

A SOLUTION TO BEZDEK'S CONJECTURE

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ABSTRACT. For a given $\lambda > 0$, a convex body in $\mathbb{R}^n$ is λ -convex if it is the intersection of (finitely or infinitely many) balls of radius $1/\lambda$. In this note, we show that among all λ -convex bodies in $\mathbb{R}^n$, $n \geq 2$, with a given inradius, the λ -convex lens (i.e., the intersection of two balls of radius $1/\lambda$) has the largest mean width. This gives an affirmative answer to the conjecture of K. Bezdek. Under an additional symmetry assumption on λ -convex bodies, we resolve the analogous inradius conjecture of Bezdek for arbitrary intrinsic volumes.

We also establish an answer to the corresponding conjecture of K. Bezdek about the circumradius. In particular, we prove that the λ -convex spindle (i.e., the intersection of all balls of radius $1/\lambda$ containing a given pair of points) is the unique minimizer of the mean width among all λ -convex bodies with a fixed circumradius.

1. INTRODUCTION

A convex body in $\mathbb{R}^n$ is a compact convex subset with non-empty interior. For a given $\lambda > 0$, a convex body $K \subset \mathbb{R}^n$ is called λ -convex if for every point $p \in \partial K$ there exists a neighborhood U_p and a ball $B_{\lambda,p}$ of radius $1/\lambda$ such that $p \in \partial B_{\lambda,p}$ and $U_p \cap K \subset B_{\lambda,p}$ (see Figure 1). By Blaschke's rolling theorem [Bla] (see also [Dr3] and references therein), every λ -convex body can be represented as an intersection of (finitely or infinitely many) balls of radius $1/\lambda$. A λ -convex lens in $\mathbb{R}^n$ is the intersection of two balls of radius $1/\lambda$.

The class of λ -convex bodies appears naturally in connection with the Kneser–Poulsen conjecture [Be], the study of the illumination conjecture [Be2], approximation of convex bodies [FV, NV]; see also [AAF, AACF, SWY] for further developments on the geometry of λ -convex bodies and related questions. Recently, the study of *reverse* isoperimetric problems in the class of λ -convex bodies garnered a lot of attention. In particular, the analogue of Ball's reverse isoperimetric inequality for λ -convex bodies was obtained in [FKV, BD2] in $\mathbb{R}^2$ and in [DT] in $\mathbb{R}^3$. We refer the reader to the nice surveys [BLN, BLNP] about topics in λ -convexity.

The classical Steiner formula asserts that for every $\varepsilon > 0$,

$$(1.1) \quad \text{Vol}(K + \varepsilon \mathbf{B}) = \sum_{j=0}^n \kappa_{n-j} V_j(K) \varepsilon^{n-j}$$

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where Vol is the Lebesgue measure on $\mathbb{R}^n$, $\mathbf{B}$ is the unit Euclidean ball in $\mathbb{R}^n$, κ_{n-j} is the volume of the unit Euclidean ball in $\mathbb{R}^{n-j}$, and $+$ denotes the Minkowski addition. The coefficients $V_j(K)$ are called the *intrinsic volumes* of K . It is known that $V_n(K)$ is the volume of K , $V_{n-1}(K)$ is (up to the scalar) the surface area of K , $V_1(K)$ is (up to a scalar) the mean width of K , and $V_0(K) = \kappa_n$ is the volume of $\mathbf{B}$. We refer readers to [Sch] for a comprehensive discussion about intrinsic volumes.

The *inradius* $r(K)$ of a convex body K is the radius of a largest ball contained in K . In this paper, we answer the conjecture posed by K. Bezdek in [Be3, Conjecture 5] about minimizing the inradius in the class of λ -convex bodies with fixed mean width. We remark that this result supplements the analogous statements for the volume due to Bezdek [Be3], and for the surface area due to the authors [DT].

Theorem A (Reverse inradius inequality for mean width). *Let $n \geq 2$ and let $\lambda > 0$. Let $K \subset \mathbb{R}^n$ be a λ -convex body and $L \subset \mathbb{R}^n$ be a λ -convex lens. If $r(K) = r(L)$, then*

$$(1.2) \quad V_1(K) \leq V_1(L),$$

with equality if and only if K is a λ -convex lens.

For $n = 2$, this result was obtained by Milka in [Mi], and it was later generalized to all 2-dimensional Alexandrov spaces of curvature bounded below by the first author in [Dr2].

The circumradius $R(K)$ of a convex body K is the radius of the smallest ball that contains K . K. Bezdek [Be3, Conjecture 10] conjectured that a λ -convex spindle has the largest circumradius among all λ -convex bodies in $\mathbb{R}^n$ with fixed mean width. A λ -convex spindle is the intersection of all balls that contain a pair of given points. These points, called the *vertices* of the spindle, must be necessarily at a distance at most $2/\lambda$ apart. Spindles appear as solutions to various optimization problems for λ -convex domains (see e.g., [Be3, BD1, BM, Dr1]).

As an application of Theorem A and the use of the λ -duality in the class of λ -convex bodies, we obtain a solution to the partial case of Bezdek's conjecture [Be3, Conjecture 10] for the mean width.

Theorem B (Reverse circumradius inequality for mean width). *Let $n \geq 2$ and let $\lambda > 0$. Let $K \subset \mathbb{R}^n$ be a λ -convex body and $S \subset \mathbb{R}^n$ be a λ -convex spindle. If $R(K) = R(S)$, then*

$$(1.3) \quad V_1(K) \geq V_1(S),$$

with equality if and only if K is a λ -convex spindle.

We note that the inequality of Linhart [Li] gives a lower bound for $V_1(K)$ in terms of the circumradius of K for any convex body $K \subset \mathbb{R}^n$, that is,

$$V_1(K) \geq 2R(K)$$

with equality if and only if K is a segment (see also [BH, Theorem 3.3]). In view of remark in [Be3, Remark 11], Theorem B extends Linhart's inequality from the classical convexity to λ -convexity.

