EDGEWORTH EXPANSIONS FOR LINEAR RANK STATISTICS USING STEIN'S METHOD

by

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Abstract

Let $\mathscr{F}_A$ be the distribution function of the standardized linear rank statistic for the matrix A, where it is assumed that the rank vector is equally distributed on the permutations. In this work, we analyze the conditions to be imposed on A so that $\mathscr{F}_A$ has first and second order Edgeworth expansions with asymptotically "sufficiently" small remainder terms. The methods used are the Stein method combined with an extension of techniques that go back to Bolthausen (1984).

The conditions obtained are very similar to the necessary and sufficient conditions found by Bickel and Robinson (1982). In this paper they investigated when the distribution function of a sum of independent and identically distributed random variables has Edgeworth expansions with asymptotically small enough residues. However, these conditions are often difficult to prove directly.

For simple linear rank statistics, however, it is possible to use a result from van Zwet (1982) to verify these assumptions. Thus we obtain conditions on A for the validity of Edgeworth expansions, which on the one hand are very easy to prove and on the other hand are much more general than all previously known conditions.

Finally, this result is applied to the special case of approximating and exact scores.

Contents

Introduction, summary and acknowledgements of the original 1987 version

Let A be an $n \times n$ - matrix of real numbers, let π be uniformly distributed on the set $\mathscr{P}_n$ of permutations of $\{1, \ldots, n\}$ and

$$T_A = \sum_{i=1}^{n} a_{i\pi(i)}.$$

In this work, we investigate when the distribution function $\mathscr{F}_A$ of

$$\mathscr{T}_A = \frac{T_A - E(T_A)}{\sqrt{\operatorname{Var}(T_A)}}$$

possesses **Edgeworth expansions of first and second order** such that the error made, when approximating $\mathscr{F}_A$ by these expansions, is asymptotically "sufficiently" small. Before we go into this in more detail, however, we would like to make a few brief comments on the statistical relevance of T_A.

These random variables are of particular interest in nonparametric statistics. They arise in the following situation:

Let $X_1, \ldots, X_n$ be independent random variables with continuous distribution functions $F_1, \ldots, F_n$. Furthermore, if $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denotes the order of the sequence $X_1, X_2, \ldots, X_n$ in increasing size, then the rank R_j of X_j is defined by $X_j = X_{R_j:n}$. The mapping

$$S_A = \sum_{i=1}^{n} a_{iR_i}$$

is now called a **linear rank statistic**. If the matrix A has coefficients of the form $a_{ij} = e_i\, d_j$, then S_A is called a **simple linear rank statistic**. This statistic can be used to test the null hypothesis $H_0 : F_1 = F_2 = \ldots = F_n$ against different classes of alternatives. The structure of the alternatives is reflected in the choice of the regression constants e_i and the scores d_j. A well-known example is the **two-sample statistic**, for which $e_1 = \ldots = e_m = 1$, $e_{m+1} = \ldots = e_n = 0$, and which is used to test H_0 against alternatives of the form $F_1 = \ldots = F_m$, $F_{m+1} = \ldots = F_n$. Further information can be found, for example, in the monograph by Hájek, Šidák and Sen [16]. Under the null hypothesis H_0, the random vector $(R_1, \ldots, R_n)$ is always uniformly distributed on the set $\mathscr{P}_n$ of permutations of $\{1, \ldots, n\}$, so that S_A is distributed as T_A.

The problem of determining critical values for these tests as simply as possible then leads, among other things, to the question of the conditions under which the distribution of $\mathscr{T}_A$ is approximated by the standard normal distribution for large n. This question and the related question of the convergence rate of this approximation have been investigated by a large number of authors. See the detailed overview of Does [14], section 1.2, and the paper of Bolthausen [6].

The desire for more accurate approximations than those provided by the normal approximation then takes us to the Edgeworth expansions of $\mathscr{F}_A$. To date, the papers of Bickel & Van Zwet [5] and Robinson [22] exist on this topic, but they only consider two-sample statistics. Furthermore, there is a paper of Does [14] (resp. [12]) for simple linear rank statistics. Does, however, only considers the case of approximating scores, i.e. $d_j = J(\frac{j}{n+1})$, where $J : (0,1) \to \mathbb{R}$ is a function that must fulfil certain regularity assumptions.

In this work, we mostly consider completely general a_{ij}. The results we obtain are presented in the following overview of the content. We also indicate the techniques used.

Stein's method plays a central role in this work. This method was introduced by C. Stein [26] to prove results of the Berry-Esséen type. We will use this method to establish Edgeworth expansions for $\mathscr{F}_A$ (i.e. determine these expansions and estimate the error of the approximation by these expansions).

To show how we proceed **principally**, we introduce in chapter 1 Edgeworth expansions for sums of independent and identically distributed random variables X_n, $n \in \mathbb{N}$. We obtain a result (cf. Theorem 1.1.3, page 11), which essentially originates from Bickel and Robinson [4]. However, their proof is based on Fourier analytical methods.

What is remarkable about this result is that it does not assume the validity of Cramér's condition (i.e. $\limsup_{|t|\to\infty} E(e^{itX_1}) < 1$), as comparable results do. Instead, the following more difficult condition is required:

There exists a constant $\mathcal{C} > 0$ such that

$$(\mathrm{I}) \qquad \sup_{z\in\mathbb{R}} \Big| \big(F_n(z+y) - F_n(z)\big) - \big(F_n(z) - F_n(z-y)\big)\Big| \le \mathcal{C}\Big(\frac{1}{n} + y^2\Big) \quad \text{for all } 0 \le y \le \frac{1}{\sqrt{n}},\ n \in \mathbb{N}.$$

Here, F_n denotes the distribution function of the standardized sum of $X_1, \dots, X_n$. In chapter 4, section 4.2, we show that (I) really follows from Cramér's condition.

When proving the Theorem 1.1.3 using Stein's method, some purely analytical results are required. We do not prove them in chapter 1 for reasons of "clarity" and since similar results are also used in chapter 3. Instead, we list these analytical auxiliary "considerations" in chapter 2 and prove them there.

In the central chapter 3 we derive Edgeworth expansions for $\mathscr{F}_A$. In order to present the results obtained in more detail, we define

$$\breve{a}_{ij} = a_{ij} - \frac{1}{n}\sum_{l=1}^{n} a_{il} - \frac{1}{n}\sum_{k=1}^{n} a_{kj} + \frac{1}{n^2}\sum_{k,l=1}^{n} a_{kl}$$

and

$$\sigma_A^2 = \frac{1}{n-1}\sum_{i,j=1}^{n} \breve{a}_{ij}^2 , \qquad \hat{a}_{ij} = \frac{1}{\sigma_A}\breve{a}_{ij} .$$

The first and second order expansions used then have the following form:

$$\begin{aligned} e_{1,A}(x) &= \Phi(x) - \psi(x)\,\frac{\lambda_{1,A}}{6}\,(x^2-1) , \\ e_{2,A}(x) &= \Phi(x) - \psi(x)\left\{\frac{\lambda_{1,A}}{6}\,(x^2-1) + \frac{\lambda_{2,A}}{24}\,(x^3-3x) + \frac{\lambda_{1,A}^2}{72}\,(x^5-10x^3+15x)\right\} . \end{aligned}$$

Here Φ denotes the distribution function and ψ the density of the standard normal distribution. Furthermore are defined

$$\lambda_{1,A} = \frac{1}{n}\sum_{i,j=1}^{n} \hat{a}_{ij}^3 \quad \text{and}$$

$$\lambda_{2,A} = \frac{1}{n}\sum_{i,j=1}^{n} \hat{a}_{ij}^4 + \frac{3}{n} - \frac{3}{n^2}\sum_{i,j,k=1}^{n} \big(\hat{a}_{ij}^2\hat{a}_{ik}^2 + \hat{a}_{ij}^2\hat{a}_{kj}^2\big) .$$

If $\mathscr{F}_A$ now fulfils a condition that has great analogies to (I), we can show the existence of a constant $K_1 > 0$ that depends only on constants from this condition, such that

$$\text{(II)} \qquad \sup_{z\in\mathbb{R}} \Big| \mathscr{F}_A(z) - e_{1,A}(z) \Big| \leq K_1 \, \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^4 \qquad \text{(cf. Theorem 3.1.10, page 57).}$$

Under somewhat more complicated conditions, the following also holds

$$\text{(III)} \qquad \sup_{z\in\mathbb{R}} \Big| \mathscr{F}_A(z) - e_{2,A}(z) \Big| \leq K_2 \, \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^5 \qquad \text{(cf. Theorem 3.1.13, page 57).}$$

It should be noted that in many typical application cases $\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^4$ is of the order of magnitude of $\frac{1}{n}$ and $\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^5$ is of the order of magnitude of $\frac{1}{n^{3/2}}$.

The proof of (II) and (III) takes up the whole chapter 3. In section 3.1 we formulate the Theorems 3.1.10 and 3.1.13 and make a few basic remarks about them. The sections 3.2, 3.3 and 3.4 contain some essential preliminary considerations for the proof of the two theorems, such as the truncation technique in section 3.4. The section 3.5 is one of the most important in this work. In this section we extend some techniques going back to Bolthausen [6], which ensure that the structure of the proof we learned in chapter 1 is also portable to linear rank statistics (under the null hypothesis H_0). In section 3.6 we use these techniques for the first time to prove another "auxiliary result". This is:

If $k \in \mathbb{N}$, there exists a constant $C(k) > 0$ that depends only on k, such that

$$\text{(IV)} \quad \sup_{z\in\mathbb{R}} \left| E\Big(| \mathscr{T}_A|^k \, 1_{(-\infty,z]}(\mathscr{T}_A) \Big) - \int_{-\infty}^{z} |x|^k \psi(x) dx \right| \leq C(k) \left(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^3 + \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{k+4} \right)$$

(cf. Theorem 3.6.1, page 104).

This result for $k = 1, 2$ is then used together with the techniques from section 3.5 and Stein's method to derive the Theorem 3.1.10 in section 3.7 and the Theorem 3.1.13 in section 3.8.

However, the assumptions of the Theorems 3.1.10 and 3.1.13 have the disadvantage that they are often difficult to prove directly. Therefore, in chapter 4 we give easily verifiable conditions for simple linear rank statistics under which (II) or (III) hold (except for small correction factors of size $(\log n)^2$ or n^ϵ, $\epsilon > 0$). These conditions originate from van Zwet [32], who used them to derive an estimate of the characteristic function of $\mathscr{F}_A$ (see Theorem 4.3.1, page 157). From this estimate, we obtain the conditions of the two theorems (modulo the above correction factors) in section 4.1 and section 4.3. It follows that

(V) $$\sup_{z\in\mathbb{R}} \Big|\mathscr{F}_A(z) - e_{1,A}(z)\Big| \le K_3\,(\log n)^2\,\frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^4$$

and for any $\epsilon > 0$

(VI) $$\sup_{z\in\mathbb{R}} \Big|\mathscr{F}_A(z) - e_{2,A}(z)\Big| \le K_4\, n^{\epsilon}\,\frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^5$$ (cf. Theorem 4.3.13, page 165).

Here, K_3 and K_4 depend only on the constants from van Zwet's conditions, and K_4 additionally depends on ϵ. If these conditions are strengthened slightly, it is even the case that $\frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^4$ is of the order of magnitude of $\frac{1}{n}$ and $\frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^5$ is of the order of magnitude of $\frac{1}{n^{3/2}}$ (cf. Lemma 3.1.18, a) in combination with Lemma 4.3.6, c)).

These results are then applied to the special case of approximating and exact scores in section 4.4 of chapter 4 (see Theorem 4.4.21, page 178 and Theorem 4.4.35, page 185). We obtain more far-reaching results than Does (cf. Remark 4.4.28, c), page 181).

I received the suggestion for this doctoral thesis from Prof. Dr. E. Bolthausen. I would like to thank him for this and for his valuable expert advice. I would also like to thank Mrs F. Siwak and Mrs G. Lindner-Raphaël for the careful preparation of the original manuscript (prepared with typewriters).

Comments on the updated version of 2025

The entire work was written with LaTeX and then translated to English with the help of an artificial intelligence machine translation tool. In addition, it was partly extended for better comprehensibility and (printing) errors were eliminated. Most of the changes were made in the sections 3.5 and 4.4.

In section 3.5 the too simple definition of the distribution of $\underline{I} = (\, I_1, \ldots, I_{16}\,)$ was corrected (cf. (3.5.7) - (3.5.10)). The distribution of (I_1, I_2, I_3, I_4) now corresponds exactly with the distribution of (I_1, I_2, I_3, I_4) in Bolthausen's paper [6] (derivable from (3.5.11)).

In section 4.4 the term "condition V_α" was introduced (cf. Lemma 4.4.8). This term allows a better and shorter formulation of the results of this section.

In addition, more precise estimates were made in the sections 1.4 and 1.5, so that the constant $\mathcal{K}$ in Theorem 1.1.3 has become smaller and slightly better, but certainly still not optimal.

Furthermore, it was added in section 3.1 that the condition (3.1.15) for $B = \hat{A}$ can also be derived from (3.1.16) with another constant (cf. Proposition 3.1.20, c) and d)).

Last but not least, the proof of (3.8.32) in subsection 3.8.3 was improved and shortened. We now use that the function r_z' is a linear combination of functions $\in \mathcal{H}$ (cf. (0.1.1)).

Essential ideas from this work have been published in [25] for first-order Edgeworth expansions and with the restriction to standardized matrices $\hat{A}$ whose elements are bounded in absolute value by one.

Moreover, the techniques of Bolthausen [6] and here can also be used to give a short proof for the normal approximation of $\mathscr{T}_A$ for large n (see Schneller [24]).

Finally, it should be noted that up-to-date bibliographies on the topic of convergence rates for combinatorial central limit theorems can be found, for example, in Chen and Fang [7] and Roos [23].

0 Notations and preliminary remarks

a) Numbers, sequences, quantities

For $x \in \mathbb{R}$ we define:

$$x^+ = \max\{x,\, 0\} \qquad \text{(positive part);}$$

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \le x\} \qquad \text{(floor function).}$$

Furthermore, if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers (with $b_n \neq 0$), we use the notations

$$a_n \sim b_n \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 1;$$

$$a_n = o(b_n) \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0;$$

$$a_n = \mathcal{O}(b_n) \quad \Leftrightarrow \quad \text{the sequence } \left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}} \text{ is bounded.}$$

In the following, we also use the symbol $\mathbb{N}_0$ for the set of natural numbers including zero.

b) Functions

In this work, $\psi(x)$ denotes the density and $\Phi(x)$ the distribution function of the standard normal distribution. In addition, we write $\Phi(h)$ or $\Phi(h(x))$ for the expected value of the function h with respect to the standard normal distribution.

Furthermore, we define

$$\mathcal{H} = \Big\{ h : \mathbb{R} \to [\,0,\,1\,] \,:\, h \text{ is monotonically decreasing and continuous from the left} \Big\}. \tag{0.1.1}$$

If $F : \mathbb{R} \to \mathbb{R}$ is a function, we set

$$||F|| \quad \text{or} \quad ||F(z)|| = \sup_{z\in\mathbb{R}} |F(z)|.$$

Next, for $y \in \mathbb{R}$ we denote by $\Delta_y F$ the function $\mathbb{R} \to \mathbb{R}$, which is defined by

$$\Delta_y F(z) = F(z+y) - F(z) \quad \text{for } z \in \mathbb{R} \tag{0.1.2}$$

$\Delta_y F$ is the **first difference** of F with respect to y. The ***k*th difference** of F with respect to y is then obtained by applying the operator Δ_y k times, i.e.

$$\begin{aligned} \Delta_y^k F(z) &= \Delta_y\big(\Delta_y \dots (\Delta_y F)\big)(z) \\ &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} F(z+jy) \quad \text{for } z \in \mathbb{R},\ k \in \mathbb{N}. \end{aligned} \tag{0.1.3}$$

With this notation, the **interpolating polynomial** of F at the points z, $z+y$, ..., $z+ky$ of degree at most $k \in \mathbb{N}_0$ can be written as

$$\begin{gathered} P_y^k(x; z, F) = F(z) + \sum_{s=1}^{k} \Delta_y^s F(z) \prod_{i=1}^{s} \frac{x - z - (i-1)y}{iy} \\ \text{(where } \textstyle\sum \dots = 0 \quad \text{for } y = 0). \end{gathered} \tag{0.1.4}$$

c) Constants

Throughout the paper, Latin letters (e.g. c, c_1, C_1, e, E, n_0, ...) and Greek letters (e.g. ϵ_0, δ, ...) are used to denote those constants that depend only on the theorem and proof in which they appear. If there are also dependencies on quantities from the theorem or proof, this is indicated in a subordinate clause or by the use of round brackets (e.g. $C(k)$).

In contrast, the letters $\mathcal{C}$, $\mathcal{K}$, $\mathcal{E}$, $\mathfrak{K}$, $\mathfrak{L}$ (including indices and embellishments) denote constants that are considered fixed for the entire work. If these constants depend on other mathematical quantities, this is also stated in a subordinate clause or indicated by round brackets.

d) Further notations

In Hölder's inequality, which is frequently used in this work, we denote the exponents with p and q, where $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. For example, an application of this inequality with the exponents p and $q = \frac{p}{p-1}$ to finite sequences $x_1, \dots, x_\nu$ gives

$$\Big(\sum_{i=1}^{\nu} |x_i|\Big)^p = \Big(\sum_{i=1}^{\nu} 1\,|x_i|\Big)^p \le \nu^{p-1} \sum_{i=1}^{\nu} |x_i|^p. \tag{0.1.5}$$

Further important notations in connection with $n \times n$ - matrices A can be found at the beginning of the section 3.1 of chapter 3.

1 Edgeworth expansions of first order for sums of independent and identically distributed random variables

1.1 Introduction and results

In the central part of this work, Edgeworth expansions for linear rank statistics are established using Stein's method (see chapter 3). Since this requires somewhat extensive considerations, we will demonstrate in this chapter how to obtain Edgeworth expansions for independent and identically distributed (iid) random variables using Stein's method. However, for the sake of clarity, we will postpone some important analytical estimates required for this to the next chapter 2.

In the following, let X_i, $i \in \mathbb{N}$, be a sequence of independent and identically distributed random variables with a common distribution function F and

(1.1.1) $$E(X_i) = 0,\ E(X_i^2) = 1 \text{ and } \beta_4 = E(X_i^4) < \infty.$$

Then, of course, $\mu_3 = E(X_i^3)$ and $\beta_3 = E(|X_i|^3)$ also exist. Furthermore, let $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ for $n \in \mathbb{N}$, F_n the distribution function of S_n and

(1.1.2) $$e_n(x) = \Phi(x) - \frac{\mu_3}{6\sqrt{n}}(x^2-1)\psi(x) \quad \text{for } x \in \mathbb{R}.$$

With these notations we show:

1.1.3 Theorem

Suppose that there exists a constant $\mathcal{C} > 0$ such that

(1.1.4) $$||\Delta_y^2 F_n|| \le \mathcal{C}\Big(\frac{1}{n} + y^2\Big) \quad \text{for all } 0 \le y \le \frac{1}{\sqrt{n}},\ n \in \mathbb{N}.$$

Then there exists a constant $\mathcal{K} > 0$ depending only on β_3, β_4 and $\mathcal{C}$ such that

$$(1.1.5) \qquad ||F_n - e_n|| \le \frac{\mathcal{K}}{n} \quad \text{for all } n \in \mathbb{N}.$$

1.1.6 Remarks

a) The proof of the theorem shows that

$$(1.1.7) \qquad \mathcal{K} = \big(\, 2 + \beta_4 \,\big)\, \mathcal{C} + \big(\, 3 + 11\, \beta_3 + 13\, \beta_4 + 9\, \beta_3\, \beta_4 \,\big)$$

can be selected. See (1.5.2) and the considerations before it.

In this proof, however, the primary aim was to present our application of Stein's method and only secondarily to minimize $\mathcal{K}$. This constant is therefore probably still far from being an optimal constant.

b) In chapter 4, section 4.2 we show, that from Cramér's condition (i.e. $\limsup_{|t|\to\infty} E(e^{itX_1}) < 1$) we get the existence of a constant $\mathcal{C} > 0$ that fulfills the condition (1.1.4).

In particular, such a $\mathcal{C} > 0$ exists if the distribution of X_1 has a density. See also Lemma 2.4.2, b) for a different approach in the case of an existing density.

c) If, on the other hand, $P(X_1 = -1) = \frac{1}{2} = P(X_1 = 1)$, neither (1.1.4) nor (1.1.5) apply, since we obtain with the help of Stirling's formula:

$$\begin{aligned}
||F_{2n} - e_{2n}|| &\ge \max\Big\{ |F_{2n}(0) - e_{2n}(0)|,\ |F_{2n}(0-) - e_{2n}(0)| \Big\} \\
&\ge \frac{1}{2}\, |F_{2n}(0) - F_{2n}(0-)| \\
&= \frac{1}{2}\, P(S_{2n} = 0) = \frac{1}{2}\, \frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{2}\, \frac{1}{\sqrt{\pi n}} \quad \text{for } n \to \infty.
\end{aligned}$$

d) From (1.1.5) we can conclude the other way round:

$$\begin{aligned}
||\Delta_y^2\, F_n|| &\le 4\, \frac{\mathcal{K}}{n} + ||\Delta_y^2\, e_n|| \\
&\le 4\, \frac{\mathcal{K}}{n} + \left(\frac{1}{4} + \frac{1}{5}\, \frac{\beta_3}{\sqrt{n}} \right) y^2 \qquad \text{(see part b) of the proof of Proposition 1.2.2)} \\
&\le \left(4\mathcal{K} + \frac{1}{4} + \frac{1}{5}\, \beta_3 \right) \left(\frac{1}{n} + y^2 \right) \qquad \text{for all } y \in \mathbb{R},\ n \in \mathbb{N}.
\end{aligned}$$

e) Bickel and Robinson [4], cf. Theorem and Note (ii), have shown an analogous result with a shorter Fourier-analytic proof. In their result, however, $\mathcal{K}$ depends on $\mathcal{C}$ and F (instead of β_3, β_4).

In the following proof of Theorem 1.1.3, let n be fixed. Furthermore, we can assume $n \geq 6$ without loss of generality. For $n \leq 5$ and $\mathcal{K}$ from (1.1.7) we trivially obtain (cf. Lemma 2.2.8, e) and $\beta_3 \geq 1$):

$$(1.1.8) \qquad ||F_n - e_n|| \leq ||F_n - \Phi|| + \frac{\beta_3}{6\sqrt{n}} \, ||\,(x^2-1)\,\psi\,|| \leq 1 + \frac{1}{15}\,\frac{\beta_3}{\sqrt{n}} \leq \frac{1}{n}\,6\,\beta_3 \leq \frac{1}{n}\,\mathcal{K}.$$

By using Stein's method, we now show that from

$$(1.1.9) \qquad ||\Delta_y^2\, F_n|| \leq \mathcal{C}\Big(\frac{1}{n} + y^2\Big) \quad \text{for } 0 \leq y \leq \frac{1}{\sqrt{n}} \quad \text{and}$$

$$(1.1.10) \qquad ||\Delta^2_{\frac{1}{\sqrt{n-1}}}\, F_{n-1}|| \leq \mathcal{C}\,\frac{2}{n-1}$$

we get the inequality

$$||F_n - e_n|| \leq \frac{\mathcal{K}}{n} \quad \text{with } \mathcal{K} \text{ according to (1.1.7)}$$

The corresponding proof is divided into four parts. The central considerations take place in the third section, while the next section contains a preliminary consideration.

1.2 Transition to smooth functions

The starting point of this section is the following trivial equation

$$F_n(z) - e_n(z) = \int\limits_{\mathbb{R}} 1_{(-\infty,\,z\,]}\,dF_n - \int\limits_{\mathbb{R}} 1_{(-\infty,\,z\,]}\,e_n'\,dx.$$

In order to apply Stein's method, we must first replace the discontinuous functions $1_{(-\infty,\,z\,]}$, $z \in \mathbb{R}$, by certain differentiable functions q_z, $z \in \mathbb{R}$. The following piecewise defined functions prove to be suitable. For $z \in \mathbb{R}$ let (cf. (2.3.2) with $\lambda = \dfrac{1}{\sqrt{n}}$)

$$(1.2.1) \qquad q_z(x) = \begin{cases} 1 & \text{for } x \leq z, \\ 1 - \dfrac{n}{2}\Big(x - z\Big)^2 & \text{for } z \leq x \leq z + \dfrac{1}{\sqrt{n}}, \\ \dfrac{n}{2}\Big(z + \dfrac{2}{\sqrt{n}} - x\Big)^2 & \text{for } z + \dfrac{1}{\sqrt{n}} \leq x \leq z + \dfrac{2}{\sqrt{n}}, \\ 0 & \text{for } z + \dfrac{2}{\sqrt{n}} \leq x\,. \end{cases}$$

We get the following result:

1.2.2 Proposition

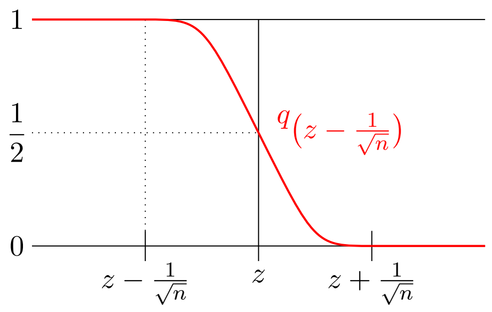
$$||F_n - e_n|| \le \sup_{z\in\mathbb{R}} \Big| \int_{\mathbb{R}} q_z \, dF_n - \int_{\mathbb{R}} q_z \, e_n' \, dx \Big| + \frac{1}{n}\Big(\frac{7}{12}\,\mathcal{C} + \frac{1}{48} + \frac{1}{60}\,\frac{\beta_3}{\sqrt{n}}\Big)\,.$$

Proof:

a) Integration by parts for Lebesgue-Stieltjes integrals (cf. e.g. [2], Theorem 5.2.3) gives

$$\Big| F_n(z) - \int_{\mathbb{R}} q_{\left(z - \frac{1}{\sqrt{n}}\right)} \, dF_n \Big|$$

$$= \Big| \int_{\mathbb{R}} 1_{(-\infty,\,z\,]} - q_{\left(z - \frac{1}{\sqrt{n}}\right)} \, dF_n \Big|$$



, hence

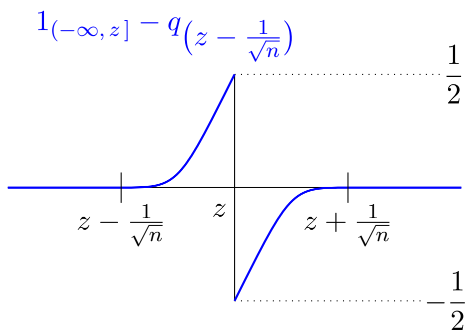


Fig. 1.2.1: Sketch of the functions used

$$= \Big| \int_{\left(z - \frac{1}{\sqrt{n}},\, z\,\right]} \frac{n}{2}\Big(x - \big(z - \frac{1}{\sqrt{n}}\big)\Big)^2 dF_n(x) - \int_{\left(z,\, z + \frac{1}{\sqrt{n}}\,\right]} \frac{n}{2}\Big(\big(z + \frac{1}{\sqrt{n}}\big) - x\Big)^2 dF_n(x) \Big|$$

$$= \Big| \frac{1}{2}\,F_n(z) - \int_{\left(z - \frac{1}{\sqrt{n}},\, z\,\right]} n\Big(x - \big(z - \frac{1}{\sqrt{n}}\big)\Big)\,F_n(x)\,dx \; +$$

$$\frac{1}{2}\,F_n(z) - \int_{\left(z,\, z + \frac{1}{\sqrt{n}}\,\right]} n\Big(\big(z + \frac{1}{\sqrt{n}}\big) - x\Big)\,F_n(x)\,dx \Big|$$

$$= \Big| F_n(z) + \int_z^{z - 1/\sqrt{n}} n\Big(\frac{1}{\sqrt{n}} - (z - x)\Big)\,F_n(x)\,dx - \int_z^{z+1/\sqrt{n}} n\Big(\frac{1}{\sqrt{n}} - (x - z)\Big)\,F_n(x)\,dx \Big|.$$

We now substitute in the first integral $y = z - x$ and in the second integral $y = x - z$. Because of $\int\limits_0^{1/\sqrt{n}} n\Big(\frac{1}{\sqrt{n}} - y\Big)\,dy = \frac{1}{2}$ we then obtain

$$= \Big| \int\limits_0^{1/\sqrt{n}} n\Big(\frac{1}{\sqrt{n}} - y\Big) \Big\{ -F_n(z+y) + 2\,F_n(z) - F_n(z-y) \Big\}\, dy \Big|$$

$$\leq \int\limits_0^{1/\sqrt{n}} n\Big(\frac{1}{\sqrt{n}} - y\Big)\, \mathcal{C}\Big(\frac{1}{n} + y^2\Big)\, dy \qquad \text{(because of (1.1.9))}$$

$$= \frac{7}{12}\,\frac{\mathcal{C}}{n}.$$

b) In addition, we have for all $y \in \mathbb{R}$

$$||\Delta_y^2\, e_n|| \leq y^2\, ||e_n''|| \qquad \text{(cf. Lemma 2.4.2, b))}$$

$$\leq y^2 \Big(||x\,\psi(x)|| + \frac{\beta_3}{6\,\sqrt{n}}\, ||(x^4 - 6x^2 + 3)\,\psi(x)|| \Big)$$

$$\leq y^2 \Big(\frac{1}{4} + \frac{1}{5}\,\frac{\beta_3}{\sqrt{n}} \Big) \qquad \text{(cf. Lemma 2.2.8, b) and h)).}$$

By proceeding in the same way as in part a) with $e_n(z)$ instead of $F_n(z)$, we get

$$\Big| e_n(z) - \int\limits_{\mathbb{R}} q_{\left(z - \frac{1}{\sqrt{n}}\right)}\, e_n'\, dx \Big|$$

$$\leq \int\limits_0^{1/\sqrt{n}} n\Big(\frac{1}{\sqrt{n}} - y\Big)\, y^2 \Big(\frac{1}{4} + \frac{1}{5}\,\frac{\beta_3}{\sqrt{n}} \Big)\, dy$$

$$= \frac{1}{n} \Big(\frac{1}{48} + \frac{1}{60}\,\frac{\beta_3}{\sqrt{n}} \Big).$$

□

1.3 The central step

According to Proposition 1.2.2, the term

$$\sup_{z\in\mathbb{R}} \Big| \int\limits_{\mathbb{R}} q_z\, dF_n - \int\limits_{\mathbb{R}} q_z\, e_n'\, dx \Big|$$

still needs to be estimated. To do this, we use Stein's method to derive an equation of the type

$$\int_{\mathbb{R}} q_z \, dF_n - \int_{\mathbb{R}} q_z \, \psi \, dx = T(q_z) + R(q_z) \tag{1.3.1}$$

for each $z \in \mathbb{R}$. For the two right-hand summands, we can then show that

$$\Big| T(q_z) - \frac{\mu_3}{6\sqrt{n}} \int_{\mathbb{R}} q_z(x) \, (3\,x - x^3) \, \psi(x) \, dx \Big| \le \frac{c_1}{n} \qquad \text{(see section 1.4)}$$

and

$$|R(q_z)| \le \frac{c_2 \, \mathcal{C} + c_3}{n} \qquad \text{(see section 1.5)}$$

holds.

In order to be able to formulate the summands $T(q_z)$ and $R(q_z)$ precisely, we define

$$f_z(x) = \psi(x)^{-1} \int_{-\infty}^{x} \big(q_z(y) - \Phi(q_z) \big) \, \psi(y) \, dy. \tag{1.3.2}$$

This function is a solution of the differential equation (or, in this context, Stein's equation)

$$f_z'(x) - x \, f_z(x) = q_z(x) - \Phi(q_z) \tag{1.3.3}$$

and also has some very useful analytical properties (cf. chapter 2).

The precise form of (1.3.1) is now:

1.3.4 Proposition

For every $z \in \mathbb{R}$ is valid:

$$\int_{\mathbb{R}} q_z \, dF_n - \int_{\mathbb{R}} q_z \, \psi \, dx = T(q_z) + R(q_z),$$

where

$$T(q_z) = \frac{\mu_3}{2\sqrt{n}} \Big(-E\big(S_n \, f_z'(S_n) \big) \Big),$$

$$R(q_z) = \frac{1}{\sqrt{n}} \, E\bigg(\int_0^1 \Big[X_n + \frac{\mu_3}{2} X_n^2 - (1-t) \, X_n^3 \Big] \Big(f_z''\big(S_{n-1}^n + t \, \frac{X_n}{\sqrt{n}} \big) - f_z''\big(S_{n-1}^n \big) \Big) \, dt \bigg).$$

Here we write $S_{n-1}^n = \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{n-1} X_i$.

Proof:

Throughout the proof, let $z \in \mathbb{R}$ be fixed. We can therefore set $q = q_z$ and $f = f_z$.

Using (1.3.3), we then obtain

$$\begin{aligned}\int\limits_{\mathbb{R}} q\,dF_n - \int\limits_{\mathbb{R}} q\,\psi\,dx &= E\Big(q(S_n) - \Phi(q)\Big)\\ &= E\Big(f'(S_n) - S_n\,f(S_n)\Big)\\ &= E\Big(f'(S_n)\Big) - E\Big(\sqrt{n}\,X_n\,f(S_n)\Big).\end{aligned}$$

For the last equation, it was used that the X_i are independent and identically distributed. It should also be noted that the above chain of equations forms the core of Stein's method.

We now consider the terms $E\Big(f'(S_n)\Big)$ and $E\Big(\sqrt{n}\,X_n\,f(S_n)\Big)$ separately. A Taylor expansion of f' and f about S_{n-1}^n then gives, if we additionally use $E(X_n) = 0$, $E(X_n^2) = 1$ and the independence of X_n and S_{n-1}^n:

$$\begin{aligned}E\Big(f'(S_n)\Big) &= E\bigg(\,f'(S_{n-1}^n) + \frac{X_n}{\sqrt{n}}\,f''(S_{n-1}^n) + \frac{X_n}{\sqrt{n}}\int\limits_0^1 \Big(\,f''\big(S_{n-1}^n + t\,\frac{X_n}{\sqrt{n}}\big) - f''(S_{n-1}^n)\Big)\,dt\bigg)\\ &= E\Big(\,f'(S_{n-1}^n)\Big) + E\bigg(\,\frac{X_n}{\sqrt{n}}\int\limits_0^1 \Big(\,f''\big(S_{n-1}^n + t\,\frac{X_n}{\sqrt{n}}\big) - f''(S_{n-1}^n)\Big)\,dt\bigg),\end{aligned}$$

$$\begin{aligned}E\Big(\sqrt{n}\,X_n\,f(S_n)\Big) &= E\bigg(\,\sqrt{n}\,X_n\,\Big[\,f(S_{n-1}^n) + \frac{X_n}{\sqrt{n}}\,f'(S_{n-1}^n) + \frac{X_n^2}{2n}\,f''(S_{n-1}^n)\\ &\qquad + \frac{X_n^2}{n}\int\limits_0^1 (1-t)\,\Big(\,f''\big(S_{n-1}^n + t\,\frac{X_n}{\sqrt{n}}\big) - f''(S_{n-1}^n)\Big)\,dt\,\Big]\bigg)\\ &= E\Big(\,f'(S_{n-1}^n)\Big) + \frac{\mu_3}{2\sqrt{n}}\,E\Big(\,f''(S_{n-1}^n)\Big)\\ &\qquad + E\bigg(\,\frac{X_n^3}{\sqrt{n}}\int\limits_0^1 (1-t)\,\Big(\,f''\big(S_{n-1}^n + t\,\frac{X_n}{\sqrt{n}}\big) - f''(S_{n-1}^n)\Big)\,dt\bigg).\end{aligned}$$

Combining these calculations yields

$$\int\limits_{\mathbb{R}} q\,dF_n - \int\limits_{\mathbb{R}} q\,\psi\,dx = \frac{\mu_3}{2\sqrt{n}}\,E\Big(-f''(S_{n-1}^n)\Big) \tag{1.3.5}$$

$$+\frac{1}{\sqrt{n}}\,E\Big(\int_0^1 \Big[X_n-(1-t)\,X_n^3\Big]\,\Big(f''\big(S_{n-1}^n+t\,\frac{X_n}{\sqrt{n}}\big)-f''(S_{n-1}^n)\Big)\,dt\Big).$$

The degree of the derivative of f in the term $\dfrac{\mu_3}{2\sqrt{n}}\,E\Big(-f''(S_{n-1}^n)\Big)$ is now being reduced again. The following calculation is used for this:

$$\begin{aligned}
&E\Big(S_n\,f'(S_n)\Big)\\
&=E\Big(\sqrt{n}\,X_n\,f'(S_n)\Big)\\
&=E\Big(\sqrt{n}\,X_n\Big[f'(S_{n-1}^n)+\frac{X_n}{\sqrt{n}}\,f''(S_{n-1}^n)+\frac{X_n}{\sqrt{n}}\int_0^1\Big(f''\big(S_{n-1}^n+t\,\frac{X_n}{\sqrt{n}}\big)-f''(S_{n-1}^n)\Big)\,dt\Big]\Big)\\
&=E\Big(f''(S_{n-1}^n)\Big)+E\Big(X_n^2\int_0^1\Big(f''\big(S_{n-1}^n+t\,\frac{X_n}{\sqrt{n}}\big)-f''(S_{n-1}^n)\Big)\,dt\Big).
\end{aligned}$$

If we now insert the corresponding expression for $E\Big(f''(S_{n-1}^n)\Big)$ into (1.3.5), we obtain the assertion of the proposition. □

1.4 Deducing the correct expansion

This section deals with the term $T(q_z)=\dfrac{\mu_3}{2\sqrt{n}}\Big(-E\big(S_n\,f_z'(S_n)\big)\Big)$. We show the following result.

1.4.1 Proposition

$$\sup_{z\in\mathbb{R}}\Big|E\Big(S_n\,f_z'(S_n)\Big)-\Phi\Big(x\,f_z'(x)\Big)\Big|\le 21,2\,\frac{\beta_3}{\sqrt{n}}.$$

Since

$$\Phi\big(x\,f_z'(x)\big)=\frac{1}{3}\int_{\mathbb{R}}q_z(x)\,(3\,x-x^3)\,\psi(x)\,dx\qquad\text{for all } z\in\mathbb{R}$$

(cf. Lemma 2.2.5, c)), we get from Proposition 1.4.1 the estimate:

$$\text{(1.4.2)}\qquad \sup_{z\in\mathbb{R}}\Big|T(q_z)+\frac{\mu_3}{6\sqrt{n}}\int_{\mathbb{R}}q_z(x)\,(3\,x-x^3)\,\psi(x)\,dx\Big|\le 10,6\,\frac{\beta_3^2}{n}\le 10,6\,\frac{\beta_4}{n}.$$

For the second inequality, the following application of Hölder's inequality is used:

$$\beta_3 = E\Big(|X_i|\,|X_i|^2\Big) \le \Big(E(X_i^2)\Big)^{1/2}\,\beta_4^{1/2} = \beta_4^{1/2}.$$

We take the most important step in proving Proposition 1.4.1 by proving the following theorem, which is interesting in itself. This theorem is also useful in the next section when estimating $R(q_z)$.

1.4.3 Theorem

$$\sup_{z\in\mathbb{R}} \left| E\Big(\,|S_n|\,1_{(-\infty,\,z\,]}(S_n)\Big) - \Phi\Big(\,|x|\,1_{(-\infty,\,z\,]}(x)\Big)\right| \le 10,6\,\frac{\beta_3}{\sqrt{n}}.$$

Proof:

We proceed as in the proof of Bolthausen's paper (cf. [6], pp. 380, 381) using the method of Stein and a recursion argument. Furthermore, in the following we assume $n \ge 75$ without loss of generality. This is possible because we have $E\big(|S_n|\big) \le 1$, $\Phi\big(|x|\big) \le 1$ and $\beta_3 \ge 1$.

For $x, z \in \mathbb{R}$ and $\lambda > 0$ we now define the functions

$$\begin{aligned} p_z^\lambda(x) &= |x|\,\Big\{\,1_{(-\infty,\,z\,]}(x) + \frac{z+\lambda-x}{\lambda}\,1_{(\,z,\,z+\lambda\,]}(x)\,\Big\}, \\ p_z^0(x) &= |x|\,1_{(-\infty,\,z\,]}(x). \end{aligned}$$

Furthermore, let

$$\begin{aligned} \mathscr{L}(n,\beta_3) \; &= \Big\{\,\underline{X} = (X_1,\dots,X_n) :\, X_1,\dots,X_n \text{ are independent and identically} \\ &\qquad \text{distributed with } E(X_i) = 0,\ E(X_i^2) = 1 \text{ and } E(|X_i|^3) = \beta_3\,\Big\}, \\ \delta(\lambda,\beta_3,n) &= \sup\Big\{\,\Big|\,E\big(p_z^\lambda(S_n)\big) - \Phi\big(p_z^\lambda\big)\,\Big| \,:\, z\in\mathbb{R},\ \underline{X}\in\mathscr{L}(n,\beta_3)\,\Big\}, \\ \delta(\beta_3,n) \; &= \delta(0,\beta_3,n). \end{aligned}$$

By using $p_{z-\lambda}^\lambda \le p_z^0 \le p_z^\lambda$ and $\sup_{x\in\mathbb{R}} |x|\,\psi(x) = \dfrac{1}{\sqrt{2\,\pi\,e}}$ (cf. Lemma 2.2.8, b)) we obtain

$$\begin{aligned} E\big(p_z^0(S_n)\big) - \Phi\big(p_z^0\big) &\le E\big(p_z^\lambda(S_n)\big) - \Phi\big(p_z^\lambda\big) + \Phi\big(p_z^\lambda\big) - \Phi\big(p_z^0\big) \\ &\le \delta(\lambda,\beta_3,n) + \frac{1}{\sqrt{2\,\pi\,e}}\int_z^{z+\lambda} \frac{z+\lambda-x}{\lambda}\,dx \qquad \text{and} \end{aligned}$$

$$-\Big(\,E\big(p_z^0(S_n)\big) - \Phi\big(p_z^0\big)\Big) \le \Phi\big(p_{z-\lambda}^\lambda\big) - E\big(p_{z-\lambda}^\lambda(S_n)\big) + \Phi\big(p_z^0\big) - \Phi\big(p_{z-\lambda}^\lambda\big)$$

$$\leq \delta(\lambda, \beta_3, n) + \frac{1}{\sqrt{2\,\pi\, e}} \int\limits_{z-\lambda}^{z} 1 - \frac{z-x}{\lambda}\, dx.$$

Together this leads to

$$\delta(\beta_3, n) \;\leq\; \delta(\lambda, \beta_3, n) + \frac{\lambda}{2}\, \frac{1}{\sqrt{2\,\pi\, e}}. \tag{1.4.4}$$

In order to estimate $\delta(\beta_3, n)$, we will first estimate $\delta(\lambda, \beta_3, n)$. To do this, we consider

$$\Big|\, E\big(p_z^\lambda(S_n)\big) - \Phi\big(p_z^\lambda\big) \,\Big| \qquad \text{for } \textbf{fixed } \lambda > 0 \quad \text{and} \quad z \in \mathbb{R}.$$

To simplify the notations we will write $p = p_z^\lambda$ in the following and define

$$d(x) = \psi(x)^{-1} \int\limits_{-\infty}^{x} \big(p(y) - \Phi(p)\,\big)\, \psi(y)\, dy.$$

Then Stein's equation is

$$d'(x) - x\, d(x) = p(x) - \Phi(p).$$

This leads to

$$\begin{aligned}
&\Big|\, E\big(p(S_n)\big) - \Phi\big(p\big) \,\Big| \\
&= \Big|\, E\big(d'(S_n)\big) - E\big(S_n\, d(S_n)\big) \,\Big| \\
&= \Big|\, E\big(d'(S_n)\big) - \sqrt{n}\, E\big(X_n\, d(S_n)\big) \,\Big| \\
&\leq E\Big(\Big|\, d'(S_n) - d'(S_{n-1}^n) \,\Big| \Big) + E\Big(X_n^2 \int\limits_0^1 \Big|\, d'\big(\, S_{n-1}^n + t\, \frac{X_n}{\sqrt{n}}\big) - d'\big(\, S_{n-1}^n\,\big) \,\Big|\, dt \Big) \\
&= A_1 + A_2.
\end{aligned}$$

To obtain the inequality in the above calculations, we used the independence of X_n and $S_{n-1}^n = \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{n-1} X_i$ and $E(X_n) = 0$, $E(X_n^2) = 1$, as in the last section.

For the estimate of A_1 we now use

$$\begin{aligned}
&|\, d'(x+y) - d'(x)\, | \\
&\leq |y| \left\{ 2 + \sqrt{\frac{2}{\pi}}\, |x| + x^2 + |x\, y| + \frac{|x|}{\lambda} \int\limits_0^1 1_{(z,\, z+\lambda]}(x + s\, y)\, ds \right\} \qquad \text{for } x, y \in \mathbb{R}
\end{aligned}$$

(cf. Lemma 2.3.5, c), second inequality), so that for A_1 follows

$$A_1 \le E\Big(\frac{|X_n|}{\sqrt{n}} \Big\{ 2 + \sqrt{\frac{2}{\pi}}\, \big|S_{n-1}^n\big| + \big(S_{n-1}^n\big)^2 + \big|S_{n-1}^n\big| \frac{|X_n|}{\sqrt{n}} + \frac{\big|S_{n-1}^n\big|}{\lambda} \int\limits_0^1 1_{(z,\, z+\lambda]}\big(S_{n-1}^n + s\,\frac{X_n}{\sqrt{n}}\big)\, ds \Big\} \Big)$$

$$\le 3,9134\, \frac{\beta_3}{\sqrt{n}} + \frac{1}{\lambda\sqrt{n}} \int\limits_0^1 \int\limits_{\mathbb{R}} |x_n|\, E\Big(\big|S_{n-1}^n\big|\, 1_{(z,\, z+\lambda]}\big(S_{n-1}^n + s\,\frac{x_n}{\sqrt{n}}\big) \Big)\, PX_n^{-1}(dx_n)\, ds.$$

Here again, the independence of X_n and S_{n-1}^n, and $E\big((S_{n-1}^n)^2\big) \le 1$, $\beta_3 \ge 1$ and $n \ge 75$ are used.

Let us denote for a short moment with $a = \sqrt{\frac{n}{n-1}}\,\big(z - s\,\frac{x_n}{\sqrt{n}}\big)$ and $b = \sqrt{\frac{n}{n-1}}\,\lambda \le \sqrt{\frac{75}{74}}\,\lambda$, then

$$\begin{aligned} & E\Big(\big|S_{n-1}^n\big|\, 1_{(z,\, z+\lambda]}\big(S_{n-1}^n + s\,\frac{x_n}{\sqrt{n}}\big) \Big) \\ & \le E\Big(\big|S_{n-1}\big|\, 1_{(a,\, a+b]}\big(S_{n-1}\big) \Big) \\ & \le \Big| E\big(p_{a+b}^0(S_{n-1})\big) - \Phi\big(p_{a+b}^0\big) \Big| + \Big| E\big(p_a^0(S_{n-1})\big) - \Phi\big(p_a^0\big) \Big| + b \sup_{z\in\mathbb{R}} |x|\,\psi(x) \\ & \le 2\,\delta(\beta_3, n-1) + 0,2437\,\lambda. \end{aligned} \tag{1.4.5}$$

This leads to

$$A_1 \le 4,1571\, \frac{\beta_3}{\sqrt{n}} + \frac{2}{\lambda}\, \frac{\beta_3}{\sqrt{n}}\, \delta(\beta_3, n-1).$$

For the estimate of A_2, on the other hand, we use

$$\begin{aligned} & |\, d'(x+y) - d'(x)\, | \\ & \le |y| \Big\{ 2 + \frac{|x|}{\lambda} \int\limits_0^1 1_{(z,\, z+\lambda]}(x + s\,y)\, ds \Big\} + |x|\, |\, d(x+y) - d(x)\, | \qquad \text{for } x, y \in \mathbb{R} \end{aligned}$$

(cf. Lemma 2.3.5, c), first inequality), from which follows

$$A_2 \le E\Big(X_n^2 \int\limits_0^1 t\, \frac{|X_n|}{\sqrt{n}} \Big\{ 2 + \frac{\big|S_{n-1}^n\big|}{\lambda} \int\limits_0^1 1_{(z,\, z+\lambda]}\big(S_{n-1}^n + s\,t\,\frac{X_n}{\sqrt{n}}\big)\, ds \Big\}\, dt \Big)$$

$$
\begin{aligned}
&+ E\Big(1_{\{|X_n| \le 1{,}4477\sqrt{n}\}} X_n^2 \, |S_{n-1}^n| \int_0^1 \Big| d\big(S_{n-1}^n + t\,\frac{X_n}{\sqrt{n}}\big) - d\big(S_{n-1}^n\big) \Big| \, dt \Big) \\
&+ E\Big(1_{\{|X_n| > 1{,}4477\sqrt{n}\}} X_n^2 \, |S_{n-1}^n| \int_0^1 \Big| d\big(S_{n-1}^n + t\,\frac{X_n}{\sqrt{n}}\big) - d\big(S_{n-1}^n\big) \Big| \, dt \Big) \\
= \; & A_3 + A_4 + A_5.
\end{aligned}
$$

Using Fubini's theorem and a calculation analogous to (1.4.5), we further obtain

$$
\begin{aligned}
A_3 &= \int_0^1 t \int_0^1 \int_{\mathbb{R}} \frac{|x_n|^3}{\sqrt{n}} \, E\Big(2 + \frac{|S_{n-1}^n|}{\lambda} \, 1_{(z,\, z+\lambda]}\big(S_{n-1}^n + s\,t\,\frac{x_n}{\sqrt{n}}\big) \Big) \, PX_n^{-1}(dx_n)\, ds\, dt \\
&\le \int_0^1 t \int_0^1 \int_{\mathbb{R}} \frac{|x_n|^3}{\sqrt{n}} \Big(2 + \frac{1}{\lambda} \Big(2\,\delta(\beta_3, n-1) + 0{,}2437\,\lambda \Big) \Big) \, PX_n^{-1}(dx_n)\, ds\, dt \\
&= \int_0^1 t\, dt \, \Big(2{,}2437 \, \frac{\beta_3}{\sqrt{n}} + \frac{2}{\lambda}\,\frac{\beta_3}{\sqrt{n}}\, \delta(\beta_3, n-1) \Big) \\
&\le 1{,}1219 \, \frac{\beta_3}{\sqrt{n}} + \frac{1}{\lambda}\,\frac{\beta_3}{\sqrt{n}}\, \delta(\beta_3, n-1).
\end{aligned}
$$

For the estimate of A_4 we use

$$
|\, d(x+y) - d(x)\,| \;\le\; |y| \sup_{0<\theta<1} |d'(x+\theta\, y)| \;\le\; |y| \Big\{ \, |x| + |y| + \sqrt{\frac{2}{\pi}} \, \Big\}
$$

(cf. the estimate of f_1' in Corollary 2.1.12, b)). We then get

$$
\begin{aligned}
A_4 &\le E\Big(1_{\{|X_n| \le 1{,}4477\sqrt{n}\}} \, \frac{|X_n|^3}{\sqrt{n}} \, |S_{n-1}^n| \int_0^1 t \Big\{ \, |S_{n-1}^n| + t\, \frac{|X_n|}{\sqrt{n}} + \sqrt{\frac{2}{\pi}} \, \Big\} \, dt \Big) \\
&\le E\Big(1_{\{|X_n| \le 1{,}4477\sqrt{n}\}} \, \frac{|X_n|^3}{\sqrt{n}} \, |S_{n-1}^n| \Big\{ \, \frac{1}{2}\, |S_{n-1}^n| + \frac{1}{3}\, 1{,}4477 + \frac{1}{2} \sqrt{\frac{2}{\pi}} \, \Big\} \Big) \\
&\le E\Big(1_{\{|X_n| \le 1{,}4477\sqrt{n}\}} \, \frac{|X_n|^3}{\sqrt{n}} \Big) \, E\Big(\, |S_{n-1}^n| \Big\{ \, \frac{1}{2}\, |S_{n-1}^n| + 0{,}8816 \, \Big\} \Big) \\
&\le 1{,}3816 \, E\Big(1_{\{|X_n| \le 1{,}4477\sqrt{n}\}} \, \frac{|X_n|^3}{\sqrt{n}} \Big) \qquad\qquad \Big(E\big((S_{n-1}^n)^2\big) \le 1 \Big).
\end{aligned}
$$

For the estimate of A_5, on the other hand, we use

$$
|\, d(x+y) - d(x)\,| \;\le\; |\, d(x+y)\,| + |\, d(x)\,| \;\le\; 2
$$

(cf. the estimate of f_1 in Corollary 2.1.12, b)). The result is

$$\begin{aligned} A_5 &\le 2\,E\Big(1_{\{|X_n|>1{,}4477\sqrt{n}\}}\,X_n^2\Big)\,E\Big(\big|S_{n-1}^n\big|\Big) \\ &\le 1,3816\,E\Big(1_{\{|X_n|>1{,}4477\sqrt{n}\}}\,\frac{|X_n|^3}{\sqrt{n}}\Big) \qquad \Big(\frac{2}{1,4477} \le 1,3816\Big). \end{aligned}$$

If we summarize the results for A_3, A_4 and A_5, we obtain

$$\begin{aligned} A_2 &\le 1,1219\,\frac{\beta_3}{\sqrt{n}} + \frac{1}{\lambda}\,\frac{\beta_3}{\sqrt{n}}\,\delta(\beta_3, n-1) + 1,3816\,\frac{\beta_3}{\sqrt{n}} \\ &= 2,5035\,\frac{\beta_3}{\sqrt{n}} + \frac{1}{\lambda}\,\frac{\beta_3}{\sqrt{n}}\,\delta(\beta_3, n-1). \end{aligned}$$

Together with the estimate of A_1 and using (1.4.4), we get for all $n \ge 75$

$$\delta(\beta_3, n) \;\le\; 6,6606\,\frac{\beta_3}{\sqrt{n}} + \frac{3}{\lambda}\,\frac{\beta_3}{\sqrt{n}}\,\delta(\beta_3, n-1) + \frac{\lambda}{2}\,\frac{1}{\sqrt{2\,\pi\,e}}.$$

If we now choose $\lambda = 16,26\,\frac{\beta_3}{\sqrt{n}}$, we obtain for all $n \ge 75$

$$\delta(\beta_3, n) \;\le\; 8,6279\,\frac{\beta_3}{\sqrt{n}} + 0,1846\,\delta(\beta_3, n-1)$$

and thus

$$\delta(\beta_3, n) \;\le\; 8,6279\,\beta_3 \sum_{i=0}^{n-75} \frac{0,1846^i}{\sqrt{n-i}} + 0,1846^{n-74}\,\delta(\beta_3, 74).$$

For further considerations, we note that for all $n \ge 75$ we have

$$\frac{n}{n-i} = \frac{n}{n-1}\,\frac{n-1}{n-2}\,\cdots\,\frac{n-i+1}{n-i} \le \Big(\frac{75}{74}\Big)^i \qquad \text{for } 0 \le i \le n-74.$$

Moreover, due to $\delta(\beta_3, 74) \le 1$, $\beta_3 \ge 1$ and $\sqrt{74} \le 8,6279$, we also have for all $n \ge 75$

$$0,1846^{n-74}\,\delta(\beta_3, 74) \le 0,1846^{n-74}\,1 \le 0,1846^{n-74}\,8,6279\,\frac{\beta_3}{\sqrt{n-(n-74)}}.$$

So all in all, using the summation formula for the geometric series leads to

$$\delta(\beta_3, n) \le 8,6279\,\frac{\beta_3}{\sqrt{n}} \sum_{i=0}^{n-74} \Big(0,1846\,\sqrt{\frac{75}{74}}\Big)^i$$

$$\leq 8,6279\,\frac{\beta_3}{\sqrt{n}}\,\sum_{i=0}^{\infty} 0,1859^i \;\leq\; 10,5981\,\frac{\beta_3}{\sqrt{n}}.$$

This proves the theorem. □

1.4.6 Remarks

a) The above proof shows that for Theorem 1.4.3 and thus all results of this section only $\beta_3 < \infty$ instead of $\beta_4 < \infty$ has to be assumed.

b) The estimate

$$\sup_{z\in\mathbb{R}} \left| E\Big(|S_n|\, 1_{(-\infty,\,z\,]}(S_n) \Big) - \Phi\Big(|x|\, 1_{(-\infty,\,z\,]}(x) \Big) \right| \leq C\,\frac{\beta_3}{\sqrt{n}},$$

where $C > 0$ is an absolute constant, can also be deduced from a more general result of Sweeting [28] (see Theorem 1, page 39 and the following example). In contrast to here, however, Sweeting uses the operator method, which goes back to Trotter [30].

The Theorem 1.4.3 also holds without the vertical bars for the absolute values of S_n and x.

1.4.7 Corollary

$$\sup_{z\in\mathbb{R}} \left| E\Big(S_n\, 1_{(-\infty,\,z\,]}(S_n) \Big) - \Phi\Big(x\, 1_{(-\infty,\,z\,]}(x) \Big) \right| \leq 10,6\,\frac{\beta_3}{\sqrt{n}}.$$

Proof:

For $z \leq 0$ the above estimate follows directly from Theorem 1.4.3. For $z \geq 0$, on the other hand, we obtain

$$\begin{aligned}
&\left| E\Big(S_n\, 1_{(-\infty,\,z\,]}(S_n) \Big) - \Phi\Big(x\, 1_{(-\infty,\,z\,]}(x) \Big) \right| \\
&= \left| E\Big(S_n\, 1_{(\,z,\,\infty\,)}(S_n) \Big) - \Phi\Big(x\, 1_{(\,z,\,\infty\,)}(x) \Big) \right| && \Big(\text{since } E(S_n) = 0 = \Phi(id_{\mathbb{R}})\Big) \\
&= \left| E\Big(|S_n|\, 1_{(\,z,\,\infty\,)}(S_n) \Big) - \Phi\Big(|x|\, 1_{(\,z,\,\infty\,)}(x) \Big) \right| && \Big(\text{since } z \geq 0\Big) \\
&= \left| E\Big(|-S_n|\, 1_{(\,-\infty,\,-z\,)}(-S_n) \Big) - \Phi\Big(|x|\, 1_{(\,-\infty,\,-z\,)}(x) \Big) \right| \\
&\leq 10,6\,\frac{\beta_3}{\sqrt{n}}.
\end{aligned}$$

The inequality is valid, because with $X_1, \ldots, X_n$ also $-X_1, \ldots, -X_n$ fulfil the conditions of Theorem 1.4.3. $\square$

For the proof of Proposition 1.4.1 we need a slightly more general version of the last corollary. This reads as follows:

1.4.8 Corollary

Let $\mathcal{H}$ be defined according to (0.1.1). Then

$$\sup_{h\in\mathcal{H}} \left| E\Big(S_n\, h(S_n) \Big) - \Phi\Big(x\, h(x) \Big) \right| \le 10,6\, \frac{\beta_3}{\sqrt{n}}.$$

Proof:

According to Corollary 1.4.7,

$$\left| E\Big(S_n\, h(S_n) \Big) - \Phi\Big(x\, h(x) \Big) \right| \le 10,6\, \frac{\beta_3}{\sqrt{n}} \tag{1.4.9}$$

is valid for the functions $h = 1_{(-\infty,\, z\,]}$, $z \in \mathbb{R}$. From this we can derive (1.4.9) for convex combinations of these functions and then, using majorized convergence, for all $h \in \mathcal{H}$.

To apply the theorem of majorized convergence, we note that for all $h \in \mathcal{H}$ the inequalities $\big|\, S_n\, h(S_n) \,\big| \le \big|\, S_n \,\big|$ and $\big|\, x\, h(x)\, \psi(x) \,\big| \le \big|\, x\, \psi(x) \,\big|$ hold due to $0 \le h \le 1$. $\square$

We now obtain Proposition 1.4.1, since $f_z'(x) = h_1(x) - h_2(x)$ with

$$h_1(x) = q_z(x) \in \mathcal{H} \quad \text{and} \quad h_2(x) = -\,\Big(x\, f_z(x) - \Phi(q_z) \Big) \in \mathcal{H}$$

(cf. Lemma 2.1.13, a) and (1.3.3)).

1.5 The estimate of $R(q_z)$

Finally, in this section, we show the following estimate:

1.5.1 Proposition

$$\sup_{z\in\mathbb{R}} \Big| R(q_z) \Big| \;\le\; \frac{1}{n} \Big(\frac{1}{2} + \frac{5}{12}\, \beta_4 \Big) \Big(\frac{2\,n}{n-1}\, \mathcal{C} + 5,2 + 21,2\, \beta_3 \Big).$$

Proof:

For $x, y, z \in \mathbb{R}$ we use

$$\Big|\,(\,f_z''-q_z'\,)(x+y)-(\,f_z''-q_z'\,)(x)\,\Big|$$

$$\leq |y|\left\{3+\frac{\sqrt{2\pi}}{4}\,|x|+x^2+\sqrt{n}\,|x|\int_0^1 1_{\left(z,\,z+\frac{2}{\sqrt{n}}\right]}(x+s\,y)\,ds\right\}$$

(cf. Lemma 2.3.6, d) with $\lambda=\dfrac{1}{\sqrt{n}}$). Thus we obtain for all $z\in\mathbb{R}$

$$\Big|\,R(q_z)\,\Big|=\frac{1}{\sqrt{n}}\left|E\bigg(\int_0^1\Big[\,X_n+\frac{\mu_3}{2}\,X_n^2-(1-t)\,X_n^3\Big]\Big(f_z''(\,S_{n-1}^n+t\,\frac{X_n}{\sqrt{n}}\,)-f_z''(\,S_{n-1}^n\,)\Big)\,dt\bigg)\right|$$

$$\leq\frac{1}{n}\,(\,A_1+A_2+A_3\,),$$

where

$$A_1=E\bigg(\int_0^1\Big[\,t\,X_n^2+t\,\frac{\beta_3}{2}\,|X_n|^3+(t-t^2)\,X_n^4\Big]\,dt\,\Big\{\,3+\frac{\sqrt{2\pi}}{4}\,|S_{n-1}^n|+(S_{n-1}^n)^2\,\Big\}\bigg),$$

$$A_2=\int_0^1\int_0^1 E\bigg(\Big[\,t\,X_n^2+t\,\frac{\beta_3}{2}\,|X_n|^3+(t-t^2)\,X_n^4\Big]$$

$$\sqrt{n}\,|S_{n-1}^n|\,1_{\left(z,\,z+\frac{2}{\sqrt{n}}\right]}\Big(\,S_{n-1}^n+s\,t\,\frac{X_n}{\sqrt{n}}\,\Big)\bigg)\,ds\,dt,$$

$$A_3=\int_0^1\int_0^1\bigg|E\bigg(\Big[\,t\,X_n^2+t\,\frac{\mu_3}{2}\,X_n^3-(t-t^2)\,X_n^4\Big]$$

$$n\,\Big(1_{\left(z+\frac{1}{\sqrt{n}},\,z+\frac{2}{\sqrt{n}}\right]}-1_{\left(z,\,z+\frac{1}{\sqrt{n}}\right]}\Big)\Big(\,S_{n-1}^n+s\,t\,\frac{X_n}{\sqrt{n}}\,\Big)\bigg)\bigg|\,ds\,dt$$

(for A_3 see also Lemma 2.3.6, b)).

Due to the independence of X_n and S_{n-1}^n, and due to $E(X_n^2)=1$, $E\big((S_{n-1}^n)^2\big)\leq 1$ and $\beta_3^2\leq\beta_4$ then follows

$$A_1=E\bigg(\frac{1}{2}\,X_n^2+\frac{1}{2}\,\frac{\beta_3}{2}\,|X_n|^3+\frac{1}{6}\,X_n^4\bigg)\,E\bigg(\,3+\frac{\sqrt{2\pi}}{4}\,|S_{n-1}^n|+(S_{n-1}^n)^2\bigg)$$

$$\leq\Big(\,4+\frac{\sqrt{2\pi}}{4}\Big)\Big(\,\frac{1}{2}+\frac{5}{12}\,\beta_4\Big).$$

Furthermore, using Fubini's theorem and Theorem 1.4.3 and Lemma 2.2.8, b), we obtain

$$A_2 = \int_0^1 \int_0^1 \int_{\mathbb{R}} \Big[\, t\, x_n^2 + t\, \frac{\beta_3}{2}\, |x_n|^3 + (t - t^2)\, x_n^4 \,\Big]$$

$$\sqrt{n}\, E\bigg(|S_{n-1}^n|\, 1_{\left(z,\, z + \frac{2}{\sqrt{n}}\right]} \Big(S_{n-1}^n + s\, t\, \frac{x_n}{\sqrt{n}} \Big) \bigg)\, P X_n^{-1}(dx_n)\, ds\, dt$$

$$\le \Big(21,2\, \beta_3 + \frac{2}{\sqrt{2\, \pi\, e}} \Big) \Big(\frac{1}{2} + \frac{5}{12}\, \beta_4 \Big).$$

Finally, by applying again Fubini's theorem and then condition (1.1.10), we get

$$A_3 \le \int_0^1 \int_0^1 \int_{\mathbb{R}} \Big[\, t\, x_n^2 + t\, \frac{\beta_3}{2}\, |x_n|^3 + (t - t^2)\, x_n^4 \,\Big]$$

$$n \left| E\Big(\Big(1_{\left(z + \frac{1}{\sqrt{n}},\, z + \frac{2}{\sqrt{n}}\right]} - 1_{\left(z,\, z + \frac{1}{\sqrt{n}}\right]} \Big) \Big(S_{n-1}^n + s\, t\, \frac{x_n}{\sqrt{n}} \Big) \right| P X_n^{-1}(dx_n)\, ds\, dt$$

$$\le \frac{2\, n}{n-1}\, \mathcal{C}\, \Big(\frac{1}{2} + \frac{5}{12}\, \beta_4 \Big).$$

□

Overall, summarizing the results of this chapter (cf. Proposition 1.2.2, Proposition 1.3.4, (1.4.2) and Proposition 1.5.1) gives

$$||F_n - e_n|| \le \frac{1}{n} \bigg\{ \Big(\frac{7}{12}\, \mathcal{C} + \frac{1}{48} + \frac{1}{60}\, \frac{\beta_3}{\sqrt{n}} \Big) + 10,6\, \beta_4 + \Big(\frac{1}{2} + \frac{5}{12}\, \beta_4 \Big) \Big(\frac{2\, n}{n-1}\, \mathcal{C} + 5,2 + 21,2\, \beta_3 \Big) \bigg\}.$$

For $n \ge 6$ $\Big(\Leftrightarrow \dfrac{2\, n}{n-1} \le \dfrac{12}{5} \Big)$ this leads to

$$||F_n - e_n|| \le \frac{1}{n} \Big\{ \big(2 + \beta_4 \big)\, \mathcal{C} + \big(3 + 11\, \beta_3 + 13\, \beta_4 + 9\, \beta_3\, \beta_4 \big) \Big\}. \tag{1.5.2}$$

Because of (1.1.8) this completes the proof of Theorem 1.1.3.

2 Some analytical tools

2.1 Elementary properties of the solution f_k of Stein's equation for the function $|x|^k\, q(x)$

As we have seen in chapter 1, some properties of the solution f_k of Stein's equation for the function $|x|^k\, q(x)$ are required when applying Stein's method. These are provided in this and the following sections.

First of all, let $q : \mathbb{R} \to \mathbb{R}$ be measurable and bounded. For $k \in \mathbb{N}_0$ and $x \in \mathbb{R}$ we **define**

$$f_k(x) = \psi(x)^{-1} \int\limits_{-\infty}^{x} \Big(|y|^k\, q(y) - \Phi\big(|v|^k\, q(v) \big) \Big)\, \psi(y)\, dy,$$

$$f'_k(x) = x\, f_k(x) + |x|^k\, q(x) - \Phi\big(|v|^k\, q(v) \big) \qquad \text{(Stein's equation)}, \tag{2.1.1}$$

$$f = f_0,\ f' = f'_0.$$

In the case of a continuous function q (see e.g. $p^{\lambda}_{z,0}$, $q^{\lambda,0}_{z}$ and r^{λ}_{z} in section 2.3), f_k is therefore differentiable and the derivative is just the above f'_k. Now the following holds:

2.1.2 Lemma

Let $q \in \mathcal{H}$ (cf. (0.1.1) for the definition of $\mathcal{H}$), $k \in \mathbb{N}_0$ and f_k, f'_k, f, f' as above.

Furthermore, for $a \in \mathbb{R}$ let

$$\eta_k(a) = \sum_{i=1}^{\lfloor k/2 \rfloor} a^{k+1-2\,i} \prod_{j=1}^{i-1} (k+1-2\,j) + \prod_{j=1}^{\lfloor k/2 \rfloor} (k+1-2\,j).$$

As usual, $\sum\limits_{\emptyset} \ldots = 0$ and $\prod\limits_{\emptyset} \ldots = 1$.

Then, for all $k \in \mathbb{N}_0$ and $x \in \mathbb{R}$,

a) $|f_k(x)| \le \eta_k(|x|)$,

b) $|f_k'(x)| \le |x|\,\eta_k(|x|) + \Phi(|y|^k)$.

Proof:

First of all, we note that we may assume $q = 1_{(-\infty,\,z\,]}$, $z \in \mathbb{R}$, without loss of generality. From this we obtain the assertion for $q \in \mathcal{H}$ by first going to the convex combinations of the functions $1_{(-\infty,\,z\,]}$, $z \in \mathbb{R}$, and then to their limits via e.g. majorized convergence.

