SPECTRAL SEQUENCES, MASSEY PRODUCTS AND HOMOLOGY OF COVERING SPACES

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Abstract. We revisit the equivariant spectral sequence considered by Papadima-Suciu, and show that all its differentials are computed by higher order Massey products. As a first application, we extend to arbitrary field coefficients results of Pajitnov relating the size of Jordan blocks for the eigenvalue 1 part of the Alexander modules to the length of nonvanishing Massey products in cohomology. We also give computable upper bounds for the mod p Betti numbers of prime power cyclic covers, and resp. for the ranks of the cohomology groups with coefficients in a prime order rank one local system. Under suitable conditions, these bounds are improvements of the ones obtained by Papadima-Suciu. We also specialize these results to the case of hyperplane arrangement complements, showing e.g., that vanishing of higher-order Massey products implies that the mod p Betti numbers of prime p tower cyclic covers are combinatorially determined.

1. Introduction

1.1. **Infinite cyclic cover.** Let X be a connected finite CW complex with a group epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, and consider the corresponding infinite cyclic cover X^ν. Fix a ground field $\mathbb{K}$, and set $R = \mathbb{K}[\mathbb{Z}] \simeq \mathbb{K}[t^{\pm 1}]$. Then the *i-th Alexander module* $H_i(X^\nu, \mathbb{K})$ of (X, ν) is a finitely generated R-module for any integer i.

If we assume that ν is induced by a fibration $f\colon X \to S^1$ with fiber F, there are many results in the literature that relate the size of the Jordan blocks for the monodromy action on $H_*(F, \mathbb{C}) \cong H_*(X^\nu, \mathbb{C})$ to the Massey products of X. For instance, Fernández–Gray–Morgan [15] used the relation between non-vanishing Massey products of length 2 and the Jordan blocks of size greater than 1 to prove that certain mapping tori do not admit the structure of a Kähler manifold. Papadima–Suciu [22] showed that if $\pi_1(X)$ is 1-formal, the eigenvalue 1 part of $H_1(F, \mathbb{C})$ is semi-simple, see also [21]. Moreover, Bazzoni–Fernández–Muñoz [1] showed that the existence of Jordan blocks of size 2 implies the existence of non-zero triple Massey products.

Without the circle fibration assumption, the size of Jordan blocks for the eigenvalue 1 part of $H_*(X^\nu, \mathbb{C})$ was studied by Papadima–Suciu in [21, Section 9]. Also, Budur and the first and third authors of this paper studied this question for compact Kähler manifolds and smooth complex quasi-projective varieties, see [3]. In particular, the torsion part of

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$H_*(X^\nu, \mathbb{C})$ is semi-simple for X a compact Kähler manifold, since such a manifold is formal over $\mathbb{C}$, cf. [9]. Finally, Pajitnov identified the length of certain non-zero higher order Massey products with the size of Jordan blocks for the eigenvalue 1 part of $H_*(X^\nu, \mathbb{C})$, see [19, Theorem 5.1], and also [20] for the twisted version.

One of the motivations for this paper is to extend Pajitnov's results [19] to positive characteristic field coefficients. In this context, formality is often not available, e.g., compact Kähler manifolds [13] and hyperplane arrangement complements [18] are in general not formal over $\mathbb{F}_p$.

There exist several spectral sequences in the literature, designed to study the (torsion submodule of the) Alexander module $H_*(X^\nu, \mathbb{K})$. Here we use a Massey type spectral sequence, which as we show in Proposition 2.2 is dual to the J-adic equivariant spectral sequence studied by Papadima–Suciu [21]. Furthermore, standard facts about spectral sequences can be used to show that the differentials of the Massey type spectral sequence are computed in terms of a special type of higher order Massey products (see Definition 3.4 and see also [16, 19, 20] for similar statements). The latter identification leads us to the following generalization of the above mentioned result of Pajitnov [19, Theorem 5.1] for arbitrary field coefficients:

Theorem 1.1. *For $i \geq 0$, the maximal size of Jordan blocks for the eigenvalue 1 part of $H_i(X^\nu, \mathbb{K})$ is one less than the length of the highest nonvanishing Massey product on degree i associated to a 1-cocycle representative η for $\nu \in H^1(X^\nu, \mathbb{K})$. In particular, if all higher order Massey products of X are trivial, the eigenvalue 1 part of $H_*(X^\nu, \mathbb{K})$ is semi-simple.*

As a byproduct, we also get the following application, extending some results from [14] and [4].