If a λ -convex body $K \subset \mathbb{R}^n$ has *inball symmetries* (that is, K possesses an additional symmetry assumption defined in Section 5), we resolve K. Bezdek's conjecture [Be3, Conjecture 5] for any j -th intrinsic volume.

Theorem C (Reverse inradius inequality for intrinsic volumes under symmetry assumption). *Let $n \geq 2$ and let $\lambda > 0$. Let $K \in \mathbb{R}^n$ be a λ -convex body that has inball symmetries and $L \subset \mathbb{R}^n$ be a λ -convex lens. If $r(K) = r(L)$, then*

$$(1.4) \quad V_j(K) \leq V_j(L) \quad \text{for any } j \in \{1, \dots, n\},$$

with equality if and only if K is a λ -convex lens.

The paper is organized as follows. In Section 2, we introduce basic notions and provide the needed background. In Sections 3 and 4, we prove Theorem A and Theorem B, respectively. Finally, imposing additional symmetry constraints on λ -convex bodies, we prove Theorem C in Section 5.

2. PRELIMINARIES

In this section, we aim to recall some general background from convex geometry and λ -convexity.

2.1. Mixed volumes. We refer the readers to [Sch] for a detailed study of mixed volumes and basic notions from convexity.

We work in the Euclidean space $\mathbb{R}^n$ with scalar product $\langle \cdot, \cdot \rangle$. We denote the unit ball centered at the origin by $\mathbf{B}$, and its boundary unit sphere as $\mathbb{S}^{n-1} = \partial\mathbf{B}$.

Let $K_1, \dots, K_m$ be convex bodies in $\mathbb{R}^n$. The fundamental theorem of Minkowski describes the behavior of the volume of the Minkowski sum of convex bodies:

$$(2.1) \quad \text{Vol}(t_1 K_1 + \dots + t_m K_m) = \sum_{i_1, \dots, i_n=1}^m V(K_{i_1}, \dots, K_{i_n}) t_{i_1} \dots t_{i_n}$$

for any $t_1, \dots, t_m \geq 0$. The coefficients $V(K_{i_1}, \dots, K_{i_n})$ in this polynomial are called *mixed volumes* which are important geometric functionals associated with convex bodies. These coefficients are nonnegative, continuous, and multilinear in their arguments. Moreover, mixed volumes are invariant under permutations, that is, if τ is a permutation on $\{1, \dots, n\}$, then

$$(2.2) \quad V(K_1, \dots, K_n) = V(K_{\tau(1)}, \dots, K_{\tau(n)}).$$

As a special case, (2.1) implies the Steiner's formula (1.1). The coefficients $V(K, \dots, K, \mathbf{B}, \dots, \mathbf{B})$ are called quermassintegrals $W_j(K)$ of a convex body K and defined as

$$W_j(K) = V(\underbrace{K, \dots, K}_{n-j}, \underbrace{\mathbf{B}, \dots, \mathbf{B}}_j)$$

for $j = 0, 1, \dots, n$. We will use their weighted versions $V_j(K)$ which are called *intrinsic volumes* of K :

$$(2.3) \quad V_j(K) = \frac{\binom{n}{j}}{\kappa_{n-j}} W_{n-j}(K) = \frac{\binom{n}{j}}{\kappa_{n-j}} V(\underbrace{K, \dots, K}_j, \underbrace{\mathbf{B}, \dots, \mathbf{B}}_{n-j}).$$

It is known that $V_n(K)$ is the n -dimensional volume of K , $2V_{n-1}(K)$ is the surface area of K , $\frac{2\kappa_{n-1}}{n\kappa_n} V_1(K)$ is the mean width of K , and finally, $V_0(K) = \kappa_n$ is just the volume of the unit Euclidean ball $\mathbf{B}$ in $\mathbb{R}^n$.

The support function of a convex body K is defined as $h_K(x) = \max_{y \in K} \langle x, y \rangle$ for every $x \in \mathbb{R}^n$.

The mixed area measure $S(K_2, \dots, K_n)(\cdot)$ of $K_2, \dots, K_n$ is a unique non-negative measure on $\mathbb{S}^{n-1}$ such that

$$V(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{K_1}(u) dS(K_2, \dots, K_n)(u).$$

Note that $S(\mathbf{B}, \dots, \mathbf{B})$ is the standard measure du on $\mathbb{S}^{n-1}$, and more generally, $S(K, \dots, K)$ is the surface area measure on ∂K .

The mixed area measure $S(K_2, \dots, K_n)(\cdot)$ is invariant under ambient isometries applied simultaneously to all convex bodies:

$$(2.4) \quad S(K_2, \dots, K_n)(\cdot) = S(\sigma K_2, \dots, \sigma K_n)(\cdot), \quad \text{for any orientation-preserving isometry } \sigma: \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

Furthermore, if each of the bodies $K_2, \dots, K_n$ is rotationally symmetric with respect to an axis parallel to a given unit direction ℓ , then the measure $S(K_2, \dots, K_n)(\cdot)$ is rotationally invariant in the following sense. Let $p \in \mathbb{S}^{n-1}$ be the point that corresponds to ℓ , and let $A \subset \mathbb{S}^{n-1}$ be a set measurable with respect to $S(K_2, \dots, K_n)(\cdot)$. Then for every rotation σ on $\mathbb{S}^{n-1}$ around p , we have

$$(2.5) \quad S(K_2, \dots, K_n)(A) = S(K_2, \dots, K_n)(\sigma A).$$

The *width* of a convex body $K \subset \mathbb{R}^n$ in the direction $u \in \mathbb{S}^{n-1}$ is defined as $h_K(u) + h_K(-u)$. The *mean width* $w(K)$ of K is the average of the widths of K , that is,

$$(2.6) \quad w(K) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} (h_K(u) + h_K(-u)) \, du = \frac{2}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} h_K(u) \, du,$$

where $|\mathbb{S}^{n-1}|$ is the $(n-1)$ -dimensional measure of $\mathbb{S}^{n-1}$. Thus, we can write $V_1(K)$ as

$$(2.7) \quad V_1(K) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} h_K(u) \, du = \frac{n\kappa_n}{2\kappa_{n-1}} w(K),$$

where we used that $|\mathbb{S}^{n-1}| = n\kappa_n$.

