HELICAL VORTEX FILAMENTS WITH COMPACTLY SUPPORTED CROSS-SECTIONAL VORTICITY FOR THE INCOMPRESSIBLE EULER EQUATIONS IN $\mathbb{R}^3$

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ABSTRACT. We revisit the vortex filament conjecture for three-dimensional inviscid and incompressible Euler flows with helical symmetry and no swirl. Using gluing arguments, we provide the first construction of a smooth helical vortex filament in the whole space $\mathbb{R}^3$ whose cross-sectional vorticity is compactly supported in $\mathbb{R}^2$ for all times. The construction extends to a multi-vortex solution comprising several helical filaments arranged along a regular polygon. Our approach yields fine asymptotics for the vorticity cores, thus improving related variational results for smooth solutions in bounded helical domains and infinite pipes, as well as non-smooth vortex patches in the whole space.

1. INTRODUCTION

We consider the incompressible Euler equations in $\mathbb{R}^3$, which govern the evolution of an incompressible and inviscid fluid with constant density. Given a smooth initial velocity field $u_0(x)$, the Cauchy problem for the Euler system in vorticity formulation reads

$$\begin{aligned} \vec{\omega}_t + (\mathbf{u}\cdot\nabla)\vec{\omega} &= (\vec{\omega}\cdot\nabla)\mathbf{u} && \text{in } \mathbb{R}^3\times(0,T), \\ \mathbf{u} = \nabla\times\vec{\psi},\ -\Delta\vec{\psi} &= \vec{\omega} && \text{in } \mathbb{R}^3\times(0,T), \\ \vec{\omega}(\cdot,0) &= \nabla\times u_0 && \text{in } \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where $\mathbf{u}:\mathbb{R}^3\times[0,T)\to\mathbb{R}^3$ represents the velocity field, $\vec{\omega}=\nabla\times\mathbf{u}:\mathbb{R}^3\times[0,T)\to\mathbb{R}^3$ denotes the vorticity field, and $\vec{\psi}:\mathbb{R}^3\times[0,T)\to\mathbb{R}^3$ is the corresponding stream function of the fluid.

We study smooth solutions of (1.1) with highly concentrated vorticity, particularly those known as *vortex filaments*, corresponding to solutions whose vorticity is sharply concentrated in a thin tube around a smooth evolving curve in $\mathbb{R}^3$. The rigorous analysis of such flows is challenging and dates back to the works of Helmholtz [27] and Kelvin.

In their classical works [10, 32], Da Rios and Levi-Civita used potential theory to formally show that if the vorticity concentrates in a tube of radius $\varepsilon>0$ around a curve $\mathcal{G}(t)$, then its motion follows a binormal curvature flow with speed of order $|\log\varepsilon|$. Specifically, for an arclength parametrization $\gamma(s,t)$, they derived

$$\partial_t\gamma = 2\bar{c}|\log\varepsilon|\,(\gamma_s\times\gamma_{ss}) \quad \text{as} \quad \varepsilon\to 0,$$

where $\bar{c}$ is a circulation constant. Rescaling time via $t=\frac{\tau}{|\log\varepsilon|}$ yields the binormal flow

$$\partial_\tau\gamma = 2\bar{c}\kappa\mathbf{B}_{\mathcal{G}(\tau)}, \tag{1.2}$$

where κ is the curvature and $\mathbf{B}_{\mathcal{G}(\tau)}$ the binormal unit vector. This result identifies (1.2) as the effective dynamics of a vortex filament, in analogy with the Helmholtz-Kirchhoff system for 2D point vortices [33, 34, 37]. Under an a priori concentration assumption, Jerrard and Seis [28] provided the first rigorous justification of (1.2).

In this context, a central open problem is the *vortex filament conjecture*, which posits that solutions of (1.1) initially concentrated near a curve evolving by (1.2) remain concentrated for some time. Although still open in general, significant progress has been achieved for certain symmetric curves by using explicit solutions of (1.2), such as translating circles and helices.

For the case of circular filaments, Fraenkel [21] rigorously constructed *vortex rings* traveling at speed $\mathcal{O}(|\log\varepsilon|)$, thereby confirming the conjecture for translating circles; see also [2, 16, 20, 35] for subsequent developments. For helical filaments, the first rigorous construction of *vortex helices* was obtained in [15], validating Joukowsky's early predictions [29], with further extensions concerning multi-helix clusters in [23] and nearly parallel helical configurations in [24], providing a rigorous justification of the model in [31]. Moreover, related works examine helical dynamics in infinite pipes and bounded helical domains [5, 7, 8, 17], including vortex patches, whereas [36] removes previous orthogonality constraints by constructing helical vortices with non-vanishing swirl. Finally, global well-posedness and long-time behaviour are addressed in [3, 25, 26], while the analysis of helical structures in models closely related to the Euler equations can be found in [9, 12, 22, 38].

1.1. **Setting of the problem.** In this work, we revisit the vortex filament conjecture for the case of translating–rotating helices, and provide a novel construction of a smooth helical filament $\vec{\omega}_\varepsilon(x,\tau)$ in the whole $\mathbb{R}^3$, whose *cross-sectional vorticity* is compactly supported in $\mathbb{R}^2$ for all times. In concrete terms, for a small $\varepsilon>0$, the three-dimensional filament will be confined within a tubular neighbourhood of size $\varepsilon>0$ around a helix for all $\tau\in\mathbb{R}$, while its cross-sectional vorticity (obtained by intersecting the filament with a plane orthogonal to its tangent vector) will correspond to a smooth vorticity profile with compact support. In particular, there exist universal constants $C_1, C_2>0$ such that for all $\tau\in\mathbb{R}$, it holds $C_1\varepsilon\le \operatorname{diam}\left(\operatorname{supp}\vec{\omega}_\varepsilon\cap\left\{\mathbb{R}^2\times\{0\}\right\}\right)\le C_2\varepsilon$. Here and in what follows, for any bounded set A, we denote $\operatorname{diam}(A)=\sup\limits_{x,y\in A}|x-y|$.

Let us be more precise. For any $h>0$ and a fixed $r_0>0$, we consider a point $P=(r_0,0)\in\mathbb{R}^2$ and define the time-evolving curve $\mathcal{G}(\tau)$ parametrised by

$$\gamma(s,\tau)=\begin{pmatrix} r_0\cos\left(\frac{s-\sigma_1\tau}{\sqrt{h^2+r_0^2}}\right)\\ r_0\sin\left(\frac{s-\sigma_1\tau}{\sqrt{h^2+r_0^2}}\right)\\ \frac{hs+\sigma_2\tau}{\sqrt{h^2+r_0^2}}\end{pmatrix}\in\mathbb{R}^3,\quad \sigma_1=\frac{2\,\bar{c}\,h}{r_0^2+h^2},\quad \sigma_2=\frac{2\,\bar{c}\,r_0^2}{r_0^2+h^2},\tag{1.3}$$

This map gives an arclength parametrization of a circular helix with radius r_0 and height h, combining rotation and simultaneous translation. In addition, with curvature $\kappa=\frac{r_0}{r_0^2+h^2}$ and torsion $\frac{h}{r_0^2+h^2}$, a direct computation shows that γ evolves by the binormal flow (1.2).

In view of our interest in helical filaments, we restrict our analysis to a particular class of symmetry, which naturally leads to the notion of helical symmetry. Defining the rotation matrices

$$R_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\quad Q_\theta=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix},\tag{1.4}$$

we say that a scalar function $f:\mathbb{R}^3\to\mathbb{R}$ and a vector field $F:\mathbb{R}^3\to\mathbb{R}^3$ have helical symmetry, if for all $h>0$ and $\theta\in\mathbb{R}$, they satisfy

$$f\left(R_\theta x', x_3+h\theta\right)=f(x),\quad F\left(R_\theta x', x_3+h\theta\right)=Q_\theta F(x),\quad x=(x',x_3)\in\mathbb{R}^3.\tag{1.5}$$

1.2. **Statement of the main result.** The solution we construct belongs to the class of flows with helical symmetry, subject to the additional requirement that the velocity field in (1.1) is orthogonal to the tangent lines of a helix [18, 19].

The main result of this work reads as follows.

Theorem 1. *Let $r_0>0$, $h>0$, and consider the helix $\mathcal{G}(\tau)$ parametrised by (1.3), as well as the rotation matrices R_θ and Q_θ in (1.4). Then, there exist a constant $c>0$ and a smooth, global-in-time solution $\vec{\omega}_\varepsilon(x,\tau)$ to (1.1) with compactly supported cross-sectional vorticity in $\mathbb{R}^2$, such that for σ_1, σ_2 as in (1.3) and all $\tau\in\mathbb{R}$, it satisfies*

$$\vec{\omega}_\varepsilon(x,\tau)=Q_{\sigma_1\tau}\vec{\omega}_\varepsilon\left(R_{-\sigma_1\tau}x', x_3+\sigma_2\tau, 0\right),\quad \vec{\omega}_\varepsilon(x,\tau)=\vec{\omega}_\varepsilon\left(x', x_3+2\pi h,\tau\right),$$

and

$$\vec{\omega}_\varepsilon\left(x, |\log\varepsilon|^{-1}\tau\right)\rightharpoonup c\delta_{\mathcal{G}(\tau)}\mathbf{T}_{\mathcal{G}(\tau)}\quad as\quad \varepsilon\to 0$$

in the sense of measures, where $\delta_{\mathcal{G}(\tau)}$ denotes a uniform Dirac delta supported on $\mathcal{G}(\tau)$ and $\mathbf{T}_{\mathcal{G}(\tau)}$ its tangent unit vector.

The construction of Theorem 1 can be extended to a multi-vortex solution arranged along the vertices of a regular polygon. For any integer $k\ge 2$ and the helix in (1.3), we consider the curves $\mathcal{G}_j(\tau)$ parametrised by

$$\gamma_j(s,\tau)=Q_{\frac{2\pi(j-1)}{k}}\gamma(s,\tau),\quad j=1,2,\dots,k.\tag{1.6}$$

We additionally obtain the following Theorem.

Theorem 2. *Let $r_0>0$, $h>0$ and $k\ge 2$ be an integer. Consider the helices $\mathcal{G}_j(\tau)$, $j=1,\dots,k$, parametrised by (1.6), and the rotation matrices R_θ and Q_θ in (1.4). Then, there exist a constant $c>0$ and a smooth, global-in-time solution $\vec{\omega}_\varepsilon(x,\tau)$ to (1.1) with cross-sectional vorticity supported in a finite union of compact balls in $\mathbb{R}^2$, such that for σ_1, σ_2 as in (1.3) and all $\tau\in\mathbb{R}$, it satisfies*

$$\vec{\omega}_\varepsilon(x,\tau)=Q_{\sigma_1\tau}\vec{\omega}_\varepsilon\left(R_{-\sigma_1\tau}x', x_3+\sigma_2\tau, 0\right),\quad \vec{\omega}_\varepsilon(x,\tau)=\vec{\omega}_\varepsilon\left(x', x_3+2\pi h,\tau\right),$$

and

$$\vec{\omega}_\varepsilon(x, |\log\varepsilon|^{-1}\tau) \rightharpoonup c\sum_{j=1}^{k} \delta_{\mathcal{G}_j(\tau)}\mathbf{T}_{\mathcal{G}_j(\tau)} \quad \textit{as} \quad \varepsilon \to 0$$

in the sense of measures, where $\delta_{\mathcal{G}_j(\tau)}$, $j = 1, \dots, k$, *denotes a uniform Dirac delta supported on* $\mathcal{G}_j(\tau)$ *and* $\mathbf{T}_{\mathcal{G}_j(\tau)}$ *its tangent unit vector.*

We make some remarks on Theorems 1 and 2, commenting on their connection to the existing literature.

Remark 1.1. *To the best of our knowledge, the construction in Theorem 1 provides the first smooth helical filament in the whole space* $\mathbb{R}^3$ *with compactly supported cross-sectional vorticity in* $\mathbb{R}^2$ *(obtained by intersecting the filament with a plane orthogonal to its tangent vector) for all times. More precisely, for* $\varepsilon > 0$ *small, the vorticity* $\vec{\omega}_\varepsilon(x,\tau)$ *remains confined within a tube of size* $\varepsilon > 0$ *around the helix* (1.3) *for all* $\tau \in \mathbb{R}$*, with* $C_1\varepsilon \le \operatorname{diam}\left(\operatorname{supp}\vec{\omega}_\varepsilon \cap \left\{\mathbb{R}^2 \times \{0\}\right\}\right) \le C_2\varepsilon$ *for some universal constants* $C_1, C_2 > 0$*. An analogous statement is also valid for Theorem 2, where the solution remains concentrated along a polygonal configuration of multiple helices for all times, with cross-sectional vorticity supported in a finite union of compact balls.*

Remark 1.2. *A related result was obtained in* [17] *for bounded helical domains, where vorticity initially concentrated near helices of pairwise distinct radii remains concentrated, with the associated cross-sectional vorticity being supported in a thin annulus. Although the authors achieve strong radial localisation in each finite interval* $[0,T]$*, only* L^1 *control is obtained along the flow direction. In contrast, our solutions exhibit strong confinement in both directions, as the vorticity remains concentrated with cross-sectional vorticity compactly supported for all times, which seems unattainable in* [17] *due to the lack of uniform-in-time estimates. Besides, the assumption of distinct radii excludes polygonal configurations as in Theorem 2. Moreover,* [8] *is concerned with related solutions in infinite pipes obtained via variational methods. However, it is unclear whether these results can be extended to the whole* $\mathbb{R}^3$ *due to the failure of the uniform ellipticity of* L_x *in* (2.2) *in the entire* $\mathbb{R}^2$*, and the breakdown of the expansion of the Biot–Savart law used. Finally, for non-smooth vortex patches in the whole* $\mathbb{R}^3$ *obtained through variational methods and bifurcation theory, we refer to* [4, 6].

1.3. **Reduction of the problem.** We exploit the invariance of the Euler equations under helical symmetry [18, 19], taking $\mathbf{u}$ and $\vec{\omega}$ to satisfy (1.5) and the "helical no-swirl" condition

$$\mathbf{u}\cdot\vec{\xi} = 0, \quad \text{where} \quad \vec{\xi}(x) = (-x_2, x_1, h) \in \mathbb{R}^3.$$

Ettinger and Titi [19] showed that solving (1.1) reduces to a 2D transport equation, establishing global well-posedness through Yudovich's method [30]. In this framework, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x' = (x_1, x_2) \in \mathbb{R}^2$ and the rescaled time $t = |\log\varepsilon|^{-1}\tau$, we consider the scalar transport equation for $w(x',\tau)$ given by

$$\begin{cases} |\log\varepsilon| w_\tau + \nabla^\perp\psi\cdot\nabla w = 0 & \text{in} \quad \mathbb{R}^2\times(-\infty,+\infty), \\ -\operatorname{div}(K\nabla\psi) = w & \text{in} \quad \mathbb{R}^2\times(-\infty,+\infty), \end{cases} \tag{1.7}$$

with the convention $(a,b)^\perp = (b,-a)$ and $K(x_1,x_2)$ denoting the matrix

$$K(x_1,x_2) = \frac{1}{h^2+x_1^2+x_2^2}\begin{pmatrix} h^2+x_2^2 & -x_1x_2 \\ -x_1x_2 & h^2+x_1^2 \end{pmatrix}.$$

For a self-contained derivation of (1.7), we refer to [15, 19].

