ON GAMMA-VECTORS AND CHOW POLYNOMIALS OF RESTRICTIONS OF REFLECTION ARRANGEMENTS

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Abstract. Simplicial arrangements are a special class of hyperplane arrangements, having the property that every chamber is a simplicial cone. It is known that the simpliciality property is preserved under taking restrictions. In this article we focus on the class of reflection arrangements and investigate two different polynomial invariants associated to them and their restrictions, the h-polynomial with its γ-vector and the Chow polynomial. We prove that all restrictions of reflection arrangements are γ-positive and give an explicit combinatorial formula of the Chow polynomial in type B. Furthermore we prove that for a special class of restrictions of arrangements of type D, called intermediate arrangements, both the h-polynomial as well as the Chow polynomial behave arithmetically, that is they interpolate linearly between the respective invariants for type B and D.

Keywords: simplicial arrangements, reflection arrangements, matroids, γ-positivity, Chow polynomials.

1. Introduction

In the study of hyperplanes arrangements, the simplicial ones constitute a special class, having the property that every chamber is a simplicial cone. The most prominent examples, called reflection arrangements, arise from finite reflection groups, i.e., finite groups that are generated by reflections, see [OT13]. Beyond reflection arrangements the simplicial ones are rare. In rank three, two infinite families and 95 sporadic examples of simplicial arrangements up to projective transformation are described in the Grünbaum-Cuntz catalog, see [Grü09; Cun12; Cun22], which is conjectured to be complete up to combinatorial isomorphism.

It is known that simpliciality is preserved under taking restrictions. An active field of research is concerned with investigating properties of restrictions of simplicial arrangements, see for instance [MS22; Cun22]. For reflection arrangements, these restrictions are typically not reflection arrangements anymore. A full overview of the restrictions for all possible reflection arrangements and their relations is illustrated in [MS22, Figure 9].

One special family arises from restrictions of reflection arrangements in type D. In this article we mainly focus on this family of arrangements often called *intermediate arrangements*. We define these arrangements in $\mathbb{R}^n$ by means of their normal vectors

$$\mathcal{D}_{n,s} := \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_i : 1 \leq i \leq s\},$$

where e_i is the i-th standard basis vector in $\mathbb{R}^n$ and $0 \leq s \leq n$. The name comes from the fact that $\mathcal{D}_{n,0} = \mathcal{D}_n$ and $\mathcal{D}_{n,n} = \mathcal{B}_n$ and so they can be viewed as interpolating between the reflection arrangements of type $\mathcal{D}_n$ and $\mathcal{B}_n$, see [OT13, Section 6.9] for more details. Note that the arrangement $\mathcal{D}_{n,s}$ is denoted $\mathcal{A}_n^s$ in [OT13].

In this article we discuss two different polynomial invariants attached to the intermediate arrangements $\mathcal{D}_{n,s}$ and their respective properties. The first being the h-polynomial and its γ-vector and the second being the Chow-polynomial. We prove that both invariants form an arithmetic sequence with respect to s, meaning that the difference of the invariants associated to $\mathcal{D}_{n,s}$ and $\mathcal{D}_{n,s}$ is independent of s.

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1.1. **The γ-vector.** The first invariant is motivated by the fact that every simplicial arrangement $\mathcal{A}$ gives rise to a simplicial complex $\Delta_{\mathcal{A}}$, triangulating a sphere. Therefore one can define the f- and h-polynomial of the arrangement in the usual way:

Given the simplicial complex $\Delta_{\mathcal{A}}$ of dimension $d-1$, the f-vector $f(\mathcal{A}) = (f_{-1}, f_0, \ldots, f_{d-1})$ consists of the entries $f_k := f_k(\mathcal{A})$ recording the number of k-dimensional simplicies of $\mathcal{A}$, for $-1 \leq k \leq d-1$. We define the *f-polynomial* $f(t) \in \mathbb{Z}[t]$ as

$$f(t) := \sum_{i=0}^{d} f_{d-1-i} t^i$$

and the *h-polynomial*

$$h(t) := f(t-1) = \sum_{i=0}^{d} h_i t^i$$

whose coefficients form the *h-vector* $h = (h_0, \ldots, h_d)$.

We known that in the case of a simplicial complex triangulating a sphere the h-vector is non-negative and symmetric, i.e., $h_i \geq 0$ and $h_i = h_{d-i}$ for all i. The latter are called the Dehn-Sommerville equations, see [Kle64; Gal05]. Therefore one can rewrite the h-polynomial in a different basis (*γ-basis*) and obtain:

$$h(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i \cdot t^i (1+t)^{d-2i}.$$

The above coefficients $\gamma_i \in \mathbb{Z}$ form the so-called *γ-vector* $\gamma = (\gamma_0, \ldots, \gamma_{\lfloor d/2 \rfloor})$ of the underlying simplicial complex $\Delta_{\mathcal{A}}$. We say that $\Delta_{\mathcal{A}}$ is γ-positive if $\gamma_i \geq 0$ for all i, see [Gal05; Ath18] for more details. The property of being γ-positive is widely studied, see [Ath18] for an overview. The main motivation is drawn from the celebrated Gal's conjecture:

Conjecture 1.1. *[Gal05, Conjecture 2.1.7] If a simplicial complex is a flag sphere, i.e., the minimal non-faces are of size two and it triangulates a sphere, then it is γ-positive.*

Gal proved this conjecture for all flag spheres up to dimension 4, see[Gal05]. It is known that simplicial arrangements yield flag spheres, so it follows that all simplicial arrangements of rank ≤ 5 are γ-positive. Moreover, γ-positivity was proven for all reflection arrangements in [Ste08].

One of our main results in this article confirms Gal's conjecture for all restrictions of reflection arrangements. This is achieved by proving arithmeticity of the γ-coefficients for fixed n and varying $0 \leq s \leq n$ for the intermediate arrangements $\mathcal{D}_{n,s}$, together with `SageMath`-computations in the exceptional cases.

Theorem A. *All restrictions of reflection arrangements are γ-positive.*

As a byproduct of the proof of this theorem we then also obtain the exact increment of the arithmetic sequence in the cases of the intermediate arrangements, see Theorem 3.16.

1.2. **The Chow polynomial.** The second invariant is motivated by an important geometric-algebraic object one can attach to a simple matroid M on the groundset E, called the *Chow ring* of M and is defined as follows:

Let $\mathcal{L}(M)$ be the lattice of flats of M, then

$$\mathrm{CH}(M) := \frac{\mathbb{Q}[x_F \mid F \in \mathcal{L}(M) \setminus \{\emptyset\}]}{I+J},$$

where I, J are the two ideals

$$I = \langle x_{F_1} x_{F_2} \mid F_1, F_2 \in \mathcal{L}(M) \setminus \{\emptyset, E\} \text{ are incomparable} \rangle$$
$$J = \left\langle \sum_{F \ni i} x_F \mid i \in E \right\rangle.$$

Since their introduction in [FY04], they have attracted a lot of attention in the last years and are at the core of much current research in algebraic and geometric combinatorics, maybe most notably as the crucial tool in the resolution of the long-standing Heron–Rota–Welsh conjectures by Adiprasito–Huh–Katz in [AHK18]. There, the authors showed that the Chow ring of a matroid satisfies the

Kähler package, i.e., Poincaré duality, the Hodge-Riemann relations and the hard Lefschetz property. Thereby, they initiated the study of combinatorial Hodge theory.

The Hilbert-Poincaré series associated to $\mathrm{CH}(M)$ is the polynomial defined as

$$H_M(t) = \sum_{k\geq 0} \dim(\mathrm{CH}(M)_k)t^k,$$

where $\mathrm{CH}(M)_k$ is the k-th graded piece of $\mathrm{CH}(M)$. We call it the *Chow polynomial* of M, see [Fer+24].

The Chow polynomial has been extensively studied in recent years. It is known to have palindromic coefficients due to Poincaré duality and conjectured to be real-rooted by June Huh and Matthew Stevens, see [Ste21, Conjecture 4.1.3 and 4.3.3]. In the uniform case this conjecture was recently confirmed by Brändén and Vecchi [BV25]. Moreover Stump in [Stu25] obtained a general combinatorial description of the Chow polynomial using an R-labeling for the lattice of flats of the matroid. He thereby reproved that the Chow polynomial has a γ-positive expansion, initially proven in [Fer+24]. Furthermore he obtained a compact expression of this polynomial in the braid arrangement case (type A). Soon after Hoster gave an explicitly combinatorial formula, based on Schubert matroids, for the Chow polynomial of uniform matroids, see [Hos24].

For any sequence $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, we denote by $\mathrm{des}(a_1, \ldots, a_n)$ the number of descents of the sequence, i.e. the number of indices i such that $a_i > a_{i+1}$. In this article we provide an explicit combinatorial formula of the Chow polynomial for the reflection arrangement of type B, by using type B analogues of the methods in [Stu25].

Theorem B. *The Chow polynomial for type B has the expansion*

$$H_{\mathcal{B}_n}(x) = \sum_{(a_1,\ldots,a_n)} \left(\prod_{i=1}^{n}(2a_i - 1)\right) x^{des(a_1,\ldots,a_n)}(x+1)^{n-1-2des(a_1,\ldots,a_n)}$$

where the sum ranges over all tuples $(a_1, \ldots, a_n)$ with $a_i \in \{1, \ldots, n+1-i\}$ such that $a_1 \leq a_2$ and also that $a_i > a_{i+1}$ implies $a_{i-1} \leq a_i$ for $i \in \{2, \ldots, n-1\}$.

Moreover we show that the Chow polynomials for the intermediate arrangements $\mathcal{D}_{n,s}$ behaves arithmetically with respect to fixed n and varying $0 \leq s \leq n$. This is done in two different ways, using on the one hand an EL-labeling of the intersection lattices as introduced in [DGP19], see Theorem 4.27, and on the other hand a recursive approach via reduced characteristic polynomials of the minors of the underlying matroids, see Theorem 4.33.

The rest of the article is organized as follows: In Section 2 we briefly recap the basic notions of hyperplane arrangements and matroids, needed for the discussion in the later sections. In Section 3 we show that the h-polynomials of the intermediate arrangements $\mathcal{D}_{n,s}$ behave arithmetically with respect to fixed n and varying $0 \leq s \leq n$ and subsequently show the γ-positivity for these arrangements. Moreover, we explicitly determine the increment of the above mentioned arithmetic sequence. Afterwards, we discuss a computational approach via `SageMath` to verify the γ-positivity of all rank 6 and 7 restrictions of the exceptional cases, which then enables us to conclude the proof of Theorem A. In Section 4 we introduce an R-labeling for the lattice of flats of matroids arising from reflection arrangements of type B. We then apply Theorem 4.2 to derive the explicit formula of their Chow polynomials, as stated in Theorem B. Thereafter we discuss two approaches for proving the arithmetical behavior of the Chow polynomials associated to the matroids underlying the $\mathcal{D}_{n,s}$-arrangements. For the first approach we investigate an EL-labeling for their lattice of flats and for the second we examine a recursive formula of their Chow polynomials in terms of reduced characteristic polynomials of minors. At the end of the article (Section A) we provide a small catalogue of Chow polynomials of $\mathcal{D}_{n,s}$-matroids up to dimension $n = 7$.

Acknowledgments

This paper stemmed from a project within a Dive into Research, which grants research experience to undergraduates. We thank Michael Cuntz, Lukas Kühne, and Raman Sanyal, who organized the Dive into Research. We are also thankful for the funding for the Dive into Research from the DFG priority program SPP 2458 Combinatorial Synergies. We want to give special thanks to Michael Cuntz, for providing us essential gamma-vector computations and Lukas Kühne and Raman Sanyal, who posed the starting questions and helped a lot with proofreading and insightful discussions. Sebastian Degen was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB-TRR 358/1 2023 – 491392403 and SPP 2458 – 539866293.

2. Preliminaries

2.1. **Notation.** Throughout the article for a given positive integer n we set $[n] := \{1, \dots, n\}$ and denote be S_n the symmetric group on n elements.

2.2. **Hyperplane arrangements.** In this subsection we review some basics theory of hyperplane arrangements, that we use frequently throughout the article. For a more detailed overview the reader may consult [OT13].

Definition 2.1. A **hyperplane in** $\mathbb{R}^n$ is a $(n-1)$-dimensional linear subspace H, which can be describe in the following way

$$H := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \alpha^t x = \alpha_1 x_1 + \dots + \alpha_n x_n = 0\},$$

for some $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}$. We call the vector α a **normal of** H and denote the hyperplane defined by α as H_α.

The closed and open positive half spaces corresponding to H_α are defined as

$$H_\alpha^{\geq} := \{x \in \mathbb{R}^n \mid \alpha^t x \geq 0\} \text{ and } H_\alpha^{>} := \{x \in \mathbb{R}^n \mid \alpha^t x > 0\}.$$

Analogously, we define the negative half spaces denoted by $H_\alpha^{\leq}$ and $H_\alpha^{<}$.

Definition 2.2. A **hyperplane arrangement** $\mathcal{A}$ in $\mathbb{R}^n$ is a finite non-empty collection of hyperplanes. We say *arrangement* for short.

Note that in this context, every hyperplane H is linear, i.e. contains the origin. In other words, we only consider so called **central** hyperplane arrangements $\mathcal{A}$. Every arrangement comes with an associated lattice, encoding the intersections of the hyperplanes.

Definition 2.3. The **lattice of intersections** of an arrangement $\mathcal{A} = \{H_1, \dots, H_m\}$ in $\mathbb{R}^n$ is the poset $(\mathcal{L}(\mathcal{A}), \preceq)$, where

$$\mathcal{L}(\mathcal{A}) := \left\{ \bigcap_{i \in I} H_i \mid I \subseteq [m] \right\}$$

and $\leq$ denotes the reverse inclusion, i.e. for all $X, Y \in \mathcal{L}(\mathcal{A})$ we have $X \preceq Y$ if $Y \subseteq X$. In the following, we abbreviate this poset by $\mathcal{L}(\mathcal{A})$.

Remark 2.4. For any arrangement $\mathcal{A}$, the poset $\mathcal{L}(\mathcal{A})$ is a geometric lattice. The **rank** of an element $X \in \mathcal{L}(\mathcal{A})$ is its codimension in $\mathbb{R}^n$ and we call the codimension of $\bigcap_{H \in \mathcal{A}} H$ the **rank** of the arrangement $\mathcal{A}$. Moreover, the **atoms** of the lattice, i.e., codimension 1 elements, are the hyperplanes in $\mathcal{A}$.

For an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ the complement $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$ is disconnected and the connected components are of special interest in the study of hyperplane arrangements:

Definition 2.5. The **chambers** of $\mathcal{A}$ are the closures of the connected components of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$. We denote the set of all chambers with $\mathcal{C}(\mathcal{A})$. An intersection of two chambers spanning a codimension-1 subspace is called a **wall** of both chambers. Every wall is contained in a unique hyperplane of $\mathcal{A}$, which is given by span of the aforementioned intersection. For a given wall we call this hyperplane the **supporting hyperplane** of the wall.

In this article, we mainly focused on a particular class of arrangements, having specially shaped chambers.

Definition 2.6. Let $\mathcal{A}$ be an arrangement. We call a chamber $C \in \mathcal{C}(\mathcal{A})$ **simplicial** if the normal vectors of all the supporting hyperplanes of all walls of C are linearly independent. If all elements of $\mathcal{C}(\mathcal{A})$ are simplicial, then the arrangement $\mathcal{A}$ is called **simplicial**.