The most fundamental inequality that relates mixed volumes is the classical Alexandrov–Fenchel inequality.

The Alexandrov–Fenchel Inequality. *Let $K_1, K_2, \dots, K_n \subset \mathbb{R}^n$ be convex bodies. Then*

$$V(K_1, K_2, K_3, \dots, K_n)^2 \geq V(K_1, K_1, K_3, \dots, K_n) \cdot V(K_2, K_2, K_3, \dots, K_n).$$

2.2. λ -convexity. In this subsection, we collect some definitions and properties related to λ -convex bodies.

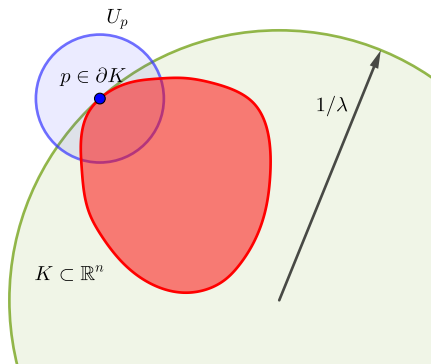
Let $\lambda > 0$. A λ -convex polytope is a λ -convex body given as the intersection of finitely many balls of radius $1/\lambda$. An $(n-1)$ -dimensional *facet* of a λ -convex polytope P is the intersection of the corresponding ball of radius $1/\lambda$ with the boundary of P . Two facets are either disjoint or intersect. In the latter case, their intersection is called a (lower-dimensional) face. The faces of dimension 0 and 1 are called vertices and edges of the polytope P , respectively. The λ -convex lens is an example of a λ -convex polytope.

We also recall that a convex body $K \subset \mathbb{R}^n$ is called λ -convex if for every point $p \in \partial K$ there exists a neighborhood U_p and a ball $B_{\lambda,p}$ of radius $1/\lambda$ such that $p \in \partial B_{\lambda,p}$ and $U_p \cap K \subset B_{\lambda,p}$ (see Figure 1). When ∂K is at least C^2 -smooth, this condition is equivalent to requiring that the principal curvatures $k_i(p)$, $i \in \{1, \dots, n-1\}$, with respect to the inward-pointing normal at every $p \in \partial K$, satisfy $k_i(p) \geq 1$.

For every λ -convex body K , we define the λ -convex dual K^λ as

$$K^\lambda = \bigcap_{x \in K} \left(\frac{1}{\lambda} \mathbf{B} + x \right).$$

It follows from this definition that K^λ is also a λ -convex body. Moreover, λ -duality is an order-reversing involution, that is, $(K^\lambda)^\lambda = K$ and $K^\lambda \subseteq M^\lambda$ if $M \subseteq K$ for any λ -convex bodies K and M . The λ -duality has been recently investigated in [AACF, BLN, FKV].

FIGURE 1. The definition of a λ -convex body

It is known (see, for example, [FKV, Proposition 2.2]) that for a λ -convex body K the following relation holds

$$K + (-K^\lambda) = \frac{1}{\lambda} B.$$

Equivalently,

$$(2.8) \quad h_K(u) + h_{K^\lambda}(-u) = \frac{1}{\lambda}, \quad \text{for any } u \in \mathbb{S}^{n-1}.$$

The relation (2.8) immediately implies that

$$(2.9) \quad r(K) + R(K^\lambda) = \frac{1}{\lambda}.$$

Finally, if L is a λ -convex lens, then $L^\lambda = S$ is a λ -convex spindle. We refer the reader to [BLN, Section 3.1] for the detailed discussion of the λ -duality.

REMARK. We note that λ -duality is different from the spherical duality between λ -convex and λ -concave domains in $\mathbb{S}^2$, which was used in [BD3] to relate the reverse isoperimetric problems in these two classes of bodies.

3. PROOF OF THEOREM A

Without loss of generality, we can assume that $\lambda = 1$. Let B be an inscribed ball for K . We can assume that this ball is centered at the origin and has the radius $r(K) = r(L) \leq 1$.

Since B is the inscribed ball for K , the intersection $\partial K \cap \partial B$ contains a finite set of points $T = \{p_0, p_1, \dots\}$, with $|T| \leq n + 1$, that does not lie in any of the open hemispheres of ∂B (this statement is folklore, see e.g., [DT, Proposition 2.9]; it can also be deduced from the Carathéodory theorem).

Let $\tilde{K}$ be a 1-convex polytope given by the intersection of the unit balls tangent to B at the points in T and containing B in their closures. By Blaschke's Rolling Theorem, $K \subseteq \tilde{K}$. Furthermore, by [DT, Proposition 2.9], B is the inscribed ball for $\tilde{K}$. Thus,

$$r(\tilde{K}) = r(K) \quad \text{and} \quad K \subseteq \tilde{K}.$$

Hence,

$$V_1(K) \leq V_1(\tilde{K})$$

by the monotonicity of the intrinsic volumes, with equality if and only if $\tilde{K} = K$.

Therefore, to establish Theorem A, it is enough to consider the case when K is a 1-convex polytope with the property that all of its facets touch the inscribed ball at the points in the set $T = \{p_0, p_1, \dots, p_{|T|}\}$, $|T| \leq n + 1$.

For the rest of this section, we further assume that the lens L is a 1-convex lens centered at the origin (and hence, sharing the same inscribed ball with K) and rotated so that it touches the inscribed ball in one of the points in T , say, p_0 .

Let $\mathcal{C} := (\underbrace{\mathbf{B}, \dots, \mathbf{B}}_{n-1})$ be an $(n-1)$ -tuple of unit Euclidean balls $\mathbf{B}$. Then our goal is to show

$$(3.1) \quad V(K, \mathcal{C}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) du \leq \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u) du = V(L, \mathcal{C}),$$

with equality if and only if K is a lens.