For R_θ denoting the rotation matrix in (1.4) and $w(x',\tau)$ solving (1.7), it was shown in [19] that the vorticity vector defined as

$$\vec{\omega}(x,\tau) = \frac{1}{h} w\left(R_{-\frac{x_3}{h}}x', \tau\right)\vec{\xi}(x), \quad x = (x', x_3) \in \mathbb{R}^3 \tag{1.8}$$

has helical symmetry and solves the Euler equations (1.1), while rotating solutions to (1.7) of the form

$$w(x',\tau) = W\left(R_{\alpha\tau}x'\right), \quad \psi(x',\tau) = \Psi\left(R_{\alpha\tau}x'\right), \quad x' = (x_1,x_2) \in \mathbb{R}^2, \tag{1.9}$$

can be obtained through the semilinear elliptic equation

$$\nabla_{\tilde{x}}\cdot(K\nabla_{\tilde{x}}\Psi) + f\left(\Psi - \frac{\alpha}{2}|\log\varepsilon||\tilde{x}|^2\right) = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \tilde{x} = R_{\alpha\tau}x'. \tag{1.10}$$

For our objectives, one must choose a suitable nonlinear function f in (1.10) so that for some $r_0 > 0$, the vorticity $W(\tilde{x}) = f\left(\Psi - \frac{\alpha}{2}|\log\varepsilon||\tilde{x}|^2\right)$ will be compactly supported around the point $P = (r_0, 0) \in \mathbb{R}^2$. In this way, the profile $w(x',\tau) = W(R_{\alpha\tau}x')$ in (1.9) will rotate rigidly around the origin with a constant rotational speed α, yielding a three-dimensional helical filament $\vec{\omega}$ in (1.8) with compactly supported cross-sections in $\mathbb{R}^2$ for all times.

Remark 1.3. *The solutions in Theorems 1 and 2 are constructed using the Inner-Outer gluing scheme, which was also employed in* [15]*. In this work, we employ a nonlinearity f with compact support in* (1.10)*, inspired by the semilinear elliptic equation $\Delta u + u_+^\gamma = 0$ in $\mathbb{R}^2$, where $u_+ = \max(0, u)$ and $\gamma > 3$. In contrast to the Liouville equation $\Delta u + e^u = 0$ in $\mathbb{R}^2$ used in* [15] *where $f(s) \sim e^s$ yields a fast-decaying vorticity, the particular choice $f(s) \sim s_+^\gamma$ gives vorticity supported within a tube of size $\varepsilon > 0$ around the helix for all times. We note that a similar nonlinearity has been used in* [13, 14] *to construct traveling vortex-pair solutions for the Euler equations in $\mathbb{R}^2$, while a nonlocal analogue was employed in* [1] *to obtain traveling and rotating solutions for the generalized inviscid SQG equation in $\mathbb{R}^2$. Nevertheless, these results cannot be adapted to our setting, since they concern bounded solutions (cf. Lemma 5.1).*

Remark 1.4. *Our constructions rely on elliptic singular perturbation methods, where a crucial step involves the linearisation around an approximate solution. In this framework, we are naturally led to study the linearised operator $\Delta + \gamma\Gamma_+^{\gamma-1}$ in $\mathbb{R}^2$, where Γ is defined in* (2.7)*. A non-degeneracy result of Dancer and Yan* [11] *indicates that the only bounded functions in the kernel of this operator in $\mathbb{R}^2$ are given by the partial derivatives of Γ. However, in our setting, the solution exhibits logarithmic growth at infinity (see Lemma 5.1), which introduces an additional radial component $Z_0 = \mathcal{O}\left((\log(2 + |y|)\right)$ in the kernel. This feature complicates our analysis, since the estimate $\int_{\mathbb{R}^2} \gamma\Gamma_+^{\gamma-1} Z_0 = O(1)$ induces a strong coupling between the Inner and Outer Equations (we refer to Section 5 and Proposition 7.1 for more details). On the other hand,* [15] *shows that the associated radial element $\tilde{Z}_0 = O(1)$ in the kernel of the linear operator $\Delta + e^u$ in $\mathbb{R}^2$ (with u solving $\Delta u + e^u = 0$ in $\mathbb{R}^2$) satisfies $\int_{\mathbb{R}^2} e^u \tilde{Z}_0 = 0$, which decouples the Inner-Outer system at main order. This obstruction implies that the construction of the helical filaments in our case is more delicate, requiring several adjustments of existing gluing methods.*

In summary, following the preceding discussion, in the remainder of this work we write x instead of $\tilde{x}$ in (1.10) for ease of notation, and focus on solving the equation

$$\nabla_x \cdot (K\nabla_x\Psi) + f\left(\Psi - \frac{\alpha}{2}|\log\varepsilon||x|^2\right) = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.11}$$

For a fixed $r_0 > 0$ and the point $P = (r_0, 0) \in \mathbb{R}^2$, our aim is to construct a stream function Ψ for (1.11) such that for some constant $c > 0$ it exhibits the asymptotic behaviour $-\nabla_x \cdot (K\nabla_x\Psi) \sim c\delta_P$ as $\varepsilon \to 0$, while we also require that the vorticity profile $W(x) = f\left(\Psi - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)$ has compact support around P. Therefore, we realise that in a small neighbourhood of the point P, the stream function Ψ must resemble a Green's function for the elliptic operator $-\nabla_x \cdot (K\nabla_x\cdot)$ in $\mathbb{R}^2$ up to lower-order corrections, capturing the locally singular structure.

1.4. **Notation.** Throughout the paper, $c > 0$ and $C > 0$ denote generic constants that may change from line to line and are independent of all relevant quantities. In addition, $B_r = B_r(0)$ denotes the open ball of radius $r > 0$ centered at the origin.

1.5. **Structure of the paper.** This work is organized as follows. In Section 2, we construct a regularized approximation of the Green's function for the operator $-\nabla_x \cdot (K\nabla_x\cdot)$ in $\mathbb{R}^2$ via a change of variables, using a radial profile Γ solving a semilinear elliptic equation. Section 3 selects a compactly supported nonlinearity f and derives error estimates for the preceding approximation. In Section 4, we apply the *Inner–Outer gluing scheme* to perturb the approximation into an exact solution, relying on the linear theories from Sections 5 and 6. Section 7 establishes existence and uniqueness for a projected linearised problem through a fixed point argument. In Section 8, we choose the rotational speed in (1.9) so that the solution of the projected problem can be lifted to a genuine solution. Finally, Section 9 outlines the proof of Theorem 2.

2. Construction of the approximate stream function

Let $r_0 > 0$, $h > 0$ and consider the point $P = (r_0, 0) \in \mathbb{R}^2$. This section is devoted to the construction of an approximate stream function Ψ satisfying

$$-\nabla_x \cdot (K\nabla_x\Psi) \sim c\delta_P \tag{2.1}$$

in a neighbourhood of the point P.

Hereafter, for ease of notation, we set

$$L_x = \nabla_x \cdot (K\nabla_x\cdot), \quad \text{with} \quad K = \frac{1}{h^2 + x_1^2 + x_2^2}\begin{pmatrix} h^2 + x_2^2 & -x_1x_2 \\ -x_1x_2 & h^2 + x_1^2 \end{pmatrix}. \tag{2.2}$$

We construct an approximate Green's function for (2.1) by studying L_x locally around $P=(r_0,0)\in\mathbb{R}^2$. Unlike the Laplacian Δ_x, this elliptic operator in divergence form has variable coefficients, producing anisotropic diffusion that degenerates for large $|x|$. Besides, the parameter h formally interpolates between the 3D axisymmetric case ($h\to 0$) and the standard planar setting ($h\to\infty$).

Unlike planar or axisymmetric settings, the helical framework lacks coordinates aligned with symmetry directions [19]. We therefore introduce the change of variables

$$x-P=A[P]z,\quad A[P]=\begin{pmatrix}\frac{h}{\sqrt{h^2+r_0^2}} & 0\\ 0 & 1\end{pmatrix},\tag{2.3}$$

originally introduced in [23]. This anisotropic scaling normalizes the coefficient matrix K in (2.2) at $P=(r_0,0)$, recasting L_x as Δ_z plus lower-order corrections in the z variable, thus simplifying the analysis of (2.1).
This is the content of the next Proposition.

Proposition 2.1. *Let z be the variable in (2.3). It holds*

$$L_x=\Delta_z+B_0,$$

where

$$\begin{aligned}B_0=&\left(\frac{h^2(r_0^2-r^2)+z_2^2(h^2+r_0^2)}{h^2(h^2+r^2)}\right)\partial_{z_1z_1}+\frac{1}{(h^2+r^2)}\left(\left(z_1\frac{h}{\sqrt{h^2+r_0^2}}+r_0\right)^2-r^2\right)\partial_{z_2z_2}\\&-2\frac{\sqrt{h^2+r_0^2}}{h(h^2+r^2)}z_2\left(z_1\frac{h}{\sqrt{h^2+r_0^2}}+r_0\right)\partial_{z_1z_2}\\&-\frac{z_1(h^2+r_0^2)+r_0h\sqrt{h^2+r_0^2}}{h^2(h^2+r^2)}\left(1+\frac{2h^2}{h^2+r^2}\right)\partial_{z_1}-\frac{z_2}{h^2+r^2}\left(1+\frac{2h^2}{h^2+r^2}\right)\partial_{z_2},\end{aligned}$$

and

$$r^2=|x|^2=r_0^2+2r_0\frac{h}{\sqrt{h^2+r_0^2}}z_1+\frac{h^2}{h^2+r_0^2}z_1^2+z_2^2.$$

Proof. The result follows from setting $(a,b)=(r_0,0)$ in ([23], Proposition 2.1). □

Due to Proposition 2.1, we realise that in the region $|z|<\delta$ for a fixed small $\delta>0$, the operator L_x is a perturbation of the usual Laplace operator in $\mathbb{R}^2$ given by

$$L_x=\Delta_z+B_0,\tag{2.4}$$

where B_0 can be expressed as

$$\begin{aligned}B_0=&\left(-2\frac{r_0h}{(h^2+r_0^2)^{\frac{3}{2}}}z_1+O(|z|^2)\right)\partial_{z_1z_1}+\mathcal{O}(|z|^2)\partial_{z_2z_2}-\left(2\frac{r_0}{h\sqrt{h^2+r_0^2}}z_2+\mathcal{O}(|z|^2)\right)\partial_{z_1z_2}\\&-\left(\frac{r_0}{h\sqrt{h^2+r_0^2}}\left(1+\frac{2h^2}{h^2+r_0^2}\right)+\mathcal{O}(|z|)\right)\partial_{z_1}-\left(\frac{z_2}{h^2+r_0^2}\left(1+\frac{2h^2}{h^2+r_0^2}\right)+\mathcal{O}\left(|z|^2\right)\right)\partial_{z_2},\end{aligned}$$

with no constant multiples of second-order derivatives appearing.
In addition, (2.1) is now equivalent to

$$-(\Delta_z+B_0)\psi=c\delta_0\quad\text{and}\quad\psi(z)=\Psi(P+A[P]z),\tag{2.5}$$

where δ_0 is a Dirac delta centered at the origin, and $A[P]$ is the matrix associated with the change of variables in (2.3). From this perspective, obtaining an approximate regularisation of the Green's function of the operator L_x in $\mathbb{R}^2$ leads to constructing a regularisation of the Green's function of the Laplace operator Δ_z in $\mathbb{R}^2$.

Accordingly, we now introduce the regularising profile associated with Δ_z in $\mathbb{R}^2$ as follows. For any $\gamma>3$ and $s_+=\max\{0,s\}$, consider the semilinear elliptic equation

$$\Delta_z\Gamma+\Gamma_+^\gamma=0\quad\text{on }\mathbb{R}^2,\quad\{\Gamma>0\}=B_1(0),\tag{2.6}$$

which admits a radially symmetric, classical solution of the form

$$\Gamma(z)=\begin{cases}\nu(|z|) & \text{if } |z|\le 1\\ \nu'(1)\log|z| & \text{if } |z|>1\end{cases},\tag{2.7}$$

where ν denotes the unique, radial ground state solution of
$$\Delta_z \nu + \nu^\gamma = 0, \quad \nu \in H_0^1(B_1), \quad \nu > 0 \text{ on } B_1.$$
Due to Hopf's lemma we get $\nu'(1) < 0$, which will be an important feature for the analysis carried out in Section 3. Moreover, for any $\varepsilon > 0$, it is easy to see that the rescaled profile
$$\Gamma_\varepsilon(z) = \Gamma\left(\frac{z}{\varepsilon}\right) \tag{2.8}$$
is C^1 on the boundary of the ball $B_\varepsilon(0)$ and satisfies the rescaled semilinear elliptic equation
$$\Delta_z \Gamma_\varepsilon(z) + \frac{1}{\varepsilon^2}\left(\Gamma_\varepsilon(z)\right)_+^\gamma = 0 \quad \text{in } \mathbb{R}^2. \tag{2.9}$$
In what follows, we restrict our attention to the region $|z| < \delta$ for a small $\delta > 0$, and define
$$\hat{\Gamma}_\varepsilon(z) = \Gamma\left(\frac{z}{\varepsilon}\right) - \nu'(1)|\log\varepsilon|, \tag{2.10}$$
which satisfies
$$-\Delta_z \hat{\Gamma}_\varepsilon(z) \rightharpoonup c\delta_0 \quad \text{as} \quad \varepsilon \to 0, \quad \text{where} \quad c = \int_{\mathbb{R}^2} \Gamma_+^\gamma.$$
Motivated by the semilinear equation (2.6), we use (2.10) as the fundamental building block and employ elliptic singular perturbation methods to construct an approximate solution of (2.5).

The following Proposition holds.

Proposition 2.2. *Given $r_0 > 0$ and $h > 0$, consider the profile $\hat{\Gamma}_\varepsilon(z)$ in (2.10) and define the approximate solution of (2.5) as*
$$\psi_{3\varepsilon}(z) = \frac{\alpha}{2}|\log\varepsilon| r_0^2 + \hat{\Gamma}_\varepsilon(z)\left(1 + c_1 z_1 + c_2|z|^2\right) + \frac{r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}} H_{1\varepsilon}(z),$$
where
$$c_1 = \frac{r_0 h}{2(h^2+r_0^2)^{\frac{3}{2}}}, \quad c_2 = \frac{3h^2 r_0^2 + r_0^4}{8(h^2+r_0^2)^3},$$
and
$$\Delta_z H_{1\varepsilon} + \frac{\mathrm{Re}(z^3)}{\varepsilon^2|z|^2}\left(\Gamma''\left(\frac{z}{\varepsilon}\right) - \frac{\Gamma'\left(\frac{z}{\varepsilon}\right)}{\frac{z}{\varepsilon}}\right) = 0.$$
It the rescaled variable $y = \frac{z}{\varepsilon}$ it holds $\psi_{3\varepsilon}(\varepsilon y_1, \varepsilon y_2) = \psi_{3\varepsilon}(\varepsilon y_1, -\varepsilon y_2)$, while for any small $\delta > 0$, in the region $|y| < \frac{\delta}{\varepsilon}$ we have
$$\varepsilon^2 L_x(\psi_{3\varepsilon})(\varepsilon y) = \Delta_y \Gamma(y) + \frac{3r_0h^2 + r_0^3}{2h(h^2+r_0)^{\frac{3}{2}}}\varepsilon y_1\left(\Gamma(y)\right)_+^\gamma + \varepsilon^2 E_*,$$
where E_* is a smooth function in $z = \varepsilon y$, uniformly bounded as $\varepsilon \to 0$.

In the original variable x, the approximate regularisation can be expressed as
$$\Psi_P(x) = \psi_{3\varepsilon}\left(A[P]^{-1}(x-P)\right),$$
where $P = (r_0, 0) \in \mathbb{R}^2$ and $A[P]$ is given in (2.3).

Proof. Let $\delta > 0$ be small. We introduce the rescaled variable $y = \frac{z}{\varepsilon}$, consider the region $|y| < \frac{\delta}{\varepsilon}$, and use (2.4) in order to compute
$$\begin{aligned}\varepsilon^2 B_0(\varepsilon y)[\hat{\Gamma}_\varepsilon] = &-\frac{2r_0h}{(h^2+r_0^2)^{\frac{3}{2}}}\varepsilon y_1 \partial_{y_1y_1}\hat{\Gamma}_\varepsilon - \frac{2r_0}{h\sqrt{h^2+r_0^2}}\varepsilon y_2 \partial_{y_1y_2}\hat{\Gamma}_\varepsilon \\ &- \frac{r_0}{h\sqrt{h^2+r_0^2}}\left(1 + \frac{2h^2}{h^2+r_0^2}\right)\varepsilon\partial_{y_1}\hat{\Gamma}_\varepsilon + \varepsilon^2 E_1,\end{aligned} \tag{2.11}$$
where E_1 is a smooth function in $z = \varepsilon y$, uniformly bounded as $\varepsilon \to 0$.