In the following, let $z \in \mathbb{R}$ be fixed and $q = 1_{(-\infty,\,z\,]}$. Then

$$\begin{aligned} f_k(x) &= \psi(x)^{-1} \int_{-\infty}^{x} \Big(|y|^k\, 1_{(-\infty,\,z\,]}(y) - \Phi\big(|v|^k\, 1_{(-\infty,\,z\,]}(v) \big) \Big)\, \psi(y)\, dy, \\ &= \psi(x)^{-1} \cdot \begin{cases} \Phi\Big(|y|^k\, 1_{(-\infty,\,x\,]}(y) \Big) - \Phi\Big(|y|^k\, 1_{(-\infty,\,z\,]}(y) \Big)\, \Phi(x) & \text{for } x \le z\,, \\ \Phi\Big(|y|^k\, 1_{(-\infty,\,z\,]}(y) \Big) \Big(1 - \Phi(x) \Big) & \text{for } x \ge z\,. \end{cases} \end{aligned}$$

With the definition

$$S_k(x) = \frac{1-\Phi(x)}{\psi(x)}\, \Phi\Big(|y|^k\, 1_{(-\infty,\,x\,]}(y) \Big),$$

we can show the estimates

$$\text{(2.1.3)} \qquad -S_k(-x) \;\le\; f_k(x) \;\le\; S_k(x) \quad \text{for all } x \in \mathbb{R},\ k \in \mathbb{N}_0.$$

We obtain the upper estimate if we replace z with x in the representations of $f_k(x)$ both in the case of $x \le z$ and in the case of $x \ge z$.

Furthermore, since $f_k(x) \ge 0$ for $x \ge z$, it is sufficient for the lower estimate to consider the case $x \le z$:

$$\begin{aligned} f_k(x) &= \psi(x)^{-1} \Big[\Phi\Big(|y|^k\, 1_{(-\infty,\,x\,]}(y) \Big) - \Phi\Big(|y|^k\, 1_{(-\infty,\,z\,]}(y) \Big)\, \Phi(x) \Big] \\ &\ge \psi(x)^{-1}\, \Phi(x) \Big[\Phi\Big(|y|^k\, 1_{(-\infty,\,x\,]}(y) \Big) - \Phi\Big(|y|^k \Big) \Big] \\ &= \psi(x)^{-1}\, \Phi(x) \Big[-\Phi\Big(|y|^k\, 1_{(\,x,\,\infty\,)}(y) \Big) \Big] \\ &= -\,\psi(-x)^{-1} \Big(1 - \Phi(-x) \Big)\, \Phi\Big(|y|^k\, 1_{(-\infty,\,-x\,]}(y) \Big) = -S_k(-x). \end{aligned}$$

This shows (2.1.3). From (2.1.3) we obtain

$$|f_k(x)| = \max\Big\{ -f_k(x),\, f_k(x) \Big\} \le \max\Big\{ S_k(-x),\, S_k(x) \Big\} \quad \text{for all } x \in \mathbb{R},\, k \in \mathbb{N}_0. \tag{2.1.4}$$

Part a) is thus verified by the proof of the following estimate:

$$0 \,\le\, S_k(x) \,\le\, \eta_k(|x|) \quad \text{for all } x \in \mathbb{R},\, k \in \mathbb{N}_0. \tag{2.1.5}$$

To prove (2.1.5), we first use that integration by parts (together with $\psi'(x) = -x\,\psi(x)$) gives the following recursion formulas:

$$\begin{aligned} &\Phi\Big(y^{k+2}\, 1_{(-\infty,\,x\,]}(y) \Big) = -x^{k+1}\,\psi(x) \,+ (\,k\,+\,1\,)\,\Phi\Big(y^k\, 1_{(-\infty,\,x\,]}(y) \Big) \qquad \text{and} \\ &\Phi\Big(|y|^{k+2} \Big) = (\,k\,+\,1\,)\,\Phi\Big(|y|^k \Big) \qquad \text{for all } x \in \mathbb{R},\, k \in \mathbb{N}_0. \end{aligned}$$

Furthermore, the following formulas follow from the definition of $\eta_k(x)$:

$$\begin{aligned} &\eta_{k+2}(x) = x^{k+1} + (k+1)\,\eta_k(x) \qquad \text{and} \\ &\eta_k(0) = \prod_{j=1}^{\lfloor k/2 \rfloor} (k+1-2\,j) \qquad \text{for all } x \in \mathbb{R},\, k \in \mathbb{N}_0. \end{aligned}$$

Thus, by induction we get

$$\Phi\Big(y^k\, 1_{(-\infty,\,x\,]}(y) \Big) = \begin{cases} -\psi(x)\,\eta_k(x) & \text{for } k = 1, 3, 5, \ldots \\[2ex] -\psi(x)\,\eta_k(x) + \Big(\psi(x) + \Phi(x) \Big)\,\eta_k(0) & \text{for } k = 0, 2, 4, \ldots \end{cases} \tag{2.1.6}$$

and

$$\Phi\Big(|y|^k \Big) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\,\eta_k(0) & \text{for } k = 1, 3, 5, \ldots \\[2ex] \eta_k(0) & \text{for } k = 0, 2, 4, \ldots\ . \end{cases} \tag{2.1.7}$$

In addition, we observe the following properties of $\eta_k(x)$:

$$\eta_k(x) = \eta_k(|x|) \qquad \text{for } \textbf{odd } k \text{ and } x \in \mathbb{R}, \tag{2.1.8}$$

$$-\,\big(\,\eta_k(x) - \eta_k(0)\,\big) = \eta_k(|x|) - \eta_k(0) \qquad \text{for } \textbf{even } k \text{ and } x \le 0. \tag{2.1.9}$$

Now (2.1.5) follows directly from (2.1.6) and (2.1.8) for **odd** k and $x \le 0$.

Moreover, from the well-known inequality (cf. e.g. C. Stein [26], inequality (2.67), p. 594)

$$\text{(2.1.10)} \qquad \frac{1-\Phi(x)}{\psi(x)} \le \frac{1}{x} \quad \text{for all } x>0,$$

it also follows that the function

$$b(y) = \frac{1-\Phi(y)}{\psi(y)} \quad \text{is monotonically decreasing all over } \mathbb{R}.$$

Using this and then (2.1.7), we get for **odd** k and $x \ge 0$ the estimate

$$S_k(x) \le \frac{1}{2\,\psi(0)}\,\Phi\big(\,|y|^k\,\big) = \eta_k(0) \le \eta_k(|x|).$$

Furthermore, to prove (2.1.5) for **even** k we still need

$$\text{(2.1.11)} \qquad \frac{\Phi(x)\,\big(1-\Phi(x)\big)}{\psi(x)} \le \min\left\{\frac{1}{4\,\psi(x)},\, \frac{\Phi(|x|)}{|x|}\right\} \le 1 \quad \text{for all } x \in \mathbb{R}.$$

The first inequality of (2.1.11) is valid because the function $g(x) = \Phi(x)\,\big(1-\Phi(x)\big)$ has its maximum at the point $x=0$ with $g(0) = \frac{1}{4}$. In addition, (2.1.10) is needed again.

For the second inequality of (2.1.11), we use $\psi(x) \ge \frac{1}{4}$ for $|x| \le 0,96$ and $\Phi(|x|) \le |x|$ for $|x| \ge 0,79$.

Because of (2.1.6) and (2.1.11) it now follows for **even** k and $x \ge 0$

$$\begin{aligned}
S_k(x) &= \frac{1-\Phi(x)}{\psi(x)}\Big[-\psi(x)\,\eta_k(|x|) + \Big(\psi(x)+\Phi(x)\Big)\,\eta_k(0)\Big] \\
&\le \frac{1-\Phi(x)}{\psi(x)}\Big[-\psi(x)\,\eta_k(|x|) + \Big(\psi(x)+\Phi(x)\Big)\,\eta_k(|x|)\Big] \\
&= \frac{1-\Phi(x)}{\psi(x)}\,\Phi(x)\,\eta_k(|x|) \le \eta_k(|x|)\,.
\end{aligned}$$

For **even** k and $x \le 0$, on the other hand, (2.1.6) and (2.1.11) and additionally (2.1.9) provide

$$\begin{aligned}
S_k(x) &= \frac{1-\Phi(x)}{\psi(x)}\Big[-\psi(x)\Big(\eta_k(x)-\eta_k(0)\Big) + \Phi(x)\,\eta_k(0)\Big] \\
&= \frac{1-\Phi(x)}{\psi(x)}\Big[\psi(x)\Big(\eta_k(|x|)-\eta_k(0)\Big) + \Phi(x)\,\eta_k(0)\Big] \\
&= \Big(1-\Phi(x)\Big)\Big(\eta_k(|x|)-\eta_k(0)\Big) + \frac{\big(1-\Phi(x)\big)\,\Phi(x)}{\psi(x)}\,\eta_k(0) \\
&\le \eta_k(|x|)-\eta_k(0)+\eta_k(0) = \eta_k(|x|).
\end{aligned}$$

This concludes the proof of (2.1.5) for all possible cases.

Multiplication of (2.1.3) with $x > 0$ or $x < 0$ also provides

$$-x\,S_k(-x) \le x\,f_k(x) \le x\,S_k(x) \qquad \text{for } x > 0 \qquad \text{and}$$

$$x\,S_k(x) \le x\,f_k(x) \le -x\,S_k(-x) \qquad \text{for } x < 0,$$

from which, in summary, for $x \in \mathbb{R}$ follows

$$-|x|\,S_k(-|x|) \;\le\; x\,f_k(x) \;\le\; |x|\,S_k(|x|).$$

Thus, by applying the definition (2.1.1) and the inequalities (2.1.10) and (2.1.5) we get

$$\begin{aligned} f_k'(x) &= x\,f_k(x) + |x|^k\,q(x) - \Phi\big(\,|v|^k\,q(v)\,\big) \\ &\le |x|\,S_k(|x|) + |x|^k \\ &\le \begin{cases} |x|\,\dfrac{1}{|x|}\,\Phi(|y|^k) + |x|^k \le \Phi(|y|^k) + |x|\,\eta_k(|x|) & \text{for } x \ne 0 \text{ and } k \ne 0, \\ |x|\,\eta_k(|x|) + |x|^k \qquad \le |x|\,\eta_k(|x|) + \Phi(|y|^k) & \text{for } x = 0 \text{ or } k = 0. \end{cases} \end{aligned}$$

On the other hand, using (2.1.1) and (2.1.5) we obtain

$$\begin{aligned} f_k'(x) &\ge -|x|\,S_k(-|x|) - \Phi(|y|^k) \\ &\ge -|x|\,\eta_k(\big|-|x|\big|) - \Phi(|y|^k) \\ &= -|x|\,\eta_k(|x|) - \Phi(|y|^k). \end{aligned}$$

All in all, this proves the part b). $\square$

For $k = 0$ we find better estimates of f_k and f_k' in the literature. These and simpler, but not as good estimates of f_k and f_k' for $k \ge 2$ are given below.

2.1.12 Corollary

Suppose that the conditions and notations of Lemma 2.1.2 apply. Then, for all $x \in \mathbb{R}$,

a) $|f(x)| \le \dfrac{\sqrt{2\pi}}{4} = 0,6267$ and $|f'(x)| \le 1$,

b) $|f_1(x)| \le 1$ and $|f_1'(x)| \le |x| + \sqrt{\dfrac{2}{\pi}}$,

c) $|f_k(x)| \le B_k\left(1+|x|^{k-1}\right)$ and $|f_k'(x)| \le 2\,B_k\left(1+|x|^k\right)$ for $k \ge 2$,

where $B_k = \left\lfloor \frac{k}{2} \right\rfloor \prod_{j=1}^{\lfloor k/2 \rfloor} (k+1-2\,j)$.

Proof:

Part a): The calculations for $|f'(x)| \le 1$ can be found in C. Stein [26], proof of the inequality (2.66) from Lemma 2.5, p. 594, 595. Alternatively, this inequality can also be found in Chen, L. H. Y., Goldstein, L. and Shao, Q. [8], Lemma 2.3, p. 16 (proof in the Appendix to Chapter 2, pp. 37, 38). The proof of $|f(x)| \le \sqrt{2\pi}/4$ can also be found there.

Part b): Due to $\eta_1 \equiv 1$ and (2.1.7), this follows from Lemma 2.1.2.

Part c): At first, we can use $|x|^l \le 1+|x|^r$ for $0<l<r$ to obtain the estimate

$$\sum_{i=1}^{\lfloor k/2 \rfloor} |x|^{k+1-2\,i} \le |x|^{k-1} + \sum_{i=2}^{\lfloor k/2 \rfloor} \left(1+|x|^{k-1}\right) = \left\lfloor \frac{k}{2} \right\rfloor |x|^{k-1} + \left\lfloor \frac{k}{2} \right\rfloor - 1 \quad \text{for } k \ge 2.$$

From this and Lemma 2.1.2 we get

$$\begin{aligned} |f_k(x)| \le \eta_k(|x|) &\le B_k\,|x|^{k-1} + \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right) \prod_{j=1}^{\lfloor k/2 \rfloor} (k+1-2\,j) + \prod_{j=1}^{\lfloor k/2 \rfloor} (k+1-2\,j) \\ &= B_k\left(|x|^{k-1}+1\right) \quad \text{for } k \ge 2. \end{aligned}$$

Finally, we additionally use $\Phi(|y|^k) \le B_k$ for $k \ge 2$ (cf. (2.1.7)) to estimate $|f_k'(x)|$. $\square$

Next, we prove monotonicity properties of some of the functions considered here.

2.1.13 Lemma

Let $q \in \mathcal{H}_c = \left\{ h \in \mathcal{H} : \text{there exists a } K = K(h) \in \mathbb{R} \text{ so that } h \text{ is continuous in } \mathbb{R} \setminus [-K,\, K] \right\}$. Furthermore, let $q(+\infty) = \lim_{x \to +\infty} q(x)$ and $q(-\infty) = \lim_{x \to -\infty} q(x)$, and

$$f^1(x) = \psi(x)^{-1} \int_{-\infty}^{x} \Big(y\,q(y) - \Phi\big(v\,q(v)\big) \Big)\,\psi(y)\,dy$$

and f as above (see (2.1.1)). Then

a) The mapping $x \to x\,f(x)$ is **monotonically increasing** with

$$\lim_{x\to-\infty} x\,f(x) = \Phi(q) - q(-\infty) \qquad \text{and} \qquad \lim_{x\to+\infty} x\,f(x) = \Phi(q) - q(+\infty),$$

b) The mapping $x \to f^1(x)$ is **monotonically increasing** with

$$\lim_{x\to-\infty} f^1(x) = -q(-\infty) \qquad \text{and} \qquad \lim_{x\to+\infty} f^1(x) = -q(+\infty).$$

Proof:

An application of l'Hospital's rule yields

$$\begin{aligned}\lim_{x\to\pm\infty} x\,f(x) &= \lim_{x\to\pm\infty} \frac{\int\limits_{-\infty}^{x} \Big(q(y) - \Phi(q)\Big)\,\psi(y)\,dy}{\frac{1}{x}\,\psi(x)} = \lim_{x\to\pm\infty} \frac{\Big(q(x) - \Phi(q)\Big)\,\psi(x)}{-\frac{1}{x^2}\,\psi(x) - \psi(x)} \\ &= \Phi(q) - q(\pm\infty)\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\pm\infty} f^1(x) &= \lim_{x\to\pm\infty} \frac{\int\limits_{-\infty}^{x} \Big(y\,q(y) - \Phi\big(v\,q(v)\big)\Big)\,\psi(y)\,dy}{\psi(x)} = \lim_{x\to\pm\infty} \frac{\Big(x\,q(x) - \Phi\big(v\,q(v)\big)\Big)\,\psi(x)}{-x\,\psi(x)} \\ &= -q(\pm\infty).\end{aligned}$$

To prove that the functions $x\,f(x)$ and $f^1(x)$ are monotonically increasing, we can assume as in the proof of Lemma 2.1.2 without loss of generality, that $q = 1_{(-\infty,\,z\,]}$ with a fixed $z \in \mathbb{R}$.

We then get

$$x\,f(x) = x\,\psi(x)^{-1} \cdot \begin{cases} \Phi(x)\,\big(1 - \Phi(z)\big) & \text{for all } x \le z, \\ \Phi(z)\,\big(1 - \Phi(x)\big) & \text{for all } x \ge z. \end{cases}$$

Differentiating this function at the point $x \ne z$ gives

$$\frac{d}{dx}\,x\,f(x) = \begin{cases} \big(1 - \Phi(z)\big)\,T(-x) & \text{for all } x < z, \\ \Phi(z)\,T(x) & \text{for all } x > z, \end{cases}$$

where

$$T(x) = (1 + x^2)\,\frac{1 - \Phi(x)}{\psi(x)} - x.$$

Since the function $x \to x\, f(x)$ is continuous in z, we obtain the desired monotonicity if we show $T(x) \geq 0$ for $x \in \mathbb{R}$.

For $x \leq 0$, however, this is clear, and for $x > 0$ this follows from the inequality

$$(2.1.14) \qquad \frac{1-\Phi(x)}{\psi(x)} \geq \frac{x}{1+x^2} \quad \text{for all } x > 0 \qquad \text{(cf. e.g. McKean [18], page 4, Problem 1).}$$

In addition, we have

$$f^1(x) = \psi(x)^{-1} \cdot \begin{cases} \Phi\big(\, y\, 1_{(-\infty,\, x\,]}(y)\,\big) - \Phi\big(\, y\, 1_{(-\infty,\, z\,]}(y)\,\big)\, \Phi(x) & \text{for all } x \leq z, \\ \Phi\big(\, y\, 1_{(-\infty,\, z\,]}(y)\,\big)\, \big(\, 1 - \Phi(x)\,\big) & \text{for all } x \geq z, \end{cases}$$

$$= \begin{cases} \psi(z)\, \dfrac{\Phi(x)}{\psi(x)} - 1 & \text{for all } x \leq z, \\ -\psi(z)\, \dfrac{1-\Phi(x)}{\psi(x)} & \text{for all } x \geq z. \end{cases}$$

Differentiating this function at the point $x \neq z$ gives

$$(2.1.15) \qquad \frac{d}{dx}\, f^1(x) = \begin{cases} \psi(z)\, \big(\, 1 - V(-x)\,\big) & \text{for all } x < z, \\ \psi(z)\, \big(\, 1 - V(x)\,\big) & \text{for all } x > z, \end{cases}$$

where

$$(2.1.16) \qquad V(x) = \frac{x\, \big(\, 1 - \Phi(x)\,\big)}{\psi(x)}.$$

Since the function $x \to f^1(x)$ is continuous in z, we obtain the desired monotonicity analogously to above if we show $V(x) \leq 1$ for $x \in \mathbb{R}$.

This is clear for $x \leq 0$, and for $x > 0$ this follows from the inequality (2.1.10), which has already been used several times. $\square$

From the lemma just proved we obtain further important estimates.

2.1.17 Corollary

Suppose that the conditions and notations of Lemma 2.1.13 apply. Furthermore, we **define**

$$(f^1)'(x) = x\, f^1(x) + x\, q(x) - \Phi\big(\, v\, q(v)\,\big) \qquad \text{(Stein's equation)}$$

analogously to (2.1.1). Then, for all $x \in \mathbb{R}$,

a) $|x\,f(x)| \le \max\big\{\, q(-\infty) - \Phi(q),\, \Phi(q) - q(+\infty) \,\big\} \le 1,$

b) $|f^1(x)| \le \max\big\{\, q(-\infty),\, q(+\infty) \,\big\} \le 1,$

c) $|(f^1)'(x)| \le |x| + \dfrac{1}{\sqrt{2\,\pi}}.$

Proof:

Parts a) and b) follow directly from Lemma 2.1.13, so that only part c) remains to be proved.

As in the proof of Lemma 2.1.2 and Lemma 2.1.13, we can assume without loss of generality, that $q = 1_{(-\infty,\, z\,]}$ with a fixed $z \in \mathbb{R}$.

Then $(f^1)'(x)$ has the representation (2.1.15) with the only difference that $x \le z$ instead of $x < z$ holds. Moreover, it follows from the (proof of) Lemma 2.1.13 that $(f^1)'(x) \ge 0$ for all $x \in \mathbb{R}$.

To prove part c), we consider the following four cases. We use the function $V(x)$ according to (2.1.16).

Case 1: $x \le 0 \le z$ or $x \le z \le 0$.

In this case, we have $(-x) \ge 0$ and therefore $V(-x) \ge 0$, from which $(f^1)'(x) \le \psi(z) \le \psi(0) = \dfrac{1}{\sqrt{2\,\pi}}$ follows.

Case 2: $0 \le z < x$ or $z < 0 \le x$.

Now $V(x) \ge 0$ is valid, from which again $(f^1)'(x) \le \psi(z) \le \psi(0) = \dfrac{1}{\sqrt{2\,\pi}}$ follows.

Case 3: $0 \le x \le z$.

In this case, we get

$$\psi(z) \le \psi(x) \le \psi(0) = \frac{1}{\sqrt{2\,\pi}} \tag{2.1.18}$$

and

$$V(-x) = \frac{(-x)\,\big(1 - \Phi(-x)\big)}{\psi(-x)} = \frac{(-x)\,\Phi\big(-(-x)\big)}{\psi(x)} = \frac{(-x)\,\Phi(x)}{\psi(x)}.$$

It follows that

$$
\begin{aligned}
(f^1)'(x) &= \psi(z)\Big(1 + \frac{x\,\Phi(x)}{\psi(x)}\Big) \\
&\le \psi(z) + |x|\,\frac{\psi(z)}{\psi(x)} && (\Phi(x) \le 1 \quad \text{and} \quad x = |x|) \\
&\le \psi(0) + |x| = |x| + \frac{1}{\sqrt{2\,\pi}} && (\frac{\psi(z)}{\psi(x)} \le 1).
\end{aligned}
$$

Case 4: $z < x \le 0$.

In case 4, the inequality chain (2.1.18) holds analogously to case 3. It follows that

$$
\begin{aligned}
(f^1)'(x) &= \psi(z)\Big(1 - x\,\frac{\big(1-\Phi(x)\big)}{\psi(x)}\Big) \\
&\le \psi(z) + |x|\,\frac{\psi(z)}{\psi(x)} && (\big(1-\Phi(x)\big) \le 1 \quad \text{and} \quad -x = |x|) \\
&\le \psi(0) + |x| = |x| + \frac{1}{\sqrt{2\,\pi}} && (\frac{\psi(z)}{\psi(x)} \le 1). \ \square
\end{aligned}
$$

2.2 A relation between the Hermite polynomials and the functions $q(x)$ and $f(x)$

Let $n \in \mathbb{N}_0$. We denote by

$$H_n(x) = (-1)^n\, e^{x^2/2}\, \frac{d^n}{dx^n}\, e^{-\,x^2/2}$$

the nth-order **Hermite polynomial**. We obtain this Hermite polynomial if we differentiate the function $\psi(x)$ n times, i.e. trivially holds

$$\frac{d^n}{dx^n}\,\psi(x) = (-1)^n\,\psi(x)\,H_n(x). \tag{2.2.1}$$

This immediately leads to the following useful formulas

$$H_{n+1}(x)\,\psi(x) = (-1)^{n+1}\Big(\frac{d^n}{dx^n}\,\psi(x)\Big)' = \Big(-H_n(x)\,\psi(x)\Big)' \tag{2.2.2}$$

and

$$\int\limits_{\mathbb{R}} H_{n+1}(y)\,\psi(y)\,dy = \Big[-H_n(x)\,\psi(x)\Big]_{-\infty}^{+\infty} = 0 \qquad \text{for all } n \in \mathbb{N}_0. \tag{2.2.3}$$

The first nine Hermite polynomials are given as examples:

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3\,x,$$

$$H_4(x) = x^4 - 6\,x^2 + 3, \quad H_5(x) = x^5 - 10\,x^3 + 15\,x,$$

$$H_6(x) = x^6 - 15\,x^4 + 45\,x^2 - 15, \quad H_7(x) = x^7 - 21\,x^5 + 105\,x^3 - 105\,x,$$

$$H_8(x) = x^8 - 28\,x^6 + 210\,x^4 - 420\,x^2 + 105.$$

Further Hermite polynomials can be specified by using the following recursion formula, which can be proved by simple calculations (using the product rule for n-fold derivatives):

$$H_{n+1}(x) = (-n)\,H_{n-1}(x) + x\,H_n(x) \qquad \text{for } n \in \mathbb{N},\ x \in \mathbb{R}. \tag{2.2.4}$$

Hermite polynomials are of interest for Edgeworth expansions because they are part of these expansions, cf. (1.1.2) in chapter 1 $\big($and (3.1.6), (3.1.7) in chapter 3$\big)$:

$$e_n(x) = \Phi(x) - \frac{\mu_3}{6\sqrt{n}}\,H_2(x)\,\psi(x) = \Phi(x) - \frac{\mu_3}{6\sqrt{n}}\,\psi''(x).$$

The next lemma is therefore an important tool that we need to determine the correct Edgeworth expansions when applying Stein's method.

2.2.5 Lemma

Let $\mathcal{H}_d = \Big\{ h \in \mathcal{H} \,:\, h \text{ is differentiable with a bounded derivative } h' \Big\}$. Furthermore, let $q \in \mathcal{H}_e = \mathcal{H}_{e1} \cup \mathcal{H}_{e2} = \Big\{ h \,:\, h \in \mathcal{H}_d \Big\} \cup \Big\{ id_{\mathbb{R}} \cdot h \,:\, h \in \mathcal{H}_d \Big\}$ and

$$f(x) = \psi(x)^{-1} \int_{-\infty}^{x} \big(q(y) - \Phi(q) \big)\,\psi(y)\,dy.$$

Then

a) $$\int_{\mathbb{R}} f(y)\,H_n(y)\,\psi(y)\,dy = -\frac{1}{n+1} \int_{\mathbb{R}} q(y)\,H_{n+1}(y)\,\psi(y)\,dy,$$

b) $$\int_{\mathbb{R}} f(y)\,H_n(y)\,\psi(y)\,dy = \int_{\mathbb{R}} f'(y)\,H_{n-1}(y)\,\psi(y)\,dy = \int_{\mathbb{R}} f''(y)\,H_{n-2}(y)\,\psi(y)\,dy.$$

In particular, we obtain the special cases:

c) $$\int_{\mathbb{R}} f'(y)\,y\,\psi(y)\,dy = \frac{1}{3} \int_{\mathbb{R}} q(y)\,\big(3\,y - y^3\big)\,\psi(y)\,dy,$$

d) $\displaystyle\int\limits_{\mathbb{R}} f''(y)\, y\, \psi(y)\, dy = \frac{1}{4} \int\limits_{\mathbb{R}} q(y) \left(-y^4 + 6\, y^2 - 3 \right) \psi(y)\, dy,$

e) $$\frac{1}{3} \int\limits_{\mathbb{R}} f'(y)\, y \left(3\, y - y^3 \right) \psi(y)\, dy + \int\limits_{\mathbb{R}} f''(y)\, y\, \psi(y)\, dy$$
$$= \frac{1}{18} \int\limits_{\mathbb{R}} q(y) \left(y^6 - 15\, y^4 + 45\, y^2 - 15 \right) \psi(y)\, dy.$$

Proof:

First of all, we note that the integrals

$$\int\limits_{\mathbb{R}} f(y)\, p(y)\, \psi(y)\, dy, \qquad \int\limits_{\mathbb{R}} f'(y)\, p(y)\, \psi(y)\, dy \qquad \text{and} \qquad \int\limits_{\mathbb{R}} f''(y)\, p(y)\, \psi(y)\, dy$$

exist for all polynomials p. This follows for the first two integrals in the case of $q \in \mathcal{H}_{e1}$ because of Corollary 2.1.12, a) and in the case of $q \in \mathcal{H}_{e2}$ because of Corollary 2.1.17, b) and c).

Differentiating Stein's equation $f'(x) = x\, f(x) + q(x) - \Phi(q)$ (see (2.1.1)) gives

$$f''(x) = f(x) + x\, f'(x) + q'(x),$$

from which, together with the above references, we can conclude that the third integral also exists.

The above references also show that

$$\lim_{x \to \pm\infty} f(x)\, H_n(x)\, \psi(x) = 0 \qquad \text{and} \tag{2.2.6}$$

$$\lim_{x \to \pm\infty} f'(x)\, H_n(x)\, \psi(x) = 0 \qquad \text{for all } n \in \mathbb{N}_0. \tag{2.2.7}$$

We now obtain the assertions of the lemma as follows:

a) $$\int\limits_{\mathbb{R}} f(y)\, H_n(y)\, \psi(y)\, dy$$
$$= -\frac{1}{n+1} \left(\int\limits_{\mathbb{R}} f(y)\, H_{n+2}(y)\, \psi(y)\, dy - \int\limits_{\mathbb{R}} y\, f(y)\, H_{n+1}(y)\, \psi(y)\, dy \right) \qquad \text{(cf. (2.2.4))}$$
$$= -\frac{1}{n+1} \left(\int\limits_{\mathbb{R}} f'(y)\, H_{n+1}(y)\, \psi(y)\, dy - \int\limits_{\mathbb{R}} y\, f(y)\, H_{n+1}(y)\, \psi(y)\, dy \right)$$

(integration by parts using (2.2.2) (with $n+1$ instead of n) and (2.2.6))

$$= -\frac{1}{n+1} \int_{\mathbb{R}} \Big(f'(y) - y\, f(y) + \Phi(q) \Big)\, H_{n+1}(y)\, \psi(y)\, dy \qquad \text{(cf. (2.2.3))}$$

$$= -\frac{1}{n+1} \int_{\mathbb{R}} q(y)\, H_{n+1}(y)\, \psi(y)\, dy \qquad \text{(cf. Stein's equation).}$$

b) Double integration by parts as in a) using (2.2.2) and (2.2.6), (2.2.7).

c) Straightforward conclusion from a) and b) with $n = 2$.

d) Straightforward conclusion from a) and b) with $n = 3$.

e)
$$\frac{1}{3} \int_{\mathbb{R}} f'(y)\, y\, \big(3\, y - y^3 \big)\, \psi(y)\, dy + \int_{\mathbb{R}} f''(y)\, y\, \psi(y)\, dy$$

$$= \int_{\mathbb{R}} f'(y) \Big[H_2(y) - \frac{1}{3}\, y\, H_3(y) \Big]\, \psi(y)\, dy \qquad \text{(cf. b) with } n = 3)$$

$$= -\frac{1}{3} \int_{\mathbb{R}} f'(y)\, H_4(y)\, \psi(y)\, dy \qquad \text{(cf. (2.2.4) with } n = 3)$$

$$= \frac{1}{18} \int_{\mathbb{R}} q(y)\, H_6(y)\, \psi(y)\, dy \qquad \text{(cf. b) with } n = 5 \text{ and a) with } n = 5). \ \square$$

We conclude this section with some easy-to-calculate estimates of terms such as $||\, H_n\, \psi\, ||$, etc.

2.2.8 Lemma

a) $||\, H_0\, \psi\, || = ||\, \psi\, || = \psi(0) = \dfrac{1}{\sqrt{2\,\pi}} = 0,39894 \le \dfrac{2}{5}$,

b) $||\, H_1\, \psi\, || = ||\, x\, \psi(x)\, || = \psi(1) = \dfrac{1}{\sqrt{2\,\pi\, e}} = 0,24197 \le \dfrac{1}{4}$,

c) $||\, x\, \big(H_1\, \psi\big)(x)\, || = ||\, x^2\, \psi(x)\, || = 2\, \psi(\sqrt{2}) = \dfrac{\sqrt{2}}{e\,\sqrt{\pi}} = 0,29353 \le \dfrac{3}{10}$,

d) $||\, x^2\, \big(H_1\, \psi\big)(x)\, || = ||\, x^3\, \psi(x)\, || = 3^{3/2}\, \psi(\sqrt{3}) = 0,46254 \le \dfrac{7}{15}$,

e) $||\, H_2\, \psi\, || = \psi(0) = \dfrac{1}{\sqrt{2\,\pi}} = 0,39894 \le \dfrac{2}{5}$,

f) $||\, H_3\, \psi\, || = \Big| \big(H_3\, \psi\big)\Big(\sqrt{3 - \sqrt{6}}\Big) \Big| = \Big| \big(H_3\, \psi\big)\big(0,74196\big) \Big| = 0,55059 \le \dfrac{5}{9}$,

g) $||\,x\,\big(H_3\,\psi\big)(x)\,|| = 2\,\psi(1) = \dfrac{\sqrt{2}}{\sqrt{\pi\,e}} = 0,48394 \le \dfrac{1}{2}$,

h) $||\,H_4\,\psi\,|| = 3\,\psi(0) = \dfrac{3}{\sqrt{2\,\pi}} = 1,19683 \le \dfrac{6}{5}$,

i) $||\,x\,\big(H_4\,\psi\big)(x)\,|| = \Big|\,1,49882\,\big(H_4\,\psi\big)\big(1,49882\big)\,\Big| = 1,05638 \le \dfrac{18}{17}$,

j) $||\,H_5\,\psi\,|| = \Big|\,\big(H_5\,\psi\big)\big(0,61671\big)\,\Big| = 2,30711 \le \dfrac{7}{3}$,

k) $||\,x\,(H_5\,\psi)(x)\,|| = ||\,1,97963\,(H_5\,\psi)(1,97963)\,|| = 1,94588 \le 2$,

l) $||\,H_6\,\psi\,|| = 15\,\psi(0) = \dfrac{15}{\sqrt{2\,\pi}} = 5,98413 \le 6$,

m) $||\,x\,(H_6\,\psi)(x)\,|| = ||\,1,27145\,(H_6\,\psi)(1,27145)\,|| = 5,14696 \le \dfrac{26}{5}$,

n) $||\,x\,(H_7\,\psi)(x)\,|| = ||\,1,71316\,(H_7\,\psi)(1,71316)\,|| = 12,83583 \le 13$,

o) $||\,H_8\,\psi\,|| = 105\,\psi(0) = \dfrac{105}{\sqrt{2\,\pi}} = 41,88894 \le 42$.

2.3 Introduction of several smooth functions

In chapter 1 we did not start from the expression $F_n(z) - \Phi(z)$ when applying Stein's method, but from $E\big(\,q(S_n)\,\big) - \Phi(q)$, where q was a "smoother" function than $q = 1_{(-\infty,\,z\,]}$. Since we have to proceed analogously in the next chapter, where we consider linear rank statistics, we will introduce the smooth functions required there in this section (or partially repeat them from the chapter 1). Subsequently we prove some properties of these smooth functions.

For $x, z \in \mathbb{R}$, $k, m \in \mathbb{N}$ and $\lambda > 0$ denote in the following

$$\text{(2.3.1)} \qquad p^{\lambda}_{z,0}(x) = \begin{cases} 1 & \text{for } x \le z, \\ 1 - \dfrac{1}{\lambda}\,\big(\,x - z\,\big) & \text{for } z \le x \le z + \lambda, \\ 0 & \text{for } z + \lambda \le x\,. \end{cases}$$

$$p^{\lambda}_{z,k}(x) = |x|^k\,p^{\lambda}_{z,0}(x).$$

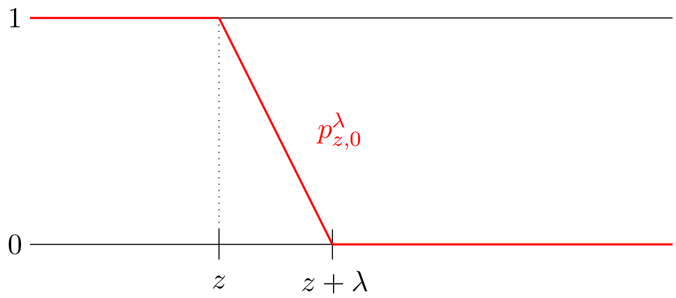


Fig. 2.3.1: Sketch of the "linear" function $p_{z,0}^\lambda$

We use the functions $p_{z,k}^\lambda$ for "Berry-Esséen proofs".

Furthermore,

$$q_z^{\lambda,0}(x) = \begin{cases} 1 & \text{for } x \le z, \\ 1 - \frac{1}{2\,\lambda^2}\,\big(x-z\big)^2 & \text{for } z \le x \le z+\lambda, \\ \frac{1}{2\,\lambda^2}\,\big(z+2\,\lambda-x\big)^2 & \text{for } z+\lambda \le x \le z+2\,\lambda, \\ 0 & \text{for } z+2\,\lambda \le x\,. \end{cases} \tag{2.3.2}$$

$$q_z^{\lambda,m}(x) = x^m\, q_z^{\lambda,0}(x).$$

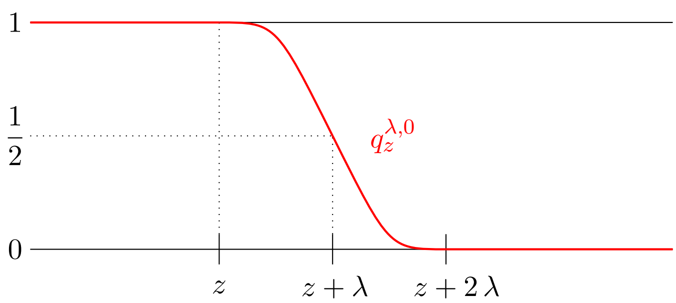


Fig. 2.3.2: Sketch of the "quadratic" funktion $q_z^{\lambda,0}$

We need the functions $q_z^{\lambda,m}$ to establish first-order Edgeworth expansions. In the iid case already considered in chapter 1, we used $\lambda = \dfrac{1}{\sqrt{n}}$ (cf. (1.2.1)).

Furthermore,

$$r_z^\lambda(x) = \begin{cases} 1 & \text{for } x \le z, \\ 1 - \dfrac{1}{6\,\lambda^3}\,\big(x-z\big)^3 & \text{for } z \le x \le z+\lambda, \\ \dfrac{1}{24\,\lambda^3}\Big\{\big(2\,(x-z) - 3\,\lambda\big)^3 & \\ \qquad\qquad - 18\,\lambda^2\,\big(x-z\big) + 39\,\lambda^3\Big\} & \text{for } z+\lambda \le x \le z+2\,\lambda, \\ \dfrac{1}{6\,\lambda^3}\,\big(z+3\,\lambda - x\big)^3 & \text{for } z+2\,\lambda \le x \le z+3\,\lambda, \\ 0 & \text{for } z+3\,\lambda \le x\,. \end{cases} \tag{2.3.3}$$

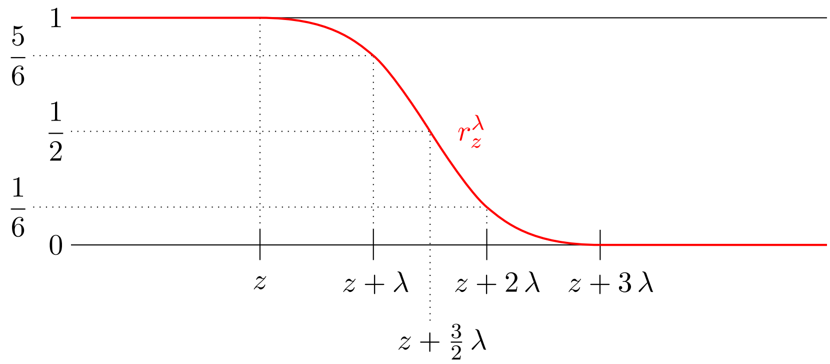


Fig. 2.3.3: Sketch of the "cubic" function r_z^λ

We need the functions r_z^λ to establish second-order Edgeworth expansions.

In addition, we define

$$\begin{aligned} d_{z,k}^\lambda(x) &= \psi(x)^{-1} \int_{-\infty}^{x} \Big(p_{z,k}^\lambda(y) - \Phi\big(p_{z,k}^\lambda\big)\Big)\,\psi(y)\,dy, \\ f_z^{\lambda,m}(x) &= \psi(x)^{-1} \int_{-\infty}^{x} \Big(q_z^{\lambda,m}(y) - \Phi\big(q_z^{\lambda,m}\big)\Big)\,\psi(y)\,dy, \\ g_z^\lambda(x) &= \psi(x)^{-1} \int_{-\infty}^{x} \Big(r_z^\lambda(y) - \Phi\big(r_z^\lambda\big)\Big)\,\psi(y)\,dy. \end{aligned} \tag{2.3.4}$$

However, in cases where there is clarity about certain indices, these will be omitted in the future.

The following results are of interest for the functions $p^\lambda_{z,k}$ and $d^\lambda_{z,k}$:

2.3.5 Lemma

Let $x, y, z \in \mathbb{R}$ and $\lambda > 0$. Then

a) $\left| p^\lambda_{z,k}(x+y) - p^\lambda_{z,k}(x) \right|$

$$\leq \begin{cases} |y| \, \dfrac{1}{\lambda} \displaystyle\int_0^1 1_{(\,z,\, z+\lambda\,]}(x + s\, y)\, ds & \text{for } k = 0, \\ |y| \left\{ \dfrac{|x|^k}{\lambda} \displaystyle\int_0^1 1_{(\,z,\, z+\lambda\,]}(x + s\, y)\, ds + k\, 2^{k-2} \left(|x|^{k-1} + |y|^{k-1} \right) \right\} & \text{otherwise.} \end{cases}$$

b) $\left| \left(d^\lambda_{z,0}\right)'(x+y) - (d^\lambda_{z,0})'(x) \right|$

$$\leq |y| \left\{ \frac{\sqrt{2\pi}}{4} + |x| + \frac{1}{\lambda} \int_0^1 1_{(\,z,\, z+\lambda\,]}(x + s\, y)\, ds \right\}.$$

c) $\left| \left(d^\lambda_{z,1}\right)'(x+y) - (d^\lambda_{z,1})'(x) \right|$

$$\leq |y| \left\{ 2 + \frac{|x|}{\lambda} \int_0^1 1_{(z,\, z+\lambda\,]}(x + s\, y)\, ds \right\} + |x| \, | \, d^\lambda_{z,1}(x+y) - d^\lambda_{z,1}(x) \, |$$

$$\leq |y| \left\{ 2 + \sqrt{\frac{2}{\pi}}\, |x| + x^2 + |x\, y| + \frac{|x|}{\lambda} \int_0^1 1_{(\,z,\, z+\lambda\,]}(x + s\, y)\, ds \right\}.$$

d) Let $k \in \mathbb{N}_0$. Then there exists a constant $C(k) > 0$ such that

$\left| \left(d^\lambda_{z,k}\right)'(x+y) - (d^\lambda_{z,k})'(x) \right|$

$$\leq C(k)\, |y| \left\{ 1 + |x|^{k+1} + \left(1 + |x| \right) |y|^k + \frac{|x|^k}{\lambda} \int_0^1 1_{(\,z,\, z+\lambda\,]}(x + s\, y)\, ds \right\}.$$

Proof:

Part a) is obtained by the following calculation:

$$\begin{aligned}
&\Big|\, p^{\lambda}_{z,k}(x+y) - p^{\lambda}_{z,k}(x) \,\Big| \\
&\le |x|^k \,\Big|\, p^{\lambda}_{z,0}(x+y) - p^{\lambda}_{z,0}(x) \,\Big| + \Big|\, |x+y|^k - |x|^k \,\Big| \,\Big|\, p^{\lambda}_{z,0}(x+y) \,\Big| \\
&\le |x|^k \, \frac{|y|}{\lambda} \int_0^1 1_{(\,z,\, z+\lambda\,]}(x + s\, y)\, ds + \Big|\, |x+y|^k - |x|^k \,\Big| .
\end{aligned}$$

In the case of $k \in \mathbb{N}$ the following holds for the second summand

$$\begin{aligned}
\Big|\, |x+y|^k - |x|^k \,\Big| &\le |y| \sup_{0<\theta<1} k\, |x + \theta\, y|^{k-1} \\
&\le |y|\, k \Big(\, |x| + |y| \,\Big)^{k-1} \\
&\le |y|\, k\, 2^{k-2} \Big(\, |x|^{k-1} + |y|^{k-1} \,\Big).
\end{aligned}$$

The last inequality is clear for $k = 1$ and $k = 2$. For $k > 2$ this inequality follows from Hölder's inequality (0.1.5) with $\nu = 2$ and $p = k - 1$.

To prove the parts b) - d), we use Stein's equation

$$\big(d^{\lambda}_{z,k}\big)'(x) = x\, d^{\lambda}_{z,k}(x) + p^{\lambda}_{z,k}(x) - \Phi\big(\, p^{\lambda}_{z,k}\,\big)$$

from which follows

$$\begin{aligned}
&\Big|\, \big(d^{\lambda}_{z,k}\big)'(x+y) - (d^{\lambda}_{z,k})'(x) \,\Big| \\
&\le |y| \,\Big|\, d^{\lambda}_{z,k}(x+y) \,\Big| + |x| \,\Big|\, d^{\lambda}_{z,k}(x+y) - d^{\lambda}_{z,k}(x) \,\Big| + \Big|\, p^{\lambda}_{z,k}(x+y) - p^{\lambda}_{z,k}(x) \,\Big| \\
&\le |y| \left\{ \Big|\, d^{\lambda}_{z,k}(x+y) \,\Big| + |x| \sup_{0<\theta<1} \Big|\, \big(d^{\lambda}_{z,k}\big)'(x + \theta\, y) \,\Big| \right\} + \Big|\, p^{\lambda}_{z,k}(x+y) - p^{\lambda}_{z,k}(x) \,\Big|
\end{aligned}$$

Using part a) and Corollary 2.1.12 with $|z|^l \le 1 + |z|^r$ for $0 < l < r$, $z \in \mathbb{R}$ then yields the assertion. □

Next, we examine the functions $q^{\lambda,m}_z$ and $f^{\lambda,m}_z$.

2.3.6 Lemma

Let $x, y, z \in \mathbb{R}$ and $\lambda > 0$. Then

a) $$\left| q_z^{\lambda,m}(x+y) - q_z^{\lambda,m}(x) \right| \le \begin{cases} |y| \, \dfrac{1}{\lambda} \displaystyle\int_0^1 1_{(\,z,\,z+2\,\lambda\,]}(x+s\,y)\,ds & \text{for } m=0, \\ |y| \left\{ 1 + \dfrac{|x|}{\lambda} \displaystyle\int_0^1 1_{(\,z,\,z+2\,\lambda\,]}(x+s\,y)\,ds \right\} & \text{for } m=1. \end{cases}$$

b) $$\big(q_z^{\lambda,0}\big)'(x+y) - (q_z^{\lambda,0})'(x) = y\,\frac{1}{\lambda^2} \int_0^1 \Big(1_{(\,z+\lambda,\,z+2\,\lambda\,]} - 1_{(\,z,\,z+\lambda\,]} \Big)(x+s\,y)\,ds.$$

c) $$\big(q_z^{\lambda,1}\big)'(x+y) - (q_z^{\lambda,1})'(x) = y\,\frac{x}{\lambda^2} \int_0^1 \Big(1_{(\,z+\lambda,\,z+2\,\lambda\,]} - 1_{(\,z,\,z+\lambda\,]} \Big)(x+s\,y)\,ds + R_z^{\lambda,1}(x,y),$$

$$\text{where} \quad \big|\, R_z^{\lambda,1}(x,y) \,\big| \le |y|\,\frac{1}{\lambda} \left\{ 1_{(\,z,\,z+2\,\lambda\,]}(x+y) + \int_0^1 1_{(\,z,\,z+2\,\lambda\,]}(x+s\,y)\,ds \right\}.$$

d) $$\left| \Big(\big(f_z^{\lambda,0}\big)'' - \big(q_z^{\lambda,0}\big)' \Big)(x+y) - \Big(\big(f_z^{\lambda,0}\big)'' - \big(q_z^{\lambda,0}\big)' \Big)(x) \right|$$

$$\le |y| \left\{ 3 + \frac{\sqrt{2\pi}}{4}\,|x| + x^2 + \frac{|x|}{\lambda} \int_0^1 1_{(\,z,\,z+2\,\lambda\,]}(x+s\,y)\,ds \right\}$$

$$\le |y| \left\{ \frac{11}{3} + \frac{5}{3}\,x^2 + \frac{|x|}{\lambda} \int_0^1 1_{(\,z,\,z+2\,\lambda\,]}(x+s\,y)\,ds \right\}.$$

e) $$\left| \Big(\big(f_z^{\lambda,1}\big)'' - \big(q_z^{\lambda,1}\big)' \Big)(x+y) - \Big(\big(f_z^{\lambda,1}\big)'' - \big(q_z^{\lambda,1}\big)' \Big)(x) \right|$$

$$\le |y| \left\{ \frac{33}{5} + \frac{32}{5}\,|x|^3 + (3+x^2)\,|y| + \frac{x^2}{\lambda} \int_0^1 1_{(\,z,\,z+2\,\lambda\,]}(x+s\,y)\,ds \right\}.$$

Proof:

By differentiating $q_z^{\lambda,0}(x)$ we obtain

$$(2.3.7) \qquad \big(q_z^{\lambda,0}\big)'(x) = \begin{cases} 0 & \text{for } x \le z \ \text{ or } \ z+2\,\lambda \le x, \\ -\dfrac{1}{\lambda^2}\,\big(\,x-z\,\big) & \text{for } z \le x \le z+\lambda, \\ -\dfrac{1}{\lambda^2}\,\big(\,z+2\,\lambda-x\,\big) & \text{for } z+\lambda \le x \le z+2\,\lambda, \end{cases}$$

and therefore

$$\left| \big(q_z^{\lambda,0}\big)'(x) \right| \le \frac{1}{\lambda}\, 1_{(\,z,\,z+2\,\lambda\,]}(x) \qquad \text{for all } x \in \mathbb{R}. \tag{2.3.8}$$

From this estimate, part a) follows analogously to part a) of the last Lemma 2.3.5.

Further differentiation of $\big(q_z^{\lambda,0}\big)'(x)$ at the points $x \neq z, z+\lambda, z+2\,\lambda$ yields

$$\big(q_z^{\lambda,0}\big)''(x) = \begin{cases} 0 & \text{for } x < z \ \text{ or } \ z+2\,\lambda < x, \\ -\dfrac{1}{\lambda^2} & \text{for } z < x < z+\lambda, \\ \dfrac{1}{\lambda^2} & \text{for } z+\lambda < x < z+2\,\lambda, \end{cases} \tag{2.3.9}$$

from which we can immediately see the validity of part b).

In addition, the product rule gives us

$$\begin{aligned} &\big(q_z^{\lambda,1}\big)'(x+y) - (q_z^{\lambda,1})'(x) \\ &= x\left[\, \big(q_z^{\lambda,0}\big)'(x+y) - (q_z^{\lambda,0})'(x) \,\right] + R_z^{\lambda,1}(x,y), \\ &\text{where} \quad R_z^{\lambda,1}(x,y) = y\,\big(q_z^{\lambda,0}\big)'(x+y) + \left[\, q_z^{\lambda,0}(x+y) - q_z^{\lambda,0}(x) \,\right]. \end{aligned}$$

If we now use the parts b) and a), and (2.3.8), we get part c).

To derive the parts d) and e), Stein's equation

$$f'(x) = x\,f(x) + q(x) - \Phi(q) \tag{2.3.10}$$

is used (with the abbreviations $f = f_z^{\lambda,m}$ and $q = q_z^{\lambda,m}$). Differentiation of (2.3.10) gives

$$\begin{aligned} f''(x) &= f(x) + x\,f'(x) + q'(x) \\ &= \big(\,1+x^2\,\big)\,f(x) + x\,\big(\,q(x) - \Phi(q)\,\big) + q'(x) \end{aligned} \tag{2.3.11}$$

and therefore

$$\begin{aligned} &\left| \,(\,f'' - q'\,)(x+y) - \big(\,f'' - q'\,\big)(x) \,\right| \\ &\le \big(\,1+x^2\,\big)\,\big|\,f(x+y) - f(x)\,\big| + \big|\,2\,x\,y + y^2\,\big|\,\big|\,f(x+y)\,\big| + |y|\,\big|\,q(x+y) - \Phi(q)\,\big| \end{aligned} \tag{2.3.12}$$

$$+\,|x|\,\Big|\,\big(\,q(x+y)-\Phi(q)\,\big)-\big(\,q(x)-\Phi(q)\,\big)\,\Big|$$

$$\le |y|\,\Big\{\,\big(\,1+x^2\,\big)\sup_{0<\theta<1}\big|\,f'(x+\theta\,y)\,\big|+\big|\,(x+y)\,f(x+y)\,\big|+|x|\,\big|\,f(x+y)\,\big|$$

$$+\,\big|\,q(x+y)-\Phi(q)\,\big|\,\Big\}+|x|\,\big|\,q(x+y)-q(x)\,\big|.$$

In the case $m=0$ we now obtain from (2.3.12) because of Corollary 2.1.12, a) and Corollary 2.1.17, a)

$$\le |y|\,\Big\{\,\big(\,1+x^2\,\big)\,1+1+\frac{\sqrt{2\pi}}{4}\,|x|+1\,\Big\}+|x|\,\big|\,q(x+y)-q(x)\,\big|.$$

Using part a) and the inequality $|x|\le 1+x^2$, part d) follows from this.

In the case $m=1$, on the other hand, we get from (2.3.12) because of Corollary 2.1.17, b) and c), and (2.1.7)

$$\le |y|\,\Big\{\,\big(\,1+x^2\,\big)\,\big(\,|x|+|y|+\frac{1}{\sqrt{2\,\pi}}\,\big)+\big(\,|x|+|y|\,\big)\,1+|x|\,1+\big(\,|x|+|y|\,\big)+\sqrt{\frac{2}{\pi}}\,\Big\}$$

$$+\,|x|\,\big|\,q(x+y)-q(x)\,\big|$$

$$\le |y|\,\Big\{\,\frac{6}{5}+4\,|x|+|x|^3+\frac{2}{5}\,x^2+3\,|y|+x^2\,|y|\,\Big\}+|x|\,\big|\,q(x+y)-q(x)\,\big|.$$

Using part a) and the inequalities $|x|\le 1+|x|^3$ and $x^2\le 1+|x|^3$, part e) follows from this. $\square$

Finally, we look at the functions r_z^λ and g_z^λ.

2.3.13 Lemma

Let $x,y,z\in\mathbb{R}$ and $\lambda>0$. Then

a) $\Big|\,r_z^\lambda(x+y)-r_z^\lambda(x)\,\Big|\;\le\;\dfrac{3}{4}\,|y|\,\dfrac{1}{\lambda}\displaystyle\int_0^1 1_{(\,z,\,z+3\,\lambda\,]}(x+s\,y)\,ds.$

b) $\big(r_z^\lambda\big)''(x+y)-(r_z^\lambda)''(x)$

$$=\,y\,\frac{1}{\lambda^3}\int_0^1\Big(-1_{(\,z+2\,\lambda,\,z+3\,\lambda\,]}+2\cdot 1_{(\,z+\lambda,\,z+2\,\lambda\,]}-1_{(\,z,\,z+\lambda\,]}\Big)(x+s\,y)\,ds.$$

c) $\displaystyle\int_z^{z+3\lambda} \big(r_z^\lambda\big)'(x)\,dx = -1, \qquad \int_z^{z+3\lambda} \frac{x-z}{\lambda}\,\big(r_z^\lambda\big)'(x)\,dx = -\frac{3}{2},$

$$\int_z^{z+3\lambda} \frac{(x-z)\,(x-z-\lambda)}{2\,\lambda^2}\,\big(r_z^\lambda\big)'(x)\,dx = -\frac{1}{2}.$$

d) $\big(r_z^\lambda\big)'(x) = \dfrac{3}{4}\,\dfrac{1}{\lambda}\,\big(\,h_1(x) - h_2(x)\,\big) \qquad \text{with } h_1, h_2 \in \mathcal{H}.$

e) $\Big|\Big(\big(g_z^\lambda\big)''' - \big(r_z^\lambda\big)''\Big)(x+y) - \Big(\big(g_z^\lambda\big)''' - \big(r_z^\lambda\big)''\Big)(x) - x\,\Big(\big(r_z^\lambda\big)'(x+y) - \big(r_z^\lambda\big)'(x)\Big)\Big|$

$$\leq\ |y|\,\Big\{\,\frac{35}{3} + \frac{32}{3}\,|x|^3 + 3\,|y| + \frac{3}{4}\,\frac{1}{\lambda}\,1_{(\,z,\,z+3\,\lambda\,]}(x+y) + \frac{3}{4}\,(x^2+2)\,\frac{1}{\lambda}\int_0^1 1_{(\,z,\,z+3\,\lambda\,]}(x+s\,y)\,ds\,\Big\}.$$

Proof:

We obtain by differentiation (cf. some parallels to the proof of Lemma 2.3.6):

$$\big(r_z^\lambda\big)'(x) = \begin{cases} 0 & \text{for } x \leq z \ \text{ or } \ z + 3\,\lambda \leq x, \\ -\dfrac{1}{2\,\lambda^3}\,\big(\,x-z\,\big)^2 & \text{for } z \leq x \leq z + \lambda, \\ \dfrac{1}{4\,\lambda^3}\Big\{\,\big(\,2\,(x-z) - 3\,\lambda\,\big)^2 - 3\,\lambda^2\,\Big\} & \text{for } z + \lambda \leq x \leq z + 2\,\lambda, \\ -\dfrac{1}{2\,\lambda^3}\,\big(\,z + 3\,\lambda - x\,\big)^2 & \text{for } z + 2\,\lambda \leq x \leq z + 3\,\lambda, \end{cases} \tag{2.3.14}$$

and

$$\big(r_z^\lambda\big)''(x) = \begin{cases} 0 & \text{for } x \leq z \ \text{ or } \ z + 3\,\lambda \leq x, \\ -\dfrac{1}{\lambda^3}\,\big(\,x-z\,\big) & \text{for } z \leq x \leq z + \lambda, \\ \dfrac{1}{\lambda^3}\,\big(\,2\,(x-z) - 3\,\lambda\,\big) & \text{for } z + \lambda \leq x \leq z + 2\,\lambda, \\ \dfrac{1}{\lambda^3}\,\big(\,z + 3\,\lambda - x\,\big) & \text{for } z + 2\,\lambda \leq x \leq z + 3\,\lambda. \end{cases} \tag{2.3.15}$$

In particular, we have

$$\big(r_z^\lambda\big)'\big(\,z + \frac{3}{2}\,\lambda\big) = -\frac{3}{4}\,\frac{1}{\lambda} \leq \big(r_z^\lambda\big)'(x) \leq 0 \qquad \text{for all } x \in [\,z,\ z + 3\,\lambda\,], \tag{2.3.16}$$

$$(2.3.17)\qquad \begin{aligned} &\big(r_z^\lambda\big)''(x) \le 0 \qquad \text{for all } x \in [\,z,\, z + \tfrac{3}{2}\,\lambda\,] \qquad \text{and} \\ &\big(r_z^\lambda\big)''(x) \ge 0 \qquad \text{for all } x \in [\,z + \tfrac{3}{2}\,\lambda,\, z + 3\,\lambda\,]. \end{aligned}$$

From (2.3.16) and (2.3.18) we immediately get

$$(2.3.18)\qquad \Big|\,\big(r_z^\lambda\big)'(x)\,\Big| \le \frac{3}{4}\,\frac{1}{\lambda}\,1_{(\,z,\,z+3\,\lambda\,]}(x) \qquad \text{for all } x \in \mathbb{R}$$

and thus part a).

Further differentiation of $\big(r_z^\lambda\big)''(x)$ at the points $x \ne z, z+\lambda, z+2\,\lambda, z+3\,\lambda$ gives

$$(2.3.19)\qquad \big(r_z^\lambda\big)'''(x) = \begin{cases} 0 & \text{for } x < z \ \text{ or } \ z + 3\,\lambda < x, \\ -\dfrac{1}{\lambda^3} & \text{for } z < x < z + \lambda, \\ \dfrac{2}{\lambda^3} & \text{for } z + \lambda < x < z + 2\,\lambda, \\ -\dfrac{1}{\lambda^3} & \text{for } z + 2\,\lambda < x < z + 3\,\lambda, \end{cases}$$

from which we can at once deduce the validity of part b).

Part c), on the other hand, is obtained by simple calculations. Furthermore, part d) is valid, since we get because of (2.3.16) and (2.3.17):

$$(2.3.20)\qquad \begin{aligned} &h_1(x) = 1_{(\,-\infty,\,z+\frac{3}{2}\,\lambda\,]}(x) + \frac{4}{3}\,\lambda\,1_{(\,-\infty,\,z+\frac{3}{2}\,\lambda\,]}(x)\,\big(r_z^\lambda\big)'(x) \in \mathcal{H} \qquad \text{and} \\ &h_2(x) = 1_{(\,-\infty,\,z+\frac{3}{2}\,\lambda\,]}(x) - \frac{4}{3}\,\lambda\,1_{(\,z+\frac{3}{2}\,\lambda,\,+\infty\,)}(x)\,\big(r_z^\lambda\big)'(x) \in \mathcal{H}. \end{aligned}$$

To derive part e), we start as in the proof of the last Lemma 2.3.6 (cf. (2.3.12)) from Stein's equation

$$(2.3.21)\qquad g'(x) = x\,g(x) + r(x) - \Phi(r)$$

(with the abbreviations $g = g_z^\lambda$ and $r = r_z^\lambda$). By differentiating both sides first follows

$$(2.3.22)\qquad g''(x) = g(x) + x\,g'(x) + r'(x).$$

Differentiating again and then substituting (2.3.22) and (2.3.21) yields

$$(2.3.23)\qquad \begin{aligned} g'''(x) &= 2\,g'(x) + x\,g''(x) + r''(x) \\ &= 2\,g'(x) + x\,g(x) + x^2\,g'(x) + x\,r'(x) + r''(x) \\ &= r''(x) + x\,r'(x) + (x^3 + 3\,x)\,g(x) + (x^2+2)\,\big(\,r(x) - \Phi(r)\,\big). \end{aligned}$$

This gives us

$$\begin{aligned} &\Big|\,\big(g''' - r''\,\big)(x+y) - \big(\,g''' - r''\,\big)(x) - x\,\big(\,r'(x+y) - r'(x)\,\big)\,\Big| \\ &\le |y|\,\big|\,r'(x+y)\,\big| + \big|\,(x+y)^3\,g(x+y) - x^3\,g(x)\,\big| + 3\,\big|\,(x+y)\,g(x+y) - x\,g(x)\,\big| \\ &\quad + \big|\,(x+y)^2\,r(x+y) - x^2\,r(x)\,\big| + 2\,\big|\,r(x+y) - r(x)\,\big| + \big|\,(x+y)^2 - x^2\,\big|\,\Phi(r). \end{aligned}$$

The various summands can now be estimated using (2.3.18) and Corollary 2.1.12, a) and Corollary 2.1.17, a) and part a):

1) $|y|\,\big|\,r'(x+y)\,\big| \le \dfrac{3}{4}\,|y|\,\dfrac{1}{\lambda}\,1_{(\,z,\,z+3\,\lambda\,]}(x+y),$

2)
$$\begin{aligned} &\big|\,(x+y)^3\,g(x+y) - x^3\,g(x)\,\big| \\ &\le |x|^3\,\big|\,g(x+y) - g(x)\,\big| + \Big|\,\big(\,(x+y)^3 - x^3\,\big)\,g(x+y)\,\Big| \\ &\le |y|\,\Big\{\,|x|^3 \sup_{0<\theta<1}\,\big|\,g'(x+\theta\,y)\,\big| + \big|\,(2\,x+y)\,(x+y)\,g(x+y)\,\big| + \big|\,x^2\,g(x+y)\,\big|\,\Big\} \\ &\le |y|\,\Big\{\,|x|^3 + 2\,|x| + |y| + \frac{\sqrt{2\pi}}{4}\,x^2\,\Big\}, \end{aligned}$$

3)
$$\begin{aligned} &3\,\big|\,(x+y)\,g(x+y) - x\,g(x)\,\big| \\ &\le 3\,|x|\,\big|\,g(x+y) - g(x)\,\big| + 3\,\big|\,y\,g(x+y)\,\big| \\ &\le |y|\,\Big\{\,3\,|x| \sup_{0<\theta<1}\,\big|\,g'(x+\theta\,y)\,\big| + 3\,\big|\,g(x+y)\,\big|\,\Big\} \\ &\le |y|\,\Big\{\,3\,|x|\,+3\,\frac{\sqrt{2\pi}}{4}\,\Big\}, \end{aligned}$$

4)
$$\begin{aligned} &\big|\,(x+y)^2\,r(x+y) - x^2\,r(x)\,\big| \\ &\le x^2\,\big|\,r(x+y) - r(x)\,\big| + \big(\,2\,|x|\,|y| + y^2\,\big)\,r(x+y) \\ &\le |y|\,\Big\{\frac{3}{4}\,x^2\,\frac{1}{\lambda}\int_0^1 1_{(\,z,\,z+3\,\lambda\,]}(x+s\,y)\,ds + 2\,|x| + |y|\,\Big\}, \end{aligned}$$

5) $2\,\big|\,r(x+y)-r(x)\,\big| \le \dfrac{3}{2}\,|y|\,\dfrac{1}{\lambda}\displaystyle\int_0^1 1_{(\,z,\,z+3\,\lambda\,]}(x+s\,y)\,ds,$

6) $\big|\,(x+y)^2-x^2\,\big|\,\Phi(r) \le \big|\,2\,x\,y+y^2\,\big|\,1 \le |y|\,\Big\{\,2\,|x|+|y|\,\Big\}.$

By adding the various estimates and using $|x| \le 1+|x|^3$ and $x^2 \le 1+|x|^3$, the assertion follows.

□

2.4 Some results on multiple differences and interpolating polynomials

In this section, some simple results on multiple differences and interpolating polynomials are summarized. It should be noted that these results are partly taken from the paper of Bickel and Robinson [4].

2.4.1 Lemma

Let $k \in \mathbb{N}$, $\lambda > 0$ and $F\,:\,\mathbb{R} \to \mathbb{R}$ be a function such that there exists a constant $C > 0$ so that for all $z \in \mathbb{R}$ and $z \le x \le z+(k+1)\,\lambda$:

$$\Big|\,F(x)-P_\lambda^k(x;z,F)\,\Big| \;\le\; C\,\Big(\,\lambda^{k+1}+(x-z)^{k+1}\,\Big).$$

Then follows for all $0 \le y \le \lambda$:

$$\big|\big|\,\Delta_y^{k+1}F\,\big|\big| \;\le\; D\,\Big(\,\lambda^{k+1}+y^{k+1}\,\Big),$$

where $\quad D \;=\; C\,\Big(\,1+\displaystyle\sum_{j=1}^{k+1}\,\binom{k+1}{j}\,j^{k+1}\,\Big) \quad$ can be selected.

Proof:

$$\begin{aligned}
\Big|\,\Delta_y^{k+1}F(z)\,\Big| &= \Big|\,\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}\,F(z+j\,y)\,\Big| && \text{(cf. (0.1.3))}\\
&\le \Big|\,\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}\,P_\lambda^k(z+j\,y;z,F)\,\Big| + \sum_{j=0}^{k+1}\binom{k+1}{j}\,C\,\Big(\,\lambda^{k+1}+(j\,y)^{k+1}\Big)\\
&\le \Big|\,\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}\,\Big(\,F(z)\,+\,\sum_{s=1}^{k}\frac{1}{s!}\,\Delta_\lambda^s F(z)\,\prod_{i=1}^{s}\big(\,j\,\frac{y}{\lambda}-i+1\,\big)\Big)\,\Big|
\end{aligned}$$

$$+\,C\left(1+\sum_{j=1}^{k+1}\binom{k+1}{j}j^{k+1}\right)\left(\lambda^{k+1}+y^{k+1}\right) \qquad \text{(cf. (0.1.4)).}$$

Since

$$\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}j^{l} \;=\; 0 \qquad \text{for all } \; l=0,1,\ldots,k,$$

the first sum disappears, and thus the assertion follows. □

Furthermore, for sufficiently smooth functions F we have

2.4.2 Lemma

Let $k\in\mathbb{N}$, $\lambda>0$ and $F:\mathbb{R}\to\mathbb{R}$ be a function.

a) If F is $k+1$ times differentiable with $\big|\big|\,F^{(k+1)}\,\big|\big|<\infty$, then

$$\Big|\,F(x)-P_\lambda^k(x;z,F)\,\Big| \;\le\; \frac{2^k}{k+1}\,\big|\big|\,F^{(k+1)}\,\big|\big|\left(\lambda^{k+1}+|x-z|^{k+1}\right) \qquad \text{for all } z,x\in\mathbb{R}.$$

b) If F is k times differentiable with $\big|\big|\,F^{(k)}\,\big|\big|<\infty$, then

$$\big|\big|\,\Delta_y^kF\,\big|\big| \;\le\; \big|\big|\,F^{(k)}\,\big|\big|\;\big|\,y\,\big|^k \qquad \text{for all } y\in\mathbb{R}.$$

c) If F is twice differentiable with $\big|\big|\,F'\,\big|\big|<\infty$ and $\big|\big|\,x\,F''(x)\,\big|\big|<\infty$, then

$$\big|\big|\,z\,\Delta_y^2F(z)\,\big|\big| \;\le\; \Big(4\,\big|\big|\,F'\,\big|\big|+\big|\big|\,x\,F''(x)\,\big|\big|\Big)\,\big|\,y\,\big|^2 \qquad \text{for all } y\in\mathbb{R}.$$

Proof:

a) According to Stoer and Bulirsch [27], Theorem (2.1.4.1), page 49, holds:

There exists a number ζ from the smallest interval I containing z, $z+\lambda$, $z+2\,\lambda,\ldots,$ $z+k\,\lambda$ and x such that

$$F(x)-P_\lambda^k(x;z,F)=\frac{F^{(k+1)}(\zeta)}{(k+1)!}\,(x-z)\,(x-z-\lambda)\,\cdots\,(x-z-k\,\lambda).$$

It follows that

$$\begin{aligned}\Big|\,F(x)-P_\lambda^k(x;z,F)\,\Big| &\le \frac{||F^{(k+1)}||}{(k+1)!}\,|x-z|\,|x-z-\lambda|\,\ldots\,|x-z-k\,\lambda|\\ &\le \frac{1}{k+1}\,||F^{(k+1)}||\,\Big(\,\lambda+|x-z|\Big)^{k+1}\end{aligned}$$

$$\leq \frac{2^k}{k+1} \,||\, F^{(k+1)} \,||\, \Big(\lambda^{k+1} + |x-z|^{k+1} \Big).$$

For the last inequality, Hölder's inequality (0.1.5) with $\nu = 2$ and $p = k+1$ was applied.

b) For $k = 1$ the assertion follows directly from the mean value theorem.