Let F_i be a facet of the 1-convex polytope K . Denote by p_i the point of tangency of F_i and the inscribed ball B . Let $\mathcal{N}_{F_i} \subset \mathbb{S}^{n-1}$ be the radial projection of F_i onto the unit sphere $\mathbb{S}^{n-1}$ with respect to the origin. Equivalently, if $Q_1, \dots, Q_{|T|} \subset \partial B$ are the Voronoi cells of points $p_1, \dots, p_{|T|} \in \partial B$, then each $\mathcal{N}_{F_i}$ is the radial projection of the corresponding Q_i onto the unit sphere $\mathbb{S}^{n-1}$.

Assume that the 1-convex lens L is placed so that its center is at the origin and one of its facets, which we call F , touches the inscribed ball B at the point p_i . If $\mathcal{N}_F \subset \mathbb{S}^{n-1}$ is a similar radial projection of F , then we have

$$(3.2) \quad F_i \subseteq F, \quad \mathcal{N}_{F_i} \subseteq \mathcal{N}_F, \quad \text{with simultaneous equality if and only if } F_i = F.$$

The first inclusion follows from Blaschke's rolling theorem [Bla], and the second inclusion follows from the symmetry and the fact that F_i and F are regions on the unit sphere.

Lemma 3.1.

$$(3.3) \quad \frac{\int_{\mathcal{N}_{F_i}} h_K(u) du}{\int_{\mathcal{N}_{F_i}} du} \leq \frac{\int_{\mathcal{N}_F} h_L(u) du}{\int_{\mathcal{N}_F} du} = \frac{V(L, \mathcal{C})}{V(\mathbf{B}, \mathcal{C})}.$$

Moreover, the equality holds if and only if $\mathcal{N}_F = \mathcal{N}_{F_i}$ (and thus, if and only if $F = F_i$).

Now, assume that Lemma 3.1 holds for every $i \in \{1, \dots, |T|\}$. Then, summing over all facets of K , we obtain

$$\begin{aligned} V(K, \mathcal{C}) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) du = \frac{1}{n} \sum_i \int_{\mathcal{N}_{F_i}} h_K(u) du \\ &\leq \frac{V(L, \mathcal{C})}{V(\mathbf{B}, \mathcal{C})} \cdot \frac{1}{n} \sum_i \int_{\mathcal{N}_{F_i}} du = \frac{V(L, \mathcal{C})}{V(\mathbf{B}, \mathcal{C})} \cdot \frac{1}{n} \int_{\mathbb{S}^{n-1}} du = V(L, \mathcal{C}), \end{aligned}$$

which yields (3.1) and finishes the proof of Theorem A. In the next subsection, we prove Lemma 3.1.

3.1. Proof of Lemma 3.1. First observe that by (3.2),

$$h_K(u) \leq h_L(u), \quad \forall u \in \mathcal{N}_{F_i}.$$

Therefore, to prove (3.3), it is enough to establish

$$(3.4) \quad \frac{\int_{\mathcal{N}_{F_i}} h_L(u) du}{\int_{\mathcal{N}_{F_i}} du} \leq \frac{\int_{\mathcal{N}_F} h_L(u) du}{\int_{\mathcal{N}_F} du}.$$

Choose the coordinates (t, ϑ) in $\mathbb{S}^{n-1}$ centred at $\frac{1}{r(K)}p_i$ such that a point $u \in \mathbb{S}^{n-1}$ can be written as

$$u = \cos(t) \frac{p_i}{r(K)} + \sin(t) \vartheta,$$

where $\vartheta \in \mathbb{S}^{n-2}$ (more precisely, $\vartheta \in p_i^\perp \cap \mathbb{S}^{n-1}$) and t is the angle between u and direction op_i . Then the surface area measure on $\mathbb{S}^{n-1}$ is $du = (\sin(t))^{n-2} dt d\vartheta$, and we set $q(t) := (\sin(t))^{n-2}$ for $t \in [0, \pi/2]$. The function q is strictly monotone increasing on its domain of definition. Also, denote the spherical radial function of $\mathcal{N}_{F_i}$ in the direction ϑ by

$$\varphi(\vartheta) := \max_{t \in [0, \pi/2]} \left\{ \cos(t) \frac{p_i}{r(K)} + \sin(t) \vartheta \in \mathcal{N}_{F_i} \right\}.$$

By rotational symmetry, the support function $h_L(u) = h_L(t)$ is independent of ϑ . Furthermore, the function $t \mapsto h_L(t)$ is strictly monotone increasing on its domain of definition $t \in [0, \pi/2]$.

Then we can rewrite

$$\int_{\mathcal{N}_{F_i}} h_L(u) du = \int_{\mathbb{S}^{n-2}} \int_0^{\varphi(\vartheta)} h_L(t) q(t) dt d\vartheta \quad \text{and} \quad \int_{\mathcal{N}_{F_i}} du = \int_{\mathbb{S}^{n-2}} \int_0^{\varphi(\vartheta)} q(t) dt d\vartheta.$$

Also, let $\mathcal{F} := \int_0^{\pi/2} h_L(t) q(t) dt / \int_0^{\pi/2} q(t) dt$ and consider

$$(3.5) \quad \mathcal{R}[\varphi(\vartheta)] := \int_0^{\varphi(\vartheta)} (h_L(t) - \mathcal{F}) q(t) dt,$$

where $\varphi(\vartheta) \in [0, \pi/2]$. Note that $\mathcal{R}[\pi/2] = 0$.

