{(7.13)}{\geqslant} \|\big(\gamma DQ_1(v), \nu^{-|\mathtt{i}|}v^u\big)\| \\ \geqslant \gamma\|\big(DQ_1(v), v^u\big)\| = \gamma\|DQ_{u,1}(\mathbf{v})\|. \tag{7.15}$$

Take $\mathbf{v} = (v^s, v^u, v) \notin \operatorname{Ker} D_{\tilde{x}}(P_{cs,2}\circ f)$. We claim that the inequality in (7.14) is strict for $\mathbf{v}$. Note first that if $\|\nu^{|\mathtt{i}|}v^s\| \leqslant \|D(P_2\circ\phi)(v)\|$, then $\|D(P_{cs,2}\circ f)(\mathbf{v})\| = \|D(P_2\circ\phi)(v)\|$ and consequently $v \notin \operatorname{Ker} D_x(P_2\circ\phi)$. Thus, by Lemma 6.4, the first inequality in (7.13) is strict and hence, the first inequality in (7.14) is also strict. On the other hand, if $\|D(P_2\circ\phi)(v)\| < \|\nu^{|\mathtt{i}|}v^s\|$, then $v^s \neq 0$, thus the second inequality in (7.14) is strict since $\nu^{|\mathtt{i}|} < \mu^{-1}$. Arguing analogously, we get that if $\mathbf{v} \notin \operatorname{Ker} D_{\tilde{x}} Q_{u,1}$ then the inequality in (7.15) is strict. Lemma 6.4 now implies that $f = A_{\mathtt{i}}\times\phi$ is $(\mathfrak{S}_1,\mathfrak{S}_2,\mu,\gamma)$-dominated on $R_{\mathtt{i}}$. $\square$

To prove the theorem, we wiil see that (PB1)-(PB3) in Definition 7.10 for $\mathfrak{D}_{\mathscr{F}}$ imply (BM1)-(BM4) in Definition 6.8 for $\mathfrak{B}_F$ for every $\theta > 0$ small enough.

Proof of (BM1)*:* As the stretching ν of A is adapted to $\mathfrak{D}_{\mathscr{F}}$, it follows that $\phi_{\mathtt{i}}$ is $(\mathfrak{X},\mu_{\mathtt{i}},\gamma_{\mathtt{i}})$-dominated on $R_{\mathtt{i}}$ with $\nu^{|\mathtt{i}|} < \mu_{\mathtt{i}}^{-1} < \gamma_{\mathtt{i}} < \nu^{-|\mathtt{i}|}$ for every $\mathtt{i}\in I$, (PB1). Lemma 7.15 implies that $A_{\mathtt{i}}\times\phi_{\mathtt{i}}$ satisfies the $(\mathfrak{S},\theta)$-cone invariance condition on $\mathbf{R}_{\mathtt{i}}$ for every $\theta > 0$. As $F|_{\mathbf{R}_{\mathtt{i}}} = A_{\mathtt{i}}\times\phi_{\mathtt{i}}$, it follows that F satisfies the $(\mathfrak{S},\theta)$-cone invariance condition on $\mathbf{R}_{\mathtt{i}}$.

Proof of (BM2)*:* Recall $H_{\mathtt{i}}$ and $V_{\mathtt{i}}$ in (7.3). Note that $R_{\mathtt{i}}$ is a $(P, Q\circ\phi_{\mathtt{i}})$-rectangle for every $\mathtt{i}\in I$, (PB2). As $F|_{\mathbf{R}_{\mathtt{i}}} = A_{\mathtt{i}}\times\phi_{\mathtt{i}}$, it follows that every $\mathbf{R}_{\mathtt{i}}$ is a $(P_{cs}, Q_u\circ F)$-rectangle. Similarly, we get

$$Q_u(\mathbf{R}_{\mathtt{i}}) = Q(R_{\mathtt{i}})\times V_{\mathtt{i}} \overset{(PB2)}{\Subset} Q(B)\times V \overset{(7.12)}{=} Q_u(Z),$$
$$(Q_u\circ F)(\mathbf{R}_{\mathtt{i}}) = (Q\circ\phi_{\mathtt{i}})(R_{\mathtt{i}})\times V = Q(B)\times V \overset{(7.12)}{=} Q_u(Z),$$
$$(P_{cs}\circ F)(\mathbf{R}_{\mathtt{i}}) = H_{\mathtt{i}}\times(P\circ\phi_{\mathtt{i}})(R_{\mathtt{i}}) \Subset H\times P(B) \overset{(7.12)}{=} P_{cs}(Z),$$

proving (BM2).

Proof of (BM3)*:* Note that $P_{cs}\colon \mathbb{R}^{d_s+d_u}\times M \to \mathbb{R}^{d_s+d_c} = \mathbb{R}^{d_{cs}}$. Thus we can consider the composition $\pi_s\circ P_{cs}$ to get

$$(\pi_s\circ P_{cs})(\mathbf{R}_{\mathtt{i}}) = \pi_s(H\times R_{\mathtt{i}}) = H = \pi_s(H\times B) = (\pi_s\circ P_{cs})(Z),$$

obtaining (BM3).

Proof of (BM4)*:* Note that we have

$$(\pi_c\circ P_{cs})(Z) \overset{(7.12)}{=} P(B) \overset{(PB3)}{\Subset} \bigcup_{\mathtt{i}\in I} P(R_{\mathtt{i}}) = \bigcup_{\mathtt{i}\in I}(\pi_c\circ P_{cs})(\mathbf{R}_{\mathtt{i}}).$$

Let Δ be a Lebesgue number of the covering $\{\operatorname{int}(\pi_c\circ P_{cs})(\mathbf{R}_{\mathtt{i}})\}_{\mathtt{i}\in I}$ of $(\pi_c\circ P_{cs})(Z)$. Taking θ_0 small enough such that $\Delta > \theta_0 \operatorname{diam} Q_u(Z)$, condition (BM4) holds for every $\theta\in(0,\theta_0]$. This completes the proof of the theorem. $\square$

8. CONNECTING SPLIT BLENDING MACHINES

We present a refined version of [2, Theorem 2.13], establishing a robust intersection between the stable sets of two split blending machines[6], see Theorem 8.5 stated in Section 8.2 and whose proof is completed in Section 8.3. These two machines model, respectively, a cs-blender and a cu-blender. The construction relies on a transition map connecting the two machines, see Section 8.1. The conditions satisfied by this transition map (see Definition 8.1) resemble those in [2, Theorem 2.13]. The core of the argument is Lemma 8.7, proved in Section 8.3.

8.1. **Transitions between split blending machines.** For what follows, recall the notation for a split blending machine $\mathfrak{B}=(\mathscr{R},Z,\mathfrak{S},\mathfrak{C},\theta)$, where $\mathscr{R}=\{R_{\mathtt{i}}\}_{\mathtt{i}\in I}$, in Definition 6.8.

Definition 8.1 (Adapted transitions between split blending machines)**.** Let $p,q,d_{c,1},d_{c,2}$ be positive integers with $d_{c,1}<p$ and $d_{c,2}<q$. For $\tau=1,2$, consider open manifolds N_τ, open subsets U_τ of N_τ, and split blending machines $\mathfrak{B}_\tau=(\mathscr{R}_\tau,Z_\tau,\mathfrak{S}_\tau,\mathfrak{C}_\tau,\theta)$ for C^1 maps $f_\tau\colon U_\tau\to N_\tau$, where

- $\mathfrak{S}_1=(P_{cs,1},Q_{u,1})$ is a (p,q)-splitting of N_1,
- $\mathfrak{S}_2=(Q_{cs,2},P_{u,2})$ is a (q,p)-splitting of N_2,
- $\mathfrak{C}_1=(P_s,P_c)$ is a $(p-d_{c,1},d_{c,1})$-splitting of $\mathbb{R}^p$, and
- $\mathfrak{C}_2=(Q_s,Q_c)$ is a $(q-d_{c,2},d_{c,2})$-splitting of $\mathbb{R}^q$.

Consider a compact set $R_\sharp$, an open neighbourhood $U_\sharp$ of $R_\sharp$, with $U_\sharp\subset U_2$, and a C^1 embedding $f_\sharp\colon U_\sharp\to N_1$. We say that the triple $(R_\sharp,U_\sharp,f_\sharp)$ is a *adapted transition between the split blending machines* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ if there are positive numbers μ_τ, γ_τ, $\tau\in\{1,2,\sharp\}$, satisfying conditions (AT1)–(AT5) below:

(AT1) $\mu_1\gamma_2>1$ and $\mu_2\gamma_1>1$,

(AT2) for $\tau=1,2$, the map f_τ satisfies the $(\mathfrak{X}_\tau,\theta,\mu_\tau,\gamma_\tau)$-cone condition on every set $R_{\tau,\mathtt{i}}$ of $\mathscr{R}_\tau\overset{\text{def}}{=}\{R_{\tau,\mathtt{i}}\}_{\mathtt{i}\in I_\tau}$, where

$$\begin{aligned}&\mathfrak{X}_1\overset{\text{def}}{=}(P_c\circ P_{cs,1},Q_c\circ Q_{u,1})\text{ is a }(d_{c,1},d_{c,2})\text{-pair of }N_1\text{ and}\\ &\mathfrak{X}_2\overset{\text{def}}{=}(Q_c\circ Q_{cs,2},P_c\circ P_{u,2})\text{ is a }(d_{c,2},d_{c,1})\text{-pair of }N_2,\end{aligned}\tag{8.1}$$

(AT3) $R_\sharp$ is a $(P_{u,2},Q_{u,1}\circ f_\sharp)$-rectangle such that

(a) $Q_{cs,2}(R_\sharp)\Subset Q_{cs,2}(Z_2)$, (b) $P_{u,2}(R_\sharp)=P_{u,2}(Z_2)$,

(c) $(P_{cs,1}\circ f_\sharp)(R_\sharp)\Subset P_{cs,1}(Z_1)$, (d) $(Q_{u,1}\circ f_\sharp)(R_\sharp)=Q_{u,1}(Z_1)$,

(AT4) $f_\sharp$ satisfies the $(\mathfrak{X}_2^t,\mathfrak{X}_1,\theta,\mu_\sharp,\gamma_\sharp)$-cone condition on $R_\sharp$,

(AT5) the Lebesgue number Δ_τ of $\mathfrak{B}_\tau$ satisfy

$$\theta\,\mathrm{diam}\,(Q_c\circ Q_{u,1})(Z_1)+\mu_\sharp^{-1}\mathrm{diam}\,(P_c\circ P_{u,2})(Z_2)<\Delta_1,$$
$$\theta\,\mathrm{diam}\,(P_c\circ P_{u,2})(Z_2)+\gamma_\sharp^{-1}\mathrm{diam}\,(Q_c\circ Q_{u,1})(Z_1)<\Delta_2.$$

Remark 8.2. The split blender machines $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in Definition 8.1 have dimensions $(p-d_{c,1},d_{c,1},q)$ and $(q-d_{c,2},d_{c,2},p)$, respectively.

[6]We follow the approach in [2], where these machines are defined and connected in an abstract way for two different dynamics, which is why their stable sets may intersect.

Remark 8.3 (Definition 8.1 versus Theorem 2.13 in [2])**.** Each condition (ATi) mimics hypothesis (i) in [2]. The main difference occurs between (AT5) and (5). Condition (5) in [2] deals with the diameter of the unstable projection of the superposition region of the blending machine. This condition is not suitable for our purposes (as these diameters cannot be small enough). In contrast, condition (AT5) depends only on the central projection, which in our setting can be done as small as required.

Remark 8.4. Bearing in mind item (ii) in Remark 6.9, if in Definition 8.1 we take $P_c = \mathrm{Id}$, $Q_c = \mathrm{Id}$ and P_s and Q_s the trivial projections, we recover the conditions in [2].

8.2. **Connecting split blending machines.** Recall the definition of the stable set $\Lambda^s(\mathfrak{B}_f)$ in (6.14). The theorem below does not explicitly involve the dimensions of the machines. The fact that these dimensions match is implicitly contained in the definition of the transition map, see also Remark 8.2.

Theorem 8.5. *For $\tau = 1, 2$, consider split blender machines $\mathfrak{B}_\tau$ for C^1 maps $f_\tau : U_\tau \to N_\tau$. Let $(R_\sharp, U_\sharp, f_\sharp)$ be an adapted transition between $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Then*

$$\Lambda^s(\mathfrak{B}_1) \cap f_\sharp(R_\sharp \cap \Lambda^s(\mathfrak{B}_2)) \neq \emptyset.$$

Moreover, this intersection is robust: given a compact neighbourhoods K_τ of $\bigcup_{\mathtt{i}\in I_\tau} R_{\tau,\mathtt{i}}$, $\tau = 1, 2$, there exist C^1 neighbourhoods $\mathscr{U}_\tau$ of f_τ, $\tau = 1, 2, \sharp$, such that

$$\Lambda^s(K_1, g_1) \cap g_\sharp(\Lambda^s(K_2, g_2)) \neq \emptyset \quad \textit{for every } g_\tau \in \mathscr{U}_\tau.$$

Proof. The proof follows modifying the inductive process in the proof of [2, Theorem 2.13]. For completeness, we will repeat the steps of this proof and explain the changes. Write

- $\mathfrak{S}_1 = (P_{cs,1}, Q_{u,1})$ is a (p, q)-splitting and $\mathfrak{S}_2 = (Q_{cs,2}, P_{u,2})$ is a (q, p)-splitting,
- $\mathfrak{C}_1 = (P_s, P_c)$ is a $(p - d_{c,1}, d_{c,1})$-splitting and $\mathfrak{C}_2 = (Q_s, Q_c)$ is a $(q - d_{c,2}, d_{c,2})$-splitting.