Corollary 1.2. *Let X be a connected complex n-dimensional algebraic variety, possibly singular. Assume that $W_0H^1(X, \mathbb{C}) = 0$, where W is the weight filtration. Then for any epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$, the size of Jordan blocks for the eigenvalue 1 part of $H_i(X^\nu, \mathbb{C})$ is at most $\min\{2i+2, 2n\}$. For $H_1(X^\nu, \mathbb{C})$, the bound can be improved to be 3. If we further assume that $W_1H^1(X, \mathbb{C}) = 0$, then the size of Jordan blocks for the eigenvalue 1 part of $H_i(X^\nu, \mathbb{C})$ is at most $\min\{i+1, n\}$. For $H_1(X^\nu, \mathbb{C})$, the bound can be improved to be 1.*

1.2. **Prime tower cyclic cover.** If p is a prime integer and r is a positive integer, by further projecting the epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$ to $\mathbb{Z}_{p^r}$ yields a corresponding finite cyclic p^r-fold cover X_r. Let $\mathbb{K} = \mathbb{F}_p$, and let $\eta_p \in C^1(X, \mathbb{F}_p)$ be the corresponding 1-cocycle (where the subscript is used to emphasize the characteristic of the field). Then by the J-adic filtration, one gets a truncated spectral sequence induced from the Massey type spectral sequence discussed above, which computes the mod p homology $H_*(X_r, \mathbb{F}_p)$ of the cover X_r, and which was considered, e.g., by Papadima–Suciu in [21, Section 7.1]. When $r = 1$, this spectral sequence specializes to the one studied by Reznikov in [23].

By our key technical result of Proposition 2.2, this truncated spectral sequence degenerates at most at the E_{p^r}-page. Moreover, the differential map on the E_1-page is given by the left cup product with $[\eta_p]$, and the E_k-page for $k \geq 2$ is computed by the higer order Massey product associated to η_p. This spectral sequence is used in Section 4 to obtain upper bounds for the Betti numbers of X_r with $\mathbb{F}_p$-coefficients. More precisely, denoting by

$$\beta_i(X,\eta_p) := \dim_{\mathbb{F}_p} H^i(H^*(X,\mathbb{F}_p),[\eta_p]\cup -)$$

the corresponding i-th Aomoto Betti number with $\mathbb{F}_p$-coefficients, we have the following.

Proposition 1.3. *Let $X_r \to X$ be the p^r-fold cover defined above. Then for any $i \geq 0$ we have*

$$b_i(X_r,\mathbb{F}_p) \leq b_i(X,\mathbb{F}_p) + (p^r-1)\cdot \beta_i(X,\eta_p). \tag{1}$$

If we further assume that $r=1$ and $p=2$, the spectral sequence degenerates at the E_2-page and one gets a long exact sequence, also known as the transfer long exact sequence

$$\cdots \to H^{i-1}(X,\mathbb{F}_2) \xrightarrow{[\eta_2]\cup} H^i(X,\mathbb{F}_2) \to H^i(Y,\mathbb{F}_2) \to H^i(X,\mathbb{F}_2) \xrightarrow{[\eta_2]\cup} H^{i+1}(X,\mathbb{F}_2) \to \cdots,$$

where Y is the associated double cover of X. Yoshinaga [25] used this long exact sequence to show that

$$b_i(Y,\mathbb{F}_2) = b_i(X,\mathbb{F}_2) + \beta_i(X,\eta_2). \tag{2}$$

In particular, $b_i(Y,\mathbb{F}_2)$ is determined by combinatorics when X is a hyperplane arrangement complement.

On the other hand, in Section 4 we show the following results for $p^r > 2$.

Proposition 1.4. *Let X_r be the p^r-fold cover of X associated to the reduction mod $\mathbb{Z}_{p^r}$ of the epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$. For $p^r > 2$, (1) holds as an equality for all i if and only if the above spectral sequence degenerates at the E_2-page. In particular, (1) holds as a strict inequality at degree i if there exist a non-trivial k-tuple Massey product on degree i associated to η_p for some $3 \leq k \leq p^r$.*

Note that when X is the complement of a hyperplane arrangement then, under the hypotheses for which (1) holds as an equality, it follows that the mod p Betti numbers $b_i(X_r,\mathbb{F}_p)$ of the cover X_r are combinatorially determined, extending the above mentioned result of Yoshinaga to arbitrary primes.

On the other hand, if there exist non-trivial higher order Massey products on X, and assuming that $H_*(X,\mathbb{Z})$ has no p-torsion, we obtain sharper bounds for the Betti numbers of rank-one local systems than the ones obtained in [21]. Specifically, we show the following.