Since $t \mapsto (h_L(t) - \mathcal{F})q(t)$ is a continuous and strictly increasing function on $[0, \pi/2]$, we obtain that $\mathcal{R}[\varphi(\vartheta)] \leq 0$ for all possible $\varphi(\vartheta) \in [0, \pi/2]$ with equality if and only if $\varphi(\vartheta) = \pi/2$ (see [DT, Lemma 4.5]). Hence,

$$\begin{aligned} \int_{\mathcal{N}_{F_i}} h_L(u) du &= \int_{\mathbb{S}^{n-2}} \int_0^{\varphi(\vartheta)} h_L(t) q(t) dt d\vartheta \leq \mathcal{F} \int_{\mathbb{S}^{n-2}} \int_0^{\varphi(\vartheta)} q(t) dt d\vartheta \\ &= \frac{\int_{\mathbb{S}^{n-2}} \int_0^{\frac{\pi}{2}} h_L(t) q(t) dt d\vartheta}{\int_{\mathbb{S}^{n-2}} \int_0^{\frac{\pi}{2}} q(t) dt d\vartheta} \int_{\mathbb{S}^{n-2}} \int_0^{\varphi(\vartheta)} q(t) dt d\vartheta = \frac{\int_{\mathcal{N}_F} h_L(u) du}{\int_{\mathcal{N}_F} du} \int_{\mathcal{N}_{F_i}} du, \end{aligned}$$

as required.

4. PROOF OF THEOREM B

Let K be a λ -convex body with circumradius $R(K)$, and let S be a λ -convex spindle such that $R(S) = R(K)$. Consider the λ -dual body K^λ and the λ -convex lens $L := S^\lambda$.

By (2.7) and (2.8),

$$(4.1) \quad V_1(K) + V_1(K^\lambda) = \frac{1}{\lambda} V_1(\mathbf{B}).$$

Also, $r(K^\lambda) = \frac{1}{\lambda} - R(K) = \frac{1}{\lambda} - R(S) = r(S^\lambda) = r(L)$ by (2.9).

Theorem A (applied for $j = 1$) implies that

$$(4.2) \quad V_1(K^\lambda) \leq V_1(L)$$

with equality if and only if K^λ is a λ -convex lens L . Thus, (4.1) and (4.2) yield that

$$V_1(K) = \frac{1}{\lambda} V_1(\mathbf{B}) - V_1(K^\lambda) \geq \frac{1}{\lambda} V_1(\mathbf{B}) - V_1(L) = V_1(L^\lambda) = V_1(S)$$

with equality if and only if K is a λ -convex spindle S . This completes the proof of Theorem B (including the equality case).

5. PROOF OF THEOREM C

For intrinsic volumes different from the volume, the surface area, and the mean width, our method allows us to derive a result similar to our Theorem A under the additional symmetry assumption as follows. Let $i(K) \in \partial K$ be the set of points at which the inscribed ball of K touches the boundary ∂K . This set can be finite or infinite. For example, if L is a lens, then $i(L)$ consists of two points (symmetric with respect to the center of L). If S is a spindle, then $i(S)$ is a circle.

Let $\text{Isom}^+(\mathbb{R}^n)$ be the group of orientation-preserving isometries in $\mathbb{R}^n$ and $\mathcal{O}(K) < \text{Isom}^+(\mathbb{R}^n)$ be the maximal subgroup of rotations around the center o of the inscribed ball to K that fixes $i(K)$. In other words, a rotation $\sigma \in \text{Isom}^+(\mathbb{R}^n)$ about o lies in $\mathcal{O}(K)$ if

$$i(K) = \sigma(i(K)) := \{\sigma(x) \mid x \in i(K)\}.$$

Definition 5.1 (Inball symmetries). We say that K has *inball symmetries* if $\mathcal{O}(K)$ is not equal to the identity group and it acts transitively on the set $i(K)$: for every $x, y \in i(K)$ there exists $\sigma \in \mathcal{O}(K)$ such that $\sigma(x) = y$.

Note that every lens has inball symmetries, and furthermore, L is invariant under the action of $\mathcal{O}(L)$. However, a general convex body K need not be invariant under the action of $\mathcal{O}(K)$.

5.1. Proof of Theorem C. The proof begins in a similar way to the proof of Theorem A. Starting with K , we can construct a 1-convex polytope $\tilde{K}$ such that all of its faces are tangent to the inscribed ball B along the finite set T , and $V_j(K) \leq V_j(\tilde{K})$ (with equality if and only if $\tilde{K} = K$) using the monotonicity of the intrinsic volumes. We assume that B is centered at the origin.

Since K has inball symmetries, we can further choose T (if there is a choice) so that $\tilde{K}$ has inball symmetries as well. Note that $\mathcal{O}(\tilde{K})$ is a subgroup of $\mathcal{O}(K)$, not necessarily proper. Since $\tilde{K}$ is a 1-convex polytope that is completely determined by $i(\tilde{K})$, we obtain that

$$(5.1) \quad \sigma(\tilde{K}) = \tilde{K}, \quad \forall \sigma \in \mathcal{O}(\tilde{K}),$$

i.e., not only are the touching points invariant under the action of $\mathcal{O}(\tilde{K})$, but the polytope itself.

Therefore, as in the proof of Theorem A, in order to establish Theorem C, we can assume that K is a 1-convex polytope that satisfies the **Standing Assumptions**: the faces of K touch the inscribed ball in some set $T = \{p_0, p_1, \dots, p_{|T|}\}$, $|T| \leq n + 1$, and K satisfies (5.1) with $\tilde{K} = K$.

Also, let L be a 1-convex lens centered at the origin (and hence, sharing the same inscribed ball with K).

For a convex body M and $t \in \{1, \dots, n\}$, we will use a calligraphic version of M , i.e., $\mathcal{M}_t$, to denote a t -tuple of copies of M , i.e., $\mathcal{M}_t = (M, \dots, M)$. In this way, for example, $V(\mathcal{M}_t, \mathcal{C}_{n-t}) = V(\underbrace{M, \dots, M}_t, \underbrace{C, \dots, C}_{n-t})$. We extend this for $t = 0$ by setting $V(\mathcal{M}_0, \mathcal{C}_n) = V(\underbrace{C, \dots, C}_n)$.

Lemma 5.2. *Let K be the 1-convex polytope that satisfies the Standing Assumption, L be the 1-convex lens, and $\mathcal{B}$ be the unit Euclidean ball in $\mathbb{R}^n$ as above. Then for every $s \in \{1, \dots, n\}$ and $t \in \{0, \dots, s-1\}$,*

$$(5.2) \quad \frac{V(\mathcal{K}_{s-t}, \mathcal{L}_t, \mathcal{B}_{n-s})}{V(\mathcal{K}_{s-t-1}, \mathcal{L}_{t+1}, \mathcal{B}_{n-s})} \leq 1.$$

Moreover, the equality holds if and only if K is a lens.