If we now assume the validity of the assertion for $k-1$, we obtain the following by reapplying the mean value theorem to $\Delta_y^{k-1} F$

$$\begin{aligned}
\Big| \, \Delta_y^k F(z) \, \Big| &= \Big| \, \Delta_y^{k-1} F(z+y) - \Delta_y^{k-1} F(z) \, \Big| \\
&\leq |\, y \,| \; ||\big(\, \Delta_y^{k-1} F \, \big)'|| \\
&\leq |\, y \,|^k \; ||F^{(k)}|| \, .
\end{aligned}$$

since $\big(\Delta_y^{k-1} F \big)' = \Delta_y^{k-1} (F)'$ (c.f. (0.1.3)).

c) We get

$$\begin{aligned}
\big| \, z \, \Delta_y^2 F(z) \, \big| &= \big| \, z\, F(z+2\,y) - 2\, z\, F(z+y) + z\, F(z) \, \big| \\
&\leq 2 \, |\, y \,| \, |\, F(z+2\,y) - F(z+y) \,| \\
&\quad + \Big| \, (z+2\,y)\, F(z+2\,y) - 2\,(z+y)\, F(z+y) + z\, F(z) \, \Big| \\
&\leq 2 \, |\, y \,|^2 \, ||\, F' \,|| \\
&\quad + \Big| \int\limits_z^{z+y} \big[\, F(s+y) - F(s) + (s+y)\, F'(s+y) - s\, F'(s) \, \big] \, ds \, \Big| \\
&= 2 \, |\, y \,|^2 \, ||\, F' \,|| \\
&\quad + \Big| \int\limits_z^{z+y} \Big(\int\limits_s^{s+y} F'(t)\, dt \Big) \, ds + \int\limits_z^{z+y} \Big(\int\limits_s^{s+y} F'(t) + t\, F''(t)\, dt \Big) \, ds \, \Big| \\
&\leq 3 \, |\, y \,|^2 \, ||\, F' \,|| + |\, y \,|^2 \Big(\, ||\, F' \,|| + ||\, x\, F''(x) \,|| \, \Big) \\
&= |\, y \,|^2 \Big(4 \, ||\, F' \,|| + ||\, x\, F''(x) \,|| \Big).
\end{aligned}$$

□

3 Edgeworth expansions for linear rank statistics using Stein's method

3.1 Descriptions and results

Let $A = (a_{ij})$ be an $n\times n-$matrix of real numbers. We first introduce some abbreviations and notations in connection with this matrix.

We define

$$a_{i.} = \frac{1}{n}\sum_{j=1}^{n} a_{ij}, \quad a_{.j} = \frac{1}{n}\sum_{i=1}^{n} a_{ij}, \quad a_{..} = \frac{1}{n^2}\sum_{i,j=1}^{n} a_{ij}$$

and

$$\mu_A = n\, a_{..}\ ,$$

$$\sigma_A^2 = \begin{cases} 0 & \text{for } n = 1, \\ \dfrac{1}{n-1}\displaystyle\sum_{i,j=1}^{n}\big(a_{ij} - a_{i.} - a_{.j} + a_{..}\big)^2 & \text{for } n \geq 2. \end{cases}$$

In the following, we will only consider matrices with $\sigma_A > 0$ and thus $n \geq 2$. This allows us to introduce the corresponding standardized matrix $\hat{A} = (\hat{a}_{ij})$, whose coefficients are

$$\hat{a}_{ij} = \frac{1}{\sigma_A}\big(a_{ij} - a_{i.} - a_{.j} + a_{..}\big).$$

We get

(3.1.1) $\qquad \hat{a}_{i.} = \hat{a}_{.j} = \hat{a}_{..} = 0 \ \text{ for all } i,\, j \ \ ; \quad \mu_{\hat{A}} = 0 \ \text{ and } \ \sigma_{\hat{A}}^2 = 1.$

In addition, we define

$$\beta_A = \sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^3, \quad \delta_A = \sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^4, \quad \eta_A = \sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^5,$$

$$D_A = \left(\frac{\delta_A}{n}\right)^{1/2}, \quad E_A = \left(\frac{\eta_A}{n}\right)^{1/3}$$

and

$$\begin{aligned} M(l,A) &= \text{set of all } (n-l)\times(n-l) - \text{matrices, which can be} \\ &\quad \text{obtained from } A \text{ by cancelling } l \text{ rows and } l \text{ columns,} \\ N(l,A) &= \bigcup \Big\{ M(r,A) \,:\, 0 \le r \le \min\{\, l, n-1 \,\} \Big\}. \end{aligned}$$

If π is uniformly distributed on the set $\mathscr{P}_n$ of permutations of $\{1,\ldots,n\}$, we denote by

$$T_A = \sum_{i=1}^{n} a_{i\pi(i)}.$$

the **linear rank statistic** for the matrix A (under the null hypothesis H_0).

According to Hájek, Šidák and Sen [16], Theorem 1 in section 3.3.1, page 57, we have

$$\text{(3.1.2)} \qquad E(T_A) = \mu_A \ \text{ and } \ \mathrm{Var}(T_A) = \sigma_A^2.$$

Thus, for the related standardized rank statistic holds

$$\mathscr{T}_A = \frac{T_A - E(T_A)}{\sqrt{\mathrm{Var}(T_A)}} = \frac{T_A - \mu_A}{\sigma_A} = T_{\hat{A}}.$$

Furthermore, F_A and $\mathscr{F}_A$ define the distribution function of T_A and $\mathscr{T}_A$, respectively.

The starting point for this chapter is the following estimate of the distance of $\mathscr{F}_A$ to the distribution function Φ of the standard normal distribution. This result is from Bolthausen [6], Theorem, page 380, and states:

3.1.3 Theorem

There exists an absolute constant $\mathcal{K}_1 > 0$ such that for all matrices A satisfying $\sigma_A > 0$

$$\text{(3.1.4)} \qquad ||\,\mathscr{F}_A - \Phi\,|| \le \mathcal{K}_1 \,\frac{\beta_A}{n}.$$

If $\big(A^{(n)}\big)_{n\ge 2}$ is a sequence of $n\times n-$matrices, this theorem gives in many typical applications the convergence rate $||\,\mathscr{F}_{A^{(n)}} - \Phi\,|| = \mathcal{O}\bigg(\dfrac{1}{\sqrt{n}}\bigg)$.

3.1.5 Remark

Three decades after the publication of Theorem 3.1.3 Chen and Fang [7] proved that $\mathcal{K}_1 \leq 451$ can be chosen. Shortly afterwards Thành [29] showed that even $\mathcal{K}_1 \leq 90$ is possible.

In the following, we will consider first and second order Edgeworth expansions of $\mathscr{F}_A$ and prove similar results for the distance between $\mathscr{F}_A$ and these expansions. We define the first and second order Edgeworth expansions of $\mathscr{F}_A$ as follows

$$e_{1,A}(x) = \Phi(x) - \psi(x)\,\frac{\lambda_{1,A}}{6}\,(x^2-1) \qquad \text{and} \tag{3.1.6}$$

$$e_{2,A}(x) = \Phi(x) - \psi(x)\left\{\frac{\lambda_{1,A}}{6}\,(x^2-1) + \frac{\lambda_{2,A}}{24}\,(x^3-3x) + \frac{\lambda_{1,A}^2}{72}\,(x^5-10x^3+15x)\right\}, \tag{3.1.7}$$

where

$$\lambda_{1,A} = \frac{1}{n}\sum_{i,j=1}^{n} \hat{a}_{ij}^3 \qquad \text{and} \tag{3.1.8}$$

$$\lambda_{2,A} = \frac{1}{n}\sum_{i,j=1}^{n} \hat{a}_{ij}^4 + \frac{3}{n} - \frac{3}{n^2}\sum_{i,j,k=1}^{n}\left(\hat{a}_{ij}^2\hat{a}_{ik}^2 + \hat{a}_{ij}^2\hat{a}_{kj}^2\right). \tag{3.1.9}$$

For $e_{1,A}$ we obtain the following result:

3.1.10 Theorem

There exist absolute constants $\mathcal{K}_2 > 0$ and $\mathcal{K}_3 > 0$ such that for all matrices A satisfying $\sigma_A > 0$ and the condition

(3.1.11) there exists a constant $\mathcal{C}_1 > 0$ such that

$$\left|\,\Delta_y^2 F_B(z)\,\right| \leq \mathcal{C}_1\left(D_A^2 + y^2\right)$$

for all $z \in \mathbb{R}$, $0 \leq y \leq D_A$ and $B \in N(8, \hat{A})$,

we have

$$||\,\mathscr{F}_A - e_{1,A}\,|| \leq \left(\mathcal{K}_2\,\mathcal{C}_1 + \mathcal{K}_3\right) D_A^2. \tag{3.1.12}$$

The corresponding result for $e_{2,A}$ is:

3.1.13 Theorem

There exist absolute constants $\mathcal{K}_4 > 0$, $\mathcal{K}_5 > 0$ and $\mathcal{K}_6 > 0$ such that for all matrices A satisfying

$\sigma_A > 0$ and the conditions

(3.1.14) there exists a constant $\mathcal{C}_2 > 0$ such that

$$\Big| F_B(x) - P^2_{E_A}(x; z, F_B) \Big| \le \mathcal{C}_2 \, \Big(E_A^3 + (x-z)^3 \Big)$$

for all $z \in \mathbb{R}$, $z \le x \le z + 3\, E_A$ and $B \in N(16, \hat{A})$,

(3.1.15) there exists a constant $\mathcal{C}_3 > 0$ such that

$$(|z| + 1) \, \big| \, \Delta_y^2 F_B(z) \, \big| \le \mathcal{C}_3 \, \big(\, E_A^2 + y^2 \, \big)$$

for all $z \in \mathbb{R}$, $0 \le y \le E_A$ and $B \in N(16, \hat{A})$,

we have

(3.1.16) $|| \, \mathscr{F}_A - e_{2,A} \, || \le \big(\mathcal{K}_4 \, \mathcal{C}_2 + \mathcal{K}_5 \, \mathcal{C}_3 + \mathcal{K}_6 \, \big) \, E_A^3.$

3.1.17 Remarks

a) Although the Theorems 3.1.10 and 3.1.13, as Theorem 3.1.3 above, are formulated for fixed A and n, their real purpose is to apply them to sequences of $n \times n-$matrices $A^{(n)}$. One then usually obtains (cf. b) below) a statement about the rate of convergence of $|| \, \mathscr{F}_{A^{(n)}} - e_{1,A^{(n)}} \, ||$ and $|| \, \mathscr{F}_{A^{(n)}} - e_{2,A^{(n)}} \, ||$ to zero.

b) In many typical applications, D_A^2 is of order $\dfrac{1}{n}$ and E_A^3 is of order $\dfrac{1}{n^{3/2}}$ (cf. e.g. Lemma 3.1.18, a) and Lemma 4.3.6, c)).

However, there are also sequences $A^{(n)}$ for which $D^2_{A^{(n)}} \not\to 0$ and $E^3_{A^{(n)}} \not\to 0$. We give a simple example:

$$A^{(n)} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix}.$$

If n is even, then $\sigma^2_{A^{(n)}} = \dfrac{2\,n}{n-1}$ and thus

$$D^2_{A^{(n)}} = \frac{1}{2} \Big(\frac{n-1}{n} \Big)^2, \quad E^3_{A^{(n)}} = \frac{1}{2^{\,3/2}} \Big(\frac{n-1}{n} \Big)^{5/2}.$$

c) In chapter 4 we will apply the two theorems to matrices of the form $a_{ij} = e_i \, d_j$. In doing so, we specify conditions that are easy to prove and under which the conditions (3.1.11) or (3.1.14) and (3.1.15) are fulfilled (but with "constants" $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, which grow slightly with n).

The following proof of the Theorems 3.1.10 and 3.1.13 is divided into seven sections. The next five sections (i.e. 3.2 - 3.6) contain numerous preparations. These ensure that we can prove Theorem 3.1.10 in section 3.7 and Theorem 3.1.13 in section 3.8 using Stein's method.

In this section 3.1 we will next show some simple, but repeatedly needed statements about quantities that were introduced at the beginning of this section. After that we will make some statements about the necessity of the conditions (3.1.11), (3.1.14) and (3.1.15).

3.1.18 Lemma

a) If $2 < k$, then

$$\sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^k \;\geq\; \frac{1}{2^{k/2} n^{(k/2)-2}}\ .$$

In particular,

$$\beta_A \geq \Big(\frac{n}{8}\Big)^{1/2}, \qquad \delta_A \geq \frac{1}{4}, \qquad \eta_A \geq \Big(\frac{1}{32\,n}\Big)^{1/2} \qquad \text{and} \qquad D_A \geq \frac{1}{2\sqrt{n}}, \qquad E_A \geq \frac{1}{2^{5/6}\sqrt{n}}.$$

b) If $2 < k < r$, then

$$\Big(\frac{1}{n}\sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^k\Big)^{1/(k-2)} \;\leq\; \Big(\frac{1}{n}\sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^r\Big)^{1/(r-2)}\ .$$

In particular, $\dfrac{\beta_A}{n} \leq D_A \leq E_A.$

c) $$\sum_{i,j,k=1}^{n} \hat{a}_{ij}^2\,\hat{a}_{ik}^2 \;\leq\; n\sum_{i,j=1}^{n}\hat{a}_{ij}^4 \quad \text{and} \quad \sum_{i,j,k=1}^{n} \hat{a}_{ij}^2\,\hat{a}_{kj}^2 \;\leq\; n\sum_{i,j=1}^{n}\hat{a}_{ij}^4\ .$$

d) $$|\,\lambda_{1,A}\,| \;\leq\; \frac{\beta_A}{n} \qquad \text{and} \qquad |\,\lambda_{2,A}\,| \;\leq\; \frac{3}{n} + 7\,\frac{\delta_A}{n} \;\leq\; 19\,\frac{\delta_A}{n}.$$

e) $$||\,e_{2,A} - e_{1,A}\,|| \leq \frac{1}{2}\,\frac{\delta_A}{n}\,\Big(=\frac{1}{2}\,D_A^2\Big) \qquad \text{and} \qquad ||\,e'_{2,A} - e'_{1,A}\,|| \leq \frac{11}{10}\,\frac{\delta_A}{n},$$

$$||\,x\,e''_{2,A} - x\,e''_{1,A}\,|| \leq \frac{9}{5}\,\frac{\delta_A}{n}.$$

f) $\|\, \mathscr{F}_A - e_{1,A} \,\| \le \|\, \mathscr{F}_A - \Phi \,\| + \dfrac{1}{6}\, \dfrac{\beta_A}{n} \,\|\, H_2\, \psi \,\| \le 1 + \dfrac{1}{15}\, \dfrac{\beta_A}{n}.$

g) $\|\, e_{1,A} \,\| \le \|\, \Phi \,\| + \dfrac{1}{6}\, \dfrac{\beta_A}{n} \,\|\, H_2\, \psi \,\| \le 1 + \dfrac{1}{15}\, \dfrac{\beta_A}{n}$ and $\|\, e'_{1,A} \,\| \le \dfrac{2}{5} + \dfrac{1}{10}\, \dfrac{\beta_A}{n},$

$\|\, x\, e''_{1,A}(x) \,\| \le \dfrac{3}{10} + \dfrac{1}{5}\, \dfrac{\beta_A}{n}$ and $\|\, e''_{1,A} \,\| \le \dfrac{1}{4} + \dfrac{1}{5}\, \dfrac{\beta_A}{n}.$

h) $\|\, \mathscr{F}_A - e_{2,A} \,\| \le 1 + \dfrac{1}{15}\, \dfrac{\beta_A}{n} + \dfrac{1}{2}\, \dfrac{\delta_A}{n}.$

i) $\|\, e_{2,A} \,\| \le 1 + \dfrac{1}{15}\, \dfrac{\beta_A}{n} + \dfrac{1}{2}\, \dfrac{\delta_A}{n}$ and $\|\, e'_{2,A} \,\| \le \dfrac{2}{5} + \dfrac{1}{10}\, \dfrac{\beta_A}{n} + \dfrac{11}{10}\, \dfrac{\delta_A}{n},$

$\|\, x\, e''_{2,A}(x) \,\| \le \dfrac{3}{10} + \dfrac{1}{5}\, \dfrac{\beta_A}{n} + \dfrac{9}{5}\, \dfrac{\delta_A}{n}$ and $\|\, e'''_{2,A} \,\| \le \dfrac{2}{5} + \dfrac{7}{18}\, \dfrac{\beta_A}{n} + \dfrac{16}{3}\, \dfrac{\delta_A}{n}.$

j) Let $e^1_{1,A}(z) = \displaystyle\int_{-\infty}^{z} x\, e'_{1,A}(x)\, dx = -\psi(z) - z^3\, \psi(z)\, \frac{\lambda_{1,A}}{6}.$ Then $\|\, e^1_{1,A} \,\| \le \dfrac{2}{5} + \dfrac{7}{90}\, \dfrac{\beta_A}{n}.$

Proof:

a) Since we assume $\sigma_A > 0$ and thus $n \ge 2$, we have $n \le 2\,(n-1)$. By applying $1 = \sigma^2_{\hat{A}}$ (cf. (3.1.1)) and Hölder's inequality (0.1.5) with $\nu = n^2$ and $p = \dfrac{k}{2} > 1$ we obtain

$$\begin{aligned} 1 = \frac{1}{n-1} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^2 &\le \frac{1}{n-1} \Big(\sum_{i,j=1}^{n} \big(|\hat{a}_{ij}|^2 \big)^{k/2} \Big)^{2/k} \cdot \big(n^2 \big)^{(k-2)/k} \\ &= \frac{1}{n-1} \Big(\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \Big)^{2/k} \cdot n^{2(k-2)/k}. \end{aligned}$$

Exponentiating this with $p = \dfrac{k}{2}$ and then applying $n - 1 \ge \dfrac{n}{2}$ leads to

$$\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \ge \frac{(n-1)^{k/2}}{n^{k-2}} \ge \frac{n^{k/2}}{2^{k/2} n^{k-2}} = \frac{1}{2^{k/2} n^{(k/2)-2}}. \tag{3.1.19}$$

b) Because of $k = r\, \dfrac{k-2}{r-2} + 2\, \dfrac{r-k}{r-2}$, Hölder's inequality with $p = \dfrac{r-2}{k-2}$, $q = \dfrac{r-2}{r-k}$ and $1 = \sigma^2_{\hat{A}}$ (cf. (3.1.1)) gives

$$\sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^k \le \Big(\sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^r \Big)^{\frac{k-2}{r-2}} \cdot \Big(\sum_{i,j=1}^{n} \hat{a}^2_{ij} \Big)^{\frac{r-k}{r-2}} \le \Big(\sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^r \Big)^{\frac{k-2}{r-2}} \cdot n^{\frac{r-k}{r-2}} \ .$$

The assertion follows from this due to $\dfrac{1}{k-2} - \dfrac{1}{r-2} = \dfrac{r-k}{(k-2)\,(r-2)}$.

c) A further application of Hölder's inequality (0.1.5) with $\nu = n$ and $p = 2$ gives

$$\sum_{i,j,k=1}^{n} \hat{a}_{ij}^2\, \hat{a}_{ik}^2 = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \hat{a}_{ij}^2 \Big)^2 \leq n \sum_{i,j=1}^{n} \hat{a}_{ij}^4.$$

The second inequality follows analogously.

d) Using part c) and part a) we obtain

$$|\,\lambda_{2,A}\,| \leq \frac{\delta_A}{n} + \frac{3}{n} + \frac{3}{n^2}\, 2\, n\, \delta_A = \frac{3}{n} + 7\, \frac{\delta_A}{n} \leq 12\, \frac{\delta_A}{n} + 7\, \frac{\delta_A}{n} = 19\, \frac{\delta_A}{n}.$$

e) Using part d), Lemma 2.2.8, f) and j), and then part b) we obtain

$$\begin{aligned}
||\, e_{2,A} - e_{1,A}\, || &= ||\, \frac{\lambda_{2,A}}{24}\, \psi(x)\, H_3(x) + \frac{\lambda_{1,A}^2}{72}\, \psi(x)\, H_5(x)\, || \\
&\leq \frac{1}{24}\, |\lambda_{2,A}| \cdot ||\, \psi(x)\, H_3(x)\, || + \frac{1}{72}\, |\lambda_{1,A}|^2 \cdot ||\, \psi(x)\, H_5(x)\, || \\
&\leq \frac{19}{24}\, \frac{\delta_A}{n}\, \frac{5}{9} + \frac{1}{72} \Big(\frac{\beta_A}{n} \Big)^2 \frac{7}{3} \leq \frac{1}{2}\, \frac{\delta_A}{n}.
\end{aligned}$$

Analogously, using Lemma 2.2.8, h) and l), and Lemma 2.2.8, k) and n) gives

$$\begin{aligned}
||\, e_{2,A}' - e_{1,A}'\, || &\leq \frac{19}{24}\, \frac{\delta_A}{n}\, \frac{6}{5} + \frac{1}{72} \Big(\frac{\beta_A}{n} \Big)^2 6 \leq \frac{11}{10}\, \frac{\delta_A}{n}, \\
||\, x\, e_{2,A}'' - x\, e_{1,A}''\, || &\leq \frac{19}{24}\, \frac{\delta_A}{n}\, 2 + \frac{1}{72} \Big(\frac{\beta_A}{n} \Big)^2 13 \leq \frac{9}{5}\, \frac{\delta_A}{n}.
\end{aligned}$$

f) The second inequality follows from Lemma 2.2.8, e).

g) The estimates follow because of Lemma 2.2.8, e), f), i), h) for the second summands (in this order) and due to Lemma 2.2.8, a), c), b) for the first summands.

h) This follows from the parts f) and e).

i) With the exception of the estimate for $||\, e_{2,A}'''\, ||$, this follows from the parts g) and e). To estimate $||\, e_{2,A}'''\, ||$, we use Lemma 2.2.8, e), j), l), o) and get

$$||\, e_{2,A}'''\, || \leq ||\, H_2\, \psi\, || + \frac{1}{6}\, \frac{\beta_A}{n}\, ||\, H_5\, \psi\, || + \frac{19}{24}\, \frac{\delta_A}{n}\, ||\, H_6\, \psi\, || + \frac{1}{72} \Big(\frac{\beta_A}{n} \Big)^2 ||\, H_8\, \psi\, ||$$

$$\le \frac{2}{5} + \frac{1}{6}\,\frac{\beta_A}{n}\,\frac{7}{3} + \frac{19}{24}\,\frac{\delta_A}{n}\,6 + \frac{1}{72}\,\frac{\delta_A}{n}\,42$$

$$= \frac{2}{5} + \frac{7}{18}\,\frac{\beta_A}{n} + \frac{16}{3}\,\frac{\delta_A}{n}.$$

j) The estimate follows from Lemma 2.2.8, a) and d). □

We now show that from (3.1.12) we obtain the condition (3.1.11) for $B = \hat{A}$ with a different constant. In the same way, we can derive the conditions (3.1.14) and (3.1.15) for $B = \hat{A}$ with other constants from (3.1.16).
However, these proofs do not apply to the submatrices $B \in N(8, \hat{A})$ (or $B \in N(16, \hat{A})$).

3.1.20 Proposition

a) If $\|\,\mathscr{F}_A - e_{1,A}\,\| \le K\,D_A^2$, then

$$(3.1.21) \qquad \left|\,\Delta_y^2 \mathscr{F}_A(z)\,\right| \le \left(\,4\,K + 1\,\right)\left(\,D_A^2 + y^2\,\right) \quad \text{for all } z, y \in \mathbb{R}.$$

b) If $\|\,\mathscr{F}_A - e_{2,A}\,\| \le K\,E_A^3$, then

$$(3.1.22) \qquad \left|\mathscr{F}_A(x) - P^2_{E_A}(x; z, \mathscr{F}_A)\right| \le \left(\,8\,K + 6\,\right)\left(E_A^3 + |x - z|^3\right) \quad \text{for all } z, x \in \mathbb{R}.$$

c) If $\|\,\mathscr{F}_A - e_{2,A}\,\| \le K\,E_A^3$, then

$$(3.1.23) \qquad \left|\,z\,\Delta_y^2 \mathscr{F}_A(z)\,\right| \le \left(\,16\,K + 11\,\right)\left(\,E_A^2 + y^2\,\right) \quad \text{for all } z \in \mathbb{R},\ 0 \le y \le E_A.$$

d) If $\|\,\mathscr{F}_A - e_{2,A}\,\| \le K\,E_A^3$, then

$$(3.1.24) \qquad \|\,\mathscr{F}_A - e_{1,A}\,\| \le \left(K + \frac{16}{15}\right) E_A^2$$

and thus

$$(3.1.25) \qquad \left|\,\Delta_y^2 \mathscr{F}_A(z)\,\right| \le \left(4\left(K + \frac{16}{15}\right) + 1\right)\left(\,E_A^2 + y^2\,\right) \quad \text{for all } z, y \in \mathbb{R}.$$

Proof:

a) Due to Lemma 2.4.2, b) (for $e_{1,A}$ and $k = 2$) and Lemma 3.1.18, g) (for $\|\,e''_{1,A}\,\|$) we get

$$\left|\,\Delta_y^2 \mathscr{F}_A(z)\,\right| \le 4\,K\,D_A^2 + \left|\,\Delta_y^2 e_{1,A}(z)\,\right|$$

$$\le 4\,K\,D_A^2 + \left(\frac{1}{4} + \frac{1}{5}\,\frac{\beta_A}{n}\right) y^2.$$

Now, if $\dfrac{\beta_A}{n} \le 1$, then (3.1.21) holds. On the other hand, if $1 < \dfrac{\beta_A}{n}$, then (3.1.21) follows using Lemma 3.1.18, b)

$$\bigl|\, \Delta_y^2 \mathscr{F}_A(z) \,\bigr| \le 1 \le \Bigl(\frac{\beta_A}{n} \Bigr)^2 \le D_A^2 .$$

b) Due to Lemma 2.4.2, a) (for $e_{2,A}$ and $k = 2$) and Lemma 3.1.18, i) (for $\|\, e'''_{2,A} \,\|$) we get

$$\begin{aligned}
\Bigl| \mathscr{F}_A(x) - P^2_{E_A}(x; z, \mathscr{F}_A) \Bigr| &\le \Bigl| \mathscr{F}_A(x) - e_{2,A}(x) \Bigr| + \Bigl| e_{2,A}(x) - P^2_{E_A}(x; z, e_{2,A}) \Bigr| \\
&\quad + \Bigl| P^2_{E_A}(x; z, e_{2,A}) - P^2_{E_A}(x; z, \mathscr{F}_A) \Bigr| \qquad \text{(cf. [4], page 502)} \\
&\le K\, E_A^3 + \frac{4}{3} \Bigl(\frac{2}{5} + \frac{7}{18}\, \frac{\beta_A}{n} + \frac{16}{3}\, \frac{\delta_A}{n} \Bigr) \Bigl(E_A^3 + |x - z|^3 \Bigr) \\
&\quad + K\, E_A^3 + 2\, K\, E_A^3\, \frac{|x - z|}{E_A} + 4\, K\, E_A^3\, \frac{|x - z|}{E_A}\, \frac{|x - z - E_A|}{2\, E_A} \\
&\le K\, E_A^3 + \Bigl(\frac{8}{15} + \frac{14}{27}\, \frac{\beta_A}{n} + \frac{64}{9}\, \frac{\delta_A}{n} \Bigr) \Bigl(E_A^3 + |x - z|^3 \Bigr) \\
&\quad + K\, E_A^3 + 2\, K\, E_A^2\, |x - z| + 2\, K\, E_A \Bigl(|x - z|^2 + |x - z|\, E_A \Bigr) \\
&\le \Bigl(8\, K + \frac{8}{15} + \frac{14}{27}\, \frac{\beta_A}{n} + \frac{64}{9}\, \frac{\delta_A}{n} \Bigr) \Bigl(E_A^3 + |x - z|^3 \Bigr).
\end{aligned}$$

For the last inequality, the following formula was used:

$$a^2\, b \le \bigl(\max\{a, b\} \bigr)^3 \le a^3 + b^3 \qquad \text{for } a, b \ge 0.$$

In the case of $E_A^3 \le \dfrac{1}{2}$, from which $\dfrac{\beta_A}{n} \le \Bigl(\dfrac{1}{2} \Bigr)^{1/3}$ and $\dfrac{\delta_A}{n} \le \Bigl(\dfrac{1}{2} \Bigr)^{2/3}$ follow (cf. Lemma 3.1.18, b)), we thus obtain (3.1.22).

On the other hand, in the case of $E_A^3 > \dfrac{1}{2}$, we get due to the inequalities $|x| \le 1 + |x|^3$ and $x^2 \le 1 + |x|^3$:

$$\begin{aligned}
\Bigl| \mathscr{F}_A(x) - P^2_{E_A}(x; z, \mathscr{F}_A) \Bigr| &\le \Bigl| \mathscr{F}_A(x) - \mathscr{F}_A(z) \Bigr| + \frac{|x - z|}{E_A} + \frac{1}{2}\, \frac{|x - z|^2 + |x - z|\, E_A}{E_A^2} \\
&\le 3 \Bigl(1 + \frac{|x - z|^3}{E_A^3} \Bigr) \\
&\le 6 \Bigl(E_A^3 + |x - z|^3 \Bigr).
\end{aligned}$$

c) Because of Lemma 2.4.2, c) (for $e_{2,A}$) holds

$$\begin{aligned}
\big|\, z\,\Delta_y^2 \mathscr{F}_A(z)\,\big| &\le |z|\,\big|\,\mathscr{F}_A(z+2y) - e_{2,A}(z+2y)\,\big| + 2\,|z|\,\big|\,\mathscr{F}_A(z+y) - e_{2,A}(z+y)\,\big| \\
&\qquad + |z|\,\big|\,\mathscr{F}_A(z) - e_{2,A}(z)\,\big| + \big|\, z\,\Delta_y^2 e_{2,A}(z)\,\big| \\
&\le 4\,|z|\,K\,E_A^3 + \Big(\,4\,\big|\big|\,e'_{2,A}\,\big|\big| + \big|\big|\,x\,e''_{2,A}(x)\,\big|\big|\,\Big)\,y^2.
\end{aligned}$$

We now assume $|z| \le \dfrac{4}{E_A}$ and $E_A \le 1$. Using Lemma 3.1.18, b) and i), we then obtain

$$\begin{aligned}
\big|\, z\,\Delta_y^2 \mathscr{F}_A(z)\,\big| &\le 4\,\frac{4}{E_A}\,K\,E_A^3 + \Big(\,4\,\frac{8}{5} + \frac{23}{10}\,\Big)\,y^2 \\
&\le 16\,K\,E_A^2 + 9\,y^2.
\end{aligned}$$

For the further calculation, let $z \in \mathbb{R}$ and $0 \le y \le E_A$ be general again:

$$\begin{aligned}
\big|\, z\,\Delta_y^2 \mathscr{F}_A(z)\,\big| &= \Big|\, z\,P\Big(\,z+y < \mathscr{T}_A \le z+2y\,\Big) - z\,P\Big(\,z < \mathscr{T}_A \le z+y\,\Big)\,\Big| \\
&\le |z|\,P\Big(\,z+y < \mathscr{T}_A \le z+2y\,\Big) + |z|\,P\Big(\,z < \mathscr{T}_A \le z+y\,\Big) \\
&\le |z|\,P\Big(\,z - 2\,E_A \le \mathscr{T}_A \le z + 2\,E_A\,\Big) \qquad (3.1.26) \\
&\le |z|\,P\Big(\,|z| - 2\,E_A \le |\mathscr{T}_A|\,\Big) \\
&= |z|\,P\Big(\,|z| \le |\mathscr{T}_A| + 2\,E_A\,\Big).
\end{aligned}$$

We now suppose $|z| \ge \dfrac{4}{E_A}$ and $E_A \le 1$. An application of Markov's inequality with the function $h(x) = |x|^3$ then yields

$$\begin{aligned}
\big|\, z\,\Delta_y^2 \mathscr{F}_A(z)\,\big| &\le |z|\,P\Big(\,|z| \le |\mathscr{T}_A| + 2\,\Big) && \text{(since } E_A \le 1) \\
&\le |z|\,\frac{1}{|z|^3}\,E\big(\,\big|\,|\mathscr{T}_A| + 2\,\big|^3\,\big) && \text{(Markov's inequality)} \\
&\le \frac{1}{|z|^2}\,E\big(\,\big(\,|\mathscr{T}_A| + 2\,\big)^4\,\big)^{3/4} && \text{(Hölder's inequality)} \\
&\le \frac{1}{|z|^2}\,E\big(\,8\,\big(\,\mathscr{T}_A^4 + 16\,\big)\,\big)^{3/4} && \text{((0.1.5) with } \nu = 2,\ p = 4) \\
&= \frac{1}{|z|^2}\,\Big(\,8\,E\big(\,\mathscr{T}_A^4\,\big) + 128\,\Big)^{3/4}
\end{aligned}$$

$$\leq \frac{E_A^2}{4^2}\, 928^{3/4} \qquad\qquad \Big(\, |z| \geq \frac{4}{E_A} \;\Rightarrow\; \frac{E_A}{4} \geq \frac{1}{|z|}\,\Big)$$

$$\leq 11\, E_A^2.$$

For the penultimate inequality, we used the estimate $E\big(\,\mathscr{T}_A^4\,\big) \leq 100$. This estimate follows from Proposition 3.2.2, d) together with $\dfrac{\delta_A}{n} \leq E_A^2 \leq 1$ (cf. Lemma 3.1.18, b)).

It remains to consider the case $E_A \geq 1$. We use the inequality (3.1.26) again:

$$\begin{aligned} \big|\, z\, \Delta_y^2 \mathscr{F}_A(z)\, \big| &\leq |z|\, P\Big(\, |z| \leq |\mathscr{T}_A| + 2\, E_A\,\Big) \\ &\leq E\big(\,|\mathscr{T}_A| + 2\, E_A\,\big) \\ &\leq E\big(\,\mathscr{T}_A^2\,\big)^{1/2} + 2\, E_A \\ &= 1 + 2\, E_A \leq 3\, E_A^2. \end{aligned}$$

d) Because of Lemma 3.1.18, e) and b) we get:

$$\begin{aligned} ||\, \mathscr{F}_A - e_{1,A}\, || &\leq ||\, \mathscr{F}_A - e_{2,A}\, || + ||\, e_{2,A} - e_{1,A}\, || \\ &\leq K\, E_A^3 + \frac{1}{2}\, D_A^2 \\ &\leq \Big(\, K\, E_A + \frac{1}{2}\Big)\, E_A^2. \end{aligned}$$

If $E_A \leq 1$, then (3.1.24) holds. On the other hand, if $1 < E_A$, then because of Lemma 3.1.18, f) and b) we obtain:

$$\begin{aligned} ||\, \mathscr{F}_A - e_{1,A}\, || &\leq 1 + \frac{1}{15}\, \frac{\beta_A}{n} \\ &\leq 1 + \frac{1}{15}\, E_A \leq \frac{16}{15}\, E_A^2. \end{aligned}$$

The statement (3.1.25) now follows from (3.1.24) analogously to the proof in part a). □

3.1.27 Remark

The question arises whether it is sufficient in the conditions of the two Theorems 3.1.10 and 3.1.13 to require the validity of the inequalities only for $\hat{A}$ instead of $B \in N(8, \hat{A})$ or $B \in N(16, \hat{A})$. I assume that this is the case. However, I cannot provide any proof of this.

3.2 Higher moments of linear rank statistics

At first, we consider the third and fourth moment of $\mathscr{T}_A$.

3.2.1 Theorem

a) $$E\big(\mathscr{T}_A^3\big) = \frac{n}{(n-1)\,(n-2)} \sum_{i,j=1}^{n} \hat{a}_{ij}^3 \qquad \text{for } n \geq 3,$$

b) $$\begin{aligned} E\big(\mathscr{T}_A^4\big) = {} & 3\,\frac{(n^2-3\,n+1)\,(n-1)}{n\,(n-2)\,(n-3)} \\ & - \frac{3}{(n-2)\,(n-3)} \sum_{i,j,k=1}^{n} \big(\,\hat{a}_{ij}^2\,\hat{a}_{ik}^2 + \hat{a}_{ij}^2\,\hat{a}_{kj}^2\,\big) \\ & + \frac{n\,(n+1)}{(n-1)\,(n-2)\,(n-3)} \sum_{i,j=1}^{n} \hat{a}_{ij}^4 \\ & + \frac{6}{n\,(n-1)\,(n-2)\,(n-3)} \sum_{i,j,k,h=1}^{n} \hat{a}_{ik}\,\hat{a}_{ih}\,\hat{a}_{jk}\,\hat{a}_{jh} \qquad \text{for } n \geq 4. \end{aligned}$$

Proof:

Hájek, Šidák and Sen [16], Problem 27 to Section 3.3, page 90, taking into account $\sum_{i,j=1}^{n} \hat{a}_{ij}^2 = n-1$ (cf. (3.1.1)). □

A somewhat more manageable formula of the third and fourth moment of $\mathscr{T}_A$, which is sufficient for our purposes, is contained in the following proposition.

3.2.2 Proposition

a) $$E\big(\mathscr{T}_A^3\big) = \frac{1}{n} \sum_{i,j=1}^{n} \hat{a}_{ij}^3 + R_{3,A},$$

$$\text{where } |R_{3,A}| \leq \frac{11}{n^2}\,\beta_A \qquad \text{for } n \geq 3.$$

b) $$E\big(\mathscr{T}_A^4\big) = 3 + \frac{3}{n} - \frac{3}{n^2} \sum_{i,j,k=1}^{n} \big(\,\hat{a}_{ij}^2\,\hat{a}_{ik}^2 + \hat{a}_{ij}^2\,\hat{a}_{kj}^2\,\big) + \frac{1}{n} \sum_{i,j=1}^{n} \hat{a}_{ij}^4 + R_{4,A},$$

$$\text{where } |R_{4,A}| \leq \frac{336}{n^2}\,\delta_A \qquad \text{for } n \geq 4.$$

c) $$\Big|\,E\big(\mathscr{T}_A^3\big)\Big| \leq 5\,\frac{\beta_A}{n} \qquad \text{for } n \geq 3.$$

d) $\left| E\left(\mathscr{T}_A^4 - 3 \right) \right| \le 97 \, \dfrac{\delta_A}{n} \qquad$ for $n \ge 4$.

Proof:

Part a) follows immediately from the previous Theorem 3.2.1, a) and the inequality

$$\text{(3.2.3)} \qquad \left| \frac{n}{(n-1)\,(n-2)} - \frac{1}{n} \right| \le \frac{11}{n^2} \qquad \text{for } n \ge 3.$$

This inequality (3.2.3) is derived from the following property of the polynomial P_1

$$P_1(x) = 8\,x^2 - 31\,x + 22 \ge 0 \qquad \text{for } x \ge 2,9395 \ ,$$

from which we get for $n \ge 3$

$$\begin{aligned} & 11\,n^3 - 33\,n^2 + 22\,n \ge 3\,n^3 - 2\,n^2 \\ \Leftrightarrow \ & \frac{11}{n^2} \ge \frac{3\,n-2}{n\,(n-1)\,(n-2)} = \frac{n}{(n-1)\,(n-2)} - \frac{1}{n}. \end{aligned}$$

Furthermore, since $\dfrac{11}{n} \le 4$ is valid for $n \ge 3$, part c) is a simple consequence of part a).

For part b) we first need

$$\text{(3.2.4)} \qquad \delta_A \ge \left(\frac{n-1}{n} \right)^2 \ge \frac{9}{16} \qquad \text{for } n \ge 4$$

(cf. (3.1.19) in the proof of Lemma 3.1.18, a)). The difference between the first two summands of the formula from part b) and the first summand of Theorem 3.2.1, b) is then

$$\begin{aligned} & \left| 3\, \frac{(n^2 - 3\,n + 1)\,(n-1)}{n\,(n-2)\,(n-3)} - \left(3 + \frac{3}{n} \right) \right| \\ = \ & 3\, \frac{3\,n - 7}{n\,(n-2)\,(n-3)} \le \frac{30}{n^2} \le \frac{54}{n^2}\, \delta_A \qquad \text{for } n \ge 4. \end{aligned}$$

The last inequality follows from (3.2.4). The penultimate inequality is derived analogously to above from the following property of the polynomial P_2

$$P_2(x) = 7\,x^2 - 43\,x + 60 \ge 0 \qquad \text{for } x \ge 4 \ ,$$

from which we get for $n \ge 4$

$$10\,n^3 - 50\,n^2 + 60\,n \ge 3\,n^3 - 7\,n^2$$

$$\Leftrightarrow \;\; \frac{10}{n^2} \geq \frac{3\,n-7}{n\,(n-2)\,(n-3)}.$$

Furthermore, for the difference between the third summand of the formula from part b) and the second summand of Theorem 3.2.1, b), the following holds due to Lemma 3.1.18, c)

$$\begin{aligned} & \left| \left(\frac{3}{(n-2)\,(n-3)} - \frac{3}{n^2} \right) \sum_{i,j,k=1}^{n} \left(\hat{a}_{ij}^2\, \hat{a}_{ik}^2 + \hat{a}_{ij}^2\, \hat{a}_{kj}^2 \right) \right| \\ \leq & \left| \left(\frac{3}{(n-2)\,(n-3)} - \frac{3}{n^2} \right) \right| 2\,n\,\delta_A \\ = & \; 6\,\frac{5\,n-6}{n\,(n-2)\,(n-3)}\,\delta_A \leq \frac{168}{n^2}\,\delta_A \qquad \text{for } n \geq 4. \end{aligned}$$

The last inequality is derived again analogously to above from

$$P_3(x) = 23\,x^2 - 134\,x + 168 \geq 0 \qquad \text{for } x \geq 4 \;,$$

from which we get for $n \geq 4$

$$\begin{aligned} & 28\,n^3 - 140\,n^2 + 168\,n \geq 5\,n^3 - 6\,n^2 \\ \Leftrightarrow \;\; & \frac{28}{n^2} \geq \frac{5\,n-6}{n\,(n-2)\,(n-3)}. \end{aligned}$$

For the difference of the terms with the sums of the fourth powers holds

$$\begin{aligned} & \left| \left(\frac{n\,(n+1)}{(n-1)\,(n-2)\,(n-3)} - \frac{1}{n} \right) \sum_{i,j=1}^{n} \hat{a}_{ij}^4 \right| \\ = & \; \frac{7\,n^2 - 11\,n + 6}{n\,(n-1)\,(n-2)\,(n-3)}\,\delta_A \leq \frac{50}{n^2}\,\delta_A \qquad \text{for } n \geq 4. \end{aligned}$$

To justify the last inequality, we consider

$$P_4(x) = 43\,x^3 - 289\,x^2 + 544\,x - 300 \geq 0 \qquad \text{for } x \geq 3,9864 \;,$$

from which we get for $n \geq 4$

$$\begin{aligned} & 50\,n^4 - 300\,n^3 + 550\,n^2 - 300\,n \geq 7\,n^4 - 11\,n^3 + 6\,n^2 \\ \Leftrightarrow \;\; & \frac{50}{n^2} \geq \frac{7\,n^2 - 11\,n + 6}{n\,(n-1)\,(n-2)\,(n-3)}. \end{aligned}$$

Finally, the fourth summand of the Theorem 3.2.1, b) disappears completely in the term $R_{4,A}$, since

$$\Big| \sum_{i,j,k,h=1}^{n} \hat{a}_{ik}\, \hat{a}_{ih}\, \hat{a}_{jk}\, \hat{a}_{jh} \Big| \le \sum_{k,h=1}^{n} \Big(\sum_{i=1}^{n} |\hat{a}_{ik}\, \hat{a}_{ih}| \Big)^2$$

$$\le \sum_{k,h=1}^{n} \Big[\Big(\sum_{i=1}^{n} \hat{a}_{ik}^2 \Big) \Big(\sum_{j=1}^{n} \hat{a}_{jh}^2 \Big) \Big] \qquad \text{(Hölder's inequality)}$$

$$= \Big(\sum_{i,k=1}^{n} \hat{a}_{ik}^2 \Big) \Big(\sum_{j,h=1}^{n} \hat{a}_{jh}^2 \Big)$$

$$= (n-1)^2 \qquad \text{(cf. (3.1.1))}$$

and therefore

$$\Big| \frac{6}{n\,(n-1)\,(n-2)\,(n-3)} \sum_{i,j,k,h=1}^{n} \hat{a}_{ik}\, \hat{a}_{ih}\, \hat{a}_{jk}\, \hat{a}_{jh} \Big|$$

$$\le \frac{6\,(n-1)}{n\,(n-2)\,(n-3)} \le \frac{36}{n^2} \le \frac{64}{n^2}\, \delta_A \qquad \text{for } n \ge 4.$$

The last inequality follows from (3.2.4). The penultimate inequality is deduced from

$$P_5(x) = 30\,x^2 - 174\,x + 216 \ge 0 \qquad \text{for } x \ge 4\ ,$$

from which we get for $n \ge 4$

$$36\,n^3 - 180\,n^2 + 216\,n \ge 6\,n^3 - 6\,n^2$$

$$\Leftrightarrow\ \frac{36}{n^2} \ge \frac{6\,(n-1)}{n\,(n-2)\,(n-3)}.$$

In summary, we obtain part b), since

$$|R_{4,A}| \le \frac{54}{n^2}\, \delta_A + \frac{168}{n^2}\, \delta_A + \frac{50}{n^2}\, \delta_A + \frac{64}{n^2}\, \delta_A = \frac{336}{n^2}\, \delta_A.$$

Last but not least, part d) is derived from part b) by the following calculations:

$$\Big| E\big(\mathscr{T}_A^4 - 3 \big) \Big| \ \le\ |\,\lambda_{2,A}\,| + |\,R_{4,A}\,| \qquad \text{(cf. (3.1.9))}$$

$$\le\ \frac{3}{n} + 7\,\frac{\delta_A}{n} + \frac{336}{n^2}\, \delta_A \qquad \text{(cf. Lemma 3.1.18, d))}$$

$$\le\ \frac{16}{3}\,\frac{\delta_A}{n} + 7\,\frac{\delta_A}{n} + 84\,\frac{\delta_A}{n} \le 97\,\frac{\delta_A}{n} \qquad \text{for } n \ge 4.$$

For the penultimate inequality, we used (3.2.4) and $\dfrac{336}{n} \le 84$ for $n \ge 4$. $\square$

The inequalities from parts c) and d) of Proposition 3.2.2 can be generalized in a certain way to higher moments as follows:

3.2.5 Proposition

Let $k \in \mathbb{N} \setminus \{1, 2\}$. Then there exists a constant $C(k) > 0$ such that for all matrices A satisfying $\sigma_A > 0$

$$\left| E\big(\mathscr{T}_A^k\big) - \Phi(x^k) \right| \leq C(k) \Big(\frac{\beta_A}{n} + \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \Big). \tag{3.2.6}$$

Proof:

If we proceed as in Hoeffding [17], page 563, from (29), to page 566, we obtain the following result:

The difference $\left| E\big(\mathscr{T}_A^k\big) - \Phi(x^k) \right|$ can be estimated by $c_1(k)$ many summands, each of which can in turn be estimated by an expression of the form

$$c_2(k)\,\frac{1}{n} \quad \text{or} \quad c_3(k) \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{r_1} \Big) \cdot \ldots \cdot \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{r_s} \Big).$$

Here, $r_1 + \ldots + r_s = k$; $r_1 \geq 2, \ldots, r_s \geq 2$ and $r_t > 2$ for **at least** one $t \in \{1, \ldots, s\}$.

Because of $\beta_A \geq \frac{1}{2}$ for $n \geq 2$ (cf. Lemma 3.1.18, a)), it follows that

$$c_2(k)\,\frac{1}{n} \leq c_4(k) \Big(\frac{\beta_A}{n} + \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \Big) \qquad \text{with } c_4(k) = 2\,c_2(k).$$

Furthermore, if $v_1, \ldots, v_u$ are those of the exponents $r_1, \ldots, r_s$ that are **really greater** than 2, and we denote their sum by $v = v_1 + \ldots + v_u \leq k$, then

$$\begin{aligned}
& c_3(k) \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{r_1} \Big) \cdot \ldots \cdot \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{r_s} \Big) \\
\leq\ & c_3(k) \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{v_1} \Big) \cdot \ldots \cdot \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{v_u} \Big) \qquad \Big(\text{since } \sum_{i,j=1}^{n} |\hat{a}_{ij}|^2 = n - 1 \leq n\Big) \\
\leq\ & c_3(k) \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{v-2\,(u-1)} \Big)^{\frac{v_1 - 2}{v - 2u}} \cdot \ldots \cdot \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{v-2\,(u-1)} \Big)^{\frac{v_u - 2}{v - 2u}}
\end{aligned}$$

(cf. Lemma 3.1.18, b))

$$= c_3(k) \left(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{v-2\,(u-1)} \right) \qquad \text{(since } (v_1 - 2) + \ldots + (v_u - 2) = v - 2u\text{)}$$

$$\leq c_3(k) \left(\frac{\beta_A}{n} + \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{k} \right) \qquad \text{(since } 3 \leq v - 2\,(u-1) \leq k\text{). } \square$$

3.3 The submatrices $B \in M(l, \hat{A})$

In this paragraph, we mainly analyze the expected value μ_B and the variance σ_B^2 for matrices $B \in M(l, \hat{A})$. We will prove that these two values for certain matrices A do not deviate "too far" from $\mu_{\hat{A}}$ and $\sigma_{\hat{A}}^2$.

The following result serves as preparation for this.

3.3.1 Lemma

If $B \in M(l, \hat{A})$, then

a) $|\,\mu_B| \leq \dfrac{l^2}{n-l} + \dfrac{\beta_A}{n-l}$;

b) $$\left| \sigma_B^2 - \frac{1}{n-l-1} \sum_{i,j=1}^{n} \hat{a}_{ij}^2 \right|$$

$$\leq \frac{1}{n-l-1} \left\{ \sum_{(i \in Z_l) \,\vee\, (j \in S_l)} \hat{a}_{ij}^2 + 5 \left(\frac{l^2}{n-l} + \frac{\beta_A}{n-l} \right)^2 + 10\,l^2 + 10\,l\,\frac{\beta_A}{n-l} \right\},$$

where $Z_l = \{\, z_1, \ldots, z_l \,\}$ ($S_l = \{\, s_1, \ldots, s_l \,\}$) denotes the set of indices of the **cancelled** rows (columns).

In addition, the symbols $\vee$ denote the logical "or" and $\wedge$ the logical "and".

Proof:

We have

(3.3.2)
$$|\,b_{..}\,| = \left| \frac{1}{(n-l)^2} \sum_{(i \in Z_l) \,\wedge\, (j \in S_l)} \hat{a}_{ij} \right| \qquad \text{(since } \hat{a}_{i.} = 0 = \hat{a}_{.j}\text{)}$$

$$\leq \frac{1}{(n-l)^2} \sum_{(i \in Z_l) \,\wedge\, (j \in S_l)} \left(1 + |\hat{a}_{ij}|^3 \right) \qquad \text{(since } |\hat{a}_{ij}| \leq 1 + |\hat{a}_{ij}|^3\text{)}$$

$$\leq \frac{l^2}{(n-l)^2} + \frac{\beta_A}{(n-l)^2}\,.$$

From this follows a). Analogously to (3.3.2) we obtain

$$\text{(3.3.3)} \qquad \sum_{i \notin Z_l} b_{i.}^2 = \sum_{i \notin Z_l} \frac{1}{(n-l)^2} \Big(\sum_{j \in S_l} \hat{a}_{ij} \Big)^2 \qquad \text{(since } \hat{a}_{i.} = 0\text{)}$$

$$\leq \sum_{i \notin Z_l} \frac{1}{(n-l)^2}\, l \sum_{j \in S_l} \hat{a}_{ij}^2 \qquad \text{((0.1.5) with } \nu = l,\ p = 2\text{)}$$

$$\leq \frac{l}{(n-l)^2} \sum_{(i \notin Z_l)\ \wedge\ (j \in S_l)} \big(1 + |\hat{a}_{ij}|^3 \big) \qquad \text{(since } \hat{a}_{ij}^2 \leq 1 + |\hat{a}_{ij}|^3\text{)}$$

$$\leq \frac{l^2}{n-l} + l\, \frac{\beta_A}{(n-l)^2}$$

and

$$\text{(3.3.4)} \qquad \sum_{j \notin S_l} b_{.j}^2 \leq \frac{l^2}{n-l} + l\, \frac{\beta_A}{(n-l)^2} \ .$$

Also is valid

$$\text{(3.3.5)} \qquad \sum_{(i \notin Z_l)\ \wedge\ (j \notin S_l)} b_{ij} \big(b_{..} - b_{i.} - b_{.j} \big)$$

$$= b_{..} \sum_{(i \notin Z_l)\ \wedge\ (j \notin S_l)} b_{ij} \ - \sum_{i \notin Z_l} b_{i.} \sum_{j \notin S_l} b_{ij} \ - \sum_{j \notin S_l} b_{.j} \sum_{i \notin Z_l} b_{ij}$$

$$= (n-l)^2\, b_{..}^2 \ - \ (n-l) \sum_{i \notin Z_l} b_{i.}^2 \ - \ (n-l) \sum_{j \notin S_l} b_{.j}^2 \ .$$

By reapplying Hölder's inequality (0.1.5) with $\nu = 3$ and $p = 2$ we further obtain

$$\text{(3.3.6)} \qquad \big(b_{..} - b_{i.} - b_{.j} \big)^2 \leq 3 \big(b_{..}^2 + b_{i.}^2 + b_{.j}^2 \big) \qquad \text{for all } i \notin Z_l \text{ and } j \notin S_l.$$

Now, using (3.3.5) and (3.3.6), we get

$$\Big| \sigma_B^2 - \frac{1}{n-l-1} \sum_{i,j=1}^{n} \hat{a}_{ij}^2 \Big|$$

$$\leq \frac{1}{n-l-1} \Big\{ \sum_{(i \in Z_l)\ \vee\ (j \in S_l)} \hat{a}_{ij}^2 + 2 \Big| \sum_{(i \notin Z_l)\ \wedge\ (j \notin S_l)} b_{ij} \big(b_{..} - b_{i.} - b_{.j} \big) \Big|$$

$$+ \sum_{(i \notin Z_l)\ \wedge\ (j \notin S_l)} \big(b_{..} - b_{i.} - b_{.j} \big)^2 \Big\}$$

$$\leq \frac{1}{n-l-1} \Big\{ \sum_{(i \in Z_l)\ \vee\ (j \in S_l)} \hat{a}_{ij}^2 + 5 \Big((n-l)^2\, b_{..}^2 + (n-l) \sum_{i \notin Z_l} b_{i.}^2 + (n-l) \sum_{j \notin S_l} b_{.j}^2 \Big) \Big\}.$$

This and (3.3.2), (3.3.3), (3.3.4) finally give part b). $\square$

A direct consequence of Lemma 3.3.1 is:

3.3.7 Corollary

There exists a $0 < \epsilon_0 \leq 1$ and an $n_0 \in \mathbb{N}$ such that for all matrices A satisfying $\sigma_A > 0$, $\dfrac{\beta_A}{n} \leq \epsilon_0$ and $n \geq n_0$ holds:

If $1 \leq l \leq 17$ and $B \in M(l, \hat{A})$, then

(3.3.8) $\quad | \, \mu_B \, | \leq 1,$

(3.3.9) $\quad \big| \, \sigma_B^2 - 1 \, \big| \leq \dfrac{1}{3}.$

Proof:

If ϵ_0 is sufficiently small and n_0 is sufficiently large, then (3.3.8) follows from part a) of the previous Lemma 3.3.1.

Furthermore, an application of Hölder's inequality (0.1.5) with $\nu \leq 2\, l\, n$ and $p = \dfrac{3}{2}$ gives

$$\begin{aligned}\sum_{(i \in Z_l) \,\vee\, (j \in S_l)} \hat{a}_{ij}^2 &\leq (2\, l\, n)^{1/3} \Big(\sum_{(i \in Z_l) \,\vee\, (j \in S_l)} |\hat{a}_{ij}|^3 \Big)^{2/3} \\ &\leq 4\, n \Big(\frac{\beta_A}{n} \Big)^{2/3} \qquad \text{for } 1 \leq l \leq 17.\end{aligned}$$

If we insert this estimate into part b) of the previous Lemma 3.3.1) and use $\displaystyle\sum_{i,j=1}^{n} \hat{a}_{ij}^2 = n - 1$, we further obtain

$$\begin{aligned}\big| \, \sigma_B^2 - 1 \, \big| &\leq \Big| \, \sigma_B^2 - \frac{1}{n-l-1} \sum_{i,j=1}^{n} \hat{a}_{ij}^2 \, \Big| + \Big| \, \frac{1}{n-l-1} \sum_{i,j=1}^{n} \hat{a}_{ij}^2 - 1 \, \Big| \\ &\leq \frac{1}{n-l-1} \Big\{ 4\, n \Big(\frac{\beta_A}{n} \Big)^{2/3} + 5 \Big(\frac{l^2}{n-l} + \frac{\beta_A}{n-l} \Big)^2 + 10\, l^2 + 10\, l \, \frac{\beta_A}{n-l} \Big\} \\ &\qquad + \Big| \, \frac{n-1}{n-l-1} - 1 \, \Big| \,,\end{aligned}$$

Thus we get (3.3.9) for sufficiently small ϵ_0 and sufficiently large n_0. $\square$

For matrices $B \in M(l, \hat{A})$ satisfying $\sigma_B^2 \geq \frac{2}{3}$ the following very useful relationship between $\hat{B}$ and $\hat{A}$ holds:

3.3.10 Lemma

Let $k \in \mathbb{N}$ and $B \in M(l, \hat{A})$ satisfying $\sigma_B^2 \geq \frac{2}{3}$. Then

$$\sum_{(i \notin Z_l) \,\wedge\, (j \notin S_l)} |\hat{b}_{ij}|^k \leq 5^k \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k. \tag{3.3.11}$$

Proof:

At first, an application of Hölder's inequality (0.1.5) with $\nu = n - l$ and $p = k$ gives

$$\begin{aligned}(n-l) \sum_{i \notin Z_l} |b_{i.}|^k &\leq \frac{1}{(n-l)^{k-1}} \sum_{i \notin Z_l} \Big(\sum_{j \notin S_l} |\hat{a}_{ij}| \Big)^k \\ &\leq \frac{1}{(n-l)^{k-1}} \sum_{i \notin Z_l} \Big((n-l)^{k-1} \sum_{j \notin S_l} |\hat{a}_{ij}|^k \Big) \leq \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k.\end{aligned}$$

Analogously follows

$$(n-l) \sum_{j \notin S_l} |b_{.j}|^k \leq \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{and} \qquad (n-l)^2 \, |b_{..}|^k \leq \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k.$$

A further application of Hölder's inequality (0.1.5) with $\nu = 4$ and $p = k$ finally leads to

$$\begin{aligned}\sum_{(i \notin Z_l) \,\wedge\, (j \notin S_l)} |\hat{b}_{ij}|^k &\leq \Big(\frac{3}{2}\Big)^{k/2} 4^{k-1} \sum_{(i \notin Z_l) \,\wedge\, (j \notin S_l)} \Big(|b_{ij}|^k + |b_{i.}|^k + |b_{.j}|^k + |b_{..}|^k \Big) \\ &\leq 5^k \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad\qquad (\text{since } \sqrt{\tfrac{3}{2}}\, 4 = 4,8990 \leq 5). \quad \square\end{aligned}$$

3.4 The truncated matrices A' and $\bar{A}$

In the following, many results can only be shown for matrices whose elements are bounded in absolute value by one. However, in order to be able to prove the two Theorems 3.1.10 and 3.1.13 for general matrices, in this section, we will first create the truncated matrices A' and $\bar{A}$ from $\hat{A}$ and then establish some useful connections between these matrices.

Let

$$a'_{ij} = \begin{cases} \hat{a}_{ij} & \text{if} \quad |\hat{a}_{ij}| \le \dfrac{1}{2}, \\[2ex] 0 & \text{if} \quad |\hat{a}_{ij}| > \dfrac{1}{2}. \end{cases}$$

In addition, let

$$\Gamma = \left\{ (i,j) \,:\, |\hat{a}_{ij}| > \frac{1}{2} \right\} ,$$

$$d_{ij} = a'_{i.} + a'_{.j} - a'_{..} ,$$

$$\bar{a}_{ij} = \hat{a}'_{ij} = \frac{1}{\sigma_{A'}} \, (a'_{ij} - d_{ij})$$

and $A' = (a'_{ij})$, $D = (d_{ij})$, $\bar{A} = (\bar{a}_{ij})$ the corresponding $n \times n-$matrices. We will first list some elementary results.

3.4.1 Lemma

Let $k, r \in \mathbb{N}$ and

$$A[k] = 2^k \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k .$$

Then

a) $|\Gamma| \le A[k]$,

b) $P(\mathscr{T}_A \ne T_{A'}) \le \dfrac{|\Gamma|}{n} \le \dfrac{A[k]}{n}$,

c) $\displaystyle\sum_{(i,j)\,\in\,\Gamma} |\hat{a}_{ij}|^r \le 2^{-r}\, A[k] \qquad \text{for } r \le k,$

d) $\displaystyle\sum_{i=1}^{n} |a'_{i.}| \le \frac{A[k]}{2\,n}, \qquad \sum_{j=1}^{n} |a'_{.j}| \le \frac{A[k]}{2\,n} \qquad \text{and} \qquad |a'_{..}| \le \frac{A[k]}{2\,n^2},$

e) $\displaystyle\sum_{i,j=1}^{n} |d_{ij}| \le \frac{3}{2}\, A[k],$

f) $\displaystyle |d_{ij}| \le \frac{3}{2} \min\left\{ 1, \frac{A[k]}{n} \right\} \qquad \text{for all } 1 \le i,j \le n,$

g) $|\mu_{A'}| \le \dfrac{A[k]}{2\,n}$,

h) $\displaystyle \left|\, \sigma^2_{A'} - 1 \,\right| \le 8\, \frac{A[k]}{n} \qquad \text{for } k \ge 2.$

Proof:

This proof is based on arguments of Bolthausen [6], page 382.

a) $\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \geq \sum_{(i,j)\,\in\,\Gamma} |\hat{a}_{ij}|^k \geq \frac{1}{2^k}\,|\Gamma|.$

b) For $i \in \{1,\ldots,n\}$ we define the following i-section of Γ:

$$\Gamma_i = \big\{\, j \in \{1,\ldots,n\} \,:\, (i,j) \in \Gamma \,\big\}.$$

Since $\pi(i)$ is **equally distributed** on $\{1,\ldots,n\}$ for all i, we obtain

$$\begin{aligned}
P(\mathscr{T}_A \neq T_{A'}) &= P(T_{\hat{A}} \neq T_{A'}) \\
&\leq P\Big(\sum_{i=1}^{n} 1_\Gamma(i,\pi(i)) \geq 1\Big) \\
&\leq E\Big(\sum_{i=1}^{n} 1_\Gamma(i,\pi(i))\Big) \\
&= \sum_{i=1}^{n} P\Big(\,\pi(i) \in \Gamma_i\,\Big) \\
&= \sum_{i=1}^{n} \frac{|\Gamma_i|}{n} = \frac{|\Gamma|}{n} \leq \frac{A[k]}{n} && \text{(cf. part a)).}
\end{aligned}$$

c) $$\begin{aligned}
\sum_{(i,j)\,\in\,\Gamma} |\hat{a}_{ij}|^r &\leq |\Gamma|^{(k-r)/k}\,\Big(\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k\Big)^{r/k} && \Big(\text{(0.1.5) with } \nu = |\Gamma|,\ p = \frac{k}{r}\Big) \\
&\leq 2^{-r}\,A[k] && \text{(cf. part a)).}
\end{aligned}$$

d) Using the notation Γ_i from the proof of part b), we obtain

$$\begin{aligned}
\sum_{i=1}^{n} |a'_{i.}| &= \sum_{i=1}^{n} |a'_{i.} - \hat{a}_{i.}| && \text{(since } \hat{a}_{i.} = 0 \text{ (cf. (3.1.1))} \\
&= \sum_{i=1}^{n} \Big|\,\frac{1}{n}\sum_{j=1}^{n} a'_{ij} - \frac{1}{n}\sum_{j=1}^{n} \hat{a}_{ij}\,\Big| \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\sum_{j\,\in\,\Gamma_i} |\hat{a}_{ij}| \\
&= \frac{1}{n}\sum_{(i,j)\,\in\,\Gamma} |\hat{a}_{ij}| \leq \frac{A[k]}{2\,n} && \text{(cf. part c) with } r = 1).
\end{aligned}$$

Similarly, $\sum_{j=1}^{n} |a'_{.j}| \leq \frac{A[k]}{2\,n}$ and $|a'_{..}| \leq \frac{A[k]}{2\,n^2}$.

e) Due to part d) is valid $\displaystyle\sum_{i,j=1}^{n} |d_{ij}| \le n \sum_{i=1}^{n} |a'_{i.}| + n \sum_{j=1}^{n} |a'_{.j}| + n^2\,|a'_{..}| \le \frac{3}{2}\,A[k].$

f) We have $|a'_{ij}| \le \dfrac{1}{2}$ and thus $|d_{ij}| \le |a'_{i.}| + |a'_{.j}| + |a'_{..}| \le \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} = \dfrac{3}{2}.$

Due to part d) also holds $|d_{ij}| \le |a'_{i.}| + |a'_{.j}| + |a'_{..}| \le \dfrac{A[k]}{2\,n} + \dfrac{A[k]}{2\,n} + \dfrac{A[k]}{2\,n^2} \le \dfrac{3}{2}\,\dfrac{A[k]}{n}.$

g) Due to part d) is valid $|\mu_{A'}| = |n\,a'_{..}| \le \dfrac{A[k]}{2\,n}.$

h)
$$\begin{aligned}
\left|\,\sigma^2_{A'} - 1\,\right| &= \left|\,\sigma^2_{A'} - \sigma^2_{\hat{A}}\,\right| && \text{(since } \sigma^2_{\hat{A}} = 1 \text{ (cf. (3.1.1))}\\
&= \frac{1}{n-1}\left|\,\sum_{i,j=1}^{n} (a'_{ij} - d_{ij})^2 - \hat{a}^2_{ij}\,\right| \\
&= \frac{1}{n-1}\left|\,\sum_{i,j=1}^{n} \left(\,a'^2_{ij} - \hat{a}^2_{ij}\,\right) - 2\,a'_{ij}\,d_{ij} + d^2_{ij}\,\right| \\
&\le \frac{1}{n-1}\left(\,\sum_{(i,j)\,\in\,\Gamma} \hat{a}^2_{ij} + \frac{5}{2}\sum_{i,j=1}^{n} |d_{ij}|\,\right) && \text{(since } |a'_{ij}| \le \tfrac{1}{2},\ |d_{ij}| \le \tfrac{3}{2} \text{ (cf. part f))}\\
&\le 4\,\frac{A[k]}{n-1} \le 8\,\frac{A[k]}{n} && \text{(cf. part c), part e) and } n \ge 2\text{).} \quad \square
\end{aligned}$$

An important consequence of Lemma 3.4.1 is:

3.4.2 Corollary

There exists a $0 < \epsilon_0 \le 1$ such that for all matrices A satisfying $\sigma_A > 0$ and $\dfrac{\beta_A}{n} \le \epsilon_0$ holds:

(3.4.3) $\quad \left|\,\sigma^2_{A'} - 1\,\right| \le \dfrac{1}{3},$

(3.4.4) $\quad |\overline{a}_{ij}| \le 1 \qquad$ for all $1 \le i, j \le n$.

Proof:

Let $\epsilon_0 = \dfrac{1}{192}$. Then we obtain (3.4.3) by applying Lemma 3.4.1, h) with $k = 3$. From this and because of Lemma 3.4.1, f) with $k = 3$, we also get (3.4.4):

$$|\overline{a}_{ij}| \le \frac{1}{\sigma_{A'}}\left(\,|a'_{ij}| + |d_{ij}|\,\right) \le \sqrt{\frac{3}{2}}\left(\frac{1}{2} + 12\,\frac{\beta_A}{n}\right) \le \sqrt{\frac{3}{2}}\,\frac{9}{16} \le 1. \qquad \square$$

Now, under the condition $\sigma^2_{A'} \ge \dfrac{2}{3}$, which follows from (3.4.3), we can derive some useful relations between the matrices $\hat{A}$ and $\bar{A}$. The first result is an analogue of Lemma 3.3.10.

3.4.5 Lemma

Let $k \in \mathbb{N}$. Then, for each matrix A satisfying $\sigma^2_{A'} \geq \frac{2}{3}$,

$$\sum_{i,j=1}^{n} |\bar{a}_{ij}|^k \leq 5^k \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k.$$

Proof:

We use $|a'_{ij}| \leq |\hat{a}_{ij}|$ for all $1 \leq i, j \leq n$ and proceed in the same way as in the proof of Lemma 3.3.10 (with $l = 0$). □

Since the expansions $e_{1,A}$ and $e_{2,A}$ contain the terms $\frac{1}{n}\sum_{i,j=1}^{n} \hat{a}^3_{ij}$, $\frac{1}{n}\sum_{i,j=1}^{n} \hat{a}^4_{ij}$, $\Big(\frac{1}{n}\sum_{i,j=1}^{n} \hat{a}^3_{ij}\Big)^2$, etc., the following estimates are also of interest for the further procedure.