Theorem 1.5. *For $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$ as before, fix $\lambda \in \mathbb{C}^*$ of prime order p, and denote by L_λ the rank one $\mathbb{C}$-local system on X defined by sending the generator of $\mathbb{Z}$ to λ. Then for any integer i we have the inequality*

$$b_i(X,L_\lambda) \leq \beta_i(X,\eta_p) + \frac{b_i(X,\mathbb{F}_p) - b_i(X,\mathbb{C})}{p-1}. \tag{3}$$

If, moreover, $H_*(X,\mathbb{Z})$ has no p-torsion, then

$$b_i(X,L_\lambda) \leq \beta_i(X,\eta_p). \tag{4}$$

If we further assume that some k-tuple Massey product on degree j associated to η_p is non-trivial for some $3 \leq k \leq p$, then

$$b_j(X, L_\lambda) < \beta_j(X, \eta_p). \tag{5}$$

Remark 1.6. The inequality (4) was proved more generally, for λ of order p^r with $r \geq 1$, by Papadima–Suciu in [21, Theorem C] by a different method.

1.3. **Hyperplane arranegements.** In Section 5, we specialize the above results to the case of hyperplane arrangement complements. It is well-known that the complement of a hyperplane arrangement is formal over $\mathbb{C}$ in the sense of Sullivan's rational homotopy theory, but in general it is not formal over $\mathbb{F}_p$. In this regard, we recall Matei's example of an arrangement complement with non-trivial triple Massey products over $\mathbb{F}_p$ [18] and indicate how Theorem 1.5 applies in this example. On the other hand, results of Cirici–Horel [7] provide many examples of hyperplane arrangement complements having all higher order Massey products trivial over $\mathbb{F}_p$. In particular, as discussed in Proposition 1.4, the mod p Betti number $b_i(X_r, \mathbb{F}_p)$ of the cover X_r are in this case combinatorially determined. As a concrete example, we discuss in detail the case of graphic arrangements.

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2. Alexander modules and the cochain algebra

Let X be a connected finite CW complex with a group epimorphism $\nu \colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$. Consider the corresponding infinite cyclic cover X^ν. Fix a ground field $\mathbb{K}$, and set $R = \mathbb{K}[\mathbb{Z}] \simeq \mathbb{K}[t^{\pm 1}]$. Then $H_i(X^\nu, \mathbb{K})$ is a finitely generated R-module for any integer i.

Consider the local system $\mathcal{L}^\nu$ on X with stalk R, and representation of the fundamental group defined by the composition:

$$\pi_1(X) \xrightarrow{\nu} \mathbb{Z} \to Aut(R),$$

with the second map being given by $1_\mathbb{Z} \mapsto t$. Here t is the automorphism of R given by multiplication by t. Then we have the R-module isomorphism $H_i(X, \mathcal{L}^\nu) \cong H_i(X^\nu, \mathbb{K})$ for any integer i.

In this section, we show that the completion of the R-module $H^i(X, \mathcal{L}^\nu)$ at the maximal ideal $(t-1)$ is determined by a cochain complex $C^*(X, \mathbb{K})$ as differential graded algebra (short as dg-algebra). In particular, by using the universal coefficient theorem, it follows that the eigenvalue 1 part of $\mathrm{Tors}_R H_i(X^\nu, \mathbb{K})$ is determined by $C^*(X, \mathbb{K})$ and the group homomorphism ν, where we use Tors_R to denote the torsion part of an R-module.

Remark 2.1. We will use simplicial cochain complexes. Let us first explain our convention for the cap product and clarify various sign conventions. Given a k-simplex $\langle v_0, \dots, v_k\rangle$ and a p-cochain α, the cap product is defined by

$$\langle v_0, \dots, v_k\rangle \cap \alpha = \alpha(\langle v_0, \dots, v_p\rangle)\langle v_p, \dots, v_k\rangle.$$

Thus, in our convention, in a cap product the chain is always on the left and the cochain is on the right. Moreover, the associativity of the cap product is of the form

$$(\omega \cap \alpha) \cap \beta = \omega \cap (\alpha \cup \beta).$$

The Leibniz rule takes the form

$$\partial(\omega \cap \alpha) = (-1)^{|\alpha|}\partial\omega \cap \alpha - (-1)^{|\alpha|}\omega \cap \delta\alpha,$$

where $|\alpha|$ denotes the degree of α. In fact, it can be derived from the definition by using the Leibniz rule for cup product as follows