Since $\hat{\Gamma}_\varepsilon$ in (2.10) is radial in y, we then calculate
$$\partial_{y_1}\hat{\Gamma}_\varepsilon = \Gamma'\frac{y_1}{|y|}, \quad \partial_{y_1y_1}\hat{\Gamma}_\varepsilon = \left(\Gamma'' - \frac{\Gamma'}{|y|}\right)\frac{y_1^2}{|y|^2} + \frac{\Gamma'}{|y|}, \quad \partial_{y_1y_2}\hat{\Gamma}_\varepsilon = \left(\Gamma'' - \frac{\Gamma'}{|y|}\right)\frac{y_1y_2}{|y|^2},$$
where Γ' designates the radial derivative of $\Gamma(y) = \Gamma_\varepsilon(z)$.

Substituting the above into (2.11) and identifying the vector $y = (y_1, y_2) \in \mathbb{R}^2$ with the complex number $y = y_1 + iy_2 \in \mathbb{C}$, by virtue of the identities

$$y_1 y_2^2 = \frac{y_1|y|^2}{4} - \frac{\mathrm{Re}(y^3)}{4}, \quad y_1^3 = \frac{3y_1|y|^2}{4} + \frac{\mathrm{Re}(y^3)}{4},$$

we further find

$$\begin{aligned}\varepsilon^2 B_0(\varepsilon y)[\hat{\Gamma}_\varepsilon] = &- \frac{r_0 h}{(h^2+r_0^2)^{\frac{3}{2}}} \frac{\Gamma' \varepsilon y_1}{|y|} - \frac{r_0^3 + 4r_0 h^2}{2h(h^2+r_0^2)^{\frac{3}{2}}} \left(\Gamma'' + \frac{\Gamma'}{|y|}\right) \varepsilon y_1 \\ &+ \frac{\varepsilon r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}} \left(\Gamma'' - \frac{\Gamma'}{|y|}\right) \frac{\mathrm{Re}(y^3)}{|y|^2} + \varepsilon^2 E_1,\end{aligned} \tag{2.12}$$

with $\mathrm{Re}(y^3)$ denoting the real part of y^3 and E_1 as in (2.11).

To proceed, we modify the approximation to eliminate the first error term in (2.12). Defining

$$\psi_{1\varepsilon}(\varepsilon y) = \frac{\alpha}{2}|\log \varepsilon| r_0^2 + (1 + c_1 \varepsilon y_1)\hat{\Gamma}_\varepsilon,$$

we observe that

$$\Delta_y(c_1 \varepsilon y_1 \hat{\Gamma}_\varepsilon) = c_1 \varepsilon y_1 \Delta_y \Gamma + 2c_1 \frac{\Gamma' \varepsilon y_1}{|y|},$$

hence choosing $c_1 = \frac{1}{2}\frac{r_0 h}{(h^2+r_0^2)^{\frac{3}{2}}}$ yields

$$\begin{aligned}\varepsilon^2 L_x(\psi_{1\varepsilon}) = \Delta_y \Gamma &- \left(\frac{3r_0 h^2 + r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}\right)\left(\Gamma'' + \frac{\Gamma'}{|y|}\right)\varepsilon y_1 + \frac{\varepsilon r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}\left(\Gamma'' - \frac{\Gamma'}{|y|}\right)\frac{\mathrm{Re}(y^3)}{|y|^2} \\ &+ \varepsilon^2 B_0(\varepsilon y)\left[c_1 \varepsilon y_1 \hat{\Gamma}_\varepsilon\right] + \varepsilon^2 E_1.\end{aligned}$$

Following a similar reasoning as before, one can show that

$$\varepsilon^2 B_0(\varepsilon y)[c_1 \varepsilon y_1 \hat{\Gamma}_\varepsilon] = -\frac{c_1 \varepsilon^2 r_0}{h\sqrt{h^2+r_0^2}}\left(1 + \frac{2h^2}{h^2+r_0^2}\right)\hat{\Gamma}_\varepsilon + \varepsilon^2 E_2,$$

where E_2 is another smooth function in εy which is uniformly bounded as $\varepsilon \to 0$, while

$$\begin{aligned}\varepsilon^2 L_x(\psi_{1\varepsilon}) = \Delta_y \Gamma &- \frac{3r_0 h^2 + r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}\left(\Gamma'' + \frac{\Gamma'}{|y|}\right)\varepsilon y_1 - \frac{\varepsilon^2(3h^2 r_0^2 + r_0^4)}{2(h^2+r_0^2)^3}\hat{\Gamma}_\varepsilon \\ &+ \frac{\varepsilon r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}\left(\Gamma'' - \frac{\Gamma'}{|y|}\right)\frac{\mathrm{Re}(y^3)}{|y|^2} + \varepsilon^2(E_1 + E_2).\end{aligned} \tag{2.13}$$

The next step for improving the approximation is to eliminate the third error term in (2.13), which is proportional to $\hat{\Gamma}_\varepsilon$. We notice that

$$\Delta_y\left(c_2 \varepsilon^2 |y|^2 \hat{\Gamma}_\varepsilon\right) = c_2 \varepsilon^2 \left(4\hat{\Gamma}_\varepsilon + 4y \cdot \nabla_y \Gamma + |y|^2 \Delta_y \Gamma\right),$$

thus we adjust the approximation by setting

$$\psi_{2\varepsilon}(\varepsilon y) = \frac{\alpha}{2}|\log \varepsilon| r_0^2 + \left(1 + c_1 \varepsilon y_1 + c_2 \varepsilon^2 |y|^2\right)\hat{\Gamma}_\varepsilon, \quad c_2 = \frac{1}{8}\frac{3h^2 r_0^2 + r_0^4}{(h^2+r_0^2)^3}.$$

With this modification, we infer that the approximate regularisation $\psi_{2\varepsilon}$ satisfies

$$\begin{aligned}\varepsilon^2 L_x(\psi_{2\varepsilon}) = \Delta_y \Gamma &- \frac{3r_0 h^2 + r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}\left(\Gamma'' + \frac{\Gamma'}{|y|}\right)\varepsilon y_1 + \frac{\varepsilon r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}\left(\Gamma'' - \frac{\Gamma'}{|y|}\right)\frac{\mathrm{Re}(y^3)}{|y|^2} \\ &+ \varepsilon^2(E_1 + E_2 + E_3),\end{aligned} \tag{2.14}$$

where E_3 has the same properties as E_1 and E_2.

To cancel the third term in (2.14), we introduce a radial correction $h_{1\varepsilon}(s)$ defined as the solution of

$$h_{1\varepsilon}'' + \frac{1}{s}h_{1\varepsilon}' - \frac{9}{s^2}h_{1\varepsilon} + \frac{s}{\varepsilon^2}\left(\Gamma''\left(\frac{s}{\varepsilon}\right) - \frac{\Gamma'\left(\frac{s}{\varepsilon}\right)}{\frac{s}{\varepsilon}}\right) = 0,$$

which can be expressed as

$$h_{1\varepsilon}(s) = s^3 \int_s^1 \frac{\mathrm{d}r}{r^7} \int_0^r \frac{t^5}{\varepsilon^2} \left(\Gamma''\left(\frac{t}{\varepsilon}\right) - \frac{\Gamma'\left(\frac{t}{\varepsilon}\right)}{\frac{t}{\varepsilon}} \right) \mathrm{d}t.$$

We remark that $\tilde{F}_\varepsilon(s) := \frac{s}{\varepsilon^2}\left(\Gamma''\left(\frac{s}{\varepsilon}\right) - \frac{\Gamma'\left(\frac{s}{\varepsilon}\right)}{\frac{s}{\varepsilon}}\right)$ is uniformly bounded as $\varepsilon \to 0$. To see this, since $\varepsilon \to 0$ corresponds to $u := \frac{s}{\varepsilon} \to \infty$, using the definition of Γ in (2.7) one can find that $u\left(\Gamma''(u) - \frac{\Gamma'(u)}{u}\right) = \mathcal{O}\left(\frac{1}{u}\right)$ as $u \to \infty$. In addition, we observe that $\lim_{u \to 0} \frac{\Gamma'(u)}{u} = \Gamma''(0)$, which in turn yields $\left(\Gamma''(u) - \frac{\Gamma'(u)}{u}\right) = \mathcal{O}(u)$ as $u \to 0$. Combining these asymptotics, it can be verified that $h_{1\varepsilon}$ is a smooth function in s, uniformly bounded as $\varepsilon \to 0$, satisfying $h_{1\varepsilon}(s) = \mathcal{O}(s^3)$ as $s \to 0$.

Identifying again the vector $(y_1, y_2) \in \mathbb{R}^2$ with $y_1 + iy_2 \in \mathbb{C}$ and turning to polar coordinates $y = |y|e^{i\theta}$, the function $H_{1\varepsilon}(z) = h_\varepsilon(|z|)\cos(3\theta)$ satisfies the equation

$$\Delta_y H_{1\varepsilon} + \frac{\varepsilon\,\mathrm{Re}(y^3)}{|y|^2}\left(\Gamma''(y) - \frac{\Gamma'(y)}{|y|}\right) = 0,$$

thus we refine the approximation of the stream function as

$$\psi_{3\varepsilon}(\varepsilon y) = \frac{\alpha}{2}|\log\varepsilon| r_0^2 + \left(1 + c_1\varepsilon y_1 + c_2\varepsilon^2|y|^2\right)\hat{\Gamma}_\varepsilon + \frac{r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}} H_{1\varepsilon}(\varepsilon y). \tag{2.15}$$

Combining the results above, straightforward computations show that

$$\varepsilon^2 L_x\left(\psi_{3\varepsilon}\right) = \Delta_y \Gamma + \frac{3r_0h^2 + r_0^3}{2h(h^2+r_0)^{\frac{3}{2}}}\varepsilon y_1 \Gamma_+^\gamma + \varepsilon^2 E_*, \tag{2.16}$$

where E_* is once again a smooth function in εy, uniformly bounded as $\varepsilon \to 0$.

Referring to the change of variables in (2.3), we can express the approximate stream function (2.15) around $P = (r_0, 0) \in \mathbb{R}^2$ as

$$\Psi_P(x) = \psi_{3\varepsilon}\left(A[P]^{-1}(x - P)\right),$$

which concludes the proof. □

2.1. **Globally defined approximate regularisation.** At this point, in order to extend the definition of the approximate stream function of Proposition 2.2 to the whole $\mathbb{R}^2$, we introduce a smooth cut-off function

$$\eta_\delta(x) = \eta\left(\frac{|z|}{\delta}\right), \quad \text{where} \quad \eta(s) = 1 \quad \text{for} \quad |s| \le \frac{1}{2}, \quad \eta(s) = 0 \quad \text{for} \quad |s| \ge 1, \tag{2.17}$$

and consider the modified approximation

$$\tilde{\Psi}_\alpha(x) = \frac{\alpha}{2}|\log\varepsilon| r_0^2 + \eta_\delta(x)\left(\left(1 + c_1 z_1 + c_2|z|^2\right)\hat{\Gamma}_\varepsilon(z) + \frac{r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}} H_{1\varepsilon}(z)\right). \tag{2.18}$$

Since the operator L_x in (2.2) is linear, it follows from (2.15) and (2.18) that

$$L_x(\tilde{\Psi}_\alpha) = \eta_\delta(x)\left(L_x\left(\psi_{3\varepsilon}\right) - E_*\right) + g(x), \tag{2.19}$$

with E_* as in (2.16) and

$$g(x) = \eta_\delta(x)E_* + L_x\left(\eta_\delta(x)\left(\psi_{3\varepsilon} - \frac{\alpha}{2}|\log\varepsilon| r_0^2\right)\right) - \eta_\delta(x)L_x\left(\psi_{3\varepsilon} - \frac{\alpha}{2}|\log\varepsilon| r_0^2\right).$$

One can verify that $g(x)$ has compact support, thus we readily get $\|g\|_{L^\infty(\mathbb{R}^2)} \le C_\delta$ for a constant depending on $\delta > 0$.

The next step towards a global improvement of the approximation (2.18) is to add a correction $H_{2\varepsilon}(x)$ in the original variable x, to cancel the bounded error term $g(x)$ in (2.19). To be precise, we let $H_{2\varepsilon}(x)$ satisfy the Poisson equation

$$L_x(H_{2\varepsilon}) + g(x) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{2.20}$$

where an application of Proposition 5.1 guarantees a positive solution to (2.20) satisfying the bound

$$|H_{2\varepsilon}| \le C_\delta\left(1 + |x|^2\right). \tag{2.21}$$

Moreover, as the solution is determined only up to an additive constant, we select the particular one satisfying

$$H_{2\varepsilon}(r_0, 0) = 0. \tag{2.22}$$

In terms of the variable $z=\varepsilon y$ and (2.3), the final global approximation reads

$$\Psi_\alpha(x)=\frac{\alpha}{2}|\log\varepsilon|r_0^2+\eta_\delta(x)\left(\left(1+c_1z_1+c_2|z|^2\right)\hat{\Gamma}_\varepsilon(z)+\frac{r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}H_{1\varepsilon}(z)\right)+H_{2\varepsilon}(x),\tag{2.23}$$

with Ψ_α being even in x_2, i.e. $\Psi_\alpha(x_1,-x_2)=\Psi_\alpha(x_1,x_2)$. This property holds since $\psi_{3\varepsilon}$ in (2.15) is even in $x_2=\varepsilon y_2$, while $H_{2\varepsilon}(x)$ inherits the same symmetry due to $g(x)$ being even in x_2.

Finally, due to Proposition 2.2, the action of the operator L_x to the final approximation Ψ_α in (2.23) is equal to

$$L_x(\Psi_\alpha)=\eta\left(\frac{|z|}{\delta}\right)\left(\Delta_z\Gamma_\varepsilon+\frac{3r_0h^2+r_0^3}{2h(h^2+r_0^2)^{\frac{3}{2}}}\frac{z_1}{\varepsilon^2}\left(\Gamma_\varepsilon\right)_+^\gamma\right),\tag{2.24}$$

with $\Gamma_\varepsilon(z)$ as in (2.8).

3. Choice of Nonlinearity and Error of Approximation

In this section, we select a nonlinearity f and estimate the error when evaluating (1.11) at Ψ_α from (2.23). We define the error operator

$$S(\Psi)(x)=L_x(\Psi)+f\left(\Psi-\frac{\alpha}{2}|\log\varepsilon||x|^2\right),\tag{3.1}$$

where exact solutions satisfy $S(\Psi)=0$. As discussed in Section 1.3, we choose f so that the vorticity $W(x)=f\left(\Psi-\frac{\alpha}{2}|\log\varepsilon||x|^2\right)$ concentrates near $P=(r_0,0)$, with $W\rightharpoonup c\delta_P$ as $\varepsilon\to 0$, yielding a 3D helical filament with compactly supported cross-section in $\mathbb{R}^2$.

Motivated by (2.9), we introduce the nonlinearity

$$f(s)=\frac{1}{\varepsilon^2}\left(s+\nu'(1)|\log\varepsilon|\right)_+^\gamma,\tag{3.2}$$

where $\gamma>3$ and $s_+=\max(s,0)$. One can verify that this nonlinearity has sufficient regularity for the construction of a smooth vorticity, since $s\mapsto s_+^\gamma\in C^{\gamma-1}$ if $\gamma\in\mathbb{Z}$ and $s\mapsto s_+^\gamma\in C^{\lfloor\gamma\rfloor,\gamma-\lfloor\gamma\rfloor}$ otherwise.