3.4.6 Lemma

Let $k \in \mathbb{N}$. Then there exist positive constants $C_1(k)$, $C_2(k)$, $C_3(k)$ and $C_4(k)$ such that for all matrices A satisfying $\sigma^2_{A'} \geq \frac{2}{3}$ holds:

a) $$\Big| \frac{1}{n}\sum_{i,j=1}^{n} \hat{a}^3_{ij} - \frac{1}{n}\sum_{i,j=1}^{n} \bar{a}^3_{ij} \Big| \leq C_1(k)\, \frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{for } k \geq 3.$$

b) $$\Big| \frac{1}{n}\sum_{i,j=1}^{n} \hat{a}^4_{ij} - \frac{1}{n}\sum_{i,j=1}^{n} \bar{a}^4_{ij} \Big| \leq C_2(k)\, \frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{for } k \geq 4.$$

c) $$\Big| \frac{1}{n^2}\sum_{i,j,r=1}^{n} \hat{a}^2_{ij}\, \hat{a}^2_{ir} - \frac{1}{n^2}\sum_{i,j,r=1}^{n} \bar{a}^2_{ij}\, \bar{a}^2_{ir} \Big| \leq C_3(k)\, \frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{for } k \geq 4.$$

The same is true for $\sum_{i,j,r=1}^{n} \hat{a}^2_{ij}\, \hat{a}^2_{rj}$.

d) $$\Big| \Big(\frac{1}{n}\sum_{i,j=1}^{n} \hat{a}^3_{ij}\Big)^2 - \Big(\frac{1}{n}\sum_{i,j=1}^{n} \bar{a}^3_{ij}\Big)^2 \Big| \leq C_4(k)\, \frac{1}{n}\sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{for } k \geq 4.$$

Proof:

a) Let $k \geq 3$ be fixed. Then an application of the triangle inequality gives

$$(3.4.7) \qquad \Big|\, \frac{1}{n} \sum_{i,j=1}^{n} \hat{a}_{ij}^3 - \frac{1}{n} \sum_{i,j=1}^{n} \overline{a}_{ij}^3 \,\Big|$$
$$\leq \Big|\, 1 - \frac{1}{\sigma_{A'}^3} \,\Big| \, \frac{\beta_A}{n} + \frac{1}{\sigma_{A'}^3} \,\Big|\, \frac{1}{n} \sum_{i,j=1}^{n} \hat{a}_{ij}^3 - \frac{1}{n} \sum_{i,j=1}^{n} \big(a'_{ij} - d_{ij} \big)^3 \,\Big|.$$

To estimate the first summand of (3.4.7), we first consider

$$\Big|\, 1 - \frac{1}{\sigma_{A'}^3} \,\Big| \leq \frac{3}{2} \max\Big\{ \frac{1}{\sigma_{A'}^5}, 1 \Big\} \big|\, \sigma_{A'}^2 - 1 \,\big| \qquad \text{(mean value theorem for } f(x) = \frac{1}{x^{3/2}})$$
$$\leq \Big(\frac{3}{2} \Big)^{7/2} 2^{k+2} \, \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{k-1} \qquad \text{(due to Lemma 3.4.1, h) and } \sigma_{A'}^2 \geq \frac{2}{3}).$$

We claim that, using the definition $c_1(k) = \Big(\frac{3}{2} \Big)^{7/2} 2^{k+2}$, we get

$$\Big|\, 1 - \frac{1}{\sigma_{A'}^3} \,\Big| \, \frac{\beta_A}{n} \leq c_1(k) \, \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k.$$

This inequality is obtained in the case $k = 3$ because of $\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{k-1} = \frac{n-1}{n} \leq 1$ and in the case $k > 3$ by a double application of Lemma 3.1.18, b), which yields

$$(3.4.8) \qquad \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{k-1} \Big) \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^3 \Big)$$
$$\leq \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^k \Big)^{(k-3)/(k-2)} \Big(\frac{1}{n} \sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^k \Big)^{1/(k-2)} = \frac{1}{n} \sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^k.$$

To estimate the second summand of (3.4.7), we calculate

$$\Big|\, \sum_{i,j=1}^{n} \hat{a}_{ij}^3 - \sum_{i,j=1}^{n} \big(a'_{ij} - d_{ij} \big)^3 \,\Big|$$
$$= \Big|\, \sum_{i,j=1}^{n} \big(\hat{a}_{ij}^3 - a_{ij}'^3 \big) + 3\, a_{ij}'^2 \, d_{ij} - 3\, a'_{ij} \, d_{ij}^2 + d_{ij}^3 \,\Big|$$
$$\leq \sum_{(i,j) \in \Gamma} |\hat{a}_{ij}|^3 + \frac{21}{4} \sum_{i,j=1}^{n} |d_{ij}| \qquad \text{(since } |a'_{ij}| \leq \frac{1}{2} \text{ and } |d_{ij}| \leq \frac{3}{2}\text{, cf. Lemma 3.4.1, f))}$$
$$\leq \Big(2^{k-3} + \frac{63}{8} \, 2^k \Big) \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{(due to Lemma 3.4.1, c) and e))}$$
$$= 2^{k+3} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k.$$

b) We obtain the claimed estimate analogously to part a) via

$$\Big| 1 - \frac{1}{\sigma_{A'}^4} \Big| \le c_2(k)\, \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{k-2} \qquad \text{for } k \ge 4.$$

c) Using Lemma 3.1.18, c) we get analogously to (3.4.7)

$$\begin{aligned}
&\Big| \frac{1}{n^2} \sum_{i,j,r=1}^{n} \hat{a}_{ij}^2\, \hat{a}_{ir}^2 - \frac{1}{n^2} \sum_{i,j,r=1}^{n} \bar{a}_{ij}^2\, \bar{a}_{ir}^2 \Big| \\
&\le \Big| 1 - \frac{1}{\sigma_{A'}^4} \Big| \frac{\delta_A}{n} + \frac{1}{\sigma_{A'}^4} \Big| \frac{1}{n^2} \sum_{i,j,r=1}^{n} \hat{a}_{ij}^2\, \hat{a}_{ir}^2 - \frac{1}{n^2} \sum_{i,j,r=1}^{n} \big(a'_{ij} - d_{ij} \big)^2 \big(a'_{ir} - d_{ir} \big)^2 \Big|.
\end{aligned}$$

The second summand of this is estimated as follows. Let $k \ge 2$, then

$$\begin{aligned}
&\frac{1}{n} \Big| \sum_{i,j,r=1}^{n} \hat{a}_{ij}^2\, \hat{a}_{ir}^2 - \sum_{i,j,r=1}^{n} \big(a'_{ij} - d_{ij} \big)^2 \big(a'_{ir} - d_{ir} \big)^2 \Big| \\
&\le \frac{1}{n} \Big| \sum_{i,j,r=1}^{n} \big(\hat{a}_{ij}^2\, \hat{a}_{ir}^2 - a'^2_{ij}\, a'^2_{ir} \big) \Big| + \frac{85}{8} \sum_{i,j=1}^{n} |d_{ij}| \qquad \Big(\text{since } |a'_{ij}| \le \frac{1}{2} \text{ and } |d_{ij}| \le \frac{3}{2}\Big) \\
&\le \frac{2}{n} \sum_{\substack{i,j,r=1 \\ (i,j) \in \Gamma}}^{n} \hat{a}_{ij}^2\, \hat{a}_{ir}^2 + \frac{85}{8} \sum_{i,j=1}^{n} |d_{ij}| \\
&\le 2 \sum_{(i,j) \in \Gamma}^{n} \hat{a}_{ij}^2 + \frac{85}{8} \sum_{i,j=1}^{n} |d_{ij}| \qquad \Big(\text{since } \sum_{l,r=1}^{n} \hat{a}_{lr}^2 = n - 1 \le n\Big) \\
&\le \Big(2^{k-1} + \frac{255}{16}\, 2^k \Big) \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad (\text{due to Lemma 3.4.1, c) and e)}) \\
&= \frac{263}{16}\, 2^k \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k.
\end{aligned}$$

d) At first, we remark that

$$|\, a^2 - b^2 \,| = |\, a + b \,|\, |\, a - b \,| = |\, 2\, a - (a - b) \,|\, |\, a - b \,| \le \big(2\, |\, a \,| + |\, a - b \,| \big)\, |\, a - b \,|$$

holds for all $a, b \in \mathbb{R}$. We use this inequality and then part a) twice, so that we get

$$\begin{aligned}
\Big| \Big(\frac{1}{n} \sum_{i,j=1}^{n} \hat{a}_{ij}^3 \Big)^2 - \Big(\frac{1}{n} \sum_{i,j=1}^{n} \bar{a}_{ij}^3 \Big)^2 \Big| &\le \Big(2 + C_1(3) \Big) \frac{\beta_A}{n}\, C_1(k-1)\, \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^{k-1} \\
&\le \Big(2 + C_1(3) \Big)\, C_1(k-1)\, \frac{1}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{for } k \ge 4.
\end{aligned}$$

For the last inequality, we again used the estimate from (3.4.8). □

Next, we show a small analytic lemma that we need to derive our most important result about the relations between the matrices $\hat{A}$ and $\bar{A}$.

3.4.9 Lemma

Let g be differentiable and let an $r \in \mathbb{R}$ be given such that

$$(3.4.10) \qquad \begin{cases} g(z) = r + g(-z) & \text{for all } z \in \mathbb{R} \quad \text{or} \\ g(z) = r - g(-z) & \text{for all } z \in \mathbb{R}\,. \end{cases}$$

Then

a) $\sup\limits_{z\in\mathbb{R}} \big|\, g(z+q) - g(z) \,\big| \leq |q|\, ||g'|| \qquad \text{for } q \in \mathbb{R},$

b) $\sup\limits_{z\in\mathbb{R}} \big|\, g(p\,z) - g(z) \,\big| \leq (p-1)\, ||id_{\mathbb{R}}\, g'|| \qquad \text{for } p \geq 1,$

c) $\sup\limits_{z\in\mathbb{R}} \big|\, g(p\,z) - g(z) \,\big| \leq \Big(\dfrac{1}{p} - 1\Big)\, ||id_{\mathbb{R}}\, g'|| \qquad \text{for } 0 < p \leq 1.$

Proof:

Part a) is a trivial conclusion from the mean value theorem and part c) follows from part b) using

$$\sup_{z\in\mathbb{R}} \big|\, g(p\,z) - g(z) \,\big| = \sup_{z\in\mathbb{R}} \big|\, g(z) - g(\frac{z}{p}) \,\big|.$$

Part b) therefore remains to be verified. Because of (3.4.10), $z > 0$ can be assumed without loss of generality. A further application of the mean value theorem then yields

$$\begin{aligned} \big|\, g(p\,z) - g(z) \,\big| &= \big|\, g(z + (p-1)\,z) - g(z) \,\big| \\ &\leq (p-1)\,z \sup_{z \,\leq\, t \,\leq\, (p-1)\,z} |\, g'(t) \,| \\ &\leq (p-1) \sup_{z \,\leq\, t \,\leq\, (p-1)\,z} |\, t\, g'(t) \,| \\ &\leq (p-1)\, ||id_{\mathbb{R}}\, g'||. \end{aligned}$$

□

3.4.11 Example

The condition (3.4.10) is fulfilled, for example, by the function $g(z) = \Phi(z)$ and all its derivatives (cf. (2.2.1)).

3.4.12 Conclusion

If a differentiable function g satisfies the condition (3.4.10) and $\sigma_{A'}^2 \geq \dfrac{2}{3}$ is valid, then

$$\Big| g\Big(\frac{z-\mu_{A'}}{\sigma_{A'}}\Big) - g(z)\Big| \leq \Big| g(z-\mu_{A'}) - g(z)\Big| + \Big| g\Big(\frac{z-\mu_{A'}}{\sigma_{A'}}\Big) - g(z-\mu_{A'})\Big|$$
$$\leq |\,\mu_{A'}\,|\,||g'|| + \sqrt{\frac{3}{2}}\,\Big|\,\sigma_{A'}^2 - 1\Big|\,||id_{\mathbb{R}}\,g'|| \qquad \text{for all } z \in \mathbb{R}.$$

To prove the last inequality, we consider the two cases $\sigma_{A'} \leq 1$ and $\sigma_{A'} \geq 1$.

In the case $\sigma_{A'} \leq 1$, Lemma 3.4.9, b) gives the following factor of $||id_{\mathbb{R}}\,g'||$

$$\Big(\frac{1}{\sigma_{A'}} - 1\Big) = \frac{1}{\sigma_{A'}}\Big(1-\sigma_{A'}\Big) \leq \sqrt{\frac{3}{2}}\Big(1-\sigma_{A'}\Big) \leq \sqrt{\frac{3}{2}}\Big(1-\sigma_{A'}^2\Big).$$

In the case $\sigma_{A'} \geq 1$, on the other hand, Lemma 3.4.9, c) gives

$$\Big(\sigma_{A'} - 1\Big) \leq \Big(\sigma_{A'}^2 - 1\Big) \leq \sqrt{\frac{3}{2}}\Big(\sigma_{A'}^2 - 1\Big). \qquad \square$$

By using the Conclusion 3.4.12 we obtain the following important result.

3.4.13 Proposition

There exist constants $C_1 > 0$ and $C_2 > 0$ such that for all matrices A satisfying $\sigma_{A'}^2 \geq \dfrac{2}{3}$ and for all functions $h \in \mathcal{H}$ (cf. (0.1.1) for the definition of $\mathcal{H}$) holds:

$$\Big| E\big(h(\mathscr{T}_A)\big) - \int_{\mathbb{R}} h(x)\,e'_{1,A}(x)\,dx \Big| \tag{3.4.14}$$
$$\leq \Big| E\Big(h\big(\sigma_{A'}\,T_{\bar{A}} + \mu_{A'}\big)\Big) - \int_{\mathbb{R}} h\big(\sigma_{A'}\,x + \mu_{A'}\big)\,e'_{1,\bar{A}}(x)\,dx \Big| + C_1\,D_A^2,$$

$$\Big| E\big(h(\mathscr{T}_A)\big) - \int_{\mathbb{R}} h(x)\,e'_{2,A}(x)\,dx \Big| \tag{3.4.15}$$
$$\leq \Big| E\Big(h\big(\sigma_{A'}\,T_{\bar{A}} + \mu_{A'}\big)\Big) - \int_{\mathbb{R}} h\big(\sigma_{A'}\,x + \mu_{A'}\big)\,e'_{2,\bar{A}}(x)\,dx \Big| + C_2\,E_A^3.$$

Proof:

To prove (3.4.14) it is sufficient to show

$$\Big| E\big(h(\mathscr{T}_A)\big) - E\Big(h\big(\sigma_{A'}\,T_{\bar{A}} + \mu_{A'}\big)\Big) - \int_{\mathbb{R}} h(x)\,e'_{1,A}(x)\,dx + \int_{\mathbb{R}} h\big(\sigma_{A'}\,x + \mu_{A'}\big)\,e'_{1,\bar{A}}(x)\,dx \Big|$$
$$\leq C_1\,D_A^2$$

for the functions $h = 1_{(-\infty,\,z\,]}$, $z \in \mathbb{R}$. Once this has been shown, we can move on to the convex combinations of these functions and then to their limits via majorized convergence.

Using the notation $H_2(x) = x^2 - 1$ (see section 2.2), we obtain

$$\begin{aligned}
& \left| P\big(\,\mathscr{T}_A \le z\,\big) - P\Big(T_{\bar{A}} \le \frac{z-\mu_{A'}}{\sigma_{A'}}\Big) - e_{1,A}(z) + e_{1,\bar{A}}\Big(\,\frac{z-\mu_{A'}}{\sigma_{A'}}\,\Big)\right| \\
= & \left| P\big(\,\mathscr{T}_A \le z\,\big) - P\big(T_{A'} \le z\,\big) - e_{1,A}(z) + e_{1,\bar{A}}\Big(\,\frac{z-\mu_{A'}}{\sigma_{A'}}\,\Big)\right| \qquad \Big(\text{since } T_{\bar{A}} = \frac{T_{A'}-\mu_{A'}}{\sigma_{A'}}\Big) \\
\le & \; P(\mathscr{T}_A \ne T_{A'}) + \left|\Phi(z) - \Phi\Big(\,\frac{z-\mu_{A'}}{\sigma_{A'}}\,\Big)\right| \\
& + \left| H_2(z)\,\psi(z) - H_2\Big(\,\frac{z-\mu_{A'}}{\sigma_{A'}}\,\Big)\,\psi\Big(\,\frac{z-\mu_{A'}}{\sigma_{A'}}\,\Big)\right| \frac{\beta_A}{6\,n} \\
& + \frac{1}{6}\,||\,H_2\,\psi\,||\,\left|\frac{1}{n}\sum_{i,j=1}^{n}\hat{a}_{ij}^3 - \frac{1}{n}\sum_{i,j=1}^{n}\bar{a}_{ij}^3\right| \\
\le & \; c_1\,\frac{\delta_A}{n} + c_2\,\Big(\,\frac{\beta_A}{n}\,\Big)^2 \le C_1\,D_A^2.
\end{aligned}$$

The penultimate inequality is obtained by applying of

- Lemma 3.4.1, b) with $k = 4$, and Lemma 3.4.6, a) with $k = 4$,
- Conclusion 3.4.12 with $g(z) = \Phi(z)$ and $g(z) = H_2(z)\,\psi(z)$,
- Lemma 3.4.1, g) and h) with $k = 3$ and $k = 4$, and Lemma 2.2.8, a), b), e), f) and g).

The last inequality, on the other hand, follows from Lemma 3.1.18, b).

The proof of (3.4.15) is completely analogous and finally yields the inequalities

$$\ldots \le c_3\,\frac{\eta_A}{n} + c_4\,\frac{\delta_A}{n}\,\frac{\beta_A}{n} + c_5\,\Big(\,\frac{\beta_A}{n}\,\Big)^3 \le C_2\,E_A^3.$$

The penultimate inequality in the estimate for (3.4.15) is obtained by applying of

- Lemma 3.4.1, b) with $k = 5$, and Lemma 3.4.6, a), b), c) and d) with $k = 5$,
- Conclusion 3.4.12 with $g(z) = \Phi(z)$, $g(z) = H_2(z)\,\psi(z)$, $g(z) = H_3(z)\,\psi(z)$ and $g(z) = H_5(z)\,\psi(z)$ (cf. section 2.2 for the definition of the Hermite polynomials H_n),
- Lemma 3.4.1, g) and h) with $k = 3$, $k = 4$ and $k = 5$,
- Lemma 3.1.18, d), and Lemma 2.2.8, a), b), e), f), g), h), i), j), l) and m).

The last inequality, on the other hand, follows again from Lemma 3.1.18, b). □

The last part of this section deals with the conditions of the Theorems 3.1.10 and 3.1.13. The aim of the following two propositions is to transfer the inequalities of (3.1.11) and (3.1.15) from the matrices $B \in N(l, \hat{A})$ to the corresponding "truncated" matrices $\widetilde{B} \in N(l, \bar{A})$.

For the remainder of this section, let $B \in N(l, \hat{A})$ be fixed and $Z_l = \{ z_1, \dots, z_l \}$ ($S_l = \{ s_1, \dots, s_l \}$) the set of indices of the cancelled rows (columns). Furthermore,

$\widetilde{B}$ is the $(n-l)\times(n-l)-$matrix, which is obtained from $\bar{A}$ by cancelling the the same rows and columns as for B,

B' is the $(n-l)\times(n-l)-$matrix, which is obtained from A' by cancelling the the same rows and columns as for B (also B' = truncated matrix B), and

$$\theta_{B'} = \sum_{i \notin Z_l} a'_{i\cdot} + \sum_{j \notin S_l} a'_{\cdot j} - (n-l)\, a'_{\cdot\cdot} \; .$$

It follows that

$$T_{\widetilde{B}} = \frac{T_{B'} - \theta_{B'}}{\sigma_{A'}}. \tag{3.4.16}$$

The next proposition provides a relation between the rth differences of F_B and $F_{\widetilde{B}}$.

3.4.17 Proposition

Let $\dfrac{n}{n-l} \le 10$, $\sigma_{A'} > 0$ and $k, r \in \mathbb{N}$.

Then there exists a constant $C > 0$, which depends only on k and r, such that

$$\left|\left| \, \Delta^r_y F_{\widetilde{B}} \, \right|\right| \le \left|\left| \, \Delta^r_{\sigma_{A'} y} F_B \, \right|\right| + \frac{C}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k \qquad \text{for all } y \in \mathbb{R}.$$

Proof:

Due to Lemma 3.4.1, b) (B' = truncated matrix B) we have

$$P(\, T_B \neq T_{B'} \,) \le \frac{|\Gamma|}{n-l} \le \frac{10}{n}\, 2^k \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k. \tag{3.4.18}$$

Thus, for all $z \in \mathbb{R}$,

$$\left|\, \Delta^r_y F_{\widetilde{B}}(z) \,\right| = \left|\, \Delta^r_{\sigma_{A'} y} F_{B'}(\sigma_{A'}\, z + \theta_{B'}) \,\right| \qquad \text{(due to (3.4.16))}$$

$$\leq \left|\, \Delta^r_{\sigma_{A'} y} F_B(\sigma_{A'}\, z + \theta_{B'}) \,\right| + 2^r\, P(\, T_B \neq T_{B'}\,)$$

$$\leq \left|\left|\, \Delta^r_{\sigma_{A'} y} F_B \,\right|\right| + \frac{C}{n} \sum_{i,j=1}^{n} |\hat{a}_{ij}|^k, \qquad \text{where } C = 10 \cdot 2^{k+r}. \qquad \square$$

Finally, with more effort, we prove a similar relation between $z\, \Delta^2_y F_{\widetilde{B}}(z)$ and $z\, \Delta^2_{\sigma_{A'} y} F_B(z)$.

3.4.19 Proposition

Let $\dfrac{n}{n-l} \leq 10$, $|\mu_B| \leq 1$, $\left|\, \sigma_B^2 - 1 \,\right| \leq \dfrac{1}{3}$ and $\left|\, \sigma_{A'}^2 - 1 \,\right| \leq \dfrac{1}{3}$.

Then there exists a constant $C > 0$ (which does not depend on anything!) such that

$$\left|\left|\, z\, \Delta^2_y F_{\widetilde{B}}(z) \,\right|\right| \leq 2 \left|\left|\, z\, \Delta^2_{\sigma_{A'} y} F_B(z) \,\right|\right| + C\left(\, E_A^2 + y^2 \,\right) \qquad \text{for all } y \geq 0.$$

Proof:

First, we show the existence of a constant C_1 such that

$$\text{(3.4.20)} \qquad \left|\, E\big(T_{B'}\, 1_{\{\, T_{B'} \leq z \,\}} \big) - E\big(T_B\, 1_{\{\, T_B \leq z \,\}} \big) \,\right| \leq C_1\, E_A^2 \qquad \text{for all } z \in \mathbb{R}.$$

To prove this, we use the triangle inequality and obtain

$$\text{(3.4.21)} \qquad \left|\, E\big(T_{B'}\, 1_{\{\, T_{B'} \leq z \,\}} \big) - E\big(T_B\, 1_{\{\, T_B \leq z \,\}} \big) \,\right|$$

$$\leq \left|\, E\big(T_{B'}\, 1_{\{\, T_{B'} \leq z \,\}} \big) - E\big(T_B\, 1_{\{\, T_{B'} \leq z \,\}} \big) \,\right| + \left|\, E\big(T_B\, 1_{\{\, T_{B'} \leq z \,\}} \big) - E\big(T_B\, 1_{\{\, T_B \leq z \,\}} \big) \,\right|$$

$$\leq E\big(\, |\, T_{B'} - T_B \,|\, \big) + E\big(\, |\, T_B \,|\, 1_{\{\, T_{B'} \neq T_B \,\}} \big)$$

$$\leq E\big(\, |\, T_{B'} - T_B \,|\, \big) + \sigma_B\, E\big(\, |\, \mathscr{T}_B \,|\, 1_{\{\, T_{B'} \neq T_B \,\}} \big) + |\, \mu_B \,|\, P\big(T_{B'} \neq T_B \,\big).$$

The first summand of (3.4.21) can be estimated as follows:

$$E\big(\, |\, T_{B'} - T_B \,|\, \big) \leq \sum_{i \notin Z_l} E\Big(\left|\, b'_{i\pi(i)} - b_{i\pi(i)} \,\right| \Big)$$

$$= \frac{1}{n-l} \sum_{\substack{(i \notin Z_l) \,\wedge\, (j \notin S_l) \\ (i,j) \,\in\, \Gamma}} |b_{ij}| \qquad \left(\text{since } P(\pi(i) = j) = \frac{1}{n-l}\right)$$

$$\begin{aligned}
&\leq \frac{10}{n} \sum_{(i,j)\,\in\,\Gamma} |\hat{a}_{ij}| && \Big(\text{since } \frac{n}{n-l} \leq 10\Big) \\
&\leq c_1 \, \frac{\delta_A}{n} && (\text{cf. Lemma 3.4.1, c)}) \\
&\leq c_1 \, E_A^2 && (\text{cf. Lemma 3.1.18, b)}).
\end{aligned}$$

Furthermore, we estimate

$$\begin{aligned}
E\big(\,|\mathscr{T}_B|^3\,\big)^{1/3} &\leq E\big(\,|\mathscr{T}_B|^4\,\big)^{1/4} && (\text{Hölder's inequality}) \\
&\leq \Big(3 + 97\,\frac{\delta_B}{n-l}\Big)^{1/4} && (\text{cf. Proposition 3.2.2, d)}) \\
&\leq \Big(3 + 97\,\frac{\delta_B}{n-l}\Big)^{1/3} && \Big(\text{since } \Big(3 + 97\,\frac{\delta_B}{n-l}\Big) \geq 1\Big) \\
&\leq c_2 + c_3 \,\Big(\frac{\delta_A}{n}\Big)^{1/3} && \Big(\text{cf. Lemma 3.3.10 and } \sigma_B^2 \geq \frac{2}{3}\Big).
\end{aligned}$$

We note that for the last inequality we used the formula

$$(a + b\,x)^{1/3} \leq a^{1/3} + b^{1/3}\,x^{1/3} \qquad \text{for all } a, b, x \geq 0.$$

For the second summand of (3.4.21) we thus obtain:

$$\begin{aligned}
\sigma_B \, E\big(\,|\,\mathscr{T}_B\,|\,1_{\{\,T_{B'} \neq T_B\,\}}\,\big) &\leq \sigma_B \, E\big(\,|\mathscr{T}_B|^3\,\big)^{1/3} \, P(\,T_B \neq T_{B'}\,)^{2/3} && \Big(\text{Hölder's ineq., } p = 3,\ q = \frac{3}{2}\Big) \\
&\leq \sqrt{\frac{4}{3}} \left(c_2 + c_3 \,\Big(\frac{\delta_A}{n}\Big)^{1/3} \right) P(\,T_B \neq T_{B'}\,)^{2/3} && \Big(\text{since } \sigma_B^2 \leq \frac{4}{3}\Big) \\
&= \sqrt{\frac{4}{3}}\, c_2 \, P(\,T_B \neq T_{B'}\,)^{2/3} + \sqrt{\frac{4}{3}}\, c_3 \,\Big(\frac{\delta_A}{n}\Big)^{1/3} P(\,T_B \neq T_{B'}\,)^{2/3} \\
&\leq c_4 \,\Big(\frac{\eta_A}{n}\Big)^{2/3} + c_5 \, \frac{\delta_A}{n} && (\text{cf. (3.4.18) for } k = 5 \text{ and } k = 4) \\
&\leq (c_4 + c_5)\, E_A^2 && (\text{cf. Lemma 3.1.18, b)}).
\end{aligned}$$

Finally, the estimate of the third summand of (3.4.21) is

$$\begin{aligned}
|\,\mu_B\,|\, P\big(\,T_{B'} \neq T_B\,\big) &\leq P\big(\,T_{B'} \neq T_B\,\big) && (\text{since } |\mu_B| \leq 1) \\
&\leq c_6 \, \frac{\delta_A}{n} \leq c_6 \, E_A^2 && (\text{cf. (3.4.18) for } k = 4).
\end{aligned}$$

As a consequence of (3.4.20), we now get a constant C_2 such that

(3.4.22) $$\Big|\, z\,P\big(\,T_{B'} \in I\,\big) - z\,P\big(\,T_B \in I\,\big)\,\Big| \le C_2\,\big(\,E_A^2 + a^2 + b^2\,\big)$$

for all $I = (\,z + a,\ z + a + b\,]$, where $z \in \mathbb{R}$ and $a, b \ge 0$.

We proof this as follows:

$$\Big|\, z\,P\big(\,T_{B'} \in I\,\big) - z\,P\big(\,T_B \in I\,\big)\,\Big|$$

$$\le \Big|\, E\big(\,T_{B'}\,1_{\{\,T_{B'} \in I\,\}}\,\big) - E\big(\,T_B\,1_{\{\,T_B \in I\,\}}\,\big)\,\Big| + \big(\,a+b\,\big)\,\Big(\,P\big(\,T_{B'} \in I\,\big) + P\big(\,T_B \in I\,\big)\,\Big)$$

$$\le 2\,C_1\,E_A^2 + \big(\,a+b\,\big)\,\Big(\,2\,P\big(\,T_B \in I\,\big) + P\big(\,T_B \ne T_{B'}\,\big)\,\Big) \qquad \text{(due to (3.4.20))}$$

$$\le 2\,C_1\,E_A^2 + \big(\,a+b\,\big)\,\bigg(\,4\,\mathcal{K}_1\,\frac{\beta_B}{n-l} + \frac{2}{\sqrt{2\,\pi}}\,\frac{b}{\sigma_B} + c_7\,\frac{\beta_A}{n}\,\bigg)$$

(cf. Theorem 3.1.3 for $\mathscr{T}_B$ and (3.4.18))

$$\le C_2\,\big(\,E_A^2 + a^2 + b^2\,\big) \qquad \text{(cf. Lemma 3.3.10, } \sigma_B^2 \ge \frac{2}{3} \text{ and } \frac{\beta_A}{n} \le E_A\text{).}$$

Since we additionally have

(3.4.23) $$|\theta_{B'}| \le c_8\,\frac{\delta_A}{n} \le c_8\,E_A^2, \qquad \text{(cf. Lemma 3.4.1, d)),}$$

we obtain for all $z \in \mathbb{R}$ and $y \in \mathbb{R}$ (with the abbreviation $z' = \sigma_{A'}\,z + \theta_{B'}$):

$$\big|\,z\,\Delta_y^2 F_{\widetilde{B}}(z)\,\big| \le \frac{1}{\sigma_{A'}}\,\Big(\,\big|\,z'\,\Delta_{\sigma_{A'}y}^2 F_{B'}(z')\,\big| + |\theta_{B'}|\,\Big) \qquad \text{(due to (3.4.16) and } \big|\,\Delta_y^2 F_{\widetilde{B}}(z)\,\big| \le 1\text{)}$$

$$\le \frac{1}{\sigma_{A'}}\,\Big(\,\big|\,z'\,\Delta_{\sigma_{A'}y}^2 F_B(z')\,\big| + C_2\,\big(\,2\,E_A^2 + 3\,\sigma_{A'}^2\,y^2\,\big) + c_8\,E_A^2\,\Big)$$

$$\le 2\,\big|\big|\,z\,\Delta_{\sigma_{A'}y}^2 F_B(z)\,\big|\big| + C\,\big(\,E_A^2 + y^2\,\big).$$

For the penultimate inequality, (3.4.22) was applied twice, once with $a = 0$, $b = \sigma_{A'}y$ (≥ 0) and once with $a = b = \sigma_{A'}y$ (≥ 0). Furthermore, (3.4.23) was used.

For the last inequality, we used the condition $\frac{2}{3} \le \sigma_{A'}^2 \le \frac{4}{3}$. $\square$

3.4.24 Remark

Note, that the condition $\dfrac{n}{n-l} \le 10$ is fulfilled in our relevant cases $l \le 8 < 10 \le n$ (cf. section 3.7) and $l \le 16 < 18 \le n$ (cf. section 3.8).

3.5 Construction of several random permutations

In order to apply Stein's method, as we used it in chapter 1, to linear rank statistics, we must, among other things, find an analogue to the independence of $S^n_{n-1} = (1/\sqrt{n})(X_1 + \ldots + X_{n-1})$ and X_n (see section 1.3).

This is done in the following by generating five specially constructed random permutations π_1, π_2, π_3, π_4, π_5 (see (3.5.12) and (3.5.13) together with Table 3.5.1) and subsequent by defining the differences $\Delta T_i = T_{i+1} - T_i$, where the T_i are the rank statistics belonging to these random permutations π_i (see the definitions in (3.5.21)). The role of S^n_{n-1} is then played by the T_i and that of X_n by the ΔT_j, $j \geq i+1$ (cf. Lemma 3.5.23).

The existing independencies between the T_i and ΔT_j can be explained and motivated to some extent using the important Table 3.5.2. This table shows the random indices that are obtained when the random permutations $\pi_1, \ldots, \pi_5$ are applied to the random indices $I_1, \ldots, I_{16}$. It is essential to note that the $I_1, \ldots, I_{16}$ are **independent** of the $\pi_1, \ldots, \pi_5$ (cf. (3.5.12) and Lemma 3.5.15, a)). Under π_1 the obtained random indices are completely given by the $J_1, \ldots, J_{16}$ (cf. (3.5.14)). Under $\pi_2, \ldots, \pi_5$ some of these obtained random indices are no longer specified explicitly, as they are not used explicitly in these cases and would also be very complex.

In the following, we look at the Table 3.5.2 from **right to left** and consider the measurability statements of Lemma 3.5.22, i.e. $\Delta T_j \in \sigma(I_1, \ldots, J_1, \ldots)$ (briefly, $\Delta T_j \hookrightarrow (I_1, \ldots, J_1, \ldots)$). First of all, according to the above remarks,

- π_5 and I_1 are independent, but **not** π_5 and (I_1, J_1).

Therefore, we swap J_1 and J_2 in the first two lines of the Table 3.5.2 when we move from π_5 to π_4. We get:

- $T_4 \hookrightarrow \pi_4$ and $(I_1, J_1) = (I_1, \pi_4(I_2)) \hookleftarrow a_{I_1 J_1}$ are independent.

For π_3 the blocks (J_1, J_2) and (J_3, J_4) are swapped in Table 3.5.2, so that

- $T_3 \hookrightarrow \pi_3$ and $(I_1, I_2, J_1, J_2) = (I_1, I_2, \pi_3(I_3), \pi_3(I_4)) \hookleftarrow \Delta T_4$ are independent.

The blocks that are swapped now become larger and larger. To be more precisely, the block size doubles with each additional step. For π_2 the blocks $(J_1, \ldots, J_4)$ and $(J_5, \ldots, J_8)$ are swapped in Table 3.5.2, so that

- $T_2 \hookrightarrow \pi_2$ and $(\, I_1, .\,.\,, I_4, J_1, .\,.\,, J_4\,) = (\, I_1, .\,.\,, I_4,\, \pi_2(\, I_5\,), .\,.\,, \pi_2(\, I_8\,)\,) \hookleftarrow \Delta T_3$ are independent.

Finally, for π_1 the blocks $(\, J_1, \ldots, J_8\,)$ and $(\, J_9, \ldots, J_{16}\,)$ are swapped in Table 3.5.2, so that

- $T_1 \hookrightarrow \pi_1$ and $(\, I_1, .\,.\,, I_8, J_1, .\,.\,, J_8\,) = (\, I_1, .\,.\,, I_8,\, \pi_1(\, I_9\,), .\,.\,, \pi_1(\, I_{16}\,)\,) \hookleftarrow \Delta T_2$ are independent.

After these considerations, we start with the formal, explicit construction (in reverse order, i.e. π_1 is defined first, then π_2 …and finally π_5).

Let

$$
\begin{aligned}
N &= \{\, 1, \ldots, n \,\} \quad \text{and} \\
M_{16} &= \Big\{\, \underline{i} = (\, i_1, \ldots, i_{16}\,) \in N^{16} \;:\; \underline{i} \text{ satisfies the equivalences:} \\
&\qquad (\text{t1}) \quad i_1 = i_2 \;\Leftrightarrow\; i_3 = i_4, \\
&\qquad (\text{u1}) \quad i_1 = i_2 \;\Leftrightarrow\; i_7 = i_8, \\
&\qquad (\text{u2}) \quad i_3 = i_4 \;\Leftrightarrow\; i_5 = i_6, \\
&\qquad (\text{u3}) \quad i_l = i_k \;\Leftrightarrow\; i_{l+6} = i_{k+2} \qquad \text{for } 1 \le l \le 2,\;\; 3 \le k \le 4, \\
&\qquad (\text{v1}) \quad i_l = i_k \;\Leftrightarrow\; i_{l+12} = i_{k+12} \qquad \text{for } 1 \le l \le 4,\;\; 1 \le k \le 4, \\
&\qquad (\text{v2}) \quad i_l = i_k \;\Leftrightarrow\; i_{l+4} = i_{k+4} \qquad \text{for } 5 \le l \le 8,\;\; 5 \le k \le 8, \\
&\qquad (\text{v3}) \quad i_l = i_k \;\Leftrightarrow\; i_{l+12} = i_{k+4} \qquad \text{for } 1 \le l \le 4,\;\; 5 \le k \le 8 \;\Big\}.
\end{aligned}
\tag{3.5.1}
$$

For each $\underline{i} = (\, i_1, \ldots, i_{16}\,) \in M_{16}$ we **fix** once and for all permutations $v(\underline{i})$, $u(\underline{i})$, $t(\underline{i})$ and $s(\underline{i})$ of N with properties as described in Table 3.5.1 on the following page.

3.5.2 Remarks on the Table 3.5.1

a) The mapping

$$
\begin{aligned}
&M_{16} \ni \underline{i} \longrightarrow v(\underline{i}) \quad \text{is well defined beause of (v1), (v2) and (v3); and} \\
&M_{16} \ni \underline{i} \longrightarrow u(\underline{i}) \quad \text{is well defined beause of (u1), (u2) and (u3); and} \\
&M_{16} \ni \underline{i} \longrightarrow t(\underline{i}) \quad \text{is well defined beause of (t1).}
\end{aligned}
$$

b) As can be seen from the Table 3.5.1, we have not defined all values of $v(\underline{i})$, $u(\underline{i})$ and $t(\underline{i})$ **explicitly**. The reason for this is that, on the one hand, we do not need these explicit definitions and, on the other hand, we would have to distinguish many cases for a complete explicit definition. The following two cases are mentioned as an example:

$$i_8 = i_9 \quad \Rightarrow \quad [v(\underline{i})](i_9) = [v(\underline{i})](i_8) = i_{12}, \quad \text{but}$$
$$i_8 \neq i_9 \quad \Rightarrow \quad [v(\underline{i})](i_9) \neq [v(\underline{i})](i_8) = i_{12}.$$

Table 3.5.1: Definition of the permutations $v(\underline{i})$, $u(\underline{i})$, $t(\underline{i})$ and $s(\underline{i})$

<table>
<tr><th></th><th>$v(\underline{i})$</th><th>$u(\underline{i})$</th><th>$t(\underline{i})$</th><th>$s(\underline{i})$</th></tr>
<tr><td>i_1
i_2</td><td>i_{13}
i_{14}</td><td>i_7
i_8</td><td>i_4
i_3</td><td>i_2
i_1</td></tr>
<tr><td>i_3
i_4</td><td>i_{15}
i_{16}</td><td>i_5
i_6</td><td>Values
$\in \{ i_1, \ldots, i_4 \}$</td><td rowspan="2"></td></tr>
<tr><td>i_5
i_6
i_7
i_8</td><td>i_9
i_{10}
i_{11}
i_{12}</td><td>Values
$\in \{ i_1, \ldots, i_8 \}$</td><td></td></tr>
<tr><td>i_9
i_{10}
i_{11}
i_{12}
i_{13}
i_{14}
i_{15}
i_{16}</td><td>Values
$\in \{ i_1, \ldots, i_{16} \}$</td><td colspan="3" rowspan="2">Remain
fixed</td></tr>
<tr><td>$N \setminus \{ i_1, \ldots, i_{16} \}$</td><td></td></tr>
</table>

For our considerations we need further useful notations:

$$M_4 = \left\{ \check{i} = (i_1, \ldots, i_4) \in N^4 : \check{i} \text{ satisfies the equivalence (t1) from (3.5.1)} \right\},$$

(3.5.3) $$M_8 = \left\{ \bar{i} = (i_1, \ldots, i_8) \in N^8 : \bar{i} \text{ satisfies the equivalences (t1), (u1), (u2) and (u3) from (3.5.1)} \right\}.$$

Moreover, for each $\underline{i} \in M_{16}$ let

(3.5.4) $\gamma(\underline{i}) = \big|\,\{i_1, i_2\}\,\big| = \big|\,\{i_3, i_4\}\,\big|,$

(3.5.5) $\theta(\underline{i}) = \big|\,\{i_1, i_2, i_3, i_4\}\,\big| = \big|\,\{i_5, i_6, i_7, i_8\}\,\big|,$

(3.5.6) $\mu(\underline{i}) = \big|\,\{\, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8 \,\}\,\big| = \big|\,\{\, i_9, i_{10}, i_{11}, i_{12}, i_{13}, i_{14}, i_{15}, i_{16} \,\}\,\big|.$

The second "=" in (3.5.4) is valid due to (t1). Similarly, the second "=" in (3.5.5) is from (u1), (u2), (u3) and the second "=" in (3.5.6) is from (v1), (v2), (v3).

Furthermore, let $\underline{I} = (\, I_1, \ldots, I_{16} \,)$ be a random element on M_{16} whose distribution can be described as follows:

(3.5.7) $(\, I_1, I_2 \,)$ is **uniformly distributed** on N^2.

(3.5.8) For each $(\, i_1, i_2 \,) \in N^2$,

$$\big(\, I_3,\, I_4 \,\big)\,\Big|\,\big(\, I_1,\, I_2 \,\big) = \big(\, i_1,\, i_2 \,\big)$$

is **uniformly distributed** on $\big\{\, (\, \eta_3, \eta_4 \,) \,:\, (\, i_1, i_2, \eta_3, \eta_4 \,) \in M_4 \,\big\}$.

(3.5.9) For each $(\, i_1, i_2, i_3, i_4 \,) \in M_4$,

$$\big(\, I_5,\, I_6,\, I_7,\, I_8 \,\big)\,\Big|\,\big(\, I_1,\, I_2,\, I_3,\, I_4 \,\big) = \big(\, i_1,\, i_2,\, i_3,\, i_4 \,\big)$$

is **uniformly distributed** on

$$\big\{\, (\, \eta_5, \eta_6,\, \eta_7, \eta_8 \,) \,:\, (\, i_1, i_2, , i_3, i_4, \eta_5, \eta_6, \eta_7, \eta_8 \,) \in M_8 \,\big\}.$$

(3.5.10) For each $(\, i_1, i_2, \ldots, i_8 \,) \in M_8$,

$$\big(\, I_9,\, I_{10},\, \ldots,\, I_{16} \,\big)\,\Big|\,\big(\, I_1,\, I_2,\, \ldots,\, I_8 \,\big) = \big(\, i_1,\, i_2,\, \ldots,\, i_8 \,\big)$$

is **uniformly distributed** on

$$\big\{\, (\, \eta_9, \eta_{10},\, \ldots,\, \eta_{16} \,) \,:\, (\, i_1, i_2,\, \ldots, i_8, \eta_9, \eta_{10},\, \ldots, \eta_{16} \,) \in M_{16} \,\big\}.$$

To summarize, the multiplication rule together with (3.5.4), (3.5.5) and (3.5.6) yields

(3.5.11) $$P\Big(\, \underline{I} = \underline{i} \,\Big) = \frac{\big(n - \mu(\underline{i})\big)!}{n!} \cdot \frac{\big(n - \theta(\underline{i})\big)!}{n!} \cdot \frac{\big(n - \gamma(\underline{i})\big)!}{n!} \cdot \frac{1}{n^2} \qquad \text{for each } \underline{i} \in M_{16}.$$

Additionally, let

(3.5.12) π_1 be a random permutation that is **uniformly distributed** on the set $\mathscr{P}_n$ of permutations of N and **independent** of $\underline{I}$.

Next, we compose π_1 and the random permutations $v(\underline{I})$, $u(\underline{I})$, $t(\underline{I})$, $s(\underline{I})$. In detail:

$$\pi_2 = \pi_1 \circ v(\underline{I}), \quad \pi_3 = \pi_2 \circ u(\underline{I}), \quad \pi_4 = \pi_3 \circ t(\underline{I}), \quad \pi_5 = \pi_4 \circ s(\underline{I}). \tag{3.5.13}$$

Finally, we define:

$$\begin{aligned} &J_1 = \pi_1(I_9), \quad J_2 = \pi_1(I_{10}), \ldots, \quad J_8 = \pi_1(I_{16}), \\ &J_9 = \pi_1(I_1), \quad J_{10} = \pi_1(I_2), \ldots, \quad J_{16} = \pi_1(I_8). \end{aligned} \tag{3.5.14}$$

Using the definition of v, u, t and s, we see in the following Table 3.5.2 how $\pi_1, \ldots, \pi_5$ map $I_1, \ldots, I_{16}$.

Tabelle 3.5.2: Values of $I_1, \ldots, I_{16}$ under $\pi_1, \ldots, \pi_5$

<table>
<tr><th></th><th>π_1</th><th>π_2</th><th>π_3</th><th>π_4</th><th>π_5</th></tr>
<tr><td>I_1</td><td>J_9</td><td>J_5</td><td>J_3</td><td>J_2</td><td>J_1</td></tr>
<tr><td>I_2</td><td>J_{10}</td><td>J_6</td><td>J_4</td><td>J_1</td><td>J_2</td></tr>
<tr><td>I_3</td><td>J_{11}</td><td>J_7</td><td>J_1</td><td rowspan="2">Random variables $\in \sigma(I_1, \ldots, I_4, J_1, \ldots, J_4)$</td><td rowspan="2">Same as π_4</td></tr>
<tr><td>I_4</td><td>J_{12}</td><td>J_8</td><td>J_2</td></tr>
<tr><td>I_5</td><td>J_{13}</td><td>J_1</td><td rowspan="4">Random variables $\in \sigma(I_1, \ldots, I_8, J_1, \ldots, J_8)$</td><td rowspan="4">Same as π_3</td><td rowspan="4">Same as π_3</td></tr>
<tr><td>I_6</td><td>J_{14}</td><td>J_2</td></tr>
<tr><td>I_7</td><td>J_{15}</td><td>J_3</td></tr>
<tr><td>I_8</td><td>J_{16}</td><td>J_4</td></tr>
<tr><td>I_9</td><td>J_1</td><td rowspan="8">Random variables $\in \sigma(I_1, \ldots, I_{16}, J_1, \ldots, J_{16})$</td><td rowspan="8">Same as π_2</td><td rowspan="8">Same as π_2</td><td rowspan="8">Same as π_2</td></tr>
<tr><td>I_{10}</td><td>J_2</td></tr>
<tr><td>I_{11}</td><td>J_3</td></tr>
<tr><td>I_{12}</td><td>J_4</td></tr>
<tr><td>I_{13}</td><td>J_5</td></tr>
<tr><td>I_{14}</td><td>J_6</td></tr>
<tr><td>I_{15}</td><td>J_7</td></tr>
<tr><td>I_{16}</td><td>J_8</td></tr>
</table>

As usual we define that $\sigma(X)$ is the σ-field generated by the random vector X and $f \in \sigma(X)$ means that f is measurable relative to $\sigma(X)$.

Further important results about the above random permutations and random indices are:

3.5.15 Lemma

a) π_1, π_2, π_3, π_4, π_5 are uniformly distributed on $\mathscr{P}_n$ and independent of $\underline{I} = (\,I_1, \ldots, I_{16}\,)$.

b) π_1 and $(\,I_1, \ldots, I_8, J_1, \ldots, J_8\,)$ are independent.

c) π_2 and $(\,I_1, \ldots, I_4, J_1, \ldots, J_4\,)$ are independent.

d) π_3 and $(\,I_1, I_2, J_1, J_2\,)$ are independent.

e) π_4 and $(\,I_1, J_1\,)$ are independent.

f) π_5 and I_1 are independent.

g) $I_1, \ldots, I_{16}$ are uniformly distributed on N.

h) $\big(I_l, \pi_k(I_l)\big)$ are uniformly distributed on N^2 for all $1 \le l \le 16$, $1 \le k \le 5$.

Proof:

a) Since π_1 is uniformly distributed on the set $\mathscr{P}_n$ and since π_1 is also independent of $\underline{I}$, (cf. (3.5.12)), it follows for all $\pi \in \mathscr{P}_n$ and $\underline{i} \in M_{16}$ that

$$\begin{aligned} P\big(\,\pi_2 = \pi,\, \underline{I} = \underline{i}\,\big) &= P\big(\,\pi_1 \circ v(\underline{i}) = \pi,\, \underline{I} = \underline{i}\,\big) \qquad\qquad \text{(cf. (3.5.13): } \pi_2 = \pi_1 \circ v(\underline{I})) \\ &= P\big(\,\pi_1 = \pi \circ v(\underline{i})^{-1},\, \underline{I} = \underline{i}\,\big) \\ &= P\big(\,\pi_1 = \pi \circ v(\underline{i})^{-1}\,\big)\, P\big(\,\underline{I} = \underline{i}\,\big) \\ &= \frac{1}{n!}\, P\big(\,\underline{I} = \underline{i}\,\big). \end{aligned}$$

Thus summation over all $\underline{i} \in M_{16}$ gives first

$$P\big(\,\pi_2 = \pi\,\big) = \frac{1}{n!}$$

and then the independence of π_2 and $\underline{I}$.

From this we obtain the assertion for π_3, then for π_4 and finally for π_5 in a completely analogous way.

b) Since $\underline{I}$ and π_1 are independent (cf. (3.5.12)), we get for all $\underline{i} = (\,i_1, \ldots, i_{16}\,) \in M_{16}$ and $\pi \in \mathscr{P}_n$:

$$\begin{aligned}
&P\Big(\,(\,I_1,\ldots,I_8,J_1,\ldots,J_8\,)=\underline{i},\ \pi_1=\pi\,\Big)\\
&=P\Big(\,(\,I_1,\ldots,I_8,\pi(I_9),\ldots,\pi(I_{16})\,)=\underline{i},\ \pi_1=\pi\,\Big) \qquad \text{(cf. (3.5.14))}\\
&=P\Big(\,\underline{I}=\big(\,i_1,\ldots,i_8,\pi^{-1}(i_9),\ldots,\pi^{-1}(i_{16})\,\big),\ \pi_1=\pi\,\Big)\\
&=P\Big(\,\underline{I}=\big(\,i_1,\ldots,i_8,\pi^{-1}(i_9),\ldots,\pi^{-1}(i_{16})\,\big)\,\Big)\,P\Big(\,\pi_1=\pi\,\Big)\\
&=P\Big(\,\underline{I}=\underline{i}\,\Big)\,P\Big(\,\pi_1=\pi\,\Big).
\end{aligned}$$

The last equation is valid, since we can use (3.5.10) for

$$\underline{i}=\big(\,i_1,\ldots,i_8,i_9,\ldots,i_{16}\,\big)\in M_{16} \text{ and } \underline{\xi}=\big(\,i_1,\ldots,i_8,\pi^{-1}(i_9),\ldots,\pi^{-1}(i_{16})\,\big)\in M_{16}.$$

Remember that $\pi^{-1}\in\mathscr{P}_n$ is bijective and thus $\underline{\xi}\in M_{16}$ due to (3.5.1), (v1), (v2) and (v3). Now summation over all $\pi\in\mathscr{P}_n$ gives first

$$P\Big(\,(\,I_1,\ldots,I_8,J_1,\ldots,J_8\,)=\underline{i}\,\Big)=P\Big(\,\underline{I}=\underline{i}\,\Big) \tag{3.5.16}$$

and then the assertion.

c) The proof of part c) is completely analogous to that of part b), if we use

- π_2 instead of π_1,
- $J_1=\pi_2(I_5)$, $J_2=\pi_2(I_6)$, $J_3=\pi_2(I_7)$, $J_4=\pi_2(I_8)$ from Table 3.5.2,
- $(\,I_1,\ldots,I_8\,)$ and π_2 are independent due to part a),
- (3.5.9) instead of (3.5.10) for

$$\big(\,i_1,\ldots,i_4,i_5,\ldots,i_8\,\big)\in M_8 \text{ and } \big(\,i_1,\ldots,i_4,\pi^{-1}(i_5),\ldots,\pi^{-1}(i_8)\,\big)\in M_8,$$

- $\big(\,i_1,\ldots,i_4,\pi^{-1}(i_5),\ldots,\pi^{-1}(i_8)\,\big)\in M_8$ due to (3.5.1), (u1), (u2) and (u3).

d) The proof of part d) is completely analogous to that of part b), if we use

- π_3 instead of π_1,
- $J_1=\pi_3(I_3)$, $J_2=\pi_3(I_4)$ from Table 3.5.2,
- $(\,I_1,\ldots,I_4\,)$ and π_3 are independent due to part a),
- (3.5.8) instead of (3.5.10) for

$$\big(\,i_1,i_2,i_3,i_4\,\big)\in M_4 \text{ and } \big(\,i_1,i_2,\pi^{-1}(i_3),\pi^{-1}(i_4)\,\big)\in M_4,$$

- $\big(\,i_1,i_2,\pi^{-1}(i_3),\pi^{-1}(i_4)\,\big)\in M_4$ due to (3.5.1), (t1).

e) The proof of part e) is completely analogous to that of part b), if we use

- π_4 instead of π_1,
- $J_1 = \pi_4(I_2)$ from Table 3.5.2,
- $(\,I_1, I_2\,)$ and π_4 are independent due to part a),
- (3.5.7) instead of (3.5.10) for

$$\big(\,i_1, i_2\,\big) \in N^2 \text{ and } \big(\,i_1, \pi^{-1}(i_2)\,\big) \in N^2.$$

f) I_1 and π_5 are independent due to part a).

g) Trivially, we have for all $1 \le l, k \le 16$ and $i_l, i_k \in N$:

$$i_l = i_k \;\Leftrightarrow\; i_l + 1 = i_k + 1 \text{ (mod } n) \;\Leftrightarrow\; i_l - 1 = i_k - 1 \text{ (mod n)}.$$

Therefore the mapping

$$\tau \,:\, M_{16} \longrightarrow M_{16} \,:\, \underline{i} \longrightarrow \underline{i} + (1, 1, \ldots, 1) \text{ (mod } n)$$

is **bijective** with the inverse mapping

$$\tau^{-1} \,:\, M_{16} \longrightarrow M_{16} \,:\, \underline{i} \longrightarrow \underline{i} - (1, 1, \ldots, 1) \text{ (mod } n).$$

Since also

$$\gamma(\underline{i}) = \gamma\big(\tau(\underline{i})\big),\ \theta(\underline{i}) = \theta\big(\tau(\underline{i})\big) \text{ and } \mu(\underline{i}) = \mu\big(\tau(\underline{i})\big),$$

we get using (3.5.11):

$$\text{(3.5.17)} \qquad P\big(\,\underline{I} = \underline{i}\,\big) = P\big(\,\underline{I} = \tau(\underline{i})\,\big) \qquad \text{for all } \underline{i} \in M_{16}.$$

From this we immediately obtain for all $1 \le l \le 16$ and $1 \le r \le n-1$:

$$\begin{aligned} P(I_l = r) &= \sum_{\substack{\underline{i} \in M_{16} \\ i_l = r}} P\big(\,\underline{I} = \underline{i}\,\big) \\ &= \sum_{\substack{\underline{i} \in M_{16} \\ i_l = r}} P\big(\,\underline{I} = \tau(\underline{i})\,\big) && \text{(cf. (3.5.17))} \\ &= \sum_{\substack{\underline{i} \in M_{16} \\ i_l = r+1}} P\big(\,\underline{I} = \underline{i}\,\big) && \Big(\,\underline{i} = \tau\big(\,\tau^{-1}(\underline{i})\,\big)\Big) \\ &= P(I_l = r+1). \end{aligned}$$

h) We get for all $(i,j) \in N^2$:

$$\begin{aligned} P\Big(\,(\,I_l, \pi_k(I_l)\,) = (\,i,j\,)\Big) &= \sum_{\pi \in \mathscr{P}_n} P\Big(\,(\,I_l, \pi(I_l)\,) = (\,i,j\,),\ \pi_k = \pi\,\Big) \\ &= \sum_{\substack{\pi \in \mathscr{P}_n \\ \pi(i)=j}} P\Big(\,I_l = i,\ \pi_k = \pi\,\Big) \\ &= \sum_{\substack{\pi \in \mathscr{P}_n \\ \pi(i)=j}} P\Big(\,I_l = i\,\Big)\, P\Big(\,\pi_k = \pi\,\Big) \qquad \text{(cf. part a))} \\ &= (n-1)!\, \frac{1}{n}\, \frac{1}{n!} = \frac{1}{n^2}. \end{aligned}$$

For the last equation, we used that I_l is uniformly distributed on N (see part g)) and π_k is uniformly distributed on $\mathscr{P}_n$ (see part a)). □

In the proof of Lemma 3.5.15, which we have just given, we also proved equalities of distributions (see e.g. (3.5.16)), which we list again separately in the following corollary.

In addition, the proof structure for the part h) of this lemma can also be used for other statements.

3.5.18 Corollary

a) $(\,I_1, \ldots, I_8, I_9, \ldots, I_{16}\,)$ and $(\,I_1, \ldots, I_8, J_1, \ldots, J_8\,)$ have the same distribution.

b) $(\,I_1, \ldots, I_4, I_5, \ldots, I_8\,)$ and $(\,I_1, \ldots, I_4, J_1, \ldots, J_4\,)$ have the same distribution.

c) $(\,I_1, I_2, I_3, I_4\,)$ and $(\,I_1, I_2, J_1, J_2\,)$ have the same distribution.

d) Let $\underline{i} = \big(\,i_1, \ldots, i_{16}\,\big) \in M_{16}$ and $\underline{j} = \big(\,j_1, \ldots, j_{16}\,\big) \in M_{16}$ have the properties

$$\begin{aligned} &\text{(3.5.19)} \qquad \text{(w1)} \quad i_l = i_k \;\Leftrightarrow\; j_{l+8} = j_{k+8} \quad \text{for } 1 \le l \le 8,\;\; 1 \le k \le 8, \\ &\qquad\qquad\quad\;\; \text{(w2)} \quad i_l = i_k \;\Leftrightarrow\; j_{l-8} = j_{k-8} \quad \text{for } 9 \le l \le 16,\;\; 9 \le k \le 16, \\ &\qquad\qquad\quad\;\; \text{(w3)} \quad i_l = i_k \;\Leftrightarrow\; j_{l+8} = j_{k-8} \quad \text{for } 1 \le l \le 8,\;\; 9 \le k \le 16. \end{aligned}$$

Furthermore, let $\underline{J} = (\,J_1, \ldots, J_{16}\,)$ and $\rho(\underline{i}) = \big|\,\{\,i_1, i_2, \ldots, i_{15}, i_{16}\,\}\,\big|$. Then

$$P\Big(\,\underline{I} = \underline{i},\ \underline{J} = \underline{j}\,\Big) = \frac{\big(n-\rho(\underline{i})\big)!}{n!} \cdot \frac{\big(n-\mu(\underline{i})\big)!}{n!} \cdot \frac{\big(n-\theta(\underline{i})\big)!}{n!} \cdot \frac{\big(n-\gamma(\underline{i})\big)!}{n!} \cdot \frac{1}{n^2}. \tag{3.5.20}$$

Proof:

a) - c) See proof of Lemma 3.5.15, especially part b), but also the parts c) and d).

d) We obtain for $\underline{i} = \big(i_1, \ldots, i_{16}\big) \in M_{16}$ and $\underline{j} = \big(j_1, \ldots, j_{16}\big) \in M_{16}$ with the properties (w1), (w2) and (w3):

$$\begin{aligned}
&P\Big(\big(I_1, \ldots, I_8, I_9, \ldots, I_{16} \big) = \big(i_1, \ldots, i_8, i_9, \ldots, i_{16} \big), \\
&\qquad \big(J_1, \ldots, J_8, J_9, \ldots, J_{16} \big) = \big(j_1, \ldots, j_8, j_9, \ldots, j_{16} \big) \Big) \\
&= \sum_{\pi \in \mathscr{P}_n} P\Big(\big(I_1, \ldots, I_8, I_9, \ldots, I_{16} \big) = \big(i_1, \ldots, i_8, i_9, \ldots, i_{16} \big), \\
&\qquad \big(\pi(I_9), \ldots, \pi(I_{16}), \pi(I_1), \ldots, \pi(I_8) \big) = \big(j_1, \ldots, j_8, j_9, \ldots, j_{16} \big),\, \pi_1 = \pi \Big) \\
&\hspace{25em} \text{(cf. (3.5.14))} \\
&= \sum_{\substack{\pi \in \mathscr{P}_n \\ \pi(i_9) = j_1, \ldots, \pi(i_{16}) = j_8 \\ \pi(i_1) = j_9, \ldots, \pi(i_8) = j_{16}}} P\Big(\big(I_1, \ldots, I_{16} \big) = \big(i_1, \ldots, i_{16} \big),\, \pi_1 = \pi \Big) \\
&= \sum_{\substack{\pi \in \mathscr{P}_n \\ \pi(i_9) = j_1, \ldots, \pi(i_{16}) = j_8 \\ \pi(i_1) = j_9, \ldots, \pi(i_8) = j_{16}}} P\Big(\big(I_1, \ldots, I_{16} \big) = \big(i_1, \ldots, i_{16} \big) \Big)\, P\Big(\pi_1 = \pi \Big) \qquad \text{(cf. (3.5.12))} \\
&= (n - \rho(\underline{i}))!\, P\Big(\big(I_1, \ldots, I_{16} \big) = \big(i_1, \ldots, i_{16} \big) \Big)\, \frac{1}{n!}.
\end{aligned}$$

By using (3.5.11) we then get the assertion (3.5.20). □

Next, we define

$$\begin{aligned}
T_i &= \sum_{j=1}^{n} a_{j\pi_i(j)} \qquad \text{for } i = 1, 2, 3, 4, 5; \\
\Delta T_i &= T_{i+1} - T_i \qquad \text{for } i = 1, 2, 3, 4.
\end{aligned} \tag{3.5.21}$$

According to our construction, the following representations and measurability statements are valid for the ΔT_i with regard to the random indices $I_1, \ldots, I_{16}$ and $J_1, \ldots, J_{16}$.

3.5.22 Lemma

a) $\Delta T_1 = \sum_{j=1}^{16} \Big(a_{I_j \pi_2(I_j)} - a_{I_j \pi_1(I_j)} \Big) \in \sigma\big(I_1, \ldots, I_{16}, J_1, \ldots, J_{16} \big)$,

b) $\Delta T_2 = \sum_{j=1}^{8} \Big(a_{I_j \pi_3(I_j)} - a_{I_j \pi_2(I_j)} \Big) \in \sigma\big(I_1, \ldots, I_8, J_1, \ldots, J_8 \big),$

c) $\Delta T_3 = \sum_{j=1}^{4} \Big(a_{I_j \pi_4(I_j)} - a_{I_j \pi_3(I_j)} \Big) \in \sigma\big(I_1, \ldots, I_4, J_1, \ldots, J_4 \big),$

d) $\Delta T_4 = a_{I_1 J_1} + a_{I_2 J_2} - a_{I_1 J_2} - a_{I_2 J_1} \in \sigma\big(I_1, I_2, J_1, J_2 \big).$

Proof:

The representations of the ΔT_i, $i = 1, 2, 3, 4$, follow from the fact that $v(\underline{i})$ leaves the numbers outside of $\{ i_1, \ldots, i_{16} \}$ fixed, $u(\underline{i})$ leaves the numbers outside of $\{ i_1, \ldots, i_8 \}$ fixed, $t(\underline{i})$ leaves the numbers outside of $\{ i_1, \ldots, i_4 \}$ fixed, and $s(\underline{i})$ leaves the numbers outside of $\{ i_1, i_2 \}$ fixed (see Table 3.5.1).

The measurability statements, on the other hand, can be found in the other Table 3.5.2. □

The following lemma summarizes the statements of the Lemmata 3.5.15 and 3.5.22. This gives us the necessary independence statements for the application of Stein's method in the following sections 3.6, 3.7 and 3.8.

3.5.23 Lemma

a) T_4 and $a_{I_1 J_1}$ are independent.

b) T_3 and $\big(a_{I_1 J_1},\, \Delta T_4 \big)$ are independent.

c) T_2 and $\big(a_{I_1 J_1},\, \Delta T_4,\, \Delta T_3 \big)$ are independent.

d) T_1 and $\big(a_{I_1 J_1},\, \Delta T_4,\, \Delta T_3,\, \Delta T_2 \big)$ are independent.

Proof:

a) Lemma 3.5.15, e).

b) Lemma 3.5.15, d) and Lemma 3.5.22, d).

c) Lemma 3.5.15, c) and Lemma 3.5.22, c), d).

d) Lemma 3.5.15, b) and Lemma 3.5.22, b), c), d). □

3.5.24 Remark

Illustration of the dependencies between the T_i and ΔT_j:

$$\begin{aligned} T_A = T_5 &= T_4 + \Delta T_4 \\ &= T_3 + \Delta T_3 + \Delta T_4 \\ &= T_2 + \Delta T_2 + \Delta T_3 + \Delta T_4 \\ &= T_1 + \Delta T_1 + \Delta T_2 + \Delta T_3 + \Delta T_4 \end{aligned}$$

For each line:

same colour = independent.

The following lemma contains some useful and repeatedly used results when applying Stein's method to the random variables T_i, ΔT_j, $a_{I_1 J_1}$, etc.

3.5.25 Lemma

a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then

$$E\big(T_A\, f(T_A)\big) = E\big(T_5\, f(T_5)\big) = n\, E\big(a_{I_1 J_1}\, f(T_5)\big).$$

b) Let $r \in \mathbb{N}$ and $l_1, l_2, \ldots, l_r \in \{1, 2, \ldots, 16\}$ and $k_1, k_2, \ldots, k_r \in \{1, 2, \ldots, 5\}$. Then

$$n\, E\Big(\,|\, a_{I_{l_1}\pi_{k_1}(I_{l_1})} \,|\,|\, a_{I_{l_2}\pi_{k_2}(I_{l_2})} \,|\, \ldots \,|\, a_{I_{l_r}\pi_{k_r}(I_{l_r})} \,|\,\Big) \leq \frac{1}{n} \sum_{i,j=1}^{n} |\, a_{ij} \,|^r.$$

c) Let $e_1, e_2, e_3, e_4 \in \mathbb{N}_0$ and $e = 1 + e_1 + e_2 + e_3 + e_4$. Then

$$n\, E\Big(\,|\, a_{I_1 J_1} \,|\,|\, \Delta T_4 \,|^{e_4}\,|\,\Delta T_3\,|^{e_3}\,|\,\Delta T_2\,|^{e_2}\,|\,\Delta T_1\,|^{e_1}\Big) \leq C\, \frac{1}{n} \sum_{i,j=1}^{n} |\, a_{ij} \,|^e,$$

where $C = 4^{e_4}\, 8^{e_3}\, 16^{e_2}\, 32^{e_1} \leq 32^{e-1}$.

Proof:

a)

$$\begin{aligned} & n\, E\big(a_{I_1 J_1}\, f(T_5)\big) \\ &= n\, E\big(a_{I_1 \pi_5(I_1)}\, f(T_5)\big) && \text{(cf. Table 3.5.2: } J_1 = \pi_5(I_1)\text{)} \\ &= n \sum_{i=1}^{n} P(I_1 = i)\, E\big(a_{i\pi_5(i)}\, f(T_5) \,\big|\, I_1 = i\big) && \text{(law of total probability)} \\ &= n \sum_{i=1}^{n} P(I_1 = i)\, E\big(a_{i\pi_5(i)}\, f(T_5)\big) && \text{(cf. Lemma 3.5.15, f))} \end{aligned}$$

$$= n \sum_{i=1}^{n} \frac{1}{n} \, E\big(a_{i\pi_5(i)} \, f(\, T_5 \,) \big) \qquad \text{(cf. Lemma 3.5.15, g))}$$

$$= E\big(T_5 \, f(\, T_5 \,) \big) \qquad (E \text{ linear and } (3.5.21))$$

$$= E\big(T_A \, f(\, T_A \,) \big) \qquad \text{(cf. Lemma 3.5.15, a)).}$$

b) An application of Hölder's inequality with r terms and of Lemma 3.5.15, h) gives

$$n \, E\Big(\,|\, a_{I_{l_1}\pi_{k_1}(I_{l_1})} \,|\, |\, a_{I_{l_2}\pi_{k_2}(I_{l_2})} \,|\, \ldots \,|\, a_{I_{l_r}\pi_{k_r}(I_{l_r})} \,|\, \Big)$$

$$\leq n \, E\Big(\,|\, a_{I_{l_1}\pi_{k_1}(I_{l_1})} \,|^r \Big)^{1/r} E\Big(\,|\, a_{I_{l_2}\pi_{k_2}(I_{l_2})} \,|^r \Big)^{1/r} \ldots E\Big(\,|\, a_{I_{l_r}\pi_{k_r}(I_{l_r})} \,|^r \Big)^{1/r}$$

$$= n \left(\frac{1}{n^2} \sum_{i,j=1}^{n} |\, a_{ij} \,|^r \right)^{1/r + 1/r + \ldots + 1/r} = \frac{1}{n} \sum_{i,j=1}^{n} |\, a_{ij} \,|^r .$$

c) $J_1 = \pi_5(I_1)$, $J_2 = \pi_5(I_2)$, $J_1 = \pi_4(I_2)$ and $J_2 = \pi_4(I_1)$ (cf. Table 3.5.2). Thus, all ΔT_i for $i = 1, 2, 3, 4$ can be represented as a sum of terms $\pm a_{I_l \pi_k(I_l)}$ (cf. Lemma 3.5.22).

Consequently, the term in this part c) can be estimated by a sum of terms from part b). The constant C indicates the number of these summands. □

The following result is important for the correct determination of the Edgeworth expansions.