Let R be a compact subset of U_2 and h a C^1 embedding from an open neighbourhood of R to U_1. We say that the pair (R, h) satisfies the *condition* $(\sharp)$ if:

($\sharp$1) R is a $(P_{u,2}, Q_{u,1} \circ h)$-rectangle;

($\sharp$2) h satisfies the $(\mathfrak{X}_2^t, \mathfrak{X}_1, \theta, \mu_\sharp, \gamma_\sharp)$-cone condition on R, where $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are as in (8.1);

($\sharp$3) it holds

(a) $Q_{cs,2}(R) \Subset Q_{cs,2}(Z_2)$, (b) $P_{u,2}(R) = P_{u,2}(Z_2)$,

(c) $(P_{cs,1} \circ h)(R) \Subset P_{cs,1}(Z_1)$, (d) $(Q_{u,1} \circ h)(R) = Q_{u,1}(Z_1)$.

Remark 8.6. By (AT3)–(AT4), the pair $(R_\sharp, f_\sharp)$ satisfies condition $(\sharp)$.

Lemma 8.7. *Consider a pair (R, h) satisfying condition $(\sharp)$. Then there are $\mathtt{i} \in I_1$ and $\mathtt{k} \in I_2$ such that the pair (R', h') given by*

$$R' \overset{\text{def}}{=} f_2\big(R_{2,\mathtt{k}} \cap R \cap h^{-1}(R_{1,\mathtt{i}})\big) \quad \textit{and} \quad h' \overset{\text{def}}{=} f_1 \circ h \circ f_2^{-1}$$

satisfies condition $(\sharp)$.

We postpone the proof of the lemma and complete the proof of the theorem assuming it. By Remark 8.6, the pair $(R_0, h_0) = (R_\sharp, f_\sharp)$ satisfies condition $(\sharp)$. Let (R_1, h_1) be the pair obtained from (R_0, h_0) using Lemma 8.7. Since this pair also satisfies condition

($\sharp$), we can argue inductively to get a sequence of pairs (R_n, h_n), where each (R_n, h_n) is obtained from (R_{n-1}, h_{n-1}) using Lemma 8.7. In this construction we have

$$h_n \stackrel{\text{def}}{=} f_1 \circ h_{n-1} \circ f_2^{-1} = f_1^n \circ f_\sharp \circ f_2^{-n} \quad \text{and} \quad R_n \stackrel{\text{def}}{=} f_2\big(R_{2,\mathtt{k}_{n-1}} \cap R_{n-1} \cap h_{n-1}^{-1}(R_{1,\mathtt{i}_{n-1}})\big).$$

where $\mathtt{i}_n \in I_1$ and $\mathtt{k}_n \in I_2$. Each set R_n is a non-empty compact subset of U_2, then $\bigcap_{n\geqslant 0} f_\sharp \circ f_2^{-n}(R_n) \neq \emptyset$. Finally, we have that

$$\bigcap_{n\geqslant 0} f_1^{-n}(R_{1,\mathtt{i}_n}) \cap f_\sharp \left(R_\sharp \cap \bigcap_{n\geqslant 0} f_2^{-n}(R_{2,\mathtt{k}_n})\right) = \bigcap_{n\geqslant 0} f_\sharp \circ f_2^{-n}(R_n) \neq \emptyset.$$

Recalling (6.14), note that

$$\bigcap_{n\geqslant 0} f_1^{-n}(R_{1,\mathtt{i}_n}) \subset \Lambda^s(\mathfrak{B}_1) \quad \text{and} \quad \bigcap_{n\geqslant 0} f_2^{-n}(R_{2,\mathtt{k}_n}) \subset \Lambda^s(\mathfrak{B}_2)$$

implies that

$$\Lambda^s(\mathfrak{B}_1) \cap f_\sharp(R_\sharp \cap \Lambda^s(\mathfrak{B}_2)) \neq \emptyset,$$

proving the first part of the theorem.

The robustness statement is standard and follows exactly as in [2]. $\square$

To complete the proof the theorem we prove the above lemma.

8.3. **Proof of Lemma 8.7.** According to (b) and (d) in condition ($\sharp$3), we have that

$$P_{u,2}(R) = P_{u,2}(Z_2) \qquad \text{and} \qquad (Q_{u,1} \circ h)(R) = Q_{u,1}(Z_1). \tag{8.2}$$

Since h satisfies the $(\mathfrak{X}_2^t, \mathfrak{X}_1, \theta, \mu_\sharp, \gamma_\sharp)$-cone condition on R (condition ($\sharp$2)), then Lemma 6.6, (8.2), and condition (AT5) imply that

$$\operatorname{diam}(P_c \circ P_{cs,1})(h(R)) \leqslant \theta \operatorname{diam}(Q_c \circ Q_{u,1})(Z_1) + \mu_\sharp^{-1} \operatorname{diam}(P_c \circ P_{u,2})(Z_2) \stackrel{\text{(AT5)}}{<} \Delta_1,$$

$$\operatorname{diam}(Q_c \circ Q_{cs,2})(R) \leqslant \theta \operatorname{diam}(P_c \circ P_{u,2})(Z_2) + \gamma_\sharp^{-1} \operatorname{diam}(Q_c \circ Q_{u,1})(Z_1) \stackrel{\text{(AT5)}}{<} \Delta_2,$$

where Δ_1 and Δ_2 are Lebesgue numbers of the coverings $\{\operatorname{int}\big(P_c \circ P_{cs,1}(R_{1,\mathtt{i}})\big)\}_{\mathtt{i}\in I_1}$ of $\overline{P_c \circ P_{cs,1}(Z_1)}$ and $\{\operatorname{int}\big(Q_c \circ Q_{cs,2}(R_{2,\mathtt{k}})\big)\}_{\mathtt{k}\in I_2}$ of $\overline{Q_c \circ Q_{cs,2}(Z_2)}$, respectively. Thus, there exist $\mathtt{i} \in I_1$ and $\mathtt{k} \in I_2$ such that

$$P_c \circ P_{cs,1}(h(R)) \subset \operatorname{int}\big(P_c \circ P_{cs,1}(R_{1,\mathtt{i}})\big) \quad \text{and} \quad Q_c \circ Q_{cs,2}(R) \subset \operatorname{int}\big(Q_c \circ Q_{cs,2}(R_{2,\mathtt{k}})\big). \tag{8.3}$$

Now we will verify that (R', h') satisfy the condition ($\sharp$) for the rectangles $R_{1,\mathtt{i}}, R_{2,\mathtt{k}}$ obtained above. We begin with condition ($\sharp$1). Consider

- the (p,q)-splittings $\mathfrak{L}_1, \mathfrak{L}_2$ of N_2 define by
$$\mathfrak{L}_i = (P_i, Q_i) \stackrel{\text{def}}{=} (P_{u,2}, Q_{cs,2}) = \mathfrak{S}_2^t, \quad i = 1,2,$$
- the (p,q)-splittings $\mathfrak{L}_3, \mathfrak{L}_4$ of N_1 defined by
$$\mathfrak{L}_j = (P_j, Q_j) \stackrel{\text{def}}{=} (P_{cs,1}, Q_{u,1}) = \mathfrak{S}_1, \quad j = 3,4,$$
- the compact sets $R_1 \stackrel{\text{def}}{=} f_2(R_{2,\mathtt{k}})$, $R_2 \stackrel{\text{def}}{=} R$ and $R_3 = R_{1,\mathtt{i}}$,
- the C^1 embeddings
  - $g_1 =$ the restriction of f_2^{-1} to an open neighbourhood of R_1,
  - $g_2 = h$,
  - $g_3 =$ the restriction of f_1 to an open neighbourhood of R_3,

  and define $G_1 \stackrel{\text{def}}{=} g_1$, $G_2 \stackrel{\text{def}}{=} g_2 \circ g_1$, and $G_3 \stackrel{\text{def}}{=} g_3 \circ g_2 \circ g_1 = h'$.

If for $i = 1,2,3$, the set R_i is a $(P_i, Q_{i+1} \circ g_i)$-rectangle such that

$$P_{i+1}\big(g_i(R_i)\big) \subset \mathrm{int}\big(P_{i+1}(R_{i+1})\big) \quad \text{and} \quad Q_{i+1}(R_{i+1}) \subset \mathrm{int}\big(Q_{i+1}(g_i(R_i))\big),$$

then Lemma 6.7 guarantees that

$$R_* \stackrel{\text{def}}{=} R_1 \cap G_1^{-1}(R_2) \cap G_2^{-1}(R_3) = f_2\big(R_{2,\mathtt{k}} \cap R \cap h^{-1}(R_{1,\mathtt{i}})\big) = R'$$

is a $(P_{u,2}, Q_{u,1} \circ h')$-rectangle with

$$P_{u,2}(R') = P_{u,2}\big(f_2(R_{2,\mathtt{k}})\big) \quad \text{and} \quad (Q_{u,1} \circ h')(R') = Q_{u,1}\big(f_1(R_{1,\mathtt{i}})\big),$$

obtaining condition ($\sharp$1), as desired. We will now check the part “if” above.

Conditions “R_i is a $(P_i, Q_{i+1} \circ g_i)$-rectangle" translate to:

(r1) $f_2(R_{2,\mathtt{k}})$ is a $(P_{u,2}, Q_{cs,2} \circ f_2^{-1})$-rectangle,

(r2) R is a $(P_{u,2}, Q_{u,1} \circ h)$-rectangle, and

(r3) $R_{1,\mathtt{i}}$ is a $(P_{cs,1}, Q_{u,1} \circ f_1)$-rectangle.

The inclusion conditions are translated as follows:

(i1) $P_2\big(g_1(R_1)\big) \subset \mathrm{int}\big(P_2(R_2)\big) \mapsto P_{u,2}\big(f_2^{-1}(f_2(R_{2,\mathtt{k}}))\big) = P_{u,2}(R_{2,\mathtt{k}}) \subset \mathrm{int}\big(P_{u,2}(R)\big)$,

(i2) $P_3\big(g_2(R_2)\big) \subset \mathrm{int}\big(P_3(R_3)\big) \mapsto P_{cs,1} \circ h(R) \subset \mathrm{int}\big(P_{cs,1}(R_{1,\mathtt{i}})\big)$,

(i3) $Q_2(R_2) \subset \mathrm{int}\big(Q_2(g_1(R_1))\big) \mapsto Q_{cs,2}(R) \subset \mathrm{int}\big(Q_{cs,2}(f_2^{-1}(f_2(R_{2,\mathtt{k}})))\big) = \mathrm{int}\big(Q_{cs,2}(R_{2,\mathtt{k}})\big)$,

(i4) $Q_3(R_3) \subset \mathrm{int}\big(Q_3(g_2(R_2))\big) \mapsto Q_{u,1}(R_{1,\mathtt{i}}) \subset \mathrm{int}\big(Q_{u,1}(h(R))\big)$.

Claim 8.8. *Conditions* (r1) – (r3) *and* (i1) – (i4) *hold.*

Proof. We just check condition (r1), conditions (r2)-(r3) follow similarly and their proofs omitted. Note that $f_2(R_{2,\mathtt{k}})$ is diffeomorphic to $R_{2,\mathtt{k}}$, which is (by definition) a $(Q_{cs,2}, P_{u,2} \circ f_2)$-rectangle. Therefore $R_{2,\mathtt{k}}$ (and hence $f_2(R_{2,\mathtt{k}})$) is diffeomorphic to

$$\big(Q_{cs,2}(R_{2,\mathtt{k}})\big) \times \big(P_{u,2} \circ f_2(R_{2,\mathtt{k}})\big) = \big(Q_{cs,2} \circ f_2^{-1}(f_2(R_{2,\mathtt{k}}))\big) \times \big(P_{u,2} \circ f_2(R_{2,\mathtt{k}})\big).$$

This precisely means that $f_2(R_{2,\mathtt{k}})$ is a $(P_{u,2}, Q_{cs,2} \circ f_2^{-1})$-rectangle.

We now check the inclusion properties. We begin proving (i2) and (i3). For that, the condition ($\sharp$3) part (a) and (c) implies that

$$P_{cs,1}(h(R)) \Subset P_{cs,1}(Z_1) \quad \text{and} \quad Q_{cs,2}(R) \Subset Q_{cs,2}(Z_2)$$

and condition (BM3) in Definition 6.8 of a blender machine implies that

$$P_s \circ P_{cs,1}(Z_1) \Subset (P_s \circ P_{cs,1})(R_{1,\mathtt{i}}) \qquad \text{and} \qquad Q_s \circ Q_{cs,2}(Z_2) \Subset (Q_s \circ Q_{cs,2})(R_{2,\mathtt{k}}).$$

Therefore

$$(P_s \circ P_{cs,1})(h(R)) \subset \mathrm{int}\,(P_s \circ P_{cs,1})(R_{1,\mathtt{i}}) \quad \text{and} \quad (Q_s \circ Q_{cs,2})(R) \subset \mathrm{int}\,(Q_s \circ Q_{cs,2})(R_{2,\mathtt{k}}). \tag{8.4}$$

Recalling the diffeomorphims $\mathfrak{C}_1 = (P_s, P_c)$ and $\mathfrak{C}_2 = (Q_s, Q_c)$, it follows that the two inclusions (8.3) and (8.4) are equivalent to

$$\mathfrak{C}_1\big(P_{cs,1}(h(R)\big) \subset \mathrm{int}\big(\mathfrak{C}_1(P_{cs,1}(R_{1,\mathtt{i}}))\big) = \mathfrak{C}_1\big(\mathrm{int}\, P_{cs,1}(R_{1,\mathtt{i}})\big) \quad \text{and}$$

$$\mathfrak{C}_2\big(Q_{cs,2}(R)\big) \subset \mathrm{int}\big(\mathfrak{C}_2(Q_{cs,2}(R_{2,\mathtt{k}}))\big) = \mathfrak{C}_2\big(\mathrm{int}(Q_{cs,2}(R_{2,\mathtt{k}}))\big).$$

Hence

$$P_{cs,1}(h(R)) \subset \operatorname{int}\big(P_{cs,1}(R_{1,\mathtt{i}})\big) \quad \text{and} \quad Q_{cs,2}(R) \subset \operatorname{int}\big(Q_{cs,2}(R_{2,\mathtt{k}})\big) \tag{8.5}$$

and inclusions (i2) and (i3) follow.