This lemma immediately yields Theorem C. Indeed, to show $V_j(K) \leq V_j(L)$ for a given $j \in \{1, \dots, n\}$, in view of (2.3), we need to show that

$$\frac{V(\mathcal{K}_j, \mathcal{B}_{n-j})}{V(\mathcal{L}_j, \mathcal{B}_{n-j})} \leq 1.$$

The following chain of the inequalities is obtained by applying Lemma 5.2 for $s = j$ and $t \in \{0, \dots, j-1\}$

$$\frac{1}{V(\mathcal{L}_j, \mathcal{B}_{n-j})} \stackrel{t=j-1}{\leq} \frac{1}{V(\mathcal{K}_1, \mathcal{L}_{j-1}, \mathcal{B}_{n-j})} \stackrel{t=j-2}{\leq} \dots \stackrel{t=1}{\leq} \frac{1}{V(\mathcal{K}_{j-1}, \mathcal{L}_1, \mathcal{B}_{n-j})} \stackrel{t=0}{\leq} \frac{1}{V(\mathcal{K}_j, \mathcal{B}_{n-j})}.$$

Multiplying the above by $V(\mathcal{K}_j, \mathcal{B}_{n-j})$, we get

$$\frac{V(\mathcal{K}_j, \mathcal{B}_{n-j})}{V(\mathcal{L}_j, \mathcal{B}_{n-j})} \leq 1$$

as needed. The equality case follows from the equality case in Lemma 5.2. This finishes the proof of Theorem C. The remainder of the section will be devoted to the proof of Lemma 5.2.

5.2. Proof of Lemma 5.2. Let $s \in \{1, \dots, n\}$. We claim that it is enough to prove Lemma 5.2 for $t = s-1$. Indeed, let $t \in \{0, \dots, s-2\}$. Then, by the Alexandrov–Fenchel inequality and the symmetry of mixed volumes, we have the following

$$\frac{V(\mathcal{K}_{s-t}, \mathcal{L}_t, \mathcal{B}_{n-s})}{V(\mathcal{K}_{s-t-1}, \mathcal{L}_{t+1}, \mathcal{B}_{n-s})} \leq \frac{V(\mathcal{K}_{s-t-1}, \mathcal{L}_{t+1}, \mathcal{B}_{n-s})}{V(\mathcal{K}_{s-t-2}, \mathcal{L}_{t+2}, \mathcal{B}_{n-s})}.$$

If $\mathcal{Q} := (\mathcal{K}_{s-t-2}, \mathcal{L}_t, \mathcal{B}_{n-s})$, then the last inequality becomes

$$\frac{V(K, K, \mathcal{Q})}{V(K, L, \mathcal{Q})} \leq \frac{V(K, L, \mathcal{Q})}{V(L, L, \mathcal{Q})}.$$

Using the Alexandrov–Fenchel inequality $s - t - 1 \geq 1$ times, we obtain the following sequence of inequalities:

$$\begin{aligned} \frac{V(\mathcal{K}_{s-t}, \mathcal{L}_t, \mathcal{B}_{n-s})}{V(\mathcal{K}_{s-t-1}, \mathcal{L}_{t+1}, \mathcal{B}_{n-s})} &\leq \frac{V(\mathcal{K}_{s-t-1}, \mathcal{L}_{t+1}, \mathcal{B}_{n-s})}{V(\mathcal{K}_{s-t-2}, \mathcal{L}_{t+2}, \mathcal{B}_{n-s})} \leq \frac{V(\mathcal{K}_{s-t-2}, \mathcal{L}_{t+2}, \mathcal{B}_{n-s})}{V(\mathcal{K}_{s-t-3}, \mathcal{L}_{t+3}, \mathcal{B}_{n-s})} \leq \dots \\ &\dots \leq \frac{V(\mathcal{K}_1, \mathcal{L}_{s-1}, \mathcal{B}_{n-s})}{V(\mathcal{L}_s, \mathcal{B}_{n-s})}. \end{aligned}$$

If the claim is true for $t = s - 1$ then

$$\frac{V(\mathcal{K}_1, \mathcal{L}_{s-1}, \mathcal{B}_{n-s})}{V(\mathcal{L}_s, \mathcal{B}_{n-s})} \leq 1.$$

This yields the required inequality for any $t \in \{0, \dots, s - 2\}$ together with the equality case.

Now we prove the case $t = s - 1$. Let

$$\mathcal{C} := (\mathcal{L}_{s-1}, \mathcal{B}_{n-s}).$$

We need to show that

$$(5.3) \quad \frac{V(\mathcal{K}_1, \mathcal{L}_{s-1}, \mathcal{B}_{n-s})}{V(\mathcal{L}_s, \mathcal{B}_{n-s})} = \frac{V(K, \mathcal{L}_{s-1}, \mathcal{B}_{n-s})}{V(L, \mathcal{L}_{s-1}, \mathcal{B}_{n-s})} = \frac{V(K, \mathcal{C})}{V(L, \mathcal{C})} \leq 1,$$

with equality if and only if K is a 1-convex lens.

Similar to the proof of Theorem A, let F_i be a facet of the 1-convex polytope K tangent to B at p_i . Let $\mathcal{N}_{F_i} \subset \mathbb{S}^{n-1}$ be the radial projection of F_i onto the unit sphere $\mathbb{S}^{n-1}$ with respect to the origin.

Assume that the 1-convex lens L is placed so that its center is at the origin and one of its facets, which we call F , touches the inscribed ball B at the point p_i . We denote $\mathcal{N}_F \subset \mathbb{S}^{n-1}$ a similar radial projection of F on B . We have the same inclusion (3.2).

The following lemma is an analogue of Lemma 3.1. We postpone its proof to the end of this section. The main difference to the earlier proof is that, a priori, the measure $S(\sigma_i \mathcal{C})(u)$ might not have a smooth monotone density, and hence we cannot use the Fubini argument that we used in the proof of Lemma 3.1. Instead, our argument will follow the approximation strategy similar to the one in [DT, Section 4].