3.5.26 Lemma

If $A = \hat{A}$, then

a) $E\big(a_{I_1 J_1} \big) = 0$,

b) $n \, E\big(a_{I_1 J_1} \, \Delta T_4 \big) = 1$,

c) $n \left\{ E\big(a_{I_1 J_1} \, \Delta T_4 \, \Delta T_3 \big) + E\Big(a_{I_1 J_1} \, \frac{(\Delta T_4)^2}{2} \Big) \right\} = \frac{1}{2} \, E(T_A^3)$,

d) $$n \left\{ E\big(a_{I_1 J_1} \, \Delta T_4 \, \Delta T_3 \, \Delta T_2 \big) + E\Big(a_{I_1 J_1} \, \Delta T_4 \, \frac{(\Delta T_3)^2}{2} \Big) \right.$$

$$\left. + \, E\Big(a_{I_1 J_1} \, \frac{(\Delta T_4)^2}{2} \, \Delta T_3 \Big) + E\Big(a_{I_1 J_1} \, \frac{(\Delta T_4)^3}{6} \Big) \right\} = \frac{1}{6} \, \big(E(T_A^4) - 3 \big).$$

Proof:

a) Since $J_1 = \pi_5(I_1)$ (cf. Table 3.5.2) and since $\big(I_1, \pi_5(I_1)\big)$ is uniformly distributed on N^2 (cf. Lemma 3.5.15, h)), we obtain, because of $A = \hat{A}$ and (3.1.1), i.e. $\hat{a}_{..} = 0$,

$$E\big(\, a_{I_1 J_1} \,\big) = \frac{1}{n^2} \sum_{i,j=1}^{n} a_{ij} = 0.$$

b)

$$\begin{aligned}
& E\big(\, a_{I_1 J_1}\, \Delta T_4 \,\big) \\
&= E\big(\, a_{I_1 J_1}\, T_5 \,\big) - E\big(\, a_{I_1 J_1}\, T_4 \,\big) && \text{(cf. (3.5.21): } \Delta T_4 = T_5 - T_4) \\
&= E\big(\, a_{I_1 J_1}\, T_5 \,\big) - E\big(\, a_{I_1 J_1} \,\big)\, E\big(\, T_4 \,\big) && \text{(cf. Lemma 3.5.23, a))} \\
&= \frac{1}{n}\, E\big(\, T_A^2 \,\big) - E\big(\, a_{I_1 J_1} \,\big)\, E\big(\, T_A \,\big) && \text{(cf. Lemma 3.5.25, a) with } f(x) = x) \\
&= \frac{1}{n} && \text{(cf. (3.1.1): } \mu_{\hat{A}} = 0,\ \sigma_{\hat{A}}^2 = 1).
\end{aligned}$$

c)

$$\begin{aligned}
& E\big(\, a_{I_1 J_1}\, \Delta T_4\, \Delta T_3 \,\big) + E\Big(\, a_{I_1 J_1}\, \frac{(\Delta T_4)^2}{2} \,\Big) \\
&= E\big(\, a_{I_1 J_1}\, T_5\, T_4 \,\big) - E\big(\, a_{I_1 J_1}\, T_4^2 \,\big) - E\big(\, a_{I_1 J_1}\, \Delta T_4\, T_3 \,\big) \\
&\quad + \frac{1}{2}\, E\big(\, a_{I_1 J_1}\, T_5^2 \,\big) - E\big(\, a_{I_1 J_1}\, T_5\, T_4 \,\big) + \frac{1}{2}\, E\big(\, a_{I_1 J_1}\, T_4^2 \,\big) \\
&= -\, \frac{1}{2}\, E\big(\, a_{I_1 J_1} \,\big)\, E\big(\, T_4^2 \,\big) - E\big(\, a_{I_1 J_1}\, \Delta T_4 \,\big)\, E\big(\, T_3 \,\big) \\
&\quad + \frac{1}{2}\, E\big(\, a_{I_1 J_1}\, T_5^2 \,\big) && \text{(cf. Lemma 3.5.23, a) and b))} \\
&= \frac{1}{2}\, E\big(\, a_{I_1 J_1}\, T_5^2 \,\big) && \text{(cf. part a) and (3.1.1): } \mu_{\hat{A}} = 0) \\
&= \frac{1}{n}\, \frac{1}{2}\, E(T_A^3) && \text{(cf. Lemma 3.5.25, a) with } f(x) = x^2).
\end{aligned}$$

d) We obtain for the first two summands:

$$\begin{aligned}
& E\big(\, a_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, \Delta T_2 \,\big) + E\Big(\, a_{I_1 J_1}\, \Delta T_4\, \frac{(\Delta T_3)^2}{2} \,\Big) \\
&= E\big(\, a_{I_1 J_1}\, \Delta T_4\, T_4\, T_3 \,\big) - E\big(\, a_{I_1 J_1}\, \Delta T_4\, T_3^2 \,\big) - E\big(\, a_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, T_2 \,\big) \\
&\quad + \frac{1}{2}\, E\big(\, a_{I_1 J_1}\, T_5\, T_4^2 \,\big) - \frac{1}{2}\, E\big(\, a_{I_1 J_1}\, T_4^3 \,\big) - E\big(\, a_{I_1 J_1}\, \Delta T_4\, T_4\, T_3 \,\big) + \frac{1}{2}\, E\big(\, a_{I_1 J_1}\, \Delta T_4\, T_3^2 \,\big)
\end{aligned}$$

$$= -\frac{1}{2} E\big(a_{I_1 J_1} \, \Delta T_4 \big) E\big(T_3^2 \big) - E\big(a_{I_1 J_1} \, \Delta T_4 \, \Delta T_3 \big) E\big(T_2 \big)$$

$$+ \frac{1}{2} E\big(a_{I_1 J_1} \, T_5 \, T_4^2 \big) - \frac{1}{2} E\big(a_{I_1 J_1} \big) E\big(T_4^3 \big) \qquad \text{(cf. Lemma 3.5.23, a), b) and c))}$$

$$= \frac{1}{2} E\big(a_{I_1 J_1} \, T_5 \, T_4^2 \big) - \frac{1}{2} \frac{1}{n} \qquad \text{(cf. part a) and b), and (3.1.1): } \mu_{\hat{A}} = 0,\ \sigma_{\hat{A}}^2 = 1).$$

For the third summand we get:

$$E\Big(a_{I_1 J_1} \, \frac{(\Delta T_4)^2}{2} \, \Delta T_3 \Big)$$

$$= E\Big(a_{I_1 J_1} \, \frac{(\Delta T_4)^2}{2} \, T_4 \Big) - E\Big(a_{I_1 J_1} \, \frac{(\Delta T_4)^2}{2} \Big) E\big(T_3 \big) \qquad \text{(cf. Lemma 3.5.23, b))}$$

$$= \frac{1}{2} E\big(a_{I_1 J_1} \, T_5^2 \, T_4 \big) - E\big(a_{I_1 J_1} \, T_5 \, T_4^2 \big) + \frac{1}{2} E\big(a_{I_1 J_1} \, T_4^3 \big) \qquad \text{(cf. (3.1.1): } \mu_{\hat{A}} = 0)$$

$$= \frac{1}{2} E\big(a_{I_1 J_1} \, T_5^2 \, T_4 \big) - E\big(a_{I_1 J_1} \, T_5 \, T_4^2 \big) + \frac{1}{2} E\big(a_{I_1 J_1} \big) E\big(T_4^3 \big) \qquad \text{(Lemma 3.5.23, a))}$$

$$= \frac{1}{2} E\big(a_{I_1 J_1} \, T_5^2 \, T_4 \big) - E\big(a_{I_1 J_1} \, T_5 \, T_4^2 \big) \qquad \text{(cf. part a)).}$$

For the fourth summand we get analogously to above:

$$E\Big(a_{I_1 J_1} \, \frac{(\Delta T_4)^3}{6} \Big)$$

$$= \frac{1}{6} \Big(E\Big(a_{I_1 J_1} \, \big(T_5^3 - 3 \, T_5^2 \, T_4 + 3 \, T_5 \, T_4^2 \big) \Big) - E\big(a_{I_1 J_1} \big) E\big(T_4^3 \big) \Big) \quad \text{(Lemma 3.5.23, a))}$$

$$= \frac{1}{6} E\big(a_{I_1 J_1} \, T_5^3 \big) - \frac{1}{2} E\big(a_{I_1 J_1} \, T_5^2 \, T_4 \big) + \frac{1}{2} E\big(a_{I_1 J_1} \, T_5 \, T_4^2 \big) \qquad \text{(cf. part a)).}$$

Lastly, a summary of the three partial results gives:

$$\Big\{ \ldots\ldots\ldots\ldots \Big\} = \frac{1}{6} E\big(a_{I_1 J_1} \, T_5^3 \big) - \frac{1}{2} \frac{1}{n}$$

$$= \frac{1}{n} \frac{1}{6} \big(E(T_A^4) - 3 \big) \qquad \text{(cf. Lemma 3.5.25, a) with } f(x) = x^3). \ \square$$

Finally, we consider conditional distributions on the permutations of N.

3.5.27 Lemma

a) Let $(i_1, \ldots, i_4,\, j_1, \ldots, j_4) \in M_8$, $\breve{\imath} = (i_i, \ldots, i_4)$ and $\theta(\breve{\imath}) = \big| \{ i_1, i_2, i_3, i_4 \} \big|$.
Also, let $\pi \in \mathscr{R}_n = \Big\{ \nu \in \mathscr{P}_n \, : \, \nu(i_1) = j_3,\ \nu(i_2) = j_4,\ \nu(i_3) = j_1,\ \nu(i_4) = j_2 \Big\}$.

Then

$$P\Big(\pi_3 = \pi \,\Big|\, \big(\,I_1,\ldots,I_4\,\big) = (\,i_1,\ldots,i_4\,),\, \big(\,J_1,\ldots,J_4\,\big) = (\,j_1,\ldots,j_4\,)\Big) = \frac{1}{(n-\theta(\breve{i}))!},$$

i.e. π_3, given $\big(\,I_1,\ldots,I_4\,\big) = (\,i_1,\ldots,i_4\,)$ and $\big(\,J_1,\ldots,J_4\,\big) = (\,j_1,\ldots,j_4\,)$, is uniformly distributed on $\mathscr{R}_n$.

b) Let $(\,i_1,\ldots,i_8,\,j_1,\ldots,j_8\,) \in M_{16}$, $\bar{i} = (\,i_i,\ldots,i_8\,)$ and $\mu(\bar{i}) = \big|\,\{\,i_1,i_2,\ldots,i_7,i_8\,\}\,\big|$.
Also, let $\pi \in \mathscr{S}_n = \Big\{\,\nu \in \mathscr{P}_n \,:\, \nu(i_1) = j_5,\,\ldots,\nu(i_4) = j_8,\,\nu(i_5) = j_1,\,\ldots,\,\nu(i_8) = j_4\,\Big\}$.
Then

$$P\Big(\pi_2 = \pi \,\Big|\, \big(\,I_1,\ldots,I_8\,\big) = (\,i_1,\ldots,i_8\,),\, \big(\,J_1,\ldots,J_8\,\big) = (\,j_1,\ldots,j_8\,)\Big) = \frac{1}{(n-\mu(\bar{i}))!},$$

i.e. π_2, given $\big(\,I_1,\ldots,I_8\,\big) = (\,i_1,\ldots,i_8\,)$ and $\big(\,J_1,\ldots,J_8\,\big) = (\,j_1,\ldots,j_8\,)$, is uniformly distributed on $\mathscr{S}_n$.

c) Let $\underline{i} = \big(i_1,\ldots,i_{16}\big) \in M_{16}$ and $\underline{j} = \big(j_1,\ldots,j_{16}\big) \in M_{16}$ with the properties (w1), (w2) and (w3) from (3.5.19). Also, let $\rho(\underline{i}) = \big|\,\{\,i_1,i_2,\ldots,i_{15},i_{16}\,\}\,\big|$ and
$\pi \in \mathscr{W}_n = \Big\{\,\nu \in \mathscr{P}_n \,:\, \nu(i_1) = j_9,\,\ldots,\nu(i_8) = j_{16},\,\nu(i_9) = j_1,\,\ldots,\,\nu(i_{16}) = j_8\,\Big\}$.
Then

$$P\Big(\pi_1 = \pi \,\Big|\, \big(\,I_1,\ldots,I_{16}\,\big) = (\,i_1,\ldots,i_{16}\,),\, \big(\,J_1,\ldots,J_{16}\,\big) = (\,j_1,\ldots,j_{16}\,)\Big) = \frac{1}{(n-\rho(\underline{i}))!},$$

i.e. π_1, given $\big(\,I_1,\ldots,I_{16}\,\big) = (\,i_1,\ldots,i_{16}\,)$ and $\big(\,J_1,\ldots,J_{16}\,\big) = (\,j_1,\ldots,j_{16}\,)$, is uniformly distributed on $\mathscr{W}_n$.

Proof:

a) We get for $\pi \in \mathscr{R}_n$:

$$P\Big(\pi_3 = \pi \,\Big|\, \big(\,I_1,\ldots,I_4\,\big) = (\,i_1,\ldots,i_4\,),\, \big(\,J_1,\ldots,J_4\,\big) = (\,j_1,\ldots,j_4\,)\Big)$$

$$= \frac{P\Big(\pi_3 = \pi,\, \big(\,I_1,..,I_4\,\big) = (\,i_1,..,i_4\,),\, \big(\,\pi_3(I_3),\,\pi_3(I_4),\,\pi_3(I_1),\,\pi_3(I_2)\,\big) = (\,j_1,\,j_2,\,j_3,\,j_4\,)\Big)}{P\Big(\big(\,I_1,\ldots,I_4\,\big) = (\,i_1,\ldots,i_4\,),\, \big(\,J_1,\ldots,J_4\,\big) = (\,j_1,\ldots,j_4\,)\Big)}$$

(cf. π_3 in Table 3.5.2)

$$= \frac{P\Big(\pi_3 = \pi,\, \big(\,I_1,\ldots,I_4\,\big) = (\,i_1,\ldots,i_4\,),\, \big(\,\pi(i_3),\,\pi(i_4),\,\pi(i_1),\,\pi(i_2)\,\big) = (\,j_1,\,j_2,\,j_3,\,j_4\,)\Big)}{P\Big(\big(\,I_1,\ldots,I_4\,\big) = (\,i_1,\ldots,i_4\,),\, \big(\,J_1,\ldots,J_4\,\big) = (\,j_1,\ldots,j_4\,)\Big)}$$

$$= \frac{P\Big(\pi_3 = \pi,\ \big(I_1, \ldots, I_4\big) = (i_1, \ldots, i_4)\Big)}{P\Big(\big(I_1, \ldots, I_4\big) = (i_1, \ldots, i_4),\ \big(J_1, \ldots, J_4\big) = (j_1, \ldots, j_4)\Big)} \qquad (\pi \in \mathscr{R}_n)$$

$$= \frac{1}{n!}\,\frac{P\Big(\big(I_1, \ldots, I_4\big) = (i_1, \ldots, i_4)\Big)}{P\Big(\big(I_1, \ldots, I_4\big) = (i_1, \ldots, i_4),\ \big(J_1, \ldots, J_4\big) = (j_1, \ldots, j_4)\Big)} \qquad \text{(Lemma 3.5.15, a)))}$$

$$= \frac{1}{n!}\,\frac{n!}{\big(n-\theta(\breve{i})\big)!} = \frac{1}{\big(n-\theta(\breve{i})\big)!} \qquad \text{(cf. (3.5.11) and Corollary 3.5.18, b)).}$$

b) Completely analogous to part a), where now Corollary 3.5.18, a) is used instead of Corollary 3.5.18, b).

c) Also completely analogous to part a), but now for the penultimate "=" in the denominator Corollary 3.5.18, d) is used instead of Corollary 3.5.18, b). □

3.6 A modification of Theorem 3.1.3

In the next two sections we need an estimate of

$$\sup_{z \in \mathbb{R}} \left| E\Big(|\mathscr{T}_A|^k\, 1_{(-\infty,\, z]}(\mathscr{T}_A)\Big) - \Phi\Big(|x|^k\, 1_{(-\infty,\, z]}(x)\Big) \right|$$

for $k = 1$ and $k = 2$ to prove the two Theorems 3.1.10 and 3.1.13. The aim of this section is therefore to provide this estimate. Since the corresponding proof can be carried out for each $k \in \mathbb{N}$ without much more work, we will consider this more general framework in the following.

The methods used in the proof are almost identical to those in Bolthausen [6]. However, it was not possible to transfer the truncation technique used there (see in particular page 383, inequality (3.6)) to the situation here, which led in the estimate obtained to the high power $k+4$ of the term $\frac{1}{n}\sum_{i,j} |\,\hat{a}_{ij}\,|^{k+4}$. For the further procedure, however, this is irrelevant. The estimate is as follows:

3.6.1 Theorem

Let $k \in \mathbb{N}$ be fixed.

Then there exists a constant $\mathcal{K}_7(k) > 0$ such that for all matrices A satisfying $\sigma_A > 0$

$$(3.6.2) \qquad \sup_{z \in \mathbb{R}} \left| E\Big(|\, \mathscr{T}_A|^k \, 1_{(-\infty,\, z\,]}(\, \mathscr{T}_A) \Big) - \Phi\Big(|x|^k \, 1_{(-\infty,\, z\,]}(x) \Big) \right|$$
$$\leq \mathcal{K}_7(k) \left(\frac{\beta_A}{n} + \frac{1}{n} \sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^{k+4} \right).$$

Proof:

At first, it should be briefly noted that this proof has a similar structure to the proof of Theorem 1.4.3.

In the following, $k \in \mathbb{N}$ is assumed to be **fixed**. Therefore the **dependence** of the constants $c_1, c_2, \ldots$ of k is **no** longer emphasized in this proof.

Furthermore, we define for $\gamma, \lambda > 0$ and $n \in \mathbb{N} \setminus \{1\}$:

$$\mathscr{M}(n,\gamma) \;= \text{set of } n{\times}n\text{-matrices } A \text{ satisfying } \sigma_A > 0,$$
$$a_{ij} = \hat{a}_{ij} \text{ for } 1 \leq i,j \leq n \;\text{ and }\; \beta_A + \sum_{i,j=1}^{n} |\, \hat{a}_{ij} \,|^{k+4} \leq \gamma;$$
$$\delta(\lambda,\gamma,n) = \sup \left\{ \left| E\big(p^{\lambda}_{z,k}(\,T_A)\big) - \Phi\big(p^{\lambda}_{z,k}\big) \right| \,:\, z \in \mathbb{R},\, A \in \mathscr{M}(n,\gamma) \right\};$$
$$\delta(\gamma,n) \;= \sup \left\{ \left| E\big(\,|T_A|^k \, 1_{(-\infty,\, z\,]}(\,T_A)\,\big) - \Phi\big(\,|x|^k \, 1_{(-\infty,\, z\,]}(x)\,\big) \right| \,:\, z \in \mathbb{R},\, A \in \mathscr{M}(n,\gamma) \right\}.$$

The functions $p^{\lambda}_{z,k}$ (cf. (2.3.1)) and $d^{\lambda}_{z,k}$ (cf. (2.3.4)) are defined as in section 2.3. Moreover, let $\sup \emptyset = 0$.

If

$$c_1 = \sup_{x \in \mathbb{R}} |x|^k \, \psi(x) = \frac{1}{\sqrt{2\,\pi}} \left(\frac{k}{e} \right)^{k/2},$$

then

$$(3.6.3) \qquad \delta(\gamma,n) \leq \delta(\lambda,\gamma,n) + \frac{c_1}{2}\,\lambda \qquad \text{(analogous to (1.4.4)).}$$

To show the assertion $\delta(\gamma,n) \leq C(k)\,\dfrac{\gamma}{n}$, we first estimate the expression

$$\left| E\big(p^{\lambda}_{z,k}(\,T_A)\big) - \Phi\big(p^{\lambda}_{z,k}\big) \right|$$

for matrices $A \in \mathscr{M}(n,\gamma)$ using Stein's method and then apply (3.6.3). For the time being, these matrices A should also have the following property:

(3.6.4) $\beta_A \le \epsilon_0\, n$ and $n \ge n_0$, where $0 < \epsilon_0 \le 1$ and $n_0 \in \mathbb{N}$ are chosen so that (3.3.8) and (3.3.9) of Corollary 3.3.7 are valid. However, we require at least $n_0 \ge 6$ (so that $n_0 - 4 \ge 2$ and thus $\mathscr{M}(n_0 - 4, \gamma)$ is defined).

Furthermore, let $z \in \mathbb{R}$ and $\lambda > 0$ be fixed in the following considerations, so that we can set $p = p^{\lambda}_{z,k}$ and $d = d^{\lambda}_{z,k}$ for simplification.

Using these abbreviations and $T_5 = T_4 + \Delta T_4$ and $T_4 = T_3 + \Delta T_3$ (cf. (3.5.21)), we obtain:

(3.6.5)
$$
\begin{aligned}
&E\big(T_A\, d(T_A)\big) \\
&= n\, E\big(a_{I_1 J_1}\, d(T_5)\big) && \text{(cf. Lemma 3.5.25, a))} \\
&= n\, E\big(a_{I_1 J_1}\, d(T_4)\big) + n\, E\Big(a_{I_1 J_1}\, \Delta T_4 \int_0^1 d'\big(T_4 + t\, \Delta T_4\big)\, dt\Big) \\
&= n\, E\big(a_{I_1 J_1}\, d(T_4)\big) + n\, E\big(a_{I_1 J_1}\, \Delta T_4\, d'(T_3)\big) \\
&\quad + n\, E\Big(a_{I_1 J_1}\, \Delta T_4 \int_0^1 \Big(d'\big(T_3 + \Delta T_3 + t\, \Delta T_4\big) - d'(T_3)\Big)\, dt\Big) \\
&= E\big(d'(T_3)\big) + n\, E\Big(a_{I_1 J_1}\, \Delta T_4 \int_0^1 \Big(d'\big(T_3 + \Delta T_3 + t\, \Delta T_4\big) - d'(T_3)\Big)\, dt\Big).
\end{aligned}
$$

The last equation follows because of Lemma 3.5.23, a) and b), and Lemma 3.5.26, a) and b).

From this calculation and Lemma 2.3.5, d) and Hölder's inequality (0.1.5) with $\nu = 2$ and $p = k + 1$ we further get:

(3.6.6)
$$
\begin{aligned}
&\Big| E\big(p(T_A)\big) - \Phi(p) \Big| \\
&= \Big| E\big(d'(T_A)\big) - E\big(T_A\, d(T_A)\big) \Big| && \text{(Stein's equation)} \\
&= \Big| E\big(d'(T_3)\big) - E\big(T_A\, d(T_A)\big) \Big| \\
&\le n\, E\Big(|a_{I_1 J_1}|\, |\Delta T_4| \int_0^1 \Big| d'\big(T_3 + \Delta T_3 + t\, \Delta T_4\big) - d'(T_3) \Big|\, dt \Big) && \text{(cf. (3.6.5))} \\
&\le c_2\, n\, E\Big(|a_{I_1 J_1}|\, |\Delta T_4| \int_0^1 |\Delta T_3 + t\, \Delta T_4| \Big[1 + |T_3|^{k+1}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \Big(1 + \big|\,T_3\,\big|\Big)\,\big|\,\Delta T_3 + t\,\Delta T_4\,\big|^k \\
&\qquad\qquad + \frac{\big|\,T_3\,\big|^k}{\lambda} \int\limits_0^1 1_{(\,z,\,z+\lambda\,]}\big(\,T_3 + s\,\Delta T_3 + s\,t\,\Delta T_4\,\big)\,ds\,\Big]\,dt\Big) \\
&\le c_2 \Big\{\, n\,E\Big(\,\big|\,a_{I_1 J_1}\,\big|\,\big|\,\Delta T_4\,\big|\,\Big(\,\big|\,\Delta T_3\,\big| + \big|\,\Delta T_4\,\big|\,\Big)\Big) \\
&\quad + n\,E\Big(\,\big|\,a_{I_1 J_1}\,\big|\,\big|\,\Delta T_4\,\big|\,\Big(\,\big|\,\Delta T_3\,\big| + \big|\,\Delta T_4\,\big|\,\Big)\,\big|\,T_3\,\big|^{k+1}\Big) \\
&\quad + n\,E\Big(\,\big|\,a_{I_1 J_1}\,\big|\,\big|\,\Delta T_4\,\big|\,2^k\,\Big(\,\big|\,\Delta T_3\,\big|^{k+1} + \big|\,\Delta T_4\,\big|^{k+1}\Big)\,\Big(1 + \big|\,T_3\,\big|\Big)\Big) \\
&\quad + n\,E\Big(\,\big|\,a_{I_1 J_1}\,\big|\,\big|\,\Delta T_4\,\big|\,\Big(\,\big|\,\Delta T_3\,\big| + \big|\,\Delta T_4\,\big|\,\Big)\,\frac{\big|\,T_3\,\big|^k}{\lambda} \\
&\qquad\qquad \cdot \int\limits_0^1 \int\limits_0^1 1_{(\,z,\,z+\lambda\,]}\big(\,T_3 + s\,\Delta T_3 + s\,t\,\Delta T_4\,\big)\,ds\,dt\Big)\Big\} \\
&= c_2\,\big(\,A_1 + A_2 + A_3 + A_4\,\big).
\end{aligned}
$$

To estimate A_1 we use Lemma 3.5.25, c) and obtain

$$
A_1 \le c_3\,\frac{\beta_A}{n} \le c_3\,\frac{\gamma}{n}. \tag{3.6.7}
$$

Because of (cf. Hölder's inequality (0.1.5) with $\nu = 2$ and $p = k+1$)

$$
\big|\,T_3\,\big|^{k+1} = \big|\,T_2 + \Delta T_2\,\big|^{k+1} \le 2^k\,\Big(\,\big|\,T_2\,\big|^{k+1} + \big|\,\Delta T_2\,\big|^{k+1}\Big)
$$

and because of Lemma 3.5.23, c) and Lemma 3.5.25, c) we get analogously

$$
A_2 \le c_4\,\Big(\,\frac{\beta_A}{n}\,E\big(\,\big|\,T_A\,\big|^{k+1}\,\big) + \frac{1}{n}\sum_{i,j=1}^{n}\big|\,a_{ij}\,\big|^{k+4}\Big).
$$

We estimate the $(k+1)$th absolute moment of T_A. Since $E\big(\,T_A^2\,\big) = 1$, we may assume $k \ge 2$. Using Proposition 3.2.5, we get:

$$
\begin{aligned}
E\big(\,\big|\,T_A\,\big|^{k+1}\,\big) &\le \begin{cases} \Phi(x^{k+1}) + c_5\,\Big(\,\dfrac{\beta_A}{n} + \dfrac{1}{n}\displaystyle\sum_{i,j=1}^{n}\big|\,a_{ij}\,\big|^{k+1}\Big) & \text{for } k \ge 3 \text{ odd,} \\ \Phi(x^{k+2}) + c_6\,\Big(\,\dfrac{\beta_A}{n} + \dfrac{1}{n}\displaystyle\sum_{i,j=1}^{n}\big|\,a_{ij}\,\big|^{k+2}\Big) & \text{for } k \ge 2 \text{ even,} \end{cases} \\
&\le c_7 + c_8\,\frac{\gamma}{n}.
\end{aligned} \tag{3.6.8}
$$

The first case "$k \geq 3$ odd" follows directly from Proposition 3.2.5. For the second case "$k \geq 2$ even" we use additionally Hölder's inequality

$$E\big(\,|\,T_A\,|^{k+1}\,\big) \leq E\big(\,|\,T_A\,|^{k+2}\,\big)^{(k+1)/(k+2)} \leq E\big(\,|\,T_A\,|^{k+2}\,\big) \quad \text{for } k \geq 2$$

and then apply Proposition 3.2.5 to $E\big(\,|\,T_A\,|^{k+2}\,\big)$. For the second inequality of (3.6.8), the inequality $|x|^{k+1} \leq |x|^3 + |x|^{k+4}$ is used for odd $k \geq 3$ and $|x|^{k+2} \leq |x|^3 + |x|^{k+4}$ for even $k \geq 2$.

For the sake of completeness, we set $c_7 = 1$, $c_8 = 0$ for $k = 1$. Due to $\epsilon_0 \leq 1$ and thus $\dfrac{\beta_A}{n} \leq \epsilon_0 \leq 1$ in (3.6.4), it follows that

$$A_2 \leq c_4 \left(c_7 \, \frac{\beta_A}{n} + c_8 \, \frac{\gamma}{n} + \frac{1}{n} \sum_{i,j=1}^{n} |\, a_{ij} \,|^{k+4} \right) \leq c_9 \, \frac{\gamma}{n}. \tag{3.6.9}$$

Because of

$$|\,T_3\,| \leq |\,T_2\,| + |\,\Delta T_2\,| \quad \text{and} \quad E\big(\,|\,T_2\,|\,\big) \leq E\big(\,|\,T_2\,|^2\,\big)^{1/2} = \sigma_A = 1$$

and again because of Lemma 3.5.23, c) and Lemma 3.5.25, c) also holds

$$A_3 \leq c_{10} \left(\frac{1}{n} \sum_{i,j=1}^{n} |\, a_{ij} \,|^{k+3} + \frac{1}{n} \sum_{i,j=1}^{n} |\, a_{ij} \,|^{k+4} \right) \leq 2\, c_{10} \, \frac{\gamma}{n}. \tag{3.6.10}$$

For the second inequality, the inequality $|x|^{k+3} \leq |x|^3 + |x|^{k+4}$ for $k \in \mathbb{N}$ is used.

To estimate A_4, we first have to estimate the term

$$\begin{aligned} A_5 = E\Big(\,|\,T_3\,|^k \, 1_{(\,z,\, z\,+\,\lambda\,]}\big(\,T_3 + s\,\Delta T_3 + s\,t\,\Delta T_4\,\big) \,\Big|\, (\,I_1, \ldots, I_4\,) = (\,i_1, \ldots, i_4\,), \\ (\,J_1, \ldots, J_4\,) = (\,j_1, \ldots, j_4\,) \Big) \end{aligned} \tag{3.6.11}$$

for all $0 \leq s, t \leq 1$ and $(\,i_1, \ldots, i_4,\, j_1, \ldots, j_4\,) \in M_8$.

Because of Lemma 3.5.27, a) is T_3, given $\big(I_1, \ldots, I_4\big) = (\,i_1, \ldots, i_4\,)$ and $\big(J_1, \ldots, J_4\big) = (\,j_1, \ldots, j_4\,)$, distributed as

$$\sum_{(i,\,j)\,\in\, S} a_{ij} + T_B, \tag{3.6.12}$$

where

$$S = \Big\{ (\,i_1,\, j_3\,),\, (\,i_2,\, j_4\,),\, (\,i_3,\, j_1\,),\, (\,i_4,\, j_2\,) \Big\} \tag{3.6.13}$$

and

(3.6.14) $\quad B$ is the $(n-l)\times(n-l)-$matrix obtained from A by **cancelling** the $l = \big|\{\, i_1, i_2, i_3, i_4 \,\}\big|$ rows $i_1, \ldots, i_4$ and the l columns $j_1, \ldots, j_4$.

Using ΔT_3, $\Delta T_4 \in \sigma\big(I_1, \ldots, I_4, J_1, \ldots, J_4\big)$ (cf. Lemma 3.5.22, c) and d)) and Hölder's inequality (0.1.5) with $\nu = 5$ and $p = k$, we now get

$$A_5 \le 5^{k-1} \bigg\{ \Big(|a_{i_1 j_3}|^k + |a_{i_2 j_4}|^k + |a_{i_3 j_1}|^k + |a_{i_4 j_2}|^k \Big)\, \alpha(\lambda, A) + \beta(\lambda, A) \bigg\},$$

where

$$\alpha(\lambda, A) = \sup \Big\{ P\big(z < T_B \le z + \lambda \big) \,:\, z \in \mathbb{R}, B \in M(l, A), 1 \le l \le 4 \Big\},$$

$$\beta(\lambda, A) = \sup \Big\{ E\big(\,|\, T_B \,|^k\, 1_{(\,z,\, z+\lambda\,]}(\,T_B\,)\big) \,:\, z \in \mathbb{R}, B \in M(l, A), 1 \le l \le 4 \Big\}.$$

To estimate these two quantities, we utilize $|\,\mu_B\,| \le 1$ and $\big|\,\sigma_B^2 - 1\,\big| \le \dfrac{1}{3}$ for $B \in M(l, A)$, $1 \le l \le 4$, which is valid according to (3.6.4).

Thus we obtain for $B \in M(l, A)$, $1 \le l \le 4$:

(3.6.15)
$$\begin{aligned}
&\sup_{z\in\mathbb{R}} P\big(z < T_B \le z + \lambda \big) \\
&\le \sup_{z\in\mathbb{R}} P\big(z < \mathscr{T}_B \le z + 2\,\lambda \big) && \text{(since } T_B = \sigma_B\, \mathscr{T}_B + \mu_B \text{ and } \tfrac{1}{\sigma_B} \le 2) \\
&\le 2\,\Big(\mathcal{K}_1\, \frac{\beta_B}{n-l} + \frac{\lambda}{\sqrt{2\,\pi}} \Big) && \text{(cf. Theorem 3.1.3)} \\
&\le c_{11}\, \Big(\frac{\beta_A}{n} + \lambda \Big) && \text{(cf. Lemma 3.3.10 and } l \le 4 < 6 \le n \;\Rightarrow\; \tfrac{n}{n-l} \le 3) \\
&\le c_{11}\, \Big(\frac{\gamma}{n} + \lambda \Big) && \text{(since } A \in \mathscr{M}(n, \gamma))
\end{aligned}$$

and

(3.6.16)
$$\begin{aligned}
&\sup_{z\in\mathbb{R}} E\big(\,|\, T_B \,|^k\, 1_{(\,z,\, z+\lambda\,]}(\,T_B\,)\big) \\
&\le \sup_{z\in\mathbb{R}} E\big(\,|\, T_B \,|^k\, 1_{(\,z,\, z+2\,\lambda\,]}(\,\mathscr{T}_B\,)\big) && \text{(since } T_B = \sigma_B\, \mathscr{T}_B + \mu_B \text{ and } \tfrac{1}{\sigma_B} \le 2) \\
&\le 2^{k-1} \Big(\frac{3}{2} \Big)^k \sup_{z\in\mathbb{R}} E\big(\,|\, \mathscr{T}_B \,|^k\, 1_{(\,z,\, z+2\,\lambda\,]}(\,\mathscr{T}_B\,)\big) && (T_B = \sigma_B\, \mathscr{T}_B + \mu_B \text{ and } \sigma_B \le \tfrac{3}{2})
\end{aligned}$$

$$\begin{aligned}
&\quad + 2^{k-1}\,|\,\mu_B\,|^k \sup_{z\,\in\,\mathbb{R}} P\big(\,z < \mathscr{T}_B \le z + 2\,\lambda\,\big) && \text{(and (0.1.5) with } \nu = 2,\ p = k)\\
&\le 3^k\,\delta\Big(\,\beta_B + \sum_{i\neq i_m,\,j\neq j_m} |\,\hat{b}_{ij}\,|^{k+4},\ n - l\,\Big)\\
&\quad + 3^k\,\lambda \sup_{x\,\in\,\mathbb{R}} |x|^k\,\psi(x) + 2^{k-1}\,c_{11}\,\Big(\,\frac{\gamma}{n} + \lambda\,\Big) && \text{(cf. (3.6.15) and } |\,\mu_B\,| \le 1)\\
&\le 3^k\,\delta\big(\,c_{12}\,\gamma,\ n - l\,\big) + c_{13}\,\Big(\,\frac{\gamma}{n} + \lambda\,\Big) && \text{(cf. Lemma 3.3.10).}
\end{aligned}$$

It follows that

$$A_5 \le c_{14}\Big\{\,\Big(\,|a_{i_1j_3}|^k + |a_{i_2j_4}|^k + |a_{i_3j_1}|^k + |a_{i_4j_2}|^k + 1\,\Big)\,\Big(\,\frac{\gamma}{n} + \lambda\,\Big) + \max_{1\,\le\,l\,\le\,4}\,\delta\big(\,c_{12}\,\gamma,\ n - l\,\big)\,\Big\}$$

and thus by reusing the inequality $|x|^{k+3} \le |x|^3 + |x|^{k+4}$ for $k \in \mathbb{N}$

$$A_4 \le c_{15}\,\frac{\gamma}{n}\,\Big[\,1 + \frac{1}{\lambda}\,\frac{\gamma}{n} + \frac{1}{\lambda}\,\max_{1\,\le\,l\,\le\,4}\,\delta\big(\,c_{12}\,\gamma,\ n - l\,\big)\,\Big]. \tag{3.6.17}$$

Overall, (3.6.7), (3.6.9), (3.6.10) and (3.6.17) yield the existence of a constant c_{16} with

$$\sup_{z\,\in\,\mathbb{R}}\,\Big|\,E\big(p^\lambda_{z,k}(\,T_A)\big) - \Phi\big(p^\lambda_{z,k}\big)\,\Big| \le c_{16}\,\frac{\gamma}{n}\,\Big[\,1 + \frac{1}{\lambda}\,\frac{\gamma}{n} + \frac{1}{\lambda}\,\max_{1\,\le\,l\,\le\,4}\,\delta\big(\,c_{12}\,\gamma,\ n - l\,\big)\,\Big] \tag{3.6.18}$$

for all matrices $A \in \mathscr{M}(n,\gamma)$, which have the property (3.6.4). For the remaining matrices $A \in \mathscr{M}(n,\gamma)$, on the other hand, we can show that

$$\begin{aligned}
&\sup_{z\,\in\,\mathbb{R}}\,\Big|\,E\big(\eta(\,T_A)\big) - \Phi\big(\eta\big)\Big|, \qquad \text{where } \eta(x) = p^\lambda_{z,k}(x) \text{ or } \eta(x) = |x|^k\,1_{(-\infty,\,z\,]}(x) && (3.6.19)\\
&\le E\big(\,|\,T_A\,|^k\,\big) + \Phi\big(\,|x|^k\,\big)\\
&\le c_{17} + c_{18}\,\frac{\gamma}{n} && \text{(analogous to (3.6.8) and cf. (2.1.7)).}
\end{aligned}$$

In the case of $\beta_A > \epsilon_0\,n$, we further obtain

$$\le \frac{c_{17}}{\epsilon_0}\,\frac{\beta_A}{n} + c_{18}\,\frac{\gamma}{n} \le \Big(\,\frac{c_{17}}{\epsilon_0} + c_{18}\,\Big)\,\frac{\gamma}{n}. \tag{3.6.20}$$

Moreover, in the case $n < n_0$ we get because of $\dfrac{\beta_A}{n} \ge \dfrac{1}{\sqrt{8\,n}}$ (cf. Lemma 3.1.18, a)):

$$\le c_{17}\,\sqrt{8\,n_0}\,\frac{\beta_A}{n_0} + c_{18}\,\frac{\gamma}{n} \le c_{17}\,\sqrt{8\,n_0}\,\frac{\beta_A}{n} + c_{18}\,\frac{\gamma}{n} \le \Big(\,c_{17}\,\sqrt{8\,n_0} + c_{18}\,\Big)\,\frac{\gamma}{n}. \tag{3.6.21}$$

Combining the estimates (3.6.18), (3.6.19), (3.6.20), (3.6.21) with (3.6.3), we obtain a constant c_{19} such that

$$\delta(\gamma,n) \le c_{19}\left\{\frac{\gamma}{n}\left[1+\frac{1}{\lambda}\frac{\gamma}{n}+\frac{1}{\lambda}\max_{1\le l\le 4}\delta\big(c_{12}\,\gamma,\, n-l\big)\right]+\lambda\right\} \qquad \text{for all } n\ge 6.$$

If we now choose $\lambda = 4\,c_{19}\,c_{12}\,\dfrac{\gamma}{n}$, we get

$$\delta(\gamma,n)\le c_{20}\,\frac{\gamma}{n}+\frac{1}{4\,c_{12}}\max_{1\le l\le 4}\delta\big(c_{12}\,\gamma,\, n-l\big) \qquad \text{for all } n\ge 6.$$

Since the inequality $n\le 3\,(n-l)$ is valid for $n\ge 6$ and $l\le 4$, we conclude

$$\frac{n}{\gamma}\,\delta(\gamma,n)\le c_{20}+\frac{3}{4}\max_{1\le l\le 4}\frac{n-l}{c_{12}\,\gamma}\,\delta\big(c_{12}\,\gamma,\, n-l\big) \qquad \text{for all } n\ge 6$$

and therefore

$$\sup_{\gamma>0}\frac{n}{\gamma}\,\delta(\gamma,n)\le c_{20}+\frac{3}{4}\max_{1\le l\le 4}\sup_{\gamma>0}\frac{n-l}{\gamma}\,\delta\big(\gamma,\, n-l\big) \qquad \text{for all } n\ge 6. \tag{3.6.22}$$

A repeated application of this inequality (3.6.22) then gives

$$\sup_{\gamma>0}\frac{n}{\gamma}\,\delta(\gamma,n)\le \sum_{s=0}^{n-6}\Big(\frac{3}{4}\Big)^{s}c_{20}+\frac{3}{4}\max_{2\le r\le 5}\sup_{\gamma>0}\frac{r}{\gamma}\,\delta\big(\gamma,\, r\big) \qquad \text{for all } n\ge 6. \tag{3.6.23}$$

If we now use the summation formula of the geometric series, we further get

$$\begin{aligned}\sup_{\gamma>0}\frac{n}{\gamma}\,\delta(\gamma,n) &\le 4\,c_{20}+\frac{3}{4}\max_{2\le r\le 5}\sup_{\gamma>0}\frac{r}{\gamma}\,\delta\big(\gamma,\, r\big)\\ &\le 4\,c_{20}+\Big(3\sqrt{3}\,c_{17}+\frac{3}{4}\,c_{18}\Big)<\infty \qquad \text{for all } n\ge 2.\end{aligned}$$

For the last inequality we proceeded as in the calculations in (3.6.19) and (3.6.21) with $n_0=6$ and $\eta(x)=|x|^k\,1_{(-\infty,\,z\,]}(x)$. □

3.6.24 Remark

For $k=1$, a slightly different proof of Theorem 3.6.1 with the remainder term $c\,\beta_A/n$ instead of $c\,\big(\beta_A/n+\eta_A/n\big)$ is sketched in [25], Proposition 5.7.

The following analogous result to Theorem 3.6.1, in which the terms $|\,\mathscr{T}_A|^k$ and $|x|^k$ are replaced by the terms $\mathscr{T}_A^k$ and x^k, can be derived from this theorem without much effort.

3.6.25 Corollary

Let $k \in \mathbb{N}$ be fixed.

Then there exists a constant $\mathcal{K}_8(k) > 0$ such that for all matrices A satisfying $\sigma_A > 0$

$$\begin{aligned}(3.6.26)\qquad & \sup_{z\in\mathbb{R}} \left| E\Big(\mathscr{T}_A^k\, 1_{(-\infty,\,z\,]}(\mathscr{T}_A)\Big) - \Phi\Big(x^k\, 1_{(-\infty,\,z\,]}(x)\Big)\right| \\ & \le \mathcal{K}_8(k)\left(\frac{\beta_A}{n} + \frac{1}{n}\sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^{k+4}\right).\end{aligned}$$

Proof:

We consider the two cases $z \le 0$ and $z \ge 0$:

$z \le 0$: $E\Big(\mathscr{T}_A^k\, 1_{(-\infty,\,z\,]}(\mathscr{T}_A)\Big) = (-1)^k\, E\Big(|\,\mathscr{T}_A|^k\, 1_{(-\infty,\,z\,]}(\mathscr{T}_A)\Big)$.

$z \ge 0$:

$$\begin{aligned} & E\Big(\mathscr{T}_A^k\, 1_{(-\infty,\,z\,]}(\mathscr{T}_A)\Big) \\ &= E\Big(\mathscr{T}_A^k\, 1_{(-\infty,\,0\,]}(\mathscr{T}_A)\Big) + E\Big(\mathscr{T}_A^k\, 1_{(\,0,\,z\,]}(\mathscr{T}_A)\Big) \\ &= E\Big(|\,\mathscr{T}_A|^k\, 1_{(-\infty,\,z\,]}(\mathscr{T}_A)\Big) + \big((-1)^k - 1\big)\, E\Big(|\,\mathscr{T}_A|^k\, 1_{(-\infty,\,0\,]}(\mathscr{T}_A)\Big).\end{aligned}$$

Since we can proceed completely analogously with the term $\Phi\Big(x^k\, 1_{(-\infty,\,z\,]}(x)\Big)$, the assertion follows from Theorem 3.6.1. □

Finally, a slightly more general version of Corollary 3.6.25 and Theorem 3.1.3 is also necessary for the further procedure.

3.6.27 Corollary

Let $k \in \mathbb{N}_0$ be fixed, and let $\mathcal{H}$ be defined according to (0.1.1). Then

$$(3.6.28)\qquad \sup_{h\in\mathcal{H}} \left| E\Big(\mathscr{T}_A^k\, h(\mathscr{T}_A)\Big) - \Phi\Big(x^k\, h(x)\Big)\right| \le \begin{cases} \mathcal{K}_1\, \dfrac{\beta_A}{n} & \text{for } k = 0, \\[2ex] \mathcal{K}_8(k)\left(\dfrac{\beta_A}{n} + \dfrac{1}{n}\displaystyle\sum_{i,j=1}^{n} |\,\hat{a}_{ij}\,|^{k+4}\right) & \text{for } k \ge 1. \end{cases}$$

Proof:

Proceed as in the proof of Corollary 1.4.8. □

3.7 Edgeworth expansions of first order

In this section we prove the Theorem 3.1.10 and a similar version with "first moment". The latter is needed in the next section to prove the Theorem 3.1.13. The main work to achieve these two goals is done by proving the following theorem.

3.7.1 Theorem

Let $m \in \{ 0,\, 1 \}$, $G_A \in \{ D_A,\, E_A \}$ be fixed and

$$e^m_{1,A}(z) = \int\limits_{-\infty}^{z} x^m \, e'_{1,A}(x)\, dx \qquad \text{(cf. Lemma 3.1.18, j) for } m = 1\text{)}.$$

Then there exist constants $\mathfrak{K}_1(m) > 0$ and $\mathfrak{K}_2(m) > 0$ such that for all matrices A satisfying $\sigma_A > 0$ and the conditions

$$\text{(3.7.2)} \qquad \sigma^2_{A'} \geq \frac{2}{3},$$

(3.7.3) there exists a constant $\mathfrak{L}(m) > 0$ such that

$$\big(\, |z|^m + 1 \, \big) \, \big| \, \Delta^2_y F_{\widetilde{B}}(z) \, \big| \leq \mathfrak{L}(m) \, \big(\, G^2_A + y^2 \, \big)$$

for all $z \in \mathbb{R}$, $0 \leq y \leq \dfrac{G_A}{\sigma_{A'}}$ and $\widetilde{B} \in N(8, \bar{A})$,

we have

$$\text{(3.7.4)} \qquad \sup_{z \in \mathbb{R}} \left| E\Big(T^m_{\bar{A}} \, 1_{(-\infty,\, z\,]}(\, T_{\bar{A}} \,) \Big) - e^m_{1,\bar{A}}(z) \right| \leq \Big(\mathfrak{K}_1(m) \, \mathfrak{L}(m) + \mathfrak{K}_2(m) \Big) \, G^2_A.$$

The case $m = 0$ and $G_A = D_A$ leads to the proof of Theorem 3.1.10 by the considerations in the following subsection 3.7.1.

The case $m = 1$ and $G_A = E_A$ is important for the next section. There we also need a slightly more general version of Theorem 3.7.1.

3.7.5 Corollary

Suppose that the conditions and notations of the previous Theorem 3.7.1 apply. Furthermore, let $\mathcal{H}$ be defined according to (0.1.1). Then

$$\text{(3.7.6)} \qquad \sup_{h \in \mathcal{H}} \left| E\Big(T^m_{\bar{A}} \, h(\, T_{\bar{A}} \,) \Big) - \int\limits_{\mathbb{R}} x^m \, h(x) \, e'_{1,\bar{A}}(x) \, dx \right| \leq \Big(\mathfrak{K}_1(m) \, \mathfrak{L}(m) + \mathfrak{K}_2(m) \Big) \, G^2_A.$$

Proof:

Proceed as in the proof of Corollary 1.4.8. $\square$

3.7.1 Proof of Theorem 3.1.10 using Theorem 3.7.1

According to Corollary 3.4.2, there exists a $0 < \epsilon_0 \leq 1$ such that the inequalities $\frac{2}{3} \leq \sigma_{A'}^2 \leq \frac{4}{3}$ hold for all matrices A satisfying $\sigma_A > 0$ and $\beta_A \leq \epsilon_0\, n$.

We thus prove the Theorem 3.1.10 by considering the following two cases:

1. Case: $\beta_A \leq \epsilon_0\, n$ and $n \geq n_0 = 10$.

First, we derive condition (3.7.3) for $m = 0$ and $G_A = D_A$ from condition (3.1.11). To do this, we use Proposition 3.4.17 (with $r = 2$ and $k = 4$) and obtain:

$$\begin{aligned} ||\, \Delta_y^2 F_{\widetilde{B}}\, || &\leq ||\, \Delta_{\sigma_{A'} y}^2 F_B\, || + c_1\, D_A^2 \\ &\leq \Big(\, \frac{4}{3}\, \mathcal{C}_1 + c_1 \Big)\, \big(\, D_A^2 + y^2\, \big) \qquad \text{for all } 0 \leq y \leq \frac{D_A}{\sigma_{A'}} \text{ and } \widetilde{B} \in N(8, \bar{A}). \end{aligned}$$

So because of $|z|^0 + 1 = 2$, we get $\mathfrak{L}(0) = 2\,\Big(\, \frac{4}{3}\, \mathcal{C}_1 + c_1 \Big)$.

Furthermore, we obtain from (3.4.14), applied to the functions $h = 1_{(\,-\infty,\, z\,]}, z \in \mathbb{R}$, that

$$||\, \mathscr{F}_A - e_{1,A}\, || \leq ||\, \mathscr{F}_{\bar{A}} - e_{1,\bar{A}}\, || + c_2\, D_A^2.$$

Therefore, the inequality (3.1.12) of Theorem 3.1.10 follows directly from the inequality (3.7.4) with $m = 0$, $G_A = D_A$ and $\mathfrak{L}(0) = 2\,\Big(\, \frac{4}{3}\, \mathcal{C}_1 + c_1 \Big)$.

2. Case: $\beta_A > \epsilon_0\, n$ or $n < n_0 = 10$.

Because of Lemma 3.1.18, b), we have

$$\text{(3.7.7)} \qquad \beta_A > \epsilon_0\, n \quad \Rightarrow \quad 1 \leq \frac{1}{\epsilon_0}\, \frac{\beta_A}{n} \leq \frac{1}{\epsilon_0}\, D_A$$

and because of Lemma 3.1.18, a) also

$$\text{(3.7.8)} \qquad n < n_0 \quad \Rightarrow \quad 1 \leq \frac{n_0}{n} \leq \frac{n_0}{n}\, 4\, \delta_A = 4\, n_0\, D_A^2.$$

Thus, it follows that

$$
\begin{aligned}
(3.7.9) \qquad || \mathscr{F}_A - e_{1,A} || &\le 1 + \frac{1}{15}\,\frac{\beta_A}{n} && \text{(cf. Lemma 3.1.18, f))} \\
&\le 1 + \frac{1}{15}\, D_A && \text{(cf. Lemma 3.1.18, b))} \\
&\le \left[\max\left\{ \frac{1}{\epsilon_0^2},\, 4\,n_0 \right\} + \frac{1}{15}\, \max\left\{ \frac{1}{\epsilon_0},\, \sqrt{4\,n_0} \right\} \right] D_A^2 .
\end{aligned}
$$

A sufficiently large $\mathcal{K}_3$ in Theorem 3.1.10 therefore ensures the validity of this theorem in the 2nd case as well. □

We need the rest of this section for the

3.7.2 Proof of Theorem 3.7.1

Due to Corollary 3.4.2, Lemma 3.4.5 (for $\beta_{\bar{A}} \le 125\,\beta_A$) and Corollary 3.3.7 there exists a $0 < \epsilon_0 \le 1$ and an $n_0 \ge 10$ such that for all matrices A satisfying $\sigma_A > 0$, $\beta_A \le \epsilon_0\, n$ and $n \ge n_0$ holds:

$$
(3.7.10) \qquad \begin{cases}
\left|\, \sigma_{A'}^2 - 1 \,\right| \le \dfrac{1}{3}, & \\[2ex]
|\bar{a}_{ij}| \le 1 & \text{for } 1 \le i, j \le n, \\[2ex]
|\, \mu_{\widetilde{B}} \,| \le 1 & \text{for } \widetilde{B} \in N(8, \bar{A}), \\[2ex]
\left|\, \sigma_{\widetilde{B}}^2 - 1 \,\right| \le \dfrac{1}{3} & \text{for } \widetilde{B} \in N(8, \bar{A}).
\end{cases}
$$

A suitable ϵ_0 would be, for example, $\epsilon_0 = \min\left\{ \epsilon_{0,1},\, \dfrac{\epsilon_{0,2}}{125} \right\}$, where $\epsilon_{0,1}$ is determined according to Corollary 3.4.2 and $\epsilon_{0,2}$ according to Corollary 3.3.7.

We now assume

$$
(3.7.11) \qquad \beta_A \le \epsilon_0\, n \quad \text{and} \quad n \ge n_0
$$

for this proof without loss of generality, so that (3.7.10) is valid.

If, on the other hand, $\beta_A > \epsilon_0\, n$ or $n < n_0$, we proceed as in the 2nd case of the proof of Theorem 3.1.10 above. We obtain for matrices A satisfying $\sigma_A > 0$ and $\sigma_{A'}^2 \ge \dfrac{2}{3}$:

$$\begin{aligned}
&\sup_{z\in\mathbb{R}} \left| E\Big(T_{\bar A}^m \, 1_{(-\infty,\,z\,]}(\,\bar A\,) \Big) - e_{1,\bar A}^m(z) \right| \\
&\le E\big(\,|\, T_{\bar A} \,|^m \,\big) + ||\, e_{1,\bar A}^m \,|| \\
&\le c_1(m) + c_2(m)\, \frac{\beta_{\bar A}}{n} && \text{(cf. Lemma 3.1.18, g), j) and } E\big(\,|\, T_{\bar A} \,|\,\big) \le 1) \\
&\le c_1(m) + c_3(m)\, \frac{\beta_A}{n} && \text{(cf. Lemma 3.4.5 and (3.7.2), i.e. } \sigma_{A'}^2 \ge \frac{2}{3}) \\
&\le \left[c_1(m) \, \max\left\{ \frac{1}{\epsilon_0^2},\, 4\, n_0 \right\} + c_3(m) \, \max\left\{ \frac{1}{\epsilon_0},\, \sqrt{4\, n_0} \right\} \right] G_A^2 && \text{(cf. Lemma 3.1.18, a), b)).}
\end{aligned}$$

A sufficiently large choice of $\mathfrak{K}_2(m)$ thus ensures the validity of the Theorem 3.7.1 for such matrices as well.

We further note that we use the definition

$$\lambda = \frac{G_A}{\sigma_{A'}}. \tag{3.7.12}$$

for this proof. It follows from (3.7.10) that

$$\sqrt{\frac{3}{4}}\, G_A \le \lambda \le \sqrt{\frac{3}{2}}\, G_A. \tag{3.7.13}$$

Since λ is therefore fixed in this proof, we will suppress in the following for the frequently used functions $q_z^{\lambda,m}$ (see (2.3.2)) and $f_z^{\lambda,m}$ (see (2.3.4)) the dependence on λ and set

$$q_{z,m} = q_z^{\lambda,m} \quad \text{and} \quad f_{z,m} = f_z^{\lambda,m}. \tag{3.7.14}$$

After these preliminary remarks, we now turn to the central part of the proof. As the proof of Theorem 1.1.3, this is divided into 4 parts or subsections.

3.7.2.1 Transition to smooth functions

In order to apply Stein's method in the next subsection 3.7.2.2, we first proceed analogously to chapter 1, section 1.2 and replace in the expression

$$\sup_{z\in\mathbb{R}} \left| E\Big(T_{\bar A}^m \, 1_{(-\infty,\,z\,]}(\,T_{\bar A}\,) \Big) - \int_{-\infty}^{z} x^m \, e_{1,\bar A}'(x)\, dx \right|$$

the functions $x^m \, 1_{(\,-\infty,\,z\,]}(x)$, $z \in \mathbb{R}$, by smoother functions. The functions $q_{z,m}$, $z \in \mathbb{R}$, prove to be suitable. We have

3.7.15 Proposition

$$\sup_{z\in\mathbb{R}} \left| E\Big(T_{\bar A}^m\, 1_{(-\infty,\, z\,]}(\, T_{\bar A}\,) \Big) - e_{1,\bar A}^m(z) \right|$$
$$\le \sup_{z\in\mathbb{R}} \left| E\Big(q_{z,m}(\, T_{\bar A}\,) \Big) - \int\limits_{\mathbb{R}} q_{z,m}(x)\, e'_{1,\bar A}(x)\, dx \right| + \Big(\mathfrak{L}(m) + C(m) \Big)\, G_A^2.$$

Proof:

a) Integration by parts for Lebesgue-Stieltjes integrals (cf. e.g. [2], Theorem 5.2.3) and use of $\dfrac{d}{dx}x^m = m$ for $m = 0$ and $m = 1$ gives

$$\left| E\Big(T_{\bar A}^m\, 1_{(-\infty,\, z\,]}(\, T_{\bar A}\,) \Big) - E\Big(q_{z-\lambda,m}(\, T_{\bar A}\,) \Big) \right|$$

$$= \left| \int\limits_{(\,z-\lambda,\ z\,]} x^m\, \frac{\big(\,x-(\,z-\lambda\,)\,\big)^2}{2\,\lambda^2}\, dF_{\bar A}(x) \ -\ \int\limits_{(\,z,\ z+\lambda\,]} x^m\, \frac{\big(\,(\,z+\lambda\,)-x\,\big)^2}{2\,\lambda^2}\, dF_{\bar A}(x) \right|$$

$$= \Bigg| z^m\, F_{\bar A}(z) \ -\ \int\limits_{z-\lambda}^{z} x^m\, \frac{\big(\,x-(\,z-\lambda\,)\,\big)}{\lambda^2}\, F_{\bar A}(x)\, dx \ -\ \int\limits_{z}^{z+\lambda} x^m\, \frac{\big(\,(\,z+\lambda\,)-x\,\big)}{\lambda^2}\, F_{\bar A}(x)\, dx$$
$$-\ m \int\limits_{z-\lambda}^{z} \frac{\big(\,x-(\,z-\lambda\,)\,\big)^2}{2\,\lambda^2}\, F_{\bar A}(x)\, dx \ +\ m \int\limits_{z}^{z+\lambda} \frac{\big(\,(\,z+\lambda\,)-x\,\big)^2}{2\,\lambda^2}\, F_{\bar A}(x)\, dx \Bigg|.$$

We now substitute in the first and third integral $y = z - x$, and in the second and fourth integral $y = x - z$. Because of $\displaystyle\int\limits_0^{\lambda} \frac{\lambda - y}{\lambda^2}\, dy = \frac{1}{2}$ we then obtain

$$= \Bigg| \int\limits_0^{\lambda} \frac{\lambda-y}{\lambda^2} \Big\{ (z+y)^m\, F_{\bar A}(z+y) - 2\, z^m\, F_{\bar A}(z) + (z-y)^m\, F_{\bar A}(z-y) \Big\}\, dy$$
$$-\ m \int\limits_0^{\lambda} \frac{\big(\,\lambda-y\,\big)^2}{2\,\lambda^2} \Big\{ F_{\bar A}(z+y) - F_{\bar A}(z-y) \Big\}\, dy \Bigg|$$

$$= \Bigg| \int\limits_0^{\lambda} \frac{\lambda-y}{\lambda^2}\, (z-y)^m\, \Delta_y^2 F_{\bar A}(z-y)\, dy \ +m \int\limits_0^{\lambda} \frac{\lambda-y}{\lambda^2}\, 2\, y \Big\{ F_{\bar A}(z+y) - F_{\bar A}(z) \Big\}\, dy$$
$$-\ m \int\limits_0^{\lambda} \frac{\big(\,\lambda-y\,\big)^2}{2\,\lambda^2} \Big\{ F_{\bar A}(z+y) - F_{\bar A}(z-y) \Big\}\, dy \Bigg|.$$

Next, we apply condition (3.7.3) to the first integral and Theorem 3.1.3 twice to each of the other two integrals. We get

$$\begin{aligned}
&\leq \int_0^\lambda \frac{\lambda - y}{\lambda^2}\, \mathfrak{L}(m)\,\big(\, G_A^2 + y^2\,\big)\, dy \,+ m \int_0^\lambda \frac{\lambda - y}{\lambda^2}\, 2\, y \,\Big\{\, 2\, \mathcal{K}_1\, \frac{\beta_{\bar A}}{n} + \frac{y}{\sqrt{2\,\pi}}\,\Big\}\, dy \\
&\quad + m \int_0^\lambda \frac{\big(\,\lambda - y\,\big)^2}{2\,\lambda^2}\,\Big\{\, 2\, \mathcal{K}_1\, \frac{\beta_{\bar A}}{n} + \frac{2\,y}{\sqrt{2\,\pi}}\,\Big\}\, dy \\
&= \frac{1}{2}\, \mathfrak{L}(m)\, G_A^2 \,+\, \mathfrak{L}(m)\, \frac{\lambda^2}{12} \,+\, m\, \Big[\, \frac{2}{3}\, \mathcal{K}_1\, \lambda\, \frac{\beta_{\bar A}}{n} \,+\, \frac{1}{\sqrt{2\,\pi}}\, \frac{\lambda^2}{6} \,+\, \frac{1}{3}\, \mathcal{K}_1\, \lambda\, \frac{\beta_{\bar A}}{n} \,+\, \frac{1}{\sqrt{2\,\pi}}\, \frac{\lambda^2}{12}\,\Big] \\
&\leq \Big(\, \mathfrak{L}(m) + c_1(m)\,\Big)\, G_A^2.
\end{aligned}$$

For the last inequality, we used $\lambda \leq \sqrt{\dfrac{3}{2}}\, G_A$ (cf. (3.7.13)) and $\dfrac{\beta_{\bar A}}{n} \leq 125\, \dfrac{\beta_A}{n} \leq 125\, G_A$ (cf. Lemma 3.4.5 and Lemma 3.1.18, b)).

b) Furthermore, we have

$$\begin{aligned}
&\Big|\, e^m_{1,\bar A}(z) - \int_{\mathbb{R}} q_{z-\lambda,m}(x)\, e'_{1,\bar A}(x)\, dx\,\Big| \\
&= \Big|\, \int_{\mathbb{R}} x^m\, 1_{(-\infty,\, z\,]}(x)\, e'_{1,\bar A}(x)\, dx - \int_{\mathbb{R}} q_{z-\lambda,m}(x)\, e'_{1,\bar A}(x)\, dx\,\Big|.
\end{aligned}$$

We estimate this as in part a) with $e_{1,\bar A}(x)$ instead of $F_{\bar A}(x)$. Using Lemma 2.4.2, c) and b) then yields

$$\begin{aligned}
&\leq \int_0^\lambda \frac{\lambda - y}{\lambda^2}\, \Big(\, m\, \Big(\, 4\, ||\, e'_{1,\bar A}\, || + ||\, x\, e''_{1,\bar A}(x)\, ||\,\Big) + \big(\, 1 - m\,\big)\, ||\, e''_{1,\bar A}\, ||\,\Big)\, y^2\, dy \\
&\quad + m \int_0^\lambda \frac{\lambda - y}{\lambda^2}\, 2\, y\, ||\, e'_{1,\bar A}\, ||\, y\, dy + m \int_0^\lambda \frac{\big(\,\lambda - y\,\big)^2}{2\,\lambda^2}\, ||\, e'_{1,\bar A}\, ||\, 2\, y\, dy \\
&= \Big(\, m\, \Big(\, 4\, ||\, e'_{1,\bar A}\, || + ||\, x\, e''_{1,\bar A}(x)\, ||\,\Big) + \big(\, 1 - m\,\big)\, ||\, e''_{1,\bar A}\, ||\,\Big)\, \frac{\lambda^2}{12} \\
&\quad + m\, \Big(\, ||\, e'_{1,\bar A}\, ||\, \frac{\lambda^2}{6} \,+\, ||\, e'_{1,\bar A}\, ||\, \frac{\lambda^2}{12}\,\Big) \\
&\leq c_2(m)\, G_A^2.
\end{aligned}$$

For the last inequality, we used $\lambda \leq \sqrt{\frac{3}{2}}\, G_A$ (cf. (3.7.13)) and Lemma 3.1.18, g) together with $\frac{\beta_{\bar{A}}}{n} \leq 125\, \frac{\beta_A}{n} \leq 125$ (cf. Lemma 3.4.5 and (3.7.11) and $\epsilon_0 \leq 1$).

c) The assertion now follows by putting the parts a) and b) together. □

3.7.2.2 Application of Stein's method

In this part, we carry out the central part of the proof of Theorem 3.7.1 using Stein's method. Similar to chapter 1, section 1.3 (cf. in particular (1.3.3)), we apply Stein's equation

$$f'_{z,m}(x) - x\, f_{z,m}(x) = q_{z,m}(x) - \Phi(q_{z,m}) \qquad \text{for } x \in \mathbb{R} \tag{3.7.16}$$

to the expression $E\big(\, q_{z,m}(\, T_{\bar{A}}\,)\big) - \Phi\big(\, q_{z,m}\,\big)$ and then expand the functions $f_{z,m}$ and $f'_{z,m}$ according to Taylor's theorem.

In these expansions, the statistics $T_1, \ldots, T_5$ (cf. (3.5.21)) come into play, which will always refer to the matrix $\bar{A}$ in this and the following parts of the proof. With this agreement we have

3.7.17 Proposition

For every $z \in \mathbb{R}$,

$$E\big(\, q_{z,m}(\, T_{\bar{A}}\,)\big) - \Phi\big(\, q_{z,m}\,\big) + \frac{1}{2}\, E\big(\, T_{\bar{A}}^3\,\big)\, E\big(\, T_{\bar{A}}\, f'_{z,m}(\, T_{\bar{A}}\,)\big) = R(z,\, m),$$

where

$$\begin{aligned} R(z,\, m) = {} & \frac{1}{2}\, E\big(\, T_{\bar{A}}^3\,\big)\, n\, E\bigg(\, \bar{a}_{I_1 J_1}\, \Delta T_4 \int_0^1 \Big(\, f''_{z,m}(\, T_3 + \Delta T_3 + t\, \Delta T_4\,) - f''_{z,m}(\, T_3\,)\Big)\, dt \bigg) \\ & - n\, E\bigg(\, \bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3 \int_0^1 \Big(\, f''_{z,m}(\, T_2 + \Delta T_2 + t\, \Delta T_3\,) - f''_{z,m}(\, T_2\,)\Big)\, dt \bigg) \\ & - n\, E\bigg(\, \bar{a}_{I_1 J_1}\, \big(\, \Delta T_4\,\big)^2 \int_0^1 (1-t)\, \Big(\, f''_{z,m}(\, T_3 + \Delta T_3 + t\, \Delta T_4\,) - f''_{z,m}(\, T_3\,)\Big)\, dt \bigg). \end{aligned}$$

Proof:

In the following, let $m \in \{\, 0,\, 1\,\}$ and $z \in \mathbb{R}$ be fixed. We can therefore set $q = q_{z,m}$ and $f = f_{z,m}$.

Using (3.7.16), we then get

$$E\big(\,q(\,T_{\bar{A}}\,)\,\big) - \Phi\big(\,q\,\big) = E\big(\,f'(\,T_{\bar{A}}\,)\,\big) - E\big(\,T_{\bar{A}}\,f(\,T_{\bar{A}}\,)\,\big).$$

We will now look at the last term on its own. Because of Lemma 3.5.25, a) we have

$$E\big(\,T_{\bar{A}}\,f(\,T_{\bar{A}}\,)\,\big) = n\,E\big(\,\bar{a}_{I_1 J_1}\,f(\,T_5\,)\,\big).$$

Next, we use Taylor's theorem and the sums $T_5 = T_4 + \Delta T_4$ and $T_4 = T_3 + \Delta T_3$ (cf. (3.5.21)). We expand f about T_4 and obtain

$$\begin{aligned}
&= n\,E\big(\,\bar{a}_{I_1 J_1}\,f(\,T_4\,)\,\big) + n\,E\big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,f'(\,T_4\,)\,\big)\\
&\quad + n\,E\bigg(\,\bar{a}_{I_1 J_1}\,\big(\,\Delta T_4\,\big)^2 \int_0^1 (1-t)\,f''(\,T_4 + t\,\Delta T_4\,)\,dt\bigg)\\
&= n\,E\big(\,\bar{a}_{I_1 J_1}\,f(\,T_4\,)\,\big) + n\,E\big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,f'(\,T_4\,)\,\big) + n\,E\bigg(\,\bar{a}_{I_1 J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,f''(\,T_3\,)\bigg)\\
&\quad + n\,E\bigg(\,\bar{a}_{I_1 J_1}\,\big(\,\Delta T_4\,\big)^2 \int_0^1 (1-t)\,\Big(\,f''(\,T_3 + \Delta T_3 + t\,\Delta T_4\,) - f''(\,T_3\,)\Big)\,dt\bigg).
\end{aligned}$$

For the first and third summand we get, because of Lemma 3.5.23, a) and b), and Lemma 3.5.26, a),

$$\begin{aligned}
&n\,E\big(\,\bar{a}_{I_1 J_1}\,f(\,T_4\,)\,\big) = n\,E\big(\,\bar{a}_{I_1 J_1}\,\big)\,E\big(\,f(\,T_4\,)\,\big) = 0,\\
&n\,E\bigg(\,\bar{a}_{I_1 J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,f''(\,T_3\,)\bigg) = n\,E\bigg(\,\bar{a}_{I_1 J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\bigg)\,E\big(\,f''(\,T_3\,)\,\big).
\end{aligned}$$

In the second summand, however, we expand f' about T_3 and use the sums $T_4 = T_3 + \Delta T_3$ and $T_3 = T_2 + \Delta T_2$ (cf. (3.5.21))

$$\begin{aligned}
&\quad n\,E\big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,f'(\,T_4\,)\,\big)\\
&= n\,E\big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,f'(\,T_3\,)\,\big)\\
&\quad + n\,E\bigg(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\Delta T_3 \int_0^1 f''(\,T_3 + t\,\Delta T_3\,)\,dt\bigg)\\
&= n\,E\big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,f'(\,T_3\,)\,\big) + n\,E\big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\Delta T_3\,f''(\,T_2\,)\,\big)
\end{aligned}$$

$$+\, n\, E\Big(\, \bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3 \int\limits_0^1 \Big(\, f''(\, T_2 + \Delta T_2 + t\, \Delta T_3\,) - f''(\, T_2\,)\Big)\, dt \Big)$$

$$= E\big(\, f'(\, T_3\,)\big) + n\, E\big(\, \bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\,\big)\, E\big(\, f''(\, T_2\,)\big)$$

$$+\, n\, E\Big(\, \bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3 \int\limits_0^1 \Big(\, f''(\, T_2 + \Delta T_2 + t\, \Delta T_3\,) - f''(\, T_2\,)\Big)\, dt \Big).$$

For the last equation, Lemma 3.5.23, b) and c), and Lemma 3.5.26, b) were applied.