For inclusions in (i4) and (i1), by (♯3) part (b) and (d) and the condition (BM2) in Definition 6.8 of a blender machine, it follows

$$\begin{aligned} &Q_{u,1}(R_{1,\mathtt{i}}) \subset \operatorname{int}(Q_{u,1}(Z_1)) \overset{\text{(d)}}{=} \operatorname{int}(Q_{u,1}(h(R))) \quad \text{and} \\ &P_{u,2}(R_{2,\mathtt{k}}) \subset \operatorname{int}(P_{u,2}(Z_2)) \overset{\text{(b)}}{=} \operatorname{int}(P_{u,2}(R)) \end{aligned} \tag{8.6}$$

and thus (i4) and (i1) hold. Thus inclusions (i1)-(i4) hold, proving the claim. □

Therefore, Claim 8.8 imply that condition (♯1) holds for (R', h'), that is,

$$R' = f_2\big(R_{2,\mathtt{k}} \cap R \cap h^{-1}(R_{1,\mathtt{i}})\big)$$

is a $(P_{u,2}, Q_{u,1} \circ h')$-rectangle. Moreover,

$$P_{u,2}(R') = P_{u,2}\big(f_2(R_{2,\mathtt{k}})\big) \quad \text{and} \quad (Q_{u,1} \circ h')(R') = Q_{u,1}\big(f_1(R_{1,\mathtt{i}})\big). \tag{8.7}$$

Note also that, by the definitions of R' and $h' = f_1 \circ h \circ f_2^{-1}$ it holds,

$$R' \subset f_2(R_{2,\mathtt{k}}) \qquad \text{and} \qquad h'(R') \subset f_1(R_{1,\mathtt{i}}). \tag{8.8}$$

From (8.7), (8.8), and (BM2) in Definition 6.8 of blender machine, it follows

(a') $Q_{cs,2}(R') \overset{(8.8)}{\subset} Q_{cs,2}\big(f_2(R_{2,\mathtt{k}})\big) \overset{\text{(BM2)}}{\Subset} Q_{cs,2}(Z_2)$,
(b') $P_{u,2}(R') \overset{(8.7)}{=} P_{u,2}\big(f_2(R_{2,\mathtt{k}})\big) \overset{\text{(BM2)}}{=} P_{u,2}(Z_2)$,
(c') $(P_{cs,1} \circ h')(R') \overset{(8.8)}{\subset} P_{cs,1}\big(f_1(R_{1,\mathtt{i}})\big) \overset{\text{(BM2)}}{\Subset} P_{cs,1}(Z_1)$, and
(d') $(Q_{u,1} \circ h')(R') \overset{(8.7)}{=} Q_{u,1}\big(f_1(R_{1,\mathtt{i}})\big) \overset{\text{(BM2)}}{=} Q_{u,1}(Z_1)$

Note that (a'),(b'), (c'), and (d') correspond, respectively, to conditions (a), (b), (c), and (d) in (♯3). Therefore, condition (♯3) holds for (R', h').

It remains to check condition (♯2) (h' satisfies the $(\mathfrak{X}_2^t, \mathfrak{X}_1, \theta, \mu_\sharp, \gamma_\sharp)$-cone condition on R'). Note that the cone conditions for f_1, f_2, and h in (AT2) in Definition 8.1 and (♯2) imply that $h' = f_1 \circ h \circ f_2^{-1}$ satisfies the $(\mathfrak{X}_2^t, \mathfrak{X}_1, \theta, \mu_1\mu_\sharp\gamma_2, \gamma_1\gamma_\sharp\mu_2)$-cone condition on R'. As $\mu_1\gamma_2 > 1$ and $\mu_2\gamma_1 > 1$ (condition (AT1) in Definition 8.1) we have that h' satisfies the $(\mathfrak{X}_2^t, \mathfrak{X}_1, \theta, \mu_\sharp, \gamma_\sharp)$-cone condition on R'. This ends the proof of the lemma. □

9. Connecting preblending machines

We introduce adapted transitions for preblending machines and show that they give rise to two connected split blending machines associated with skew-product tower maps, see Theorem 9.2.

9.1. **Transitions between preblending machines.**

We introduce transitions between preblender machines, following the spirit of Definition 8.1. Conditions (at1)-(at5) below mimic conditions (AT1)-(AT5) in that definition.

Definition 9.1 (Adapted transitions between preblender machines)**.** Let M be a manifold of dimension $d \geqslant 2$ and N_1 and N_2 open subsets of M. Let d_c be a positive integer with $d_c < d$. For $\tau = 1, 2$, consider

- families $\mathscr{F}_\tau = \{\phi_{\tau,i} \colon i = 1, \dots, k_\tau\}$, $k_\tau \geqslant 2$, of C^1 local diffeomorphisms of M, and
- preblending machines $\mathfrak{D}_\tau = (\mathscr{R}_\tau, B_\tau, \mathfrak{X}_\tau)$ for $\mathscr{F}_\tau$, where $\mathfrak{X}_1 = (P_1, Q_1)$ is a $(d_c, d-d_c)$-splitting of N_1 and $\mathfrak{X}_2 = (Q_2, P_2)$ is a $(d-d_c, d_c)$-splitting of N_2.

Let $R_\sharp \subset N_2$ be a compact set, $U_\sharp$ an open neighbourhood of $R_\sharp$, and $\phi_\sharp \colon U_\sharp \to N_1$ a C^1 embedding. The triple $(R_\sharp, U_\sharp, \phi_\sharp)$ is a *adapted transition to the preblending machines* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ if there are positive numbers μ_τ, γ_τ, $\tau = 1, 2, \sharp$, satisfying conditions (at1)-(at5) below:

(at1) $\mu_1\gamma_2 > 1$ and $\mu_2\gamma_1 > 1$,
(at2) for $\tau = 1,2$ the map $\phi_{\tau,\mathtt{i}}$ is $(\mathfrak{X}_\tau, \mu_\tau, \gamma_\tau)$-dominated on every set $R_{\tau,\mathtt{i}}$ of $\mathscr{R}_\tau \stackrel{\text{def}}{=} \{R_{\tau,\mathtt{i}}\}_{\mathtt{i}\in I_\tau}$,
(at3) $R_\sharp$ is a $(P_2, Q_1 \circ \phi_\sharp)$-rectangle satisfying

$$Q_2(R_\sharp) \Subset Q_2(B_2), \qquad P_2(R_\sharp) = P_2(B_2),$$
$$(P_1 \circ \phi_\sharp)(R_\sharp) \Subset P_1(B_1), \qquad (Q_1 \circ \phi_\sharp)(R_\sharp) = Q_1(B_1),$$

(at4) $\phi_\sharp$ is $(\mathfrak{X}_2^t, \mathfrak{X}_1, \mu_\sharp, \gamma_\sharp)$-dominated on $R_\sharp$,
(at5) the Lebesgue numbers Δ_1 and Δ_2 of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ satisfy

$$\mu_\sharp^{-1} \operatorname{diam}(P_2(B_2)) < \Delta_1 \quad \text{and} \quad \gamma_\sharp^{-1} \operatorname{diam}(Q_1(B_1)) < \Delta_2.$$

9.2. **Connecting split blending machines through preblending machines.** The next result is a direct consequece of Theorem 8.5. To state it, recall Definitions 7.1, 7.6, and 7.13 of a horseshoe map, adapted stretching, and a skew-product tower map.

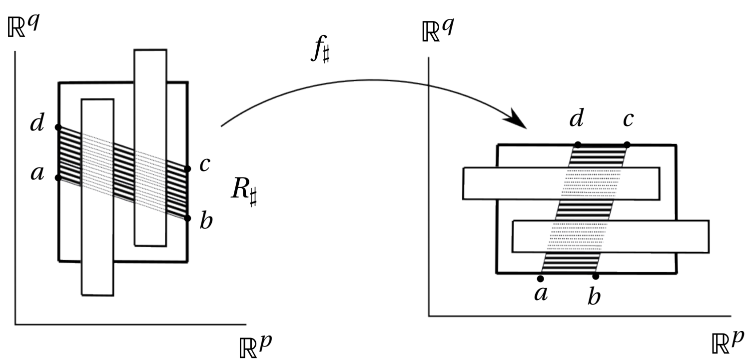

FIGURE 9. Connected preblending machines

Theorem 9.2. *Let M be a manifold of dimension $d \geqslant 2$ and N_1 and N_2 open subsets of M. Let d_c be a positive integer with $d_c < d$. Consider:*

- *Families $\mathscr{F}_\tau = \{\phi_{\tau,i} \colon i = 1, \dots, k_\tau\}$, $\tau = 1,2$ and $k_\tau \geqslant 2$, and $\mathscr{F}_\sharp = \{\phi_\sharp\}$, of C^1 local diffeomorphisms of M.*
- *Preblending machines $\mathfrak{D}_\tau = (\mathscr{R}_\tau, B_\tau, \mathfrak{X}_\tau)$ for $\mathscr{F}_\tau$, $\tau = 1,2$, where $\mathfrak{X}_1$ is a $(d_c, d-d_c)$-splitting of N_1, $\mathfrak{X}_2$ is a $(d-d_c, d_c)$-splitting of N_2, and $\mathscr{R}_\tau = \{R_{\tau,\mathtt{i}}\}_{\mathtt{i}\in I_\tau}$.*
- *A compact set $R_\sharp$ and a neighbourhood $U_\sharp$ of it such that $(R_\sharp, U_\sharp, \phi_\sharp)$ is an adapted transition between $\mathfrak{D}_1$ and $\mathfrak{D}_2$.*
- *Horseshoe maps $A_1, A_2, A_\sharp$ of type (k_1, d_s, d_u, ν_1), (k_2, d_u, d_s, ν_2), and $(1, d_s, d_u, \nu_\sharp)$, whose legs are $\{\mathsf{H}_{1,\mathtt{i}}\}_{\mathtt{i}\in\Sigma_{k_1}^*}$, $\{\mathsf{H}_{2,\mathtt{i}}\}_{\mathtt{i}\in\Sigma_{k_2}^*}$, and $\{\mathsf{H}_\sharp\}$.*

Assume that the stretching ν_1, ν_2 *of* A_1, A_2 *are adapted to* $\mathfrak{D}_1, \mathfrak{D}_2$ *and the stretching* $\nu_\sharp$ *of* $A_\sharp$ *is* $(\mathfrak{X}_2^t, \mathfrak{X}_1)$*-adapted to* $\phi_\sharp$ *on* $R_\sharp$*.*

Consider the skew-product tower maps F_τ *associated to* $\mathfrak{T}_\tau = (A_\tau \ltimes \mathscr{F}_\tau, \mathscr{R}_\tau, I_\tau)$, $\tau = 1, 2, \sharp$, *where:*

- $\mathscr{R}_\tau = \{\mathbf{R}_{\tau,\mathtt{i}}\}_{\mathtt{i}\in I_\tau}$, *where* $\mathbf{R}_{\tau,\mathtt{i}} = \mathsf{H}_{\tau,\mathtt{i}} \times R_{\tau,\mathtt{i}}$, $\tau = 1, 2$, *and*
- $\mathscr{R}_\sharp = \{\mathbf{R}_\sharp\}$, *where* $\mathbf{R}_\sharp = \mathsf{H}_\sharp \times R_\sharp$,

and the split blending machines $\mathfrak{B}_\tau = \mathfrak{B}_{\mathfrak{D}_{\mathscr{F}_\tau}, F_\tau}$ *given by Theorem 7.14.*

Then for every sufficiently small $\theta > 0$ *holds*

$$\Lambda^s(\mathfrak{B}_1) \cap F_\sharp(\mathbf{R}_\sharp \cap \Lambda^s(\mathfrak{B}_2)) \neq \emptyset, \quad \textit{where} \quad F_\sharp = A_\sharp \times \phi_\sharp \quad \textit{and} \quad \mathbf{R}_\sharp = \mathsf{H}_\sharp \times R_\sharp.$$

Moreover, for every compact neighbourhood K_τ *of* $\mathbf{R}_{\mathfrak{T}_\tau} = \bigcup_{\mathtt{i}\in I_\tau} \mathbf{R}_{\tau,\mathtt{i}}$ *there is a* C^1 *neighbourhood* $\mathscr{U}_\tau$ *of* F_τ, $\tau = 1, 2, \sharp$, *such that*

$$\Lambda^s(K_1, G_1) \cap G_\sharp(\Lambda^s(K_2, G_2)) \neq \emptyset, \quad \textit{for every } G_\tau \in \mathscr{U}_\tau,\ \tau = 1, 2, \sharp.$$

The split blending machines $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in Theorem 9.2 have dimensions $(d_s, d_c, d - d_c + d_u)$ and $(d_u, d - d_c, d_c + d_s)$, respectively.

Proof of Theorem 9.2. We will apply Theorem 8.5 to a pair of split blending machines with dimensions $(p - d_{c,1}, d_{c,1}, q)$ and $(q - d_{c,2}, d_{c,2}, p)$ where

- $p = d_s + d_c$, $q = d_u + d - d_c$, $d_{c,1} = d_c$, and $d_{c,2} = d - d_c$,
- the standard projections

$$P\colon \mathbb{R}^p = \mathbb{R}^{d_c} \times \mathbb{R}^{d_s} \to \mathbb{R}^{d_c} = \mathbb{R}^{d_{c,1}} \quad\text{and}\quad Q\colon \mathbb{R}^q = \mathbb{R}^{d-d_c} \times \mathbb{R}^{d_u} \to \mathbb{R}^{d-d_c} = \mathbb{R}^{d_{c,2}}, \tag{9.1}$$

- $\mathbf{N}_1 = \mathbb{R}^{d_s+d_u} \times N_1$ and $\mathbf{N}_2 = \mathbb{R}^{d_s+d_u} \times N_2$,

to the split blending machines $\mathfrak{B}_\tau = (\mathscr{R}_\tau, Z_\tau, \mathfrak{S}_\tau, \mathfrak{C}_\tau, \theta)$ for the tower maps $F_\tau\colon \bigcup_{R\in\mathscr{R}_\tau} R \to \mathbf{N}_\tau$, $\tau = 1, 2$, provided by Theorem 7.14, the map $F_\sharp\colon \mathbf{R}_\sharp \to \mathbf{N}_1$, the set $\mathbf{R}_\sharp \subset \mathbf{N}_2$, and the positive numbers μ_τ, γ_τ, $\tau = 1, 2, \sharp$.