In the sequel, for given $r_0>0$, $h>0$, and recalling that $\nu'(1)<0$, we fix the rotational speed in (1.9) as

$$\alpha=-\frac{\nu'(1)}{2\left(h^2+r_0^2\right)}+\tilde{\alpha},\quad\text{where}\quad\tilde{\alpha}=\mathcal{O}\left(|\log\varepsilon|^{-1}\right).\tag{3.3}$$

The rigorous justification of this particular choice will be carried out in Section 8, while currently our aim is to derive error estimates for $S\left(\Psi_\alpha\right)$.

We have the following Proposition.

Proposition 3.1. *Consider $r_0>0$, $h>0$, and let α be the rotational speed in (3.3). For the approximate stream function Ψ_α in (2.23) and the function f in (3.2), it holds that for sufficiently small $\delta>0$, there exists a constant $C>0$ such that for all $\varepsilon>0$ small, the error function in (3.1) satisfies*

$$\varepsilon^2S\left(\Psi_\alpha\right)\le C\varepsilon|y|\Gamma_+^\gamma,$$

where $y=\frac{z}{\varepsilon}$ and $\operatorname{supp}\Gamma_+^\gamma\subset B_1(0)$.

Proof. To estimate the error of approximation

$$\varepsilon^2S(\Psi_\alpha)=\varepsilon^2L_x(\Psi_\alpha)+\left(\Psi_\alpha-\frac{\alpha}{2}|\log\varepsilon||x|^2+\nu'(1)|\log\varepsilon|\right)_+^\gamma,$$

it is necessary to consider three different regions in the $y=\frac{z}{\varepsilon}$ variable as follows.
Case 1: $|y|\le 1$.
Due to (2.17), in this region we have $\eta_\delta=1$, thus using (2.24) we find

$$\begin{aligned}\varepsilon^2S\left(\Psi_\alpha\right)=&\ \Delta_y\Gamma+\left(\frac{3r_0h^2+r_0^3}{2h\left(h^2+r_0^2\right)^{\frac{3}{2}}}\right)\varepsilon y_1\Gamma_+^\gamma\\&+\left[\frac{\alpha}{2}|\log\varepsilon|r_0^2+\left(\Gamma(y)-\nu'(1)|\log\varepsilon|\right)\left(1+c_1\varepsilon y_1+c_2\varepsilon^2|y|^2\right)+\frac{r_0^3}{2h\left(h^2+r_0^2\right)^{\frac{3}{2}}}H_{1\varepsilon}(\varepsilon y)\right.\\&\left.+H_{2\varepsilon}(x)-\frac{\alpha}{2}|\log\varepsilon||x|^2+\nu'(1)|\log\varepsilon|\right]_+^\gamma.\end{aligned}$$

Using the asymptotic expansions

$$H_{2\varepsilon}(x) = H_{2\varepsilon}(P) + \varepsilon\left(A[P]y\right)\cdot\nabla H_{2\varepsilon}(P) + \mathcal{O}\left(\varepsilon^2|y|^2\right), \quad H_{1\varepsilon} = \mathcal{O}(\varepsilon^3|y|^3) \quad \text{as} \quad \varepsilon\to 0, \tag{3.4}$$

a Taylor approximation around $\Gamma(y)$ yields

$$\begin{aligned}\varepsilon^2 S\left(\Psi_\alpha\right) = \Delta_y\Gamma + \left(\frac{3r_0h^2+r_0^3}{2h\left(h^2+r_0^2\right)^{\frac{3}{2}}}\right)\varepsilon y_1\Gamma_+^\gamma + \Gamma_+^\gamma + \gamma\Gamma_+^{\gamma-1}\Bigg[\varepsilon y_1|\log\varepsilon|\left(-c_1\nu'(1) - \frac{\alpha r_0 h}{\sqrt{h^2+r_0^2}}\right)\\ + \varepsilon y_1\left(c_1\Gamma(y) + \frac{h}{\sqrt{h^2+r_0^2}}\partial_{y_1}H_{2\varepsilon}(P)\right) + \mathcal{O}\left(\varepsilon^2|\log\varepsilon||y|^2\right)\Bigg] + \mathcal{O}\left(\gamma(\gamma-1)\Gamma_+^{\gamma-2}\varepsilon^2|y|^2\right),\end{aligned}$$

where we used (2.22), the fact that $H_{2\varepsilon}(x)$ is even in x_2 and $|x|^2 = r_0^2 + 2r_0\frac{h}{\sqrt{h^2+r_0^2}}\varepsilon y_1 + \frac{h^2}{h^2+r_0^2}\varepsilon^2y_1^2 + \varepsilon^2y_2^2$.

Employing (2.6) and (3.3), one can directly verify that

$$\varepsilon^2 S(\Psi_\alpha)(\varepsilon y) = \mathcal{O}\left(\varepsilon|y|\Gamma_+^\gamma\right). \tag{3.5}$$

Case 2: $1 < |y| < \frac{\delta}{\varepsilon}$.
In this intermediate region we have that $\eta_\delta(x)\in(0,1]$ and $\Gamma(y) = \nu'(1)\log|y| < 0$, thus from (2.7) we obtain $L_x(\Psi_\alpha) = 0$. Using (3.3) and (3.4), the error function takes the form

$$\begin{aligned}\varepsilon^2 S\left(\Psi_\alpha\right) &= \Bigg[\eta_\delta(\nu'(1)\log(|y|) - \nu'(1)|\log\varepsilon|)\left(1 + c_1\varepsilon y_1 + c_2\varepsilon^2|y|^2\right) + \varepsilon y_1\frac{h}{\sqrt{h^2+r_0^2}}\partial_{y_1}H_{2\varepsilon}(P)\\ &\quad + \mathcal{O}(\varepsilon^2|y|^2) - \frac{\alpha\, r_0 h}{\sqrt{h^2+r_0^2}}\varepsilon y_1|\log\varepsilon| - \frac{\alpha}{2}\varepsilon^2|\log\varepsilon|\left(\frac{h^2}{h^2+r_0^2}y_1^2 + y_2^2\right) + \nu'(1)|\log\varepsilon|\Bigg]_+^\gamma\\ &= \left(\nu'(1)|\log\varepsilon|\tilde{E}(y)\right)_+^\gamma,\end{aligned}$$

where

$$\begin{aligned}\tilde{E}(y) = \eta_\delta\left(\frac{\log(|y|)}{|\log\varepsilon|}\left(1 + c_1\varepsilon y_1 + c_2\varepsilon^2|y|^2\right) - \left(1 + c_1\varepsilon y_1 + c_2\varepsilon^2|y|^2\right)\right) + \mathcal{O}(\delta)\\ + \mathcal{O}\left(\frac{\delta}{|\log\varepsilon|}\right) + \mathcal{O}\left(\frac{1}{|\log\varepsilon|}\right) + \frac{\varepsilon^2}{4(h^2+r_0^2)}\left(\frac{h^2}{h^2+r_0^2}y_1^2 + y_2^2\right) + 1.\end{aligned}$$

Due to the expansion $\frac{\log(|y|)}{|\log\varepsilon|} = 1 + \mathcal{O}\left(\frac{\log\delta}{|\log\varepsilon|}\right)$, a careful analysis shows that for $\delta > 0$ sufficiently small, there exists $\varepsilon_0 > 0$ such that $\tilde{E}(y) > 0$ for all $\varepsilon\in(0,\varepsilon_0)$. Since $\nu'(1)|\log\varepsilon| < 0$, it then follows immediately that

$$\varepsilon^2 S\left(\Psi_\alpha\right) = 0, \quad \text{for all} \quad 1 < |y| < \frac{\delta}{\varepsilon}. \tag{3.6}$$

Case 3: $|y| \geq \frac{\delta}{\varepsilon}$.
In this unbounded region we have $\eta_\delta(x) = 0$, thus using (2.21) we infer

$$\varepsilon^2 S\left(\Psi_\alpha\right) = \left(\frac{\alpha}{2}|\log\varepsilon|r_0^2 + H_{2\varepsilon}(x) - \frac{\alpha}{2}|\log\varepsilon||x|^2 + \nu'(1)|\log\varepsilon|\right)_+^\gamma = 0, \tag{3.7}$$

since for sufficiently small $\varepsilon > 0$ the dominant term is given by $-\frac{\alpha}{2}|\log\varepsilon||x|^2$.

Collecting (3.5),(3.6) and (3.7), we globally write

$$\varepsilon^2 S(\Psi_\alpha) \leq C\varepsilon|y|\Gamma_+^\gamma,$$

which concludes the proof. □

4. Inner-Outer Gluing Scheme

In this section, we consider the approximate stream function Ψ_α in (2.23) satisfying the error estimates of Proposition 3.1, and look for an exact solution Ψ of the equation

$$S\left(\Psi\right)(x) = L_x\left(\Psi\right) + f\left(\Psi - \frac{\alpha}{2}|\log\varepsilon||x|^2\right) = 0 \quad \text{in} \quad \mathbb{R}^2, \quad f(s) = \frac{1}{\varepsilon^2}\left(s + \nu'(1)|\log\varepsilon|\right)_+^\gamma, \tag{4.1}$$

as a small perturbation around the approximate solution constructed. We recall that the rotational speed is fixed as $\alpha = \frac{-\nu'(1)}{2(h^2+r_0^2)} + \mathcal{O}\left(|\log\varepsilon|^{-1}\right)$ in (3.3).

Using the *Inner–Outer gluing scheme*, which is a perturbative method for concentration problems in nonlinear PDEs, we seek a solution as a small perturbation of Ψ_α of the form

$$\Psi = \Psi_\alpha + \varphi(x), \quad \varphi(x) = \tilde{\eta}_\delta(x)\phi_{in}\left(\frac{z}{\varepsilon}\right) + \phi_{out}(x), \tag{4.2}$$

where

$$\tilde{\eta}_\delta(x) = \eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right), \tag{4.3}$$

with η defined in (2.17). Notice that the cut-off $\tilde{\eta}_\delta(x)$ considered here is shorter than the one used in (2.23), with this choice localising the contribution of the perturbation ϕ_{in} in a ball of radius $\frac{\delta}{|\log\varepsilon|^2}$ around the point of concentration.

We emphasise that the specific structure of the perturbation $\varphi(x)$ is fundamental to the gluing scheme. In particular, the perturbation splits into an inner function ϕ_{in} in the expanded variable $y = \frac{z}{\varepsilon}$, designed to provide the local correction near the concentration point, as well as a function $\phi_{out}(x)$ in the original variable x, which accounts for the cancellation of the far-field error.

By substituting (4.2) into (4.1) and linearising with respect to φ, the problem (4.1) reads

$$S(\Psi_\alpha + \varphi) = \mathcal{E}_\alpha + \mathcal{L}_{\Psi_\alpha}[\varphi] + N_{\Psi_a}[\varphi] = 0 \quad \text{in} \quad \mathbb{R}^2,$$

where

$$\begin{aligned}
&\mathcal{E}_\alpha = S(\Psi_\alpha),\\
&\mathcal{L}_{\Psi_\alpha}[\varphi] = L_x[\varphi] + f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)\varphi,\\
&N_{\Psi_\alpha}[\varphi] = f\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2 + \varphi\right) - f\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right) - f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)\varphi.
\end{aligned}$$

The problem can be equivalently rearranged as

$$\begin{aligned}
S(\Psi_\alpha + \varphi) &= 0 \quad \text{in} \quad \mathbb{R}^2,\\
S(\Psi_\alpha + \varphi) &= \tilde{\eta}_\delta\left[L_x[\phi_{in}] + f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)(\phi_{in} + \phi_{out}) + S(\Psi_\alpha) + N_{\Psi_\alpha}(\varphi)\right]\\
&\quad + (1-\tilde{\eta}_\delta)\left[f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)\phi_{out} + S(\Psi_\alpha) + N_{\Psi_\alpha}(\varphi)\right]\\
&\quad + L_x[\phi_{out}] + L_x[\tilde{\eta}_\delta\phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}].
\end{aligned}$$

We then realise that Ψ in (4.2) is an exact solution of (4.1) if and only if (ϕ_{in}, ϕ_{out}) in the decomposition of φ solve the coupled system of equations

$$L_x[\phi_{in}] + f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)(\phi_{in} + \phi_{out}) + S(\Psi_\alpha) + N_{\Psi_\alpha}(\varphi) = 0, \quad |z| < \frac{2\delta}{|\log\varepsilon|^2}, \tag{4.4}$$

and

$$\begin{aligned}
&L_x[\phi_{out}] + (1-\tilde{\eta}_\delta)\left[f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)\phi_{out} + S(\Psi_\alpha) + N_{\Psi_\alpha}(\varphi)\right]\\
&\qquad + L_x[\tilde{\eta}_\delta\phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}] = 0, \quad \text{in } \mathbb{R}^2.
\end{aligned} \tag{4.5}$$

In this context, we refer to (4.4) as the *Inner problem* and to (4.5) as the *Outer problem*.

For our analysis, it is relevant to recast (4.4) in the expanded variable $y = \frac{z}{\varepsilon}$, where $|y| < \frac{2\delta}{\varepsilon|\log\varepsilon|^2}$. According to Proposition 2.1, for any smooth function $\bar{\phi}(z) = \phi(y)$ in this region, one can observe that $\varepsilon^2 L_x(\bar{\phi}(\varepsilon y)) = \Delta_y\bar{\phi} + \tilde{B}_0(y)[\bar{\phi}]$, where $\tilde{B}_0 = \varepsilon^2 B_0(\varepsilon y)$, see (2.4).

In addition, for $|y| < \frac{2\delta}{\varepsilon|\log\varepsilon|^2}$ we can expand

$$\varepsilon^2 f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right) = \gamma\Gamma_+^{\gamma-1} + b_0(y),$$

where

$$\begin{aligned}
b_0(y) = \gamma(\gamma-1)\Gamma_+^{\gamma-2}&\Bigg[\varepsilon|\log\varepsilon|y_1\left(-c_1\nu'(1) - \frac{\alpha r_0 h}{\sqrt{h^2+r_0^2}}\right) + \varepsilon y_1\left(c_1\Gamma(y) + \frac{h}{\sqrt{h^2+r_0^2}}\partial_{y_1}H_{2\varepsilon}(P)\right)\\
&+ \mathcal{O}\left(\varepsilon^2|\log\varepsilon||y|^2\right)\Bigg] + \mathcal{O}\left(\varepsilon^2|y|^2\Gamma_+^{\gamma-3}\right),
\end{aligned} \tag{4.6}$$

and $b_0(y_1, y_2) = b_0(y_1, -y_2)$.

Arguing in a similar manner as above, the nonlinear quadratic term has the expansion

$$\mathcal{N}_{\Psi_\alpha}(\varphi) = \varepsilon^2 N_{\Psi_\alpha}(\tilde{\eta}_\delta \phi_{in} + \phi_{out}) = \left(\gamma \Gamma_+^{\gamma-1} + b_0(y)\right) \mathcal{O}\left(\varphi^2\right),$$

hence if we multiply equation (4.4) by ε^2, the Inner problem can be expressed as

$$\Delta_y \phi_{in} + \gamma \Gamma_+^{\gamma-1} \phi_{in} + \bar{B}[\phi_{in}] + \mathcal{N}_{\Psi_\alpha}(\tilde{\eta}_\delta \phi_{in} + \phi_{out}) + \varepsilon^2 S(\Psi_\alpha) + \left(\gamma \Gamma_+^{\gamma-1} + b_0(y)\right) \phi_{out} = 0 \quad \text{in } B_\rho, \tag{4.7}$$

where $\rho = \frac{2\delta}{\varepsilon|\log\varepsilon|^2}$ and $\bar{B}[\phi_{in}](y) = \varepsilon^2 B_0(\varepsilon y)[\phi_{in}] + b_0(y)\phi_{in}(y)$, as in (2.4) and (4.6).