$$\begin{aligned}
&\partial(\omega \cap \alpha) \cap \beta = (\omega \cap \alpha) \cap \delta\beta = \omega \cap (\alpha \cup \delta\beta)\\
=&\omega \cap \big((-1)^{|\alpha|}\delta(\alpha \cup \beta) - (-1)^{|\alpha|}\delta\alpha \cup \beta\big) = (-1)^{|\alpha|}\omega \cap \delta(\alpha \cup \beta) - (-1)^{|\alpha|}\omega \cap (\delta\alpha \cup \beta)\\
=&(-1)^{|\alpha|}\partial\omega \cap (\alpha \cup \beta) - (-1)^{|\alpha|}(\omega \cap \delta\alpha) \cap \beta = (-1)^{|\alpha|}(\partial\omega \cap \alpha) \cap \beta - (-1)^{|\alpha|}(\omega \cap \delta\alpha) \cap \beta.
\end{aligned}$$

The following construction and notations will be used in the formulation of the main results of this section below.

Since $\nu\colon \pi_1(X) \to \mathbb{Z}$ can be regarded as an element of $H^1(X, \mathbb{Z})$ and $S^1 = \mathbb{R}/\mathbb{Z} = K(\mathbb{Z}, 1)$, there is a continuous map $f_\nu\colon X \to S^1$ whose induced homomorphism on fundamental groups is ν. Let $q\colon \mathbb{R} \to S^1$ be the covering map, and consider the following Cartesian diagram

$$\begin{tikzcd}
X^\nu \arrow[r, "f'_\nu"] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "q"] \\
X \arrow[r, "f_\nu"] & S^1.
\end{tikzcd} \tag{6}$$

Since every finite CW complex is homotopy equivalent to a finite simplicial complex, by the simplicial approximation theorem we can assume that X is a finite simplicial complex and f_ν is a map of simplicial complexes. Without loss of generality, we assume that the simplicial structure on $S^1 = \mathbb{R}/\mathbb{Z}$ is given by vertices $\{0 = 1, \frac{1}{N}, \dots, \frac{N-1}{N}\}$ $(N \geq 2)$ and edges $\{[0, \frac{1}{N}], \dots, [\frac{N-1}{N}, 1]\}$. For $0 \leq i \leq N-1$, denote the vertex $\frac{i}{N}$ by x_i, and denote the edge $[\frac{i}{N}, \frac{i+1}{N}]$ by γ_{i+1}. As covering spaces, we endow $\mathbb{R}$ and X^ν with the induced simplicial structures from S^1 and X, respectively. For the remaining part of the proof, we will only work with simplicial chains and cochains.

Let η_0 be the 1-cochain on S^1 defined by

$$\eta_0(\gamma_i) = \begin{cases} 0 & \text{when } i \neq N, \\ 1 & \text{when } i = N. \end{cases} \tag{7}$$

Since S^1 is one-dimensional, η_0 is also a 1-cocycle. Let

$$\eta = f_\nu^*(\eta_0) \ \text{ and } \ \eta' = p^*(\eta) \tag{8}$$

be the induced 1-cocyles on X and X^ν, respectively.

Let us next define, for any $m \in \mathbb{Z}_{>0}$, a twisted chain complex of $R/(s^{m+1})$-modules, denoted $C_*(X, \eta, m)$, by

$$\big(C_*(X) \otimes_{\mathbb{K}} R/(s^{m+1}), \partial + s\eta\big),$$

where $s = t-1$, $(C_*(X), \partial)$ is the simplicial chain complex of X, and with boundary map defined by

$$(\partial + s\eta)(\Delta) = \partial\Delta + s(\Delta \cap \eta),$$

for a simplex Δ of X.

Proposition 2.2. *Under the above notations, $(\partial + s\eta)^2 = 0$. In particular, $C_*(X, \eta, m)$ is a chain complex. Moreover, for any integer i there is an isomorphism of $R/(s^{m+1})$-modules*

$$H_i\big(X, \mathcal{L}^\nu \otimes_R R/(s^{m+1})\big) \cong H_i\big(C_*(X, \eta, m)\big). \tag{9}$$

Proof. We prove the first statement by explicit computation. For a simplicial chain ω, we have

$$\begin{aligned}
(\partial + s\eta)^2\omega &= \partial^2\omega + s\partial(\omega \cap \eta) + s\partial\omega \cap \eta + s^2\omega \cap (\eta \cup \eta)\\
&= \partial^2\omega - s\partial\omega \cap \eta + s\omega \cap \delta\eta + s\partial\omega \cap \eta + s^2\omega \cap (\eta \cup \eta)\\
&= \partial^2\omega + s\omega \cap \delta\eta + s^2\omega \cap (\eta \cup \eta)\\
&= 0.
\end{aligned}$$

Here we use the fact that η is a 1-cocycle on X, and also $\eta \cup \eta = 0$ since η is the pull-back of a 1-cocycle on S^1 (cf. (8)).