Let us now recall the elements of the split blending machines $\mathfrak{B}_\tau$:

- $Z_\tau = \mathsf{D} \times B_\tau$, $\tau = 1, 2$, where $\mathsf{D} = [0,1]^{d_s+d_u}$.
- The splitting $\mathfrak{S}_1 = (P_{cs,1}, Q_{u,1})$ of $\mathbf{N}_1$ is given (in coordinates) by

$$P_{cs,1} = (\Pi_s, P_1) \quad\text{and}\quad Q_{u,1} = (Q_1, \Pi_u),$$

  where Π_s and Π_u are the canonical projections from $\mathbb{R}^{d_s+d_u} = \mathbb{R}^{d_s} \times \mathbb{R}^{d_u}$ into $\mathbb{R}^{d_s}$ and $\mathbb{R}^{d_u}$.[7]
- The splitting $\mathfrak{S}_2 = (Q_{cs,2}, P_{u,2})$ of $\mathbf{N}_2$ is given (in coordinates) by

$$Q_{cs,2} = (\Pi_u, Q_2) \quad\text{and}\quad P_{u,2} = (P_2, \Pi_s).$$

- The splittings $\mathfrak{C}_1 = (\pi_s, P)$ and $\mathfrak{C}_2 = (\pi_u, Q)$, where π_s and π_u are the standard projections

$$\pi_s\colon \mathbb{R}^{d_s} \times \mathbb{R}^{d_c} \to \mathbb{R}^{d_s} \qquad\text{and}\qquad \pi_u\colon \mathbb{R}^{d_u} \times \mathbb{R}^{d-d_c} \to \mathbb{R}^{d_u}.$$

[7] There is a slight abuse of notation here, as the domain of the coordinate maps are different. We use simplified notation similar to the one in (7.12) for the sake of brevity.

We now are ready to check that conditions (AT1)-(AT5) hold. Note that (AT1) and (at1) are the same.

To obtain (AT2) consider the pairs $\widehat{\mathfrak{X}}_1$ and $\widehat{\mathfrak{X}}_2$ given by

$$\widehat{\mathfrak{X}}_1 = (P \circ P_{cs,1}, Q \circ Q_{u,1}) \quad \text{and} \quad \widehat{\mathfrak{X}}_2 = (Q \circ Q_{cs,2}, P \circ P_{u,2}).$$

Denote by $\varpi_\tau \colon \mathbf{N}_\tau \to N_\tau$ the standard projection and note that $\widehat{\mathfrak{X}}_\tau = \mathfrak{X}_\tau \circ \varpi_\tau$, $\tau = 1,2$. From this relation and $\varpi_\tau \circ (F_\tau|_{\mathbf{R}_{\tau,\mathtt{i}}}) = (\phi_{\tau,\mathtt{i}}|_{R_{\tau,\mathtt{i}}}) \circ \varpi_\tau$, it follows from (at2) that F_τ satisfies the $(\widehat{\mathfrak{X}}_\tau, \theta, \mu_\tau, \gamma_\tau)$-cone condition for $\tau = 1,2$. Hence (AT2) holds. Similarly, we get that (at4) implies (AT4).

To get (AT5) note that from the definition of P in (9.1) it follows that

$$P \circ P_{u,2}(Z_2) = P\big(P_2(B_2), \Pi_s(\mathsf{D}_2)\big) = P_2(B_2).$$

Similarly, $Q \circ Q_{u,1}(Z_1) = Q_1(B_1)$, it follows from (at5) that

$$\begin{aligned} \mu_\sharp^{-1} \operatorname{diam}(P \circ P_{u,2})(Z_2) &= \mu_\sharp^{-1} \operatorname{diam}\big(P_2(B_2)\big) < \Delta_1, \\ \gamma_\sharp^{-1} \operatorname{diam}(Q \circ Q_{u,1})(Z_1) &= \gamma_\sharp^{-1} \operatorname{diam}\big(Q_1(B_1)\big) < \Delta_2. \end{aligned}$$

Therefore condition (AT5) follows for every $\theta > 0$ small enough.

It remains to show (AT3), for that recall the notation in Section 7.1 for horseshoe maps. Observe that $\mathsf{H}_\sharp$ is a leg of the $(1, d_s, d_u, \nu_\sharp)$-horseshoe map $A_\sharp$, therefore

$$\begin{aligned} \mathsf{H}_\sharp &= [0,1]^{d_s} \times V_\sharp^u, \quad \text{where } V_\sharp^u \Subset [0,1]^{d_u}, \\ \mathsf{V}_\sharp &= A_\sharp(\mathsf{H}_\sharp) = H_\sharp^s \times [0,1]^{d_u}, \quad \text{where } H_\sharp^s \Subset [0,1]^{d_s}. \end{aligned} \tag{9.2}$$

Recalling that $\mathbf{R}_\sharp = \mathsf{H}_\sharp \times R_\sharp$, it follows that $F_\sharp(\mathbf{R}_\sharp) = \mathsf{V}_\sharp \times \phi_\sharp(R_\sharp)$. Using (at3) we get the following relations:

$$\begin{aligned} Q_{cs,2}(\mathbf{R}_\sharp) &= V_\sharp^u \times Q_2(R_\sharp) \overset{\text{(at3)}}{\Subset} [0,1]^{d_u} \times Q_2(B_2) = Q_{cs,2}(Z_2), \\ P_{u,2}(\mathbf{R}_\sharp) &= P_2(R_\sharp) \times [0,1]^{d_s} \overset{\text{(at3)}}{=} P_2(B_2) \times [0,1]^{d_s} = P_{u,2}(Z_2), \\ (P_{cs,1} \circ F_\sharp)(\mathbf{R}_\sharp) &= H_\sharp^s \times (P_1 \circ \phi_\sharp)(R_\sharp) \overset{\text{(at3)}}{\Subset} [0,1]^{d_s} \times P_1(B_1) = P_{cs,1}(Z_1), \\ (Q_{u,1} \circ F_\sharp)(\mathbf{R}_\sharp) &= (Q_1 \circ \phi_{\sharp,1})(R_\sharp) \times [0,1]^{d_u} = Q_1(B_1) \times [0,1]^{d_u} = Q_{u,1}(Z_1). \end{aligned}$$

As $R_\sharp$ is a $(P_2, Q_1 \circ \phi_\sharp)$-rectangle, it follows from the equation above that $\mathbf{R}_\sharp$ is a $(P_{u,2}, Q_{u,1} \circ F_\sharp)$-rectangle satisfying the claimed properties. This proves that (AT3) holds, concluding the proof of the theorem. □

10. Proof of Theorem A

In view of Lemmas 3.5 and 3.6, it is enough to prove Theorem A for diffeomorphisms f having simple contours Υ. Theorem 5.10 provides a pair of robust cycles of type $(\iota(\Upsilon), \iota(\Upsilon)+1)$ and $(\iota(\Upsilon)+1, \iota(\Upsilon)+2)$ for a perturbation g of f preserving the contour was obtained. Thus it remains to obtain robust cycles of co-index two.

Remark 10.1 (The hyperbolic sets in the robust cycle)**.** Consider the points $\mathtt{A}_{\mathtt{i}_k}$ and $\mathtt{D}'_{\mathtt{j}_k}$, $k = 2,3$, in Proposition 5.1, and the corresponding periodic points (that we assume to be hyperbolic) given by Remark 4.4, $A_k = A_{(\mathtt{i}_k{}^{\mathbb{Z}},\mathtt{A}_{\mathtt{i}_k})}$ and $D'_k = D'_{(\mathtt{j}_k{}^{\mathbb{Z}},\mathtt{D}'_{\mathtt{j}_k})}$ for a diffeomorphism g with a simple contour. We construct a pair of hyperbolic sets $\Lambda_\mathtt{q}$ and $\Lambda_\mathtt{p}$ with u-indices $\iota(\Upsilon)$ and $\iota(\Upsilon)+2$, such that

$$A_k \in \Lambda_\mathtt{q},\ D'_k \in \Lambda_\mathtt{p}, \quad \text{thus, by Remark 5.9,} \quad W^\mathrm{s}(\Lambda_\mathtt{q}, g) \pitchfork W^\mathrm{u}(\Lambda_\mathtt{p}, g) \neq \emptyset. \tag{10.1}$$

Thus, to prove the theorem, it remains to obtain perturbations h of g such that

$$W^\mathrm{u}(\Lambda_\mathtt{q}, h) \cap W^\mathrm{s}(\Lambda_\mathtt{p}, h) \neq \emptyset$$

is robust one (see Section 10.2). The construction of the sets $\Lambda_\mathtt{q}$ and $\Lambda_\mathtt{p}$ relies on the preblending machines constructed in Section 10.1.

10.1. **Connecting expanding preblending machines.** Consider the family $\mathfrak{F} = \{f_\mathtt{x} : \mathtt{x} \in \mathscr{C}\}$ of local maps of f in (3.3) and its quotient family $\mathscr{F}$ in (3.10). Proposition 10.3 below provides an adapted perturbation g of f (with respect to $\mathfrak{F}$) whose family of local maps $\mathscr{G}$ have a pair of expanding preblending machines (Definition 7.10), one for $\mathscr{G}$ and the other for $\mathscr{G}^{-1}$, that are "connected" in the sense below. Recall also the adapted transitions between preblending machines in Definition 9.1.

Definition 10.2 (Connected preblending machines)**.** Let $\mathscr{G}_1 = \mathscr{G}$, $\mathscr{G}_2 = \mathscr{G}^{-1}$ with expanding preblending machines $\mathfrak{D}_\tau = (\mathscr{R}_\tau, B_\tau, \mathfrak{X}_\tau)$, $\tau = 1,2$, defined on disjoint open sets N_1, N_2. We say that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are *connected* if there are $\sharp \in \Sigma_T^*$, a compact set $R_\sharp$, and an open set $U_\sharp$ such that $R_\sharp \subset U_\sharp \subset N_2$ and $(R_\sharp, U_\sharp, \psi_\sharp)$ is an adapted transition from $\mathfrak{D}_2$ to $\mathfrak{D}_1$.

Proposition 10.3. *Consider a diffeomorphism f with a simple contour and its quotient family $\mathscr{F}$. Then there exist an arbitrarily small adapted perturbation g of f preserving the contour whose quotient family $\mathscr{G}$ has expanding preblending machines $\mathfrak{D}_1$ (for $\mathscr{G}$) and $\mathfrak{D}_2$ (for $\mathscr{G}^{-1}$) which are connected.*

The proof of this proposition has two steps: the construction of the preblending machines (Section 10.1.1) and their connection (Section 10.1.2).

10.1.1. *Choice of the preblending machines.* Associated to f, consider the quotient dynamics in Section 3.4. For simplicity, we will assume that the numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ in (3.11) are all positive.

The preblending machines are $\mathfrak{D}_1 = \mathfrak{D}_\mathtt{p} = (\mathscr{R}_\mathtt{p}, B_\mathtt{p}, \mathfrak{X}_\mathtt{p})$ and $\mathfrak{D}_2 = \mathfrak{D}_\mathtt{q} = (\mathscr{R}_\mathtt{q}, B_\mathtt{q}, \mathfrak{X}_\mathtt{q})$ are associated with a perturbation $\mathscr{G}$ of $\mathscr{F}$ and defined as follows. First, $\mathfrak{X}_\mathtt{q}, \mathfrak{X}_\mathtt{p}$ are the projections given by

$$\mathfrak{X}_\mathtt{r} = (P_\mathtt{r}, Q_\mathtt{r}), \quad P_\mathtt{r}, Q_\mathtt{r} : \mathtt{U}_\mathtt{r} \to \mathbb{R}, \quad P_\mathtt{r}(x,y) = x, \quad Q_\mathtt{r}(x,y) = y, \quad \mathtt{r} = \mathtt{q}, \mathtt{p}. \tag{10.2}$$

The other elements are obtained from the rectangles in Section 5.5 as follows. Consider $\vartheta_0, \delta_0 > 0$, the adapted perturbation $\mathscr{G}$ of $\mathscr{F}$, the words $\mathtt{i}_2, \mathtt{i}_3, \mathtt{j}_2, \mathtt{j}_3$, and the rectangles

$$\mathtt{H}(\delta) \times \mathtt{V}_{\mathtt{j}_2,\mathtt{j}_3}, \qquad \mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3} \times \mathtt{V}(\vartheta), \qquad \text{and} \qquad \mathtt{R}_{\mathtt{i}_k}, \quad \mathtt{R}_{\mathtt{j}_k}, \quad k = 2,3$$

in Claims 5.12 and 5.13. We let

$$\begin{aligned} &\mathscr{R}_{\mathtt{p},\mathtt{j}_2,\mathtt{j}_3} \stackrel{\text{def}}{=} \{\mathtt{R}_{\mathtt{j}_2}, \mathtt{R}_{\mathtt{j}_3}\}, \qquad B_{\mathtt{p},\mathtt{j}_2,\mathtt{j}_3}(\delta) \stackrel{\text{def}}{=} \mathtt{H}(\delta) \times \mathtt{V}_{\mathtt{j}_2,\mathtt{j}_3}, \qquad \mathfrak{X}_\mathtt{p} = \{P_\mathtt{p}, Q_\mathtt{p}\}, \\ &\mathscr{R}_{\mathtt{q},\mathtt{i}_2,\mathtt{i}_3} \stackrel{\text{def}}{=} \{\mathtt{R}_{\mathtt{i}_2}, \mathtt{R}_{\mathtt{i}_3}\}, \qquad B_{\mathtt{q},\mathtt{i}_2,\mathtt{i}_3}(\vartheta) \stackrel{\text{def}}{=} \mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3} \times \mathtt{V}(\vartheta), \qquad \mathfrak{X}_\mathtt{q} = \{P_\mathtt{q}, Q_\mathtt{q}\}, \end{aligned} \tag{10.3}$$

and denote

$$\mathfrak{D}_{\mathtt{p}} = \mathfrak{D}_{\mathtt{p}}(\delta) = \mathfrak{D}_{\mathtt{p}}(\delta, \mathtt{j}_2, \mathtt{j}_3) \stackrel{\text{def}}{=} (\mathscr{R}_{\mathtt{p},\mathtt{j}_2,\mathtt{j}_3}, B_{\mathtt{p},\mathtt{j}_2,\mathtt{j}_3}(\delta), \mathfrak{X}_{\mathtt{p}}),$$
$$\mathfrak{D}_{\mathtt{q}} = \mathfrak{D}_{\mathtt{q}}(\vartheta) = \mathfrak{D}_{\mathtt{q}}(\vartheta, \mathtt{i}_2, \mathtt{i}_3) \stackrel{\text{def}}{=} (\mathscr{R}_{\mathtt{q},\mathtt{i}_2,\mathtt{i}_3}, B_{\mathtt{q},\mathtt{i}_2,\mathtt{i}_3}(\vartheta), \mathfrak{X}_{\mathtt{q}}). \tag{10.4}$$

In Lemma 10.4 below, we see that $\mathfrak{D}_{\mathtt{p}}$ and $\mathfrak{D}_{\mathtt{q}}$ are indeed preblending machines.