The most prominent examples of simplicial arrangements arise from *finite reflection groups*, i.e., finite groups that are generated by orthogonal reflections in hyperplanes in $\mathbb{R}^n$. The irreducible finite reflection groups can be classified into types, consisting of the infinite families $A_n, B_n, D_n, I_2(m)$ as well as the exceptional cases $E_6, E_7, E_8, F_4, H_3, H_4$; see [Hum90]. Let W be a finite reflection group acting on $\mathbb{R}^n$, then the collection $\mathcal{A}(W)$ of reflecting hyperplanes associated to the reflections in W is called the **reflection arrangement of** W.

Example 2.7. The reflection arrangements in $\mathbb{R}^n$ corresponding to the reflection groups of type A, B and D are given by the following sets of normal vectors of the reflecting hyperplanes:

- $\mathcal{A}_{n-1} := \{e_i - e_j \mid 1 \leq i < j \leq n\}$,
- $\mathcal{B}_n := \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$ and
- $\mathcal{D}_n := \{e_i \pm e_j \mid 1 \leq i < j \leq n\}$.

Here e_i denote the standard basis vectors of $\mathbb{R}^n$, for $1 \leq i \leq n$.

Lastly, we review some common operations on arrangements.

Definition 2.8. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. Any subset of $\mathcal{A}$ is called a **subarrangement** of $\mathcal{A}$. The **deletion** of a hyperplane $H \in \mathcal{A}$ is the subarrangement $\mathcal{A} \setminus \{H\} \subseteq \mathcal{A}$ in $\mathbb{R}^n$. The **restriction** to a hyperplane $H \in \mathcal{A}$ is an arrangement in $H \cong \mathbb{R}^{n-1}$ given by:

$$\mathcal{A}^H := \{H \cap H' \mid H' \in \mathcal{A} \setminus \{H\}\}.$$

Remark 2.9. If $\mathcal{A}$ is a simplicial arrangement, then its restriction to any hyperplane $H \in \mathcal{A}$ is a simplicial arrangement as well. In this way, we can construct new simplicial arrangements from already known ones. For reflection arrangements, a full overview of their restrictions and relations is illustrated in [MS22, Figure 9].

2.3. **Matroids.** In this subsection we briefly recap some general matroid theoretical notions and concepts that are used in the discussion of Chow polynomials, see Section 4. We refer to [Oxl11] for an overview of matroid theory.

Definition 2.10. A **matroid** M is a pair (E, r), consisting of a finite set E and a function $r : 2^E \to \mathbb{Z}$, called the **rank function** satisfying the following axioms for all $A, B \in 2^E$:

(1) $0 \leq r(A) \leq |A|$,
(2) If $A \subseteq B$, then $r(A) \leq r(B)$, and
(3) $r(A \cap B) + r(A \cup B) \leq r(A) + r(B)$.

Let $M = (E, r)$ be a matroid. We call $r(M) := r(E)$ the **rank** of M. A subset $A \in 2^E$ is called **independent** if $r(A) = |A|$, otherwise it is called **dependent**. A dependent subset of size 1 is called **loop** and one of size 2 with rank 1 is called **parallel element**. If there are no loops and parallel elements, M is a **simple** matroid. Finally, we call a subset $F \in 2^E$ a **flat** of M if $r(F \cup e) > r(F)$ for all $e \in E \setminus F$.

Every matroid M can be equivalently described by its collection of independent sets, dependent sets, flats and hyperplanes, see [Oxl11]. These equivalent descriptions are often called **cryptomorphic characterizations** of M.

The collection of flats of a matroid admits a poset lattice structure.

Definition 2.11. Let $M = (E, r)$ be a matroid. Then the collection of flats of M forms a lattice with respect to inclusion, called the **lattice of flats** of M. We denote this lattice by $\mathcal{L}(M)$. For two flats $F, G \in \mathcal{L}(M)$ the **join** and the **meet** are defined as

$$F \wedge G := F \cap G \quad \text{and} \quad F \vee G := \mathrm{cl}(F \cup G),$$

where $\mathrm{cl} : 2^E \to 2^E$ is a function called the **closure**, assigning every $A \subseteq E$ the set

$$\mathrm{cl}(A) := \{x \in E \mid r(A \cup x) = r(A)\}.$$

Note that from the definition of the closure function we see that a subset $F \subseteq E$ is a flat of M if F is its own closure, i.e., $\mathrm{cl}(F) = F$.

Remark 2.12. One can show that the lattice of flats of matroid even constitutes a geometric lattice. Moreover, there is even a one-to-one correspondence between simple matroids and geometric lattices, see [Oxl11, Section 1.7] for a detailed discussion.

Now we review an important class of matroids, which provides the bridge to hyperplane arrangement theory. For our discussion in Section 4 we only use these kind of matroids.

Definition 2.13. Let $\mathbb{K}$ be field and $\mathcal{A} = \{H_e\}_{e \in E}$ be an arrangement in $\mathbb{K}^n$, for some finite set E. This yields a matroid $M(\mathcal{A}) = (E, r)$ defined via the rank function

$$r(S) := n - \dim\left(\bigcap_{e \in S} H_e\right).$$

We call matroids arising in this way **representable over** $\mathbb{K}$.

Note that the lattice of flats of a matroid corresponding to an arrangement coincides with the lattice of intersection of the arrangement, see [Oxl11, Section 6]. This means that the matroid is simple, since there exists no loops or parallel elements.

There are some operations we can perform on a given matroid to obtain new matroids.

Definition 2.14. Let $M = (E, r_M)$ be matroid. The **restriction of** M **to a subset** $S \subseteq E$ is the matroid $M|_S := (S, r_{M|_S})$ with rank function

$$r_{M|_S} := r_M(A) \quad \text{for all } A \subseteq S.$$

Restricting to $E \setminus S$ is often called the **deletion of** S **from** M.

The **contraction of** M **by a subset** $T \subseteq E$ is the matroid $M/T := (E \setminus T, r_{M/T})$ with rank function

$$r_{M/T}(A) := r_M(A \cup T) - r_M(A) \quad \text{for all } A \subseteq E \setminus T.$$

Remark 2.15. If a matroid arises from an arrangement as in Theorem 2.13 the deletion and restriction operation of the arrangement as in Theorem 2.8 correspond to the deletion and contraction operation of the matroid respectively.

Lastly we would like to review a polynomial invariant attached to a given matroid, which is used for the discussion later on in Section 4.4.

Definition 2.16. Let $M = (E, r)$ be a matroid of rank k. The **characteristic polynomial** χ_M **of** M is defined as

$$\chi_M(t) := \sum_{S \subseteq E} (-1)^{|S|} t^{k - r(S)} \in \mathbb{R}[t].$$

Note that $t = 1$ is always a root of the characteristic polynomial χ_M. Therefore we also define the **reduced characteristic polynomial of** M given by $\overline{\chi}_M(t) := \frac{\chi_M(t)}{t-1}$.

3. Gamma positivity of restrictions of reflection arrangements

3.1. **Simplicial arrangements and simplicial complexes.** In this subsection, we recall some results concerning simplicial complexes associated to simplicial arrangements. In particular, we review different ways of computing the h-polynomial of the associated complex and discuss some known γ-positivity statements.

Given a simplicial hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ with collection of chambers $\mathcal{C}(\mathcal{A})$. A *cone* of $\mathcal{A}$ is an intersection of a subset of chambers from $\mathcal{C}(\mathcal{A})$. The *dimension* of a cone corresponds to the dimension of the subspace it spans.

Let R be the set of rays of $\mathcal{A}$ (i.e. the 1-dimensional cones) and define

$$\Delta_{\mathcal{A}} := \{\sigma \subseteq R : \sigma \text{ is the set of rays in some cone of } \mathcal{A}\}$$

which is a simplicial complex. Alternatively, we can intersect all cones of $\mathcal{A}$ with the sphere S^{d-1} to obtain the faces of a topological simplicial complex which is isomorphic to $\Delta_{\mathcal{A}}$. This shows that $\Delta_{\mathcal{A}}$ triangulates a $(d-1)$-sphere and hence its h-polynomial has non-negative and symmetric coefficients. Thus, it makes sense to speak of the γ-vector of a simplicial arrangement.

Note that in this special case, positivity and symmetry are relatively easy to prove directly. Indeed, the positivity of the h-vector follows from the following formula, which is a reformulation of [Bre94, Theorem 2.3.]:

Lemma 3.1. *Let $\mathcal{A}$ be a simplicial hyperplane arrangement with chambers $\mathcal{C}$ and choose some $C_0 \in \mathcal{C}$. Then*

$$h(t) = \sum_{C \in \mathcal{C}} t^{sep(C)}$$

where $sep(C)$ is the number of walls of C which correspond to hyperplanes separating C from C_0.

This also easily implies symmetry of the h-vector, since we can apply the above formula with either C_0 or its negative $-C_0$ as the base region, noting that the walls which separate a region C from C_0 are exactly the walls which do not separate C from $-C_0$. Altogether, we see that simplicial arrangements have a well-defined γ-vector.

We can compute the h-polynomial using a special directed graph attached to a reflection arrangement.

Definition 3.2. Let $\mathcal{A}$ be a simplicial arrangement and $\mathcal{C}$ its collection of chambers. The **tope graph** of $\mathcal{A}$ is the simple undirected graph $T = (V, E)$, where $V = \mathcal{C}$ is the set of chambers of $\mathcal{A}$ and $\{C_1, C_2\} \in E$ whenever the corresponding chambers C_1, C_2 share a common wall.

Let us fix a fundamental chamber $C_0 \in \mathcal{C}$ and orient all hyperplanes in $\mathcal{A}$ such that C_0 is in the positive halfspace of every hyperplane. So we can represent it via a sign-vector of the form $(+, \dots, +)$. Similarly, we can represent all the other chambers by sign vectors. Therefore, the above graph T can be interpreted in the following way: Vertices correspond to sign vectors and two vertices are joined by an edge if there is a sign flip in the position associated to the shared hyperplane between them. With this interpretation, we can direct the tope graph as follows.

Definition 3.3. Let $\mathcal{A}$ be an simplicial arrangement and $T = (V, E)$ its topegraph. We define the **directed topegraph**, again denote by $T = (V, E)$, by directing an egde towards the vertex having the greater amount of negative signs in its sign-vector analogue.

Utilizing this notion of a directed topegraph we obtain the following result, which is a direct consequence of [PRW08, Proposition 2.1.].

Lemma 3.4. *Let $\mathcal{A}$ be an simplicial arrangement and $T = (V, E)$ its directed topegraph. Then we get*

$$h_{\mathcal{A}}(t) = \sum_{v \in V} t^{\mathrm{indeg}(v)},$$

where $\mathrm{indeg}(v)$ *denotes the number of incoming edges to v, i.e., the in-degree of v.*

Then the fundamental question is which simplicial arrangements are γ-positive? A first promising result is this:

Theorem 3.5. *[Ste08, Theorem 1.2.] All reflection arrangements are γ-positive.*

However, all known proofs of this fact use the classification of finite reflection groups and work on a case-by-case basis. Another interesting result which will be important later is also the following:

Theorem 3.6 ([Gal05])**.** *All simplicial arrangements of rank ≤ 5 are γ-positive.*

For arrangements of type A, B, D there is a combinatorial interpretation of the entries of the γ-vector. First, for the arrangement of type A, the coefficient $\gamma_k(\mathcal{A}_n)$ is the number of permutations $\sigma \in S_n$ with exactly k descents but no double descents and no final descent, see [PRW08, Section 11] for a proof.

The types B and D were treated in [Ste08]. First, we need the notion of a **peak**. For any permutation $u \in S_n$, we denote by $p(u)$ the number of peaks, i.e., the number of positions $i \in \{1, \dots, n\}$ such that $u_{i-1} < u_i$ and $u_i > u_{i+1}$, with the convention that $u_0 = 0$ and $u_{n+1} = n + 1$. The following result is the assembly of [Ste08, Corollary A.2.] and [Ste08, Corollary A.5.].

Theorem 3.7. *We have*

$$h_{\mathcal{B}_n}(t) = \sum_{u \in S_n} (4t)^{p(u)} (1+t)^{n-2p(u)} \text{ and}$$
$$h_{\mathcal{D}_n}(t) = \frac{1}{2} \sum_{u \in S_n} \phi(u) (4t)^{p(u)} (1+t)^{n-2p(u)},$$

where

$$\phi(u) = \begin{cases} 2 & \text{if } u_1 < u_2 < u_3 \\ 0 & \text{if } u_2 < u_1 < u_3 \\ 1 & \text{otherwise} \end{cases}$$

For type B, this implies

$$\gamma_k = 4^k \cdot |\{u \in S_n : p(u) = k\}|$$

giving the aforementioned combinatorial interpretation. The type D is similar, though slightly more cumbersome because of the function ϕ.

3.2. **Arithmeticity of the γ-vectors for $\mathcal{D}_{n,s}$-arrangements.** The intermediate arrangements $\mathcal{D}_{n,s}$ can be viewed as an interpolation between the arrangements of type D and B. In the spirit of this, we show that the γ-vectors of the $\mathcal{D}_{n,s}$-arrangements interpolate linearly between $\gamma(\mathcal{D}_n)$ and $\gamma(\mathcal{B}_n)$. In other words, the vectors $\gamma(\mathcal{D}_{n,s})$ form an arithmetic sequence with respect to s, meaning that the increment $\gamma(\mathcal{D}_{n,s+1}) - \gamma(\mathcal{D}_{n,s})$ is independent of s. We also determine an explicit expression for this increment in terms of the number of permutations with a certain number of maxima.

Lemma 3.8. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and H an additional hyperplane not in $\mathcal{A}$. For each k, let n_k be the number of faces of dimension k in the restriction of $\mathcal{A} \cup \{H\}$ to H which were not contained in $\mathcal{A}$. Then*

$$f_k(\mathcal{A} \cup \{H\}) = f_k(\mathcal{A}) + n_k + n_{k-1}.$$

Proof. Let Δ and Δ' be the simplicial complexes of $\mathcal{A}$ and $\mathcal{A}\cup\{H\}$, respectively. Adding the hyperplane H to $\mathcal{A}$ creates some new faces in Δ' which were not present in Δ. To count these new faces, we only need to consider those faces of Δ' with nontrivial intersection with H. These split into two classes, namely those faces of Δ' which are contained in H and the pairs of faces of Δ' which were created by subdividing a face of Δ. In the second case, each pair which was subdivided increases the number of faces in Δ' by one. On the other hand, not all faces of Δ' which are contained in H are new. Let N_k be the set of faces of dimension k in $\Delta' \setminus \Delta$ which are contained in H. We claim that there is a bijection between N_k and the faces of dimension $k+1$ in Δ which are subdivided by adding H. In one direction, let $\sigma \in \Delta$ be a face of dimension $k+1$ such that σ is subdivided by adding H. Then $\tau = \sigma \cap H$ is an element of N_k. Conversely, given $\tau \in N_k$, we write

$$\tau = \bigcap_{H' \in I_0} H' \cap \bigcap_{H' \in I_>} H'^{>} \cap \bigcap_{H' \in I_<} H'^{<}$$

where $H'^{>}$ is the open positive halfspace of a hyperplane H' and $I_0 \sqcup I_> \sqcup I_<$ is a partition of the hyperplanes in $A \cup \{H\}$, according to whether τ is contained in a hyperplane or in its positive or negative halfspace. Since τ is contained in H, we must have $H \in I_0$. Then

$$\sigma = \bigcap_{H' \in I_0 \setminus \{H\}} H' \cap \bigcap_{H' \in I_>} H'^{>} \cap \bigcap_{H' \in I_<} H'^{<}$$

is a face in $\mathcal{A}$. The face σ properly contains τ, since the equality $\sigma = \tau$ would imply that τ was already a face of $\mathcal{A}$, in contradiction to our assumption. This implies that the dimension of σ must be $k+1$ and that H subdivides σ.