As far as the Outer problem in (4.5) is concerned, to simplify notation we write

$$L_x[\phi_{out}] + G_{out}(\phi_{in}, \phi_{out}) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{4.8}$$

where

$$\begin{aligned} G_{out}(\phi_{in}, \phi_{out}) = (1-\tilde{\eta}_\delta)\left[f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right)\phi_{out} + S(\Psi_\alpha) + N_{\Psi_\alpha}(\tilde{\eta}_\delta \phi_{in} + \phi_{out})\right] \\ + L_x[\tilde{\eta}_\delta \phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}]. \end{aligned}$$

4.1. **Solving the Outer problem.** To address the coupled system of equations in (4.7) and (4.8), we begin by solving the Outer Problem in (4.8). For $\beta \in (0,1)$, we seek a solution

$$\phi_{out} \in \mathcal{B}_{100} := \left\{ f \in C^{1,\beta}(\mathbb{R}^2) : \|(1+|x|^2)^{-1} f\|_\infty \leq 100 \right\}, \tag{4.9}$$

where $\mathcal{B}_{100}$ is a closed ball of the space $X := \left\{ f \in C^{1,\beta}(\mathbb{R}^2) : \|(1+|x|^2)^{-1} f\|_\infty < \infty \right\}$.

Regarding ϕ_{in}, we assume a priori that in the rescaled variable $y = \frac{z}{\varepsilon}$ it satisfies

$$(1+|y|)|D_y \phi_{in}(y)| + |\phi_{in}(y)| \leq \frac{C\varepsilon}{1+|y|^\sigma}, \quad \phi_{in}(y_1, y_2) = \phi_{in}(y_1, -y_2), \tag{4.10}$$

for some $\sigma > 0$. These assumptions are motivated by the estimates of $\varepsilon^2 S(\Psi_\alpha)$ in Proposition 3.1 and the fact that $\varepsilon^2 S(\Psi_\alpha)$ is even in y_2.

Under these assumptions, the following lemma holds.

Lemma 4.1. *Let ϕ_{out} as in (4.9) and ϕ_{in} satisfy (4.10). For $\delta > 0$ sufficiently small and the cut-off functions η_δ, $\tilde{\eta}_\delta$ defined in (2.17) and (4.3), there exists a small $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$G_{out}(\phi_{in}, \phi_{out}) = L_x[\tilde{\eta}_\delta \phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}].$$

Proof. Since $(1-\tilde{\eta}_\delta) \neq 0$ if and only if $|z| > \frac{\delta}{2|\log\varepsilon|^2}$, we employ the estimate $\varepsilon^2 S(\Psi_\alpha) \leq C\varepsilon|y|\Gamma_+^\gamma$ of Proposition 3.1 to deduce that

$$(1-\tilde{\eta}_\delta) S(\Psi_\alpha) = 0, \quad (1-\tilde{\eta}_\delta) f'\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2\right) = 0,$$

since $\operatorname{supp} S(\Psi_\alpha)(z) \subset B_\varepsilon(0)$. It remains to examine the term $(1-\tilde{\eta}_\delta) N_{\Psi_\alpha}(\tilde{\eta}_\delta \phi_{in} + \phi_{out})$, where we distinguish between three separate regions in the z variable, naturally suggested by the cut-offs $\tilde{\eta}_\delta$ and η_δ.

Case 1: $\frac{\delta}{2|\log\varepsilon|^2} < |z| \leq \frac{\delta}{|\log\varepsilon|^2}$.

In this region we have $\eta_\delta = 1$ and $\tilde{\eta}_\delta \neq 0$, thus using (3.3), (3.4) and the variable $y = \frac{z}{\varepsilon}$, we get

$$\begin{aligned} &(1-\tilde{\eta}_\delta) N_{\Psi_\alpha} = (1-\tilde{\eta}_\delta)\, f\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2 + \tilde{\eta}_\delta \phi_{in} + \phi_{out}\right) \\ &= \frac{1-\tilde{\eta}_\delta}{\varepsilon^2}\Bigg[\Big(\nu'(1)\log(|y|) - \nu'(1)|\log\varepsilon|\Big)\left(1 + c_1\varepsilon y_1 + c_2\varepsilon^2|y|^2\right) + \frac{h}{\sqrt{h^2+r_0^2}}\varepsilon y_1 \partial_{y_1} H_{2\varepsilon}(P) + \mathcal{O}(\varepsilon^2|y|^2) \\ &\qquad + \frac{\nu'(1) r_0 h}{2(h^2+r_0^2)^{\frac{3}{2}}}\varepsilon y_1 |\log\varepsilon| + \mathcal{O}(\varepsilon|y|) + \frac{\nu'(1)\varepsilon^2|\log\varepsilon|}{4(h^2+r_0^2)}\left(\frac{h^2}{h^2+r_0^2} y_1^2 + y_2^2\right) + \tilde{\eta}_\delta \phi_{in}(y) + \phi_{out}(x)\Bigg]_+^\gamma. \end{aligned}$$

Using (4.9) and (4.10), we then have

$$\begin{aligned} (1-\tilde{\eta}_\delta) N_{\Psi_\alpha} = \frac{1}{\varepsilon^2}\Bigg[\nu'(1)|\log\varepsilon|\Bigg(\left(\frac{\log(|y|)}{|\log\varepsilon|} - 1\right)\left(1 + c_1\varepsilon y_1 + c_2\varepsilon^2|y|^2\right) \\ + \mathcal{O}\left(\frac{\delta}{|\log\varepsilon|}\right) + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon^{1+\sigma}|\log\varepsilon|^{2\sigma-1}) + \mathcal{O}\left(\frac{1+|x|^2}{|\log\varepsilon|}\right)\Bigg)\Bigg]_+^\gamma. \end{aligned}$$

Similarly to the proof of Proposition 3.1, choosing $\delta > 0$ sufficiently small yields $(1-\tilde{\eta}_\delta)N_{\Psi_\alpha} = 0$ for all $\varepsilon > 0$ small.

Case 2: $\frac{\delta}{|\log\varepsilon|^2} < |z| < \delta$.

In this intermediate region it holds $\eta_\delta \neq 0$ and $\tilde{\eta}_\delta = 0$, so we instead find

$$\begin{aligned}(1-\tilde{\eta}_\delta)N_{\Psi_\alpha} &= f\left(\Psi_\alpha - \frac{\alpha}{2}|\log\varepsilon||x|^2 + \phi_{out}\right)\\ &= \frac{1}{\varepsilon^2}\Bigg[\nu'(1)|\log\varepsilon|\left(\frac{r_0^2}{2(h^2+r_0^2)} + \eta_\delta\left(1+c_1\varepsilon y_1 + c_2\varepsilon^2|y|^2\right)\left(\frac{\log(|y|)}{|\log\varepsilon|}-1\right)\right.\\ &\quad \left.+\,\mathcal{O}\left(\frac{\delta}{|\log\varepsilon|}\right) + \mathcal{O}(\delta) + \frac{\varepsilon^2}{4(h^2+r_0^2)}\left(\frac{h^2}{h^2+r_0^2}y_1^2 + y_2^2\right) + \mathcal{O}\left(\frac{1+|x|^2}{|\log\varepsilon|}\right)+1\right)\Bigg]_+^\gamma.\end{aligned}$$

Using an analogous argument as before, we get that for sufficiently small $\delta > 0$ it holds $(1-\tilde{\eta}_\delta)N_{\Psi_\alpha} = 0$ for all $\varepsilon > 0$ small.

Case 3: $|z| \geq \delta$.

In this unbounded region we have $\eta_\delta = 0$ and $\tilde{\eta}_\delta = 0$, thus using (4.9) and (2.21) yields

$$(1-\tilde{\eta}_\delta)\,N_{\Psi_\alpha} = \frac{1}{\varepsilon^2}\left(\frac{\alpha}{2}|\log\varepsilon| r_0^2 + H_{2\varepsilon}(x) - \frac{\alpha}{2}|\log\varepsilon||x|^2 + \phi_{out}(x)\right)_+^\gamma = 0,$$

since for sufficiently small $\varepsilon > 0$ the dominant term is $-\frac{\alpha}{2}|\log\varepsilon||x|^2 < 0$. □

A direct Corollary of Lemma 4.1 is that if ϕ_{out} and ϕ_{in} satisfy (4.9) and (4.10) respectively, then the Outer problem (4.8) reduces to

$$L_x[\phi_{out}] + L_x[\tilde{\eta}_\delta\phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}] = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{4.11}$$

In other words, choosing a priori the appropriate topologies for ϕ_{in} and ϕ_{out} leads to a significant simplification, as the structure of the Poisson equation (4.11) allows to use directly Proposition 5.1 to establish the existence of a solution, without invoking a fixed point scheme. In particular, since the term $L_x[\tilde{\eta}_\delta\phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}]$ is independent of ϕ_{out}, the previous conclusion holds provided that this term has sufficiently fast decay at infinity.

To verify this, a substitution in (2.4) gives

$$\begin{aligned}L_x\left[\tilde{\eta}_\delta\phi_{in}\right] - \tilde{\eta}_\delta L_x\left[\phi_{in}\right] = \nabla_z\eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right)\cdot\nabla_z\phi_{in}\left(\frac{z}{\varepsilon}\right) + \eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right)\Delta_z\phi_{in}\left(\frac{z}{\varepsilon}\right)\\ + \phi_{in}\left(\frac{z}{\varepsilon}\right)\Delta_z\eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right) + B_0\left[\eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right)\phi_{in}\left(\frac{z}{\varepsilon}\right)\right]\\ - \eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right)B_0\left[\phi_{in}\left(\frac{z}{\varepsilon}\right)\right].\end{aligned} \tag{4.12}$$

One can deduce that $L_x[\tilde{\eta}_\delta\phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}]$ is compactly supported, hence there exists a constant $\bar{\nu} > 2$ such that

$$\left|L_x[\tilde{\eta}_\delta\phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}]\right| < \frac{C}{1+|x|^{\bar{\nu}}}.$$

Furthermore, for $\sigma > 0$ as in (4.10), direct computations yield

$$\sup_{x\in\mathbb{R}^2}\left|L_x\left[\tilde{\eta}_\delta\phi_{in}\right] - \tilde{\eta}_\delta L_x\left[\phi_{in}\right]\right| < C\varepsilon^{1+\sigma^*},$$

for some arbitrarily small $0 < \sigma^* < \sigma$. For instance, using (4.10) and the variable $y = \frac{z}{\varepsilon}$, we can estimate

$$\left|\nabla_z\eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right)\cdot\nabla_z\phi_{in}\left(\frac{z}{\varepsilon}\right)\right| \leq \frac{C|\log\varepsilon|^2}{\varepsilon}\eta'\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right)\nabla_y\phi_{in} \leq o(1)\frac{\varepsilon^{1+\sigma^*}}{1+|x|^{\bar{\nu}}},$$

for $o(1)\to 0$ as $\varepsilon \to 0$, $\bar{\nu} > 2$ and $0 < \sigma^* < \sigma$ arbitrarily small, with similar estimates for the remaining terms in (4.12).

As a result, Proposition 5.1 establishes the existence of a solution $\phi_{out} \in C^{1,\beta}(\mathbb{R}^2)$ for any $0 < \beta < 1$ to the Poisson equation (4.11), with the bound

$$|\phi_{out}| \leq C\varepsilon^{1+\sigma^*}\left(1+|x|^2\right). \tag{4.13}$$

Finally, since the solution is given up to a constant, we choose

$$\phi_{out}(r_0, 0) = 0. \tag{4.14}$$

As it will be made evident in Proposition 7.1, the property (4.14) for ϕ_{out} is crucial for establishing existence and uniqueness of a solution of the Inner–Outer system in (4.4) and (4.11), as it yields sufficient decoupling of the equations and allows to employ a fixed point argument.

5. Inner and Outer Linear Theories

5.1. **Linear Outer Theory.** Recalling the elliptic differential operator in divergence form L_x defined in (2.2), we consider the Poisson equation

$$L_x[\phi_{out}] + g(x) = 0 \quad \text{in } \mathbb{R}^2, \tag{5.1}$$

where g is a bounded function satisfying the decay condition

$$\|g\|_{\bar{\nu}} := \sup_{x\in\mathbb{R}^2} \left(1+|x|^{\bar{\nu}}\right)|g(x)| < +\infty,$$

for some $\bar{\nu} > 2$.

The following Proposition holds.

Proposition 5.1. *There exists a solution $\phi_{out}(x)$ to (5.1) belonging to the class $C^{1,\beta}(\mathbb{R}^2)$ for any $0<\beta<1$, which defines a linear operator $\phi_{out} = \mathcal{T}^o(g)$ of g and satisfies the bound*

$$|\phi_{out}(x)| \le C\|g\|_{\bar{\nu}}(1+|x|^2),$$

for a constant $C>0$.

Proof. See ([15], Proposition 7.1). □

5.2. **Linear Inner Theory.** In this section, for $y = \frac{z}{\varepsilon}$ and $|h(y)| \le \frac{C}{1+|y|^{2+\sigma}}$ for some $\sigma > 0$, we study the linearised problem

$$\Delta_y \phi_{in}(y) + \gamma\Gamma_+^{\gamma-1}\phi_{in}(y) + h(y) = 0 \quad \text{in } \mathbb{R}^2, \tag{5.2}$$

where Γ is given in (2.7).

To solve this equation, we work in a topology of decaying functions endowed with Hölder-type norms, that capture both the decay at infinity and the Hölder regularity of the functions involved.

Following this direction, for $\sigma > 0$ and $\beta \in (0,1)$ it is relevant to define the norms

$$\begin{aligned} \|h\|_{2+\sigma} &= \sup_{y\in\mathbb{R}^2}(1+|y|)^{2+\sigma}|h(y)|, \\ \|h\|_{2+\sigma,\beta} &= \|h\|_{2+\sigma} + \sup_{y\in\mathbb{R}^2}(1+|y|)^{2+\sigma+\beta}[h]_{B_1(y),\beta}, \end{aligned} \tag{5.3}$$

where for any $A \subset \mathbb{R}^2$ we use the standard semi-norm notation

$$[h]_{A,\beta} = \sup_{\ell_1\neq\ell_2,\ell_1,\ell_2\in A} \frac{|h(\ell_1)-h(\ell_2)|}{|\ell_1-\ell_2|^\beta}.$$

A well-known result of Dancer and Yan [11] establishes that the linear operator $\Delta_y + \gamma\Gamma_+^{\gamma-1}$ in (5.2) is non-degenerate, in the sense that its kernel in $L^\infty(\mathbb{R}^2)$ is spanned by the functions $Z_i(y) = \frac{\partial\Gamma}{\partial y_i}$, $i=1,2$, which is consistent with (2.6) being invariant under translations. Nevertheless, in the absence of suitable orthogonality conditions, the solution ϕ_{in} of (5.2) grows logarithmically fast at infinity, so the previous non-degeneracy result fails in our setting. In particular, owing to the logarithmic growth, the kernel of the linear operator $\Delta_y + \gamma\Gamma_+^{\gamma-1}$ in $\mathbb{R}^2$ is now spanned by

$$Z_0 = \frac{2}{\gamma-1}\Gamma(y) + y\cdot\nabla_y\Gamma(y), \quad Z_i(y) = \partial_{y_i}\Gamma(y),\ i=1,2. \tag{5.4}$$

To identify the radial element Z_0, we observe that for any $\lambda,\zeta>0$, the equation (2.9) is invariant under the scaling $y\mapsto\lambda y, \Gamma\mapsto\lambda^\zeta\Gamma$. From this perspective, we define the rescaled profile $\Gamma_\lambda(y) = \lambda^\zeta\Gamma(\lambda y)$, where balancing the coefficients of the linear and nonlinear terms gives $\zeta = \frac{2}{\gamma-1}$. Differentiating at $\lambda=1$, one then obtains the generator of the scaling

$$Z_0 = \partial_\lambda\left(\lambda^{\frac{2}{\gamma-1}}\Gamma(\lambda y)\right)\Big|_{\lambda=1} = \frac{2}{\gamma-1}\Gamma(y) + y\cdot\nabla_y\Gamma(y).$$

We remark that for $i=1,2$, it holds $Z_i = \mathcal{O}\left(\frac{1}{1+|y|}\right)$ as $|y|\to\infty$, whereas $Z_0 = \mathcal{O}\left(\log(2+|y|)\right)$ as $|y|\to\infty$. To the best of our knowledge, this is the first work in which the element Z_0 in the kernel of the linear operator $\Delta_y + \Gamma_+^{\gamma-1}$ in $\mathbb{R}^2$ cannot be discarded from the analysis, as related works were restricted to bounded solutions.