With this terminology in (5.1), we have

$$\mathtt{i}_k = \mathtt{i}_{(m_k,n_k)}, \qquad \mathtt{j}_k = \mathtt{j}_{(r_k,t_k)}, \qquad \text{where} \quad m_3 > m_2,\ n_3 > n_2,\ r_3 > r_2,\ t_3 > t_2.$$

Lemma 10.4. *Given the words $\mathtt{j}_k = \mathtt{j}_{(r_k,t_k)}$, and $\mathtt{i}_k = \mathtt{i}_{(m_k,n_k)}$, $k = 2,3$, in Proposition 5.1, there are $\delta_0, \vartheta_0 > 0$ such that for every $\delta \in (0,\delta_0)$ and every $\vartheta \in (0,\vartheta_0)$ the following holds:*

- *$\mathfrak{D}_{\mathtt{p}} = \mathfrak{D}_{\mathtt{p}}(\delta, \mathtt{j}_2, \mathtt{j}_3)$ is an expanding preblender machine for $\mathscr{G}$ with constants $\mu_{\mathtt{p}} \in (1,2)$ and $\gamma_{\mathtt{p}} \approx \alpha_2^{r_2}\beta_2^{t_2}$, and Lebesgue number $\Delta_{\mathtt{p}} \in (\delta/2, \delta)$;*
- *$\mathfrak{D}_{\mathtt{q}} = \mathfrak{D}_{\mathtt{q}}(\vartheta, \mathtt{i}_2, \mathtt{i}_3)$ is an expanding preblending machine for $\mathscr{G}^{-1}$ with constants $\mu_{\mathtt{q}} \in (1,2)$ and $\gamma_{\mathtt{q}} \approx (\alpha_1^{m_2}\beta_1^{n_2})^{-1}$, and Lebesgue number $\Delta_{\mathtt{q}} \in (\vartheta/2, \vartheta)$.*

Proof. We prove the lemma for $\mathfrak{D}_{\mathtt{p}}$, the proof for $\mathfrak{D}_{\mathtt{q}}$ is similar and hence omitted. We need to check conditions (PB1)–(PB3) in Definition 7.10. To check the domination condition (PB1), it is enough to take $\mu_{\mathtt{j}_2} = \mu_{\mathtt{j}_3} = 1/2$ and $\gamma_{\mathtt{j}_2} = \gamma_{\mathtt{j}_3} = \kappa$, for some κ with $5 < \kappa < \min\{\eta_{\mathtt{j}_2}, \eta_{\mathtt{j}_3}\}$, where $\eta_{\mathtt{j}_k}$ is as in Claim 5.13. Thus, condition (PB1) follows from item (2) in Claim 5.13.

To check (PB2), note that, for $k = 2,3$ it holds

- $\mathtt{R}_{\mathtt{j}_k}$ is a $(P_{\mathtt{p}}, Q_{\mathtt{p}} \circ \psi_{\mathtt{j}_k})$-rectangle by construction,
- $Q_{\mathtt{p}}(\mathtt{R}_{\mathtt{j}_k}) \Subset Q_{\mathtt{p}}(B_{\mathtt{p}})$ follows from $\mathtt{V}_{\mathtt{j}_k} \subset \mathtt{V}_{\mathtt{j}_2,\mathtt{j}_3}$,
- $(Q_{\mathtt{p}} \circ \psi_{\mathtt{j}_k})(\mathtt{R}_{\mathtt{j}_k}) = Q_{\mathtt{p}}(B_{\mathtt{p}})$ follows from 5.13, and
- $(P_{\mathtt{p}} \circ \psi_{\mathtt{j}_k})(\mathtt{R}_{\mathtt{j}_k}) \Subset P_{\mathtt{p}}(B_{\mathtt{q}})$ follows from Claim 5.13.

Condition (PB3) follows from the definitions of the elements of $\mathfrak{D}_{\mathtt{p}}$ and item (1) in Claim 5.13 that can be read as $\overline{P_{\mathtt{p}}(B_{\mathtt{p}})} \subset \mathrm{int}(P_{\mathtt{p}}(\mathtt{R}_{\mathtt{j}_2})) \cup \mathrm{int}(P_{\mathtt{p}}(\mathtt{R}_{\mathtt{j}_3}))$.

The assertion about the expanding constants follows again by Claim 5.13.

Recalling the construction of the rectangles $\mathtt{R}_{\mathtt{i}_k}$ in (5.11) and having in mind the inclusion in item (1) in Claim 5.13, one deduces that the Lebesgue number $\Delta_{\mathtt{p}}$ of $\mathfrak{D}_{\mathtt{p}}$ satisfies $\Delta_{\mathtt{p}} \in (\delta/2, \delta)$, recall Definition 7.10, ending the proof of the lemma. □

Remark 10.5. In the construction of the preblending machines above:

- The fixed points of $\psi_{\mathtt{q}}$ and $\psi_{\mathtt{p}}$ are not modified and the condition $\psi_{\mathtt{c}}(0,0) = (0,0)$ (in local charts) is preserved. Thus the cycle structure kept unchanged.
- The machines coexist independently and persist after perturbations, see Remark 7.11. Note that δ and ϑ can be chosen independently and arbitrarily small. The words $\mathtt{j}_k$ and $\mathtt{i}_k$ can be chosen independently and of arbitrarily large length.
- The points $\mathtt{A}_{\mathtt{i}_k}$ and $\mathtt{D}'_{\mathtt{j}_k}$ belong to the corresponding rectangles of the machines.

10.1.2. *Connecting preblending machines.* To connect the preblending machines $\mathfrak{D}_{\mathtt{p}} = \mathfrak{D}_{\mathtt{p}}(\delta, \mathtt{j}_1, \mathtt{j}_2)$ and $\mathfrak{D}_{\mathtt{q}} = \mathfrak{D}_{\mathtt{q}}(\vartheta, \mathtt{i}_1, \mathtt{i}_2)$ in Lemma 10.4 we unfold the cycle map $\psi_{\mathtt{c}}$.

Lemma 10.6. *There exists an adapted perturbation g of f such that, for the corresponding quotient family $\mathscr{G}$, the preblending machines $\mathfrak{D}_{\mathtt{q}}$ and $\mathfrak{D}_{\mathtt{p}}$ are connected.*

Proof. Recalling Notation 3.2 for a prefix of a word, consider the orbits of the preblending machines given by:

$$\begin{aligned}\mathscr{O}(\mathfrak{D}_{\mathtt{p}}(\delta,\mathtt{j}_2,\mathtt{j}_3)) &\stackrel{\text{def}}{=} \{\psi_{\mathtt{j}}(\mathtt{R}_{\mathtt{j}_2}),\ \mathtt{j} \text{ a prefix of } \mathtt{j}_2\} \cup \{\psi_{\mathtt{j}}(\mathtt{R}_{\mathtt{j}_3}),\ \mathtt{j} \text{ a prefix of } \mathtt{j}_3\},\\ \mathscr{O}(\mathfrak{D}_{\mathtt{q}}(\vartheta,\mathtt{i}_2,\mathtt{i}_3)) &\stackrel{\text{def}}{=} \{\psi_{\mathtt{i}}(\mathtt{R}_{\mathtt{i}_2}),\ \mathtt{i} \text{ a prefix of } \mathtt{i}_3\} \cup \{\psi_{\mathtt{i}}(\mathtt{R}_{\mathtt{i}_3}),\ \mathtt{i} \text{ a prefix of } \mathtt{i}_3\}.\end{aligned} \tag{10.5}$$

These sets are compact and disjoint from $\{\mathtt{P},\mathtt{Q}\}$. This allows to select small neighbourhoods $\mathtt{V}_\mathtt{c}$ and $\mathtt{O}_\mathtt{c}$ of $\mathtt{P}$ contained in $\mathtt{U}_\mathtt{c} \subset \mathtt{U}_\mathtt{p}$ which are disjoint from the orbits $\mathscr{O}(\mathfrak{D}_\mathtt{p})$ and $\mathscr{O}(\mathfrak{D}_\mathtt{q})$ such that $\overline{\mathtt{V}}_\mathtt{c} \subset \mathtt{O}_c \subset \overline{\mathtt{O}_\mathtt{c}} \subset \mathtt{U}_\mathtt{c}$. Then there is small $\tau > 0$ such that for every $\bar\omega$ with $|\bar\omega| < \tau$ there is a local diffeomorphism, called a *translations of* $\psi_\mathtt{c}$, such that

$$\psi_{\mathtt{c},\bar\omega}\colon \mathtt{U}_\mathtt{c} \to \mathtt{U}_\mathtt{q}, \qquad \psi_{\mathtt{c},\bar\omega}(x,y) = \begin{cases} \psi_\mathtt{c}(x,y)+\bar\omega, & \text{if } (x,y)\in \mathtt{V}_\mathtt{c},\\ \psi_\mathtt{c}(x,y), & \text{if } (x,y)\notin \overline{\mathtt{O}}_\mathtt{c}.\end{cases} \tag{10.6}$$

To connect the machines $\mathfrak{D}_\mathtt{q}$ and $\mathfrak{D}_\mathtt{p}$, we consider compositions of the form

$$\psi_\# \stackrel{\text{def}}{=} \psi_\mathtt{d}\circ\psi_\mathtt{q}^w\circ\psi_{\mathtt{c},\bar\omega_{v,w}}\circ\psi_\mathtt{p}^v\circ\psi_\mathtt{a},$$

for arbitrary large w, v and a small vector $\bar\omega_{v,w}$. We begin by choosing w and v. Fixed large w, the vertical $\Delta_\mathtt{p}^{\text{ver}}(w)$ and horizontal $\Delta_\mathtt{p}^{\text{hor}}(w)$ sizes of $\psi_\mathtt{q}^{-w}(\psi_\mathtt{d}^{-1}(B_\mathtt{p}))$ satisfies

$$\Delta_\mathtt{p}^{\text{ver}}(w) \approx \beta_2^{-w}|\mathtt{V}_{\mathtt{j}_2,\mathtt{j}_3}| \qquad\text{and}\qquad \Delta_\mathtt{p}^{\text{hor}}(w)\approx \beta_1^{-w}\delta. \tag{10.7}$$

Similarly, the vertical $\Delta_\mathtt{q}^{\text{ver}}(v)$ and horizontal $\Delta_\mathtt{q}^{\text{hor}}(v)$ sizes of $\psi_\mathtt{p}^v(\psi_\mathtt{a}(B_\mathtt{q}))$ satisfy

$$\Delta_\mathtt{q}^{\text{ver}}(v)\approx\alpha_2^v\vartheta \qquad\text{and}\qquad \Delta_\mathtt{q}^{\text{hor}}(v)\approx\alpha_1^v|\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}|. \tag{10.8}$$

We will chose v, w arbitrarily large such that

$$\Delta_\mathtt{q}^{\text{ver}}(v) > \Delta_\mathtt{p}^{\text{ver}}(w) \qquad\text{and}\qquad \Delta_\mathtt{q}^{\text{hor}}(v) < \Delta_\mathtt{p}^{\text{hor}}(w).$$

In view of (10.7) and (10.8), to get these inequalities it is enough to find large v, w with

$$\alpha_2^v\beta_2^w \geqslant \frac{|\mathtt{V}_{\mathtt{j}_2,\mathtt{j}_3}|}{\vartheta} \qquad\text{and}\qquad \alpha_1^v\beta_1^w \leqslant \frac{\delta}{|\mathtt{H}_{\mathtt{i}_2,\mathtt{i}_3}|}. \tag{10.9}$$

Note that there are arbitrarily large v, w as in (10.9) such that:

$$\frac{4\vartheta}{\delta} \leqslant \alpha_2^v\beta_2^w \leqslant \frac{4\vartheta}{\delta}\alpha_2^{-2}\beta_2^2 \stackrel{\text{def}}{=} \kappa \tag{10.10}$$

Therefore

$$\alpha_1^v\beta_1^w = \left(\frac{\alpha_1}{\alpha_2}\right)^v\alpha_2^v\beta_2^w\left(\frac{\beta_1}{\beta_2}\right)^w \approx \left(\frac{\alpha_1}{\alpha_2}\right)^v\left(\frac{\beta_1}{\beta_2}\right)^w\kappa\to 0 \quad\text{as}\quad v,w\to\infty. \tag{10.11}$$

Moreover, for v, w big enough and every $|\bar\omega| \leqslant \tau$ it holds

$$\psi_\mathtt{p}^v(\psi_\mathtt{a}(B_\mathtt{q}))\subset \mathtt{V}_\mathtt{c} \quad\text{and}\quad \psi_\mathtt{q}^{-w}(\psi_\mathtt{d}^{-1}(B_\mathtt{p}))\subset\psi_\mathtt{c}(\mathtt{O}_\mathtt{c}) = \psi_{\mathtt{c},\bar\omega}(\mathtt{O}_\mathtt{c}). \tag{10.12}$$