Overall, taking into account Lemma 3.5.15, a) and Lemma 3.5.26, c), we have

$$(3.7.18) \qquad E\big(\, T_{\bar{A}}\, f(\, T_{\bar{A}}\,)\big)$$

$$= E\big(\, f'(\, T_{\bar{A}}\,)\big) + E\big(\, f''(\, T_{\bar{A}}\,)\big)\, \frac{1}{2}\, E\big(\, T_{\bar{A}}^3\,\big)$$

$$+\, n\, E\Big(\, \bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3 \int\limits_0^1 \Big(\, f''(\, T_2 + \Delta T_2 + t\, \Delta T_3\,) - f''(\, T_2\,)\Big)\, dt \Big)$$

$$+\, n\, E\Big(\, \bar{a}_{I_1 J_1}\, (\, \Delta T_4\,)^2 \int\limits_0^1 (1-t)\, \Big(\, f''(\, T_3 + \Delta T_3 + t\, \Delta T_4\,) - f''(\, T_3\,)\Big)\, dt \Big).$$

The degree of the derivative of f in the term $E\big(\, f''(\, T_{\bar{A}}\,)\big)$ will now be reduced again. If we argue as in (3.6.5) with f' instead of d and $\bar{A}$ instead of A, we get

$$E\big(\, T_{\bar{A}}\, f'(\, T_{\bar{A}}\,)\big)$$

$$= E\big(\, f''(T_{\bar{A}}\,)\big) + n\, E\Big(\, \bar{a}_{I_1 J_1}\, \Delta T_4 \int\limits_0^1 \Big(\, f''(\, T_3 + \Delta T_3 + t\, \Delta T_4\,) - f''(\, T_3\,)\Big)\, dt \Big).$$

By inserting the corresponding expression for $E\big(\, f''(T_{\bar{A}}\,)\big)$ into the equation (3.7.18), we obtain the assertion of the proposition. □

3.7.2.3 Deducing the correct expansion

This subsection deals with the term $\frac{1}{2}\, E\big(\, T_{\bar{A}}^3\,\big)\, E\big(\, T_{\bar{A}}\, f'_{z,m}(T_{\bar{A}})\big)$. We prove the following

3.7.19 Proposition

$$\sup_{z \in \mathbb{R}} \left|\, \frac{1}{2}\, E\big(\, T_{\bar{A}}^3\,\big)\, E\big(\, T_{\bar{A}}\, f'_{z,m}(\, T_{\bar{A}}\,)\big) - \frac{1}{2}\, \lambda_{1,\bar{A}}\, \Phi\big(\, x\, f'_{z,m}(x)\big) \right| \leq C(m)\, G_A^2.$$

This gives us the correct Edgeworth expansion in Proposition 3.7.17 except for an error of size $C(m)\,G_A^2$, since according to Lemma 2.2.5, c) and (2.2.2) holds:

$$\Phi\big(\,x\,f'_{z,m}(x)\,\big) = \frac{1}{3}\int\limits_{\mathbb{R}} q_{z,m}(x)\,\Big[\,(x^2-1)\,\psi(x)\,\Big]'\,dx.$$

Proof of Proposition 3.7.19:

Let $z \in \mathbb{R}$ be fixed. An application of the triangle inequality and of Lemma 3.1.18, d) yields

$$\begin{aligned}
&\Big|\,E\big(\,T_{\bar A}^3\,\big)\,E\big(\,T_{\bar A}\,f'_{z,m}(\,T_{\bar A}\,)\,\big) - \lambda_{1,\bar A}\,\Phi\big(\,x\,f'_{z,m}(x)\,\big)\,\Big| \\
&\le \Big|\,E\big(\,T_{\bar A}^3\,\big) - \lambda_{1,\bar A}\,\Big|\,E\big(\,\big|\,T_{\bar A}\,f'_{z,m}(\,T_{\bar A}\,)\,\big|\,\big) \\
&\quad + \frac{\beta_{\bar A}}{n}\,\Big|\,E\big(\,T_{\bar A}\,f'_{z,m}(\,T_{\bar A}\,)\,\big) - \Phi\big(\,x\,f'_{z,m}(x)\,\big)\,\Big| \\
&= A_1 + A_2.
\end{aligned}$$

Because of Proposition 3.2.2, a) and because of $|f'_{z,0}(x)| \le 1$ (cf. Corollary 2.1.12, a)) and $|f'_{z,1}(x)| \le |x| + \dfrac{1}{\sqrt{2\,\pi}}$ (cf. Corollary 2.1.17, c)), we obtain for A_1:

$$\begin{aligned}
A_1 &\le \frac{c_1(m)}{n^2}\,\beta_{\bar A} \\
&\le \frac{c_2(m)}{\sqrt{n}}\,\frac{\beta_A}{n} && \text{(cf. Lemma 3.4.5; } \sigma_{A'}^2 \ge \tfrac{2}{3} \text{ and } \sqrt{n} \le n) \\
&\le c_3(m)\,\Big(\,\frac{\beta_A}{n}\,\Big)^2 \le c_3(m)\,G_A^2 && \text{(cf. Lemma 3.1.18, a) and b)).}
\end{aligned}$$

To estimate A_2, the following remains to be shown due to $\dfrac{\beta_{\bar A}}{n} \le 125\,G_A$:

$$\Big|\,E\big(\,T_{\bar A}\,f'_{z,m}(\,T_{\bar A}\,)\,\big) - \Phi\big(\,x\,f'_{z,m}(x)\,\big)\,\Big| \le c_4(m)\,\frac{\beta_{\bar A}}{n}. \tag{3.7.20}$$

In the case $m = 0$ we use (3.7.16) and obtain $f'_{z,0}(x) = h_1(x) - h_2(x)$ with

$$h_1(x) = q_{z,0}(x) \in \mathcal{H} \quad \text{and} \quad h_2(x) = -\,\big(\,x\,f_{z,0}(x) - \Phi(q_{z,0})\,\big) \in \mathcal{H}$$

(cf. Lemma 2.1.13, a)). If we also take into account that we have $|\bar a_{ij}| \le 1$ for all $1 \le i,j \le n$ (cf. (3.7.10)) and therefore the inequality

$$(3.7.21) \qquad \frac{1}{n} \sum_{i,j=1}^{n} |\, \overline{a}_{ij} \,|^{k+4} \le \frac{\beta_{\overline{A}}}{n},$$

then we get (3.7.20) from Corollary 3.6.27, applied with $k = 1$ to h_1, h_2.

In the case $m = 1$ we also use (3.7.16) and obtain $f'_{z,1}(x) = x\, h_3(x) - x\, h_4(x) - h_5(x)$ with

$$h_3(x) = q_{z,0}(x) \in \mathcal{H},\ h_4(x) = -f_{z,1}(x) \in \mathcal{H} \quad \text{and} \quad h_5(x) = \Phi(q_{z,1}) \in \mathcal{H}$$

(cf. Lemma 2.1.13, b)). Thus (3.7.20) follows from Corollary 3.6.27, applied with $k = 2$ to h_3, h_4 and with $k = 1$ to h_5. □

3.7.2.4 The estimate of $R(z,\, m)$

In this subsection we finally show for the $R(z,\, m)$ from Proposition 3.7.17:

3.7.22 Proposition

$$\sup_{z \in \mathbb{R}} \big|\, R(z,\, m) \,\big| \le \Big(C_1(m)\, \mathfrak{L}(m) + C_2(m) \Big)\, G_A^2.$$

Proof:

In the following, let $\overline{I} = (\, I_1, \ldots, I_8 \,)$, $\breve{I} = (\, I_1, \ldots, I_4 \,)$ and analogously $\overline{i} = (\, i_1, \ldots, i_8 \,) \in M_8$, $\breve{i} = (\, i_1, \ldots, i_4 \,) \in M_4$. The symbols $\overline{J}$, $\breve{J}$, $\overline{j}$ and $\breve{j}$ are defined in the same way.

To prove Proposition 3.7.22, we show that there exist constants $c_1(m)$, $c_2(m)$, $c_3(m)$ and $c_4(m)$ such that

$$(3.7.23) \qquad \Big| E\Big(f''_{z,m}(\, T_2 + \Delta T_2 + t\, \Delta T_3 \,) - f''_{z,m}(\, T_2 \,) \,\Big|\, \overline{I} = \overline{i},\, \overline{J} = \overline{j} \Big) \Big|$$
$$\le \Big(c_1(m)\, \mathfrak{L}(m) + c_2(m) \Big)\, E\Big(|\, \Delta T_2 \,| + |\, \Delta T_3 \,| \,\Big|\, \overline{I} = \overline{i},\, \overline{J} = \overline{j} \Big)$$

for all $z \in \mathbb{R}$, $0 \le t \le 1$ and $\overline{i}, \overline{j} \in M_8$ with $(\overline{i},\, \overline{j}) \in M_{16}$,

$$(3.7.24) \qquad \Big| E\Big(f''_{z,m}(\, T_3 + \Delta T_3 + t\, \Delta T_4 \,) - f''_{z,m}(\, T_3 \,) \,\Big|\, \breve{I} = \breve{i},\, \breve{J} = \breve{j} \Big) \Big|$$
$$\le \Big(c_3(m)\, \mathfrak{L}(m) + c_4(m) \Big)\, E\Big(|\, \Delta T_3 \,| + |\, \Delta T_4 \,| \,\Big|\, \breve{I} = \breve{i},\, \breve{J} = \breve{j} \Big)$$

for all $z \in \mathbb{R}$, $0 \le t \le 1$ and $\breve{i}, \breve{j} \in M_4$ with $(\breve{i},\, \breve{j}) \in M_8$.

Because of Lemma 3.5.25, c) and Proposition 3.2.2, c) and

$$\Big(\frac{\beta_{\bar A}}{n}\Big)^2 \le c_5\Big(\frac{\beta_A}{n}\Big)^2 \le c_5\, G_A^2 \quad \text{and} \quad \frac{\delta_{\bar A}}{n} \le c_6\,\frac{\delta_A}{n} = c_6\, D_A^2 \le c_6\, G_A^2$$

(cf. Lemma 3.4.5 and Lemma 3.1.18, b)), Proposition 3.7.22 then follows from (3.7.23) and (3.7.24).

Since the proofs of (3.7.23) and (3.7.24) are very similar, we will only deal with the case (3.7.23) as an example.

For that we fix in the following the quantities $m \in \{0,\, 1\}$, $z \in \mathbb{R}$, $0 \le t \le 1$ and $\bar\imath, \bar\jmath \in M_8$ with $(\bar\imath,\, \bar\jmath) \in M_{16}$. We can therefore set $q = q_{z,m}$ and $f = f_{z,m}$.

Because of Lemma 3.5.27, b) is T_2, given $\overline{I} = \bar\imath$ and $\overline{J} = \bar\jmath$, distributed as

$$\text{(3.7.25)} \qquad \sum_{(i,j)\,\in S} \bar a_{ij} + T_{\widetilde B},$$

where

$$\text{(3.7.26)} \qquad S = \Big\{ (\, i_s,\, j_{s+4}\,) \,:\, 1 \le s \le 4 \Big\} \cup \Big\{ (\, i_s,\, j_{s-4}\,) \,:\, 5 \le s \le 8 \Big\}$$

and

(3.7.27) $\widetilde B$ is the $(n-l)\times(n-l)-$matrix obtained from $\bar A$ by **cancelling** the $l = \big|\{\, i_1, \dots, i_8\,\}\big|$ rows $i_1, \dots, i_8$ and the l columns $j_1, \dots, j_8$.

With the abbreviations

$$\begin{aligned} a &= \sum_{(i,j)\,\in S} \bar a_{ij}, \\ p &= E\Big(\Delta T_2 \,\Big|\, \overline{I} = \bar\imath, \overline{J} = \bar\jmath\Big) && (\Delta T_2 \in \sigma\big(\overline{I},\, \overline{J}\big), \text{ cf. Lemma 3.5.22, b)}), \\ q &= E\Big(\Delta T_3 \,\Big|\, \overline{I} = \bar\imath, \overline{J} = \bar\jmath\Big) && (\Delta T_3 \in \sigma\big(\overline{I},\, \overline{J}\big), \text{ cf. Lemma 3.5.22, c)}) \end{aligned}$$

remains to be shown:

$$\text{(3.7.28)} \qquad \begin{aligned} &\Big| E\Big(f''\big(T_{\widetilde B} + a + p + t\, q\big) - f''\big(T_{\widetilde B} + a\big)\Big)\Big| \\ &\le \Big(c_7(m)\, \mathfrak{L}(m) + c_8(m)\Big)\Big(|\,p\,| + |\,q\,|\Big). \end{aligned}$$

If we estimate the left-hand side of the inequality (3.7.28) using Lemma 2.3.6, b) - e) (with $x = T_{\widetilde{B}} + a$ and $y = p + t\,q$), we get

$$\begin{aligned}
&\Big| E\Big(f''\big(T_{\widetilde{B}} + a + p + t\,q \big) - f''\big(T_{\widetilde{B}} + a \big) \Big) \Big| \\
&\le E\Big(\Big| \left(f'' - q' \right) \left(T_{\widetilde{B}} + a + p + t\,q \right) - \left(f'' - q' \right) \left(T_{\widetilde{B}} + a \right) \Big| \Big) \\
&\quad + \Big| E\Big(q'\big(T_{\widetilde{B}} + a + p + t\,q \big) - q'\big(T_{\widetilde{B}} + a \big) \Big) \Big| \\
&\le c_9(m) \Big(|\,p\,| + |\,q\,| \Big) \Bigg\{ 1 + E\Big(\big| T_{\widetilde{B}} + a \big|^{2+m} \Big) + m \Big(|\,p\,| + |\,q\,| \Big) E\Big(1 + \big(T_{\widetilde{B}} + a \big)^2 \Big) \\
&\quad + \frac{1}{\lambda} \int_0^1 E\Big(\big| T_{\widetilde{B}} + a \big|^{1+m} 1_{(\,z,\; z\, + 2\,\lambda\,]}\big(T_{\widetilde{B}} + a + s\,p + s\,t\,q \big) \Big)\, ds \\
&\quad + \frac{1}{\lambda^2} \int_0^1 \Big| E\Big(\big(T_{\widetilde{B}} + a \big)^m \big(1_{(\,z + \lambda,\; z\, + 2\,\lambda\,]} - 1_{(\,z,\; z\, + \lambda\,]} \big) \\
&\qquad\qquad \big(T_{\widetilde{B}} + a + s\,p + s\,t\,q \big) \Big) \Big|\, ds \\
&\quad + \frac{m}{\lambda} \Bigg(P\big(z < T_{\widetilde{B}} + a + p + t\,q \le z\, + 2\,\lambda \big) \\
&\qquad\qquad + \int_0^1 P\big(z < T_{\widetilde{B}} + a + s\,p + s\,t\,q \le z\, + 2\,\lambda \big)\, ds \Bigg) \Bigg\} \\
&= c_9(m) \Big(|\,p\,| + |\,q\,| \Big) \Big(1 + A_1 + A_2 + A_3 + A_4 + A_5 \Big).
\end{aligned}$$

It therefore remains to estimate the A_i, $1 \le i \le 5$.

Due to $|\overline{a}_{ij}| \le 1$ for all $1 \le i, j \le n$, we first obtain

$$|\,a\,| \le 8, \quad |\,p\,| \le 16 \quad \text{and} \quad |\,q\,| \le 8.$$

Furthermore, since $|\,\mu_{\widetilde{B}}\,| \le 1$ and $\dfrac{2}{3} \le \sigma^2_{\widetilde{B}} \le \dfrac{4}{3}$ (cf. (3.7.10)), we also get

$$E\big(T^2_{\widetilde{B}} \big) = \sigma^2_{\widetilde{B}} + \mu^2_{\widetilde{B}} \le \frac{7}{3} \quad \text{and}$$

$$(3.7.29)\qquad E\big(\,|\,T_{\widetilde{B}}\,|^3\,\big) = E\big(\,|\,\sigma_{\widetilde{B}}\,\mathscr{T}_{\widetilde{B}} + \mu_{\widetilde{B}}\,|^3\,\big)$$

$$\le 4\,|\,\sigma_{\widetilde{B}}\,|^3\,E\big(\,|\,\mathscr{T}_{\widetilde{B}}\,|^3\,\big) + 4\,|\,\mu_{\widetilde{B}}\,|^3 \qquad ((0.1.5)\text{ with } \nu = 2,\ p = 3)$$

$$\le 4\left(\frac{4}{3}\right)^{3/2} E\big(\,|\,\mathscr{T}_{\widetilde{B}}\,|^4\,\big)^{3/4} + 4 \qquad (\text{Hölder's inequality})$$

$$\le 7\left(97\,\frac{\delta_{\widetilde{B}}}{n-l} + 3\right)^{3/4} + 4 \qquad (\text{cf. Proposition 3.2.2, d)}).$$

Next, we use the estimate

$$(3.7.30)\qquad \frac{\delta_{\widetilde{B}}}{n-l} \le 5^4\,\frac{\delta_{\bar{A}}}{n-l} \qquad (\text{cf. Lemma 3.3.10; } \sigma^2_{\widetilde{B}} \ge \frac{2}{3})$$

$$\le 10 \cdot 5^4 \cdot \frac{\delta_{\bar{A}}}{n} \qquad (\text{cf. Remark 3.4.24} \Rightarrow \frac{n}{n-l} \le 10)$$

$$\le 2 \cdot 5^5 \cdot \frac{\beta_{\bar{A}}}{n} \qquad (|\bar{a}_{ij}| \le 1 \text{ for all } 1 \le i,j \le n \Rightarrow \delta_{\bar{A}} \le \beta_{\bar{A}})$$

$$\le 2 \cdot 5^8 \cdot \frac{\beta_A}{n} \qquad (\text{cf. Lemma 3.4.5; } \sigma^2_{A'} \ge \frac{2}{3}).$$

Because of (3.7.11) and $\epsilon_0 \le 1$, we now conclude that $\dfrac{\beta_A}{n} \le 1$ and thus $E\big(\,|\,T_{\widetilde{B}}\,|^3\,\big) \le c_{10}$. Therefore, a repeated application of Hölder's inequality (0.1.5) with $\nu = 2$ and $p = 2$ ($p = 3$) shows that $A_1 \le c_{11}$ and $A_2 \le c_{12}$.

Furthermore, we obtain for all $w \in \{\,0,1,2\,\}$ and $b \in \mathbb{R}$

$$(3.7.31)\qquad E\Big(\,|\,T_{\widetilde{B}} + a\,|^w\,1_{(\,z,\ z\,+2\,\lambda\,]}\big(\,T_{\widetilde{B}} + b\,\big)\Big)$$

$$= E\Big(\,|\,\sigma_{\widetilde{B}}\,\mathscr{T}_{\widetilde{B}} + \mu_{\widetilde{B}} + a\,|^w\,1_{(\,z,\ z\,+2\,\lambda\,]}\big(\,\sigma_{\widetilde{B}}\,\mathscr{T}_{\widetilde{B}} + \mu_{\widetilde{B}} + b\,\big)\Big)$$

$$\le 3^{w-1}\,|\,\sigma_{\widetilde{B}}\,|^w\,E\Big(\,|\,\mathscr{T}_{\widetilde{B}}\,|^w\,1_{(\,z,\ z\,+2\,\lambda\,]}\big(\,\sigma_{\widetilde{B}}\,\mathscr{T}_{\widetilde{B}} + \mu_{\widetilde{B}} + b\,\big)\Big)$$

$$+\ 3^{w-1}\Big(\,|\,\mu_{\widetilde{B}}\,|^w + |\,a\,|^w\,\Big)\,E\Big(\,1_{(\,z,\ z\,+2\,\lambda\,]}\big(\,\sigma_{\widetilde{B}}\,\mathscr{T}_{\widetilde{B}} + \mu_{\widetilde{B}} + b\,\big)\Big)$$

$$\le 3^{w-1}\,|\,\sigma_{\widetilde{B}}\,|^w\left(c_{13}(w)\,\frac{\beta_A}{n} + \frac{1}{\sqrt{2\,\pi}}\,\frac{2\,\lambda}{\sigma_{\widetilde{B}}}\right)$$

$$+\ 3^{w-1}\Big(\,|\,\mu_{\widetilde{B}}\,|^w + |\,a\,|^w\,\Big)\left(c_{13}(0)\,\frac{\beta_A}{n} + \frac{1}{\sqrt{2\,\pi}}\,\frac{2\,\lambda}{\sigma_{\widetilde{B}}}\right)$$

$$\le c_{14}(w)\left\{\frac{\beta_A}{n} + \lambda\right\}$$

$$\le c_{15}(w)\,\lambda.$$

The first inequality is clear for $w = 0$ and $w = 1$. For $w = 2$ it follows from Hölder's inequality (0.1.5) with $\nu = 3$ and $p = 2$.

The second inequality is a consequence of Theorem 3.6.1, Theorem 3.1.3 and argumentations similar to those in (3.7.30). In (3.7.30) we estimate a term using the exponent $e = 4$ (i.e. $\delta_{\widetilde{B}}$). However, similar estimates also hold for the exponents $e = 3$ and $e = w + 4$.

The third inequality uses again $|\,\mu_{\widetilde{B}}\,| \leq 1$ and $\frac{2}{3} \leq \sigma^2_{\widetilde{B}} \leq \frac{4}{3}$ (cf. (3.7.10)), and $|\,a\,| \leq 8$.

The fourth and last inequality follows from $\frac{\beta_A}{n} \leq G_A$ (cf. Lemma 3.1.18, b)) and $G_A \leq \sqrt{\frac{4}{3}}\,\lambda$ (cf. (3.7.13)).

Therefore, we also have $A_3 \leq c_{16}$ and $A_5 \leq c_{17}$.

Finally, we estimate A_4. For the case $m = 0$ we get

$$\text{(3.7.32)} \qquad \Big|\, E\Big(\big(1_{(\,z+\lambda,\; z\,+2\,\lambda\,]} - 1_{(\,z,\; z\,+\lambda\,]} \big) \big(T_{\widetilde{B}} + a + s\,p + s\,t\,q \,\big) \Big) \Big|$$

$$\leq \sup_{\varrho\,\in\,\mathbb{R}} \;|\,\Delta^2_\lambda F_{\widetilde{B}}(\varrho)\,| \qquad\qquad (\varrho = z - a - s\,p - s\,t\,q)$$

$$\leq \mathfrak{L}(0)\,\big(\,G_A^2 + \lambda^2\,\big) \qquad\qquad \text{(cf. condition (3.7.3), fulfilled due to (3.7.12)).}$$

Thus, because of $G_A^2 \leq \frac{4}{3}\,\lambda^2$ (cf. (3.7.13)), we have

$$A_4 \leq \frac{7}{3}\,\mathfrak{L}(0).$$

For the case $m = 1$, on the other hand,

$$\text{(3.7.33)} \qquad \Big|\, E\Big(T_{\widetilde{B}}\,\big(1_{(\,z+\lambda,\; z\,+2\,\lambda\,]} - 1_{(\,z,\; z\,+\lambda\,]} \big) \big(T_{\widetilde{B}} + a + s\,p + s\,t\,q \,\big) \Big) \Big|$$

has to be estimated. To do this, we define again $\varrho = z - a - s\,p - s\,t\,q$ and obtain then by applying integration by parts for Lebesgue-Stieltjes integrals (cf. e.g. [2], Theorem 5.2.3)

$$= \Big|\, \int\limits_{(\,\varrho+\lambda,\;\varrho+2\,\lambda\,]} x\;\,dF_{\widetilde{B}}(x) \;-\; \int\limits_{(\,\varrho,\;\varrho+\lambda\,]} x\;\,dF_{\widetilde{B}}(x) \Big|$$

$$= \Big|\, \Delta^2_\lambda\big(\,\varrho\,F_{\widetilde{B}}(\varrho)\,\big) - \int\limits_{\varrho+\lambda}^{\varrho+2\,\lambda} F_{\widetilde{B}}(x)\,dx + \int\limits_{\varrho}^{\varrho+\lambda} F_{\widetilde{B}}(x)\,dx \,\Big|$$

$$\leq \Bigg| \varrho\, \Delta_\lambda^2 F_{\widetilde{B}}(\varrho) + 2\,\lambda\, F_{\widetilde{B}}(\,\varrho + 2\,\lambda\,) - 2\,\lambda\, F_{\widetilde{B}}(\,\varrho + \lambda\,)$$

$$- \int\limits_0^\lambda F_{\widetilde{B}}(\,v + \varrho + \lambda\,)\, dv + \int\limits_0^\lambda F_{\widetilde{B}}(\,v + \varrho\,)\, dv \Bigg| \qquad \text{(subst. } x = v + \varrho + t\,\lambda,\ t = 0\,;1)$$

$$\leq |\,\varrho\,| \left| \Delta_\lambda^2 F_{\widetilde{B}}(\varrho) \right| + 2\,\lambda \left| F_{\widetilde{B}}(\,\varrho + 2\,\lambda\,) - F_{\widetilde{B}}(\,\varrho + \lambda\,) \right|$$

$$+ \int\limits_0^\lambda \left| F_{\widetilde{B}}(\,v + \varrho + \lambda\,) - F_{\widetilde{B}}(\,v + \varrho\,) \right| dv.$$

If we now use condition (3.7.3) for the first summand and Theorem 3.1.3 as in (3.7.31) for the other two summands, we further get

$$\leq \mathfrak{L}(1)\,\big(\,G_A^2 + \lambda^2\,\big) + 3\,\lambda \Big\{\, c_{13}(0)\,\frac{\beta_A}{n} + \frac{1}{\sqrt{2\,\pi}}\,\frac{\lambda}{\sigma_{\widetilde{B}}}\,\Big\}$$

$$\leq \Big(\,\frac{7}{3}\,\mathfrak{L}(1) + c_{18}\,\Big)\,\lambda^2.$$

For the last inequality, again $\frac{2}{3} \leq \sigma_{\widetilde{B}}^2$ and $\frac{\beta_A}{n} \leq G_A$ (cf. Lemma 3.1.18, b)) and $G_A \leq \sqrt{\frac{4}{3}}\,\lambda$ (cf. (3.7.13)) were applied.

By using the triangle inequality, the estimates of (3.7.33) and (3.7.32) (with $\mathfrak{L}(1)$ instead of $\mathfrak{L}(0)$), and $|\,a\,| \leq 8$, we obtain $A_4 \leq c_{19}\,\mathfrak{L}(1) + c_{20}$ for the case $m = 1$. □

3.8 Edgeworth expansions of second order

The aim of this section is to prove Theorem 3.1.13.

To do this, we choose analogously to the proof of Theorem 3.7.1 (cf. (3.7.10)) a $0 < \epsilon_0 \leq 1$ and an $n_0 \geq 18$ such that that for all matrices A satisfying $\sigma_A > 0$, $\beta_A \leq \epsilon_0\, n$ and $n \geq n_0$ holds:

$$(3.8.1) \qquad \begin{cases} |\,\sigma_{A'}^2 - 1\,| \leq \frac{1}{3}, & \\ |\bar{a}_{ij}| \leq 1 & \text{for } 1 \leq i, j \leq n, \\ |\,\mu_Q\,| \leq 1 & \text{for } Q \in N(16, \hat{A}) \cup N(16, \bar{A}), \\ |\,\sigma_Q^2 - 1\,| \leq \frac{1}{3} & \text{for } Q \in N(16, \hat{A}) \cup N(16, \bar{A}). \end{cases}$$

As in the argumentation to (3.7.10), this follows from Corollary 3.4.2, Lemma 3.4.5 and Corollary 3.3.7.

In this section, therefore, we now assume without loss of generality that

$$\text{(3.8.2)} \qquad \frac{\beta_A}{n} \leq E_A \leq \epsilon_0 \quad \text{and} \quad n \geq n_0 \qquad \text{(cf. Lemma 3.1.18, b)).}$$

If, on the other hand, $E_A > \epsilon_0$ or $n < n_0$, we first proceed analogously to (3.7.7) and (3.7.8):

$$\text{(3.8.3)} \qquad E_A > \epsilon_0 \quad \Rightarrow \quad 1 \leq \frac{E_A}{\epsilon_0} \qquad \text{and}$$

$$\text{(3.8.4)} \qquad n < n_0 \quad \Rightarrow \quad 1 \leq \frac{n_0}{n} \leq \frac{n_0}{n}\, 4\, \delta_A = 4\, n_0\, D_A^2 \leq 4\, n_0\, E_A^2 \qquad \text{(cf. Lemma 3.1.18, a), b)).}$$

From this we get similar to (3.7.9) for matrices A with $E_A > \epsilon_0$ or $n < n_0$:

$$\begin{aligned} \text{(3.8.5)} \qquad & \| \mathscr{F}_A - e_{2,A} \| \\ & \leq 1 + \frac{1}{15} \frac{\beta_A}{n} + \frac{1}{2} \frac{\delta_A}{n} && \text{(cf. Lemma 3.1.18, h))} \\ & \leq 1 + \frac{1}{15} E_A + \frac{1}{2} E_A^2 && \text{(cf. Lemma 3.1.18, b))} \\ & \leq \left[\max\left\{ \frac{1}{\epsilon_0^3},\, 8\sqrt{n_0^3} \right\} + \frac{1}{15} \max\left\{ \frac{1}{\epsilon_0^2},\, 4\, n_0 \right\} + \frac{1}{2} \max\left\{ \frac{1}{\epsilon_0},\, 2\sqrt{n_0} \right\} \right] E_A^3. \end{aligned}$$

A sufficiently large $\mathcal{K}_6$ in Theorem 3.1.13 therefore ensures the validity of this theorem in the cases $E_A > \epsilon_0$ and $n < n_0$ as well.

Due to its length, we divide the following proof of Theorem 3.1.13 again into four parts, similar to the proof of Theorem 3.7.1 (or Theorem 1.1.3 for iid random variables).

3.8.1 Transition to smooth functions and replacing A with $\bar{A}$

In order to apply Stein's method in the next section 3.8.2, we start analogously to subsection 3.7.2.1 (or section 1.2) and replace in the expression

$$\sup_{z \in \mathbb{R}} \left| E\Big(1_{(-\infty,\, z\,]}(\, \mathscr{T}_A\,) \Big) - \int_{\mathbb{R}} 1_{(\,-\infty,\, z\,]}(x)\, e_{2,A}'(x)\, dx \right|$$

the functions $1_{(\,-\infty,\, z\,]}(x)$, $z \in \mathbb{R}$, by smoother functions. The functions r_z^η, $z \in \mathbb{R}$ (cf. (2.3.3)) with

(3.8.6) $\eta = E_A$

prove to be suitable. We have

3.8.7 Proposition

$$|| \mathscr{F}_A - e_{2,A} || \leq 3 \sup_{z \in \mathbb{R}} \left| E\Big(r_z^\eta(\mathscr{T}_A) \Big) - \int_{\mathbb{R}} r_z^\eta(x)\, e'_{2,A}(x)\, dx \right| + \Big(C_1\, \mathcal{C}_2 + C_2 \Big)\, E_A^3.$$

Proof:

In this proof, let $r_z = r_z^\eta$. Then for all $z \in \mathbb{R}$ holds

$$\begin{aligned}
&\frac{1}{2} \Big| \big(\mathscr{F}_A - e_{2,A} \big)(z + 2\,\eta) + \big(\mathscr{F}_A - e_{2,A} \big)(z + \eta) \Big| \\
&= \Big| -\ \big(\mathscr{F}_A - e_{2,A} \big)(z) - \frac{3}{2}\, \Delta_\eta^1 \big(\mathscr{F}_A - e_{2,A} \big)(z) - \frac{1}{2}\, \Delta_\eta^2 \big(\mathscr{F}_A - e_{2,A} \big)(z) \Big| \\
&= \left| \int_z^{z+3\eta} r'_z(x)\, P_\eta^2\big(x;\, z,\, \mathscr{F}_A - e_{2,A} \big)\, dx \right| \qquad \text{(cf. Lemma 2.3.13, c))} \\
&\leq \left| \int_z^{z+3\eta} r'_z(x) \Big(P_\eta^2\big(x;\, z,\, \mathscr{F}_A \big) - \mathscr{F}_A(x) \Big)\, dx \right| \\
&\quad + \left| \int_z^{z+3\eta} r'_z(x) \Big(\mathscr{F}_A(x) - e_{2,A}(x) \Big)\, dx \right| \\
&\quad + \left| \int_z^{z+3\eta} r'_z(x) \Big(e_{2,A}(x) - P_\eta^2\big(x;\, z,\, e_{2,A} \big) \Big)\, dx \right| \\
&\leq \sup_{z \in \mathbb{R}} \left| \int_{\mathbb{R}} r'_z(x)\, \mathscr{F}_A(x)\, dx - \int_{\mathbb{R}} r'_z(x)\, e_{2,A}(x)\, dx \right| \\
&\quad + \Big(28\, \mathcal{C}_2 + \frac{4}{3} \cdot 28\, || e'''_{2,A} || \Big)\, \eta^3 \int_z^{z+3\eta} \big| r'_z(x) \big|\, dx.
\end{aligned}$$

For the last inequality, we used condition (3.1.14) together with (3.8.6) and Lemma 2.4.2, a) for $F = e_{2,A}$. Furthermore, we exploited that r'_z is identical to zero outside of $[\, z,\, z + 3\,\eta \,]$ (cf. (2.3.14)).

The next step is

$$\leq \sup_{z\in\mathbb{R}} \left| E\Big(r_z(\mathscr{T}_A)\Big) - \int_{\mathbb{R}} r_z(x)\, e'_{2,A}(x)\, dx \right| + \Big(28\,\mathcal{C}_2 + 229 \Big)\, \eta^3 =: R$$

We obtained the first summand by an integration by parts for Lebesgue-Stieltjes integrals (cf. e.g. [2], Theorem 5.2.3) using $r_z(+\infty) = 0$, $\mathscr{F}_A(-\infty) = 0$ and $e_{2,A}(-\infty) = 0$.

For the second summand, we used the first integral of Lemma 2.3.13, c) again, taking into account the fact that r'_z is negative on the entire interval $[\,z,\, z + 3\,\eta\,]$. In addition, we utilized $||\, e'''_{2,A}\, || \leq 6,13$ (cf. Lemma 3.1.18, i) and b), and $E_A \leq \epsilon_0 \leq 1$).

Using the formula $\Delta_y^{k+1} G(x) = \Delta_y^k G(x+y) - \Delta_y^k G(x)$, we now get from

$$\frac{1}{2} \left| \big(\mathscr{F}_A - e_{2,A}\big)(\,z + \eta\,) + \big(\mathscr{F}_A - e_{2,A}\big)(\,z\,) \right| \leq R \qquad \text{for all } z \in \mathbb{R}$$

the inequalities

$$\frac{1}{2} \left| \Delta_\eta^1\big(\mathscr{F}_A - e_{2,A}\big)(\,z + \eta\,) + \Delta_\eta^1\big(\mathscr{F}_A - e_{2,A}\big)(\,z\,) \right| \leq 2\,R \qquad \text{and}$$

$$\frac{1}{2} \left| \Delta_\eta^2\big(\mathscr{F}_A - e_{2,A}\big)(\,z + \eta\,) + \Delta_\eta^2\big(\mathscr{F}_A - e_{2,A}\big)(\,z\,) \right| \leq 4\,R \qquad \text{for all } z \in \mathbb{R}.$$

If we then apply the formula

$$\begin{aligned} \big|\, G(x) \,\big| &= \frac{1}{2} \,\big|\, G(x) + G(x+\eta) + G(x) - G(x+\eta) \,\big| \\ &\leq \frac{1}{2} \,\big|\, G(x+\eta) + G(x) \,\big| + \frac{1}{2} \,\big|\, G(x+\eta) - G(x) \,\big| \end{aligned}$$

successively to the functions $G = \mathscr{F}_A - e_{2,A}$, $G = \Delta_\eta^1\big(\mathscr{F}_A - e_{2,A}\big)$ and $G = \Delta_\eta^2\big(\mathscr{F}_A - e_{2,A}\big)$, we obtain

$$\begin{aligned} \Big| \big(\mathscr{F}_A - e_{2,A}\big)(\,z\,) \Big| &\leq R + \frac{1}{2} \Big| \Delta_\eta^1\big(\mathscr{F}_A - e_{2,A}\big)(\,z\,) \Big| \\ &\leq 2\,R + \frac{1}{4} \Big| \Delta_\eta^2\big(\mathscr{F}_A - e_{2,A}\big)(\,z\,) \Big| \\ &\leq 3\,R + \frac{1}{8} \Big| \Delta_\eta^3\big(\mathscr{F}_A - e_{2,A}\big)(\,z\,) \Big| \\ &\leq 3\,R + \frac{1}{8} \Big| \Delta_\eta^3\, \mathscr{F}_A(z) \Big| + \frac{1}{8} \Big| \Delta_\eta^3\, e_{2,A}(z) \Big|. \end{aligned}$$

Using condition (3.1.14) and Lemma 2.4.1 for the second summand and Lemma 2.4.2, b) and $||\, e'''_{2,A}\, || \leq 6,13$ (cf. reasoning above) for the third summand, we finally get the proposition. □

For the further proof, we must now replace the matrix A with the truncated matrix $\bar{A}$. The following two propositions serve this purpose. Before doing so, it should be noted that in the entire section 3.8 we always use the definition

$$\lambda = \frac{E_A}{\sigma_{A'}} = \frac{\eta}{\sigma_{A'}}. \tag{3.8.8}$$

We then get because of $\frac{2}{3} \leq \sigma_{A'}^2 \leq \frac{4}{3}$ (cf. (3.8.1) and (3.8.2))

$$\sqrt{\frac{3}{4}}\, E_A \leq \lambda \leq \sqrt{\frac{3}{2}}\, E_A. \tag{3.8.9}$$

We point out the analogies between (3.8.8) and (3.7.12) as well as (3.8.9) and (3.7.13).

With these terms λ and η we now have:

3.8.10 Proposition

$$\begin{aligned} &\sup_{z\in\mathbb{R}} \left| E\Big(r_z^{\eta}(\,\mathscr{T}_A\,)\Big) - \int_{\mathbb{R}} r_z^{\eta}(x)\, e'_{2,A}(x)\, dx \right| \\ \leq &\sup_{z\in\mathbb{R}} \left| E\Big(r_z^{\lambda}(\,T_{\bar{A}}\,)\Big) - \int_{\mathbb{R}} r_z^{\lambda}(x)\, e'_{2,\bar{A}}(x)\, dx \right| + C\, E_A^3. \end{aligned}$$

Proof:

The proposition follows from (3.4.15) with $h = r_z^{\eta}$ taking into account

$$r_z^{\eta}(a\,x+b) = r_{(z-b)/a}^{\eta/a}(x) \quad \text{for all } z, x, b \in \mathbb{R} \text{ and } a > 0. \tag{3.8.11}$$

Since $a = \sigma_{A'}$ in (3.4.15), we get $\frac{\eta}{a} = \lambda$. □

The conditions (3.1.14) and (3.1.15) must also be transferred to the matrix $\bar{A}$. In the following, a weaker version of (3.1.14) is sufficient.

3.8.12 Proposition

a) From (3.1.14) follows:

$$\left|\, \Delta_y^3 F_{\widetilde{B}}(z) \,\right| \leq \Big(C_1\, \mathcal{C}_2 + C_2 \Big)\, \big(E_A^3 + y^3\, \big) \tag{3.8.13}$$

for all $z \in \mathbb{R}$, $0 \leq y \leq \lambda$ and $\widetilde{B} \in N(16, \bar{A})$.

b) From (3.1.15) follows:

$$(3.8.14) \qquad \big(\,|z|+1\,\big)\,\big|\,\Delta_y^2 F_{\widetilde{B}}(z)\,\big| \le \Big(\,C_3\,\mathcal{C}_3 + C_4\,\Big)\;\big(\,E_A^2 + y^2\,\big)$$

for all $z \in \mathbb{R}$, $0 \le y \le \lambda$ and $\widetilde{B} \in N(16, \bar{A})$.

Proof:

a) This follows from Lemma 2.4.1 and Proposition 3.4.17 (with $r = 3$ and $k = 5$).

b) We obtain this part from Proposition 3.4.17 (with $r = 2$ and $k = 4$), Lemma 3.1.18, b) (for $D_A^2 \le E_A^2$) and Proposition 3.4.19 (only for this Proposition 3.4.19 we need from (3.8.1) the inequalities $|\,\mu_Q\,| \le 1$ and $\big|\,\sigma_Q^2 - 1\,\big| \le \dfrac{1}{3}$ for $Q \in N(16, \hat{A})$). □

Given (3.8.13) and (3.8.14), we now have to estimate the expression

$$(3.8.15) \qquad \sup_{z\,\in\,\mathbb{R}} \left|\, E\Big(\, r_z^\lambda(\,T_{\bar{A}}\,)\Big) - \int_{\mathbb{R}} r_z^\lambda(x)\, e'_{2,\bar{A}}(x)\, dx \,\right|.$$

In these estimates, we suppress the λ of the functions r_z^λ (cf. (2.3.3)) and g_z^λ (cf. (2.3.4)) similar to (3.7.14) in the last section 3.7. We set:

$$(3.8.16) \qquad r_z = r_z^\lambda \quad \text{and} \quad g_z = g_z^\lambda.$$

3.8.2 Application of Stein's method

In order to estimate (3.8.15), we apply, as in the last section 3.7 (see in particular (3.7.16)), Stein's equation

$$(3.8.17) \qquad g_z'(x) - x\, g_z(x) = r_z(x) - \Phi(r_z) \qquad \text{for } x \in \mathbb{R}$$

to the expression $E\big(\, r_z(\,T_{\bar{A}}\,)\big) - \Phi\big(\, r_z\,\big)$ and then expand the functions g_z and g_z' according to Taylor's theorem.

In these expansions, the statistics $T_1, \ldots, T_5$ (cf. (3.5.21)) come into play, which, as in the proof of Theorem 3.7.1, will always refer to the matrix $\bar{A}$. With this agreement we have

3.8.18 Proposition

For every $z \in \mathbb{R}$,

$$E\big(r_z(T_{\bar{A}})\big) - \Phi\big(r_z\big) + \frac{1}{2}\,E\big(T_{\bar{A}}^3\big)\,E\big(T_{\bar{A}}\,g_z'(T_{\bar{A}})\big)$$
$$+ \left\{\frac{1}{6}\Big(E\big(T_{\bar{A}}^4\big) - 3\Big) - \frac{1}{4}\,E\big(T_{\bar{A}}^3\big)^2\right\} E\big(T_{\bar{A}}\,g_z''(T_{\bar{A}})\big) = R(z)$$

where

$$\begin{aligned}
R(z) = {} & \left\{\frac{1}{6}\Big(E\big(T_{\bar{A}}^4\big) - 3\Big) - \frac{1}{4}\,E\big(T_{\bar{A}}^3\big)^2\right\} n\,E\bigg(\bar{a}_{I_1J_1}\,\Delta T_4 \int_0^1 \Big(g_z'''(T_3 + \Delta T_3 + t\,\Delta T_4) \\
& \qquad\qquad - g_z'''(T_3)\Big)\,dt\bigg) \\
& + \frac{1}{2}\,E\big(T_{\bar{A}}^3\big)\,n\,E\bigg(\bar{a}_{I_1J_1}\,\Delta T_4\,\Delta T_3 \int_0^1 \Big(g_z'''(T_2 + \Delta T_2 + t\,\Delta T_3) - g_z'''(T_2)\Big)\,dt\bigg) \\
& + \frac{1}{2}\,E\big(T_{\bar{A}}^3\big)\,n\,E\bigg(\bar{a}_{I_1J_1}\,\big(\Delta T_4\big)^2 \int_0^1 (1-t)\,\Big(g_z'''(T_3 + \Delta T_3 + t\,\Delta T_4) - g_z'''(T_3)\Big)\,dt\bigg) \\
& - n\,E\bigg(\bar{a}_{I_1J_1}\,\frac{\big(\Delta T_4\big)^3}{2} \int_0^1 (1-t)^2\,\Big(g_z'''(T_3 + \Delta T_3 + t\,\Delta T_4) - g_z'''(T_3)\Big)\,dt\bigg) \\
& - n\,E\bigg(\bar{a}_{I_1J_1}\,\frac{\big(\Delta T_4\big)^2}{2}\,\Delta T_3 \int_0^1 \Big(g_z'''(T_2 + \Delta T_2 + t\,\Delta T_3) - g_z'''(T_2)\Big)\,dt\bigg) \\
& - n\,E\bigg(\bar{a}_{I_1J_1}\,\Delta T_4\,\big(\Delta T_3\big)^2 \int_0^1 (1-t)\,\Big(g_z'''(T_2 + \Delta T_2 + t\,\Delta T_3) - g_z'''(T_2)\Big)\,dt\bigg) \\
& - n\,E\bigg(\bar{a}_{I_1J_1}\,\Delta T_4\,\Delta T_3\,\Delta T_2 \int_0^1 \Big(g_z'''(T_1 + \Delta T_1 + t\,\Delta T_2) - g_z'''(T_1)\Big)\,dt\bigg).
\end{aligned}$$

Proof:

In the following, let $z \in \mathbb{R}$ be fixed and $r = r_z$, $g = g_z$. Using (3.8.17), we then get

$$E\big(r(T_{\bar{A}})\big) - \Phi\big(r\big) = E\big(g'(T_{\bar{A}})\big) - E\big(T_{\bar{A}}\,g(T_{\bar{A}})\big).$$

We will now look at the last term on its own. Because of Lemma 3.5.25, a) we have

$$E\big(T_{\bar{A}}\,g(T_{\bar{A}})\big) = n\,E\big(\bar{a}_{I_1J_1}\,g(T_5)\big).$$

Next, we use Taylor's theorem and the sums $T_5 = T_4 + \Delta T_4$ and $T_4 = T_3 + \Delta T_3$ (cf. (3.5.21)). We expand g about T_4 and obtain

$$= n\,E\big(\,\bar{a}_{I_1J_1}\,g(\,T_4\,)\,\big) + n\,E\Big(\,\bar{a}_{I_1J_1}\,\Delta T_4\,g'(\,T_4\,)\,\Big) + n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,g''(\,T_4\,)\,\Big)$$

$$+\,n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^3}{2}\int_0^1 (1-t)^2\,g'''(\,T_4 + t\,\Delta T_4\,)\,dt\Big)$$

$$\begin{aligned}
&= n\,E\big(\,\bar{a}_{I_1J_1}\,g(\,T_4\,)\,\big) + n\,E\Big(\,\bar{a}_{I_1J_1}\,\Delta T_4\,g'(\,T_4\,)\,\Big) + n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,g''(\,T_4\,)\,\Big)\\
&\quad + n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^3}{6}\,g'''(\,T_3\,)\Big)\\
&\quad + n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^3}{2}\int_0^1 (1-t)^2\,\Big(\,g'''(\,T_3 + \Delta T_3 + t\,\Delta T_4\,) - g'''(\,T_3\,)\,\Big)\,dt\Big).
\end{aligned} \tag{3.8.19}$$

For the first and fourth summand of (3.8.19) we get, because of Lemma 3.5.23, a) and b), and Lemma 3.5.26, a),

$$n\,E\big(\,\bar{a}_{I_1J_1}\,g(\,T_4\,)\,\big) = n\,E\big(\,\bar{a}_{I_1J_1}\,\big)\,E\big(\,g(\,T_4\,)\,\big) = 0,$$

$$n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^3}{6}\,g'''(\,T_3\,)\,\Big) = n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^3}{6}\,\Big)\,E\big(\,g'''(\,T_3\,)\,\big).$$

In the third summand of (3.8.19), however, we expand g'' about T_3 and use the sums $T_4 = T_3 + \Delta T_3$ and $T_3 = T_2 + \Delta T_2$ (cf. (3.5.21))

$$\begin{aligned}
&n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,g''(\,T_4\,)\,\Big)\\
&= n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,g''(\,T_3\,)\,\Big)\\
&\quad + n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,\Delta T_3\int_0^1 g'''(\,T_3 + t\,\Delta T_3\,)\,dt\Big)\\
&= n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,g''(\,T_3\,)\,\Big) + n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,\Delta T_3\,g'''(\,T_2\,)\,\Big)\\
&\quad + n\,E\Big(\,\bar{a}_{I_1J_1}\,\frac{\big(\,\Delta T_4\,\big)^2}{2}\,\Delta T_3\int_0^1 \Big(\,g'''(\,T_2 + \Delta T_2 + t\,\Delta T_3\,) - g'''(\,T_2\,)\,\Big)\,dt\Big)
\end{aligned}$$

$$= n\,E\Big(\,\bar{a}_{I_1 J_1}\,\frac{(\,\Delta T_4\,)^2}{2}\,\Big)\,E\big(\,g''(\,T_3\,)\,\big) + n\,E\Big(\,\bar{a}_{I_1 J_1}\,\frac{(\,\Delta T_4\,)^2}{2}\,\Delta T_3\,\Big)\,E\big(\,g'''(\,T_2\,)\,\big)$$

$$+\; n\,E\Big(\,\bar{a}_{I_1 J_1}\,\frac{(\,\Delta T_4\,)^2}{2}\,\Delta T_3 \int_0^1 \Big(\,g'''(\,T_2 + \Delta T_2 + t\,\Delta T_3\,) - g'''(\,T_2\,)\Big)\,dt\,\Big).$$

For the last equation, Lemma 3.5.23, b) and c) were applied.

In the second summand of (3.8.19) we expand g' also about T_3 and use the sums $T_4 = T_3 + \Delta T_3$ and $T_3 = T_2 + \Delta T_2$ (cf. (3.5.21)) again:

$$\begin{aligned}
&n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,g'(\,T_4\,)\Big)\\
&= n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,g'(\,T_3\,)\Big) + n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\Delta T_3\,g''(\,T_3\,)\Big)\\
&\quad + n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,(\,\Delta T_3\,)^2 \int_0^1 (1-t)\,g'''(\,T_3 + t\,\Delta T_3\,)\,dt\Big)\\
&= n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,g'(\,T_3\,)\Big) + n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\Delta T_3\,g''(\,T_3\,)\Big)\\
&\quad + n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\frac{(\,\Delta T_3\,)^2}{2}\,g'''(\,T_2\,)\Big)\\
&\quad + n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,(\,\Delta T_3\,)^2 \int_0^1 (1-t)\,\Big(\,g'''(\,T_2 + \Delta T_2 + t\,\Delta T_3\,) - g'''(T_2)\Big)\,dt\,\Big).
\end{aligned} \tag{3.8.20}$$

For the first and third summand of (3.8.20) we get, because of Lemma 3.5.23, b) and c), and Lemma 3.5.26, b),

$$n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,g'(\,T_3\,)\Big) = n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\Big)\,E\big(\,g'(\,T_3\,)\,\big) = E\big(\,g'(\,T_3\,)\,\big),$$

$$n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\frac{(\,\Delta T_3\,)^2}{2}\,g'''(\,T_2\,)\Big) = n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\frac{(\,\Delta T_3\,)^2}{2}\,\Big)\,E\big(\,g'''(\,T_2\,)\,\big).$$

Finally, in the second summand of (3.8.20), we expand g'' about T_2 and use the sums $T_3 = T_2 + \Delta T_2$ and $T_2 = T_1 + \Delta T_1$ (cf. (3.5.21))

$$\begin{aligned}
&n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\Delta T_3\,g''(\,T_3\,)\Big)\\
&= n\,E\Big(\,\bar{a}_{I_1 J_1}\,\Delta T_4\,\Delta T_3\,g''(\,T_2\,)\Big)
\end{aligned}$$

$$+ \, n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, \Delta T_2 \int_0^1 g'''(\, T_2 + t\, \Delta T_2\,)\, dt \Big)$$

$$= n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, g''(\, T_2\,) \Big) + n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, \Delta T_2\, g'''(\, T_1\,) \Big)$$

$$+ \, n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, \Delta T_2 \int_0^1 \Big(g'''(\, T_1 + \Delta T_1 + t\, \Delta T_2\,) - g'''(\, T_1\,) \Big)\, dt \Big)$$

$$= n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3 \Big)\, E\big(g''(\, T_2\,) \big) + n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, \Delta T_2 \Big)\, E\big(g'''(\, T_1\,) \big)$$

$$+ \, n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, \Delta T_2 \int_0^1 \Big(g'''(\, T_1 + \Delta T_1 + t\, \Delta T_2\,) - g'''(\, T_1\,) \Big)\, dt \Big).$$

For the last equation, Lemma 3.5.23, c) and d) were applied.

We summarize the previous results taking into account Lemma 3.5.15, a):

$$\begin{aligned}
& E\big(T_{\bar{A}}\, g(\, T_{\bar{A}}\,) \big) \\
&= E\big(g'(\, T_{\bar{A}}\,) \big) + E\big(g''(\, T_{\bar{A}}\,) \big) \Big\{ n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3 \Big) + n\, E\Big(\bar{a}_{I_1 J_1}\, \frac{(\, \Delta T_4\,)^2}{2} \Big) \Big\} \\
&\quad + E\big(g'''(\, T_{\bar{A}}\,) \big) \Big\{ n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \Delta T_3\, \Delta T_2 \Big) + n\, E\Big(\bar{a}_{I_1 J_1}\, \Delta T_4\, \frac{(\, \Delta T_3\,)^2}{2} \Big) \\
&\qquad\qquad + n\, E\Big(\bar{a}_{I_1 J_1}\, \frac{(\, \Delta T_4\,)^2}{2}\, \Delta T_3 \Big) + n\, E\Big(\bar{a}_{I_1 J_1}\, \frac{(\, \Delta T_4\,)^3}{6} \Big) \Big\} \\
&\quad + N(z),
\end{aligned}$$

where $N(z)$ is the sum of the last four terms of $R(z)$ (without minus sign!). An application of Lemma 3.5.26, c) and d) further yields

$$\begin{aligned}
\text{(3.8.21)} \qquad & E\big(r(\, T_{\bar{A}}\,) \big) - \Phi\big(r \big) + \frac{1}{2}\, E\big(T_{\bar{A}}^3 \big)\, E\big(g''(\, T_{\bar{A}}\,) \big) \\
&\qquad + \Big\{ \frac{1}{6} \big(E(T_{\bar{A}}^4) - 3 \big) \Big\}\, E\big(g'''(\, T_{\bar{A}}\,) \big) = -N(z).
\end{aligned}$$

The degree of the derivative of g'' in the term $E\big(g''(\, T_{\bar{A}}\,) \big)$ and of g''' in the term $E\big(g'''(\, T_{\bar{A}}\,) \big)$ will now be reduced again.

If we argue as for (3.7.18) with g' instead of f, we get

$$\begin{aligned}
&E\big(T_{\bar{A}}\, g'(T_{\bar{A}})\big)\\
&= E\big(g''(T_{\bar{A}})\big) + E\big(g'''(T_{\bar{A}})\big)\,\frac{1}{2}\,E\big(T_{\bar{A}}^3\big)\\
&\quad + n\,E\bigg(\bar{a}_{I_1 J_1}\,\Delta T_4\,\Delta T_3 \int_0^1 \Big(g'''(T_2 + \Delta T_2 + t\,\Delta T_3) - g'''(T_2)\Big)\,dt\bigg)\\
&\quad + n\,E\bigg(\bar{a}_{I_1 J_1}\,(\Delta T_4)^2 \int_0^1 (1-t)\Big(g'''(T_3 + \Delta T_3 + t\,\Delta T_4) - g'''(T_3)\Big)\,dt\bigg).
\end{aligned}$$

Next, we insert the formula just derived for $E\big(g''(T_{\bar{A}})\big)$ into the third term of (3.8.21) and obtain

$$\begin{aligned}
\frac{1}{2}\,E\big(T_{\bar{A}}^3\big)\,E\big(g''(T_{\bar{A}})\big) &= \frac{1}{2}\,E\big(T_{\bar{A}}^3\big)\,E\big(T_{\bar{A}}\,g'(T_{\bar{A}})\big)\\
&\quad - \Big(\frac{1}{2}\,E\big(T_{\bar{A}}^3\big)\Big)^2\,E\big(g'''(T_{\bar{A}})\big) - P_1(z),
\end{aligned} \tag{3.8.22}$$

where $P_1(z)$ is the sum of the second and third term of $R(z)$.

If we further argue as in (3.6.5) with g'' instead of d and $\bar{A}$ instead of A, and use Lemma 3.5.15, a) again, we get

$$\begin{aligned}
&E\big(T_{\bar{A}}\, g''(T_{\bar{A}})\big)\\
&= E\big(g'''(T_{\bar{A}})\big) + n\,E\bigg(\bar{a}_{I_1 J_1}\,\Delta T_4 \int_0^1 \Big(g'''\big(T_3 + \Delta T_3 + t\,\Delta T_4\big) - g'''(T_3)\Big)\,dt\bigg).
\end{aligned}$$

From this it finally follows for the two terms from (3.8.21) and (3.8.22), which contain the value $E\big(g'''(T_{\bar{A}})\big)$, that

$$\begin{aligned}
&\Big\{\frac{1}{6}\Big(E\big(T_{\bar{A}}^4\big) - 3\Big) - \frac{1}{4}\,E\big(T_{\bar{A}}^3\big)^2\Big\}\,E\big(g'''(T_{\bar{A}})\big)\\
&= \Big\{\frac{1}{6}\Big(E\big(T_{\bar{A}}^4\big) - 3\Big) - \frac{1}{4}\,E\big(T_{\bar{A}}^3\big)^2\Big\}\,E\big(T_{\bar{A}}\,g''(T_{\bar{A}})\big) - P_2(z),
\end{aligned}$$

where $P_2(z)$ is the first term of $R(z)$.

Overall, we have thus proved the assertion of Proposition 3.8.18. □

3.8.3 Deducing the correct expansion

This subsection deals with the two terms

$$\frac{1}{2}\,E\big(\,T_{\bar A}^3\,\big)\,E\big(\,T_{\bar A}\,g_z'(T_{\bar A})\,\big) \quad \text{and} \quad \bigg\{\,\frac{1}{6}\,\Big(\,E\big(\,T_{\bar A}^4\,\big)-3\,\Big)-\frac{1}{4}\,E\big(\,T_{\bar A}^3\,\big)^2\,\bigg\}\,E\big(\,T_{\bar A}\,g_z''(T_{\bar A})\,\big).$$

We prove the following

3.8.23 Proposition

Let $\lambda_{1,\bar A}$ be defined according to (3.1.8) and $\lambda_{2,\bar A}$ according to (3.1.9) with $\bar A$ instead of A or $\hat A$. Then:

a)
$$\sup_{z\in\mathbb{R}}\,\bigg|\,\frac{1}{2}\,E\big(\,T_{\bar A}^3\,\big)\,E\big(\,T_{\bar A}\,g_z'(\,T_{\bar A}\,)\,\big)-\frac{1}{2}\,\lambda_{1,\bar A}\,\Phi\big(\,x\,g_z'(x)\,\big)$$
$$+\,\frac{1}{12}\,\lambda_{1,\bar A}^2\int_{\mathbb{R}} x\,g_z'(x)\,(\,3\,x-x^3\,)\,\psi(x)\,dx\,\bigg|\le\Big(\,C_1\,\mathcal{C}_3+C_2\,\Big)\,E_A^3.$$

b)
$$\sup_{z\in\mathbb{R}}\,\bigg|\,\bigg\{\,\frac{1}{6}\,\Big(\,E\big(\,T_{\bar A}^4\,\big)-3\,\Big)-\frac{1}{4}\,E\big(\,T_{\bar A}^3\,\big)^2\,\bigg\}\,E\big(\,T_{\bar A}\,g_z''(T_{\bar A})\,\big)$$
$$-\,\bigg\{\,\frac{1}{6}\,\lambda_{2,\bar A}-\frac{1}{4}\,\lambda_{1,\bar A}^2\,\bigg\}\,\Phi\big(\,x\,g_z''(x)\,\big)\,\bigg|\le\Big(\,C_3\,\mathcal{C}_3+C_4\,\Big)\,E_A^3.$$

Using Lemma 2.2.5, c), d) and e), we obtain the correct Edgeworth expansion in Proposition 3.8.18 except for an error of size $\Big(\,(\,C_1+C_3\,)\,\mathcal{C}_3+\big(\,C_2+C_4\,\big)\,\Big)\,E_A^3$.

Proof of Proposition 3.8.23:

a) Let $z\in\mathbb{R}$ be fixed. We apply the triangle inequality and Lemma 3.1.18, d), and get

$$\bigg|\,E\big(\,T_{\bar A}^3\,\big)\,E\big(\,T_{\bar A}\,g_z'(\,T_{\bar A}\,)\,\big)-\lambda_{1,\bar A}\,\Phi\big(\,x\,g_z'(x)\,\big)+\frac{1}{6}\,\lambda_{1,\bar A}^2\int_{\mathbb{R}} x\,g_z'(x)\,(\,3\,x-x^3\,)\,\psi(x)\,dx\,\bigg|$$
$$=\bigg|\,E\big(\,T_{\bar A}^3\,\big)\,E\big(\,T_{\bar A}\,g_z'(\,T_{\bar A}\,)\,\big)-\lambda_{1,\bar A}\int_{\mathbb{R}} x\,g_z'(x)\,e_{1,\bar A}'(x)\,dx\,\bigg|$$
$$\le\Big|\,E\big(\,T_{\bar A}^3\,\big)-\lambda_{1,\bar A}\,\Big|\,E\Big(\,\big|\,T_{\bar A}\,g_z'(\,T_{\bar A}\,)\,\big|\,\Big)$$
$$+\,\frac{\beta_{\bar A}}{n}\,\bigg|\,E\big(\,T_{\bar A}\,g_z'(\,T_{\bar A}\,)\,\big)-\int_{\mathbb{R}} x\,g_z'(x)\,e_{1,\bar A}'(x)\,dx\,\bigg|$$
$$=A_1+A_2.$$

Because of Proposition 3.2.2, a) and because of $|\,g_z'\,| \le 1$ (cf. Corollary 2.1.12, a)), we obtain for A_1:

$$\begin{aligned} A_1 &\le \frac{11}{n^2}\,\beta_{\bar A} \\ &\le \frac{c_1}{n}\,\frac{\beta_A}{n} && \text{(cf. Lemma 3.4.5; } \sigma^2_{A'} \ge \tfrac{2}{3}) \\ &\le c_2\Big(\frac{\beta_A}{n}\Big)^3 \le c_2\,E_A^3 && \text{(cf. Lemma 3.1.18, a) and b)).} \end{aligned}$$

To estimate A_2, the following remains to be shown due to $\dfrac{\beta_{\bar A}}{n} \le 125\,E_A$:

$$\Big|\,E\big(T_{\bar A}\,g_z'(T_{\bar A})\big) - \int_{\mathbb{R}} x\,g_z'(x)\,e'_{1,\bar A}(x)\,dx\,\Big| \le \Big(c_3\,\mathcal{C}_3 + c_4\Big)\,E_A^2. \tag{3.8.24}$$

For this proof we use (3.8.17) and obtain $g_z'(x) = h_1(x) - h_2(x)$ with

$$h_1(x) = r_z(x) \in \mathcal{H} \quad \text{and} \quad h_2(x) = -\,\big(x\,g_z(x) - \Phi(r_z)\big) \in \mathcal{H} \tag{3.8.25}$$

(cf. Lemma 2.1.13, a)). Thus (3.8.24) follows from Corollary 3.7.5 with $m = 1$ and $G_A = E_A$. The condition (3.7.3) of Corollary 3.7.5 is fulfilled due to (3.8.14).

b) Let $z \in \mathbb{R}$ be fixed. A further application of the triangle inequality and of Lemma 3.1.18, d) yields

$$\begin{aligned} &\Big|\Big\{\frac16\Big(E\big(T_{\bar A}^4\big) - 3\Big) - \frac14\,E\big(T_{\bar A}^3\big)^2\Big\}\,E\big(T_{\bar A}\,g_z''(T_{\bar A})\big) \\ &\qquad - \Big\{\frac16\,\lambda_{2,\bar A} - \frac14\,\lambda_{1,\bar A}^2\Big\}\,\Phi\big(x\,g_z''(x)\big)\Big| \\ &\le \Big|\Big[\Big\{\frac16\Big(E\big(T_{\bar A}^4\big) - 3\Big) - \frac14\,E\big(T_{\bar A}^3\big)^2\Big\} - \Big\{\frac16\,\lambda_{2,\bar A} - \frac14\,\lambda_{1,\bar A}^2\Big\}\Big]\,E\big(T_{\bar A}\,g_z''(T_{\bar A})\big)\Big| \\ &\qquad + \Big|\Big\{\frac16\,\lambda_{2,\bar A} - \frac14\,\lambda_{1,\bar A}^2\Big\}\Big(E\big(T_{\bar A}\,g_z''(T_{\bar A})\big) - \Phi\big(x\,g_z''(x)\big)\Big)\Big| \\ &\le \Big(\frac16\,\Big|E\big(T_{\bar A}^4\big) - 3 - \lambda_{2,\bar A}\Big| + \frac14\,\Big|E\big(T_{\bar A}^3\big)^2 - \lambda_{1,\bar A}^2\Big|\Big)\,E\big(\big|T_{\bar A}\,g_z''(T_{\bar A})\big|\big) \\ &\qquad + \Big(\frac{19}{6}\,\frac{\delta_{\bar A}}{n} + \frac14\Big(\frac{\beta_{\bar A}}{n}\Big)^2\Big)\,\Big|E\big(T_{\bar A}\,g_z''(T_{\bar A})\big) - \Phi\big(x\,g_z''(x)\big)\Big|. \end{aligned} \tag{3.8.26}$$

Because of Proposition 3.2.2, b), we now have

$$\frac{1}{6}\Big|\,E\big(\,T_{\bar A}^4\,\big)-3-\lambda_{2,\bar A}\,\Big| = \frac{1}{6}\Big|\,R_{4,\bar A}\,\Big| \le \frac{1}{6}\,\frac{336}{n^2}\,\delta_{\bar A} = \frac{56}{n}\,\frac{\delta_{\bar A}}{n}. \tag{3.8.27}$$

Furthermore, if we apply the inequality

$$|\,a^2-b^2\,| = |\,a-b\,|\,|\,a+b\,| \le |\,a-b\,|\,\big(\,|\,a\,|+|\,b\,|\,\big) \qquad \text{for } a,b\in\mathbb{R},$$

and use Proposition 3.2.2, a) and c), and once more Lemma 3.1.18, d), we get

$$\begin{aligned} \frac{1}{4}\Big|\,E\big(\,T_{\bar A}^3\,\big)^2-\lambda_{1,\bar A}^2\,\Big| &\le \frac{1}{4}\Big|\,E\big(\,T_{\bar A}^3\,\big)-\lambda_{1,\bar A}\,\Big|\Big(\,\big|\,E\big(\,T_{\bar A}^3\,\big)\,\big|+|\,\lambda_{1,\bar A}\,|\,\Big) \\ &\le \frac{1}{4}\,\frac{11}{n^2}\,\beta_{\bar A}\,\Big(\,5\,\frac{\beta_{\bar A}}{n}+\frac{\beta_{\bar A}}{n}\,\Big) \\ &\le \frac{17}{n}\,\Big(\,\frac{\beta_{\bar A}}{n}\,\Big)^2. \end{aligned} \tag{3.8.28}$$

We now insert (3.8.27) and (3.8.28) into (3.8.26). Because of Lemma 3.4.5 and Lemma 3.1.18, b), we then obtain for (3.8.26) further

$$\begin{aligned} &\le \bigg(\,\frac{56}{n}\,\frac{\delta_{\bar A}}{n}+\frac{17}{n}\,\Big(\,\frac{\beta_{\bar A}}{n}\,\Big)^2\,\bigg)\,E\big(\,|\,T_{\bar A}\,g_z''(T_{\bar A})\,|\,\big) \\ &\quad + \bigg(\,\frac{19}{6}\,\frac{\delta_{\bar A}}{n}+\frac{1}{4}\,\Big(\,\frac{\beta_{\bar A}}{n}\,\Big)^2\,\bigg)\,\Big|\,E\big(\,T_{\bar A}\,g_z''(T_{\bar A})\,\big)-\Phi\big(\,x\,g_z''(x)\,\big)\,\Big| \\ &\le \Big(\,\frac{c_5}{n}\,E_A^2\,\Big)\,E\big(\,|\,T_{\bar A}\,g_z''(T_{\bar A})\,|\,\big)+\Big(\,c_6\,E_A^2\,\Big)\,\Big|\,E\big(\,T_{\bar A}\,g_z''(T_{\bar A})\,\big)-\Phi\big(\,x\,g_z''(x)\,\big)\,\Big| \\ &= A_3+A_4. \end{aligned} \tag{3.8.29}$$

To estimate A_3, $|\,g_z''\,|$ must be estimated. To do this, we differentiate the equation (3.8.17) on both sides and obtain

$$g_z''(x) = g_z(x)+x\,g_z'(x)+r_z'(x) \qquad \text{for } x\in\mathbb{R}. \tag{3.8.30}$$

Due to Corollary 2.1.12, a) and due to inequality (2.3.18) and (3.8.9), we then get

$$|\,g_z''(x)\,| \le \frac{\sqrt{2\pi}}{4}+|\,x\,|+\sqrt{\frac{3}{4}}\,\frac{1}{E_A} \qquad \text{for } x\in\mathbb{R}.$$

From this follows

$$E\big(\,|\,T_{\bar A}\,g_z''(T_{\bar A})\,|\,\big) \le \Big(\,\frac{2}{3}+\frac{7}{8}\,\frac{1}{E_A}\,\Big)\,E\big(\,|\,T_{\bar A}\,|\,\big)+E\big(\,|\,T_{\bar A}\,|^2\,\big) \le \frac{5}{3}+\frac{7}{8}\,\frac{1}{E_A}$$

and therefore

$$
\begin{aligned}
A_3 &\le \frac{c_5}{n}\, E_A^2 \Big(\frac{5}{3} + \frac{7}{8}\,\frac{1}{E_A} \Big) \\
&\le \frac{5}{3}\,\frac{c_5}{\sqrt{n}}\, E_A^2 + \frac{7}{8}\,\frac{c_5}{n}\, E_A && \text{(since } \sqrt{n} \le n) \\
&\le \frac{5}{3}\, 2^{5/6}\, c_5\, E_A^3 + \frac{7}{8}\, 2^{5/3}\, c_5\, E_A^3 \le 6\, c_5\, E_A^3 && \text{(cf. Lemma 3.1.18, a)).}
\end{aligned}
$$

To estimate A_4, we show

$$
\text{(3.8.31)} \qquad \Big| E\big(T_{\bar{A}}\, g_z''(T_{\bar{A}}) \big) - \Phi\big(x\, g_z''(x) \big) \Big| \le \Big(c_7\, \mathcal{C}_3 + c_8 \Big)\, E_A.
$$

To do this, we use the following representation of $x\, g_z''(x)$, where $x \in \mathbb{R}$:

$$
\begin{aligned}
x\, g_z''(x) &= x \Big(g_z(x) + x\, g_z'(x) + r_z'(x) \Big) && \text{(cf. (3.8.30))} \\
&= x\, g_z(x) + x^2 \big(x\, g_z(x) + r_z(x) - \Phi(r_z) \big) + x\, r_z'(x) && \text{(cf. (3.8.17))} \\
&= \Phi(r_z) - \Big(- \big(x\, g_z(x) - \Phi(r_z) \big) \Big) - x^2 \Big(- \big(x\, g_z(x) - \Phi(r_z) \big) \Big) + x^2\, r_z(x) + x\, r_z'(x).
\end{aligned}
$$

We note that as in (3.8.25), $\Big(- \big(x\, g_z(x) - \Phi(r_z) \big) \Big) \in \mathcal{H}$ and $r_z(x) \in \mathcal{H}$.