We obtain the following lemma.

Lemma 5.1. *Given $\sigma > 0$ and $\beta \in (0,1)$, consider the norms defined in (5.3). Then, there exist a constant $C > 0$ and a solution $\phi_{in} = \mathcal{T}[h]$ of equation (5.2) for each h with $\|h\|_{2+\sigma} < +\infty$, that defines a linear operator of h and satisfies the estimate*

$$\begin{aligned}&(1+|y|)|\nabla\phi_{in}(y)| + |\phi_{in}(y)|\\ &\le C\left[\log(2+|y|)\left|\int_{\mathbb{R}^2} hZ_0\right| + (1+|y|)\sum_{i=1}^{2}\left|\int_{\mathbb{R}^2} hZ_i\right| + (1+|y|)^{-\sigma}\|h\|_{2+\sigma}\right].\end{aligned}$$

In addition, if $\|h\|_{2+\sigma,\beta} < +\infty$, it holds

$$\begin{aligned}&(1+|y|^{2+\beta})[D_y^2\phi_{in}]_{B_1(y),\beta} + \left(1+|y|^2\right)\left|D_y^2\phi_{in}(y)\right|\\ &\le C\left[\log(2+|y|)\left|\int_{\mathbb{R}^2} hZ_0\right| + (1+|y|)\sum_{i=1}^{2}\left|\int_{\mathbb{R}^2} hZ_i\right| + (1+|y|)^{-\sigma}\|h\|_{2+\sigma,\beta}\right].\end{aligned}$$

Proof. Assuming that h is a complex-valued function, we turn to polar coordinates $y = re^{i\theta}$, $r = |y|$, and decompose ϕ_{in} and h in Fourier series as

$$h(y) = \sum_{k=-\infty}^{\infty} h_k(r)e^{ik\theta}, \quad \phi_{in}(y) = \sum_{k=-\infty}^{\infty} \phi_{in,k}(r)e^{ik\theta}.$$

It is then immediate that equation (5.2) decouples into modes, yielding the infinitely many ordinary differential equations

$$L_k[\phi_{in,k}] + h_k(r) = 0, \quad r \in (0,\infty), \quad k \in \mathbb{Z}, \tag{5.5}$$

where

$$L_k := \partial_{rr} + \frac{1}{r}\partial_r - \frac{k^2}{r^2} + \gamma\Gamma_+^{\gamma-1}.$$

For $k = 0$, we have that the function $z_0(r) = \frac{2}{\gamma-1}\Gamma(r) + r\Gamma'(r)$ satisfies $L_0[z_0] = 0$, thus if $\xi_0 \in (0,1)$ denotes the unique root of z_0, the function defined as

$$\phi_{in,0}(r) = -z_0(r)\int_{\xi_0}^{r}\frac{\mathrm{d}s}{sz_0(s)^2}\int_0^s h_0(\rho)z_0(\rho)\rho\,\mathrm{d}\rho$$

is a smooth solution of (5.5).

Since $\int_0^\infty h_0(\rho)z_0(\rho)\rho\,\mathrm{d}\rho = \frac{1}{2\pi}\int_{\mathbb{R}^2} h(y)Z_0(y)\,\mathrm{d}y$, we deduce that

$$|\phi_{in,0}(r)| \le C\Big[\log(2+r)\Big|\int_{\mathbb{R}^2} h(y)Z_0(y)\,\mathrm{d}y\Big| + (1+r)^{-\sigma}\|h\|_{2+\sigma}\Big].$$

For $|k| = 1$, we observe that for $z_k(r) = -\Gamma'(r)$ it holds $L_k[z_k] = 0$, so the function given by

$$\phi_{in,k}(r) = \Gamma'(r)\int_r^\infty\frac{\mathrm{d}s}{s\Gamma'(s)^2}\int_0^s h_k(\rho)\Gamma'(\rho)\rho\,\mathrm{d}\rho$$

solves (5.5) for $k = -1, 1$, and satisfies

$$|\phi_{in,k}(r)| \le C\Big[(1+r)\sum_{j=1}^{2}\Big|\int_{\mathbb{R}^2} h(y)Z_j(y)\,\mathrm{d}y\Big| + (1+r)^{-(1+\sigma)}\|h\|_{2+\sigma}\Big].$$

For $k = 2$, standard asymptotics give the existence of a function $z_2(r)$ such that $L_2[z_2] = 0$ and $z_2(r) = \mathcal{O}(r^2)$ as $r \to 0$ and $r \to \infty$, while for $|k| \ge 2$ we have that the function

$$\bar\phi_{in,k}(r) = \frac{4}{k^2}z_2(r)\int_0^r\frac{\mathrm{d}s}{sz_2(s)^2}\int_0^s |h_k(\rho)|z_2(\rho)\rho\,\mathrm{d}\rho$$

is a positive supersolution for equation (5.5). As such, we get the existence of a unique solution $\phi_{in,k}$ with $|\phi_{in,k}(r)| \le \bar\phi_{in,k}(r)$, so that

$$|\phi_{in,k}(r)| \le \frac{C}{k^2}(1+r)^{-\sigma}\|h\|_{2+\sigma}, \quad |k| \ge 2.$$

Combining the above, it is clear that the function

$$\phi_{in}(y) = \sum_{k=-\infty}^{\infty}\phi_{in,k}(r)e^{ik\theta}$$

solves (5.2) and defines a linear operator of h , while adding the individual estimates for each mode we infer that

$$|\phi_{in}(y)| \;\le\; C\left[\log(2+|y|)\left|\int_{\mathbb{R}^2} hZ_0\right| + (1+|y|)\sum_{i=1}^{2}\left|\int_{\mathbb{R}^2} hZ_i\right| + (1+|y|)^{-\sigma}\|h\|_{2+\sigma}\right]. \tag{5.6}$$

To proceed, assuming that $\|h\|_{2+\sigma,\beta} < +\infty$, we now turn our attention to proving similar estimates for the first-order and second-order derivatives of ϕ_{in}. To do so, we let $R = |y| \gg 1$, fix a direction $\vec{e} = \frac{y}{|y|} \in \mathbb{S}^1$ so that $y = R\vec{e}$, and define the rescaled functions

$$\phi_R(\ell) = R^{\sigma}\phi_{in}\left(y + R\ell\right), \quad h_R(\ell) = R^{2+\sigma} h\left(y+R\ell\right).$$

In the neighbourhood $U_y := \{y + R\ell : |\ell| < \frac{1}{2}\}$, a direct calculation shows that

$$\Delta_\ell \phi_R(\ell) + R^2\gamma\Gamma_+^{\gamma-1}(y+R\ell)\phi_R(\ell) + h_R(\ell) = 0.$$

Setting

$$\mu_i := \Big|\int_{\mathbb{R}^2} hZ_i\Big|, \quad i = 0,1,2,$$

we employ (5.6) and a standard elliptic estimate to obtain

$$\|\nabla_\ell \phi_R\|_{L^\infty\left(B_{\frac{1}{4}}(0)\right)} + \|\phi_R\|_{L^\infty\left(B_{\frac{1}{2}}(0)\right)} \;\le\; C\Big[\mu_0 R^{\sigma}\log R + \sum_{i=1}^{2}\mu_i R^{1+\sigma} + \|h\|_{2+\sigma}\Big], \tag{5.7}$$

where we used that $\|h_R\|_{L^\infty\left(B_{\frac{1}{2}}(0)\right)} \le C\|h\|_{2+\sigma}$.

In addition, since it also holds $[h_R(\ell)]_{B_{\frac{1}{2}}(0),\beta} \le C\|h\|_{2+\sigma,\beta}$, an application of interior Schauder estimates together with the bounds in (5.7) yields

$$\|D_\ell^2\phi_R\|_{L^\infty\left(B_{\frac{1}{4}}(0)\right)} + [D_\ell^2\phi_R]_{B_{\frac{1}{4}}(0),\beta} \;\le\; C\Big[\mu_0 R^{\sigma}\log R + \sum_{i=1}^{2}\mu_i R^{1+\sigma} + \|h\|_{2+\sigma,\beta}\Big]. \tag{5.8}$$

Collecting together (5.6), (5.7) and (5.8) we get the desired bounds, hence the proof is completed. □

6. Projected Linearised Inner Problem

In the current section, we fix a sufficiently small $\delta > 0$, and for $\rho = \frac{2\delta}{\varepsilon|\log\varepsilon|^2}$ and scalars $d_j, j = 0,1,2$, we focus on the projected linearised equation

$$\Delta_y\phi_{in} + \gamma\Gamma_+^{\gamma-1}\phi_{in} + \bar{B}[\phi_{in}] + h(y) = \sum_{j=0}^{2} d_j\gamma\Gamma_+^{\gamma-1}Z_j \quad \text{in} \quad B_\rho. \tag{6.1}$$

For convenience, we recall that the functions $Z_j, j = 1,2$ are defined in (5.4), while

$$\bar{B}[\phi_{in}] = b_0(y)\phi_{in} + \tilde{B}_0[\phi_{in}], \tag{6.2}$$

with $b_0(y) = \mathcal{O}\left(\varepsilon|y|\Gamma_+^{\gamma-2}\right)$ and $\tilde{B}_0 = \varepsilon^2 B_0(\varepsilon y)$, see (2.4) and (4.6).

For $\sigma > 0$ and $\beta \in (0,1)$, we also adapt the norms in (5.3) to be defined on any open set $A \subset \mathbb{R}^2$, by introducing

$$\|h\|_{2+\sigma,A} = \sup_{y\in A}(1+|y|)^{2+\sigma}|h(y)|, \quad \|h\|_{2+\sigma,\beta,A} = \sup_{y\in A}(1+|y|)^{2+\sigma+\beta}[h]_{B_1(y)\cap A} + \|h\|_{2+\sigma,A}, \tag{6.3}$$

while for $\phi_{in} \in C^{2,\beta}\left(\mathbb{R}^2\right)$ we write

$$\|\phi_{in}\|_{*,\sigma,A} = \|D_y^2\phi_{in}\|_{2+\sigma,\beta,A} + \|D_y\phi_{in}\|_{1+\sigma,A} + \|\phi_{in}\|_{\sigma,A}. \tag{6.4}$$

In what follows, for the sake of simplicity we omit the subscript when $A = \mathbb{R}^2$.

We prove the following Proposition.

Proposition 6.1. *For any $\sigma > 0$ and $\beta \in (0,1)$, consider the norms defined in (6.3) and (6.4) respectively. Then, for all $\varepsilon > 0$ sufficiently small, $\rho = \frac{2\delta}{\varepsilon|\log\varepsilon|^2}$ and h such that $\|h\|_{2+\sigma,\beta,B_\rho} < +\infty$, there exist numbers $\delta > 0$ and $C > 0$ such that for any differential operator $\bar{B}$ as in (6.2), equation (6.1) has a solution $\phi_{in} = T[h]$ for certain scalars $d_i = d_i[h]$, $i = 0,1,2$, that defines a linear operator of h and satisfies*

$$\|\phi_{in}\|_{*,\sigma,B_\rho} \;\le\; C\|h\|_{2+\sigma,\beta,B_\rho}.$$

Moreover, the functionals d_i have the form

$$
\begin{aligned}
d_0[h] &= \gamma_0 \int_{B_\rho} hZ_0 + \mathcal{O}\left(\frac{\log(2+\rho)}{1+\rho^\sigma}\right) \|h\|_{2+\sigma,\beta,B_\rho},\\
d_i[h] &= \gamma_i \int_{B_\rho} hZ_i + \mathcal{O}\left(\frac{1}{1+\rho^{1+\sigma}}\right) \|h\|_{2+\sigma,\beta,B_\rho},\ i=1,2,
\end{aligned}
$$

where $\gamma_i^{-1} = \int_{\mathbb{R}^2} \gamma \Gamma_+^{\gamma-1} Z_i^2$, $i=0,1,2$.

Proof. We begin by considering a standard linear extension operator $h \mapsto \tilde{h}$ to the whole $\mathbb{R}^2$, such that the support of $\tilde{h}$ is contained in $B_{2\rho}$ and $\|\tilde{h}\|_{\sigma,\beta} \leq C\|h\|_{\sigma,\beta,B_\rho}$, where $C>0$ is independent of ρ. Similarly, we assume that the coefficients of $\bar{B}$ are of class $C^1(\mathbb{R}^2)$, with compact support in $B_{2\rho}$. With these in mind, we consider the auxiliary projected problem in the full space $\mathbb{R}^2$ given by

$$
\Delta_y \phi_{in} + \gamma \Gamma_+^{\gamma-1} \phi_{in} + \bar{B}[\phi_{in}] + \tilde{h}(y) = \sum_{j=0}^{2} d_j \gamma \Gamma_+^{\gamma-1} Z_j \quad \text{in } \mathbb{R}^2, \tag{6.5}
$$

where $d_i = d_i[h, \phi_{in}]$ are the scalars defined as

$$
d_i = \gamma_i \int_{\mathbb{R}^2} (\bar{B}[\phi_{in}] + \tilde{h}(y)) Z_i, \quad \gamma_i^{-1} = \int_{\mathbb{R}^2} \gamma \Gamma_+^{\gamma-1} Z_i^2, \quad i=1,2. \tag{6.6}
$$

On the one hand, for $i=1,2$ we estimate

$$
\bar{B}[Z_i] \leq C\varepsilon\left(|y||D_y^2 Z_i| + |D_y Z_i|\right) + \mathcal{O}\left(\varepsilon |y| \Gamma_+^{\gamma-2}\right) Z_i \leq \frac{C\varepsilon}{1+|y|^2},
$$

thus using that $\int_{\mathbb{R}^2} \frac{1}{1+|y|^{2+\sigma}} < +\infty$ we get

$$
\int_{\mathbb{R}^2} \bar{B}[\phi_{in}] Z_i = \int_{\mathbb{R}^2} \phi_{in} \bar{B}[Z_i] \leq C\varepsilon \|\phi_{in}\|_{*,\sigma}.
$$

On the other hand, the case $i=0$ needs to be treated separately, since $Z_0(y) = \mathcal{O}\left(\log\left(2+|y|\right)\right)$ as $|y| \to \infty$. Specifically, we find

$$
\bar{B}[Z_0] \leq \frac{C\varepsilon}{1+|y|},
$$

hence

$$
\begin{aligned}
\int_{\mathbb{R}^2} \bar{B}[\phi_{in}] Z_0 &\leq C\varepsilon \int_{B_{2\rho}} \left(|y||D_y^2 \phi_{in}| + |D_y \phi_{in}|\right) Z_0 + \int_{B_{2\rho}} \mathcal{O}\left(\varepsilon |y| \Gamma_+^{\gamma-2}\right) Z_0\\
&\leq C\varepsilon^\sigma |\log\varepsilon|^{2\sigma-1} \|\phi_{in}\|_{*,\sigma}.
\end{aligned}
$$

Moreover, one can also derive the estimates

$$
\int_{\mathbb{R}^2\setminus B_{2\rho}} \tilde{h} Z_0 = \mathcal{O}\left(\frac{\log(2+\rho)}{1+\rho^\sigma}\right) \|h\|_{2+\sigma,\beta,B_\rho}, \quad \int_{\mathbb{R}^2\setminus B_{2\rho}} \tilde{h} Z_i = \mathcal{O}\left(\frac{1}{1+\rho^{1+\sigma}}\right) \|h\|_{2+\sigma,\beta,B_\rho},\ i=1,2,
$$

and

$$
\bar{B}[\phi_{in}] \leq \frac{C\varepsilon}{1+|y|^{1+\sigma}} \|\phi_{in}\|_{*,\sigma},
$$

which further yields

$$
\left\|\bar{B}[\phi_{in}]\right\|_{2+\sigma,\beta} \leq \frac{C}{|\log\varepsilon|^2} \|\phi_{in}\|_{*,\sigma,B_\rho}.
$$

We consider the Banach space $\mathcal{X} := \{\phi_{in} \in C^{2,\beta}(\mathbb{R}^2) \ : \|\phi_{in}\|_{*,\sigma} < +\infty\}$ and observe that in order to find a solution of (6.5), it suffices to solve the equation

$$
\phi_{in} = \mathcal{A}[\phi_{in}] + \mathcal{H}, \quad \phi_{in} \in \mathcal{X}, \tag{6.7}
$$

where

$$
\mathcal{A}[\phi_{in}] = \mathcal{T}\Big[\bar{B}[\phi_{in}] - \sum_{i=0}^{2} d_i[0,\phi_{in}] \gamma \Gamma_+^{\gamma-1} Z_i\Big], \quad \mathcal{H} = \mathcal{T}\Big[\tilde{h} - \sum_{i=0}^{2} d_i[\tilde{h},0] \gamma \Gamma_+^{\gamma-1} Z_i\Big],
$$

and $\mathcal{T}$ is the linear operator constructed in Lemma 5.1.