Note that there are sequences $(v_k),(w_k)\to\infty$ satisfying (10.10), (10.11), and (10.12). This choice of v_k and w_k implies that there are vectors $\bar\omega_k \stackrel{\text{def}}{=} \bar\omega_{v_k,w_k}$, with $||\bar\omega_k|| \approx \beta_1^{-v_k}\alpha_2^{w_k}$, whose associated translations $\psi_{\mathtt{c},k} \stackrel{\text{def}}{=} \psi_{\mathtt{c},\bar\omega_{v_k,w_k}}$ of $\psi_\mathtt{c}$ are such that $\psi_\mathtt{q}^{-w_k}(\psi_\mathtt{d}^{-1}(B_\mathtt{p}))$ and

$\psi_{\mathtt{c},\bar{\omega}_{v_k,w_k}}(\psi_{\mathtt{p}}^{v_k}(\psi_{\mathtt{a}}(B_{\mathtt{q}})))$ intersect in a Markovian way, as depicted in Figure 10. Call this intersection $\widetilde{\mathtt{R}}_{\sharp,k}$. Let

$$
\begin{aligned}
\mathtt{R}_{\sharp,k} &\stackrel{\text{def}}{=} \psi_{\mathtt{a}}^{-1}\circ\psi_{\mathtt{p}}^{-v_k}\circ\psi_{\mathtt{c},k}^{-1}(\widetilde{\mathtt{R}}_{\sharp,k})\subset \mathtt{B}_{\mathtt{q}},\\
U_\sharp &\stackrel{\text{def}}{=} \operatorname{int}(B_{\mathtt{p}}),\\
\psi_{\sharp,k} &\stackrel{\text{def}}{=} \psi_{\mathtt{d}}\circ\psi_{\mathtt{q}}^{w_k}\circ\psi_{\mathtt{c},k}\circ\psi_{\mathtt{p}}^{v_k}\circ\psi_{\mathtt{a}}.
\end{aligned}
\tag{10.13}
$$

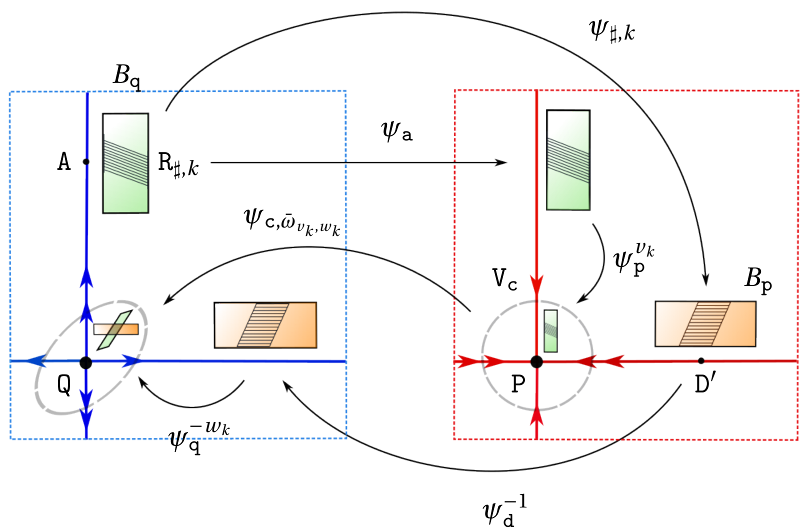

FIGURE 10. Markovian intersection between $\psi_{\mathtt{q}}^{-w_k}(\psi_{\mathtt{d}}^{-1}(B_{\mathtt{p}}))$ and $\psi_{\mathtt{c},\bar{\omega}_{v_k,w_k}}(\psi_{\mathtt{p}}^{v_k}(\psi_{\mathtt{a}}(B_{\mathtt{q}})))$

The following claim implies the lemma.

Claim 10.7. *For every k large enough, the triple $(\mathtt{R}_{\sharp,k},U_\sharp,\psi_{\sharp,k})$ is an adapted transition between $\mathfrak{D}_{\mathtt{q}}$ and $\mathfrak{D}_{\mathtt{p}}$.*

Proof. We check that conditions (at1)-(at5) in Definition 9.1 hold. We will omit the subscript k. Observe first that, by construction, the pair verifies condition (at3). Let $\mu_{\mathtt{p}},\gamma_{\mathtt{p}}$ and $\mu_{\mathtt{q}},\gamma_{\mathtt{q}}$ be the expanding constants of the preblender machines $\mathfrak{D}_{\mathtt{p}}$ and $\mathfrak{D}_{\mathtt{q}}$. Taking

$$\mu_1=\mu_{\mathtt{p}},\quad \gamma_1=\gamma_{\mathtt{p}}\quad\text{and}\quad \mu_2=\mu_{\mathtt{q}},\quad \gamma_2=\gamma_{\mathtt{q}},$$

it follows from Lemma 10.4 that $\mu_1\gamma_2>1$ and $\mu_2\gamma_1>1$, proving (at1). The domination condition in (at2) follows similarly.

It remains to check conditions (at4) and (at5) hold with the constants

$$\mu_\sharp=\mu_{\sharp,k}\approx(\alpha_1^{v_k}\beta_1^{w_k})^{-1},\qquad \gamma_\sharp=\gamma_{\sharp,k}\approx\alpha_2^{v_k}\beta_2^{w_k}.$$

These choices are done to guarantee that $\psi_{\sharp,k}$ is $\big((P_{\mathtt{q}},Q_{\mathtt{q}}),(P_{\mathtt{p}},Q_{\mathtt{p}}),\mu_{\sharp,k},\gamma_{\sharp,k}\big)$-dominated on $R_{\sharp,k}$, providing (at4).

To get (at5), recall that the Lebesgue numbers $\Delta_{\mathtt{p}}$ and $\Delta_{\mathtt{q}}$ of $\mathfrak{D}_{\mathtt{p}}(\delta)$ and $\mathfrak{D}_{\mathtt{q}}(\vartheta)$ satisfies $\Delta_{\mathtt{p}} \in (\delta/2, \delta)$ and $\Delta_{\mathtt{q}} \in (\vartheta/2, \vartheta)$, see Lemma 10.4. Moreover,

$$\operatorname{diam} P_{\mathtt{p}}(B_{\mathtt{p}}(\delta)) = \operatorname{diam}(\mathtt{H}(\delta)) = 2\delta \qquad \text{and} \qquad \operatorname{diam} Q_{\mathtt{q}}(B_{\mathtt{q}}(\vartheta)) = \operatorname{diam}(\mathtt{V}(\vartheta)) = 2\vartheta.$$

By (10.10), the values $\gamma_{\sharp,k}$ are uniformly bounded and by (10.11), $\mu_{\sharp,k}^{-1} \to 0$ as $k \to \infty$. Therefore, for sufficiently large k,

$$\mu_{\sharp,k}^{-1} \operatorname{diam} P_{\mathtt{p}}(B_{\mathtt{p}}(\delta)) \approx \mu_{\sharp,k}^{-1} 2\delta < \Delta_{\mathtt{q}} \quad \text{and} \quad \gamma_{\sharp,k}^{-1} \operatorname{diam} Q_{\mathtt{q}}(B_{\mathtt{q}}(\vartheta)) = \gamma_{\sharp,k}^{-1} 2\vartheta < \Delta_{\mathtt{p}},$$

where the inequalities follow from (10.10). This completes the proof of the claim. □

The proof of the lemma is now complete. □

The proof of the propostion is now complete. □

10.2. **End of the proof of Theorem A.** Consider an adapted perturbation g of f as in Proposition 10.3 and its quotient family $\mathscr{G}$ having connected expanding preblending machines $\mathfrak{D}_2 = \mathfrak{D}_{\mathtt{q}}$ for $\mathscr{G}^{-1}$ and $\mathfrak{D}_1 = \mathfrak{D}_{\mathtt{p}}$ for $\mathscr{G}$. First, by Theorem 7.14, the tower map F associated to $\mathscr{G}$ in (7.11) has a pair of split blending machines associated with these preblending machines. These machines provides the hyperbolic sets $\Lambda_{\mathtt{p}}$ and $\Lambda_{\mathtt{q}}$ in Remark 10.1 satisfying (10.1).

We now see that these split blending machines satisfy the hypotheses Theorem 9.2, hence the tower map F has a C^1 robust cycle of co-index to these split blending machines and the same holds for the sets $\Lambda_{\mathtt{p}}$ and $\Lambda_{\mathtt{q}}$.

We now explain the objects involved in our application of Theorem 9.2, Given g as above and its local dynamics $\{g_{\mathtt{x}} : \mathtt{x} \in \mathscr{C}\}$, we consider the family of affine maps

$$A_{\mathtt{x}} = g_{\mathtt{x}}|_{E^{\mathrm{sss}} \oplus E^{\mathrm{uuu}}}. \tag{10.14}$$

As g is an adapted perturbation, these maps are constant in the sets $U_{\mathtt{x}}$. Given a word $\mathtt{i} \in \Sigma_T^*$, we define compositions $A_{\mathtt{i}}$ as in (3.5).

Associated with the preblending machines

$$\begin{aligned} \mathfrak{D}_1 &\stackrel{\text{def}}{=} \mathfrak{D}_{\mathtt{p}} = (\mathscr{R}_{\mathtt{p}}, B_{\mathtt{p}}, \mathfrak{X}_{\mathtt{p}}) = (\mathscr{R}_{\mathtt{p},\mathtt{j}_2,\mathtt{j}_3}, B_{\mathtt{p},\mathtt{j}_2,\mathtt{j}_3}(\delta), \mathfrak{X}_{\mathtt{p}}),\\ \mathfrak{D}_2 &\stackrel{\text{def}}{=} \mathfrak{D}_{\mathtt{q}} = (\mathscr{R}_{\mathtt{q}}, B_{\mathtt{q}}, \mathfrak{X}_{\mathtt{q}}) = (\mathscr{R}_{\mathtt{q},\mathtt{i}_2,\mathtt{i}_3}, B_{\mathtt{q},\mathtt{i}_2,\mathtt{i}_3}(\vartheta), \mathfrak{X}_{\mathtt{q}}), \end{aligned} \tag{10.15}$$

whose ingredients are specified in (10.3), we consider the two legs horseshoe maps $\mathbb{A}_1$ and $\mathbb{A}_2$ whose maps are $(A_{\mathtt{i}_k})_{k=2,3}$ and $(A_{\mathtt{j}_k})_{k=2,3}$, respectively, defined by (10.14) on the corresponding rectangles. The indices of these maps are $I_1 = \{\mathtt{i}_2, \mathtt{i}_3\}$ and $I_2 = \{\mathtt{j}_2, \mathtt{j}_3\}$.

We also note that $\mathbb{A}_1$ is $\mathfrak{X}_{\mathtt{p}}$-adapted to $\mathfrak{D}_{\mathtt{p}}$, and that $\mathbb{A}_2^{-1}$ is $\mathfrak{X}_{\mathtt{q}}$-adapted to $\mathfrak{D}_{\mathtt{q}}$. All the assertions about adapted stretching follows from domination and partial hyperbolicity.

Finally, consider the triple $(R_{\sharp,k}, U_{\sharp,k}, \psi_{\sharp,k})$ in Claim 10.7, see also (10.13). In what follows, k is fixed and hence omitted. We also consider the one-leg horseshoe map $\mathbb{A}_\sharp = (A_\sharp)$ with index $I_\sharp = \{\sharp\}$.

We now detail the ingredients in our application of Theorem 9.2:

- the open sets $N_1, N_2 \subset M$ where $N_1 \stackrel{\text{def}}{=} \operatorname{int}(U_{\mathtt{p}})$ and $N_2 \stackrel{\text{def}}{=} \operatorname{int}(U_{\mathtt{q}})$;
- the $(1,1)$-splitting $\mathfrak{X}_1 \stackrel{\text{def}}{=} \mathfrak{X}_{\mathtt{p}}$ and $\mathfrak{X}_2 \stackrel{\text{def}}{=} \mathfrak{X}_{\mathtt{q}}$ of N_1 and N_2;
- the families $\mathscr{F}_1 \stackrel{\text{def}}{=} \mathscr{G}$, $\mathscr{F}_2 \stackrel{\text{def}}{=} \mathscr{G}^{-1}$ and $\mathscr{F}_\sharp \stackrel{\text{def}}{=} \{\psi_\sharp\}$;

- the preblenders in $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in (10.15),
- the adapted transition $(R_\sharp, U_\sharp, \psi_\sharp)$ between $\mathfrak{D}_1$ and $\mathfrak{D}_2$ in (10.13),
- the two-legs horseshoe maps $\mathbb{A}_1$ ($\mathfrak{X}_\mathrm{p}$-adapted to $\mathfrak{D}_\mathrm{p}$) and $\mathbb{A}_2^{-1}$ ($\mathfrak{X}_\mathrm{q}$-adapted to $\mathfrak{D}_\mathrm{q}$),
- the one-leg horseshoe map $\mathbb{A}_\sharp$, with domain $\mathbf{R}_\sharp = \mathsf{H}_\sharp \times R_\sharp$, whose stretching is $(\mathfrak{X}_2^t, \mathfrak{X}_1)$-adapted to $\psi_\sharp$ on $R_\sharp$.