Thus, due to Corollary 3.6.27 for $k = 0$ and $k = 2$, and due to (3.7.21) and $\dfrac{\beta_{\bar{A}}}{n} \le 125\, E_A$, it suffices to carry out the following estimate for the proof of (3.8.31):

$$
\text{(3.8.32)} \quad
\begin{aligned}
\Big| E\big(T_{\bar{A}}\, r_z'(T_{\bar{A}}) \big) - \Phi\big(x\, r_z'(x) \big) \Big| &\le \Big| E\big(T_{\bar{A}}\, r_z'(T_{\bar{A}}) \big) - \int_{\mathbb{R}} x\, r_z'(x)\, e_{1,\bar{A}}'\, dx \Big| \\
&\quad + \frac{\beta_{\bar{A}}}{6\, n} \int_{\mathbb{R}} \big| r_z'(x) \big| \Big| x\, (\, 3\, x - x^3\,)\, \psi(x) \Big|\, dx.
\end{aligned}
$$

We now utilize some properties of r_z' from Lemma 2.3.13, d) and from (2.3.18). Because of Corollary 3.7.5 (with $m = 1$ and $G_A = E_A$), which we can use due to (3.8.14), and because of Lemma 2.2.8, g), we then further obtain

$$
\begin{aligned}
&\le 2 \Big(\frac{3}{4}\, \frac{1}{\lambda} \Big) \Big(c_9\, \mathcal{C}_3 + c_{10} \Big)\, E_A^2 + \frac{\beta_{\bar{A}}}{6\, n} \big((\, z + 3\, \lambda\,) - z\, \big) \Big(\frac{3}{4}\, \frac{1}{\lambda} \Big) \frac{1}{2} \\
&\le \Big(c_{11}\, \mathcal{C}_3 + c_{12} \Big)\, E_A.
\end{aligned}
$$

For the last inequality, we used (3.8.9), i.e. $\dfrac{3}{4}\,\dfrac{1}{\lambda} \le \sqrt{\dfrac{3}{4}}\,\dfrac{1}{E_A}$, and again $\dfrac{\beta_{\bar{A}}}{n} \le 125\, E_A$.

In total, (3.8.31) and thus part b) is proven. □

3.8.4 The estimate of $R(z)$

In this subsection we finally show for the $R(z)$ from Proposition 3.8.18:

3.8.33 Proposition

$$\sup_{z\in\mathbb{R}} \big|\, R(z) \,\big| \;\le\; \Big(C_1\,\mathcal{C}_2 + C_2\,\mathcal{C}_3 + C_3 \Big)\, E_A^3 .$$

Proof:

In the following, let $\underline{I} = (\, I_1, \ldots, I_{16}\,)$, $\overline{I} = (\, I_1, \ldots, I_8\,)$, $\breve{I} = (\, I_1, \ldots, I_4\,)$ and analogously $\underline{i} = (\, i_1, \ldots, i_{16}\,) \in M_{16}$, $\bar{i} = (\, i_1, \ldots, i_8\,) \in M_8$, $\breve{i} = (\, i_1, \ldots, i_4\,) \in M_4$. The symbols $\underline{J}$, $\overline{J}$, $\breve{J}$, $\underline{j}$, $\bar{j}$ and $\breve{j}$ are defined in the same way.

To prove Proposition 3.8.33, we show that there exist constants $c_1, \ldots, c_9$ such that

$$\begin{aligned}(3.8.34)\qquad & \Big|\, E\Big(g_z'''(\, T_1 + \Delta T_1 + t\,\Delta T_2\,) - g_z'''(\, T_1\,) \,\Big|\, \underline{I} = \underline{i},\, \underline{J} = \underline{j} \Big) \Big| \\ & \le\; \Big(c_1\,\mathcal{C}_2 + c_2\,\mathcal{C}_3 + c_3 \Big)\, E\Big(\big|\, \Delta T_1 \,\big| + \big|\, \Delta T_2 \,\big| \,\Big|\, \underline{I} = \underline{i},\, \underline{J} = \underline{j} \Big)\end{aligned}$$

for all $z \in \mathbb{R}$, $0 \le t \le 1$ and $\underline{i}, \underline{j} \in M_{16}$, which have the properties (w1), (w2), (w3) from (3.5.19),

$$\begin{aligned}(3.8.35)\qquad & \Big|\, E\Big(g_z'''(\, T_2 + \Delta T_2 + t\,\Delta T_3\,) - g_z'''(\, T_2\,) \,\Big|\, \overline{I} = \bar{i},\, \overline{J} = \bar{j} \Big) \Big| \\ & \le\; \Big(c_4\,\mathcal{C}_2 + c_5\,\mathcal{C}_3 + c_6 \Big)\, E\Big(\big|\, \Delta T_2 \,\big| + \big|\, \Delta T_3 \,\big| \,\Big|\, \overline{I} = \bar{i},\, \overline{J} = \bar{j} \Big)\end{aligned}$$

for all $z \in \mathbb{R}$, $0 \le t \le 1$ and $\bar{i}, \bar{j} \in M_8$ with $(\bar{i},\, \bar{j}) \in M_{16}$,

$$\begin{aligned}(3.8.36)\qquad & \Big|\, E\Big(g_z'''(\, T_3 + \Delta T_3 + t\,\Delta T_4\,) - g_z'''(\, T_3\,) \,\Big|\, \breve{I} = \breve{i},\, \breve{J} = \breve{j} \Big) \Big| \\ & \le\; \Big(c_7\,\mathcal{C}_2 + c_8\,\mathcal{C}_3 + c_9 \Big)\, E\Big(\big|\, \Delta T_3 \,\big| + \big|\, \Delta T_4 \,\big| \,\Big|\, \breve{I} = \breve{i},\, \breve{J} = \breve{j} \Big)\end{aligned}$$

for all $z \in \mathbb{R}$, $0 \le t \le 1$ and $\breve{i}, \breve{j} \in M_4$ with $(\breve{i},\, \breve{j}) \in M_8$.

Because of Lemma 3.5.25, c) and Proposition 3.2.2, c) and d), and because of

$$\frac{\delta_{\bar{A}}}{n}\,\frac{\beta_{\bar{A}}}{n} \le c_{10}\,\frac{\delta_A}{n}\,\frac{\beta_A}{n} \le c_{10}\,E_A^3 \quad\text{ and }\quad \Big(\frac{\beta_{\bar{A}}}{n}\Big)^3 \le c_{11}\,\Big(\frac{\beta_A}{n}\Big)^3 \le c_{11}\,E_A^3$$

(cf. Lemma 3.4.5 and Lemma 3.1.18, b)), Proposition 3.8.33 then follows from (3.8.34), (3.8.35) and (3.8.36).

Since the proofs of (3.8.34), (3.8.35) and (3.8.36) are very similar, we will only deal with the case (3.8.34) as an example.

For that we fix in the following the quantities $z \in \mathbb{R}$, $0 \le t \le 1$ and $\underline{i}, \underline{j} \in M_{16}$, which have the properties (w1), (w2), (w3) from (3.5.19).

Because of Lemma 3.5.27, c) is T_1, given $\underline{I} = \underline{i}$ and $\underline{J} = \underline{j}$, distributed as

$$\sum_{(i,j)\in S} \bar{a}_{ij} + T_{\widetilde{B}}, \tag{3.8.37}$$

where

$$S = \Big\{ (\, i_s,\, j_{s+8}\,) \,:\, 1 \,\le\, s \,\le\, 8 \Big\} \cup \Big\{ (\, i_s,\, j_{s-8}\,) \,:\, 9 \,\le\, s \,\le\, 16 \Big\} \tag{3.8.38}$$

and

(3.8.39) $\widetilde{B}$ is the $(n-l) \times (n-l)-$matrix obtained from $\bar{A}$ by **cancelling** the $l = \big|\, \{\, i_1, \ldots, i_{16}\,\} \,\big|$ rows $i_1, \ldots, i_{16}$ and the l columns $j_1, \ldots, j_{16}$.

With the abbreviations

$$a = \sum_{(i,j)\in S} \bar{a}_{ij},$$

$$p = E\Big(\, \Delta T_1 \,\Big|\, \underline{I} = \underline{i}, \underline{J} = \underline{j} \Big) \qquad (\Delta T_1 \in \sigma\big(\underline{I},\, \underline{J}\,\big), \text{ cf. Lemma 3.5.22, a)}),$$

$$q = E\Big(\, \Delta T_2 \,\Big|\, \underline{I} = \underline{i}, \underline{J} = \underline{j} \Big) \qquad (\Delta T_2 \in \sigma\big(\underline{I},\, \underline{J}\,\big), \text{ cf. Lemma 3.5.22, b)})$$

remains to be shown:

$$\begin{aligned} &\Big| E\Big(g_z'''\big(T_{\widetilde{B}} + a + p + t\,q \,\big) - g_z'''\big(T_{\widetilde{B}} + a \,\big) \Big) \Big| \\ &\le\ \Big(c_{12}\,\mathcal{C}_2 + c_{13}\,\mathcal{C}_3 + c_{14} \Big) \Big(\,|\, p \,| + |\, q \,|\, \Big). \end{aligned} \tag{3.8.40}$$

If we estimate the left-hand side of the inequality (3.8.40) using Lemma 2.3.13, b) and e) (with $x = T_{\widetilde{B}} + a$ and $y = p + t\,q$), we get

$$\begin{aligned}
&\Big| E\Big(g_z'''(\,T_{\widetilde{B}} + a + p + t\,q\,) - g_z'''(\,T_{\widetilde{B}} + a\,)\Big)\Big| \\
&\le E\bigg(\bigg|\,(\,g_z''' - r_z''\,)\,(\,T_{\widetilde{B}} + a + p + t\,q\,) - (\,g_z''' - r_z''\,)\,(\,T_{\widetilde{B}} + a\,) \\
&\quad - (\,T_{\widetilde{B}} + a\,)\,\Big(r_z'(\,T_{\widetilde{B}} + a + p + t\,q\,) - r_z'(\,T_{\widetilde{B}} + a\,)\Big)\bigg|\bigg) \\
&\quad + \Big| E\Big(r_z''(\,T_{\widetilde{B}} + a + p + t\,q\,) - r_z''(\,T_{\widetilde{B}} + a\,)\Big)\Big| \\
&\quad + \Big| E\Big((\,T_{\widetilde{B}} + a\,)\,\Big(r_z'(\,T_{\widetilde{B}} + a + p + t\,q\,) - r_z'(\,T_{\widetilde{B}} + a\,)\Big)\Big)\Big| \\
&\le c_{15}\,\Big(\,|\,p\,| + |\,q\,|\,\Big)\,\bigg\{\,1 + E\Big(\,|\,T_{\widetilde{B}} + a\,|^3\Big) + \Big(\,|\,p\,| + |\,q\,|\,\Big) \\
&\quad + \frac{1}{\lambda}\,P\Big(\,z\,<\,T_{\widetilde{B}} + a + p + t\,q\,\le\,z + 3\,\lambda\,\Big) \\
&\quad + \frac{1}{\lambda}\int_0^1 E\Big(\Big[\,(\,T_{\widetilde{B}} + a\,)^2 + 2\,\Big]\,1_{(\,z,\,z\,+3\,\lambda\,]}(\,T_{\widetilde{B}} + a + s\,p + s\,t\,q\,)\Big)\,ds \\
&\quad + \frac{1}{\lambda^3}\int_0^1 \Big| E\Big(\,(\,1_{(\,z + 2\,\lambda,\,z\,+3\,\lambda\,]} - 2\cdot 1_{(\,z + \lambda,\,z\,+2\,\lambda\,]} + 1_{(\,z,\,z\,+\lambda\,]}\,) \\
&\qquad\qquad\qquad\qquad (\,T_{\widetilde{B}} + a + s\,p + s\,t\,q\,)\Big)\Big|\,ds \\
&\quad + \int_0^1 \Big| E\Big(\,(\,T_{\widetilde{B}} + a\,)\,r_z''(\,T_{\widetilde{B}} + a + s\,p + s\,t\,q\,)\Big)\Big|\,ds\,\bigg\} \\
&= c_{15}\,\Big(\,|\,p\,| + |\,q\,|\,\Big)\,\Big(\,1 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6\Big).
\end{aligned}$$

As in the proof of Proposition 3.7.22, it therefore remains to estimate the A_i, $1 \le i \le 6$.

Due to $|\bar{a}_{ij}| \le 1$ for all $1 \le i, j \le n$, we first obtain

$$|\,a\,| \le 16, \quad |\,p\,| \le 32 \quad \text{and} \quad |\,q\,| \le 16,$$

from which $A_2 = |\,p\,| + |\,q\,| \le 48$ follows.

If we additionally use the estimates (3.7.29) and (3.7.30) from the proof of Proposition 3.7.22, and if we use as in this proof $\dfrac{\beta_A}{n} \le \epsilon_0$ (cf. (3.8.2)) and $\epsilon_0 \le 1$, then we get $A_1 \le c_{16}$.

Moreover, if we proceed as in the proof of (3.7.31) (with $G_A = E_A$, $3\,\lambda$ instead of $2\,\lambda$, and $|\,a\,| \le 16$ instead of $|\,a\,| \le 8$) and use $E_A \le \sqrt{\frac{4}{3}}\,\lambda$ (cf. (3.8.9)), we also get $A_3 \le c_{17}$ and $A_4 \le c_{18}$.

Furthermore, the relationship (3.8.13) yields

$$\begin{aligned}
&\Big|\,E\Big(\,\big(\,1_{(\,z+2\,\lambda,\;z\,+3\,\lambda\,]} - 2\cdot 1_{(\,z+\lambda,\;z\,+2\,\lambda\,]} + 1_{(\,z,\;z\,+\lambda\,]}\,\big)\,\big(\,T_{\widetilde{B}} + a + s\,p + s\,t\,q\,\big)\Big)\,\Big| \\
&\qquad \le \sup_{\varrho\,\in\,\mathbb{R}}\;|\,\Delta_\lambda^3 F_{\widetilde{B}}(\varrho)\,| \qquad\qquad (\varrho = z - a - s\,p - s\,t\,q) \\
&\qquad \le \Big(\,c_{19}\,\mathcal{C}_2 + c_{20}\,\Big)\,\big(\,E_A^3 + \lambda^3\,\big),
\end{aligned}$$

so that due to $E_A^3 \le \Big(\dfrac{4}{3}\Big)^{3/2}\,\lambda^3$ (cf. (3.8.9)) we obtain

$$A_5 \le 3\,\Big(\,c_{19}\,\mathcal{C}_2 + c_{20}\,\Big).$$

Finally, A_6 has to be estimated. Because of

$$\text{(3.8.41)} \qquad r_z''(x+b) = r_{z-b}''(x) \quad \text{for all } z, x, b \in \mathbb{R} \qquad \text{(cf. (3.8.11) with } a = 1\text{)},$$

it is enough to show

$$\text{(3.8.42)} \qquad \sup_{\varrho\,\in\,\mathbb{R}}\,\Big|\,E\Big(\,\big(\,T_{\widetilde{B}} + a\,\big)\,r_\varrho''\big(\,T_{\widetilde{B}}\,\big)\Big)\,\Big| \le c_{21}\,\mathcal{C}_3 + c_{22}.$$

By using (2.3.15) we get for all $\varrho \in \mathbb{R}$

$$\begin{aligned}
&\Big|\,E\Big(\,\big(\,T_{\widetilde{B}} + a\,\big)\,r_\varrho''\big(\,T_{\widetilde{B}}\,\big)\Big)\,\Big| \\
&= \Bigg|\,\frac{1}{\lambda^3}\,\Bigg\{\,\int_{\varrho}^{\varrho+\lambda}\big(\,x + a\,\big)\,\big(\,\varrho - x\,\big)\,dF_{\widetilde{B}}(x) + 2\int_{\varrho+\lambda}^{\varrho+2\,\lambda}\big(\,x+a\,\big)\,\big(\,x - \varrho - \frac{3}{2}\,\lambda\,\big)\,dF_{\widetilde{B}}(x) \\
&\quad + \int_{\varrho+2\,\lambda}^{\varrho+3\,\lambda}\big(\,x+a\,\big)\,\big(\,\varrho + 3\,\lambda - x\,\big)\,dF_{\widetilde{B}}(x)\,\Bigg\}\,\Bigg|.
\end{aligned}$$

Next, we apply integration by parts for Lebesgue-Stieltjes integrals (cf. e.g. [2], Theorem 5.2.3) to all three integrals. Because of

$$\frac{d}{dx}\,\big(\,x+a\,\big)\,\big(\,\varrho - x\,\big) = \frac{d}{dx}\,\big(\,\varrho\,x - x^2 + a\,\varrho - a\,x\,\big) = -2\,x + \varrho - a,$$

$$\frac{d}{dx}\,(\,x+a\,)\,(\,x-\varrho-\frac{3}{2}\,\lambda\,)=\frac{d}{dx}\,(\,x^2-\varrho\,x-\frac{3}{2}\,\lambda\,x+a\,x-a\,\varrho-\frac{3}{2}\,\lambda\,a\,)=2\,x-\varrho-\frac{3}{2}\,\lambda+a,$$
$$\frac{d}{dx}\,(\,x+a\,)\,(\,\varrho+3\,\lambda-x\,)=\frac{d}{dx}\,(\,\varrho\,x+3\,\lambda\,x-x^2+a\,\varrho+3\,\lambda\,a-a\,x\,)=-2\,x+\varrho+3\,\lambda-a$$

and since each of the terms $\pm(\,\varrho+\lambda+a\,)\,\lambda\,F_{\widetilde{B}}(\,\varrho+\lambda\,)$ and $\pm(\,\varrho+2\,\lambda+a\,)\,\lambda\,F_{\widetilde{B}}(\,\varrho+2\,\lambda\,)$ cancels each other out, we further obtain

$$\begin{aligned}
&=\Bigg|\,\frac{1}{\lambda^3}\,\Bigg\{\int\limits_{\varrho}^{\nu+\lambda}\big(\,2\,x-\varrho+a\,\big)\,F_{\widetilde{B}}(x)\,dx-2\int\limits_{\varrho+\lambda}^{\varrho+2\,\lambda}\big(\,2\,x-\varrho-\frac{3}{2}\,\lambda+a\,\big)\,F_{\widetilde{B}}(x)\,dx\\
&\quad+\int\limits_{\varrho+2\,\lambda}^{\varrho+3\,\lambda}\big(\,2\,x-\varrho-3\,\lambda+a\,\big)\,F_{\widetilde{B}}(x)\,dx\Bigg\}\Bigg|\\
&=\Bigg|\,\frac{1}{\lambda^3}\,\Bigg\{\int\limits_{0}^{\lambda}\big(\,2\,v+\varrho+a\,\big)\,F_{\widetilde{B}}(\,v+\varrho\,)\,dv-2\int\limits_{0}^{\lambda}\big(\,2\,v+\varrho+\frac{1}{2}\,\lambda+a\,\big)\,F_{\widetilde{B}}(\,v+\varrho+\lambda\,)\,dv\\
&\quad+\int\limits_{0}^{\lambda}\big(\,2\,v+\varrho+\lambda+a\,\big)\,F_{\widetilde{B}}(\,v+\varrho+2\,\lambda\,)\,dv\Bigg\}\Bigg|\qquad(\text{subst. }x=v+\varrho+t\,\lambda,\ t=0\,;1\,;2)\\
&=\Bigg|\,\frac{1}{\lambda^3}\,\Bigg\{\int\limits_{0}^{\lambda}(\,v+\varrho\,)\,\Delta_\lambda^2F_{\widetilde{B}}(\,v+\varrho\,)\,dv+a\int\limits_{0}^{\lambda}\Delta_\lambda^2F_{\widetilde{B}}(\,v+\varrho\,)\,dv\\
&\quad+\int\limits_{0}^{\lambda}v\,\Big(\,F_{\widetilde{B}}(\,v+\varrho+2\,\lambda\,)-F_{\widetilde{B}}(\,v+\varrho+\lambda\,)\Big)\,dv\\
&\quad-\int\limits_{0}^{\lambda}v\,\Big(\,F_{\widetilde{B}}(\,v+\varrho+\lambda\,)-F_{\widetilde{B}}(\,v+\varrho\,)\Big)\,dv\\
&\quad+\lambda\int\limits_{0}^{\lambda}\Big(\,F_{\widetilde{B}}(\,v+\varrho+2\,\lambda\,)-F_{\widetilde{B}}(\,v+\varrho+\lambda\,)\Big)\,dv\Bigg\}\Bigg|\\
&\le c_{21}\,\mathcal{C}_3+c_{22}.
\end{aligned}$$

For the inequality, (3.8.14) and $|\,a\,|\le 16$ and $E_A\le\sqrt{\frac{4}{3}}\,\lambda$ (cf. (3.8.9)) were used to estimate the first and second integral.

To estimate the other three integrals, Theorem 3.1.3 was applied as in (3.7.31) (or (3.7.33)).

All in all, we obtain the assertion of Proposition 3.8.33 and thus the Theorem 3.1.13. $\square$

4 Applications of the two theorems from the chapter 3

4.1 Preparations

In this chapter, we give easily provable conditions for simple linear rank statistics under which the requirements (3.1.11), (3.1.14) and (3.1.15) are fulfilled (but with "constants" $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, which grow slightly with n). We then obtain the estimates of the Theorems 3.1.10 and 3.1.13 (cf. Theorem 4.3.13).

The conditions originate from van Zwet [32], who used them to derive an estimate of the characteristic function of $\mathscr{F}_A$ (cf. Theorem 4.3.1). In order to be able to deduce our requirements (3.1.11), (3.1.14) and (3.1.15) from this result, we must derive relations between the expressions on the left-hand side of these requirements and the characteristic functions of the distribution functions used there. This is done in the following lemma.

To formulate this lemma, we first choose a distribution function U on $\mathbb{R}$ that has a density u with the following properties:

(4.1.1) $\quad u$ is infinitely differentiable,

(4.1.2) $\quad$ the support of u is contained in the intervall $[\,-1,\,1\,]$.

The functions U and u are fixed for the entire chapter. From the two properties of u we can conclude that all derivatives of $u(x)$ and $x\,u(x)$ are integrable and therefore holds (cf. Feller [15], Chapter XV, Section 4, Lemmas 2 and 4):

$$(4.1.3) \qquad \begin{cases} \left|\,\hat{U}(t)\,\right| = \mathcal{O}(\,|\,t\,|^{-m}) \quad \text{for } |\,t\,| \to \infty \quad \text{and for all } m \in \mathbb{N}, \\[2ex] \left|\,\hat{U}'(t)\,\right| = \mathcal{O}(\,|\,t\,|^{-m}) \quad \text{for } |\,t\,| \to \infty \quad \text{and for all } m \in \mathbb{N}. \end{cases}$$

In the following, $\hat{G}$ denotes the characteristic function of the (distribution) function G.

From (4.1.3) we conclude

$$(4.1.4) \qquad \int\limits_{\mathbb{R}} \left| t \right|^m \left| \hat{U}(t) \right| dt < \infty \quad \text{and} \quad \int\limits_{\mathbb{R}} \left| t \right|^m \left| \hat{U}'(t) \right| dt < \infty \qquad \text{for all } m \in \mathbb{N}_0,$$

$$(4.1.5) \qquad \int\limits_{\mathbb{R}} \left| \hat{U}(t) \right|^m dt < \infty \quad \text{and} \quad \int\limits_{\mathbb{R}} \left| \hat{U}'(t) \right|^m dt < \infty \qquad \text{for all } m \in \mathbb{N}.$$

The following also follows from (4.1.2)

$$(4.1.6) \qquad \left| \hat{U}'(t) \right| \leq 1 \qquad \text{for all } t \in \mathbb{R}.$$

After these preparations, we now prove the lemma announced above. The proof used here is based on a technique introduced by von Bahr [31] (cf. in particular Section 3).

4.1.7 Lemma

Let F be a distribution function on $\mathbb{R}$ and

$$U_\theta(x) = U\Big(\frac{x}{\theta}\Big) \qquad \text{for } x \in \mathbb{R} \text{ and } \theta > 0.$$

Then

a) $$\left| \Delta_y^2 F(z) \right| \leq \left(y^2 + \theta \right) \int\limits_{\mathbb{R}} \left(1 + |\, t \,| \right) \left| \hat{F}(t)\, \hat{U}_\theta(t) \right| dt$$

for all $z \in \mathbb{R}$, $y > 0$ and $\theta > 0$.

b) $$\Big| F(x) - P_\lambda^2(x; z, F) \Big| \leq \theta \left(1 + \frac{|\, x - z \,|^3}{\lambda^3} \right) \int\limits_{\mathbb{R}} \left(2 + t^2 \right) \left| \hat{F}(t)\, \hat{U}_\theta(t) \right| dt$$

for all $x, z \in \mathbb{R}$ and $\theta \geq \lambda^3 > 0$.

c) If F has an existing first moment, then

$$\begin{aligned} \left| z\, \Delta_y^2 F(z) \right| \leq{}& \left(y^2 + \theta \right) \int\limits_{\mathbb{R}} \left(1 + |\, t \,| \right) \Big\{ \left| \hat{F}'(t)\, \hat{U}_\theta(t) \right| + \left| \hat{F}(t)\, \hat{U}_\theta'(t) \right| \Big\}\, dt \\ &+ 2 \left(y^2 + \theta^2 \right) \int\limits_{\mathbb{R}} \left| \hat{F}(t)\, \hat{U}_\theta(t) \right| dt \end{aligned}$$

for all $z \in \mathbb{R}$, $y > 0$ and $\theta > 0$.

Proof:

We note in advance that

$$(4.1.8) \qquad \Big| \int_{u+\alpha}^{v-\alpha} e^{-its}\, ds - \int_{u}^{v} e^{-its}\, ds \Big| = \Big| \int_{v-\alpha}^{v} e^{-its}\, ds + \int_{u}^{u+\alpha} e^{-its}\, ds \Big| \le 2\,|\alpha|$$

is valid for all $u, v, \alpha, t \in \mathbb{R}$.

In the following, let F_θ be the convolution of F and U_θ. Because of (4.1.2) then holds

$$(4.1.9) \qquad F_\theta(x-\theta) \le F(x) \le F_\theta(x+\theta) \qquad \text{for all } x \in \mathbb{R} \text{ and } \theta > 0.$$

Furthermore, F_θ has a density f_θ that is square-integrable. This follows from the square-integrability of $\hat{f}_\theta = \hat{F}\,\hat{U}_\theta$ (cf. (4.1.5)) and from the Plancherel identity (cf. Feller [15], Chapter XV, Section 3, Example c)).

We now obtain the assertions of the lemma as follows:

a) If $y \ge 2\,\theta$, then $z + y - \theta \ge z + \theta$ and thus

$$\begin{aligned}
&F(z+2\,y) - 2\,F(z+y) + F(z)\\
&\le F_\theta(z+2\,y+\theta) - 2\,F_\theta(z+y-\theta) + F_\theta(z+\theta) \qquad \text{(cf. (4.1.9))}\\
&= \Big(F_\theta(z+2\,y+\theta) - F_\theta(z+y-\theta) \Big) - \Big(F_\theta(z+y-\theta) - F_\theta(z+\theta) \Big)\\
&\le \Big| \int_{\mathbb{R}} \Big(1_{(\,z+y-\theta,\; z+2\,y+\theta\,]}(t) - 1_{(\,z+\theta,\; z+y-\theta\,]}(t) \Big)\, f_\theta(t)\, dt \Big|.
\end{aligned}$$

An application of Plancherel's theorem for scalar products (see Bhattacharya, Rao [3], Theorem 4.2) further leads to

$$\begin{aligned}
&= \frac{1}{2\,\pi} \Big| \int_{\mathbb{R}} \Big(\int_{z+y-\theta}^{z+2y+\theta} e^{-its}\, ds \;-\; \int_{z+\theta}^{z+y-\theta} e^{-its}\, ds \Big)\, \hat{f}_\theta(t)\, dt \Big|\\
&\le \frac{1}{2\,\pi} \int_{\mathbb{R}} \frac{1}{|\,t\,|} \Big| e^{-it(z+2y)} - 2\,e^{-it(z+y)} + e^{-itz} \Big|\, |\,\hat{f}_\theta(t)\,|\, dt \;+\\
&\quad \frac{1}{2\,\pi} \int_{\mathbb{R}} \Big\{ \Big| \int_{z+y-\theta}^{z+2y+\theta} e^{-its}\, ds - \int_{z+y}^{z+2y} e^{-its}\, ds \Big| + \Big| \int_{z+\theta}^{z+y-\theta} e^{-its}\, ds - \int_{z}^{z+y} e^{-its}\, ds \Big| \Big\}\, |\,\hat{f}_\theta(t)\,|\, dt.
\end{aligned}$$

For the first integral, we now apply Lemma 2.4.2, b) for each $t \in \mathbb{R}$, once for the function $g_{re}(w) = \cos(tw)$ and once for $g_{im}(w) = \sin(tw)$. For the second integral, we use the inequality (4.1.8). Thus, together with $\hat{f}_\theta = \hat{F}\,\hat{U}_\theta$, we further obtain

$$\leq y^2 \int\limits_{\mathbb{R}} |\,t\,|\,\big|\,\hat{F}(t)\,\hat{U}_\theta(t)\,\big|\,dt + \theta \int\limits_{\mathbb{R}} \big|\,\hat{F}(t)\,\hat{U}_\theta(t)\,\big|\,dt$$

$$\leq \big(\,y^2 + \theta\,\big) \int\limits_{\mathbb{R}} \big(\,1 + |\,t\,|\,\big)\,\big|\,\hat{F}(t)\,\hat{U}_\theta(t)\,\big|\,dt := R.$$

Furthermore, from $y \geq 2\,\theta$ also follows $z + 2y - \theta \geq z + y + \theta$, so that we get by completely analogous considerations

$$-\,\Big(\,F(z + 2\,y) - 2\,F(z + y) + F(z)\,\Big)$$

$$\leq \Big(\,F_\theta(z + y + \theta) - F_\theta(z - \theta)\,\Big) - \Big(\,F_\theta(z + 2\,y - \theta) - F_\theta(z + y + \theta)\,\Big) \qquad \text{(cf. (4.1.9))}$$

$$\leq \frac{1}{2\,\pi}\,\Bigg|\,\int\limits_{\mathbb{R}} \Big(\int\limits_{z+y+\theta}^{z+2y-\theta} e^{-its}\,ds \;-\; \int\limits_{z-\theta}^{z+y+\theta} e^{-its}\,ds\Big)\,\hat{f}_\theta(t)\,dt\,\Bigg| \qquad \text{(Plancherel's theorem).}$$

The last expression can be estimated in the same way as above by R.

If, on the other hand, $y < 2\,\theta$, then $z + y - \theta < z + \theta$ and thus

$$F(z + 2\,y) - 2\,F(z + y) + F(z)$$

$$\leq \Big(\,F_\theta(z + 2\,y + \theta) - F_\theta(z + y - \theta)\,\Big) + \Big(\,F_\theta(z + \theta) - F_\theta(z + y - \theta)\,\Big) \qquad \text{(cf. (4.1.9))}$$

$$\leq \frac{1}{2\,\pi}\,\Bigg|\,\int\limits_{\mathbb{R}} \Big(\int\limits_{z+y-\theta}^{z+2y+\theta} e^{-its}\,ds \;+\; \int\limits_{z+y-\theta}^{z+\theta} e^{-its}\,ds\Big)\,\hat{f}_\theta(t)\,dt\,\Bigg| \qquad \text{(Plancherel's theorem)}$$

$$= \frac{1}{2\,\pi}\,\Bigg|\,\int\limits_{\mathbb{R}} \Big(\int\limits_{z+y-\theta}^{z+2y+\theta} e^{-its}\,ds \;-\; \int\limits_{z+\theta}^{z+y-\theta} e^{-its}\,ds\Big)\,\hat{f}_\theta(t)\,dt\,\Bigg| \qquad \text{(reversing the integration limits).}$$

This expression has already been estimated in detail by R above.

Finally, from $y < 2\,\theta$ also follows $z + 2y - \theta < z + y + \theta$, so we get analogously to above

$$-\,\Big(\,F(z + 2\,y) - 2\,F(z + y) + F(z)\,\Big)$$

$$\leq \Big(\,F_\theta(z + y + \theta) - F_\theta(z - \theta)\,\Big) + \Big(\,F_\theta(z + y + \theta) - F_\theta(z + 2\,y - \theta)\,\Big) \leq R \qquad \text{(cf. (4.1.9)).}$$

Overall, the validity of part a) follows from the 4 cases considered.

b) Let $x \geq z + \lambda$. Then

$$
\begin{aligned}
& F(x) - P_\lambda^2(x; z, F) \\
& \leq F_\theta(x+\theta) - F_\theta(z-\theta) - \frac{x-z}{\lambda} \Big(F_\theta(z+\lambda-\theta) - F_\theta(z+\theta) \Big) \\
& \quad - \frac{1}{2} \frac{x-z}{\lambda} \frac{x-z-\lambda}{\lambda} \Big(F_\theta(z+2\lambda-\theta) - 2\, F_\theta(z+\lambda+\theta) + F_\theta(z-\theta) \Big) \\
& \leq \frac{1}{2\pi} \Bigg| \int_{\mathbb{R}} \Bigg\{ \int_{z-\theta}^{x+\theta} e^{-its}\, ds \;-\; \frac{x-z}{\lambda} \int_{z+\theta}^{z+\lambda-\theta} e^{-its}\, ds \\
& \qquad\qquad - \frac{1}{2} \frac{x-z}{\lambda} \frac{x-z-\lambda}{\lambda} \Bigg(\int_{z+\lambda+\theta}^{x+2\lambda-\theta} e^{-its}\, ds \;-\; \int_{z-\theta}^{x+\lambda+\theta} e^{-its}\, ds \Bigg) \Bigg\} \, \hat{f}_\theta(t)\, dt \Bigg| \\
& \leq \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{|\,t\,|} \Big| e^{-itx} - P_\lambda^2(x; z, e^{-it\cdot}) \Big| \, \big|\, \hat{f}_\theta(t) \,\big| \, dt \; + \\
& \qquad + \frac{1}{2\pi} \int_{\mathbb{R}} 2\,\theta \Big(1 + \frac{|x-z|}{\lambda} + \frac{|x-z|}{\lambda} \frac{|x-z-\lambda|}{\lambda} \Big) \big|\, \hat{f}_\theta(t) \,\big| \, dt \qquad \text{(cf. (4.1.8))}.
\end{aligned}
$$

We now apply Lemma 2.4.2, a) to the first integral and the inequality

$$
1 + \eta + \eta\,(1+\eta) = (1+\eta)^2 \leq (1+\eta)^3 \leq 4\,(1+\eta^3) \quad \text{for } \eta > 0 \quad ((0.1.5),\ \nu = 2,\ p = 3)
$$

to the second integral. This gives

$$
\begin{aligned}
& \leq \Big(\lambda^3 + |x-z|^3 \Big) \int_{\mathbb{R}} t^2 \,\big|\, \hat{f}_\theta(t) \,\big| \, dt \; + \; 2\,\theta \Big(1 + \frac{|x-z|^3}{\lambda^3} \Big) \int_{\mathbb{R}} \big|\, \hat{f}_\theta(t) \,\big| \, dt \\
& \leq \theta \Big(1 + \frac{|x-z|^3}{\lambda^3} \Big) \int_{\mathbb{R}} \big(2 + t^2 \big) \,\big|\, \hat{F}(t)\, \hat{U}_\theta(t) \,\big| \, dt \qquad \text{(cf. condition } \theta \geq \lambda^3 > 0).
\end{aligned}
$$

Since we obtain these estimates completely analogously (as in part a)) for $x \leq z$ and $z \leq x \leq z + \lambda$ as well as $-\Big(F(x) - P_\lambda^2(x; z, F) \Big)$, part b) follows.

c) Let $y \geq 2\,\theta$ and $z \geq 0$. Then

$$
\begin{aligned}
& z \Big(F(z+2\,y) - 2\,F(z+y) + F(z) \Big) \\
& \leq z \Big(F_\theta(z+2\,y+\theta) - 2\,F_\theta(z+y-\theta) + F_\theta(z+\theta) \Big) \qquad \text{(cf. (4.1.9))}
\end{aligned}
$$

$$\leq \Big| \int_{\mathbb{R}} z \Big(1_{(\, z+y-\theta,\; z+2\,y+\theta\,]}(t) - 1_{(\, z+\theta,\; z+y-\theta\,]}(t) \Big)\, f_\theta(t)\, dt \Big|$$

$$\leq \Big| \int_{\mathbb{R}} t \Big(1_{(\, z+y-\theta,\; z+2\,y+\theta\,]}(t) - 1_{(\, z+\theta,\; z+y-\theta\,]}(t) \Big)\, f_\theta(t)\, dt \Big|$$

$$+ \Big| \int_{z+y-\theta}^{z+2\,y+\theta} (t-z)\, f_\theta(t)\, dt \Big| + \Big| \int_{z+\theta}^{z+y-\theta} (t-z)\, f_\theta(t)\, dt \Big|$$

$$\leq \Big| \int_{\mathbb{R}} \Big(1_{(\, z+y-\theta,\; z+2\,y+\theta\,]}(t) - 1_{(\, z+\theta,\; z+y-\theta\,]}(t) \Big) \big(t\, f_\theta(t) \big)\, dt \Big|$$

$$+ \,(\, 2\,y+\theta\,) \int_{z+y-\theta}^{z+2\,y+\theta} f_\theta(t)\, dt \;+\; (\, y+\theta\,) \int_{z+\theta}^{z+y-\theta} f_\theta(t)\, dt \qquad (f_\theta \text{ is a density!})$$

$$\leq \frac{1}{2\,\pi} \Big| \int_{\mathbb{R}} \Big(\int_{z+y-\theta}^{z+2y+\theta} e^{-its}\, ds \;-\; \int_{z+\theta}^{z+y-\theta} e^{-its}\, ds \Big) \big(\widehat{id_{\mathbb{R}}\, f_\theta} \,\big)(t)\, dt \Big|$$

$$+ \,(\, 2\,y+\theta\,)\, \frac{1}{2\,\pi} \Big| \int_{\mathbb{R}} \Big(\int_{z+y-\theta}^{z+2y+\theta} e^{-its}\, ds \Big)\, \hat{f}_\theta(t)\, dt \Big|$$

$$+ \,(\, y+\theta\,)\, \frac{1}{2\,\pi} \Big| \int_{\mathbb{R}} \Big(\int_{z+\theta}^{z+y-\theta} e^{-its}\, ds \Big)\, \hat{f}_\theta(t)\, dt \Big|$$

$$\leq \big(y^2+\theta \big) \int_{\mathbb{R}} (\, 1+|\, t\,|\,)\, \Big| \big(\widehat{id_{\mathbb{R}}\, f_\theta} \,\big)(t) \Big|\, dt$$

$$+ \Big\{ (\, 2\,y+\theta\,)(\, y+2\,\theta\,) + (\, y+\theta\,)(\, y+2\,\theta\,) \Big\}\, \frac{1}{2\,\pi} \int_{\mathbb{R}} \big|\, \hat{f}_\theta(t) \,\big|\, dt.$$

For the last inequality, the first summand was estimated as in the proof of part a). If we now use

$$\hat{f}_\theta = \hat{F}\, \hat{U}_\theta, \quad \big(\widehat{id_{\mathbb{R}}\, f_\theta} \,\big) = \frac{1}{i}\, (\hat{f}_\theta)' \quad \text{and}$$

$$(\, 2\,y+\theta\,)(\, y+2\,\theta\,) + (\, y+\theta\,)(\, y+2\,\theta\,) = 3\,y^2 + 8\,\theta\, y + 4\,\theta^2 \leq 7\,y^2 + 8\,\theta^2,$$

the asserted estimate in part c) follows for the considered case.

All other cases are derived using completely analogous considerations. □

4.2 A conclusion from Cramér's condition

This section is to be understood as an insertion and serves to illustrate some basic ideas of the further procedure in the simpler case of independent and identically distributed (iid) random variables. More precisely, we show how to obtain the assumption (1.1.4) of Theorem 1.1.3 from Cramér's condition. In the next section 4.3, an estimate of the characteristic function of a simple linear rank statistic based on van Zwet will play the role of Cramér's condition.

In this section, let X_i, $i \in \mathbb{N}$, be a sequence of independent and identically distributed random variables with a common distribution function F and

$$(4.2.1) \qquad E(X_i) = 0 \quad \text{and} \quad E(X_i^2) = 1.$$

Furthermore, F_n denotes the distribution function of $S_n = \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{n} X_i$ for $n \in \mathbb{N}$. Then we get:

4.2.2 Theorem

Suppose that

$$(4.2.3) \qquad \limsup_{|\,t\,| \to \infty} \big|\, \hat{F}(t) \,\big| < 1 \qquad \text{("Cramér's condition").}$$

Then there exists a constant $\mathcal{C}' > 0$ depending only on F such that

$$(4.2.4) \qquad ||\Delta_y^2 F_n|| \le \mathcal{C}' \Big(\frac{1}{n} + y^2 \Big) \quad \text{for all } 0 \le y \le \frac{1}{\sqrt{n}},\ n \in \mathbb{N}.$$

Proof:

At first, it should be noted that the constants c_1, c_2, δ and η used in this proof depend on F and nothing else.

According to Lemma 4.1.7, a) with $\theta = \dfrac{1}{n}$ and $F = F_n$ remains to be shown:

$$\int_{\mathbb{R}} \big(1 + |\,t\,| \big) \, \big|\, \hat{F}_n(t) \, \hat{U}_{\frac{1}{n}}(t) \,\big| \, dt \le c_1 \quad \text{for all } n \in \mathbb{N}.$$

Because of

$$\big|\, \hat{F}_n(t) \, \hat{U}_{\frac{1}{n}}(t) \,\big| \le 1 \qquad \text{and} \qquad 1 \le |\,t\,| \ \text{ for } \ t \notin \,]-1, 1\,[\ ,$$

and because of

$$\hat{F}_n(t) = \Big(\,\hat{F}\big(\,\frac{t}{\sqrt{n}}\,\big)\Big)^n \qquad \text{and} \qquad \hat{U}_{\frac{1}{n}}(t) = \hat{U}\big(\,\frac{t}{n}\,\big),$$

it is sufficient to prove the existence of a constant c_2 such that

$$\int\limits_{\mathbb{R}} |\,t\,|\,\Big|\,\Big(\,\hat{F}\big(\,\frac{t}{\sqrt{n}}\,\big)\Big)^n\,\Big|\,\Big|\,\hat{U}\big(\,\frac{t}{n}\,\big)\,\Big|\,dt \le c_2 \quad \text{for all } n \in \mathbb{N}. \tag{4.2.5}$$

To do this, we first expand the characteristic function $\hat{F}$ about 0 according to Taylor's theorem taking into account (4.2.1). From this follows the existence of a $\delta > 0$, so that

$$\big|\,\hat{F}(t)\,\big| \le 1 - \frac{t^2}{4} \quad \text{for } |\,t\,| \le \delta$$

(cf. e.g. [20], Chapter I, Theorem 2 incl. proof, page 11). We now split the above integral into the integration regions $|\,t\,| \le \delta\sqrt{n}$ and $|\,t\,| > \delta\sqrt{n}$. The estimate over the first region gives

$$\begin{aligned}\int\limits_{|\,t\,| \le \delta\sqrt{n}} |\,t\,|\,\Big|\,\hat{F}\big(\,\frac{t}{\sqrt{n}}\,\big)\,\Big|^n\,\Big|\,\hat{U}\big(\,\frac{t}{n}\,\big)\,\Big|\,dt &\le \int\limits_{|\,t\,| \le \delta\sqrt{n}} |\,t\,|\,\Big(1 - \frac{t^2}{4\,n}\Big)^n\,dt \\ &\le \int\limits_{\mathbb{R}} |\,t\,|\,\exp\Big(-\frac{t^2}{4}\Big)\,dt = 4.\end{aligned}$$

For the last inequality, $1 - x \le e^{-x}$ for $x \in \mathbb{R}$ was used.

For the estimate over the other region, we use that there is an $\epsilon > 0$ and a $\rho > 0$ due to Cramér's condition, so that

$$\big|\,\hat{F}(t)\,\big| \le 1 - \epsilon \quad \text{for } |\,t\,| > \rho.$$

Since from this $\big|\,\hat{F}(t)\,\big| < 1$ follows for all $t \neq 0$ (cf. Feller [15], Chapter XV, Section 1, Lemma 4) and since $\big|\,\hat{F}\,\big|$ (in the case of $\delta < \rho$) is continuous on the compact interval $[\,\delta,\,\rho\,]$, there exists an $\eta > 0$ such that

$$\big|\,\hat{F}(t)\,\big| \le 1 - \eta \quad \text{for } |\,t\,| > \delta.$$

It follows that

$$\int\limits_{|\,t\,| > \delta\sqrt{n}} |\,t\,|\,\Big|\,\hat{F}\big(\,\frac{t}{\sqrt{n}}\,\big)\,\Big|^n\,\Big|\,\hat{U}\big(\,\frac{t}{n}\,\big)\,\Big|\,dt \le \big(\,1 - \eta\,\big)^n \int\limits_{|\,t\,| > \delta\sqrt{n}} |\,t\,|\,\Big|\,\hat{U}\big(\,\frac{t}{n}\,\big)\,\Big|\,dt.$$

Now we make the substitution $v = \dfrac{t}{n}$ and obtain furthermore

$$= \big(1-\eta\big)^n n^2 \int\limits_{|v|>\frac{\delta}{\sqrt{n}}} |v|\,|\hat{U}(v)|\,dv$$

$$\leq \big(1-\eta\big)^n n^2 \int\limits_{\mathbb{R}} |v|\,|\hat{U}(v)|\,dv \to 0 \quad \text{for } n\to\infty \qquad \text{(cf. (4.1.4))}.$$

Therefore (4.2.5) is valid. □

If we summarize the above Theorem 4.2.2 and the Theorem 1.1.3, we obtain the following result, which goes back to Cramér [11], Theorem 25 in Chapter VII (cf. also the original articles [9] and [10]):

4.2.6 Corollary

Suppose that

$$E(X_i^4) \leq \infty \quad \text{and} \quad \limsup_{|t|\to\infty} \left|\hat{F}(t)\right| < 1.$$

Then there exists a constant $\mathcal{K}' > 0$ depending only on F such that

$$||F_n - e_n|| \leq \frac{\mathcal{K}'}{n} \quad \text{for all } n \in \mathbb{N}.$$

4.3 Use of a result from van Zwet

In this section we consider simple linear rank statistics. We use a result of van Zwet and Lemma 4.1.7 to derive the conditions (3.1.11), (3.1.14) and (3.1.15) in a similar way as in the previous section 4.2. However, the resulting "constants" $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ will grow slightly with n.

In the following $e_1,\ldots,e_n$ (regression constants) and $d_1,\ldots,d_n$ (scores) are sequences of real numbers and

$$\overline{e} = \frac{1}{n}\sum_{i=1}^n e_i, \qquad \overline{d} = \frac{1}{n}\sum_{j=1}^n d_j$$

and $a_{ij} = e_i\,d_j$ for $i,j=1,\ldots,n$. For the matrix $A=(a_{ij})$, the same notations as in chapter 3 shall apply. Furthermore, let λ be the Lebesgue measure on $\mathbb{R}$.

The result used from van Zwet is now:

4.3.1 Theorem

Suppose that there exist positive constants e, E, d, D and δ such that

(4.3.2) $$\sum_{i=1}^{n} \left| e_i - \bar{e} \right|^r \geq en, \qquad \sum_{i=1}^{n} \left| e_i - \bar{e} \right|^k \leq En$$

for some $k > 2$ and $0 < r < k$.

(4.3.3) $$\sum_{j=1}^{n} \left| d_j - \bar{d} \right|^m \geq dn, \qquad \sum_{j=1}^{n} \left| d_j - \bar{d} \right|^s \leq Dn$$

for some $s > 2$ and $0 < m < s$.

(4.3.4) $$\lambda\Big(\bigcup_{j=1}^{n} \Big\{ x \in \mathbb{R} : \left| x - d_j \right| < \zeta \Big\} \Big) \geq \delta n \zeta$$

for some $\zeta \geq n^{-3/2} \log\, n$.

Then there exist positive constants a_1, a_2, a_3 and a_4 depending only on e, E, d, D, δ and r, k, m, s such that

(4.3.5) $$\left| \hat{F}_{\hat{A}}(t) \right| \leq a_1\, n^{-a_2 \log n} \quad \text{for} \quad a_3 \log n \leq \left| t \right| \leq a_4\, n^{3/2} \qquad (\text{note } \hat{F}_{\hat{A}} = \hat{\mathscr{F}}_A\ !).$$

Proof (by citations):

This theorem is a direct consequence of Theorem 2.1 and the comments from Section 3 of van Zwet [32]. Some of these comments also follow from the next Lemma 4.3.6. □

The estimate (4.3.5) will play in this section 4.3 the role that Cramér's condition (4.2.3) played in the previous section 4.2, i.e. with the help of (4.3.5) we will derive the conditions (3.1.11), (3.1.14) and (3.1.15) for $B = \hat{A}$.

However, in order to be able to prove these conditions also for the submatrices $B \in N(8, \hat{A})$ (or $B \in N(16, \hat{A})$), we need the validity of the estimate (4.3.5) for these submatrices as well. The following proposition, which establishes this validity for a sufficient number of matrices B, serves this purpose.

Before that, we show a simple lemma that is useful for the proof of this proposition and the further procedure.

4.3.6 Lemma

a) Let E, k and t be positive constants. Then

from $$\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^k \le En$$

follows $$\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t \le E^{t/k}\, n^{1 + \left((t/k) - 1\right)^+}.$$

b) Let e, E, r, k and t be positive constants such that $0 < r < k$. Then

from $$\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^r \ge en \quad \text{and} \quad \sum_{i=1}^{n} \left| e_i - \overline{e} \right|^k \le En$$

follows $$\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t \ge e'n$$

with a positive constant e' depending only on e, E, r, k and t.

In particular, we can assume $r = m = 2$ in (4.3.2) and (4.3.3) without loss of generality.

c) From (4.3.2) and (4.3.3) follows that there exist constants $C_1 > 0$ and $C_2 > 0$ depending only on e, E, d, D and r, k, m, s such that

$$D_A^2 \le C_1\, n^{-1 + \left((4/k) - 1\right)^+ + \left((4/s) - 1\right)^+}, \tag{4.3.7}$$

$$E_A^3 \le C_2\, n^{-(3/2) + \left((5/k) - 1\right)^+ + \left((5/s) - 1\right)^+}. \tag{4.3.8}$$

Proof:

a) **1. Case:** $t \le k$.

Hölder's inequality (0.1.5) with $\nu = n$ and $p = \dfrac{k}{t}$ gives

$$\begin{aligned} \sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t &\le \Big(\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^k \Big)^{t/k} n^{1-(t/k)} \\ &\le E^{t/k}\, n. \end{aligned}$$

2. Case: $t \geq k$.

$$\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t \leq \max_{1 \leq i \leq n} \left| e_i - \overline{e} \right|^{t-k} \sum_{i=1}^{n} \left| e_i - \overline{e} \right|^k$$
$$\leq \Big(\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^k \Big)^{1+(t-k)/k}$$
$$\leq E^{t/k} n^{t/k}.$$

b) **1. Case:** $r \leq t$.

From Hölder's inequality (0.1.5) with $\nu = n$ and $p = \dfrac{t}{r}$ we obtain

$$en \leq \sum_{i=1}^{n} \left| e_i - \overline{e} \right|^r \leq \Big(\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t \Big)^{r/t} n^{1-(r/t)}$$

and therefore

$$e^{t/r} n \leq \sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t.$$

2. Case: $r > t$.

Hölder's inequality with $p = \dfrac{k-t}{k-r}$ and $q = \dfrac{k-t}{r-t}$ gives

$$en \leq \sum_{i=1}^{n} \left| e_i - \overline{e} \right|^{t/p} \left| e_i - \overline{e} \right|^{r-(t/p)}$$
$$\leq \Big(\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t \Big)^{1/p} \Big(\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^k \Big)^{1/q} \qquad \text{(since } \Big(r - \frac{t}{p} \Big) q = k\text{)}$$
$$\leq \Big(\sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t \Big)^{1/p} (En)^{1/q}.$$

Because of $\dfrac{p}{q} = p - 1$, it follows that

$$e^p E^{1-p} n \leq \sum_{i=1}^{n} \left| e_i - \overline{e} \right|^t.$$

c) Due to part b) we can assume $r = m = 2$ without loss of generality. Therefore we get

$$\sigma_A^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(e_i - \overline{e} \right)^2 \sum_{j=1}^{n} \left(d_j - \overline{d} \right)^2 \geq edn.$$

Using part a) we then obtain

$$
\begin{aligned}
D_A^2 &= \frac{1}{n}\,\frac{1}{\sigma_A^4}\sum_{i=1}^{n}\big(\,e_i-\bar{e}\,\big)^4\sum_{j=1}^{n}\big(\,d_j-\bar{d}\,\big)^4 \\
&\le \frac{E^{\,4/k}D^{\,4/s}}{e^{\,2}\;d^2}\,n^{-\,1+\big((4/k)\,-\,1\big)^+\,+\,\big((4/s)\,-\,1\big)^+}
\end{aligned}
$$

and analogously

$$
E_A^3 \le \frac{E^{\,5/k}D^{\,5/s}}{e^{\,5/2}\,d^{\,5/2}}\,n^{-\,(3/2)\,+\,\big((5/k)\,-\,1\big)^+\,+\,\big((5/s)\,-\,1\big)^+}. \qquad \square
$$

The announced proposition is now:

4.3.9 Proposition

Suppose that the conditions and notations of the Theorem 4.3.1 apply.

Furthermore, we choose a $0 < \epsilon_0 \le 1$ and an $n_0 \in \mathbb{N}$ such that for all matrices A satisfying $\sigma_A > 0$, $\beta_A \le \epsilon_0\, n$ and $n \ge n_0$ holds:

$$
\big|\,\sigma_B^2 - 1\,\big| \le \frac{1}{3} \quad \text{for } B \in N(17, \hat{A})
$$

(cf. Corollary 3.3.7). Then there exist positive constants b_1, b_2, b_3 and b_4 depending only on e, E, d, D, δ and r, k, m, s such that in the case of

$$
n \ge n_1 = \max\Big\{\,n_0,\,\frac{68}{\delta},\,19\,\Big\} \quad \text{and} \quad \beta_A \le \epsilon_0\, n
$$

we have

$$
\big|\,\hat{F}_B(t)\,\big| \;\le\; b_1\, n^{-b_2 \log n} \qquad \text{for} \quad b_3 \log n \le \big|\,t\,\big| \le b_4\, n^{3/2} \quad \text{and} \quad B \in N(17, \hat{A}). \tag{4.3.10}
$$

Proof:

a) In the first part of the proof, we show that the conditions (4.3.2), (4.3.3) and (4.3.4) also hold for matrices $\sigma_A\, B$ with $B \in N(17, \hat{A})$ (but with other constants e', E', d', D', δ', which depend only on e, E, d, D, δ and r, k, m, s, and with another value ζ', which depends only on ζ). Because of Lemma 4.3.6, b) we can assume $r = 2 = m$ without loss of generality.

   The matrices $\sigma_A\, B$, where $B \in M(l, \hat{A})$, have the form

$$
\sigma_A\, B = (g_i f_j)_{1\,\le\, i,j\,\le\, n-l}
$$

with $g_i \in \{\, e_k - \bar{e} \,:\, k = 1, \ldots, n \,\}$ and $f_j \in \{\, d_k - \bar{d} \,:\, k = 1, \ldots, n \,\}$ (without repetitions!). Furthermore, we define

$$\bar{g} = \frac{1}{n-l} \sum_{i=1}^{n-l} g_i \quad \text{and} \quad \bar{f} = \frac{1}{n-l} \sum_{j=1}^{n-l} f_j .$$

Using these abbreviations, we now determine the items e', E', d', D', δ' and ζ'.

i) Computation of E' and D':

An application of Hölder's inequality (0.1.5) with $\nu = 2$ and $p = k$ to $\big(\,|\,g_i\,| + |\,\bar{g}\,|\,\big)^k$ and with $\nu = n - l$ and $p = k$ to $\big(\,|\,g_1\,| + \ldots + |\,g_{n-l}\,|\,\big)^k$ gives

$$\begin{aligned}
\sum_{i=1}^{n-l} \big|\, g_i - \bar{g} \,\big|^k &\le 2^{k-1} \sum_{i=1}^{n-l} \big|\, g_i \,\big|^k + 2^{k-1}\,(n-l)\,\frac{1}{(n-l)^k} \Big(\sum_{i=1}^{n-l} |\, g_i \,| \Big)^k \\
&\le 2^k \sum_{i=1}^{n-l} \big|\, g_i \,\big|^k \\
&\le 2^k \sum_{i=1}^{n} \big|\, e_i - \bar{e} \,\big|^k \\
&\le 2^k\, En && \text{(due to (4.3.2))} \\
&\le 10\,(2^k\, E)\,(n-l) \qquad \text{for } 0 \le l \le 17.
\end{aligned}$$

For the last inequality, we used that $n \le 10\,(n-l)$ holds for $n \ge 19$ and $0 \le l \le 17$. Therefore $E' = 10\,(2^k\, E)$ is suitable.

We proceed in the same way with the f_j and obtain $D' = 10\,(2^s\, D)$.

ii) Computation of e' and d':

Since we assume $\sigma_B^2 \ge \dfrac{2}{3}$ and since we have $(n-1) \le 18\Big(\,(n-1) - l\,\Big)$ for $n \ge 19$ and $0 \le l \le 17$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n-l} \big(\, g_i - \bar{g} \,\big)^2 \sum_{j=1}^{n-l} \big(\, f_j - \bar{f} \,\big)^2 &= (n-l-1)\, \sigma_A^2\, \sigma_B^2 \\
&\ge \frac{1}{18}\,(n-1)\, \sigma_A^2\, \frac{2}{3} \\
&= \frac{1}{27} \sum_{i=1}^{n} \big(\, e_i - \bar{e} \,\big)^2 \sum_{j=1}^{n} \big(\, d_j - \bar{d} \,\big)^2 \\
&\ge \frac{1}{27}\, edn^2 \qquad \text{for } 0 \le l \le 17.
\end{aligned}$$

For the last inequality, (4.3.2) and (4.3.3) were used.

It follows that

$$\begin{aligned}\sum_{i=1}^{n-l} \big(g_i - \bar{g} \big)^2 &\geq \frac{1}{27}\, edn^2 \Big(\sum_{j=1}^{n-l} \big(f_j - \bar{f} \big)^2 \Big)^{-1} \\ &\geq \frac{1}{27}\, edn^2 \Big((D')^{2/s} \, (n-l) \Big)^{-1} \\ &\geq \frac{1}{27}\, ed \, (D')^{-\,2/s} \, (n-l).\end{aligned}$$

For the penultimate inequality, Lemma 4.3.6, a) with $k = s > 2$, $E = D'$ (cf. part i)) and $t = 2$ was used.

Therefore $e' = \dfrac{1}{27}\, ed \, (D')^{-\,2/s}$ and analogously $d' = \dfrac{1}{27}\, ed \, (E')^{-\,2/k}$ are suitable.

iii) Computation of δ' and ζ' :

The translation invariance of λ yields

$$\begin{aligned}&\lambda\Big(\bigcup_{j=1}^{n} \Big\{ x \in \mathbb{R} \, : \, \big| \, x - d_j \, \big| < \zeta \Big\} \Big) \\ = \; &\lambda\Big(\bigcup_{j=1}^{n} \Big\{ x \in \mathbb{R} \, : \, \big| \, x - (\, d_j - \bar{d} \,) \, \big| < \zeta \Big\} \Big) \\ \leq \; &\lambda\Big(\bigcup_{j=1}^{n-l} \Big\{ x \in \mathbb{R} \, : \, \big| \, x - f_j \, \big| < \zeta \Big\} \Big) + l \, 2\zeta.\end{aligned}$$

For $n \geq \dfrac{68}{\delta}$ it follows that

$$\begin{aligned}&\lambda\Big(\bigcup_{j=1}^{n-l} \Big\{ x \in \mathbb{R} \, : \, \big| \, x - f_j \, \big| < \zeta \Big\} \Big) \\ \geq \; &\lambda\Big(\bigcup_{j=1}^{n} \Big\{ x \in \mathbb{R} \, : \, \big| \, x - d_j \, \big| < \zeta \Big\} \Big) - l \, 2\zeta \\ \geq \; &\delta n \zeta - l \, 2\zeta \qquad\qquad \text{(because of (4.3.4))} \\ \geq \; &\big(\, \delta n - 34 \, \big) \, \zeta \\ \geq \; &\frac{\delta}{2} \, n \, \zeta \qquad \text{for } 0 \leq l \leq 17.\end{aligned}$$

Since

$$7\,\frac{\log\ n}{n^{3/2}} \geq \frac{\log\ (n-l)}{(n-l)^{3/2}} \qquad \text{for } n \geq 19 \text{ and } 0 \leq l \leq 17,$$

we get all in all

$$\begin{aligned}
&\lambda\Big(\bigcup_{j=1}^{n-l}\Big\{\, x \in \mathbb{R} \,:\, \big|\, x - f_j \,\big| < 7\,\zeta \,\Big\}\Big) \\
&\geq \lambda\Big(\bigcup_{j=1}^{n-l}\Big\{\, x \in \mathbb{R} \,:\, \big|\, x - f_j \,\big| < \zeta \,\Big\}\Big) \\
&\geq \frac{\delta}{2}\, n\, \zeta \\
&= \frac{\delta}{14}\, n\, (7\,\zeta) \\
&\geq \frac{\delta}{14}\, (n-l)\, (7\,\zeta) \qquad \text{for } 0 \leq l \leq 17
\end{aligned}$$

and

$$\zeta' = 7\,\zeta \geq 7\,\frac{\log\ n}{n^{3/2}} \geq \frac{\log\ (n-l)}{(n-l)^{3/2}} \qquad \text{for } n \geq \max\Big\{\, \frac{68}{\delta},\, 19 \,\Big\} \text{ and } 0 \leq l \leq 17.$$

Thus, $\delta' = \dfrac{\delta}{14}$ and $\zeta' = 7\,\zeta$ are suitable.

b) The existence of positive constants a_1', a_2', a_3' and a_4' depending only on e, E, d, D, δ and r, k, m, s such that

$$\big|\, \hat{F}_{\widehat{\sigma_A\, B}}(t) \,\big| \;\leq\; a_1'\,(n-l)^{-a_2' \log\ (n-l)} \qquad \text{for} \quad a_3' \log\ (n-l) \leq \big|\, t \,\big| \leq a_4'\,(n-l)^{3/2}$$
$$\text{and} \quad B \in M(l, \hat{A}),\ 0 \leq l \leq 17$$

now follows from part a) and Theorem 4.3.1. Using $\widehat{\sigma_A\, B} = \hat{B}$ we get from this that

$$\begin{aligned}
\big|\, \hat{F}_B(t) \,\big| \;&\leq \Big|\, E\Big(\exp\big\{ it(T_B - \mu_B) \big\} \Big) \,\Big| && \Big(\text{since } \big|\exp\big(\, it\mu_B\,\big)\big| \leq 1\Big) \\
&= \big|\, \hat{F}_{\hat{B}}(\sigma_B\, t) \,\big| && (\text{since } T_{\hat{B}} = (T_B - \mu_B)/\sigma_B) \\
&\leq a_1'\,(n-l)^{-a_2' \log\ (n-l)} && \text{for } \frac{a_3' \log\ (n-l)}{\sigma_B} \leq \big|\, t \,\big| \leq \frac{a_4'\,(n-l)^{3/2}}{\sigma_B} \\
& && \text{and} \quad B \in M(l, \hat{A}),\ 0 \leq l \leq 17.
\end{aligned}$$

Because of

- $n \leq 10\,(n-l)$ for $n \geq 19$ and $0 \leq l \leq 17$,
- $\log\ (n-l) \geq \dfrac{1}{5}\ \log\ n$ for $n \geq 19$ and $0 \leq l \leq 17$,

- $\sqrt{\frac{3}{4}} \leq \frac{1}{\sigma_B} \leq \sqrt{\frac{3}{2}}$, and
- $f(x) = c_1\, x^{-\,c_2\,\log x} = c_1\, e^{-\,c_2\,(\log x)^2}$ for $x > 0$

we finally obtain Proposition 4.3.9 with

$$b_1 = a_1', \qquad b_2 = \frac{a_2'}{25}, \qquad b_3 = \sqrt{\frac{3}{2}}\, a_3' \quad \text{and} \quad b_4 = \sqrt{\frac{3}{4}}\left(\frac{1}{10}\right)^{3/2} a_4'. \qquad \square$$

As Lemma 4.1.7, c) shows, we need an additional estimate of the **derivative** of $\hat{F}_B$, when proving the condition (3.1.15).

This estimate is given in the following proposition.

4.3.11 Proposition

Suppose that the conditions and notations of the previous Proposition 4.3.9 apply.

Furthermore, let b_1, b_2, b_3 and b_4 be the same positive constants as in this Proposition 4.3.9.

Then in the case of $n \geq n_1$ and $\beta_A \leq \epsilon_0\, n$ holds:

$$(4.3.12) \qquad \left|\, \hat{F}_B'(t) \,\right| \;\leq\; b_1\, n^{-b_2 \log n + \frac{1}{2}} \qquad \text{for} \quad b_3 \log n \leq \left|\, t \,\right| \leq b_4\, n^{3/2} \quad \text{and} \quad B \in N(16, \hat{A}).$$

Proof:

For $B \in M(l, \hat{A})$, we denote by B_{jk} the matrix that we obtain from B by cancelling the jth row and the kth column. Thus, $B_{jk} \in M(l+1, \hat{A})$.

So we have:

$$\begin{aligned}
\left|\, \hat{F}_B'(t) \,\right| \;&=\; \left| E\Big(T_B \, \exp\big\{itT_B\big\} \Big) \right| \\
&\leq \sum_{j=1}^{n-l} \left| E\Big(b_{j\pi(j)} \, \exp\big\{itT_B\big\} \Big) \right| \\
&\leq \frac{1}{n-l} \sum_{j,k=1}^{n-l} \left| E\Big(b_{j\pi(j)} \, \exp\big\{itT_B\big\} \,\Big|\, \pi(j) = k \Big) \right| \qquad \Big(\text{since } P\big(\,\pi(j) = k\,\big) = \frac{1}{n-l}\Big) \\
&= \frac{1}{n-l} \sum_{j,k=1}^{n-l} \left| E\Big(b_{jk} \, \exp\big\{itb_{jk}\big\} \, \exp\big\{itT_{B_{jk}}\big\} \Big) \right|.
\end{aligned}$$

The last equation follows, since $P\big(\pi = \,.\, \big|\, \pi(j) = k\big)$ is the uniform distribution on the set of permutations $\pi_0 \in \mathscr{P}_{n-l}$ with $\pi_0(j) = k$. Due to $\big|\exp\big(itb_{jk}\big)\big| \le 1$, we further get

$$\le \frac{1}{n-l} \sum_{j,k=1}^{n-l} \big|\, b_{jk} \,\big| \Big| E\Big(\exp\{itT_{B_{jk}}\}\Big) \Big|$$

$$\le b_1\, n^{-b_2 \log n}\, \frac{1}{n-l} \sum_{j,k=1}^{n-l} \big|\, b_{jk} \,\big| \qquad \text{(note that } B_{jk} \in N(17, \hat{A})\text{)}$$

$$\le b_1\, n^{-b_2 \log n + \frac{1}{2}} \qquad \text{for} \quad b_3 \log n \le \big|\, t \,\big| \le b_4\, n^{3/2}.$$

The Proposition 4.3.9 was used for the penultimate inequality. For the last inequality, we first applied Hölder's inequality (0.1.5) with $\nu = (n-l)^2$ and $p = 2$, and then (3.1.1), i.e.

$$\frac{1}{n-l} \sum_{j,k=1}^{n-l} \big|\, b_{jk} \,\big| \le \frac{1}{n-l} \Big((n-l)^2 \sum_{j,k=1}^{n-l} b_{jk}^2 \Big)^{1/2}$$

$$\le \Big(\sum_{j,k=1}^{n} \hat{a}_{jk}^2 \Big)^{1/2} \le n^{1/2}.$$

□

After all these preparations, we can now derive from the conditions of van Zwet the requirements (3.1.11), (3.1.14), (3.1.15) and thus the results of the Theorems 3.1.10 and 3.1.13 (but with "constants" $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, which grow slightly with n).