Clearly, the two constructions $\sigma \mapsto \tau$ and $\tau \mapsto \sigma$ are inverse to each other, which proves the claimed bijection. Finally, observe that the k-faces in $\mathcal{A} \cup \{H\}$ consist of all existing faces of $\mathcal{A}$ plus the new k-faces which are contained in H (which are $|N_k| = n_k$ many) plus the new k-faces which arose by subdividing a k-face in $\mathcal{A}$. The latter come in pairs, but every pair increases the number of k-faces only by one, since it replaces an existing k-face of $\mathcal{A}$. By the above, this set of pairs of k-faces is in bijection to the new $(k-1)$-faces which are contained in H, which are $|N_{k-1}| = n_{k-1}$ many. This gives the desired formula. □

For any $k = 1, \ldots, n$, let $H_k = \{x_k = 0\}$ be the k-th coordinate hyperplane. For any $S \subseteq \{1, \ldots, n\}$ we define $\mathcal{D}_{n,S}$ to be the hyperplane arrangement consisting of the hyperplanes in $\mathcal{D}_n$ plus the coordinate hyperplanes H_k for all $k \in S$.

Lemma 3.9. *Let $S, S' \subseteq \{1, \ldots, n\}$ be subsets which both do not contain the index $i \in \{1, \ldots, n\}$ and let σ be a face of $\mathcal{D}_{n,S}$ which is contained in the coordinate hyperplane H_i. Then σ is also a face of $\mathcal{D}_{n,S'}$.*

Proof. We write

$$\sigma = \bigcap_{H \in I_0} H \cap \bigcap_{H \in I_>} H^{>} \cap \bigcap_{H \in I_<} H^{<}$$

where $I_0 \sqcup I_> \sqcup I_<$ is a partition of the hyperplanes in $\mathcal{D}_{n,S}$. First, we want to add the coordinate hyperplanes H_k for $k \in S' \setminus S$ to one of the index sets. Note that it holds

$$\begin{aligned} H_i \cap \{x_k = 0\} &= H_i \cap \{x_k + x_i = 0\} \\ H_i \cap \{x_k > 0\} &= H_i \cap \{x_k + x_i > 0\} \\ H_i \cap \{x_k < 0\} &= H_i \cap \{x_k + x_i < 0\} \end{aligned}$$

for any $k \neq i$. Thus, since σ is contained in H_i, we can add any hyperplane H_k to the same index set $I_0, I_>$ or $I_<$ in which $\{x_k \pm x_i = 0\}$ is contained. Doing this for all $k \in S' \setminus S$ does not change σ and it shows that σ is also a face in $\mathcal{D}_{n,S \cup S'}$. Next, we want to modify the expression for σ to not use the coordinate hyperplanes $\{x_k = 0\}$ for any $k \in S \setminus S'$. Firstly, we can delete any hyperplane H_k from I_0 since

$$\{x_k = 0\} \cap \{x_i = 0\} = \{x_i + x_k = 0\} \cap \{x_i - x_k = 0\}.$$

Next, we want to delete the hyperplanes H_k from $I_>$ and $I_<$. To this end, define

$$W = \bigcap_{H \in I_0} H.$$

We claim that $W \subseteq H_i$. Suppose not, then $H_i \cap W$ would be a proper linear subspace of W, while also containing σ. But σ is an open subset of W, which cannot be contained in a proper linear subspace of W. This means that we can modify the set of hyperplanes in $I_>$ and $I_<$ without changing the containment of σ in H_i. Since $x_k > 0$ and $x_k + x_i > 0$ are equivalent inside H_i for any $k \neq i$, we can delete any hyperplane H_k from $I_>$ without changing the set defined by these inequalities. Analogously, we can delete any hyperplane H_k from $I_<$. Doing this for all $k \in S \setminus S'$ shows that σ is a face of $\mathcal{D}_{n,S'}$. □

Lemma 3.10. *Let $S, S' \subseteq \{1, \dots, n\}$ be subsets which both contain the index $i \in \{1, \dots, n\}$ and let σ be a face of $\mathcal{D}_{n,S}$ which is contained in the coordinate hyperplane $\{x_i = 0\}$. Then σ is also a face of $\mathcal{D}_{n,S'}$.*

Proof. We can write $\sigma = \sigma' \cap H_i$ where σ' is a face in $\mathcal{D}_{n,S \setminus \{i\}}$. By Lemma 3.9, σ' is a face in $\mathcal{D}_{n,S' \setminus \{i\}}$ and hence σ is a face in $\mathcal{D}_{n,S'}$. □

Theorem 3.11. *For each n and k, the values $\gamma_k(\mathcal{D}_{n,s})$ form an arithmetic sequence with respect to s, meaning that the increment $\gamma_k(\mathcal{D}_{n,s+1}) - \gamma_k(\mathcal{D}_{n,s})$ is independent of s.*

Proof. It is sufficient to prove that $f_k(\mathcal{D}_{n,i})$ increases arithemtically with i, since for fixed n, the transformations to the h-vector and to the γ-vector are both linear. Going from $\mathcal{D}_{n,i-1}$ to $\mathcal{D}_{n,i}$, we add the hyperplane H_i which subdivides some faces of $\mathcal{D}_{n,i-1}$ and creates some new faces contained in H_i. Let N_k^i be the set of faces of dimension k in the restriction of $\mathcal{D}_{n,i}$ to H_i which are not contained in $\mathcal{D}_{n,i-1}$. By Lemma 3.8 we have

$$f_k(\mathcal{D}_{n,i}) = f_k(\mathcal{D}_{n,i-1}) + |N_k^i| + |N_{k-1}^i|$$

so it suffices to show that the cardinality of N_k^i is independent of i. Note that we have $\mathcal{D}_{n,i} = \mathcal{D}_{n,\{1,\dots,i\}}$. Let $\hat{N}_k^i$ be the set of faces of dimension k in the restriction of $\mathcal{D}_{n,\{i\}}$ to H_i which are not contained in $\mathcal{D}_n$. We claim that there is a bijection between N_k^i and $\hat{N}_k^i$. In one direction, let $\sigma \in N_k^i$, which is a face in $\mathcal{D}_{n,\{i\}}$ by Theorem 3.10. Suppose that σ was already contained as a face in $\mathcal{D}_n$, then by Theorem 3.9 it would be a face in $\mathcal{D}_{n,\{1,\dots,i-1\}}$ as well, contradiction to σ being new in $\mathcal{D}_{n,\{1,\dots,i\}}$. This shows that $\sigma \in \hat{N}_k^i$. The other direction works analogously. Finally, note that the cardinality of $\hat{N}_k^i$ is the same for any i, since we can relabel the hyperplane H_i to any H_j. This proves the theorem. □

Corollary 3.12. *The arrangements $\mathcal{D}_{n,s}$ are γ-positive.*

Proof. By Theorem 3.11, we have

$$\gamma_k(\mathcal{D}_{n,i}) \geq \min\{\gamma_k(\mathcal{D}_{n,n}), \gamma_k(D_n)\}$$

for all $i = 0, \dots, n$. In fact, we know from the explicit formula in Theorem 3.7 that $\gamma_k(\mathcal{D}_n) \leq \gamma_k(\mathcal{B}_n)$, so we have $\gamma_k(\mathcal{D}_{n,i}) \geq \gamma_k(\mathcal{D}_n)$. Since we also know from Theorem 3.7 that the γ-vector for D_n is positive, the claim follows. □

Remark 3.13. Note that this proof can be generalized. Consider the following setting: Let $\mathcal{A}$ be a hyperplane arrangement and let $H_1, \dots, H_s$ be hyperplanes not in $\mathcal{A}$ such that for all $1 \leq i \neq j \leq s$ there exists a set $S_{i,j} \subset \mathcal{A}$ such that $H_i \cap H_j = \bigcap_{H \in S_{i,j}} H$.

Then it can be shown that for all $k \in \mathbb{N}_0$

$$f_k(\mathcal{A} \cup \{H_1, \dots, H_s\}) = f_k(\mathcal{A}) + \sum_{i=1}^{s} f_k(\mathcal{A} \cup \{H_i\}) - f_k(\mathcal{A}).$$

In the following, we express the increment in terms of the number of so-called maxima of a permutation.

Definition 3.14. Let $u = (u_1, \dots, u_n) \in S_n$ be a permutation in one-line notation and set $u_0 = u_{n+1} = 0$. Then $m(u)$ is defined as the number of **maxima**, i.e. the number of $i \in \{1, \dots, n\}$ such that $u_{i-1} < u_i > u_{i+1}$.

The notion of a maximum differs from that of a peak only by the conditions on u_0 and u_{n+1}.

Definition 3.15. Let $u = (u_1, \dots, u_n) \in S_n$ be a permutation and set $u_0 = 0$, $u_{n+1} = n + 1$. Then $p(u)$ is the number of **peaks**, i.e. the number of $i \in \{1, \dots, n\}$ such that $u_{i-1} < u_i > u_{i+1}$.

We find the increment of the γ-vector by computing $h(\mathcal{B}_n, t) - h(\mathcal{D}_{n,n-1}, t)$. We prove the following result after proving a few auxiliary lemmata.

Proposition 3.16. *The following holds*

$$h(\mathcal{B}_n, t) - h(\mathcal{D}_{n,n-1}, t) = \frac{1}{2} \sum_{u' \in S_{n-1}} (4t)^{m(u')}(1 + t)^{n-2m(u')}.$$

As a direct consequence of Theorem 3.16, we obtain the explicit increments of the entries of the γ-vector.

Corollary 3.17. *For any $n \in \mathbb{N}$, $i \in [i]$ and $k \in \{0, \dots, \lfloor \frac{n}{2} \rfloor\}$*

$$\gamma_k(\mathcal{D}_{n,i}) - \gamma_k(\mathcal{D}_{n,i-1}) = \frac{1}{2} \cdot |\{u \in S_{n-1} : m(u) = k\}| \cdot 4^k.$$

Definition 3.18. For $u \in S_n$ we define the **horizontal flip**, denoted by $\mathrm{hf}(u)$, as the permutation $(n + 1 - u_1, \dots, n + 1 - u_n)$.

Definition 3.19. Let $u = (u_1, \dots, u_n) \in S_n$ be a permutation. We call $i \in \{2, 3, \dots, n - 1\}$ a **valley** if $u_{i-1} > u_i < u_{i+1}$.

Lemma 3.20. *Consider $u \in S_n$ such that $u_1 = n$, and denote $u' = (u_2, \dots, u_n) \in S_{n-1}$. Then $p(u) = m(hf(u'))$.*

Proof. Between every two maxima of u' is a valley which turns into a peak after a horizontal flip. Since for peaks we let $u_{n+1} = n + 1$ and $u_1 = n$ by assumption, u_1 and u_{n+1} are both greater than u_i for all $2 \le i \le n$. Hence, there are no other peaks in u on positions $2, \dots, n$.

For any permutation, every valley is between two maxima. So, there is one less valley than maxima in u'. Since there is always a peak in u at the position 1, the claim follows. □

Let $u \in S_n$ be a permutation and $\sigma \in \{1, -1\}^n$ be a sign vector. Then we call $w = (\sigma_1 u_1, \dots, \sigma_n u_n)$ a **signed permutation**. See the appendix of [Ste08] for the correspondence between signed permutations and element of the Weyl group of type B and D, which in turn correspond to regions in the hyperplane arrangements. For a signed permutation w, we denote by R_w^B the associated region in $\mathcal{B}_n$, and if the number of -1 in σ is even, we denote by R_w^D the corresponding region in $\mathcal{D}_n$, that is the region which contains the point $(\sigma_{v_1} v_1, \dots, \sigma_{v_n} v_n)$, where $v = u^{-1}$ as permutations.

In the following, let us fix the base region R_0 corresponding to $(1, 2, \dots, n)$. We analyze the contribution of regions to the h-vector with respect to R_0.

Definition 3.21. For a region R in a hyperplane arrangement $\mathcal{A}$ we denote by $h(\mathcal{A}, R)$ the power of t with which R contributes to the h polynomial with respect to the base region R_0.

By Theorem 3.1, $h(\mathcal{A}, R)$ is the number of walls of R in $\mathcal{A}$ which separate R from the point $(1, 2, \dots, n)$.

Definition 3.22. For a signed permutation w, we denote by $D_B(w)$ the set of $i \in [n]$ such that $w_{i-1} > w_i$, where we set $w_0 = 0$. If the number of minus signs in w is even, we denote by $D_D(w)$ a subset of $[n]$, such that $1 \in D_D(w)$ if $-w_1 > w_2$ and for $i > 1$, $i \in D_D(w)$ if $w_{i-1} > w_i$.

Then $|D_B(w)| = h(B, R_w)$ and $|D_D(w)| = h(D, R_w)$, see appendix in [Ste08].

The hyperplane $\{x_n = 0\}$ contains wall of exactly those regions whose canonical point $(\sigma_{v_1} v_1, \dots, \sigma_n v_n)$ satisfies $v_n = 1$, that is $u_1 = (v^{-1})_1 = n$. Hence, any region $R_{\sigma u}^B$ such that $u_1 \neq n$ is in $\mathcal{D}_{n,n-1}$ as well, since the hyperplane $\{x_n = 0\}$ does not contain any of its walls. Then $h(\mathcal{D}_{n,n-1}, R_{\sigma u}^B) = h(B_n, R_{\sigma u}^B)$, since this is a question of how many walls of $R_{\sigma u}^B$ separate it from the point $(1, 2, \dots, n)$.

Let $u \in S_n$ be such that $u_1 = n$, let $\sigma, \tau \in \{1, -1\}^n$ be such that they differ only in the first coordinate and $\sigma_1 = 1$, $\tau_1 = -1$. Then there is the region $R := R_{\sigma u} \cup R_{\tau u}$ in $\mathcal{D}_{n,n-1}$, since $R_{\sigma u}$ and $R_{\tau u}$ share a wall in the hyperplane $\{x_n = 0\}$.

Lemma 3.23. *In the setting above, $h(\mathcal{B}_n, R^B_{\sigma u}) = h(\mathcal{B}_n, R^B_{\tau u}) = h(\mathcal{D}_{n,n-1}, R)$.*

Proof. Since the interior of every region in $\mathcal{D}_n$ is intersected exactly by one coordinate hyperplane and R is intersected by $\{x_n = 0\}$, the region R is also in $\mathcal{D}_n$. It corresponds in $\mathcal{D}_n$ to one of the signed permutations σu, τu, let us denote this permutation w. We only need to compare the following three sets: $D_B(\sigma u)$, $D_B(\tau u)$ and $D_D(w)$. Using Theorem 3.22, we see that those sets can differ only in the containment of 1 and 2.

Since $u_1 = n$, $1 \notin D_B(\sigma u)$, $2 \in D_B(\sigma u)$, $1 \in D_B(\tau u)$, $2 \notin D_{\mathcal{B}}(\tau u)$. If R corresponds to σu, $D_D(\sigma u)$ contains 2 and not 1. Otherwise, $D_D(\tau u)$ contains 1 and not 2. Each of the sets contains exactly one of 1 and 2, so their sizes are equal, as desired. □

Proof of Proposition 3.16. In the view of the two lemmas above and Theorem 3.7

$$
\begin{aligned}
h(\mathcal{D}_{n,n-1}, t) = &\sum_{R \in \mathcal{D}_{n,n-1}} t^{h(\mathcal{D}_{n,n-1},R)} = \\
&= \sum_{u \in S_n, u_1 \neq n} (4t)^{p(u)}(1+t)^{n-2p(u)} + \frac{1}{2} \sum_{u \in S_n, u_1 = n} (4t)^{p(u)}(1+t)^{n-2p(u)},
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{u \in S_n, u_1 = n} (4t)^{p(u)}(1+t)^{n-2p(u)} &= \sum_{u' \in S_{n-1}} (4t)^{m(\mathrm{hf}(u'))}(1+t)^{n-2m(\mathrm{hf}(u'))} \\
&= \sum_{u' \in S_{n-1}} (4t)^{m(u')}(1+t)^{n-2m(u')}.
\end{aligned}
$$

Moreover

$$
h(\mathcal{B}_n, t) = \sum_{u \in S_n, u_1 \neq n} (4t)^{p(u)}(1+t)^{n-2p(u)} + \sum_{u \in S_n, u_1 = n} (4t)^{p(u)}(1+t)^{n-2p(u)},
$$

hence we obtain the desired result. □

We end this subsection with some illustrative examples of γ-vectors in small dimensions.