Consider the tower maps F_ρ associated to $(\mathbb{A}_\rho \ltimes \mathscr{F}_\rho, \mathscr{R}_\rho, I_\rho)$, $\rho = 1, 2, \sharp$, see (7.11), and the split blending machines $\mathfrak{B}_\tau$ associated to F_τ and $\mathfrak{D}_\tau$ for $\tau = 1, 2$, given by Theorem 7.14. Now, by Theorem 9.2,

$$\Lambda^\mathrm{s}(\mathfrak{B}_1, F_1) \cap F_\sharp(\mathbf{R}_\sharp \cap \Lambda^\mathrm{s}(\mathfrak{B}_2, F_2)) \neq \emptyset.$$

Moreover, this intersection holds for C^1 neighbourhoods of F_1 and F_2. Noting that F_2 is associated to a local inverse map, there exist a C^1 neighbourhood $\mathscr{U}$ of $F_\mathscr{G}$ such that for every $G \in \mathscr{U}$ we get

$$\Lambda^\mathrm{s}(\mathfrak{B}_{1,G}, G) \cap \Lambda^\mathrm{u}(\mathfrak{B}_{2,G}, G) \neq \emptyset$$

where $\mathfrak{B}_{\tau,G}$ is the continuation of $\mathfrak{B}_\tau$, $\tau = 1, 2$.

Finally, we observe that there exists a C^1-neighbourhood $\mathscr{V}$ of g and a correspondence $h \in \mathscr{V} \to G_h \in \mathscr{U}$, such that for each $h \in \mathscr{V}$, there exist hyperbolic sets $\Lambda_{1,h}$ and $\Lambda_{2,h}$ of u-indices $\ell + 2$ and ℓ, respectively, corresponding to the continuations of the sets $\mathfrak{B}_{1,G_h}$ and $\mathfrak{B}_{2,G_h}$This complete the proof of the theorem.

11. EXAMPLES OF NON-ESCAPING HETERODIMENSIONAL CYCLES

We present two classes of examples exhibiting non-escaping cycles. The first is obtained by adapting the nonhyperbolic diffeomorphisms of Abraham-Smale [1] and Shub [36], following Manning's construction [26]. The second arises in the context of matrix cocycles in $\mathrm{GL}(3,\mathbb{R})$. These constructions are outlined only briefly.

11.1. **Shub-Manning and Abraham-Smale-Manning's maps.** We begin with a variation of the *double derived from Anosov (DA)* diffeomorphism in [26, Example 3, pag 36]. We closely follow the construction in [15, Section 2].

Consider a (linear) Anosov diffeomorphism $L\colon \mathbb{T}^2 \to \mathbb{T}^2$ having two fixed points $\theta_1 \neq \theta_2$, a hyperbolic splitting, $T\mathbb{T}^2 = E^\mathrm{s}_L \oplus E^\mathrm{u}_L$, and the corresponding stable and unstable foliations $W^\mathrm{s}(\cdot, L)$ and $W^\mathrm{u}(\cdot, L)$. For $i = 1, 2$, denote by $W^\mathrm{u}_\pm(\theta_i, L)$ the connected component of $W^\mathrm{u}(\theta_i, L) \setminus \{\theta_i\}$. Similarly, for the components $W^\mathrm{s}_\pm(\theta_i, L)$ of $W^\mathrm{s}_L(\theta_i, L) \setminus \{\theta_i\}$. We observe that for any combination $(i, j) \in (\pm, \pm)$ it holds

$$W^\mathrm{u}_i(\theta_1, L) \pitchfork W^\mathrm{s}_j(\theta_2, L) \neq \emptyset. \tag{11.1}$$

The DA map g is obtained by perturbing L in two steps (these perturbations are not C^1 small). First, we get a classical DA map g_1 with a nontrivial transitive hyperbolic attractor Λ_1. This maps is obtained by a homotopic deformation of L around of the fixed point θ_1, breaking it into three fixed points θ_1^ℓ, $\theta_1^\mathrm{c} = \theta_1$, and θ_1^r, where θ_1 is a source and $\theta_1^{\ell,r}$ are saddles. This perturbation preserves the stable foliation of L. The attractor of g_1 is $\Lambda_1 = \mathbb{T}^2 \setminus W^\mathrm{u}(\theta_1, g_1)$ which contains θ_1^ℓ, θ_1^r.

In this construction, the map g_1 has a dominating splitting $T\mathbb{T}^2 = E^\mathrm{c}_{g_1} \oplus E^\mathrm{u}_{g_1}$, where $E^\mathrm{c}_{g_1} = E^\mathrm{s}_L$, and there are invariant foliations $W^\mathrm{c}(\cdot, g_1)$ and $W^\mathrm{u}(\cdot, g_1)$ tangent to $E^\mathrm{c}_{g_1}$ and

$E^{\mathrm{u}}_{g_1}$, respectively, such that $W^{\mathrm{u}}(\cdot, g_1)$ is uniformly expanding and $W^{\mathrm{c}}(\cdot, g_1) = W^{\mathrm{s}}(\cdot, L)$. This construction implies that equation (11.1) reads as follows

$$W^{\mathrm{u}}_i(\theta_1, g_1) \pitchfork W^{\mathrm{c}}_j(\theta_2, g_1) \neq \emptyset. \tag{11.2}$$

Note that $W^{\mathrm{u}}_{\pm}(\theta_1, g_1)$ are the separatrices of the strong unstable manifold of θ_1 and $W^{\mathrm{c}}_{\pm}(\theta_2, g_1)$ are the separatrices of the stable manifold of θ_2.

The map g is obtained from g_1, in a similar way, by a homotopic deformation around of the fixed point θ_2 giving rise to three fixed points $\theta_2^{\ell}, \theta_2^{\mathrm{c}} = \theta_2$, where θ_2 is a sink and $\theta_2^{\ell,r}$ are saddles. As above, this perturbation preserves the unstable foliation of g_1. Therefore there exists an invariant (dominated) splitting $T\mathbb{T}^2 = E^{\mathrm{c}_1}_g \oplus E^{\mathrm{c}_2}_g$ and invariant foliations $W^{\mathrm{c}_1}(\cdot, g)$ and $W^{\mathrm{c}_2}(\cdot, g)$ tangent to $E^{\mathrm{c}_1}_g$ and $E^{\mathrm{c}_2}_g$. Equation (11.2) now reads as follows

$$W^{\mathrm{c}_2}_i(\theta_1, g) \pitchfork W^{\mathrm{c}_1}_j(\theta_2, g) \neq \emptyset. \tag{11.3}$$

Note that $W^{\mathrm{c}_2}_{\pm}(\theta_1, g)$ and $W^{\mathrm{c}_1}_{\pm}(\theta_2, g)$ are the separatrices of the strong unstable and stable manifolds of θ_1 and θ_2, respectively.

Moreover, any diffeomorphisms C^1 close to g satisfies the corresponding equation (11.3). This ends the definition of the DA map g.

Consider now an Axiom A diffeomorphism $h\colon M \to M$ of a compact manifold M with two different fixed points p and q in a basic set Λ of its spectral decomposition. Consider the skew-product

$$\Phi\colon M \times \mathbb{T}^2 \to M \times \mathbb{T}^2, \quad \Phi(x, y) = (h(x), f_x(y)), \tag{11.4}$$

where $(f_x)_{x\in M}$ is a smooth family of diffeomorphisms such that

$$f_x\colon \mathbb{T}^2 \to \mathbb{T}^2, \qquad f_x = \begin{cases} L, & \text{if } x \notin B_\varrho(q), \\ g, & \text{if } x \in B_{\varrho/2}(q), \end{cases} \tag{11.5}$$

where $\varrho > 0$ is small and $B_\varrho(q)$ denotes the open ball centred at q with radius ϱ. On the set $B_\varrho(q) \setminus B_{\varrho/2}(q)$, the family f_x is an homotopic deformation between the maps L to g with one-dimensional foliations tangent to a dominated splitting, see Figure 11.

We assume that the contraction/expansion in the fibers are dominated by ones on the base, as a consequence the set $\Lambda \times \mathbb{T}^2$ is a partially hyperbolic with a splitting $T_{\Lambda\times\mathbb{T}^2} M \times \mathbb{T}^2 = E^{\mathrm{s}} \oplus E^{\mathrm{c}_1} \oplus E^{\mathrm{c}_2} \oplus E^{\mathrm{u}}$, where the fibers of the bundle $E^{\mathrm{c}} = E^{\mathrm{c}}_1 \oplus E^{\mathrm{c}}_2$ are tangent to the normally hyperbolic invariant foliation $W^{\mathrm{c}}(x, y) \stackrel{\text{def}}{=} \{x\} \times \mathbb{T}^2$, $x \in \mathbb{T}^2$. Moreover, each central leaf is sub foliated by invariant foliations $W^{\mathrm{c}_1}, W^{\mathrm{c}_2}$ tangent to $E^{\mathrm{c}_1}, E^{\mathrm{c}_2}$, respectively.

As a consequence, there exists a C^1 neighbourhood $\mathscr{U}$ of Φ such that every map in $\mathscr{U}$ is conjugate to skew-product diffeomorphisms as in (11.4) with the same basis h and fiber dynamics $\tilde{f}_x$ C^1 close to f_x, see [24, Chapter 8]. Moreover, Φ (restricted to $\Lambda \times \mathbb{T}^2$) is robustly transitive, see [14, Proposition 7.2]. In what follows, we use the perturbations in $\mathscr{U}$ and the associated skew-products interchangeably.

Consider the fixed points of saddle type $Q \stackrel{\text{def}}{=} \{(q, \theta_1)\}$ and $P \stackrel{\text{def}}{=} \{(q, \theta_2)\}$ of u-indices three and one, respectively. We claim that, after a small perturbation, these points form a non-escaping cycle.

By robust transitivity, and after an arbitrarily small perturbation, the points Q and P form a cycle. This is an standard consequence of the Connecting Lemma in [23]. Condition (11.3), involving the separatrices of the fixed saddles θ_1 and θ_2 of the map

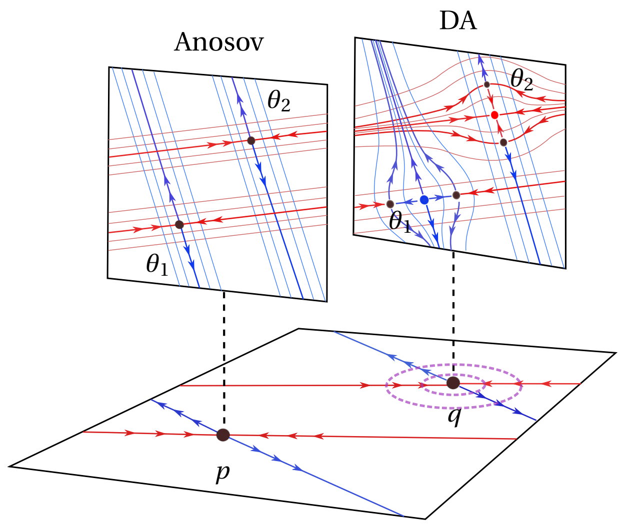

FIGURE 11. Homotopic deformation from $\sigma \times L$ to Φ

g, ensures that items (NE1)-(NE2) in Definition 2.1 are satisfied for the intersection $W^{\mathrm{u}}(Q,\Phi) \pitchfork W^{\mathrm{s}}(P,\Phi)$. These separatrices behaves as those of the map f_0 in the toy model in Section 2.2; see Figure 1.)

The previous construction applies to the *Shub-Manning maps*, where h in (11.4) is a (transitive) Anosov map in $\mathbb{T}^2$. It also applies to the *Abraham-Smale-Manning maps*, taking $h\colon \mathbb{S}^2 \to \mathbb{S}^2$ a (global) horseshoe map and Λ is a horseshoe[8].

11.2. **Non-escaping cycles arising from** GL$(3,\mathbb{R})$-**cocycles.** In [3], the set of uniformly hyperbolic SL$(2,\mathbb{R})$-cocycles is characterized, and its boundary is described either by the presence of parabolic elements or by the existence of two maps in the cocycle, one of which sends the unstable direction of the first into the stable direction of the second; see [3, Theorem 4.1]. Considering the projectivization of the cocycles, this property can be viewed as a heterodimensional cycle of the corresponding skew-product, see [18, Section 1.2]. Note that to obtain co-index two cycles, one needs to consider dimensions higher than two. We focus on GL$(3,\mathbb{R})$-cocycles (with SL$(3,\mathbb{R})$ being a particular case).

Consider $A, B \in \mathrm{GL}(3,\mathbb{R})$ such that A has basis of unitary eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with

$$A(\mathbf{e}_i) = \lambda_i \mathbf{e}_i, \quad \text{where} \quad 0 < \lambda_1 < \lambda_2 < \lambda_3, \qquad \text{and} \qquad B(\mathbf{e}_3) = \mathbf{e}_1, \quad B(\mathbf{e}_1) \neq \mathbf{e}_3. \tag{11.6}$$

Given $L \in \mathrm{GL}(3,\mathbb{R})$, denoting $[\mathbf{v}]$ the class of a nonzero vector $\mathbf{v} \in \mathbb{R}^3$ in $\mathbb{P}(\mathbb{R}^3)$, consider its projectivization

$$f_L\colon \mathbb{P}(\mathbb{R}^3) \to \mathbb{P}(\mathbb{R}^3), \quad f_L([\mathbf{v}]) \stackrel{\text{def}}{=} [L(\mathbf{v})].$$

Condition (11.6) implies that (see Figure 12)

---

[8]This map can be defined as a *one-step skew product*. Indeed, if $R_1, R_2 \subset M$ is a Markov partition of Λ, such that $p \in R_1$ and $q \in R_2$, then replacing $B_\varrho(q)$ in (11.5) by a small neighbourhood of R_2 disjoint from R_1, the restriction of Φ to $\Lambda \times \mathbb{T}^2$ is a one-step skew product.

- $[\mathbf{e}_3]$ is a sink of f_A and has a strong stable manifold,
- $[\mathbf{e}_2]$ is a saddle f_A, and
- $[\mathbf{e}_1]$ is a source of f_A and has a strong unstable manifold.