Due to (6.6) and the symmetries of Z_j, $j=0,1,2$, it follows that

$$\big\|\mathcal{A}[\phi_{in}]\big\|_{*,\sigma} \leq C\frac{\delta}{|\log\varepsilon|^2}\|\phi_{in}\|_{*,\sigma},$$

and likewise

$$\|\mathcal{H}\|_{*,\sigma} \leq C\|h\|_{2+\sigma,\beta,B_\rho}.$$

As a consequence, using the Contraction Mapping Theorem in $\mathcal{X}$ we deduce that the fixed point problem (6.7) admits a unique solution for all sufficiently small $\varepsilon>0$, which defines a linear operator of h and satisfies

$$\|\phi_{in}\|_{*,\sigma} \;\leq\; C\|h\|_{2+\sigma,\beta,B_\rho}.$$

To this end, we conclude the proof by setting $T[h]=\phi_{in}\big|_{B_\rho}$. ☐

7. Resolution of the Inner-Outer Gluing System

Having established the existence of a solution ϕ_{out} for the Outer problem (4.11) in Section 4.1, our current objective is to employ the linear theories developed in Sections 5 and 6 to find a solution ϕ_{in} for the Inner problem

$$\Delta_y\phi_{in}+\gamma\Gamma_+^{\gamma-1}\phi_{in}+\bar{B}[\phi_{in}]+\mathcal{N}\left(\tilde{\eta}_\delta\phi_{in}+\phi_{out}\right)+\varepsilon^2 S(\Psi_\alpha)+\left(\gamma\Gamma_+^{\gamma-1}+b_0(y)\right)\phi_{out}=0, \quad \text{in } B_\rho.$$

For clarity, we recall that $\rho=\frac{2\delta}{\varepsilon|\log\varepsilon|^2}$, $\tilde{\eta}_\delta=\eta\left(\frac{|\log\varepsilon|^2|z|}{\delta}\right)$ with η as in (2.17), and $\bar{B}$ was defined in (6.2).

As it will become clear later on, it is convenient to decompose $\phi_{in}=\phi_1+\phi_2$ as follows. For the functions Z_j given in (5.4) and scalars $d_j, j=0,1,2$ as in Proposition 6.1, we ask that ϕ_1 solves the auxiliary projected problem

$$\Delta_y\phi_1+\gamma\Gamma_+^{\gamma-1}\phi_1+\bar{B}[\phi_1]+\bar{B}[\phi_2]+H(\phi_1+\phi_2,\phi_{out})=\sum_{j=0}^{2}d_j\gamma\Gamma_+^{\gamma-1}Z_j \quad \text{in} \quad B_\rho, \tag{7.1}$$

with

$$H(\phi_1+\phi_2,\phi_{out}):=\mathcal{N}\left(\tilde{\eta}_\delta\phi_{in}+\phi_{out}\right)+\varepsilon^2 S(\Psi_\alpha)+\left(\gamma\Gamma_+^{\gamma-1}+b_0(y)\right)\phi_{out}.$$

Regarding ϕ_2, we require that it satisfies the linear equation

$$\Delta_y\phi_2+\gamma\Gamma_+^{\gamma-1}\phi_2+d_0\gamma\Gamma_+^{\gamma-1}Z_0=0 \quad \text{in} \quad \mathbb{R}^2. \tag{7.2}$$

Using these assumptions, one can realise that the Inner-Outer system in (4.7) and (4.11) admits a solution of the form $(\phi_{in},\phi_{out})=(\phi_1+\phi_2,\phi_{out})$, provided we further establish the coupled conditions

$$d_1\left[\bar{B}[\phi_2]+H(\phi_1+\phi_2,\phi_{out})\right]=0, \quad d_2\left[\bar{B}[\phi_2]+H(\phi_1+\phi_2,\phi_{out})\right]=0.$$

We defer the analysis of these conditions to Section 8 and begin by examining the equation for ϕ_2 in (7.2). Turning to polar coordinates $y=re^{i\theta}, r=|y|$, it is easy to see that a smooth solution to this problem is given by

$$\phi_2(r)=-d_0Z_0(r)\int_{\xi_0}^{r}\frac{\mathrm{d}s}{sZ_0(s)^2}\int_0^s\gamma\Gamma_+^{\gamma-1}(\rho)Z_0^2(\rho)\rho\,\mathrm{d}\rho=:d_0\hat{\phi}_2(r), \tag{7.3}$$

where ξ_0 is the unique root of Z_0 in $(0,1)$.

An important remark is that since $\operatorname{supp}\Gamma_+^{\gamma-1}\subset B_1(0)$, for any $s>1$ we get

$$\int_0^s\gamma\Gamma_+^{\gamma-1}(\rho)Z_0(\rho)^2\rho\,\mathrm{d}\rho=\int_0^1\gamma\Gamma_+^{\gamma-1}(\rho)Z_0(\rho)^2\rho\,\mathrm{d}\rho.$$

With this in mind, for any $r>0$ we deduce that

$$|\phi_2(r)|\leq C|d_0|\log(2+r). \tag{7.4}$$

The following Proposition is the main result of this section.

Proposition 7.1. *Let $\sigma>0$ and $\beta\in(0,1)$. For $\varepsilon>0$ sufficiently small, there exist constants $0<\sigma^*<\sigma$, functions $\phi_{out}(x)$, $\phi_1(y)$, $\phi_2(y)$ solving (4.11), (7.1) and (7.2) respectively, and scalars d_j, $j=1,2$, such that*

$$\left\|(1+|x|^2)^{-1}\phi_{out}\right\|_\infty\leq C\varepsilon^{1+\sigma^*}, \quad \|\phi_1\|_{*,\sigma,B_\rho}<C\varepsilon, \quad |d_0|<C\varepsilon^{1+\sigma}|\log\varepsilon|^{2+2\sigma}, \tag{7.5}$$

where $y=\frac{z}{\varepsilon}$, $\rho=\frac{2\delta}{\varepsilon|\log\varepsilon|^2}$ and $\|\cdot\|_{*,\sigma,B_\rho}$ is defined in (6.4).

Proof. The problem can be formulated as finding a pair (ϕ_1, d_0) solving the fixed point problem

$$(\phi_1, d_0) = \tilde{\mathcal{F}}(\phi_1, d_0), \tag{7.6}$$

where

$$\phi_1 = T[H(\phi_1 + \phi_2, \phi_{out})] \in \mathcal{X}_* := \{\phi_{in} \in C^{2,\beta}(\mathbb{R}^2) \,:\, \|\phi_{in}\|_{*,\sigma,B_\rho} < +\infty\},$$
$$d_0 = \gamma_0 \int_{R^2} \left(H(\phi_1 + \phi_2, \phi_{out}) + \bar{B}[\phi_2]\right) Z_0, \quad \gamma_0^{-1} = \int_{\mathbb{R}^2} \gamma \Gamma_+^{\gamma-1} Z_0^2.$$

In the above, T is the linear operator built in Proposition 6.1, $\phi_2(y) = d_0 \hat{\phi}_2(y)$ is given in (7.3), while $\phi_{out}(x)$ was obtained in Section 4.1, satisfying (4.13) and (4.14).

By means of the Banach's fixed point Theorem, we want to find a solution in the ball

$$\mathcal{B} = \{(\phi_1, d_0) \in \mathcal{X}_* \times \mathbb{R} : \phi_1(y_1, y_2) = \phi_1(y_1, -y_2),\ \|\phi_1\|_{*,\sigma,B_\rho} \le \hat{C}\varepsilon,\ |d_0| \le \hat{C}\varepsilon^{1+\sigma} |\log \varepsilon|^{2+2\sigma}\}, \tag{7.7}$$

for some constant $\hat{C} > 0$ to be chosen.

To carry out the fixed point argument, we firstly estimate

$$\begin{aligned} |H(\phi_1 + \phi_2, \phi_{out})| &\le C\Gamma_+^{\gamma-2} \left(|\tilde{\eta}_\delta \phi_1|^2 + |\tilde{\eta}_\delta \phi_2|^2 + |\phi_{out}|^2\right) + \varepsilon^2 S(\Psi_\alpha) + \left(\gamma \Gamma_+^{\gamma-1} + b_0(y)\right) |\phi_{out}| \\ &\le C\varepsilon |y| \Gamma_+^\gamma, \end{aligned}$$

thus we immediately obtain

$$\|H(\phi_1 + \phi_2, \phi_{out})\|_{2+\sigma,\beta,B_\rho} \le C\varepsilon.$$

In addition, using (7.4) one finds

$$\begin{aligned} \left|\bar{B}[\phi_2]\right| &\le C\varepsilon \left(|y||D_y^2 \phi_2| + |D_y \phi_2|\right) + \mathcal{O}\left(\varepsilon |y| \Gamma_+^{\gamma-2}\right) \phi_2 \\ &\le \frac{C\varepsilon |d_0|}{1+|y|} + \mathcal{O}\left(\varepsilon |y| \Gamma_+^{\gamma-2}\right) |d_0| \log(2+|y|), \end{aligned}$$

so we further have

$$\left\|\bar{B}[\phi_2]\right\|_{2+\sigma,\beta,B_\rho} \le C\varepsilon.$$

Invoking Proposition 6.1, we readily get

$$\left\|\phi_1\left[H(\phi_1 + \phi_2, \phi_{out}) + \bar{B}[\phi_2]\right]\right\|_{*,\sigma,B_\rho} \le \left\|H(\phi_1 + \phi_2, \phi_{out}) + \bar{B}[\phi_2]\right\|_{2+\sigma,\beta,B_\rho} \le C_1 \varepsilon, \tag{7.8}$$

for some $C_1 > 0$.

Next, we compute

$$\begin{aligned} \int_{B_\rho} \bar{B}[\phi_2] Z_0 &\le C\varepsilon |d_0| \int_{B_\rho} \frac{\log(2+|y|)}{1+|y|} + \int_{B_\rho} \mathcal{O}\left(\varepsilon |y| \Gamma_+^{\gamma-2}\right) |d_0| \log(2+|y|) Z_0 \\ &\le \mathcal{O}\left(\frac{|d_0|}{|\log \varepsilon|}\right), \end{aligned} \tag{7.9}$$

and

$$\begin{aligned} \int_{B_\rho} H\left(\phi_1 + \phi_2, \phi_{out}\right) Z_0 \le C \int_{B_\rho} \Gamma_+^{\gamma-2} \left(|\tilde{\eta}_\delta(\phi_1 + \phi_2)|^2 + |\phi_{out}|^2\right) Z_0 + C \int_{B_\rho} \varepsilon^2 S(\Psi_\alpha) Z_0 \\ + C \int_{B_\rho} \left(\gamma \Gamma_+^{\gamma-1} + \mathcal{O}\left(\varepsilon |y| \Gamma_+^{\gamma-2}\right)\right) |\phi_{out}| Z_0. \end{aligned}$$

Since Z_0 is a radial function, using Proposition 3.1 we observe that

$$\int_{B_\rho} \varepsilon^2 S(\Psi_\alpha) Z_0 = \mathcal{O}\left(\varepsilon^2 |\log \varepsilon|\right), \qquad \int_{B_\rho} \Gamma_+^{\gamma-2} \left(|\tilde{\eta}_\delta(\phi_1 + \phi_2)|^2 + |\phi_{out}|^2\right) Z_0 = \mathcal{O}\left(\varepsilon^2\right).$$

However, due to $\int_{B_\rho} \gamma \Gamma_+^{\gamma-1} Z_0 = \mathcal{O}(1)$, the bound for ϕ_{out} in (4.13) gives

$$\int_{B_\rho} \left(\gamma \Gamma_+^{\gamma-1} + b_0(y)\right) Z_0 |\phi_{out}| = \mathcal{O}(\varepsilon^{1+\sigma^*}),$$

for some $0 < \sigma^* < \sigma$.

This estimate reveals a strong coupling between the Inner and Outer problems, since for $0 < \sigma^* < \sigma$ we have that $\varepsilon^{1+\sigma^*} \gg \varepsilon^{1+\sigma} |\log \varepsilon|^{1+2\sigma}$ for all small $\varepsilon > 0$. This is precisely where the reduced form of the Outer problem (4.11) provided by Lemma 4.1 becomes vital, as to overcome this issue we use (4.14) together with the expansion

$$\phi_{out}(x) = \phi_{out}(P) + \mathcal{O}\left(\varepsilon|y|\|\phi_{out}\|_\infty\right) \quad \text{as} \quad \varepsilon \to 0,$$

which is valid in the region $\left\{x \in \mathbb{R}^2 : |x-P| < \frac{1}{|\log\varepsilon|^2}\right\}$, to eventually deduce that

$$\int_{B_\rho} H\left(\phi_1+\phi_2, \phi_{out}\right) Z_0 \le C\varepsilon^2 |\log\varepsilon|. \tag{7.10}$$

Combining (7.9) and (7.10), we write

$$\int_{B_\rho} \left(H(\phi_1+\phi_2,\phi_{out}+\bar{B}[\phi_2]\right) Z_0 = \mathcal{O}\left(\frac{|d_0|}{|\log\varepsilon|}\right),$$

while employing Proposition 6.1 again, one obtains

$$\begin{aligned} |d_0| &= \gamma_0 \int_{\mathbb{R}^2} \left(H\left(\phi_1+\phi_2,\phi_{out}\right) + \bar{B}[\phi_2]\right) Z_0 \\ &\le \mathcal{O}\left(\frac{|d_0|}{|\log\varepsilon|}\right) + \mathcal{O}\left(\varepsilon^\sigma |\log\varepsilon|^{1+2\sigma}\right) \left\|H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right\|_{2+\sigma,\beta,B_\rho} \\ &\le C_2 \varepsilon^{1+\sigma} |\log\varepsilon|^{2+2\sigma}, \end{aligned} \tag{7.11}$$

for some $C_2 > 0$.

Collecting (7.8) and (7.11), it is clear that $\tilde{\mathcal{F}}(\mathcal{B}) \subset \mathcal{B}$, as long as we choose the constant $\hat{C} > 0$ in (7.7) sufficiently large, independent of $\varepsilon > 0$.

To complete the proof, it remains to establish that $\tilde{\mathcal{F}}$ in (7.6) is a contraction mapping in $\mathcal{B}$. For this purpose, we consider the pairs $(\phi_1^1, d_0^1), (\phi_1^2, d_0^2) \in \mathcal{B}$, together with the corresponding ϕ_2^1, ϕ_2^2 in the decompositions of ϕ_{in}^1 and ϕ_{in}^2.