For the second part, we construct an isomorphism of complexes of $R/(s^{m+1})$-modules:

$$\Phi : \big(C_*(X) \otimes_{\mathbb{K}} R/(s^{m+1}), \partial + s\eta\big) \to \big(C_*(X^\nu) \otimes_R R/(s^{m+1}), \partial_\nu\big), \tag{10}$$

where ∂ and ∂_ν are the boundary maps of the simplicial chain complexes $C_*(X)$ and $C_*(X_\nu)$, respectively.

We first define Φ as a graded $R/(s^{m+1})$-module isomorphism, and then check its compatibility with the boundary maps. Let $\Delta \hookrightarrow X$ be a simplex of X. Since f'_ν is a map of simplicial complexes, there is a unique lift $\Delta' \hookrightarrow X^\nu$ such that $f'_\nu(\Delta'^\circ) \subset [0, 1)$, where Δ'° denotes the interior of Δ'. In particular, for a vertex v of X, $v' \in X^\nu$ is the lifting such that $f'_\nu(v') \in [0, \frac{N-1}{N}]$. Let $\Phi(\Delta \otimes 1) = \Delta' \otimes 1$. Notice that the set of Δ' defined above form a R-module basis of $C_*(X^\nu)$. Hence Φ is a graded $R/(s^{m+1})$-module isomorphism.

Next, we prove that Φ is compatible with the boundary maps. We need to check that

$$\partial_\nu\Phi(\Delta) = \Phi\big((\partial + s\eta)\Delta\big),$$

where Δ is a k-simplex of X. If $f_\nu(\Delta) \neq [\frac{N-1}{N}, 1]$, then $\Delta \cap \eta = 0$ and the above equality follows from the functoriality of chain complexes. So we only need to consider the case when $f_\nu(\Delta) = [\frac{N-1}{N}, 1]$.

We choose the total ordering of vertices of S^1 by $x_0 < x_1 < \cdots < x_{N-1}$, and choose a total ordering of vertices of X, such that $f\colon X \to S^1$ preserves the ordering of the vertices. Furthermore, choose a total ordering of vertices of X^ν such that $p\colon X^\nu \to X$ preserves the ordering of the vertices. Write $\Delta = \langle v_0, \dots, v_k\rangle$ according to the chosen ordering of vertices of X.

Since f_ν preserves the order of the vertices of X and S^1, and moreover $f_\nu(\Delta) = [\frac{N-1}{N}, 1]$, there exists $j \in \{0, \dots, k-1\}$ such that , without any loss of generality, we may assume that

$$f(v_0) = \cdots = f(v_j) = \frac{N-1}{N} \quad \text{and} \quad f(v_{j+1}) = \cdots = f(v_k) = 1.$$

Consider the case when $j \geq 1$. In this case,

$$\Delta \cap \eta = (\langle v_0, v_1\rangle \cap \eta)\langle v_1, \dots, v_k\rangle = \left(\left\langle \frac{N-1}{N}, \frac{N-1}{N}\right\rangle \cap \eta_0\right)\langle v_1, \dots, v_k\rangle = 0.$$

Denote the action of $1 \in \mathbb{Z}$ on X^ν by σ. Then

$$\begin{aligned}
&\Phi\big((\partial + s\eta)\Delta\big) = \Phi(\partial\Delta)\\
=&\sum_{0\leq i\leq j}(-1)^i\Phi\big(\langle v_0, \dots, \widehat{v_i}, \dots, v_j, \dots v_k\rangle\big) + \sum_{j+1\leq i\leq k}(-1)^i\Phi\big(\langle v_0, \dots, v_j, \dots, \widehat{v_i}, \dots, v_k\rangle\big)\\
=&\sum_{0\leq i\leq j}(-1)^i\left\langle v'_0, \dots, \widehat{v'_i}, \dots, v'_j, \sigma(v'_{j+1}), \dots \sigma(v'_k)\right\rangle\\
&+\sum_{j+1\leq i\leq k}(-1)^i\left\langle v'_0, \dots, v'_j, \sigma(v'_{j+1}), \dots, \widehat{\sigma(v'_i)}, \dots, \sigma(v'_k)\right\rangle\\
=&\partial_\nu\langle v'_0, \dots, v'_j, \sigma(v'_{j+1}), \dots \sigma(v'_k)\rangle\\
=&\partial_\nu\Phi(\Delta).
\end{aligned}$$