Example 3.24. Let $n \in \{3, \dots, 6\}$ and $s \in \{0, \dots, 6\}$. By using `SageMath` we compute the following table of γ-vectors for the arrangements $\mathcal{D}_{n,s}$:

n		$s=0$	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$
3	γ_0	1	1	1	1			
	γ_1	8	12	16	20			
4	γ_0	1	1	1	1	1		
	γ_1	40	48	56	64	72		
	γ_2	16	32	48	64	80		
5	γ_0	1	1	1	1	1	1	
	γ_1	152	168	184	200	216	232	
	γ_2	366	464	592	720	848	976	
6	γ_0	1	1	1	1	1	1	1
	γ_1	524	556	588	620	652	684	716
	γ_2	3440	4144	4848	5552	6256	6960	7664
	γ_3	832	1344	1856	2368	2880	3392	3904

The code used for these computations is available in the GitHub repository [Deg+25].

3.3. **The exceptional cases.** In this subsection we discuss the restrictions of rank ≥ 6 of the exceptional reflections types E_7 and E_8. In particular, we computationally verify the γ-positivity of those restrictions and then finally prove our first main theorem (Theorem A).

For the exceptional reflection types E_7 and E_8, there exist four restrictions of rank ≥ 6. More precisely, E_7 has one restriction of rank 6, and E_8 has two restrictions of rank 6 and one of rank 7. Following [MS22, Figure 9] we denote these four arrangements by $(46,1)$, $(63,1)$, $(68,1)$ and $(91,1)$, respectively. Here the first entry in each pair encodes the number of hyperplanes of the corresponding restriction.

Proposition 3.25. *The γ-vectors of the restrictions $(46,1)$, $(63,1)$, $(68,1)$ and $(91,1)$ are given by the vectors in Table 1. In particular, these arrangements are γ-positive.*

Proof. This proof is purely based on computations using the computer algebra system `SageMath`. For the first three simplicial arrangements we start by computing their undirected topegraphs using the algorithm from [Kör+25a, Appendix A] and direct them as described in Section 3.1. Now we are able to apply Theorem 3.4 to compute the h-polynomials associated to the three arrangements. Lastly we expand these h-polynomials in the γ-basis and obtain the first three γ-coefficient vectors displayed in Table 1. For the simplicial arrangement $(91,1)$ the aforementioned procedure fails due to the computational complexity of its topegraph. Michael Cuntz computed the f-vector of this restriction by using the structure of the associated Weyl groupoid and exploiting its symmetries, see [CH15] for more details. Thus we obtain the γ-coefficient vector in Table 1 by transforming the f-vector into the h-vector and again using the γ-expansion. $\square$

$\mathcal{A}$	$(46,1)$	$(63,1)$	$(68,1)$	$(91,1)$
γ_0	1	1	1	1
γ_1	3468	14064	8604	119940
γ_2	24048	112080	152560	1822512
γ_3	9536	52352	59712	2403008

TABLE 1. γ-vectors of the restrictions $(46,1)$, $(63,1)$, $(68,1)$ and $(91,1)$.

Remark 3.26. The `SageMath`-implementations of directing the tope graphs, computing their h-polynomials via Theorem 3.4 and expanding these in the γ-basis are available in the GitHub repository [Deg+25]. Furthermore the tope graphs of the first three restrictions in Theorem 3.25 can be found as database files in the GitHub repository [Kör+25b].

Now with all preparations done, we can finally prove the first main result of the article.

Theorem 3.27 (Theorem A)**.** *All restrictions of reflection arrangements are γ-positive.*

Proof. By Theorem 3.6 all simplicial arrangements of rank ≤ 5 are γ-positive, so we only have to consider those restrictions of reflection arrangements that are of rank ≥ 6. According to [MS22, Figure 9] the only restrictions of reflection arrangements of rank ≥ 6 are the $\mathcal{D}_{n,s}$ for $n \geq 6$ and the four exceptional restrictions of E_7 and E_8, namely $(46,1)$, $(63,1)$, $(68,1)$ and $(91,1)$. Now the γ-positivity for the $\mathcal{D}_{n,s}$ cases is covered by Theorem 3.12 and for the restrictions $(46,1)$, $(63,1)$, $(68,1)$ and $(91,1)$ by Theorem 3.25, which concludes the proof. $\square$

4. Chow polynomials of reflections arrangements

4.1. **Chow polynomials of matroids.** In this subsection we briefly review a general combinatorial description of the Chow polynomial associated to a simple matroid, introduced by Stump in [Stu25], which was already mentioned in the introduction.

In order to state this combinatorial description we first define a special labeling of a finite graded poset with a unique top and bottom element.

Let $(P,\preceq)$ be a bounded poset with the top element $\hat{1}$ and the bottom element $\hat{0}$. For $x,y \in P$ we denote by $x \prec y$ the strict relation and by $x \prec\!\!\cdot\, y$ the cover relation. Furthermore we set $\mathrm{Cov}(P) \subset P \times P$ to be the set of cover relations in P.

Definition 4.1. An **edge-labeling** of P is a map $\lambda\colon \mathrm{Cov}(P) \to \Lambda$ where Λ is another poset. Given such a labeling, each saturated chain $c = (x \prec z_1 \prec \ldots \prec z_r \prec y)$ between two elements $x \preceq y$ is called **increasing** if $\lambda(c) = (\lambda(x, z_1), \ldots, \lambda(z_r, y))$ is increasing, and **decreasing** if $\lambda(c)$ is strictly decreasing. Finally, we call λ an R**-labeling** if for all $x \prec y$ in P there exist a unique increasing saturated chain between them.

Now let $\mathcal{L}(M)$ be the lattice of flats of a matroid M with integer-valued R-labeling λ. To a maximal chain $\mathcal{F} = \{F_0 \prec \cdots \prec F_n\}$ we associate the sequence $\lambda(\mathcal{F}) = (\lambda_1, \ldots, \lambda_n)$ of edge labels $\lambda_i = \lambda(F_{i-1} \prec F_i)$, where n denotes the rank of M.

The following theorem is an important tool for the subsequent discussion.

Theorem 4.2. *[Stu25, Theorem 1.1.] The Chow polynomial $H_M(t)$ can be computed as*

$$H_M(t) = \sum_{\mathcal{F}} t^{des(\lambda_\mathcal{F})}(t+1)^{n-1-2des(\lambda_\mathcal{F})},$$

where the sum ranges over all maximal saturated chains $\mathcal{F}$ in $\mathcal{L}(M)$ such that for $\lambda_\mathcal{F} = (\lambda_1, \ldots, \lambda_n)$ we have that $\lambda_1 < \lambda_2$ and that $\lambda_i > \lambda_{i+1}$ implies $\lambda_{i-1} < \lambda_i$ for $i \in \{2, \ldots n-1\}$.

For matroids arising from arrangements of type $\mathcal{A}_{n-1}$, i.e., *braid matroids*, there exists an explicit formula using permutation statistics.

Theorem 4.3. *[Stu25, Theorem 3.1.] The Chow polynomial of the braid matroid is*

$$H_{\mathcal{A}_{n-1}}(t) = \sum_{(a_1,\ldots,a_n)} a_1 \cdots a_n t^{des(a_1,\ldots,a_n)}(t+1)^{n-1-2des(a_1,\ldots,a_n)},$$

where the sum ranges over all tuples $(a_1, \ldots, a_n)$ with $a_i \in \{1, \ldots, n+1-i\}$ such that $a_1 \leq a_2$ and $a_i > a_{i+1}$ implies $a_{i-1} \leq a_i$ for $i \in \{2, \ldots, n-1\}$.

We end with some examples of Chow polynomials corresponding to reflection arrangements of type A and B. They are computed using Theorem 4.3 and Theorem 4.2 for type A and B respectively.

Example 4.4. For small n, we get the following polynomials:

Type A	**Type** B
$H_{\mathcal{A}_2}(t) = t+1$	$H_{\mathcal{B}_2}(t) = t+1$
$H_{\mathcal{A}_3}(t) = t^2+8t+1$	$H_{\mathcal{B}_3}(t) = t^2+14t+1$
$H_{\mathcal{A}_4}(t) = t^3+41t^2+41t+1$	$H_{\mathcal{B}_4}(t) = t^3+99t^2+99t+1$

4.2. **The Chow polynomial in type** B**.** In this subsection we derive an explicit formula of the Chow polynomial for matroids arising from the arrangements of type B. Our discussion uses type B analogues of the approach in Theorem 4.3, by defining an R-labeling for the intersection lattices of $\mathcal{B}_n$-arrangements and applying Theorem 4.2.

Definition 4.5. Define Π_n^B as the set of all partitions π of $\langle n\rangle := \{-n, \ldots, n\}$ with the properties

(1) If $B \in \pi$ then $-B \in \pi$
(2) If $i \in B$ and $-i \in B$ for some i and some $B \in \pi$, then $0 \in B$.

This defines a poset by setting $x \prec y$ if and only if for each $B \in x$ there is $B' \in y$ with $B \subseteq B'$.

For any $\pi \in \Pi_n^B$, the block containing 0 is called the **zero block**. We define the rank function $r : \Pi_n^B \to \mathbb{N}$ by

$$r(\pi) = n - \frac{|\pi| - 1}{2}$$

which turns Π_n^B into a graded poset. Furthermore, we get the following identification, which is a reformulation of [DGP19, Theorem 7]:

Lemma 4.6. *The intersection lattice for the hyperplane arrangement of type $\mathcal{B}_n$ is isomorphic to Π_n^B.*

We can explicitly describe the cover relations in Π_n^B. To this end, let $\pi, \pi' \in \Pi_n^B$ such that $\pi \prec \pi'$. There are two possible cases:

- There is a block $B \in \pi$ such that $B \cup B_0 \cup (-B) \in \pi'$, where B_0 is the zero block in π, and all other blocks in π' are exactly the blocks of π which are not equal to $B, B_0, (-B)$.
- There are two blocks $B, B' \in \pi$, neither of which is the zero block, such that $B \cup B' \in \pi'$ and $(-B) \cup (-B') \in \pi'$ and all other blocks in π' are exactly the blocks of π which are not equal to $B, B', (-B), (-B')$.

Next, we define a labeling for all cover relations in Π_n^B, motivated by the usual max-of-min labeling on the (unsigned) partition lattice. Let again $\pi, \pi' \in \Pi_n^B$ be such that $\pi \prec \pi'$. Then there exist two distinct blocks $B_1, B_2 \in \pi$ which are not the zero block and which are both contained in the same block B' in π'. We define the label of the cover relation $\pi \prec \pi'$ as

$$\lambda(\pi, \pi') = \max\{\min |B_1|, \min |B_2|\}$$

where $|B| = \{|b| : b \in B\}$ is the set of absolute values of elements in the block B. Note that this is well defined in both cases of cover relations described above, since it does not matter if we choose the pair B_1, B_2 or $-B_1, -B_2$.

Lemma 4.7. *Let $x, y \in \Pi_n^B$ with $x \prec y$. For any saturated chain $x = x_0 \prec \ldots \prec x_k = y$ the set of labels*

$$\{\lambda(x_i, x_{i+1}) : i = 0, \ldots, k-1\}$$

is the same. We denote this set by $\lambda(x, y)$.

Proof. Let $x = \{B_1, \ldots, B_k\}$ and $y = \{B_1', \ldots, B_l'\}$ and consider a saturated chain $x = x_0 \prec \ldots \prec x_k = y$. For $j = 1, \ldots, k$, define $b_j = \min |B_j|$ and for $i = 1, \ldots, l$ define

$$M_i = \{b_j : B_j \subseteq B_i'\}$$

which is the set of absolute minima of all blocks in x which are merged together to obtain the block B_i' in Y. For each $\mu \in M_i \setminus \{\min M_i\}$, there must be some step in the chain where the block containing μ is merged with another block whose absolute minimum is smaller than μ. At this step, μ appears as a label. Conversely, at any merge going from some x_i to x_{i+1}, we use the absolute minimum of one of the blocks of x_i as a label. But the absolute minimum of any block of x_i already appears as the absolute minimum of a block of x, i.e. as one of the values b_j. Furthermore, it cannot be equal to $\min M_j$ for any $j = 1, \ldots, l$. Thus, we have shown that the set of labels is

$$\bigcup_{i=1}^{l} M_i \setminus \{\min M_i\}$$

which is independent of the chain we used. □

For any saturated chain c with $x = x_0 \prec \ldots \prec x_k = y$, denote by

$$\lambda(c) = (\lambda(x_0, x_1), \ldots, \lambda(x_{k-1}, x_k))$$

the sequence of labels along c.

For any permutation σ of $[k]$, the inversion sequence $\mathrm{Inv}(\sigma) = (a_1, \ldots, a_k)$ is given by

$$a_i = |\{j \geq i : \sigma(j) \leq \sigma(i)\}|.$$

Lemma 4.8. *Let $x, y \in \Pi_n^B$ with $x \prec y$ and let $\lambda(x, y) = \{b_1, \ldots, b_k\}$ with $b_1 < \cdots < b_k$. For any permutation σ of $[k]$ with inversion sequence $Inv(\sigma) = (a_1, \ldots, a_k)$, the number of saturated chains c starting from x and ending in y with $\lambda(c) = (b_{\sigma(1)}, \ldots, b_{\sigma(k)})$ is equal to $\prod_{i=1}^{k}(2a_i - 1)$.*

Proof. For any $i \in \{1, \ldots, k\}$, let $\sigma^{(i)} = (\sigma(1), \ldots, \sigma(i))$ be the partial permutation obtained by taking the first i entries of σ. Let $m(\sigma^{(i)})$ be the number of saturated chains C starting from X with $\lambda(C) = (b_{\sigma(1)}, \ldots, b_{\sigma(i)})$. We claim that

$$m(\sigma^{(i)}) = m(\sigma^{(i-1)}) \cdot (2a_i - 1). \tag{1}$$

For any saturated chain c with $\lambda(c) = (b_{\sigma(1)}, \ldots, b_{\sigma(i-1)})$, we want to count the number of ways in which c can be extended so that the next label is $b_{\sigma(i)}$. In other words, if $\pi \in \Pi_n^B$ is the endpoint of c, then we want to count the number of ways we can merge blocks of π such that $b_{\sigma(i)}$ is the label assigned to the cover relation. One of the blocks involved in such a merge must contain $b_{\sigma(i)}$, say B (which is not the zero block). The first case is that we merge B to its negative and the zero block, which gives the label $b_{\sigma(i)}$. The second case is that we merge B to some block B' which is not $-B$ or the zero block. In this case, let $b_{\sigma(j)}$ be the absolute minimum of B', then we must have $j > i$

and $b_{\sigma(j)} < b_{\sigma(i)}$. The first property comes from the fact that all values $b_{\sigma(j)}$ with $j < i$ have already appeared as a label in C, meaning that they are no longer the absolute minima in their respective blocks. The second property comes from the fact that we want

$$b_{\sigma(i)} = \max\{b_{\sigma(i)}, b_{\sigma(j)}\}$$

and it is equivalent to $\sigma(j) < \sigma(i)$. Note that every index j with $j > i$ and $\sigma(j) > \sigma(i)$ corresponds to exactly two possible merges (merging B with B' or B with $-B'$). Thus, the number of ways to merge blocks of π such that the label is $b_{\sigma(i)}$ is

$$2 \cdot |\{j > i : \sigma(j) < \sigma(i)\}| + 1 = 2a_i - 1$$

which implies Equation (1). Applying Equation (1) recursively finally yields $m(\sigma) = \prod_{i=1}^{k}(2a_i - 1)$. □