We recall that for $j = 1, 2$ it holds

$$d_0^j = \gamma_0 \int_{R^2} \left(H(\phi_1^j + \phi_2^j, \phi_{out}[\phi_1^j+\phi_2^j]) + \bar{B}[\phi_2^j]\right) Z_0, \quad \gamma_0^{-1} = \int_{\mathbb{R}^2} \gamma \Gamma_+^{\gamma-1} Z_0^2.$$

Defining $\phi_{out}^j = \phi_{out}[\phi_1^j + \phi_2^j]$ and $H^j = H\left(\phi_1^j+\phi_2^j, \phi_{out}[\phi_1^j+\phi_2^j]\right)$ for $j = 1, 2$, we use (4.12) to derive the estimate

$$\|\phi_{out}^1 - \phi_{out}^2\|_\infty \le C\left(\varepsilon^\sigma|\log\varepsilon|^\mu \|\phi_1^1-\phi_1^2\|_{*,\sigma,B_\rho} + |\log\varepsilon|^\mu |d_0^1 - d_0^2|\right), \tag{7.12}$$

for some $\mu > 0$.

Furthermore, since

$$H^1 - H^2 = \mathcal{N}\left(\tilde{\eta}_\delta(\phi_1^1+\phi_2^1)+\phi_{out}^1\right) - \mathcal{N}\left(\tilde{\eta}_\delta(\phi_1^2+\phi_2^2)+\phi_{out}^2\right) + \left(\gamma\Gamma_+^{\gamma-1} + b_0(y)\right)\left(\phi_{out}^1 - \phi_{out}^2\right),$$

we further obtain

$$\begin{aligned} |H^1 - H^2| < C\Gamma_+^{\gamma-2} &\left(\tilde{\eta}_\delta\left(|\phi_1^1-\phi_1^2|^2 + |\phi_2^1-\phi_2^2|^2\right) + |\phi_{out}^1-\phi_{out}^2|^2\right) \\ &+ \left(\gamma\Gamma_+^{\gamma-1} + \mathcal{O}(\varepsilon|y|\Gamma_+^{\gamma-2})\right)|\phi_{out}^1 - \phi_{out}^2|, \end{aligned}$$

which allows to conclude that

$$\begin{aligned} \|H^1-H^2\|_{2+\sigma,\beta,B_\rho} \le C &\left(\|\phi_{out}^1-\phi_{out}^2\|_\infty + \|\phi_1^1-\phi_1^2\|_{*,\sigma,B_\rho}^2 + |d_0^1-d_0^2|^2 + \|\phi_{out}^1-\phi_{out}^2\|_\infty^2\right) \\ &+ C\varepsilon\left(\|\phi_1^1-\phi_1^2\|_{*,\sigma,B_\rho} + |d_0^1-d_0^2| + \|\phi_{out}^1-\phi_{out}^2\|_\infty\right). \end{aligned} \tag{7.13}$$

Moreover, due to (7.3) and (7.4) we have

$$\bar{B}[\phi_2^1-\phi_2^2] = \mathcal{O}(|d_0^1-d_0^2|)\bar{B}[\hat{\phi}_2], \quad \text{where} \quad \bar{B}[\hat{\phi}_2] \le \frac{C\varepsilon}{2+|y|} + \mathcal{O}\left(\varepsilon|y|\Gamma_+^{\gamma-2}\right)\hat{\phi}_2,$$

hence we get

$$\left\|\bar{B}[\phi_2^1-\phi_2^2]\right\|_{2+\sigma,\beta,B_\rho} \le \frac{\mathcal{O}\left(|d_0^1-d_0^2|\right)}{\varepsilon^\sigma|\log\varepsilon|^{2+2\sigma}}. \tag{7.14}$$

On the other hand, it holds

$$
\begin{aligned}
|d_0^1 - d_0^2| &\le \mathcal{O}\left(\frac{\log(2+\rho)}{1+\rho^\sigma}\right)\left\|H^1 - H^2 + \bar{B}[\phi_2^1 - \phi_2^2]\right\|_{2+\sigma,\beta,B_\rho} + \mathcal{O}\left(\frac{|d_0^1 - d_0^2|}{|\log\varepsilon|}\right)\\
&\le C\varepsilon^\sigma|\log\varepsilon|^{1+2\sigma}\Bigg[\|\phi_{out}^1 - \phi_{out}^2\|_\infty + \|\phi_1^1 - \phi_1^2\|^2_{*,\sigma,B_\rho} + |d_0^1 - d_0^2|^2 + \|\phi_{out}^1 - \phi_{out}^2\|^2_\infty \\
&\quad + \varepsilon\Big(\|\phi_1^1 - \phi_1^2\|_{*,\sigma,B_\rho} + |d_0^1 - d_0^2| + \|\phi_{out}^1 - \phi_{out}^2\|_\infty\Big)\Bigg] + \mathcal{O}\left(\frac{|d_0^1 - d_0^2|}{|\log\varepsilon|}\right),
\end{aligned}
\tag{7.15}
$$

thus making use of Proposition 6.1, (7.13), (7.14) and (7.15), we find

$$
\begin{aligned}
&\left\|T\left[H^1 - H^2 + \bar{B}[\phi_2^1 - \phi_2^2]\right]\right\|_{*,\sigma,B_\rho} \le C\left\|H^1 - H^2 + \bar{B}[\phi_2^1 - \phi_2^2]\right\|_{2+\sigma,\beta,B_\rho}\\
&\le C\Bigg[\|\phi_{out}^1 - \phi_{out}^2\|^2_\infty + \|\phi_1^1 - \phi_1^2\|^2_{*,\sigma,B_\rho} + |d_0^1 - d_0^2|^2 + \|\phi_{out}^1 - \phi_{out}^2\|_\infty\\
&\qquad + \varepsilon\left(\|\phi_1^1 - \phi_1^2\|_{*,\sigma,B_\rho} + |d_0^1 - d_0^2| + \|\phi_{out}^1 - \phi_{out}^2\|_\infty\right)\\
&\qquad + \frac{1}{|\log\varepsilon|}\left(\|\phi_1^1 - \phi_1^2\|_{*,\sigma,B_\rho} + |d_0^1 - d_0^2| + \|\phi_{out}^1 - \phi_{out}^2\|_\infty\right)\Bigg].
\end{aligned}
\tag{7.16}
$$

To this end, combining (7.12), (7.15) and (7.16) one can verify that $\tilde{\mathcal{F}}$ is a contraction mapping in $\mathcal{B}$ for all sufficiently small $\varepsilon > 0$, which yields the existence of the desired fixed point. □

8. Solving the Reduced Problem

In Section 7, we have established the existence of a solution $(\phi_{in}, \phi_{out}) = (\phi_1 + \phi_2, \phi_{out})$ to the coupled system of equations

$$
\begin{cases}
\Delta_y\phi_{in} + \gamma\Gamma_+^{\gamma-1}\phi_{in} + \bar{B}[\phi_{in}] + H(\phi_1+\phi_2, \phi_{out}) = \sum_{j=1}^{2} d_j\gamma\Gamma_+^{\gamma-1}Z_j \quad \text{in} \quad B_\rho,\\
L_x[\phi_{out}] + L_x[\tilde{\eta}_\delta\phi_{in}] - \tilde{\eta}_\delta L_x[\phi_{in}] = 0 \quad \text{in } \mathbb{R}^2.
\end{cases}
$$

For the reader's convenience, we recall that $\rho = \frac{2\delta}{\varepsilon|\log\varepsilon|^2}$, $\bar{B}$ is given in (6.2) and $Z_j, j = 1,2$ are defined in (5.4), while the solution (ϕ_{in}, ϕ_{out}) and the relevant estimates are contained in Proposition 7.1 and (7.5).

To conclude the proof of Theorem 1, it remains to obtain an actual solution to the coupled Inner-Outer system in (4.7) and (4.11), by further requiring that

$$
d_1\left[H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right] = 0, \quad d_2\left[H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right] = 0, \tag{8.1}
$$

where

$$
\begin{aligned}
&d_j = \gamma_j\int_{\mathbb{R}^2}\left(H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right)Z_j, \quad \gamma_j^{-1} = \int_{\mathbb{R}^2}\gamma\Gamma_+^{\gamma-1}Z_j^2, \quad j = 1,2,\\
&H(\phi_1+\phi_2,\phi_{out}) = \mathcal{N}\left(\tilde{\eta}_\delta\phi_{in} + \phi_{out}\right) + \varepsilon^2 S(\Psi_\alpha) + \left(\gamma\Gamma_+^{\gamma-1} + \mathcal{O}\big(\varepsilon|y|\Gamma_+^{\gamma-2}\big)\right)\phi_{out}.
\end{aligned}
$$

To attain these conditions, we claim that the rotational speed α in (1.9) must be chosen as in (3.3).

To see this, using Proposition 6.1 we initially observe that for $j = 1,2$, it holds

$$
\begin{aligned}
d_j\left[H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right] &= \gamma_j\int_{B_\rho}\left(H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right)Z_j\\
&\quad + \mathcal{O}\left(\varepsilon^{1+\sigma}|\log\varepsilon|^{2+2\sigma}\right)\left\|H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right\|_{2+\sigma,\beta,B_\rho}\\
&= \gamma_j\int_{B_\rho}\left(H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right)Z_j + \mathcal{O}\left(\varepsilon^{1+\sigma}\right),
\end{aligned}
$$

for $\sigma > 0$ as in (7.5).

As a result, by continuity we get that the conditions in (8.1) reduce to

$$
\gamma_j\int_{B_\rho}\left(H(\phi_1+\phi_2,\phi_{out}) + \bar{B}[\phi_2]\right)Z_j = \mathcal{O}\left(\varepsilon^{1+\sigma}\right), \quad j = 1,2.
$$

Moreover, due to the estimate

$$\gamma_j \int_{B_\rho} \left(\mathcal{N}\left(\tilde{\eta}_\delta \phi_{in} + \phi_{out}\right) + \left(\gamma \Gamma_+^{\gamma-1} + \mathcal{O}\big(\varepsilon |y| \Gamma_+^{\gamma-2}\big)\right) \phi_{out} \right) Z_j = \mathcal{O}(\varepsilon^{1+\sigma}),$$

we realise that it remains to ensure that

$$\int_{B_\rho} \varepsilon^2 S(\Psi_\alpha) Z_j = \mathcal{O}(\varepsilon^{1+\sigma}), \quad j = 1, 2.$$

In light of Proposition 3.1, the error of approximation satisfies $\varepsilon^2 S(\Psi_\alpha)(\varepsilon y_1, \varepsilon y_2) = \varepsilon^2 S(\Psi_\alpha)(\varepsilon y_1, -\varepsilon y_2)$ and has the expansion

$$\begin{aligned} \varepsilon^2 S(\Psi_\alpha) = \frac{3r_0 h^2 + r_0^2}{2h(h^2 + r_0^2)^{\frac{3}{2}}} \varepsilon y_1 \Gamma_+^{\gamma} + \gamma \Gamma_+^{\gamma-1} \Bigg[\varepsilon y_1 |\log \varepsilon| \left(-c_1 \nu'(1) - \frac{\alpha r_0 h}{\sqrt{h^2 + r_0^2}} \right) \\ + \varepsilon y_1 \left(c_1 \Gamma + \frac{h}{\sqrt{h^2 + r_0^2}} \partial_{y_1} H_{2\varepsilon}(P) \right) + \mathcal{O}\left(\varepsilon^2 |\log \varepsilon| |y|^2\right) \Bigg] + \mathcal{O}\left(\varepsilon^2 |\log \varepsilon|^2 |y|^2 \Gamma_+^{\gamma-2}\right). \end{aligned} \tag{8.2}$$

Since $Z_2 = \frac{\partial \Gamma}{\partial y_2}$ is odd in y_2, it is then immediate due to symmetry that

$$\int_{B_\rho} \varepsilon^2 S(\Psi_\alpha) Z_2 = 0.$$

Upon testing (8.2) with Z_1, to satisfy the condition $\int_{B_\rho} \varepsilon^2 S(\Psi_\alpha) Z_1 = \mathcal{O}(\varepsilon^{1+\sigma})$ we choose

$$\alpha = \frac{-\nu'(1)}{2(h^2 + r_0^2)} + \tilde{\alpha},$$

where $\tilde{\alpha}$ takes the form

$$\tilde{\alpha} = \frac{1}{|\log \varepsilon|} \left[\frac{\int_{B_\rho} - \left[\frac{3r_0 h^2 + r_0^2}{2h(h^2+r_0^2)^{\frac{3}{2}}} \Gamma_+^{\gamma} + \gamma \Gamma_+^{\gamma-1} \left(c_1 \Gamma + \frac{h}{\sqrt{h^2+r_0^2}} \partial_{y_1} H_{2\varepsilon}(P) \right) \right] y_1 Z_1}{\int_{B_\rho} \gamma \Gamma_+^{\gamma-1} y_1 Z_1} \right] (1 + o(1)),$$

with $o(1) \to 0$ as $\varepsilon \to 0$. This concludes the proof of Theorem 1.

9. Proof of Theorem 2

In this section, we provide some ideas concerning the proof of Theorem 2. We recall that the equation we need to solve reads

$$-\nabla_x \cdot (K \nabla_x \Psi) = f\left(\Psi - \frac{\alpha}{2} |\log \varepsilon| |x|^2\right) \quad \text{in } \mathbb{R}^2, \quad f(s) = \frac{1}{\varepsilon^2}\left(s + \nu'(1) |\log \varepsilon|\right)_+^{\gamma}, \tag{9.1}$$

where $\gamma > 3$, $s_+ = \max(0, s)$, and $K(x)$ is defined in (2.2).

In polar coordinates $x = re^{i\theta}, r = |x|$, and considering the standard basis $e_r = (\cos\theta, \sin\theta)$, $e_\theta = (-\sin\theta, \cos\theta)$, we have that $K e_r = \frac{h^2}{h^2+r^2} e_r$ and $K e_\theta = e_\theta$, thus equation (9.1) can be equivalently written as

$$-\frac{h^2}{r} \partial_r \left(\frac{r}{h^2 + r^2} \partial_r \Psi \right) - \frac{1}{r^2} \partial_{\theta\theta} \Psi = f\left(\Psi - \frac{\alpha}{2} |\log \varepsilon| r^2\right) \quad \text{in } (r, \theta) \in (0, \infty) \times [0, 2\pi). \tag{9.2}$$

For any integer $k \geq 2$, it is then immediate that equation (9.2) is invariant under the rotation $\theta \mapsto \theta + \frac{2\pi}{k}$ and even symmetry $\theta \mapsto -\theta$, due to the spherical symmetry of (9.1).

To obtain the solution predicted by Theorem 2, we set $z = |z| e^{i\theta}$ and work in the function space of dihedral symmetry

$$\hat{\mathcal{X}} := \left\{ \Psi \in L^{\infty}_{\text{loc}}\left(\mathbb{R}^2\right) : \Psi\left(z e^{\frac{i2\pi}{k}}\right) = \Psi(z), \;\; \Psi(z) = \Psi(\bar{z}), \;\; z = (z_1, z_2) \in \mathbb{R}^2 \right\}.$$

We then introduce the profile

$$\tilde{H}_{1\varepsilon}(z) = \hat{\Gamma}\left(\frac{z}{\varepsilon}\right)\left(1 + c_1 z_1 + c_2 |z|^2\right) + \frac{r_0^3}{2h(h^2 + r_0^2)^{\frac{3}{2}}} H_{1\varepsilon}(z),$$

with $\hat{\Gamma}\left(\frac{z}{\varepsilon}\right)$, c_1, c_2 and $H_{1\varepsilon}(z)$ given in Proposition 2.2, and introduce the approximate solution

$$\Psi_\alpha(x)=\frac{\alpha}{2}|\log\varepsilon|r_0^2+\sum_{j=1}^{k}\eta_\delta\left(x-r_0e^{\frac{i2\pi(j-1)}{k}}\right)\tilde{H}_{1\varepsilon}\left(ze^{\frac{i2\pi(j-1)}{k}}\right)+H_{2\varepsilon}(x)\in\hat{\mathcal{X}},$$

where the cut-off function η_δ is defined in (2.17).

The remainder of the proof of Theorem 2 is identical to that of Theorem 1, hence we omit the details.

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