Consider the case when $j = 0$. In this case, $\Delta \cap \eta = \langle v_1, \dots, v_k\rangle$. Hence,

$$\begin{aligned}
\Phi((\partial + s\eta)\Delta) &= \sum_{1\leq i\leq k}(-1)^i\Phi\big(\langle v_0, v_1, \dots, \widehat{v_i}, \dots v_k\rangle\big) + \Phi\big(\langle v_1, \dots, v_k\rangle\big) + s\Phi\big(\langle v_1, \dots, v_k\rangle\big)\\
&= \sum_{1\leq i\leq k}(-1)^i\Phi\big(\langle v_0, v_1, \dots, \widehat{v_i}, \dots v_k\rangle\big) + t\Phi\big(\langle v_1, \dots, v_k\rangle\big)\\
&= \sum_{1\leq i\leq k}(-1)^i\left(\left\langle v'_0, \sigma(v'_1), \dots, \widehat{\sigma(v'_i)}, \dots \sigma(v'_k)\right\rangle\right) + t\big(\langle v'_1, \dots, v'_k\rangle\big)\\
&= \sum_{1\leq i\leq k}(-1)^i\left(\left\langle v'_0, \sigma(v'_1), \dots, \widehat{\sigma(v'_i)}, \dots \sigma(v'_k)\right\rangle\right) + \big(\langle \sigma(v'_1), \dots, \sigma(v'_k)\rangle\big)\\
&=\partial_\nu\big\langle v'_0, \sigma(v'_1), \dots, \sigma(v'_k)\big\rangle\\
&=\partial_\nu\Phi(\Delta).
\end{aligned}$$

□

Let $C^*(X)$ be the simplicial cochain complex of X. Noting that $C_*(X^\nu)$ is a complex of free R-modules, we denote by $C^*_R(X^\nu)$ its dual complex of R-modules. Given $m \in \mathbb{Z}_{>0}$, taking duals (as complexes of free $R/(s^{m+1})$-modules) of (10), we have an isomorphism of complexes

$$\Psi : \big(C^*(X) \otimes_{\mathbb{K}} R/(s^{m+1}), \delta + (\eta \cup -) \otimes s\big) \to \big(C^*_R(X^\nu) \otimes_R R/(s^{m+1}), \delta_\nu\big),$$

where δ and δ_ν are the differentials (coboundary maps) in $C^*(X)$ and $C^*_R(X^\nu)$, respectively. Since the complex $(C^*_R(X^\nu), \delta_\nu)$ computes the cohomology of $\mathcal{L}^\nu$, the complex $(C^*_R(X^\nu) \otimes_R R/(s^{m+1}), \delta_\nu)$ computes the cohomology of $\mathcal{L}^\nu \otimes_R R/(s^{m+1})$. Let

$$C^*(X, \eta, m) := (C^*(X) \otimes_{\mathbb{K}} R/(s^{m+1}), \delta + (\eta \cup -) \otimes s).$$

Thus, we have the following.

Corollary 2.3. *For any integer $i \geq 0$, there is an isomorphism of $R/(s^{m+1})$-modules*

$$H^i\big(X, \mathcal{L}^\nu \otimes_R R/(s^{m+1})\big) \cong H^i\big(C^*(X, \eta, m)\big). \tag{11}$$

Denote the completion of R at the maximal ideal $(t-1)$ by $\widehat{R}$, that is,

$$\widehat{R} = \varprojlim_m R/(s^{m+1}).$$

Note that $\widehat{R} = \mathbb{K}[[s]]$ is a principal ideal domain and a regular local ring. There is a natural $\widehat{R}$-isomorphism for any finitely generated R-module A,

$$A \otimes_R \widehat{R} \cong \widehat{A},$$

where $\widehat{A}$ is the $(t-1)$-adic completion of A.

Let

$$C^*(X, \eta, \infty) = \Big(C^*(X) \otimes_{\mathbb{K}} \widehat{R}, \delta + (\eta \cup -) \otimes s\Big)$$

be the complex of $\widehat{R}$-modules, which is also the inverse limit of $C^*(X, \eta, m)$ as $m \to \infty$. Since the left side of the isomorphism (11) is also functorial on m, we can take the inverse limit of the cohomology groups. Since the cohomology groups are finite dimensional $\mathbb{K}$-vector spaces, the Mittag-Leffler condition is automatically satisfied. Thus we have the following.

Corollary 2.4. *As $\widehat{R}$-modules,*

$$H^i\big(X, \mathcal{L}^\nu \otimes_R \widehat{R}\big) \cong H^i\big(C^*(X, \eta, \infty)\big).$$

Using the universal coefficient theorem, we then get the following.