Lemma 5.3.

$$(5.4) \quad \frac{\int_{\mathcal{N}_{F_i}} h_K(u) dS(\sigma_i \mathcal{C})(u)}{\int_{\mathcal{N}_{F_i}} dS(\sigma_i \mathcal{C})(u)} \leq \frac{\int_{\mathcal{N}_F} h_L(u) dS(\sigma_i \mathcal{C})(u)}{\int_{\mathcal{N}_F} dS(\sigma_i \mathcal{C})(u)} = \frac{V(L, \mathcal{C})}{V(B, \mathcal{C})}.$$

where σ_i is the rotation about the origin that moves L so that one of its faces touches the inscribed ball B at p_i . Moreover, the equality holds if and only if $\mathcal{N}_F = \mathcal{N}_{F_i}$ (and thus, if and only if $F = F_i$).

By definition and our choice of p_0 , each σ_i is an element of $\mathcal{O}(K)$ that rotates p_0 to p_i . We put $\sigma_0 = id$.

We know that $\sigma_i(K) = K$, and thus $h_{\sigma_i(K)} = h_K$ and $\mathcal{N}_{F_i} = \mathcal{N}_{\sigma_i(F_i)}$. Hence, by this invariance and (2.4),

$$(5.5) \quad \begin{aligned} \int_{\mathcal{N}_{F_i}} h_K(u) dS(\mathcal{C})(u) &= \int_{\mathcal{N}_{\sigma_i(F_i)}} h_{\sigma_i(K)}(u) dS(\sigma_i\mathcal{C})(u) = \int_{\mathcal{N}_{F_i}} h_K(u) dS(\sigma_i\mathcal{C})(u), \\ \int_{\mathcal{N}_{F_i}} dS(\mathcal{C})(u) &= \int_{\mathcal{N}_{F_i}} dS(\sigma_i\mathcal{C})(u). \end{aligned}$$

Then, summing over all facets of K and using (5.5) and Lemma 5.3, we obtain:

$$\begin{aligned} V(K, \mathcal{C}) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) dS(\mathcal{C})(u) = \frac{1}{n} \sum_i \int_{\mathcal{N}_{F_i}} h_K(u) dS(\mathcal{C})(u) = \frac{1}{n} \sum_i \int_{\mathcal{N}_{F_i}} h_K(u) dS(\sigma_i\mathcal{C})(u) \\ &\leq \frac{V(L, \mathcal{C})}{V(\mathbf{B}, \mathcal{C})} \cdot \frac{1}{n} \sum_i \int_{\mathcal{N}_{F_i}} dS(\sigma_i\mathcal{C})(u) = \frac{V(L, \mathcal{C})}{V(\mathbf{B}, \mathcal{C})} \cdot \frac{1}{n} \sum_i \int_{\mathcal{N}_{F_i}} dS(\mathcal{C})(u) \\ &= \frac{V(L, \mathcal{C})}{V(\mathbf{B}, \mathcal{C})} \cdot \frac{1}{n} \int_{\mathbb{S}^{n-1}} dS(\mathcal{C})(u) = V(L, \mathcal{C}), \end{aligned}$$

which yields (5.3) and finishes the proof of Lemma 5.2.

5.3. Proof of Lemma 5.3. For brevity, let us denote $\mu := S(\sigma_i\mathcal{C})$. Similar to Lemma 3.1, it is enough to establish

$$(5.6) \quad \frac{\int_{\mathcal{N}_{F_i}} h_L(u) d\mu(u)}{\int_{\mathcal{N}_{F_i}} d\mu(u)} \leq \frac{\int_{\mathcal{N}_F} h_L(u) d\mu(u)}{\int_{\mathcal{N}_F} d\mu(u)} =: A_n.$$

We need to show that

$$(5.7) \quad \mathcal{R}[X] := \int_X (h_L(u) - A_n) du \leq 0 \quad \text{for} \quad X = \mathcal{N}_{F_i} \subset \mathbb{S}^{n-1}.$$

Note that $\mathcal{R}[\mathcal{N}_F] = 0$. Furthermore, by construction, $\mathcal{N}_F$ is equal to the closed hemisphere in $\mathbb{S}^{n-1}$ centered at p_i , while $\mathcal{N}_{F_i}$ is a closed convex polytope in this hemisphere.

Let us denote $g(u) := h_L(u) - A_n$. The function g and the measure μ are both rotationally invariant with respect to the rotations in $\mathbb{S}^{n-1}$ about p_i .

For $\vartheta \in [0, 1]$, consider a pair of oriented hyperplanes passing through op_i and making the (oriented) angle $2\pi\vartheta$. Denote by H_ϑ the wedge of angle $2\pi\vartheta$ between these hyperplanes. Finally, define $C_\vartheta := H_\vartheta \cap \mathcal{N}_F$. Pick $k \in \mathbb{N}_{>0}$ and let $C_{1/k}^j$, $j \in \{1, \dots, k\}$, be a cyclically ordered set of k spherical wedges with angle $2\pi/k$ that tile $\mathcal{N}_F$. We have

$$\mathcal{R}[\mathcal{N}_F] = \int_{\mathcal{N}_F} g(u) d\mu(u) = 0, \quad \bigcup_{j=1}^k C_{1/k}^j = \mathcal{N}_F.$$

Hence, by rotational symmetry,

$$(5.8) \quad \int_{C_{1/k}} g(u) d\mu(u) = 0, \quad \forall k \in \mathbb{N}_{>0}, \forall C_{1/k}.$$