Lemma 4.9. *The map λ is an R-labeling of Π_n^B.*

Proof. Let $x, y \in \Pi_n^B$ be two partitions with $x \prec y$ and let $\lambda(x, y) = \{b_1, \dots, b_k\}$ with $b_1 < \cdots < b_k$. First, we show that there exists a saturated increasing chain from x to y. Let B' be the block in Y containing b_1 and let $b = \min |B'|$. Let x' be the partition obtained from x by merging the block containing b_1 with the block containing b (and all other merges to ensure that $x' \in \Pi_n^B$). Then x' covers x and $\lambda(x, x') = b_1$. Furthermore we get $\lambda(x', y) = \{b_2, \dots, b_k\}$, so the existence of the chain follows from induction on k. For uniqueness, note that by Lemma 4.7, any saturated chain from x to y must have exactly the labels $\lambda(x, y)$. By Lemma 4.8, the number of saturated chains c starting from x and ending in y which are increasing (i.e. with $\lambda(c) = (b_1, \dots, b_k)$) is equal to 1, since the inversion sequence of the identity permutation is $(1, \dots, 1)$. This proves existence and uniqueness of c. □

Lemma 4.10. *Let σ be a permutation of $[n]$ with inversion sequence $Inv(\sigma) = (a_1, \dots, a_n)$. Then for all $i \in \{1, \dots, n-1\}$ it holds*

$$a_i > a_{i+1} \iff \sigma(i) > \sigma(i+1)$$

and

$$a_i \leq a_{i+1} \iff \sigma(i) < \sigma(i+1).$$

Proof. First, assume that $a_i > a_{i+1}$ and assume for a contradiction that $\sigma(i) \leq \sigma(i+1)$. Then any $j \geq i+1$ with $\sigma(j) \leq \sigma(i)$ which is counted in a_i also satisfies $\sigma(j) \leq \sigma(i+1)$, meaning that it is counted in a_{i+1} as well. Furthermore, in a_i we count $j = i$ but not $j = i+1$, whereas we do count $j = i+1$ in a_{i+1}. This implies $a_i \leq a_{i+1}$, contradiction to our assumption. Conversely, assume that $\sigma(i) > \sigma(i+1)$. Then any $j \geq i+1$ with $\sigma(j) \leq \sigma(i+1)$ which is counted in a_{i+1} also satisfies $\sigma(j) \leq \sigma(i)$, meaning that it is counted in a_i as well. Furthermore, in a_i we count $j = i$ which is not counted in a_{i+1}. This implies $a_i > a_{i+1}$. Lastly, the second equivalence follows from the first by contraposition. □

Lemma 4.11. *[Sta97, Proposition 1.3.12] The map $\sigma \mapsto Inv(\sigma)$ is a bijection between permutations of $[n]$ and sequences $(a_1, \dots, a_n)$ with $a_i \in \{1, \dots, n+1-i\}$.*

Now with all preparations done, we can finally prove the second main result of the article.

Theorem 4.12 (Theorem B)**.** *The Chow polynomial for type B has the expansion*

$$H_{\mathcal{B}_n}(x) = \sum_{(a_1,\dots,a_n)} \left(\prod_{i=1}^{n}(2a_i - 1)\right) x^{des(a_1,\dots,a_n)}(x+1)^{n-1-2des(a_1,\dots,a_n)}$$

where the sum ranges over all tuples $(a_1, \dots, a_n)$ with $a_i \in \{1, \dots, n+1-i\}$ such that $a_1 \leq a_2$ and also that $a_i > a_{i+1}$ implies $a_{i-1} \leq a_i$ for $i \in \{2, \dots, n-1\}$.

Proof. We have

$$\begin{aligned}
H_{\mathcal{B}_n}(x) &= \sum_{\mathcal{F}} x^{des(\lambda(\mathcal{F}))}(x+1)^{n-1-2des(\lambda(\mathcal{F}))} \\
&= \sum_{\sigma\in S_n} \sum_{\substack{\mathcal{F} \\ \lambda(\mathcal{F})=(b_{\sigma(1)},\ldots,b_{\sigma(n)})}} x^{des(\sigma)}(x+1)^{n-1-2des(\sigma)} \\
&= \sum_{\sigma\in S_n} \left(\prod_{i=1}^{n}(2a_i-1)\right) x^{des(\sigma)}(x+1)^{n-1-2des(\sigma)} \\
&= \sum_{(a_1,\ldots,a_n)} \left(\prod_{i=1}^{n}(2a_i-1)\right) x^{des(a_1,\ldots,a_n)}(x+1)^{n-1-2des(a_1,\ldots,a_n)}.
\end{aligned}$$

The first equality is Theorem 4.2 and the sum is indexed by the maximal saturated chains $\mathcal{F}$ with no double descents and no descent in the first position. In the second step, we split the sum according to the labels of the maximal chains and observe that $b_{\sigma(i)} > b_{\sigma(i+1)}$ is equivalent to $\sigma(i) > \sigma(i+1)$, allowing us to replace the descents of $\lambda(\mathcal{F})$ by the descents of σ. In the third step we use Lemma 4.8. In the last step we use Lemma 4.11 and Lemma 4.10 to switch the index set of the summation from permutations to inversion sequences and to replace the descents of σ by the descents of $(a_1,\ldots,a_n)$. $\square$

4.3. **Chow polynomial arithmeticity for $\mathcal{D}_{n,s}$-arrangements via EL-labelings.** This subsection describes the first of two approaches to show that the Chow polynomials associated to matroids arising from $\mathcal{D}_{n,s}$-arrangements satisfy an arithmeticity property w.r.t. fixed n and varying s, see Section 4.4 for the second one. Our approach in this part utilizes the usage of EL-labelings of the their intersection lattices and is inspired by the discussions in [DGP19].

Recall from Theorem 4.5 that the intersection lattice of $\mathcal{B}_n$- arrangements can be viewed in the following way:

Define Π_n^B as the set of all partitions π of $\langle n\rangle := \{-n,\ldots,n\}$ with the properties

(1) If $B\in\pi$ then $-B\in\pi$
(2) If $i\in B$ and $-i\in B$ for some i and some $B\in\pi$, then $0\in B$.

This defines a poset by setting $x\prec y$ if for each $B\in x$ there is a $B'\in y$ with $B\subseteq B'$. For any $\pi\in\Pi_n^B$, the block containing 0 is called the **zero block**. In the following we denote the zero block by a subscript 0.

Remark 4.13. Due to [DGP19, Theorem 7] the above intersection lattice can be translated to a lattice $\Pi_{n,\Sigma}$ of signed partitions of $[[n]] := \{\bar{1},1,\ldots,\bar{n},n\}$ as introduced in [DGP19]. Therefore we set $\Sigma := \{0\}$, $-k\in\langle n\rangle$ becomes $\bar{k}\in[[n]]$ and a negative block $-R$ translates to $\bar{R}$.

Next we would like to apply the EL-labeling theory in [DGP19] to our setting. In order to do so, we have to translate their concepts regarding the lattice $\Pi_{n,\Sigma}$ into the language of the lattice Π_n^B, using the correspondence described in Theorem 4.13.

The following definitions and results are to our setting adjusted version of [DGP19]. We start with a slight modification of Theorem 4.1 to simplify the below discussion.

Definition 4.14. Let P be a bounded poset with top element $\hat{1}$ and bottom element $\hat{0}$. Let $\mathrm{Cov}(P)\subset P\times P$ be the set of cover relations in P. An **edge-labeling** of P is a map $\lambda\colon \mathrm{Cov}(P)\to\Lambda$ where Λ is a poset. Given such a labeling, each maximal chain $c=(x\prec z_1\prec\ldots\prec z_r\prec y)$ between two elements $x\preceq y$ is called **increasing** if $\lambda(c)=(\lambda(x,z_1),\ldots,\lambda(z_r,y))$ is strictly increasing, and **decreasing** if $\lambda(c)$ is decreasing.

Note that the difference between Theorem 4.1 and this definition lies in the notion of *increasing*, which now requires a strictly increasing sequence of labels, and the according changes of the notion of *decreasing*.

Definition 4.15. Let P be a bounded ranked poset. We say a maximal chain with labeling $(a_1,\ldots,a_k)$ lexicographically precedes a maximal chain with labeling $(b_1,\ldots,b_k)$ if $a_i<b_i$ in the first coordinate where the labels differ. An **edge lexicographic labeling**, or **EL-labeling** for short, of P is an edge labeling such that in each closed interval $[x,y]\subseteq P$ there is a unique increasing maximal chain and this chain lexicographically precedes all other maximal chains of $[x,y]$.

Remark 4.16. Every EL-labeling is an R-labeling.

The next result will be important later on in this note, when we discuss an EL-labeling for intersection lattices of the $\mathcal{D}_{n,s}$ arrangements.

Lemma 4.17. *[DGP19, Lemma 16] Let P be a bounded and ranked poset, having EL-labeling λ. Let $Q \subseteq P$ be a ranked subposet of P containing $\hat{0}$ and $\hat{1}$, with rank function given by restricting the one of P. Then, if for all $x \preceq y$ in Q the unique increasing maximal chain in $[x, y] \subseteq P$ is also contained in Q, the edge labeling λ restricted to Q is an EL-labeling of Q.*

We are now able to reformulate the EL-labeling proposed in [DGP19].

Definition 4.18. For a block R of some partition $\pi \in \Pi_n^B$, the **representative** of R, denoted by $r(R)$, is defined as the minimum element $i \in [n]$ such that $i \in R$ or $-i \in R$. A block $R \subseteq \langle n \rangle$ is called **normalized** if $r(R)$ belongs to R.

Example 4.19. Consider blocks $R = \{2, -4, 5\}$ and $-R = \{-2, 4, -5\}$. Then $r(R) = r(-R) = 2$ and R is normalized while $-R$ is not.

There are three types of cover relations, that is, edges in the Hasse diagram, we have to distinguish.

Definition 4.20. An edge $\mathcal{E}_{xy}$ of the Hasse diagram of Π_n^B is called:

- **signed** if $x_0 \subsetneq y_0$, where these denote the zero blocks of x and y, respectively.
- **coherent** if $R \cup R' \in y$ for some normalized non-zero blocks $R, R' \in x, R \neq R'$;
- **non-coherent** if $R \cup -R' \in y$ for some normalized non-zero blocks $R, R' \in x, R \neq R'$

An edge $\mathcal{E}_{xy}$ is called **unsigned** if it is either coherent or non-coherent.

Using all of the above mentioned notions we are able to state a slightly modified version of the edge-labeling in [DGP19].

Definition 4.21. We define the following edge labeling λ of Π_n^B.

- Let $\mathcal{E}_{xy}$ be an unsigned edge and let $R, R' \in x$ such that $R \cup R' \in y$ with representatives i and j of R and R', respectively. Then:
$$\lambda(x, y) = \begin{cases} (0, \max(i, j)) & \text{if } \mathcal{E}_{xy} \text{ coherent;} \\ (2, \min(i, j)) & \text{if } \mathcal{E}_{xy} \text{ is non-coherent.} \end{cases}$$
- Let $\mathcal{E}_{xy}$ be a signed edge. Then
$$\lambda(x, y) = (1, 1).$$

Labels are ordered lexicographically.

The next result follows from of [DGP19, Theorem 7] and [DGP19, Theorem 18].

Theorem 4.22. *The labeling λ of the definition above is an EL-labeling of Π_n^B.*

In the rest of this subsection we show that the above labeling is also an EL-labeling for the intersection lattices of $\mathcal{D}_n$- and $\mathcal{D}_{n,s}$-arrangements and examine their corresponding Chow polynomials in the spirit of Theorem 4.2.

We start by explicitly describing the intersection lattices of arrangements $\mathcal{D}_n$ and $\mathcal{D}_{n,s}$. According to [DGP19, Theorem 7] the intersection lattice $\mathcal{L}(\mathcal{D}_n)$ is isomorphic to the subposet of $\mathcal{L}(\mathcal{B}_n)$ consisting of all elements $x \in \mathcal{L}(\mathcal{B}_n)$ such that $|x_0| \neq 2$. In other words, we take $\mathcal{L}(\mathcal{B}_n)$ and delete all partitions and their connecting edges that contain a block of the form $\{k, 0, -k\}$ for $k \in [n]$. From the bijection between $\mathcal{L}(\mathcal{B}_n)$ Π_n^B, it follows that the intersection lattice of $\mathcal{D}_{n,s}$ is isomorphic to the subposet of $\mathcal{L}(\mathcal{B}_n)$ consisting of all partitions of $\mathcal{L}(\mathcal{B}_n)$ without those that contain a block of the form $\{k, 0, -k\}$ for $k \in \{s + 1, \ldots, n\}$.

The following theorem establishes that the EL-labeling defined in Theorem 4.21 is preserved when we go from $\mathcal{L}(\mathcal{B}_n)$ to $\mathcal{L}(\mathcal{D}_{n,s})$.

Theorem 4.23. *The EL-labeling λ from Definition 4.21 is an EL-labeling for the intersection lattice $\mathcal{L}(\mathcal{D}_{n,s})$, for $s \in \{0, \ldots, n\}$.*

Figure 1 below illustrates the EL-labeling for the intersection lattice of the reflection arrangement $\mathcal{D}_3$. In order to prove the above theorem we need the following three technical lemmata.

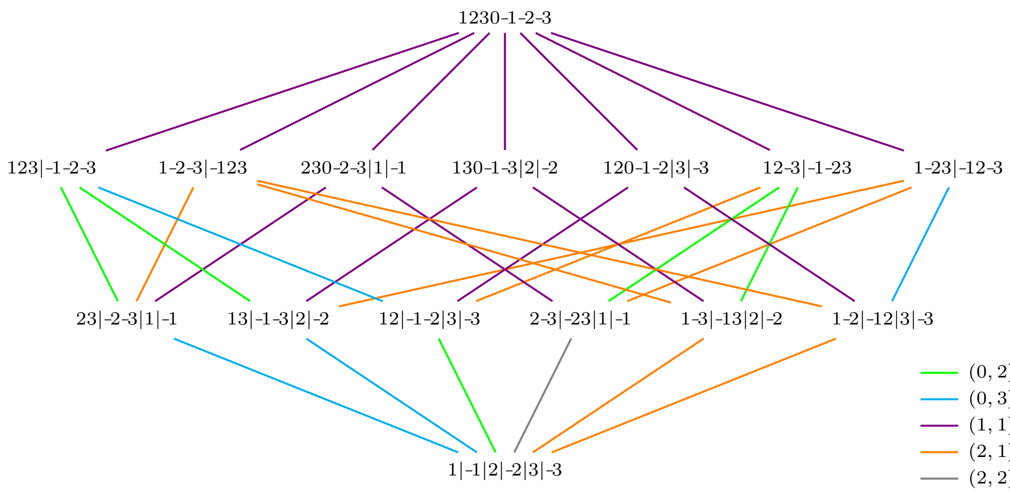

FIGURE 1. $\mathcal{L}(\mathcal{D}_3)$ with the EL-labeling.