Corollary 2.5. *As $\widehat{R}$-modules, the eigenvalue 1 part of $\mathrm{Tors}_R H_i(X^\nu, \mathbb{K})$ is isomorphic to*

$$\mathrm{Tors}_{\widehat{R}} H^{i+1}\big(C^*(X, \eta, \infty)\big),$$

where $\mathrm{Tors}_{\widehat{R}}$ means taking the torsion part as $\widehat{R}$-modules.

Remark 2.6. When X is a manifold and $\mathbb{K} = \mathbb{C}$, $C^*(X, \mathbb{K})$ can be replaced by the de Rham complex and all the results in this section are already presented in [3, Section 3] (see also [11] and [5]). More generally, when X is a finite CW complex and $\mathrm{char}(\mathbb{K}) = 0$, we can substitute the de Rham complex by Sullivan's model. So the interesting case for this section is when $\mathrm{char}(\mathbb{K}) > 0$.

3. Spectral sequences and Massey products

The cohomology class $[\eta_0] \in H^1(S^1, \mathbb{K})$ corresponding to (7) is the generator dual to the fundamental class in $H_1(S^1, \mathbb{K})$. Moreover, $\nu\colon \pi_1(X) \to \mathbb{Z} = \pi_1(S^1)$ induces a homomorphism $\nu^\vee : H^1(S^1, \mathbb{K}) \to H^1(X, \mathbb{K})$, which is equal to f_ν^*. Denoting $\nu^\vee([\eta_0])$ by ν, we have

$$[\eta] = f_\nu^*([\eta_0]) = \nu^\vee[\eta_0] = \nu \in H^1(X, \mathbb{K}). \tag{12}$$

Since $[\eta]$ is the pullback of a cohomology class in $H^1(S^1, \mathbb{K})$, we note that $[\eta] \cup [\eta] = 0$. Then we get the following Aomoto complex associated to $[\eta]$ by (left) cup product

$$(H^*(X, \mathbb{K}), [\eta]\cdot)\colon \ \cdots \to H^{i-1}(X, \mathbb{K}) \xrightarrow{[\eta]\cup} H^i(X, \mathbb{K}) \xrightarrow{[\eta]\cup} H^{i+1}(X, \mathbb{K}) \to \cdots .$$

Definition 3.1. With the above notations, we define *the i-th Aomoto Betti number with $\mathbb{K}$-coefficients* as

$$\beta_i^{\mathbb{K}}(X, \eta) \coloneqq \dim_{\mathbb{K}} H^i(H^*(X, \mathbb{K}), [\eta] \cup -).$$

As suggested by Corollary 2.5, we will be mainly concerned here with understanding the complex

$$(C^*(X, \mathbb{K}) \otimes_{\mathbb{K}} \widehat{R}, \delta \otimes \mathrm{id} + (\eta \cup -) \otimes s).$$

This complex can be viewed as a double complex

$$A^{i,j} \coloneqq C^{i+j}(X, \mathbb{K}) \otimes \mathbb{K}s^i \cong C^{i+j}(X, \mathbb{K})$$

with the vertical map given by $d \otimes \mathrm{id}$ and the horizontal map given by left cup product with η. Note that $A^{i,j} = 0$ for $i + j < 0$ or $i < 0$. Then by the stupid filtration, we get a spectral sequence with the first page

$$E_1^{i,j} = H^{i+j}(C^*(X, \mathbb{K})) = H^{i+j}(X, \mathbb{K})$$

with $d_1^{i,j}(\alpha) = [\eta] \cup \alpha$ for any $\alpha \in E_1^{i,j}$. Consequently, the second page of the spectral sequence is given by

$$E_2^{i,j} = H^{i+j}\big(H^*(X, \mathbb{K}), [\eta] \cup -\big).$$

Remark 3.2. Let $J = (t-1)$ be the augmention ideal of R generated by $t - 1$. The successive powers of J determines a decreasing filtration on R, called the J-adic filtration. The J-adic completion of R can be identified with $\widehat{R} \coloneqq \mathbb{K}[[s]]$ with $s = t - 1$. The spectral sequence induced by the J-adic filtration has been studied by Papadima–Suciu in [21], and it is dual to the above spectral sequence by Corollary 2.5. In fact, the 0-page of the above spectral sequence is given as follows