The function $g(u)$ is strictly monotonically increasing along any geodesic ray in $\mathcal{N}_F$ starting at p_i . The same is true for the function $h_L(u)$. Therefore, along any ray starting at p_i there exists a unique point $u_0 \in \mathcal{N}_F$ where $g(u_0) = 0$. This easily follows from the fact that $g(u)$ is rotationally invariant and $\mathcal{R}[\mathcal{N}_F] = 0$. All such points u_0 form an $(n-2)$ -dimensional sphere $S^0 \subset \mathcal{N}_F$ centered at p_i with the spherical radius r_0 satisfying $g(r_0) = 0$ (or equivalently, $h_L(r_0) = A_n$). This sphere splits $\mathcal{N}_F$ into two regions: the open ball D^- containing p_i , and a spherical annulus D^+ , so that $\mathcal{N}_F = D^- \sqcup S^0 \sqcup D^+$. The function $g(u)$ is negative for every $u \in D^-$ and is positive for every $u \in D^+$. The same conclusion is true for $C_{1/k} \cap D^-$ and $C_{1/k} \cap D^+$. Therefore, for every geodesic ball $D_t \subset \mathbb{S}^{n-1}$ of spherical radius t centered at p_i we have that

$$(5.9) \quad \int_{D_t \cap C_{1/k}} g(u) d\mu(u) \leq 0, \quad \text{with equality if and only if } t = \frac{\pi}{2}.$$

Indeed, if $D_t \cap C_{1/k} \subset (D^- \sqcup S^0) \cap C_{1/k}$, then the conclusion is obviously true because g is negative in D^- . If $D_t \cap C_{1/k} \supset (D^- \sqcup S^0) \cap C_{1/k}$ and $\int_{D_t \cap C_{1/k}} g(u) d\mu(u) \geq 0$, then

$$\int_{C_{1/k}} g(u) d\mu(u) = \int_{D_t \cap C_{1/k}} g(u) d\mu(u) + \int_{(D^+ \setminus D_t) \cap C_{1/k}} g(u) d\mu(u) > 0$$

because g is positive in D^+ ; this is a contradiction to (5.8). Hence (5.9) is true.

Construct a sequence $C_{1/k}^j$ of wedges, $j \in \{1, \dots, k\}$, such that $C_{1/k}^j$'s have disjoint interiors, and a sequence $D_{t_{k,j}}$ of balls such that for every $k \in \mathbb{N}$,

$$\mathcal{N}_F = \bigcup_{j=1}^k C_{1/k}^j, \quad \mathcal{N}_{F_i} \subset \bigcup_{j=1}^k (D_{t_{k,j}} \cap C_{1/k}^j),$$

and such that

$$\bigcup_{j=1}^k (D_{t_{k,j}} \cap C_{1/k}^j) \xrightarrow[k \rightarrow \infty]{} \mathcal{N}_{F_i}$$

in the Hausdorff metric on $\mathbb{S}^{n-1}$. More specifically, for each $C_{1/k}^j$, we define $t_{k,j}$ to be the smallest radius such that the ball $D_{t_{k,j}}$ contains $\mathcal{N}_{F_i} \cap C_{1/k}^j$. This approximation is similar to the one in [DT].

Therefore, since the function g is bounded on $\mathcal{N}_F$, we obtain

$$\int_{\mathcal{N}_{F_i}} g(u) d\mu(u) = \lim_{k \rightarrow \infty} \sum_{j=1}^k \int_{D_{t_{k,j}} \cap C_{1/k}^j} g(u) d\mu(u) \stackrel{(5.9)}{\leq} 0,$$

which is the desired inequality (5.7).

Let us analyze the equality case. If $\mathcal{N}_{F_i} \subset D^- \sqcup S^0$, then $\int_{\mathcal{N}_{F_i}} g(u) d\mu(u) < 0$. Thus, we can assume that $\mathcal{N}_{F_i}$ intersects D^+ . If $\mathcal{N}_F \neq \mathcal{N}_{F_i}$, then there exists a sufficiently large k_0 and $j_0 \in \{1, \dots, k_0\}$, such that

$$(5.10) \quad \mathcal{N}_F \cap C_{1/k_0}^{j_0} \supsetneq D_{t_{k_0,j_0}} \cap C_{1/k_0}^{j_0} \supsetneq \mathcal{N}_{F_i} \cap C_{1/k_0}^{j_0} \supsetneq D^- \cap C_{1/k_0}^{j_0}.$$

For $k \geq k_0$, we can split the union $\mathcal{U}_k := \bigcup_{j=1}^k (D_{t_{k,j}} \cap C_{1/k}^j)$ into two subsets $\mathcal{X}_k$ and $\mathcal{Y}_k$ with disjoint interiors such that

$$\mathcal{X}_k := \overline{C_{1/k_0}^{j_0} \cap \mathcal{U}_k}, \quad \mathcal{Y}_k := \overline{\mathcal{U}_k} \setminus \mathcal{X}_k.$$

By passing to a subsequence if necessary, we can assume that for $k \geq k_0$ if $C_{1/k}^j \cap C_{1/k_0}^{j_0} \neq \emptyset$, then $C_{1/k}^j \subseteq C_{1/k_0}^{j_0}$ (for example, we can choose $k = 2^s \cdot k_0$ for $s = 1, 2, \dots$). In this way, for $k \geq k_0$, we have $\int_{\mathcal{Y}_k} g(u) d\mu(u) \leq 0$ and

$$\begin{aligned} \sum_{j=1}^k \int_{D_{t_k,j} \cap C_{1/k}^j} g(u) d\mu(u) &= \int_{\mathcal{X}_k} g(u) d\mu(u) + \int_{\mathcal{Y}_k} g(u) d\mu(u) \\ &\stackrel{(5.10)}{\leq} \int_{D_{t_{k_0},j_0} \cap C_{1/k_0}^{j_0}} g(u) d\mu(u) =: E <^{(5.9)(5.10)} 0 \end{aligned}$$

Therefore, $\int_{\mathcal{N}_{F_i}} g(u) d\mu(u) = \lim_{k \rightarrow \infty} \sum_{j=1}^k \int_{D_{t_k,j} \cap C_{1/k}^j} g(u) d\mu(u) \leq E < 0$, thus we have the strict inequality. Thus we obtained that the only way we can achieve equality is when $\mathcal{N}_F = \mathcal{N}_{F_i}$. This finishes the proof of Lemma 5.2.

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