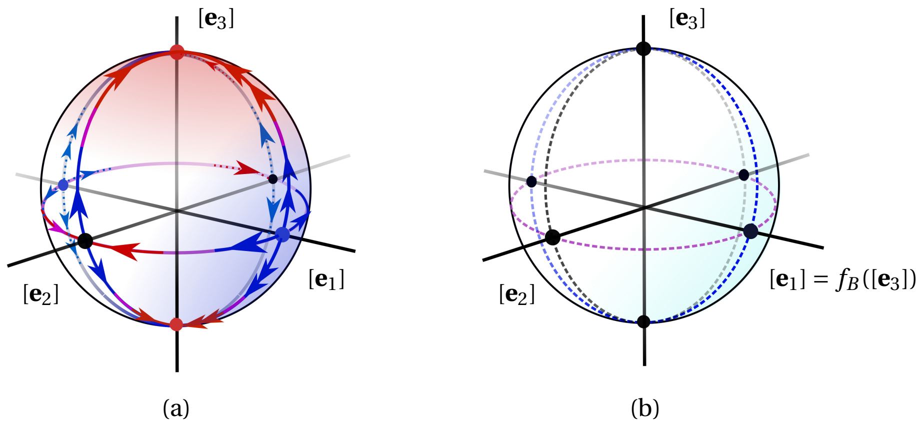

FIGURE 12. (a) Dynamics of f_A. (b) Dynamics of f_B

Letting $f_0 \stackrel{\text{def}}{=} f_A$ and $f_1 \stackrel{\text{def}}{=} f_B$, we consider the skew-product as in (2.8)

$$F\colon \Sigma_2 \times \mathbb{P}(\mathbb{R}^3) \to \Sigma_2 \times \mathbb{P}(\mathbb{R}^3), \quad (\underline{i}, x) = (\sigma(\underline{i}), f_{i_0}(x)), \quad \text{with} \quad \underline{i} = (\dots i_{-1}.i_0 i_1 \dots).$$

Comparing with Section 2.2, here $[\mathbf{e}_3]$ and $[\mathbf{e}_1]$ play the roles of P and Q, respectively. As in (2.9), we consider the “saddles” $\mathbf{E}_1 = (0^{\mathbb{Z}}, [\mathbf{e}_1])$ and $\mathbf{E}_3 = (0^{\mathbb{Z}}, [\mathbf{e}_3])$ with different "u-indices”. Note that $B(\mathbf{e}_3) = \mathbf{e}_1$ implies $f_1([\mathbf{e}_3]) = [\mathbf{e}_1]$ (as condition (d) in Section 2.2) and the global structure implies that $W^s([\mathbf{e}_3], f_0)$ intersects $W^u([\mathbf{e}_1], f_0)$, providing a cycle of co-index two between $\mathbf{E}_1$ and $\mathbf{E}_3$ for F which is of non-escaping type.

The definition of f_0 implies the existence of the strong stable bundle of $[\mathbf{e}_3]$ and the strong unstable bundle of $[\mathbf{e}_1]$. We also have a dominated splittings and the corresponding strong manifolds, we now provide additional information about these objects.

Denote by $[\mathbf{e}_i, \mathbf{e}_j]$ the equivalence classes in $\mathbb{P}(\mathbb{R}^3)$ of the vectors in the plane generated by $\mathbf{e}_i$ and $\mathbf{e}_j$, $i, j \in \{1,2,3\}$, $i < j$. The A-invariance of these planes and a direct calculation implies that both the strong stable bundle of $[\mathbf{e}_3]$ and the strong unstable bundle $[\mathbf{e}_1]$ for f_0 are tangent to $[\mathbf{e}_1, \mathbf{e}_3]$. Thus we get the inclusion

$$\big([\mathbf{e}_1, \mathbf{e}_3] \setminus \{[\mathbf{e}_1], [\mathbf{e}_2]\}\big) \subset W^{\mathrm{ss}}([\mathbf{e}_3], f_0) \cap W^{\mathrm{uu}}([\mathbf{e}_1], f_0).$$

This condition is different from (NE2) in the definition of non-escaping cycle and also from the one in the toy model in Section 2.2, where the strong invariant manifolds are different. This difficulty is bypassed using the map B. From (11.6), we get that

$$B([\mathbf{e}_1, \mathbf{e}_3]) \neq [\mathbf{e}_1, \mathbf{e}_3],$$

which means that the f_1 maps $W^{\mathrm{uu}}([\mathbf{e}_1], f_0)$ outside of $W^{\mathrm{ss}}([\mathbf{e}_3], f_0)$. The fact that f^{-1} maps $W^{\mathrm{ss}}([\mathbf{e}_3], f_0)$ disjointly from $W^{\mathrm{uu}}([\mathbf{e}_1], f_0)$ also follows from (11.6).

The fact that Df_1 maps the dominated splitting of $[\mathbf{e}_3]$ into a splitting that is in general position with respect to the one of $[\mathbf{e}_1]$ also follows from (11.6). Obtaining a similar condition to (e) in Section 2.2. This completes our sketch.

Remark 11.1. Our construction provides a small skew-product perturbation with robust cycles of co-index two and one. However, this perturbation it is not associated to a cocycle. A general question is if these perturbations can be obtained associated to a cocycle. Answering this question is beyond the scope of our paper.

REFERENCES

[1] ABRAHAM, R. AND SMALE, S. *Nongenericity of* Ω*-stability*. Global Analysis (Proc. Sympos. Pure Math., Vols. XIV, XV, XVI, Berkeley, Calif., 1968), volume XIV-XVI of Proc. Sympos. Pure Math., pages 5–8, Amer. Math. Soc., Providence, RI, 1970. 2, 4, 57

[2] ASAOKA, M. *Stable intersection of Cantor sets in higher dimension and robust homoclinic tangency of the largest codimension.* Trans. Amer. Math. Soc., 375(2):873–908, 2022. 5, 6, 11, 12, 29, 31, 32, 33, 34, 35, 36, 44, 45, 46

[3] AVILA, A., BOCHI, J., AND YOCCOZ, J.-C. *Uniformly hyperbolic finite-valued* $\mathrm{SL}(2,\mathbb{R})$*-cocycles.* Comment. Math. Helv., 85(4):813–884, 2010. 4, 59

[4] BARRIENTOS, P. G., FAKHARI, A., MALICET, D., AND SARIZADEH, A. *Expanding actions: minimality and ergodicity.* Stochastics and Dynamics, 17(4):1750031, 2017. 41

[5] BARRIENTOS, P. G., KI, Y., AND RAIBEKAS, A. *Symbolic blender-horseshoes and applications.* Nonlinearity, 27(12):2805–2839, 2014. 5, 36, 41

[6] BARRIENTOS, P. G. AND RAIBEKAS, A. *Robust tangencies of large codimension.* Nonlinearity, 30(12):4369–4399, 2017. 6, 41

[7] BOCHI, J., BONATTI, C., AND DÍAZ, L. J. *Robust criterion for the existence of nonhyperbolic ergodic measures.* Comm. Math. Phys., 344(3):751–795, 2016. 34, 35

[8] BONATTI, C., *Survey: Towards a global view of dynamical systems, for the* C^1*-topology.* Ergodic Theory Dynam. Systems, 31(4):959–993, 2011. 4, 5

[9] BONATTI, C., CROVISIER, S., DÍAZ, L. J., AND WILKINSON, A. *What is...a blender?.* Notices Amer. Math. Soc., 63(10):1175–1178, 2016. 4

[10] BONATTI, C. AND DÍAZ, L. J. *Persistent nonhyperbolic transitive diffeomorphisms.* Ann. of Math. (2),143(2):357–396, 1996. 4

[11] BONATTI, C. AND DÍAZ, L. *Robust heterodimensional cycles and* C^1*-generic dynamics.* J. Inst. Math. Jussieu, 7(3):469–525, 2008. 2, 3, 4, 5, 7, 11, 12, 16, 19, 20

[12] BONATTI, C. AND DÍAZ, L. J. *Abundance of* C^1*-robust homoclinic tangencies.* Trans. Amer. Math. Soc., 364(10):5111–5148, 2012. 22

[13] BONATTI, C., DÍAZ, L. J., AND KIRIKI, S. *Stabilization of heterodimensional cycles.* Nonlinearity, 25(4):931, 2012. 3

[14] BONATTI, C., DÍAZ, L. J., AND VIANA, M. *Dynamics beyond uniform hyperbolicity.* volume 102 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, 2005. A global geometric and probabilistic perspective, Mathematical Physics, III. 2, 58

[15] CARVALHO, M. AND PÉREZ, S. A. *Equilibrium states for a class of skew products.* Ergodic Theory Dynam. Systems, 40(11):3030–3050, 2020. 57

[16] DÍAZ, L. J. *Robust nonhyperbolic dynamics and heterodimensional cycles.* Ergodic Theory Dynam. Systems, 15(2):291–315, 1995. 7

[17] DÍAZ, L. J., ESTEVES S., AND ROCHA, J. *Skew product cycles with rich dynamics: from totally non-hyperbolic dynamics to fully prevalent hyperbolicity.* Dynamical Systems. An International Journal, 31(1):1-40, 2016. 19

[18] DÍAZ, L. J., GELFERT, K., AND RAMS, M. *Mingled hyperbolicities: ergodic properties and bifurcation phenomena (an approach using concavity).* Discrete Contin. Dyn. Syst., 42(11):5309–5376, 2022. 4, 59

[19] DÍAZ, L. J., RIOS, I. L., AND VIANA, M. *The intermittency route to chaotic dynamics.* In Global analysis of dynamical systems, 309–327, Inst. Phys., Bristol, 2001. 3

[20] DÍAZ, L. J. AND ROCHA, J. *Large measure of hyperbolic dynamics when unfolding heteroclinic cycles.* Nonlinearity, 10(4):857–884, 1997. 7

[21] GONCHENKO, S. V., SHILNIKOV, L. P., AND TURAEV, D. V. *On dynamical properties of multidimensional diffeomorphisms from Newhouse regions. I.* Nonlinearity, 21(5):923–972, 2008. 14

[22] GORODETSKIĬ, A. S., ILYASHENKO, YU. S., KLEPTSYN, V. A., AND NALSKI, M. B. *Nonremovability of zero Lyapunov exponents.* Funktsional. Anal. i Prilozhen., 39(1):27–38, 95, 2005. 3, 9

[23] HAYASHI, S. *Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjectures for flows.* Ann. of Math. (2), 145(1):81–137, 1997. 58

[24] HIRSCH, M. W., PUGH, C. C., AND SHUB, M. *Invariant manifolds.* Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin–New York, 1977. 7, 58

[25] LI, D. AND TURAEV, D. *Persistence of heterodimensional cycles.* Invent. Math., 236(3)1413–1504, 2024. 2, 3, 4, 5, 7, 11, 14

[26] MANNING, A. *On zeta functions and Anosov diffeomorphisms.* Unpublished manuscript, University of Warwick, June 1972. 4, 57

[27] NASSIRI, M. AND PUJALS, E. R. *Robust transitivity in Hamiltonian dynamics.* Ann. Sci. Éc. Norm. Supér. (4), 45(2):191–239, 2012. 41

[28] NEWHOUSE, S. E. *Nondensity of Axiom A (a) on $\mathbb{S}^2$.* Global Analysis, 1:191–202, 1970. 2

[29] NEWHOUSE, S. E. *Diffeomorphisms with infinitely many sinks.* Topology, 13: 9–18, 1974. 2

[30] NEWHOUSE, S. E. *Lectures on dynamical systems.* In Dynamical systems (C.I.M.E. Summer School, Bressanone, 1978), Progr. Math., 8:1–114, Birkhäuser, Boston, 1980. 6

[31] NEWHOUSE, S. E., PALIS, J., AND TAKENS, F. *Bifurcations and stability of families of diffeomorphisms.* Inst. Hautes Études Sci. Publ. Math., (57):5–71, 1983. 3 3

[32] PALIS, J. *A global view of dynamics and a conjecture on the denseness of finitude of attractors.* Astérisque, 261:339–351, 2000. 1

[33] PALIS, J. AND TAKENS, F. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations, volume 35 of Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1993. Fractal dimensions and infinitely many attractors. 11

[34] SAIKI, Y., TAKAHASI, H. AND YORKE, J. A. *Hausdorff dimension of Cantor intersections and robust heterodimensional cycles for heterochaos horseshoe maps.* SIAM J. Appl. Dyn. Syst., 22(3):1852–1876, 2023. 5

[35] SHILNIKOV, L. P. *On a problem of Poincaré–Birkhoff.* Math. USSR Sb., 3:353–371, 1969. 14

[36] SHUB, M. *Topological transitive diffeomorphism in $\mathbb{T}^4$.* Lecture Notes in Math., 2065:39–40, Springer, Berlin, 1971. 4, 57

[37] SIMON, C. P. *A 3-dimensional Abraham–Smale example.* Proc. Amer. Math. Soc., 34:629–630, 1972. 2, 4

(P. G. Barrientos) INSTITUTO DE MATEMÁTICA E ESTATÍSTICA UFF, RUA MARCOS WALDEMAR DE FREITAS REIS, S/N - CAMPUS DO GRAGOATÁ, NITERÓI, RIO DE JANEIRO, BRAZIL

Email address: pgbarrientos@id.uff.br

(L. J. Díaz) DEPARTAMENTO DE MATEMÁTICA PUC-RIO, MARQUÊS DE SÃO VICENTE 225, GÁVEA, RIO DE JANEIRO 225453-900, BRAZIL

Email address: lodiaz@puc-rio.br

(Y. Ki) INSTITUTO DE MATEMÁTICA E ESTATÍSTICA UFF, RUA MARCOS WALDEMAR DE FREITAS REIS, S/N - CAMPUS DO GRAGOATÁ, NITERÓI, RIO DE JANEIRO, BRAZIL

Email address: yuriki@id.uff.br

(C. Lizana) DEPARTAMENTO DE MATEMÁTICA. INSTITUTO DE MATEMÁTICA E ESTATÍSTICA. UNIVERSIDADE FEDERAL DA BAHIA. AV. MILTON SANTOS S/N, 40170-110. SALVADOR, BAHIA, BRAZIL

Email address: clizana@ufba.br

(S. A. Pérez) INSTITUTO DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE VALPARAÍSO, BLANCO VIEL 596, CERRO BARÓN, VALPARAÍSO, CHILE

Email address: sebastian.perez.o@pucv.cl