Lemma 4.24. *Let $x \in \mathcal{L}(\mathcal{D}_n)$ and $y \in \mathcal{L}(\mathcal{B}_n) \setminus \mathcal{L}(\mathcal{D}_n)$ with $x \prec y$. Then $\mathcal{E}_{xy}$ is a signed edge.*

Proof. We know that $y \in \mathcal{L}(\mathcal{B}_n) \setminus \mathcal{L}(\mathcal{D}_n)$, therefore y contains a block of the form $\{k, 0, -k\}$, for some $k \in [n]$. At the same time, x cannot contain such a block. Hence, the only possible way to go from x to y is via merging $\{k\}$ and $\{-k\}$ which creates the signed block $\{k, 0, -k\}$. Thus $\mathcal{E}_{xy}$ is signed. □

Lemma 4.25. *Let $x \in \mathcal{L}(\mathcal{B}_n) \setminus \mathcal{L}(\mathcal{D}_n)$ and $y \in \mathcal{L}(\mathcal{D}_n)$ with $x \prec y$. Then $\mathcal{E}_{xy}$ is a signed edge.*

Proof. We know that $x \in \mathcal{L}(\mathcal{B}_n) \setminus \mathcal{L}(\mathcal{D}_n)$, therefore x contains a block of the form $\{k, 0, -k\}$ for some $k \in [n]$. At the same time, y cannot contain such a block. Hence, the only possible ways to go from x to y is via merging $\{k, 0, -k\}$ with some blocks A and $-A$ of x. This creates the signed block $A \cup -A \cup \{k, 0, -k\}$ Thus $\mathcal{E}_{xy}$ is signed. □

Lemma 4.26. *Let $1 \leq u < s \leq n$. Moreover let $x \in \mathcal{L}(\mathcal{D}_{n,u}) \setminus \mathcal{L}(\mathcal{D}_n)$ and $y \in \mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,u})$ with either $rk(y) = rk(x) + 1$ or $rk(x) = rk(y) + 1$. Then in either case there is no edge between x and y.*

Proof. First assume that $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$. Suppose there exists an edge between x and y, denoted $\mathcal{E}_{xy}$. We know that $y \in \mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,u})$, therefore y contains a block of the form $\{k, 0, -k\}$ for some $k \in \{u+1, \ldots, s\}$. At the same time x cannot contain this block. Hence the only possible way to go from x to y is via merging $\{k\}$ and $\{-k\}$ which creates the signed block $\{k, 0, -k\}$ in y. However, x already contains a signed block $\{j, 0, -j\}$ with $j \in \{1, \ldots, u\}$. Since there can be at most one block with sign 0, this merge is not possible, and therefore $\mathcal{E}_{xy}$ cannot exist. Similarly for the other case. □

Proof of Theorem 4.23. Let $s \in \{0, \ldots, n\}$. We know that the labeling from Theorem 4.21 is an EL-labeling for $\mathcal{L}(\mathcal{B}_n)$ and that $\mathcal{L}(\mathcal{D}_{n,s})$ is a subposet of $\mathcal{L}(\mathcal{B}_n)$. By Theorem 4.17, it is sufficient to show that for all $x \preceq y$ in $\mathcal{L}(\mathcal{D}_{n,s})$ the unique increasing maximal chain in $[x, y] \subseteq \mathcal{L}(\mathcal{B}_n)$ is also contained in $\mathcal{L}(\mathcal{D}_{n,s})$. Let $x, y \in \mathcal{L}(\mathcal{D}_{n,s})$ with $x \preceq y$. If $[x, y] \subseteq \mathcal{L}(\mathcal{D}_{n,s})$ we are done. For the other case, let c be a maximal chain in $[x, y] \subseteq \mathcal{L}(\mathcal{B}_n)$ that is not fully contained in $\mathcal{L}(\mathcal{D}_{n,s})$. We will show that $\lambda(c)$ contains the label $(1, 1)$ at least twice, hence the chain cannot be increasing. Now there exists an element $z \in \mathcal{L}(\mathcal{B}_n) \setminus \mathcal{L}(\mathcal{D}_{n,s})$ such that $c = (x, \ldots, z) \cup (z, \ldots, y)$. In particular, there are $x_1, z_1 \in (x, \ldots, z)$ such that $x_1 \in \mathcal{L}(\mathcal{D}_n)$, $z_1 \in \mathcal{L}(\mathcal{B}_n) \setminus \mathcal{L}(\mathcal{D}_{n,s})$ and $x_1 \prec z_1$, by Theorem 4.26. Now using Theorem 4.24 it follows that $\mathcal{E}_{x_1 z_1}$ is a signed edge, hence $\lambda(x_1, z_1) = (1, 1)$. Similarly, by Theorem 4.26 there exist $z_2, y_1 \in (z, \ldots, y)$ such that $z_2 \in \mathcal{L}(\mathcal{B}_n) \setminus \mathcal{L}(\mathcal{D}_{n,s})$, $y_1 \in \mathcal{L}(\mathcal{D}_n)$ and $z_2 \prec y_1$. Using Theorem 4.25 we get that $\mathcal{E}_{z_2 y_1}$ is a signed edge, therefore $\lambda(z_2, y_1) = (1, 1)$. So the label $(1, 1)$ appears at least two times. Consequently, the unique increasing maximal chain in $[x, y]$ is contained in $\mathcal{L}(\mathcal{D}_{n,s})$. □

Further observations on the lattice and the EL-labeling lead to another interesting result:

Proposition 4.27. *The Chow polynomial is arithmetic on the arrangements $\mathcal{D}_{n,s}$. That is,*

$$H_{\mathcal{D}_{n,n}(t)} - H_{\mathcal{D}_{n,n-1}}(t) = \cdots = H_{\mathcal{D}_{n,1}}(t) - H_{\mathcal{D}_{n,0}}(t).$$

In particular

$$H_{\mathcal{D}_{n,s}}(t) = \frac{s}{n} H_{\mathcal{B}_n}(t) + \frac{n-s}{n} H_{\mathcal{D}_n}(t).$$

In order to prove this, we need a few more lemmata. The first lemma further investigates the structure of the lattice of flats of the $\mathcal{D}_{n,s}$ arrangements.

Lemma 4.28. *A maximal chain in $\mathcal{L}(\mathcal{D}_{n,s})$ is either completely contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$ or has exactly one edge from $\mathcal{L}(\mathcal{D}_n)$ to $\mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ and one edge from $\mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ back to $\mathcal{L}(\mathcal{D}_n)$.*

Proof. Let c be a maximal chain in $\mathcal{L}(\mathcal{D}_{n,s})$ and suppose that c is not fully contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$. So, there exists an element in $\mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ that is contained in c. By Theorem 4.26, there are no edges between $\mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ and $\mathcal{L}(\mathcal{D}_{n,u}) \setminus \mathcal{L}(\mathcal{D}_{n,u-1})$ for any $1 \leq u < s$. Since $\hat{0}$ is contained in $\mathcal{L}(\mathcal{D}_n)$ there must exist $x \in \mathcal{L}(\mathcal{D}_n)$ and $y \in \mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ such that $\mathcal{E}_{xy}$ is an edge in c. Using the same argument and the fact that $\hat{1}$ is contained in $\mathcal{L}(\mathcal{D}_n)$, it follows that there must exist $y' \in \mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ and $x' \in \mathcal{L}(\mathcal{D}_n)$ such that $\mathcal{E}_{y'x'}$ is in the chain c. Further note that the only possibility to go from $\mathcal{L}(\mathcal{D}_n)$ to $\mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ is via merging the singleton blocks $\{s\}$ and $\{-s\}$ creating the signed block $\{s, 0, -s\}$. Clearly, this merge can only happen once in one chain, hence $\mathcal{E}_{xy}$ is the only edge between $\mathcal{L}(\mathcal{D}_n)$ and $\mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$. As a direct consequence, there can also be no second edge in c going from $\mathcal{L}(\mathcal{D}_{n,s}) \setminus \mathcal{L}(\mathcal{D}_{n,s-1})$ to $\mathcal{L}(\mathcal{D}_n)$. □

Lemma 4.29. *Let c be a maximal chain in $\mathcal{L}(\mathcal{D}_{n,s})$ which is not fully contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$. Then every edge in c that merges blocks containing s or $-s$ is signed.*

Proof. Let c be a maximal chain in $\mathcal{L}(\mathcal{D}_{n,s})$ which is not fully contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$. Assume that c contains an unsigned edge which merges blocks containing s or $-s$ and let x and y be its end points. In the partition x, s and $-s$ are not contained in the zero block, since otherwise the edge to y would be signed. In particular, this means that elements of c coming before x (including x) are contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$. But in y and all subsequent partitions in c, there is at least some other element $k \neq s$ in the same block as s. If $-s$ is contained in this block, the edge would be signed, hence also $k \neq -s$. But then all partitions after x are contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$ as well, contradicting our assumption. □

To prove our proposition, we want to map maximal chains in $\mathcal{L}(\mathcal{D}_{n,u})$ not fully contained in $\mathcal{L}(\mathcal{D}_{n,u-1})$ bijectively to maximal chains in $\mathcal{L}(\mathcal{D}_{n,s})$ not fully contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$ while preserving descents and ascents. To this end, we define for $1 \leq u < s \leq n$ the following bijection:

$$\varphi_{u,s} \colon [n] \to [n], \quad \varphi_{u,s}(k) = \begin{cases} k & \text{if } k < u \text{ or } k > s, \\ k-1 & \text{if } u < k \leq s, \\ s & \text{if } k = u. \end{cases}$$

Further we define $\varphi_{u,s}(-k) \coloneqq -\varphi_{u,s}(k)$. For a partition $\pi \in \mathcal{L}(\mathcal{D}_{n,u})$ we write $\varphi_{u,s}(\pi)$ for the partition in $\mathcal{L}(\mathcal{D}_{n,s})$ where $\varphi_{u,s}$ was applied to every block in π. In particular, the zero block $\{u, 0, -u\}$ is mapped to the zero block $\{s, 0, -s\}$. Thus, $\varphi_{u,s}$ maps any partition from $\mathcal{L}(\mathcal{D}_{n,u})$ to a partition in $\mathcal{L}(\mathcal{D}_{n,s})$. Note that for $a_1, a_2 \in [n] \setminus \{u\}$, $a_1 < a_2$ implies $\varphi_{u,s}(a_1) < \varphi_{u,s}(a_2)$. Hence, if R is a block with representative $k \neq u$, then $\varphi(k)$ is the representative of $\varphi_{u,s}(R)$.

In the following we write $\varphi = \varphi_{u,s}$ for short. The next lemma establishes that this bijection preserves the descents of particular maximal chains.

Lemma 4.30. *Let $z_1 \prec z_2 \prec z_3$ be elements of a maximal chain in $\mathcal{L}(\mathcal{D}_{n,s})$ which is not fully contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$. Let λ be the EL-labeling introduced in Theorem 4.21. Then $\lambda(z_1, z_2) \geq \lambda(z_2, z_3)$ implies $\lambda(\varphi(z_1), \varphi(z_2)) \geq \lambda(\varphi(z_2), \varphi(z_3))$.*

Proof. First note that by definition of φ the edge $\mathcal{E}_{z_1 z_2}$ is coherent, non-coherent or signed if and only if $\mathcal{E}_{\varphi(z_1),\varphi(z_2)}$ is coherent, non-coherent or signed, respectively. This holds analogously for $\mathcal{E}_{z_2 z_3}$ and $\mathcal{E}_{\varphi(z_2),\varphi(z_3)}$. It follows that $\lambda(z_1, z_2)$ and $\lambda(\varphi(z_1), \varphi(z_2))$ as well as $\lambda(z_2, z_3)$ and $\lambda(\varphi(z_2), \varphi(z_3))$ are always equal in the first component. Now suppose that $\lambda(z_1, z_2) \geq \lambda(z_2, z_3)$. The only cases we need to consider are those when $\mathcal{E}_{z_1 z_2}$ and $\mathcal{E}_{z_2 z_3}$ are both of the same type, namely both signed, coherent or non-coherent. For all the other cases the order is already determined by the first component of the label, which is invariant under φ. If both edges are signed, then $\mathcal{E}_{\varphi(z_1),\varphi(z_2)}$ and $\mathcal{E}_{\varphi(z_2),\varphi(z_3)}$ are also signed, hence they also have label $(1, 1)$ and $\lambda(\varphi(z_1), \varphi(z_2)) \geq \lambda(\varphi(z_2), \varphi(z_3))$ holds. So now assume that $\mathcal{E}_{z_1 z_2}$ and $\mathcal{E}_{z_2 z_3}$ are both coherent. Consider $\mathcal{E}_{z_1 z_2}$ and let $R_1, R_1' \in z_1$ such that $R_1 \cup R_1' \in z_2$ with representatives a_1 and a_1' of R_1 and R_1'. Analogously for $\mathcal{E}_{z_2z_3}$ let $R_2, R_2' \in z_2$ such that $R_2 \cup R_2' \in x_3$ with representatives a_2 and a_2' of R_2 and R_2'. Then

$$\lambda(z_1, z_2) = (0, \max(a_1, a_1')) \geq (0, \max(a_2, a_2')) = \lambda(z_2, z_3).$$

Due to Theorem 4.29 none of the representatives a_1, a_1', a_2, a_2' equals u. Therefore, by definition of φ, it follows that

$$\lambda(\varphi(z_1), \varphi(z_2)) = (0, \max(\varphi(a_1), \varphi(a_1'))) \geq (0, \max(\varphi(a_2), \varphi(a_2'))) = \lambda(\varphi(z_2), \varphi(z_3)).$$

For $\mathcal{E}_{z_1z_2}$ and $\mathcal{E}_{z_2z_3}$ both being non-coherent the statement follows similarly. □

Using these lemmata, we can now conclude the following property and prove our proposition.

Lemma 4.31. *Let $1 \leq u \neq s \leq n$ and let λ be the EL-labeling from Theorem 4.21. Then there exists a bijective mapping that maps any maximal chain in $\mathcal{L}(\mathcal{D}_{n,u})$ that is not fully contained in $\mathcal{L}(\mathcal{D}_{n,u-1})$ to a maximal chain in $\mathcal{L}(\mathcal{D}_{n,s})$ that is not fully contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$ with descents at exactly the same positions in the chain and vice versa.*

Proof. First assume that $u < s$. Let $c = (\hat{0}, c_1, \ldots, c_r, \hat{1})$ be an arbitrary maximal chain in $\mathcal{L}(\mathcal{D}_{n,u})$ that is not fully contained in $\mathcal{L}(\mathcal{D}_{n,u-1})$. By Theorem 4.30, it follows that $\varphi(c) = (\varphi(\hat{0}), \varphi(c_1), \ldots,$ $\varphi(c_r), \varphi(\hat{1}))$ is a chain in $\mathcal{L}(\mathcal{D}_{n,s})$ that has descents at exactly the same positions as c and this chain is not fully contained in $\mathcal{L}(\mathcal{D}_{n,s-1})$ because c contains a partition with the block $\{u, 0, -u\}$ which is mapped under φ to the block $\{s, 0, -s\}$. Further, since by definition φ is a bijection and maps $\{-u, 0, u\}$ to $\{-s, 0, s\}$, this mapping between chains is bijective. For $u > s$ the bijection is φ^{-1}. The other direction works similarly. □

Proof of Theorem 4.27. Using the formula from Theorem 4.2 we know that the Chow polynomial is computed using maximal chains fulfilling certain conditions. Hence, the change of the Chow polynomial from $H_{\mathcal{D}_{n,s-1}}(t)$ to $H_{\mathcal{D}_{n,s}}(t)$ is uniquely determined by the new maximal chains that get added when going from $\mathcal{L}(\mathcal{D}_{n,s-1})$ to $\mathcal{L}(\mathcal{D}_{n,s})$. We want to show that the contribution of these new maximal chains is independent of s. So let $1 \leq u < s \leq n$ arbitrary. The maximal chains that newly contribute to the Chow polynomial $H_{D_{n,u}(t)}$ are those that were not fully contained in $\mathcal{L}(\mathcal{D}_{n,u-1})$. Similarly for $H_{D_{n,s}}(t)$. However, according to Theorem 4.30 for each new maximal chain c in $\mathcal{L}(\mathcal{D}_{n,u})$ there exactly one new maximal chain $\varphi(c)$ in $\mathcal{L}(\mathcal{D}_{n,s})$ with descents at the same positions. Moreover, Theorem 4.30 ensures that if a chain c in $\mathcal{L}(\mathcal{D}_{n,u})$ satisfies the additional conditions in Theorem 4.2 then $\varphi(c)$ also does. Hence, for each new maximal chain in $\mathcal{L}(\mathcal{D}_{n,u})$ that contributes to the Chow polynomial $H_{D_{n,u}}(t)$ there is a new maximal chain in $\mathcal{L}(\mathcal{D}_{n,s})$ that contributes to the Chow polynomial $H_{D_{n,s}}(t)$ in the same way. Therefore, the contribution of all new maximal chains in $\mathcal{L}(\mathcal{D}_{n,u})$ is the same as the contribution of all new maximal chains in $\mathcal{L}(\mathcal{D}_{n,s})$. Consequently, $H_{\mathcal{D}_{n,s}}(t) - H_{\mathcal{D}_{n,s-1}}(t) = H_{\mathcal{D}_{n,u}}(t) - H_{\mathcal{D}_{n,u-1}}(t)$ for $u, s \in \{1, \ldots, n\}$. □

Remark 4.32. Note that this proof can be generalized. Consider $\mathcal{A}$ and $H_1, \ldots, H_s$ as in Theorem 3.13. Moreover, assume there are R-labelings on $\mathcal{L}(\mathcal{A} \cup \{H_1\})$, …, $\mathcal{L}(\mathcal{A} \cup \{H_s\})$ that restrict to the same R-labeling of $\mathcal{L}(\mathcal{A})$. Then it can be proved that

$$H_{\mathcal{A}\cup\{H_1,\ldots,H_s\}}(t) = H_{\mathcal{A}}(t) + \sum_{i=1}^{s} H_{\mathcal{A}\cup\{H_i\}}(t) - H_{\mathcal{A}}(t).$$

Since the proof depends solely on the properties of the lattices of flats of the arrangements, there are further generalizations of this kind to lattices.