We obtain the following result, which is central to this chapter:

4.3.13 Theorem

Let $A = (a_{ij})$ be an $n{\times}n-$matrix, whose elements have the form $a_{ij} = e_i\, d_j$. In addition, let $\mathscr{F}_A$, $e_{1,A}$, $e_{2,A}$ and D_A, E_A be defined as in chapter 3.

Furthermore, suppose that there exist positive constants e, E, d, D and δ such that

(4.3.2) $$\sum_{i=1}^{n} \big|\, e_i - \overline{e} \,\big|^r \ge en, \qquad \sum_{i=1}^{n} \big|\, e_i - \overline{e} \,\big|^k \le En$$
for some $k > 2$ and $0 < r < k$.

(4.3.3) $$\sum_{j=1}^{n} \big|\, d_j - \overline{d} \,\big|^m \ge dn, \qquad \sum_{j=1}^{n} \big|\, d_j - \overline{d} \,\big|^s \le Dn$$
for some $s > 2$ and $0 < m < s$.

$$\lambda\Big(\bigcup_{j=1}^{n}\Big\{\, x\in\mathbb{R}\,:\,\big|\,x-d_j\,\big|<\zeta\,\Big\}\Big)\;\geq\;\delta n\zeta \tag{4.3.4}$$

for some $\zeta \geq n^{-3/2}\,\log\; n$.

a) Then there exist constants $\mathcal{E}_1 > 0$ and $\mathcal{E}_2 > 0$ depending only on e, E, d, D, δ and r, k, m, s such that

$$\begin{aligned} (4.3.14)\qquad ||\,\mathscr{F}_A - e_{1,A}\,|| \;&\leq\; \mathcal{E}_1\,(\log n)^2\,D_A^2 \\ &\leq\; \mathcal{E}_2\,(\log n)^2\,n^{-\,1+\,\big((4/k)\,-\,1\big)^+\,+\,\big((4/s)\,-\,1\big)^+}. \end{aligned}$$

b) Let $\epsilon > 0$. Then there exist constants $\mathcal{E}_3 > 0$ and $\mathcal{E}_4 > 0$ depending only on e, E, d, D, δ, r, k, m, s and ϵ such that

$$\begin{aligned} (4.3.15)\qquad ||\,\mathscr{F}_A - e_{2,A}\,|| \;&\leq\; \mathcal{E}_3\,n^{\epsilon}\,E_A^3 \\ &\leq\; \mathcal{E}_4\,n^{-\,(3/2)\,+\,\epsilon\,+\,\big((5/k)\,-\,1\big)^+\,+\,\big((5/s)\,-\,1\big)^+}. \end{aligned}$$

Proof:

a) At first, it should be noted that we can assume

$$n \geq n_1 \text{ and } \beta_A \leq \epsilon_0\, n \qquad \text{with } n_1,\, \epsilon_0 \text{ from Proposition 4.3.9}$$

without loss of generality. To justify this, we proceed as in (3.7.7), (3.7.8) and (3.7.9).

Because of Theorem 3.1.10, Lemma 4.1.7, a) and inequality (4.3.7), it is now sufficient to show the existence of a constant c_1 depending only on e, E, d, D, δ and r, k, m, s such that

$$\int_{\mathbb{R}} \big(\,1+|\,t\,|\,\big)\,\big|\,\hat{F}_B(t)\,\hat{U}_{D_A^2}(t)\,\big|\,dt \;\leq\; c_1\,(\log n)^2 \qquad \text{for all } B\in N(8,\hat{A}). \tag{4.3.16}$$

Thereby U is defined as in the section 4.1 (see in particular (4.1.1) and (4.1.2)) of this chapter 4. Because of

$$\int_{\mathbb{R}} \big(\,1+|\,t\,|\,\big)\,\big|\,\hat{F}_B(t)\,\hat{U}_{D_A^2}(t)\,\big|\,dt \;\leq\; 2+2\int_{\mathbb{R}} |\,t\,|\,\big|\,\hat{F}_B(t)\,\hat{U}_{D_A^2}(t)\,\big|\,dt,$$

it is also sufficient to prove (4.3.16) with $|\,t\,|$ instead of $1+|\,t\,|$.

Using the constants b_1, b_2, b_3, b_4 from Proposition 4.3.9 and the abbreviation $\rho = D_A^2$ we obtain

$$\int_{\mathbb{R}} |\,t\,|\,\big|\,\hat{F}_B(t)\,\hat{U}_\rho(t)\,\big|\,dt$$

$$\leq 2\left\{\int_0^{b_3\log n} t\,dt + \int_{b_3\log n}^{b_4 n^{3/2}} t\,\big|\,\hat{F}_B(t)\,\big|\,dt + \int_{b_4 n^{3/2}}^{\infty} t\,\big|\,\hat{U}_\rho(t)\,\big|\,dt\right\}$$

$$\leq b_3^2\,(\log n)^2 \,+\, b_1\,n^{-b_2\log n}\Big[\,b_4^2\,n^3 - b_3^2\,(\log n)^2\,\Big] \,+\, 2\int_{b_4 n^{3/2}}^{\infty} t\,\big|\,\hat{U}_\rho(t)\,\big|\,dt.$$

Because of $b_2 > 0$, it follows that

$$\lim_{n\to\infty} n^{-b_2\log n}\Big[\,b_4^2\,n^3 - b_3^2\,(\log n)^2\,\Big]$$

$$= \lim_{n\to\infty} \exp\big(-b_2\,(\log n)^2\,\big)\Big[\,b_4^2\,\exp\big(\,3\log n\,\big) - b_3^2\,(\log n)^2\,\Big] \,=\, 0,$$

so that only the third summand remains to be analyzed:

$$\int_{b_4 n^{3/2}}^{\infty} t\,\big|\,\hat{U}(\rho\,t)\,\big|\,dt \qquad (\hat{U}_\rho(t) = \hat{U}(\rho\,t))$$

$$= \frac{1}{\rho^2}\int_{b_4 n^{3/2}\rho}^{\infty} v\,\big|\,\hat{U}(v)\,\big|\,dv \qquad \text{(substitution } v = t\,\rho\text{)}.$$

Furthermore, since the inequality $\rho \geq \dfrac{1}{4n}$ is valid due to Lemma 3.1.18, a), we get with the definition $c_2 = \dfrac{1}{4}\,b_4$

$$\leq 16\,n^2\int_{c_2 n^{1/2}}^{\infty} v\,\big|\,\hat{U}(v)\,\big|\,dv$$

$$\leq 16\,n^2\int_{c_2 n^{1/2}}^{\infty} \frac{v^6}{c_2^5\,n^{5/2}}\,\big|\,\hat{U}(v)\,\big|\,dv$$

$$\leq \frac{16}{c_2^5\,n^{1/2}}\int_{\mathbb{R}} v^6\,\big|\,\hat{U}(v)\,\big|\,dv \to 0 \qquad \text{as } n\to\infty \qquad \text{(cf. (4.1.4))}.$$

Thus, the part a) is proved.

b) Analogous to the proof of part a) we can assume

$$n \geq n_1 \text{ and } \frac{\beta_A}{n} \leq E_A \leq \epsilon_0 \quad \text{with } n_1,\, \epsilon_0 \text{ from Proposition 4.3.9}$$

without loss of generality. To justify this, we proceed as in (3.8.3), (3.8.4) and (3.8.5).

Because of Theorem 3.1.13, Lemma 4.1.7, inequality (4.3.8) and

$$\left|\, \hat{U}'_{E_A^2}(t) \,\right| = E_A^2 \left|\, \hat{U}'(E_A^2\, t) \,\right| \leq E_A^2 \leq \epsilon_0^2 \leq 1 \qquad \text{and}$$

$$\left|\, \hat{F}'_B(t) \,\right| = \left|\, E\big(T_B \, \exp\{ itT_B \} \big) \,\right| \leq E\big(\left| T_B \right| \big) \leq \big(\sigma_B^2 \big)^{1/2} \leq \sqrt{\frac{4}{3}}$$

$$\text{for all } B \in N(16, \hat{A}),$$

it is then sufficient to show the existence of constants c_3, c_4, c_5 and c_6 depending only on e, E, d, D, δ, r, k, m, s and ϵ such that

$$\text{(4.3.17)} \qquad \int_{\mathbb{R}} |\, t \,|\, \left|\, \hat{F}'_B(t)\, \hat{U}_{E_A^2}(t) \,\right| dt \;\leq\; c_3\, (\log n)^2,$$

$$\text{(4.3.18)} \qquad \int_{\mathbb{R}} |\, t \,|\, \left|\, \hat{F}_B(t)\, \hat{U}'_{E_A^2}(t) \,\right| dt \;\leq\; c_4\, (\log n)^2,$$

$$\text{(4.3.19)} \qquad \int_{\mathbb{R}} |\, t \,|\, \left|\, \hat{F}_B(t)\, \hat{U}_{E_A^2}(t) \,\right| dt \;\leq\; c_5\, (\log n)^2,$$

$$\text{(4.3.20)} \qquad \int_{\mathbb{R}} t^2 \left|\, \hat{F}_B(t)\, \hat{U}_{E_A^3\, n^{\epsilon/2}}(t) \,\right| dt \;\leq\; c_6\, (\log n)^3 \qquad \text{for all } B \in N(16, \hat{A}).$$

The statements (4.3.17), (4.3.18) and (4.3.19) are proved analogously to (4.3.16) using the Propositions 4.3.9 and 4.3.11 and

$$E_A^2 \geq 2^{-5/3}\, n^{-1} \text{ (cf. 3.1.18, a)), } \left|\, \hat{U}'_{E_A^2}(t) \,\right| \leq 1 \text{ and } \left|\, \hat{F}'_B(t) \,\right| \leq \sqrt{\frac{4}{3}}.$$

It remains to verify (4.3.20). As in part a) we obtain with the abbreviation $\theta = E_A^3\, n^{\epsilon/2}$:

$$\int_{\mathbb{R}} t^2 \left|\, \hat{F}_B(t)\, \hat{U}_\theta(t) \,\right| dt$$

$$\leq \frac{2}{3}\, b_3^3\, (\log n)^3 \;+\; \frac{2}{3}\, b_1\, n^{-b_2 \log n} \left[b_4^3\, n^{9/2} - b_3^3\, (\log n)^3 \right]$$

$$+\ 2 \int\limits_{b_4 n^{3/2}}^{\infty} t^2 \,\big|\, \hat{U}_\theta(t) \,\big|\, dt.$$

Analogous to part a), we get for the third summand

$$\int\limits_{b_4 n^{3/2}}^{\infty} t^2 \,\big|\, \hat{U}(\theta\, t) \,\big|\, dt \qquad\qquad (\hat{U}_\theta(t) = \hat{U}(\theta\, t))$$

$$= \frac{1}{\theta^3} \int\limits_{b_4 n^{3/2}\theta}^{\infty} v^2 \,\big|\, \hat{U}(v) \,\big|\, dv \qquad\qquad \text{(substitution } v = t\,\theta).$$

Furthermore, since the inequality $\theta \geq \dfrac{n^{\epsilon/2}}{\sqrt{32}\, n^{3/2}}$ is valid due to Lemma 3.1.18, a), we get with the definition $c_7 = \dfrac{1}{\sqrt{32}}\, b_4$

$$\leq \frac{(\sqrt{32})^3\, n^{9/2}}{n^{3\epsilon/2}} \int\limits_{c_7 n^{\epsilon/2}}^{\infty} v^2 \,\big|\, \hat{U}(v) \,\big|\, dv$$

$$\leq \frac{(\sqrt{32})^3\, n^{9/2}}{n^{3\epsilon/2}} \int\limits_{c_7 n^{\epsilon/2}}^{\infty} \frac{v^{\,2\,+\,(9/\epsilon)}}{\big(\, c_7\, n^{\epsilon/2}\,\big)^{9/\epsilon}} \,\big|\, \hat{U}(v) \,\big|\, dv$$

$$\leq \frac{(\sqrt{32})^3}{c_7^{9/\epsilon}\, n^{3\epsilon/2}} \int\limits_{\mathbb{R}} |\, v\,|^{\,2\,+\,(9/\epsilon)} \,\big|\, \hat{U}(v) \,\big|\, dv \to 0 \qquad \text{as } n \to \infty \qquad\qquad \text{(cf. (4.1.4))}.$$

Thus, (4.3.20) is proved. All together, since $n^{\epsilon/2}(\log n)^3 \leq n^\epsilon$ for sufficiently large n, the part b) is also proved. □

4.3.21 Remarks

a) According to Lemma 3.1.18, e) follows from (4.3.15)

$$||\, \mathscr{F}_A - e_{1,A} \,|| \;\leq\; ||\, \mathscr{F}_A - e_{2,A} \,|| \;+\; ||\, e_{2,A} - e_{1,A} \,||$$

$$\leq\; \mathcal{E}_3\, n^\epsilon\, E_A^3 \;+\; \frac{1}{2}\, D_A^2.$$

Therefore, in the case of

$$\Big[\, \big((5/k)\, -\, 1\big)^+ \,-\, \big((4/k)\, -\, 1\big)^+ \Big] \,+\, \Big[\, \big((5/s)\, -\, 1\big)^+ \,-\, \big((4/s)\, -\, 1\big)^+ \Big] \,<\, \frac{1}{2}, \tag{4.3.22}$$

we even have

$$(4.3.23) \qquad ||\,\mathscr{F}_A - e_{1,A}\,|| \;\le\; \mathcal{E}_2'\, n^{-1+\left((4/k)-1\right)^+ + \left((4/s)-1\right)^+}$$

due to Lemma 4.3.6, c), where $\mathcal{E}_2' > 0$ depends only on e, E, d, D, δ and r, k, m, s.

b) If, compared to Lemma 3.1.18, a), we additionally have

$$E_A^3 \ge C\, n^{-3/2+\eta}, \text{ where } C, \eta > 0,$$

then the proof of (4.3.20) shows that (4.3.15) can be strengthened as follows:

$$(4.3.24) \qquad \begin{aligned} ||\,\mathscr{F}_A - e_{2,A}\,|| \;&\le\; \mathcal{E}_3'\,(\log n)^3\, E_A^3 \\ &\le\; \mathcal{E}_4'\,(\log n)^3\, n^{-(3/2)+\left((5/k)-1\right)^+ + \left((5/s)-1\right)^+}, \end{aligned}$$

where $\mathcal{E}_3' > 0$ and $\mathcal{E}_4' > 0$ depend only on e, E, d, D, δ, r, k, m, s and C, η.

c) We refer again to the Remark 3.1.17, a). This remark also applies analogously to the Theorem 4.3.13 and the two remarks above.

d) In the two-sample case, i.e. in the case $e_1 = \ldots = e_m = 1$ and $e_{m+1} = \ldots = e_n = 0$, the corresponding condition of (4.3.2) is:

$$(4.3.25) \qquad \text{There exists an } \eta > 0 \text{ such that } \eta \le \frac{m}{n} \le 1-\eta.$$

Since (4.3.2) follows from this even for any $0 < r < k$, the $(\log n)^2$ can always be omitted in the two-sample case in (4.3.14) (cf. Remark a) above).

Furthermore, it is known from the literature that in the two-sample case the n^ϵ from (4.3.15) (or $(\log n)^3$ from (4.3.24)) can either be omitted completely or replaced by smaller terms. We refer to the papers of Bickel and van Zwet ([5], Corollary 2.1, and second remark, p. 949 centre) and Robinson ([22], Theorem and, to weaken his condition (c), the paper [21], p. 861, Remark, and p. 852, Condition (B), (C)).

e) The condition (4.3.4) is necessary to prevent the scores $d_1, \ldots, d_n$ from being too close together in a few points and thus giving the distribution of $\mathscr{T}_A$ a grid character that is too strong (such as S_n or X_i in the Remark 1.1.6, c)).

As an example where this happens and therefore the convergence rates of the theorem do not hold, the following sequence of $n\times n-$matrices $A^{(n)} = \left(e_i^{(n)}\, d_j^{(n)}\right)$ is given:

$$e_i^{(n)} = \begin{cases} 1 & \text{for} \quad i \le \left\lfloor \frac{n}{2} \right\rfloor \\ 0 & \text{for} \quad i > \left\lfloor \frac{n}{2} \right\rfloor \end{cases}, \qquad d_j^{(n)} = \begin{cases} -1 & \text{for} \quad j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 1 & \text{for} \quad j > \left\lfloor \frac{n}{2} \right\rfloor \end{cases},$$

("Median test", cf. [16], Section 4.1.1, page 97).

For $n \ge 2$ and $k \ge 0$ then holds

$$\frac{n}{3^k} \le \sum_{i=1}^{n} \left| e_i^{(n)} - \bar{e}^{(n)} \right|^k \le n, \qquad \left(\frac{2}{3}\right)^k n \le \sum_{j=1}^{n} \left| d_j^{(n)} - \bar{d}^{(n)} \right|^k \le 2^k n.$$

On the other hand, however, we get (analogously to the Remark 1.1.6, c))

$$\begin{aligned} & \| \mathscr{F}_{A^{(4n)}} - e_{1,A^{(4n)}} \| \quad \text{bzw.} \quad \| \mathscr{F}_{A^{(4n)}} - e_{2,A^{(4n)}} \| \\ \ge\ & \frac{1}{2} P\big(\mathscr{T}_{A^{(4n)}} = 0 \big) \\ =\ & \frac{1}{2} P\big(T_{A^{(4n)}} = 0 \big) && (\text{since } \mu_{A^{(4n)}} = 0 \text{ and } 0 \cdot \sigma_{A^{(4n)}} = 0) \\ =\ & \frac{1}{2} \frac{\binom{2n}{n}^2}{\binom{4n}{2n}} \sim \frac{1}{\sqrt{2\pi n}} \qquad \text{for } n \to \infty && (\text{Stirling's formula}). \end{aligned}$$

4.4 Applications to the case of exact and approximating scores

Let $J : (0,\, 1) \to \mathbb{R}$ be an integrable function. Furthermore, we denote by $U_{j:n}$ the jth order statistic in a random sample of size n from the uniform distribution on (0, 1).

In the following we consider the scores

(4.4.1) $\qquad d_j = E\big(J(U_{j:n}) \big), \qquad j = 1, \dots, n \qquad$ ("exact scores")

and

(4.4.2) $\qquad d_j = J\Big(\dfrac{j}{n+1} \Big), \qquad j = 1, \dots, n \qquad$ ("approximating scores").

We investigate which conditions must be placed on J in each of the two upper cases so that the scores d_j fulfil the conditions (4.3.3) and (4.3.4).

In addition, we analyze for which J the expansions $e_{1,A}$ and $e_{2,A}$ can be replaced by the following somewhat simpler expansions (with integrals)

$$e^I_{1,A}(x) = \Phi(x) - \psi(x)\,\frac{\xi_{1,A}}{6}\,(x^2-1) \quad \text{and}$$
$$e^I_{2,A}(x) = \Phi(x) - \psi(x)\left\{\frac{\xi_{1,A}}{6}\,(x^2-1) + \frac{\xi_{2,A}}{24}\,(x^3-3x) + \frac{\xi^2_{1,A}}{72}\,(x^5-10x^3+15x)\right\},$$

where

$$(4.4.3) \qquad \hat{e}_i = \frac{e_i - \bar{e}}{\sqrt{\sum_{l=1}^{n}\left(e_l-\bar{e}\right)^2}}\,, \qquad \hat{J} = \frac{J - \int_0^1 J(t)\,dt}{\sqrt{\int_0^1 J^2(t)\,dt - \left(\int_0^1 J(t)\,dt\right)^2}}$$

and

$$(4.4.4) \qquad \xi_{1,A} = \sum_{i=1}^{n} \hat{e}_i^3 \int_0^1 \hat{J}^3(t)\,dt,$$

$$(4.4.5) \qquad \xi_{2,A} = \left(\sum_{i=1}^{n} \hat{e}_i^4 \left\{\int_0^1 \hat{J}^4(t)\,dt\, -\, 3\right\}\right) - \frac{3}{n}\left\{\int_0^1 \hat{J}^4(t)\,dt\, -\, 1\right\}.$$

It should be mentioned that the type of conditions placed on J in this section can also be found in Albers, Bickel and van Zwet [1], Bickel and van Zwet [5] and Does [12], [13], [14].

We start with a few simple but useful preliminary considerations.

4.4.6 Lemma

Let $\beta > 1$ and $n \in \mathbb{N}$. Then

$$(4.4.7) \qquad \sum_{j=1}^{n}\left(\frac{j}{n+1}\left(1-\frac{j}{n+1}\right)\right)^{-\beta} \le C(\beta)\,n^{\beta},$$

where $C(\beta) = 4^{\beta}\sum_{j=1}^{\infty}\frac{1}{j^{\beta}} < \infty$.

Proof:

$$\sum_{j=1}^{n}\left(\frac{j}{n+1}\left(1-\frac{j}{n+1}\right)\right)^{-\beta} = \sum_{j=1}^{n}\left(\frac{(n+1)^2}{j\,(n+1-j)}\right)^{\beta}$$

$$= (n+1)^\beta \sum_{j=1}^{n} \left(\frac{1}{j} + \frac{1}{n+1-j} \right)^\beta .$$

An application of Hölder's inequality (0.1.5) with $\nu = 2$ and $p = \beta$ further yields

$$\begin{aligned}
&\le (n+1)^\beta \sum_{j=1}^{n} 2^{\beta-1} \left(\left(\frac{1}{j}\right)^\beta + \left(\frac{1}{n+1-j}\right)^\beta \right) \\
&= (n+1)^\beta \, 2^{\beta-1+1} \sum_{j=1}^{n} \frac{1}{j^\beta} \\
&\le (2n)^\beta \, 2^\beta \sum_{j=1}^{\infty} \frac{1}{j^\beta} \qquad \text{(because of } n+1 \le 2n) \\
&= C(\beta)\, n^\beta . \qquad \square
\end{aligned}$$

The following lemma contains an important definition for this section and some basic consequences.

4.4.8 Lemma

Let $g : (0,\, 1) \to \mathbb{R}$ be continuously differentiable and $0 < \alpha < 1$.

Then g satisfies the **condition $\boldsymbol{V_\alpha}$**, if a $\Gamma > 0$ exists such that

(4.4.9) $\qquad |\, g'(t)\,| \le \Gamma \big(t\,(1-t) \big)^{-1-\alpha} \qquad$ for all $t \in (0,\, 1)$.

For functions g satisfying the condition V_α holds:

a) There exists a constant $C > 0$ depending only on g, Γ and α such that

(4.4.10) $\qquad |\, g(t)\,| \le C \big(t\,(1-t) \big)^{-\alpha} \qquad$ for all $t \in (0,\, 1)$.

b) g^s is integrable for every $s > 0$ with $\alpha < \dfrac{1}{s} \;\Leftrightarrow\; s\alpha < 1$.

Proof:

Since the function $f(t) = \Gamma \big(t\,(1-t) \big)^{-1-\alpha}$ is axially symmetric to the line $t = 1/2$, it is sufficient to consider the intervall $(0,\, 1/2\,]$ instead of the intervall $(0,\, 1)$.

a) For $x \in (0,\, 1/2\,]$ we have

$$\begin{aligned} g(x) - g(1/2) &= \int_{1/2}^{x} g'(t)\,dt = \int_{x}^{1/2} -g'(t)\,dt \le \int_{x}^{1/2} |\,g'(t)\,|\,dt \\ &\le \Gamma \int_{x}^{1/2} \big(\,t\,(1-t)\,\big)^{-1-\alpha}\,dt \le \Gamma\left(\frac{1}{2}\right)^{-1-\alpha} \int_{x}^{1/2} t^{-1-\alpha}\,dt \\ &= \Gamma\,\frac{2^{1+\alpha}}{\alpha}\left(x^{-\alpha} - 2^{\alpha}\right) \le \Gamma\,\frac{2^{1+\alpha}}{\alpha}\,x^{-\alpha} \le \Gamma\,\frac{2^{1+\alpha}}{\alpha}\,\big(\,x\,(1-x)\,\big)^{-\alpha}. \end{aligned}$$

It follows that

$$g(x) \le \left[\Gamma\,\frac{2^{1+\alpha}}{\alpha} + |\,g(1/2)\,|\,4^{-\alpha}\right]\big(\,x\,(1-x)\,\big)^{-\alpha} = C\,\big(\,x\,(1-x)\,\big)^{-\alpha}.$$

Since $-g(x) \le C\,\big(\,x\,(1-x)\,\big)^{-\alpha}$ can be derived similarly, we get $|\,g(x)\,| \le C\,\big(\,x\,(1-x)\,\big)^{-\alpha}$.

b) This follows from (4.4.10) and since $\int_{0}^{1/2} t^{-\beta}\,dt < \infty$ for $0 < \beta < 1$. □

We now consider the case of the approximating scores (cf. (4.4.2)). The following elementary lemma is an important help for this case.

4.4.11 Lemma

Let $g : (0,\,1) \to \mathbb{R}$ be continuously differentiable and $0 < \alpha < 1$ such that the **condition $\boldsymbol{V_\alpha}$** (cf. (4.4.9)) is satisfied for g.

Then there exist constants $C_1 > 0$ and $C_2 > 0$ depending only on g and α such that

$$\left|\,\sum_{j=1}^{n} g\Big(\frac{j}{n+1}\Big) - n\int_{0}^{1} g(t)\,dt\,\right| \le C_1\,n^{\alpha} \qquad \text{for all } n \in \mathbb{N}, \tag{4.4.12}$$

$$\left|\,\sum_{j=1}^{n} \left|\,g\Big(\frac{j}{n+1}\Big)\right| - n\int_{0}^{1} |\,g(t)\,|\,dt\,\right| \le C_2\,n^{\alpha} \qquad \text{for all } n \in \mathbb{N}. \tag{4.4.13}$$

Proof:

We get for all $n \ge 2$:

$$\left|\,\frac{1}{n}\sum_{j=1}^{n} g\Big(\frac{j}{n+1}\Big) - \int_{0}^{1} g(t)\,dt\,\right|$$

$$\leq \int_0^{1/(n+1)} |\, g(t)\,|\, dt + \int_{n/(n+1)}^{1} |\, g(t)\,|\, dt + \frac{1}{n}\Big|\, g\Big(\frac{n}{n+1}\Big)\Big|$$

$$+ \Big|\frac{1}{n} - \frac{1}{n+1}\Big| \sum_{j=1}^{n-1} \Big|\, g\Big(\frac{j}{n+1}\Big)\Big| + \Big| \int_{1/(n+1)}^{n/(n+1)} g(t)\, dt - \frac{1}{n+1}\sum_{j=1}^{n-1} g\Big(\frac{j}{n+1}\Big)\Big|$$

$$= S_1 + S_2 + S_3 + S_4 + S_5.$$

For the estimates of S_1 to S_4 we use (4.4.10) from Lemma 4.4.8, a) and for the estimate of S_5 we use (4.4.9).

In addition to the constant C from (4.4.10), we obtain further constants c_1, c_2, c_3 and c_4 depending only on g, Γ and α such that

$$S_1 \leq C\, 2^\alpha \int_0^{1/(n+1)} t^{-\alpha}\, dt = C\, \frac{2^\alpha}{1-\alpha}\Big(\frac{1}{n+1}\Big)^{1-\alpha} \leq c_1\, n^{\alpha-1},$$

$$S_2 \leq c_1\, n^{\alpha-1} \qquad \text{(analogous due to the axial symmetry of } \big(t\,(1-t)\big)^{-\alpha} \text{ to } t = \tfrac{1}{2}\text{)},$$

$$S_3 \leq C\,\frac{1}{n}\Big(\frac{n}{n+1}\Big(1-\frac{n}{n+1}\Big)\Big)^{-\alpha} \leq C\,2^\alpha\,\frac{1}{n}\Big(\frac{1}{n+1}\Big)^{-\alpha} \leq c_2\, n^{\alpha-1},$$

$$S_4 \leq C\,\frac{1}{n^2}\sum_{j=1}^{n-1}\Big(\frac{j}{n+1}\Big(1-\frac{j}{n+1}\Big)\Big)^{-\alpha}$$

$$\leq C\,\frac{1}{n^2}\sum_{j=1}^{n-1}\Big(\frac{j}{n+1}\Big(1-\frac{j}{n+1}\Big)\Big)^{-1-\alpha} \qquad \text{(since } \Big(\frac{j}{n+1}\Big(1-\frac{j}{n+1}\Big)\Big) \leq 1\text{)}$$

$$\leq c_3\, n^{\alpha-1} \qquad \text{(due to Lemma 4.4.6 with } \beta = 1+\alpha\text{)},$$

$$S_5 \leq \sum_{j=1}^{n-1} \int_{j/(n+1)}^{(j+1)/(n+1)} \Big|\, g(t) - g\Big(\frac{j}{n+1}\Big)\Big|\, dt$$

$$\leq \frac{1}{n^2}\sum_{j=1}^{n-1} \sup\Big\{\, |\, g'(t)\,| \,:\, \frac{j}{n+1} < t < \frac{j+1}{n+1}\Big\} \qquad \text{(mean value theorem)}$$

$$\leq \frac{\Gamma}{n^2}\sum_{j=1}^{n}\Big(\frac{j}{n+1}\Big(1-\frac{j}{n+1}\Big)\Big)^{-1-\alpha} \qquad \Big(\big(t\,(1-t)\big)^{-1-\alpha} \text{ monotone for } t \lessgtr \tfrac{1}{2}\Big)$$

$$\leq c_4 \, n^{\alpha-1} \qquad \text{(due to Lemma 4.4.6 with } \beta = 1+\alpha\text{)}.$$

Now, since we can choose Γ as a function of g and α by defining

$$\Gamma = \Gamma(g,\alpha) := \min\Big\{ \widetilde{\Gamma} \, : \, |\, g'(t) \,| \leq \widetilde{\Gamma} \, \big(t\,(1-t) \big)^{-1-\alpha} \text{ for all } t \in (0,\,1) \Big\}, \tag{4.4.14}$$

the estimate (4.4.12) follows.

The proof of (4.4.13) is completely analogous except that the reverse triangle inequality is additionally used for the estimate of S_5. □

The lemma just proved is now used to give estimates for the sums from (4.3.3) for the approximating scores d_j from (4.4.2).

4.4.15 Corollary

Let $J : (0,\,1) \to \mathbb{R}$ be continuously differentiable and $0 < \alpha < 1$ such that the **condition $\boldsymbol{V_\alpha}$** (cf. (4.4.9)) is satisfied for J. In addition, let $w \in \mathbb{N}$ with $\alpha < \dfrac{1}{w}$.

Furthermore, we define

$$\bar{J} = \frac{1}{n} \sum_{j=1}^{n} J\Big(\frac{j}{n+1} \Big).$$

If $\int_0^1 J(t)\,dt = 0$, then there exist constants $C_1 > 0$ and $C_2 > 0$ depending only on J, α and w such that

$$\bigg| \sum_{j=1}^{n} \Big(J\Big(\frac{j}{n+1}\Big) - \bar{J} \Big)^{w} - n \int_0^1 J^w(t)\,dt \bigg| \leq C_1 \, n^{w\alpha} \qquad \text{for all } n \in \mathbb{N}, \tag{4.4.16}$$

$$\bigg| \sum_{j=1}^{n} \Big| J\Big(\frac{j}{n+1}\Big) - \bar{J} \Big|^{w} - n \int_0^1 |\, J^w(t) \,|\,dt \bigg| \leq C_2 \, n^{w\alpha} \qquad \text{for all } n \in \mathbb{N}. \tag{4.4.17}$$

Proof:

At first, we note that $0 < w\alpha < 1$ and that J^w satisfies the condition $V_{w\alpha}$. This follows for $w \geq 2$ because of $\big(J^w\big)' = w\,J^{w-1}\,J'$ from (4.4.10) and (4.4.9).

To prove (4.4.16), we therefore apply Lemma 4.4.11 to $g = J^w$ and obtain

$$\left| \sum_{j=1}^{n} J^w\Big(\frac{j}{n+1}\Big) - n \int_0^1 J^w(t)\,dt \right| \le c_1\, n^{w\alpha}. \tag{4.4.18}$$

Like the following constants, c_1 only depends on J, α and w.

In addition, we have

$$\begin{aligned} &\left| \sum_{j=1}^{n} \Big(J\Big(\frac{j}{n+1}\Big) - \bar{J}\Big)^w - \sum_{j=1}^{n} J^w\Big(\frac{j}{n+1}\Big) \right| \\ &\le \sum_{j=1}^{n} \left| \Big(J\Big(\frac{j}{n+1}\Big) - \bar{J}\Big)^w - J^w\Big(\frac{j}{n+1}\Big) \right|. \end{aligned} \tag{4.4.19}$$

An application of the mean value theorem to $f(\theta) = (x - \theta\, y)^w$, where $x = J\Big(\frac{j}{n+1}\Big)$ and $y = \bar{J}$, further gives

$$\begin{aligned} &\le w\,|\,\bar{J}\,| \sum_{j=1}^{n} \sup_{0<\theta<1} \left| J\Big(\frac{j}{n+1}\Big) - \theta \bar{J} \right|^{w-1} \\ &\le w\,|\,\bar{J}\,| \sum_{j=1}^{n} \left(\left| J\Big(\frac{j}{n+1}\Big) \right| + |\,\bar{J}\,| \right)^{w-1}. \end{aligned}$$

Now, for $w > 2$ we use Hölder's inequality (0.1.5) with $\nu = 2$ and $p = w - 1$. For the remaining cases $w = 2$ and $w = 1$, however, the next step is clear. We obtain

$$\begin{aligned} &\le w\, 2^{w-2} \left(|\,\bar{J}\,| \sum_{j=1}^{n} \left| J\Big(\frac{j}{n+1}\Big) \right|^{w-1} + n\,|\,\bar{J}\,|^w \right) \\ &\le c_2 \left(n^{\alpha} + n^{w\alpha - w + 1} \right) \\ &\le 2\, c_2\, n^{w\alpha}. \end{aligned}$$

The penultimate inequality is valid, since we get the inequalities

$$|\,\bar{J}\,| \le c_3\, n^{\alpha-1} \quad \text{and} \quad \sum_{j=1}^{n} \left| J\Big(\frac{j}{n+1}\Big) \right|^{w-1} \le c_4\, n$$

due to $\int_0^1 J(t)\,dt = 0$ and Lemma 4.4.11 with $g = J$ and $g = J^{w-1}$. Thus (4.4.16) is proved.

The proof of (4.4.17) is completely analogous if we additionally use the reverse triangle inequality in the above estimate (4.4.19). □

4.4.20 Remark

Corollary 4.4.15 improves analogous results of Does [14] (cf. e.g. Lemmas 3.2.1, 2.2.1 and the proof of Lemma 3.4.1), where the additional condition

$$\limsup_{t \to 0,1} t\,(1-t) \left| \frac{J''(t)}{J'(t)} \right| < 2$$

is required.

After the above considerations, we can now state the results for the case of the approximating scores. They are summarized in the following theorem.

4.4.21 Theorem

Let $J : (0,\, 1) \to \mathbb{R}$ be a nonconstant and continuously differentiable function, and

$$n_1(J) = \sup \left\{ n \in \mathbb{N} \, : \, \sum_{j=1}^{n} \left(J\Big(\frac{j}{n+1} \Big) - \bar{J} \right)^2 \, = \, 0 \right\}.$$

In addition, let $A = (a_{ij})$ be an $n \times n-$matrix such that

$$n > n_1(J), \; a_{ij} = e_i \, d_j \text{ and } d_j = J\Big(\frac{j}{n+1} \Big).$$

Furthermore, suppose that the following two conditions hold:

(4.3.2) $\displaystyle\sum_{i=1}^{n} \left| \, e_i - \bar{e} \, \right|^r \, \geq \, en, \qquad \sum_{i=1}^{n} \left| \, e_i - \bar{e} \, \right|^k \, \leq \, En$

for some $k > 2$, $0 < r < k$ and $e > 0$, $E > 0$.

(4.4.22) J satisfies the **condition $\boldsymbol{V_\alpha}$** (cf. (4.4.9)) for some $0 < \alpha < \dfrac{1}{s}$,

where $s \in \mathbb{N}$ and $s \geq 3$.

a) Then there exists a constant $\mathcal{E}_5 > 0$ depending only on e, E, r, k and J, α, s such that

(4.4.23) $$\left|\left| \, \mathscr{F}_A - e_{1,A} \, \right|\right| \, \leq \, \mathcal{E}_5 \, (\log n)^2 \, n^{-1 + \left((4/k) - 1 \right)^+ + \left((4/s) - 1 \right)^+}.$$

b) Let $\epsilon > 0$. Then there exists a constant $\mathcal{E}_6 > 0$ depending only on e, E, r, k, J, α, s and

ϵ such that

(4.4.24) $$||\mathscr{F}_A - e_{2,A}|| \le \mathcal{E}_6\, n^{-(3/2)+\epsilon+\big((5/k)-1\big)^+ + \big((5/s)-1\big)^+}.$$

c) Let $k \ge 3$. Then there exists a constant $\mathcal{E}_7 > 0$ depending only on e, E, r, k and J, α, s such that

(4.4.25) $$||\mathscr{F}_A - e^I_{1,A}|| \le \mathcal{E}_7 \max\Big\{ n^{-(3/2)+3\alpha},\ (\log n)^2\, n^{-1+\big((4/k)-1\big)^+ + \big((4/s)-1\big)^+} \Big\}.$$

d) Let $\epsilon > 0$, $k \ge 4$ and $s \ge 4$. Then there exists a constant $\mathcal{E}_8 > 0$ depending only on e, E, r, k, J, α, s and ϵ such that

(4.4.26) $$||\mathscr{F}_A - e^I_{2,A}|| \le \mathcal{E}_8 \max\Big\{ n^{-(3/2)+3\alpha},\ n^{-(3/2)+\epsilon+\big((5/k)-1\big)^+ + \big((5/s)-1\big)^+} \Big\}.$$

Proof:

For a) and b):

To show a) and b), we use the Theorem 4.3.13. For this, we must first prove (4.3.3). We apply Corollary 4.4.15 to the function $\widetilde{J} = J - \int_0^1 J(t)\,dt$, once with $w = 2$ and once with $w = s$. Note that from (4.4.22) follows $\alpha < \frac{1}{s} \le \frac{1}{3} < \frac{1}{2}$. We then get:

There exist positive constants d, D depending only on J, α and s such that

$$\sum_{j=1}^{n} \big| d_j - \bar{d}\big|^2 \ge dn \quad \text{for } n > n_1(J), \qquad \sum_{j=1}^{n} \big| d_j - \bar{d}\big|^s \le Dn \quad \text{for } n \in \mathbb{N}.$$

Next we prove (4.3.4). Since J is not constant and continuously differentiable, there exists an intervall $\Psi = [\,t_1,\, t_2\,]$ with $0 < t_1 < t_2 < 1$ such that

$$0 < c_1 = \min_{t\in\Psi} |\, J'(t)\,| \le c_2 = \max\Big\{ 1,\ \max_{t\in\Psi} |\, J'(t)\,| \Big\} < \infty.$$

Furthermore, let $n_0 \in \mathbb{N}$ such that $t_2 - t_1 \ge \dfrac{1}{n_0+1}$ and $n \ge n_0$.

Now, for $x \in \Psi$ there exists an $j_0 \in \{1, \dots, n\}$ such that $\dfrac{j_0}{n+1} \in \Psi$ and $\Big| x - \dfrac{j_0}{n+1} \Big| < \dfrac{1}{n+1}$. Due to the mean value theorem we get

$$\Big| J(x) - J\Big(\frac{j_0}{n+1}\Big) \Big| \le \Big| x - \frac{j_0}{n+1} \Big| \cdot c_2 < \frac{c_2}{n+1} < \frac{c_2}{n}$$

and then with the definitions $\zeta = \dfrac{c_2}{n} \left(\geq \dfrac{1}{n} \geq n^{-3/2} \log\ n \right)$ and $\delta = \dfrac{c_1 \cdot (t_2 - t_1)}{c_2}$:

$$\lambda\Big(\bigcup_{j=1}^{n} \Big\{ x \in \mathbb{R} \,:\, \big| x - d_j \big| < \zeta \Big\} \Big) \geq \lambda\Big(J(\,\Psi\,) \Big) \geq c_1\,(t_2 - t_1) = \delta n \zeta \qquad \text{for } n \geq n_0.$$

For c) and d):

Since the proofs of c) and d) are completely analogous, we only show d).

Because of b) it is sufficient to prove the following estimate:

$$(4.4.27) \qquad ||\, e_{2,A} - e^{I}_{2,A} \,|| \;\leq\; c_3\, n^{-\,(3/2)\,+\,3\,\alpha} \qquad \text{for } n > n_1(J).$$

Like the following constants, c_3 only depends on e, E, r, k and J, α.

According to Corollary 4.4.15, applied to $\hat{J}$ (cf. (4.4.3)) and $w = 2$, we now obtain for

$$b_n = \frac{1}{n-1} \sum_{j=1}^{n} \Big(\hat{J}\Big(\frac{j}{n+1} \Big) - (\overline{\hat{J}}) \Big)^2$$

the inequality

$$\big|\, b_n - 1 \,\big| \leq c_4\, n^{2\alpha\,-\,1}.$$

From this we get

$$\left| \frac{\dfrac{1}{n} \sum\limits_{j=1}^{n} \Big(J\Big(\dfrac{j}{n+1} \Big) - \bar{J} \Big)^3}{\Big\{ \dfrac{1}{n-1} \sum\limits_{j=1}^{n} \Big(J\Big(\dfrac{j}{n+1} \Big) - \bar{J} \Big)^2 \Big\}^{3/2}} - \int\limits_0^1 \hat{J}^3(t)\, dt \right|$$

$$= \left| \frac{\dfrac{1}{n} \sum\limits_{j=1}^{n} \Big(\hat{J}\Big(\dfrac{j}{n+1} \Big) - (\overline{\hat{J}}) \Big)^3}{b_n^{3/2}} - \int\limits_0^1 \hat{J}^3(t)\, dt \right|$$

$$\leq \frac{1}{b_n^{3/2}} \left| \frac{1}{n} \sum_{j=1}^{n} \Big(\hat{J}\Big(\frac{j}{n+1} \Big) - (\overline{\hat{J}}) \Big)^3 - \int\limits_0^1 \hat{J}^3(t)\, dt \right| + \Big(\int\limits_0^1 |\hat{J}|^3(t)\, dt \Big) \left| \frac{1}{b_n^{3/2}} - 1 \right|$$

$$\leq c_5\, n^{3\alpha\,-\,1} + c_6\, n^{2\alpha\,-\,1} \qquad \text{(Corollary 4.4.15 for } \hat{J} \text{ and } w = 3)$$

$$\leq c_7\, n^{3\alpha\,-\,1}.$$

Analogously we obtain using Corollary 4.4.15 for $\hat{J}$ and $w = 4 \leq s$

$$\left| \frac{\frac{1}{n} \sum\limits_{j=1}^{n} \left(J\left(\frac{j}{n+1}\right) - \bar{J} \right)^4}{\left\{ \frac{1}{n-1} \sum\limits_{j=1}^{n} \left(J\left(\frac{j}{n+1}\right) - \bar{J} \right)^2 \right\}^2} - \int\limits_0^1 \hat{J}^4(t)\, dt \right| \leq c_8\, n^{4\alpha - 1}.$$

Because of

$$\sum_{i=1}^{n} |\hat{e}_i|^3 \leq \frac{c_9}{n^{1/2}}, \qquad \sum_{i=1}^{n} \hat{e}_i^4 \leq \frac{c_{10}}{n} \qquad \text{and} \qquad n^{4\alpha - 2} \leq n^{3\alpha - (3/2)},$$

we then get (4.4.27). $\square$

4.4.28 Remarks

a) The Remarks 4.3.21, a) and b) apply analogously to Theorem 4.4.21, i.e. in the case of

$$\text{(4.3.22)} \qquad \left[((5/k) - 1)^+ - ((4/k) - 1)^+ \right] + \left[((5/s) - 1)^+ - ((4/s) - 1)^+ \right] < \frac{1}{2},$$

we can omit the $(\log n)^2$ in (4.4.23) and (4.4.25).

Furthermore, the n^ϵ in (4.4.24) and (4.4.26) can be replaced by $(\log n)^3$, if

$$E_A^3 \geq C\, n^{-3/2 + \eta}, \text{ where } C, \eta > 0.$$

The constants $\mathcal{E}_6$ and $\mathcal{E}_8$ then also depend on C and η.

b) If J is not constant and if J fulfils the condition (4.4.22), then follows $\int\limits_0^1 (\widetilde{J})^2(t)\, dt > 0$ (see $\widetilde{J}$ in the proof above) and therefore $n_1(J) < \infty$ according to Corollary 4.4.15.

c) Does [12] (Theorem 3.1.1 and Theorem 2.1) and [14] shows (4.4.26) with the convergence rate $\mathcal{O}(n^{-1})$ and under stronger conditions for the regression constants e_i and for the function J. In particular, he requires for J that

$$\limsup_{t \to 0,1} t\,(1-t) \left| \frac{J''(t)}{J'(t)} \right| < 2 \qquad \text{and} \qquad |\, J'''(t)\,| \leq \Gamma \left(t\,(1-t) \right)^{-3-\alpha},$$

where $0 < \alpha < \frac{1}{14}$. But the last inequality implies (4.4.22) with $s = 14$.

d) For the two most popular score functions $J(t) = t$ (Wilcoxon-Scores) and $J(t) = \Phi^{-1}(t)$ (van der Waerden-Scores) there is a $\Gamma > 0$ with

$$|\,J'(t)\,| \leq \Gamma\,\big(\,t\,(1-t)\,\big)^{-1} \qquad \text{for all } t \in (0,\,1),$$

so that (4.4.22) is fulfilled for any $\alpha > 0$ and $s \in \mathbb{N}$ with $\alpha < \dfrac{1}{s}$.

Let us now look at the exact scores. We need the following elementary statements about order statistics of uniformly distributed random variables on (0, 1).

4.4.29 Lemma

Let $U_{j:n}$ denote the jth order statistic in a random sample of size n from the uniform distribution on (0, 1) and let $g : (0,\,1) \to \mathbb{R}$ be an integrable function. Then

a) $E\big(U_{j:n}\big) = \dfrac{j}{n+1} \qquad \text{for all } 1 \leq j \leq n,$

b) $\dfrac{1}{n} \displaystyle\sum_{j=1}^{n} E\big(\,g(\,U_{j:n}\,)\,\big) = \int_0^1 g(t)\,dt.$

Proof:

According to [16], Chapter 3, Section 3.1, Theorem 4, page 38 and 39, $U_{j:n}$ has the density

$$f_{j:n}(t) = n\binom{n-1}{j-1}\,t^{j-1}\,(1-t)^{n-j}\,1_{(0,\,1)}(t) \qquad \text{for } t \in \mathbb{R}$$

and the expected value $E\big(\,U_{j:n}\,\big) = \dfrac{j}{n+1}$ for all $1 \leq j \leq n$. Therefore only b) remains to be shown:

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} E\big(\,g(\,U_{j:n}\,)\,\big) &= \frac{1}{n}\sum_{j=1}^{n} n\int_0^1 g(t)\binom{n-1}{j-1}\,t^{j-1}\,(1-t)^{n-j}\,dt \\
&= \int_0^1 g(t)\sum_{i=0}^{n-1}\binom{n-1}{i}\,t^{i}\,(1-t)^{(n-1)-i}\,dt \qquad\qquad (i = j-1) \\
&= \int_0^1 g(t)\,\Big(\,t+(1-t)\,\Big)^{n-1}\,dt = \int_0^1 g(t)\,dt. \qquad \square
\end{aligned}$$

The following lemma, which is an analogue of (4.4.18) for the exact scores, is needed for the change from $e_{1,A}$ and $e_{2,A}$ to $e^{I}_{1,A}$ and $e^{I}_{2,A}$ in the case of the exact scores.

4.4.30 Lemma

Let $J : (0,\, 1) \to \mathbb{R}$ be continuously differentiable and $0 < \alpha < 1$ such that the **condition $\boldsymbol{V_\alpha}$** (cf. (4.4.9)) is satisfied for J. In addition, let $w \in \mathbb{N}$ with $\alpha < \dfrac{1}{w}$.

Then there exists a constant $C > 0$ depending only on J, α and w (and all $l \le w$) such that

$$(4.4.31) \qquad \Big| \sum_{j=1}^{n} \Big(E\big(J(U_{j:n}) \big) \Big)^{w} - n \int_0^1 J^w(t)\, dt \Big| \le C\, n^{w\alpha} \qquad \text{for all } n \in \mathbb{N}.$$

Proof:

Due to Lemma 4.4.29, b) with $g = J$, we can assume $w \ge 2$ without loss of generality. For all $1 \le j \le n$ we then obtain using the binomial formula

$$\begin{aligned} & E\big(J^w(U_{j:n}) \big) - \Big(E\big(J(U_{j:n}) \big) \Big)^{w} \\ = & \sum_{l=0}^{w-2} \binom{w}{l} \Big(E\big(J(U_{j:n}) \big) \Big)^{l} E\Big(\big[J(U_{j:n}) - E\big(J(U_{j:n}) \big) \big]^{w-l} \Big). \end{aligned}$$

Furthermore, we get for all $1 \le j \le n$ and $0 \le l \le w-2$

$$\begin{aligned} & E\Big(\big| J(U_{j:n}) - E\big(J(U_{j:n}) \big) \big|^{w-l} \Big) \\ \le\; & 2^{w-l-1} \Big\{ E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1} \Big) \Big|^{w-l} \Big) + \Big| J\Big(\frac{j}{n+1} \Big) - E\big(J(U_{j:n}) \big) \Big|^{w-l} \Big\} \\ & \hfill ((0.1.5) \text{ with } \nu = 2 \text{ and } p = w - l) \\ =\; & 2^{w-l-1} \Big\{ E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1} \Big) \Big|^{w-l} \Big) + \Big| E\Big(J\Big(\frac{j}{n+1} \Big) - J(U_{j:n}) \Big) \Big|^{w-l} \Big\} \\ \le\; & 2^{w-l}\, E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1} \Big) \Big|^{w-l} \Big) \end{aligned}$$

and

$$\begin{aligned} & \big| E\big(J(U_{j:n}) \big) \big|^{l} \\ \le\; & E\big(\big| J(U_{j:n}) \big|^{l} \big) && \text{(Hölder's inequality)} \\ \le\; & 2^{l-1} \Big\{ E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1} \Big) \Big|^{l} \Big) + \Big| J\Big(\frac{j}{n+1} \Big) \Big|^{l} \Big\} \\ & \hfill ((0.1.5) \text{ with } \nu = 2 \text{ and } p = l). \end{aligned}$$

If we use Lemma 4.4.29, b) with $g = J^w$, we obtain from the above considerations

$$
\begin{aligned}
(4.4.32)\qquad & \Big| \sum_{j=1}^{n} \Big(E\big(J(U_{j:n}) \big) \Big)^w - n \int_0^1 J^w(t)\,dt \Big| \\
& \le \sum_{j=1}^{n} \Big| E\big(J^w(U_{j:n}) \big) - \Big(E\big(J(U_{j:n}) \big) \Big)^w \Big| \\
& \le \sum_{l=0}^{w-2} \binom{w}{l} \Big\{ \sum_{j=1}^{n} \Big| E\big(J(U_{j:n}) \big) \Big|^l E\Big(\Big| J(U_{j:n}) - E\big(J(U_{j:n}) \big) \Big|^{w-l} \Big) \Big\} \\
& \le 2^{w-1} \sum_{l=0}^{w-2} \binom{w}{l} \Big\{ \sum_{j=1}^{n} E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1}\Big) \Big|^l \Big) E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1}\Big) \Big|^{w-l} \Big) \\
& \qquad\qquad + \sum_{j=1}^{n} \Big| J\Big(\frac{j}{n+1}\Big) \Big|^l E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1}\Big) \Big|^{w-l} \Big) \Big\}.
\end{aligned}
$$

Now let h be the function that is given by

$$h'(t) = \Gamma(J,\alpha)\,\big(t\,(1-t) \big)^{-1-\alpha} \quad \text{for all } t \in (0,\,1) \quad \text{and} \quad h\Big(\frac{1}{2}\Big) = 0$$

(for the definition of $\Gamma(J,\alpha)$ see (4.4.14)). Then, for all $1 \le j \le n$ we have

$$
\begin{aligned}
& \Big| J(U_{j:n}) - J\Big(\frac{j}{n+1}\Big) \Big| \\
& = \frac{|\,J'(\zeta)\,|}{|\,h'(\zeta)\,|} \Big| h(U_{j:n}) - h\Big(\frac{j}{n+1}\Big) \Big| && \text{(generalized mean value theorem)} \\
& \le \Big| h(U_{j:n}) - h\Big(\frac{j}{n+1}\Big) \Big| && \text{(cf. (4.4.14): } |\,J'(\zeta)\,| \le |\,h'(\zeta)\,|)
\end{aligned}
$$

for a ζ such that $\min\Big\{ U_{j:n},\, \frac{j}{n+1} \Big\} \le \zeta \le \max\Big\{ U_{j:n},\, \frac{j}{n+1} \Big\}$.

h also fulfils the *Condition R_w* from Albers, Bickel & van Zwet [1] (see page 150 above) due to $\alpha < \frac{1}{w}$, so that Lemma A2.3 (with $m = 1$, $k_1 = l$, $r_1 = w$ and $N = n$) from this paper yields for $1 \le l \le w$ and $1 \le j \le n$:

$$
\begin{aligned}
(4.4.33)\qquad & E\Big(\Big| J(U_{j:n}) - J\Big(\frac{j}{n+1}\Big) \Big|^l \Big) \\
& \le E\Big(\Big| h(U_{j:n}) - h\Big(\frac{j}{n+1}\Big) \Big|^l \Big)
\end{aligned}
$$

$$\leq c_1 \left\{ \frac{1}{n^{l/2}} \left(\frac{j}{n+1} \left(1 - \frac{j}{n+1} \right) \right)^{-(l/2) - l\alpha} \right.$$
$$\left. + \frac{1}{n^{(l+1)/2}} \left(\frac{j}{n+1} \left(1 - \frac{j}{n+1} \right) \right)^{-((l+1)/2) - l\alpha} \right\}.$$

Here, c_1 only depends on $\Gamma(J, \alpha)$, α, l and therefore J, α, l, but **not** on j.

Because of (4.4.10) we also get

$$(4.4.34) \qquad \left| J\Big(\frac{j}{n+1} \Big) \right|^l \leq c_2 \left(\frac{j}{n+1} \left(1 - \frac{j}{n+1} \right) \right)^{-l\alpha} \quad \text{for all } t \in (0,\, 1),$$

where c_2 (argumentation similar to (4.4.14)) also depends only on J, α and l.

If we now apply the estimates (4.4.33) and (4.4.34) to the terms in (4.4.32) and then use the inequality (4.4.7) with the following five parameters

$$\beta = \frac{w}{2} + w\alpha > 1, \quad \beta = \frac{w+1}{2} + w\alpha > 1, \quad \beta = \frac{w+2}{2} + w\alpha > 1,$$
$$\beta = \frac{w-l}{2} + w\alpha > 1 \quad \text{and} \quad \beta = \frac{w-l+1}{2} + w\alpha > 1$$

(note $w \geq 2$ and $w - l \geq 2$!), then follows

$$\left| \sum_{j=1}^{n} \Big(E\big(J(U_{j:n}) \big) \Big)^w - n \int_0^1 J^w(t)\, dt \right| \leq 2^{w-1} \sum_{l=0}^{w-2} \binom{w}{l} c_3\, n^{w\alpha},$$

where c_3 depends only on J, α and w (and all $l \leq w$). □

The results for the case of the exact scores are now summarized in the following theorem.

4.4.35 Theorem

Let $J : (0,\, 1) \to \mathbb{R}$ be a nonconstant, continuously differentiable and integrable function and $d_j = E\big(J(U_{j:n}) \big)$ for $j = 1, \ldots, n$ and

$$n_2(J) = \sup \left\{ n \in \mathbb{N} : \sum_{j=1}^{n} \big(d_j - \bar{d} \big)^2 = 0 \right\}.$$

In addition, let $A = (a_{ij})$ be an $n \times n-$matrix such that

$$n > n_2(J) \qquad \text{and} \qquad a_{ij} = e_i d_j.$$

Furthermore, suppose that the following two conditions hold:

(4.3.2) $$\sum_{i=1}^{n} \left| e_i - \bar{e} \right|^r \geq en, \qquad \sum_{i=1}^{n} \left| e_i - \bar{e} \right|^k \leq En$$

for some $k > 2$, $0 < r < k$ and $e > 0$, $E > 0$.

(4.4.36) There exists an $s > 2$ such that $\int_0^1 |\, J(t) \,|^s \, dt < \infty$.

a) Then there exists a constant $\mathcal{E}_9 > 0$ depending only on e, E, r, k and J, s such that

(4.4.37) $$\|\, \mathscr{F}_A - e_{1,A} \,\| \leq \mathcal{E}_9 \, (\log n)^2 \, n^{-1 + \left((4/k)-1\right)^+ + \left((4/s)-1\right)^+}.$$

b) Let $\epsilon > 0$. Then there exists a constant $\mathcal{E}_{10} > 0$ depending only on e, E, r, k, J, s and ϵ such that

(4.4.38) $$\|\, \mathscr{F}_A - e_{2,A} \,\| \leq \mathcal{E}_{10} \, n^{-(3/2) + \epsilon + \left((5/k)-1\right)^+ + \left((5/s)-1\right)^+}.$$

c) Let $k \geq 3$ and suppose that J satisfies the **condition $\boldsymbol{V_\alpha}$** (cf. (4.4.9)) for some $0 < \alpha < \dfrac{1}{3}$.

Then there exists a constant $\mathcal{E}_{11} > 0$ depending only on e, E, r, k and J, α, s such that

(4.4.39) $$\|\, \mathscr{F}_A - e^I_{1,A} \,\| \leq \mathcal{E}_{11} \, \max\left\{ n^{-(3/2) + 3\alpha}, \, (\log n)^2 \, n^{-1 + \left((4/k)-1\right)^+ + \left((4/s)-1\right)^+} \right\}.$$

d) Let $\epsilon > 0$, $k \geq 4$ and suppose that J satisfies the **condition $\boldsymbol{V_\alpha}$** (cf. (4.4.9)) for some $0 < \alpha < \dfrac{1}{4}$.

Then there exists a constant $\mathcal{E}_{12} > 0$ depending only on e, E, r, k, J, α, s and ϵ such that

(4.4.40) $$\|\, \mathscr{F}_A - e^I_{2,A} \,\| \leq \mathcal{E}_{12} \, \max\left\{ n^{-(3/2) + 3\alpha}, \, n^{-(3/2) + \epsilon + \left((5/k)-1\right)^+ + \left((5/s)-1\right)^+} \right\}.$$

Proof:

For a) and b):

To show a) and b), we use the Theorem 4.3.13. We must therefore prove the existence of positive constants d, D and δ depending only on J and s such that

(4.4.41) $$\sum_{j=1}^{n} \left| d_j - \bar{d} \right|^s \leq Dn \qquad \text{for all } n \in \mathbb{N},$$

(4.4.42) $$\sum_{j=1}^{n} \big(d_j - \bar{d} \big)^2 \geq dn \qquad \text{for all } n > n_2(J),$$

(4.4.43) $$\lambda\Big(\bigcup_{j=1}^{n} \Big\{ x \in \mathbb{R} : \big| x - d_j \big| < \zeta \Big\} \Big) \geq \delta n \zeta$$
for some $\zeta \geq n^{-3/2} \log\ n$ and all $n \in \mathbb{N}$.

In the following we assume $\bar{d} = 0$ without loss of generality. This is possible since we can pass from J to $\widetilde{J} = J - \int_0^1 J(t)\,dt$ because of $\bar{d} = \int_0^1 J(t)\,dt$ (cf. Lemma 4.4.29, b) with $g = J$).

Furthermore, we can assume in the following that there are $\gamma,\, \theta > 0$ and $\theta < a < b < 1 - \theta$ with

(4.4.44) $$J(x) \geq \gamma \quad \text{and} \quad J'(x) \geq \gamma \qquad \text{for } x \in T = [\, a - \theta,\, b + \theta\,].$$

The other three possible cases $J(x) \leq -\gamma$, $J'(x) \leq -\gamma$ and $J(x) \geq \gamma$, $J'(x) \leq -\gamma$ and $J(x) \leq -\gamma$, $J'(x) \geq \gamma$ are handled completely analogously (note that $J^2 = (-J)^2$ and consider $d_j - d_{j+1}$ for negative J').

Now, for $a \leq \frac{j}{n} \leq b$ we obtain

(4.4.45) $$P\big(U_{j:n} \leq a - \theta \big) \leq e^{-2n\theta^2} \quad \text{and} \quad P\big(U_{j:n} \geq b + \theta \big) \leq e^{-2n\theta^2}.$$

For a justification of (4.4.45) see e.g. Okamoto [19], Theorem 1 on page 33 and

$$P\big(U_{j:n} \leq a - \theta \big) = P\Big(b(n,\, a - \theta) \geq j \Big) \leq P\Big(\frac{1}{n}\, b(n,\, a - \theta) \geq (\, a - \theta\,) + \theta \Big),$$

where $b(n,\, x)$ is a binomially distributed random variable with the parameters n and x.

After these preliminary considerations, we will now show the three statements above:

Proof of (4.4.41):

$$\sum_{j=1}^{n} |\, d_j\,|^s \leq \sum_{j=1}^{n} E\big(|\, J(\, U_{j:n}\,)\,|^s \big) = n \int_0^1 |\, J(t)\,|^s\, dt.$$

For "=", Lemma 4.4.29, b) with $g = |J|^s$, $s > 2$, was used. $|J|^s$ is integrable according to condition (4.4.36).

Proof of (4.4.42):

There exists an $n' \in \mathbb{N}$ such that for all $n \geq n'$ and j with $a \leq \dfrac{j}{n} \leq b$ holds:

$$\begin{aligned} E\Big((J1_T)(U_{j:n}) \Big) &\geq \gamma\, P\big(U_{j:n} \in T \big) && \text{(cf. (4.4.44))} \\ &\geq \gamma \Big(1 - 2\, e^{-2n\theta^2} \Big) && \text{(cf. (4.4.45))} \\ &\geq \frac{\gamma}{2} \end{aligned}$$

and

$$(4.4.46) \qquad \Big| E\Big((J1_{T^c})(U_{j:n}) \Big) \Big|$$

$$\begin{aligned} &\leq \Big(E\big(J^2(U_{j:n}) \big) \Big)^{1/2} \Big(P\big(U_{j:n} \notin T \big) \Big)^{1/2} && \text{(Hölder's inequality with } p = q = 2) \\ &\leq \Big(n \int_0^1 J^2(t)\, dt \Big)^{1/2} \Big(2\, e^{-2n\theta^2} \Big)^{1/2} && \text{(cf. Lemma 4.4.29, b) with } g = J^2) \\ &\leq \frac{\gamma}{4}. \end{aligned}$$

It follows for all $n \geq \max\Big\{ n', \dfrac{2}{b-a} \Big\}$ that

$$\begin{aligned} \sum_{j=1}^{n} d_j^2 &\geq \sum_{a \leq \frac{j}{n} \leq b} \Big\{ E\Big((J1_T)(U_{j:n}) \Big) + E\Big((J1_{T^c})(U_{j:n}) \Big) \Big\}^2 \\ &\geq \sum_{na \leq j \leq nb} \Big\{ \frac{\gamma}{2} - \frac{\gamma}{4} \Big\}^2 \\ &\geq \Big(\frac{\gamma}{4} \Big)^2 \big((b-a)\, n - 1 \big) \\ &\geq \Big(\frac{\gamma}{4} \Big)^2 \frac{b-a}{2}\, n. \end{aligned}$$

From this we get (4.4.42).

Proof of (4.4.43):

Let $V_{j,n} = \Big\{ U_{j:n} \in T,\, U_{j+1:n} \in T \Big\}$. We then obtain, as above, an $n'' \in \mathbb{N}$ such that for all $n \geq n''$ and j with $a \leq \dfrac{j}{n} \leq b - \dfrac{1}{n}$ and $1 \leq l \leq n$ holds:

$$\begin{aligned} P\big(V_{j,n}^c\big) &\le P\big(U_{j:n} \notin T\big) + P\big(U_{j+1:n} \notin T\big) \\ &\le 4\,e^{-2n\theta^2} && \text{(cf. (4.4.45))} \\ &\le \frac{1}{9n} \end{aligned}$$

and

$$\begin{aligned} &\Big| E\Big(J(U_{l:n})\, 1_{V_{j,n}^c} \Big) \Big| \\ &\le \Big(E\big(J^2(U_{l:n}) \big) \Big)^{1/2} \Big(P\big(V_{j,n}^c\big) \Big)^{1/2} && \text{(Hölder's inequality with } p = q = 2\text{)} \\ &\le \Big(n \int_0^1 J^2(t)\, dt \Big)^{1/2} \Big(4\, e^{-2n\theta^2} \Big)^{1/2} && \text{(cf. Lemma 4.4.29, b) with } g = J^2\text{)} \\ &\le \frac{\gamma}{9n}. \end{aligned}$$

It follows for all $n \ge n''$ and j with $a \le \dfrac{j}{n} \le b - \dfrac{1}{n}$ that

$$\begin{aligned} &d_{j+1} - d_j \\ &= E\Big(\Big(J(U_{j+1:n}) - J(U_{j:n}) \Big) 1_{V_{j,n}} \Big) + E\Big(\Big(J(U_{j+1:n}) - J(U_{j:n}) \Big) 1_{V_{j,n}^c} \Big) \\ &\ge \gamma\, E\Big(\big(U_{j+1:n} - U_{j:n} \big) 1_{V_{j,n}} \Big) - \frac{2\gamma}{9n} && \text{(mean value theorem)} \\ &= \gamma\, E\Big(U_{j+1:n} - U_{j:n} \Big) - \gamma\, E\Big(\big(U_{j+1:n} - U_{j:n} \big) 1_{V_{j,n}^c} \Big) - \frac{2\gamma}{9n} \\ &\ge \gamma\, E\Big(U_{j+1:n} - U_{j:n} \Big) - \gamma\, P\big(V_{j,n}^c\big) - \frac{2\gamma}{9n} && \text{(since } 0 \le U_{j+1:n} - U_{j:n} \le 1\text{)} \\ &\ge \frac{\gamma}{n+1} - \frac{\gamma}{9n} - \frac{2\gamma}{9n} && \text{(due to Lemma 4.4.29, a))} \\ &\ge \frac{\gamma}{2n} - \frac{\gamma}{3n} = \frac{\gamma}{6n}. \end{aligned}$$

From this we get (4.4.43) first for sufficiently large $n \ge n_0$ (with e.g. $\delta = b - a$ and $\zeta = \dfrac{\gamma}{12n}$) and then for all $n \in \mathbb{N}$.

For c) and d):

Since the proofs of c) and d) are completely analogous, we only show d).

Because of b) it is sufficient to prove the following estimate:

$$(4.4.47) \qquad ||\, e_{2,A} - e^I_{2,A} \,|| \;\le\; c_1\, n^{-(3/2)+3\alpha} \qquad \text{for } n > n_2(J).$$

Here, c_1 depends only on e, E, r, k and J, α.

But now the proof of (4.4.47) is analogous to that of (4.4.27) if we use Lemma 4.4.30 for $w = 2, 3, 4$ and Lemma 4.4.29, b) instead of Corollary 4.4.15 for $w = 2, 3, 4$. □

4.4.48 Remarks

a) Because of Lemma 4.4.8, b) the condition (4.4.36) is automatically satisfied with $s = 3$ in part c) and with $s = 4$ in part d).

b) The Remark 4.4.28, a) applies analogously to the Theorem 4.4.35.

c) If J is nonconstant, continuous and $\int\limits_0^1 |\, J(t)\,|^p \, dt < \infty$ for a $p > 1$, then $n_2(J) < \infty$.

   This follows as in the "Proof of (4.4.42)", if Hölder's inequality in (4.4.46) is applied with this $p > 1$ instead of $p = 2$.

d) In the Remark 4.4.28, d) it has already been mentioned that the functions $J(t) = t$ (Wilcoxon-Scores, since $E\big(U_{j:n}\big) = \dfrac{j}{n+1}$) and $J(t) = \Phi^{-1}(t)$ (Fisher-Yates-Terry-Hoeffding-Scores) satisfy the **condition $\boldsymbol{V_\alpha}$** (cf. (4.4.9)) for any $\alpha > 0$.

Comparison of different numberings

In the following list, citations from [25], which refer to the original version of 1987, are compared with the changed citations from this updated version of 2025.

Original version of 1987	Updated version of 2025
Lemma 2.1.5	Lemma 2.1.13
Proposition 4.3.9 pages 148 - 151	Proposition 4.3.9 pages 160 - 164
Theorem 4.4.13(a), (b), (d) Theorem 4.4.23(a), (b)	Theorem 4.4.21, a), b), d) Theorem 4.4.35, a), b)

The order of the chapters and sections is the same in the Ph. D. thesis of 1987 and the updated version of 2025.

Only the chapter numbers have been added to the numbering of the sections in this updated version. For example, section 3 from chapter 1 is now section 1.3.

Index of keywords and symbols

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