4.4. **Chow polynomial arithmeticity for $D_{n,s}$-arrangements via characteristic polynomials.** This subsection describes a second approach to show that the Chow polynomials associated to matroids arising from the intermediate arrangements $\mathcal{D}_{n,s}$ satisfy an arithmeticity property for a fixed n and varying s, see Section 4.3 for the first one. Our approach in this part uses a recursive formula for Chow polynomials of matroids in terms of their minors and the corresponding reduced characteristic polynomials.

Theorem 4.33. *The Chow polynomial is arithmetic on the intermediate arrangements, that is $H_{\mathcal{D}_{n,s+1}}(t) - H_{\mathcal{D}_{n,s}}(t) = H_{\mathcal{D}_{n,1}}(t) - H_{\mathcal{D}_n}(t)$. In particular $H_{\mathcal{D}_{n,s}}(t) = \frac{s}{n} H_{\mathcal{B}_n}(t) + \frac{n-s}{n} H_{\mathcal{D}_n}(t)$.*

For a lattice $\mathcal{L}$ and elements $A, B \in \mathcal{L}$ we denote by $\mathcal{L}^{[A,B]}$ the restriction of $\mathcal{L}$ to the interval $[A, B]$, that is

$$\mathcal{L}^{[A,B]} := \{C \in \mathcal{L} | A \leq C \leq B\}.$$

For a graded lattice $\mathcal{L}$ we denote the unique minimal element by 0 and the maximal element by 1. Then we define its characteristic polynomial by

$$\chi_{\mathcal{L}}(t) := \sum_{A \in \mathcal{L}} \mu_{\mathcal{L}}(A) t^{rk(A)},$$

where $\mu_{\mathcal{L}} : \mathcal{L} \to \mathbb{Z}$ is the Möbius function recursively defined as

$$\mu_{\mathcal{L}}(0) = 1 \text{ and } \mu_{\mathcal{L}}(A) = -\sum_{0 \leq B < A} \mu_{\mathcal{L}}(B) \text{ for all } A \in \mathcal{L} \setminus \{0\}.$$

We define the reduced characteristic polynomial of $\mathcal{L}$ as

$$\overline{\chi}_{\mathcal{L}}(t) := \frac{\chi_{\mathcal{L}}(t)}{t-1}.$$

Note that if $\mathcal{L}$ is a lattice of flats of a matroid, which is the case in the proof below, then the definitions above coincide with Theorem 2.16.
To simplify the notation, for $A, B \in \mathcal{L}$ we denote by $\overline{\chi}_{\mathcal{L}}^{[A,B]}(t)$ the reduced characteristic polynomial of the lattice $\mathcal{L}^{[A,B]}$. We omit the lower index $\mathcal{L}$ when it is clear from the context.

In the proof of Theorem 4.33 we use the following result of [Fer+24]:

Theorem 4.34. *Let M be a loopless matroid. Then, the Chow polynomial of M satisfies*

$$H_M(t) = \sum_{F \in \mathcal{L}(M),\ F \neq \emptyset} \overline{\chi}_{M|_F} H_{M/F}(t),$$

where $M|_F$ is the restriction of M to F and M/F the contraction of M to F.

Recursively applying this to a matroid M yields the following corollary.

Corollary 4.35. *The Chow polynomial of a loopless matroid satisfies:*

$$H_M(t) = \sum_{\mathcal{C}} \prod_j \overline{\chi}_{\mathcal{L}(M)}^{[C_j, C_j+1]}(t),$$

where the sum runs over all flags of flats in M starting at $\emptyset$ and ending in M and $\overline{\chi}_{\mathcal{L}(M)}^{[C_j,C_j+1]}(t)$ denotes the reduced characteristic polynomial of the lattice of flats restricted to $[C_j, C_{j+1}]$.

We use the notation $\mathcal{D}_{n,S}$ for $S \subseteq [n]$ for the matroid corresponding to the hyperplane arrangement

$$\mathcal{D}_{n,S} := \bigcup_{H \in \mathcal{D}_n} H \cup \bigcup_{j \in S} H_j,$$

where $H_j := \{x_j = 0\}$. Note that the arrangements $\mathcal{D}_{n,S_1}, \mathcal{D}_{n,S_1}, \mathcal{D}_{n,s}$ have isomorphic intersection lattices for subsets $S_1, S_2 \subset [n]$ of the same cardinality s, in particular $H_{\mathcal{D}_{n,S_1}}(t) = H_{\mathcal{D}_{n,S_2}}(t)$. To prove Theorem 4.33 it is thus enough to show that

$$H_{\mathcal{D}_{n,S\cup\{i\}}}(t) - H_{\mathcal{D}_{n,S}}(t) = H_{\mathcal{D}_{n,S'\cup\{i\}}}(t) - H_{\mathcal{D}_{n,S'}}(t)$$

for any $S \subset S'$ and $i \in [n] \backslash S'$. First we introduce several technical lemmas about the lattice of flats of the $\mathcal{D}_{n,S}$ arrangements.

Lemma 4.36. *$\mathcal{L}(\mathcal{D}_{n,S})^{[H_i,1]} \simeq \mathcal{L}(\mathcal{B}_{n-1})$, where the isomorphism is given by restricting the arrangement to the hyperplane H_i. In particular for any $S_1, S_2 \subseteq [n]$ and and flats $F, G \in \mathcal{L}(\mathcal{D}_{n,S_1}) \cap \mathcal{L}(\mathcal{D}_{n,S_2})$ for which there exists an $i \in S_1 \cap S_2$ such that $i \in F$, we have*

$$\mathcal{L}(\mathcal{D}_{n,S_1})^{[F,G]} \simeq \mathcal{L}(\mathcal{D}_{n,S_2})^{[F,G]}.$$

Proof. The intersection of the hyperplane $\{x_k \pm x_i = 0\}$ with H_i is the hyperplane H_k (viewed as a hyperplane on the smaller subspace). The restriction of a hyperplane $\{x_k \pm x_l = 0\}$ to H_i for $k, l \neq i$ can be identified with the hyperplane $\{x_k \pm x_l = 0\}$ in the arrangement $\mathcal{B}_{n-1}$. The second part follows since the isomorphism is preserved under further restriction and localization. □

Lemma 4.37. *Let $S_1, S_2 \subseteq [n]$ and F be a flat contained in $\mathcal{L}(\mathcal{D}_{n,S_1}) \cap \mathcal{L}(\mathcal{D}_{n,S_2})$ such that $F \cap (S_1 \backslash S_2 \cup S_2 \backslash S_1) = \emptyset$. Then the lattices $\mathcal{L}(\mathcal{D}_{n,S_1})^{[\emptyset,F]}$ and $\mathcal{L}(\mathcal{D}_{n,S_2})^{[\emptyset,F]}$ are isomorphic.*

Proof. This follows since the flats of the localized arrangement can not contain j for $j \in S_1 \backslash S_2 \cup S_2 \backslash S_1$ and thus they are defined by intersections of the same set of hyperplanes. □

A particular case of interest in the previous lemma is when $i \in F$ for some $i \in S_1 \cap S_2$ and $F \nsubseteq H_j$ for any $j \neq i$. Furthermore, the isomorphism above descends to an isomorphism of any contraction of the lattices above (that is restrictions of arrangements).

Lemma 4.38. *Consider an embedding of the lattice of $\mathcal{D}_{n,S}$ into the lattice of $\mathcal{D}_{n,S\cup i}$ for $i \notin S$, given by the corresponding subspaces. Then the flats F that are not contained in the image of the embedding are exactly those, for which i is contained in F, but does not contain any k for $k \neq i$.*

Proof. Assume that F is a flat in $\mathcal{L}(\mathcal{D}_{n,S\cup i}) \setminus \mathcal{L}(\mathcal{D}_{n,S})$ with $k \in F$ for some $k \neq i$. Note that the subspace corresponding to F is an intersection of a set of hyperplanes containing H_i. But then since the subspace is contained in H_k we can replace H_i in the intersection with the hyperplanes $\{x_i + x_k = 0\} \cap \{x_i - x_k = 0\}$, so F is contained in the image of $\mathcal{L}(\mathcal{D}_{n,S})$, contradiction. In the other direction, we have to show that if an intersection of hyperplanes in $\mathcal{D}_{n,S}$ is contained in H_i, then it also has to be contained in some other coordinate hyperplane. Note that the subspace corresponding to F can not be written as an intersection of hyperplanes which are all of the form $\{x_k \pm x_l = 0\}$, $\{x_k = 0\}$ for $k, l \neq i$, since all of them contain the x_i-axis, which would contradict $i \notin F$. Thus, the subspace corresponding to F has to be contained in $\{x_k \pm x_i\}$ for some k. But since it is also contained in H_i, it has to be contained in H_k as well. □

Now we can come to evaluating the difference between the Chow polynomials, using the formula in Theorem 4.35. Consider

$$H_{\mathcal{D}_{n,S\cup\{i\}}}(t) - H_{\mathcal{D}_{n,S}}(t).$$

We want to show this is equal to $H_{\mathcal{D}_{n,S'\cup\{i\}}}(t) - H_{\mathcal{D}_{n,S'}}(t)$ for $S \subset S'$ and $i \in [n]\backslash S'$. Expand the difference as the sum of product of reduced characteristic polynomials over all different flags of flats. For a flag of flats

$$\mathcal{C} := \{\emptyset = C_0 \subset C_1 \subset \cdots \subset C_k\}$$

in $\mathcal{D}_{n,S}$ we can consider the corresponding flag of flats in $\mathcal{D}_{n,S\cup i}$ defining the same subspaces. Summing over all such flags in $\mathcal{L}(\mathcal{D}_{n,S})$ consider the difference

$$I'_S := \sum_{\mathcal{C}} \left(\prod_{j=0}^{k-1} \overline{\chi}_{\mathcal{D}_{n,S\cup i}}^{[C_j, C_{j+1}]} - \prod_{j=0}^{k-1} \overline{\chi}_{\mathcal{D}_{n,S}}^{[C_j, C_{j+1}]} \right),$$

where we denote the corresponding flats in $\mathcal{D}_{n,S\cup i}$ also by C_j to streamline the notation.
Now we can consider the set J_S of flags of flats $\mathcal{C}$, such that there is no corresponding flag in $\mathcal{D}_{n,S}$, that is such that there exists a flat C_j in $\mathcal{C}$ not contained in $\mathcal{L}(\mathcal{D}_{n,S}) \subset \mathcal{L}(\mathcal{D}_{n,S\cup i})$. Such flags give a summand

$$I''_S := \sum_{J_S} \prod_{j} \overline{\chi}_{\mathcal{D}_{n,S\cup i}}^{[C_j, C_{j+1}]}.$$

By Theorem 4.35 we get

$$H_{\mathcal{D}_{n,S\cup\{i\}}}(t) - H_{\mathcal{D}_{n,S}}(t) = I'_S + I''_S \tag{2}$$

Lemma 4.39. *The sum I''_S does not depend on S, that is for any $S' \supset S$ with $i \notin S'$ we have $I''_S = I''_{S'}$.*

Proof. For each flag $\mathcal{C}$ in J_S we can consider the largest possible index j such that C_j does not lie in the sublattice $\mathcal{L}(\mathcal{D}_{n,S})$. By Theorem 4.38, these flats C_k correspond exactly to intersections of hyperplanes, contained in H_i, but not any other coordinate hyperplanes. By Theorem 4.37 applied to $S_1 = S \cup i$ and $S_2 = S' \cup i$ we have that the lattices $\mathcal{L}(\mathcal{D}_{n,S\cup i})$ and $\mathcal{L}(\mathcal{D}_{n,S'\cup i})$ restricted to $[C_k, C_{k+1}]$ for $k < j$ are isomorphic, thus they have the same characteristic polynomials. Similarly, by Theorem 4.36 the respective lattices are isomorphic when restricted to $[C_k, C_{k+1}]$ for $k \geq j$, since C_j is contained in the coordinate hyperplane H_i. Lastly, there is a one to one correspondence between flags in J_S and $J_{S'}$, as each flat C_k in the flag in $\mathcal{L}(\mathcal{D}_{n,S'})$ satisfying $l \in C_k$ for $l \in S'\backslash S$ also satisfies $i \in C_k$. This is because each flag in $J_{S'}$ contains a flat whose corresponding subspace is contained in H_i and no other coordinate hyperplane. Since the subspace corresponding to C_k is also contained in H_l it must be above this flat in the flag, so in particular it is also contained in H_i. □

Lemma 4.40. *The sum I'_S does not depend on S, that is for any $S \subset S'$ with $i \notin S'$ we have $I'_S = I'_{S'}$.*

Proof. Consider a flag of flats $\mathcal{C}$ in $\mathcal{D}_{n,S}$. First we observe that for a fixed flag $\mathcal{C}$ there is at most one j for which the polynomials $\overline{\chi}_{\mathcal{D}_{n,S\cup i}}^{[C_j,C_{j+1}]}$ and $\overline{\chi}_{\mathcal{D}_{n,S}}^{[C_j,C_{j+1}]}$ differ. By Theorem 4.38 for each j the subspace corresponding to C_j is either not contained in H_i, or it is contained in at least 2 coordinate hyperplanes (one of which is H_i). We consider the flat C_j with the smallest possible index j, that satisfies the latter condition. Then by Theorem 4.36

$$\overline{\chi}_{\mathcal{D}_{n,S\cup i}}^{[C_k,C_{k+1}]} = \overline{\chi}_{\mathcal{D}_{n,S}}^{[C_k,C_{k+1}]} \text{ for } k \geq j.$$

Furthermore, by Theorem 4.37 applied to $S_1 = S$ and $S_2 = S \cup i$ for

$$\overline{\chi}_{\mathcal{D}_{n,S\cup i}}^{[C_k,C_{k+1}]} = \overline{\chi}_{\mathcal{D}_{n,S}}^{[C_k,C_{k+1}]} \text{ for } k < j-1.$$

Thus the only possible interval $[C_k, C_{k+1}]$ on which the lattices can be non-isomorphic is $[C_{j-1}, C_j]$. Thus we can write the term in the sum I'_S corresponding to $\mathcal{C}$ as

$$\left(\overline{\chi}_{\mathcal{D}_{n,S\cup i}}^{[C_{j-1},C_j]}(t) - \overline{\chi}_{\mathcal{D}_{n,S}}^{[C_{j-1},C_j]}(t)\right) \cdot \prod_{k\neq j} \overline{\chi}_{\mathcal{D}_{n,S}}^{[C_{k-1},C_k]}(t).$$

By applying Theorem 4.36 to $\mathcal{L}(\mathcal{D}_{n,S})$, $\mathcal{L}(\mathcal{D}_{n,S'})$ restricted to $[C_{k-1}, C_k]$ for $k \geq j$ we get that $\overline{\chi}_{\mathcal{D}_{n,S}}^{[C_{k-1},C_k]}(t) = \overline{\chi}_{\mathcal{D}_{n,S'}}^{[C_{k-1},C_k]}(t)$. Similarly, Theorem 4.37 for $S_1 = S$ and $S_2 = S'$ implies that $\overline{\chi}_{\mathcal{D}_{n,S}}^{[C_{k-1},C_k]}(t) = \overline{\chi}_{\mathcal{D}_{n,S'}}^{[C_{k-1},C_k]}(t)$ for $k < j$. We claim that for

$$\beta_S^{[C_{j-1},C_j]}(t) := \overline{\chi}_{\mathcal{D}_{n,S\cup i}}^{[C_{j-1},C_j]}(t) - \overline{\chi}_{\mathcal{D}_{n,S}}^{[C_{j-1},C_j]}(t)$$

we have $\beta_S^{[C_{j-1},C_j]}(t) = \beta_{S'}^{[C_{j-1},C_j]}(t)$, which follows from the next lemma. To finish the proof of Theorem 4.40 we note that by the previous arguments the contribution of flags not contained in $\mathcal{L}(\mathcal{D}_{n,S})$ to the sum $I'_{S'}$ is zero. This follows since they cannot contain two consecutive flats C_j, C_{j+1} such that $j \notin C_j$ for any j but the subspace corresponding to C_{j+1} is contained in the intersection $H_k \cap H_l$ for some $k \neq l$, which are the only intervals where the restriction of the lattices $\mathcal{L}(\mathcal{D}_{n,S'\cup i})$ and $\mathcal{L}(\mathcal{D}_{n,S'})$ can be non-isomorphic. □

We have established that in Theorem 4.40 that $I'_S = I'_{S'}$ and in Theorem 4.39 that $I''_S = I''_{S'}$, so by Equation (2) the equality of the Chow polynomials must hold and Theorem 4.33 follows.

Lemma 4.41. *Fix a flag $\mathcal{C}$ and flats C_{j-1}, C_j as in the proof of Theorem 4.40. The equality $\beta_S^{[C_{j-1},C_j]}(t) = \beta_{S'}^{[C_{j-1},C_j]}(t)$ holds.*

Proof. To simplify the notation, for $S \subset [n]$ we define $L_S = \mathcal{L}(\mathcal{D}_{n,S})^{[C_j,C_{j+1}]}$, $M_S^i = \mathcal{L}(\mathcal{D}_{n,S\cup i})^{[C_j,C_{j+1}]}$. Consider the embeddings

$$\iota_{S,i} : \mathcal{L}(\mathcal{D}_{n,S})^{[C_j,C_{j+1}]} \to \mathcal{L}(\mathcal{D}_{n,S\cup i})^{[C_j,C_{j+1}]} \text{ and } \iota_{S,S'} : \mathcal{L}(\mathcal{D}_{n,S\cup i})^{[C_j,C_{j+1}]} \to \mathcal{L}(\mathcal{D}_{n,S'\cup i})^{[C_j,C_{j+1}]}.$$

For $S \subset [n]$ define an integer-valued function $\tilde{\beta}_S^i$ on the lattice of flats $\mathcal{L}(\mathcal{D}_{n,S\cup i})^{[C_j,C_{j+1}]}$ as

$$\tilde{\beta}_S^i(F) := \begin{cases} \mu_{\left(\mathcal{L}(\mathcal{D}_{n,S\cup i})^{[C_j,C_{j+1}]}\right)}(F) - \mu_{\left(\mathcal{L}(\mathcal{D}_{n,S})^{[C_j,C_{j+1}]}\right)}(F) \text{ for } F \in \mathcal{L}(\mathcal{D}_{n,S})^{[C_j,C_{j+1}]} \\ \mu_{\left(\mathcal{L}(\mathcal{D}_{n,S\cup i})^{[C_j,C_{j+1}]}\right)}(F) \text{ for } F \in \mathcal{L}(\mathcal{D}_{n,S\cup i})^{[C_j,C_{j+1}]} \setminus \mathcal{L}(\mathcal{D}_{n,S})^{[C_j,C_{j+1}]} \end{cases}$$

Note that the characteristic polynomial of a lattice is given by a sum over the lattice above of the values of the Möbius function with weights $t^{\mathrm{rk}(F)-\mathrm{rk}(C_j)}$ and the reduced characteristic polynomial is obtained by dividing this sum by $t-1$. Therefore $\beta_S^{[C_{j-1},C_j]}(t)$ is obtained by a linear operation applied to $\tilde{\beta}_S^i$. To show the equality $\beta_S^{[C_{j-1},C_j]}(t) = \beta_{S'}^{[C_{j-1},C_j]}(t)$ it is thus enough to show the equality of $\tilde{\beta}_S^i$ and $\tilde{\beta}_{S'}^i$ on the lattice $\mathcal{L}(\mathcal{D}_{n,S'\cup i})^{[C_j,C_{j+1}]}$, where we view $\tilde{\beta}_S^i$ as a function on the lattice $\mathcal{L}(\mathcal{D}_{n,S'\cup i})^{[C_j,C_{j+1}]}$ via a "pushforward operation" induced by $\iota_{S,S'}$, which assigns the value $\tilde{\beta}_S^i(F)$ for a flat F lying in the image of $\iota_{S,S'}$ and zero otherwise.

We now examine the properties of those two functions. First note that by Theorem 4.38 and an analogous property for the respective Möbius functions we have

$$\sum_{G\subseteq F} \tilde{\beta}_{S'}^i(G) = 0 \tag{3}$$

for any F such that $i \notin F$ or $j \notin F$ for some $j \neq i$. Similarly,

$$\sum_{G \subseteq F} \tilde{\beta}^i_S(G) = 0 \tag{4}$$

for any F such that $j \notin F$ for $j \in (S' \cup i) \setminus S$. Furthermore by Theorem 4.37 the functions $\tilde{\beta}^i_S$ and $\tilde{\beta}^i_{S'}$ agree on F such that $j \notin F$ for $j \notin S \cup i$, since the lattices of the contracted matroids are isomorphic and thus the equality holds for two pairs of Möbius functions. Furthermore, $\tilde{\beta}^i_S(F) = \tilde{\beta}^i_S(F) = 0$ for any F such that $i \notin F$. Thus the equality of the two Möbius functions follows by induction from Equation (3) and Equation (4). This proves Theorem 4.41. □

Remark 4.42. Consider $\mathcal{A}$ and $H_1, \ldots, H_s$ as in Theorem 3.13. The proof of Theorem 4.33 can be generalised using the same arguments as given above to show that

$$H_{\mathcal{A} \cup \{H_1, \ldots, H_s\}}(t) = H_{\mathcal{A}}(t) + \sum_{i=1}^{s} H_{\mathcal{A} \cup \{H_i\}}(t) - H_{\mathcal{A}}(t).$$

Since the proof of Theorem 4.33 depends solely on the properties of the lattices of flats of the arrangements, there are further generalizations of this kind to lattices. It seems plausible to show similar arithmeticity statements on intermediate arrangements for some other valuative invariants of matroids.

References

[AHK18] K. Adiprasito, J. Huh, and E. Katz. "Hodge theory for combinatorial geometries". In: *Ann. of Math.* 188.2.2 (2018), pp. 381–452.

[Ath18] C. A. Athanasiadis. "Gamma-positivity in combinatorics and geometry". In: *Séminaire Lotharingien de Combinatoire* 77 (2018), Article B77i.

[Bre94] F. Brenti. "q-Eulerian polynomials arising from Coxeter groups". In: *European Journal of Combinatorics* 15 (1994), pp. 417–441.

[BV25] P. Brändén and L. Vecchi. *Chow polynomials of uniform matroids are real-rooted.* 2025. arXiv: `2501.07364 [math.CO]`.

[CH15] M. Cuntz and I. Heckenberger. "Finite Weyl groupoids". In: *J. Reine Angew. Math.* 702 (2015), pp. 77–108.

[Cun12] M. Cuntz. "Simplicial Arrangements with up to 27 Lines". In: *Discrete & Computational Geometry* 48 (2012).

[Cun22] M. Cuntz. "A greedy algorithm to compute arrangements of lines in the projective plane". In: *Discrete Comput. Geom.* 68.1 (2022), pp. 107–124.

[Deg+25] S. Degen, L. Henetmayr, M. Mišinová, P. Pielasa, and F. Rieg. *Chow polynomials and gamma-vectors of restrictions of reflection arrangements.* `https://github.com/sdegen95/Chow-polynomials-and-gamma-vectors-of-restrictions-of-reflection-arrangements`. GitHub. Sept. 2025.

[DGP19] E. Delucchi, N. Girard, and G. Paolini. "Shellability of posets of labeled partitions and arrangements defined by root systems". In: *The Electronic Journal of Combinatorics* 26 (2019).

[Fer+24] L. Ferroni, J. P. Matherne, M. Stevens, and L. Vecchi. "Hilbert–Poincaré series of matroid Chow rings and intersection cohomology". In: *Advances in Mathematics* 449 (2024).

[FY04] E. M. Feichtner and S. Yuzvinsky. "Chow rings of toric varieties defined by atomic lattices". In: *Inventiones Mathematicae* 155.3 (2004).

[Gal05] S. R. Gal. "Real Root Conjecture Fails for Five- and Higher-Dimensional Spheres". In: *Discrete Camp: Computational Geometry* 34.2 (2005), pp. 269–284.

[Grü09] B. Grünbaum. "A catalogue of simplicial arrangements in the real projective plane". In: *Ars Mathematica Contemporanea* 2 (2009).

[Hos24] E. Hoster. *The Chow and augmented Chow polynomials of uniform matroids.* 2024. arXiv: `2410.22329 [math.CO]`.

[Hum90] J. E. Humphreys. *Reflection groups and Coxeter groups.* Vol. 29. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1990.

[Kle64] V. Klee. “A combinatorial proof of Poincar´e’s duality theorem”. In: *Can. J. Math.* 16 (1964), pp. 517–531.

[Kör+25a] V. Körber, T. Schnieders, J. Stricker, and J. Walizadeh. *Hamiltonian Cycles in Simplicial and Supersolvable Hyperplane Arrangements*. 2025. arXiv: `2508.14538 [math.CO]`.

[Kör+25b] V. Körber, T. Schnieders, J. Stricker, and J. Walizadeh. *Hamiltonian Cycles in Simplicial Arrangements*. `https://github.com/Tobias271828/Hamilton-Cycles-in-Simplicial-Arrangements`. GitHub. Mar. 2025.

[MS22] S. Manecke and R. Sanyal. *Inscribable Fans II: Inscribed zonotopes, simplicial arrangements, and reflection groups*. 2022. arXiv: `2203.11062 [math.MG]`.

[OT13] P. Orlik and H. Terao. “Arrangements of Hyperplanes”. In: *Springer Science & Business Media* (2013).

[Oxl11] J. Oxley. “Matroid Theory”. In: *Oxford Graduate Text in Mathematics* 2nd edition (2011).

[PRW08] A. Postnikov, V. Reiner, and L. Williams. “Faces of generalized permutohedra”. In: *Documenta Mathematica* 13 (2008).

[Sta97] R. P. Stanley. *Enumerative Combinatorics, Volume 1*. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1997.

[Ste08] J. R. Stembridge. “Coxeter cones and their h-vectors”. In: *Advances in Mathematics* 217.5 (2008).

[Ste21] M. Stevens. *Real-rootedness conjectures in matroid theory*. `https://mcs0042.github.io/bachelor.pdf`. 2021.

[Stu25] C. Stump. “Chow and augmented Chow polynomials as evaluations of Poincaré-extended ab-indices”. In: *Advances in Mathematics* 482 (2025), p. 110618.

Appendix A. Small catalogue of Chow polynomials of $\mathcal{D}_{n,s}$- arrangements

We provide a small catalogue of Chow polynomials of matroids arising from $\mathcal{D}_{n,s}$ arrangements up to dimension $n = 7$.

The Chow polynomials in below Tables 2 to 4 were computed using the computer algebra system `SageMath`, based on the previously introduced EL-labeling (Theorem 4.21) and the formula from Theorem 4.2. The code used for these computations is available in the GitHub repository [Deg+25].

s	$H_{\mathcal{D}_{n,s}}(t)$
0	$t+1$
1	$t+1$
2	$t+1$

s	$H_{\mathcal{D}_{n,s}}(t)$
0	t^2+8t+1
1	$t^2+10t+1$
2	$t^2+12t+1$
3	$t^2+14t+1$

s	$H_{\mathcal{D}_{n,s}}(t)$
0	$t^3+59t^2+59t+1$
1	$t^3+69t^2+69t+1$
2	$t^3+79t^2+79t+1$
3	$t^3+89t^2+89t+1$
4	$t^3+99t^2+99t+1$

Table 2. Chow polynomials for $n = 2, 3, 4$.

s	$H_{\mathcal{D}_{n,s}}(t)$
0	$t^4+382t^3+1722t^2+382t+1$
1	$t^4+430t^3+2010t^2+430t+1$
2	$t^4+478t^3+2298t^2+478t+1$
3	$t^4+526t^3+2586t^2+526t+1$
4	$t^4+574t^3+2874t^2+574t+1$
5	$t^4+622t^3+3162t^2+622t+1$

s	$H_{\mathcal{D}_{n,s}}(t)$
0	$t^5+2515t^4+35956t^3+35956t^2+2515t+1$
1	$t^5+2771t^4+40932t^3+40932t^2+2771t+1$
2	$t^5+3027t^4+45908t^3+45908t^2+3027t+1$
3	$t^5+3283t^4+50884t^3+50884t^2+3283t+1$
4	$t^5+3539t^4+55860t^3+55860t^2+3539t+1$
5	$t^5+3795t^4+60836t^3+60836t^2+3795t+1$
6	$t^5+4051t^4+65812t^3+65812t^2+4051t+1$

Table 3. Chow polynomials for $n = 5, 6$.

s	$H_{\mathcal{D}_{n,s}}(t)$
0	$t^6+17824t^5+665107t^4+1980008t^3+665107t^2+17824t+1$
1	$t^6+19362t^5+742263t^4+2229164t^3+742263t^2+19362t+1$
2	$t^6+20900t^5+819419t^4+2478320t^3+819419t^2+20900t+1$
3	$t^6+22438t^5+896575t^4+2727476t^3+896575t^2+22438t+1$
4	$t^6+23976t^5+973731t^4+2976632t^3+973731t^2+23976t+1$
5	$t^6+25514t^5+1050887t^4+3225788t^3+1050887t^2+25514t+1$
6	$t^6+27052t^5+1128043t^4+3474944t^3+1128043t^2+27052t+1$
7	$t^6+28590t^5+1205199t^4+3724100t^3+1205199t^2+28590t+1$

Table 4. Chow polynomials for $n = 7$