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \ , \\
C^2(X,\mathbb{K}) & \xrightarrow{(-\cup\eta)\cdot s} & C^3(X,\mathbb{K})\cdot s & \xrightarrow{(-\cup\eta)\cdot s} & C^4(X,\mathbb{K})\cdot s^2 & & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^1(X,\mathbb{K}) & \xrightarrow{(-\cup\eta)\cdot s} & C^2(X,\mathbb{K})\cdot s & \xrightarrow{(-\cup\eta)\cdot s} & C^3(X,\mathbb{K})\cdot s^2 & & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
C^0(X,\mathbb{K}) & \xrightarrow{(-\cup\eta)\cdot s} & C^1(X,\mathbb{K})\cdot s & \xrightarrow{(-\cup\eta)\cdot s} & C^2(X,\mathbb{K})\cdot s^2 & & \cdots \\
 & & \delta\uparrow & & \delta\uparrow & & \\
 & & C^0(X,\mathbb{K})\cdot s & \xrightarrow{(-\cup\eta)\cdot s} & C^1(X,\mathbb{K})\cdot s^2 & & \cdots \\
 & & & & \delta\uparrow & & \\
 & & & & C^0(X,\mathbb{K})\cdot s^2 & & \cdots \\
 & & & & & & \ddots
\end{array}$$

while the the 0-page of the spectral sequence induced by the J-adic filtration is given by

$$\begin{array}{cccccccc}
\ddots & \vdots & & \vdots & & \vdots & & \vdots \ . \\
 & C_0(X,\mathbb{K})\cdot s^3 & \xleftarrow{(-\cap\eta)\cdot s} & C_1(X,\mathbb{K})\cdot s^2 & \xleftarrow{(-\cap\eta)\cdot s} & C_2(X,\mathbb{K})\cdot s & \xleftarrow{(-\cap\eta)\cdot s} & C_3(X,\mathbb{K}) \\
 & & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial \\
 & & & C_0(X,\mathbb{K})\cdot s^2 & \xleftarrow{(-\cap\eta)\cdot s} & C_1(X,\mathbb{K})\cdot s & \xleftarrow{(-\cap\eta)\cdot s} & C_2(X,\mathbb{K}) \\
 & & & & & \downarrow\partial & & \downarrow\partial \\
 & & & & & C_0(X,\mathbb{K})\cdot s & \xleftarrow{(-\cap\eta)\cdot s} & C_1(X,\mathbb{K}) \\
 & & & & & & & \downarrow\partial \\
 & & & & & & & C_0(X,\mathbb{K})
\end{array}$$

For the differential maps on the pages E_k with $k \geq 2$, we have to deal with a special kind of $(k+1)$-tuple Massey products, determined by η. See the survey paper [17] for an overview of higher order Massey products. Since later on we will use Cirici-Horel's results [7] about higher order Massey products, we follow here the terminology from their paper.

Definition 3.3. Let $\omega \in H^i(X,\mathbb{K})$. A defining system for $\{\overbrace{[\eta],\cdots,[\eta]}^{k},\omega\}$ is a collection of elements of $\{\alpha_1,\cdots,\alpha_k\}$ in $C^i(X,\mathbb{K})$ such that for the following $(k+1)\times(k+1)$-matrix

$$C=(c_{i,j})=\begin{pmatrix} \eta & 0 & \cdots & 0 & 0 \\ 0 & \eta & \cdots & 0 & \alpha_k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \eta & \alpha_2 \\ 0 & 0 & \cdots & 0 & \alpha_1 \end{pmatrix}$$

we have $[\alpha_1] = \omega$ and

$$\delta(c_{i,j}) = \sum_{q=i}^{j-1} (-1)^{\deg c_{i,q}} c_{i,q} c_{q+1,j}.$$

Here $[\alpha_1]$ denotes the cohomology class of the cocycle α_1. The $(k+1)$-*tuple Massey product* for $\{\overbrace{[\eta], \cdots, [\eta]}^{k}, \omega\}$ is defined to be the set of cohomology classes of

$$\sum_{q=1}^{k} (-1)^{\deg c_{1,q}} c_{1,q} c_{q+1,k+1}.$$

In our case, the above equations are equivalent (up to signs) to the following

$$\delta\alpha_1 = 0, [\alpha_1] = \omega, \delta\alpha_2 = \eta \cup \alpha_1, \cdots, \delta\alpha_k = \eta \cup \alpha_{k-1},$$

which in [16, 19, 20] is called a k-chain starting from ω, and the $(k+1)$-tuple Massey product is $\eta \cup \alpha_k$. There is some indeterminacy of this Massey product arising from the choice of $\{\alpha_1, \cdots, \alpha_k\}$ in $C^i(X, \mathbb{K})$. In fact, the indeterminacy of the $(k+1)$-tuple Massey product of η and ω comes from the corresponding k-tuple Massey product of η and some $\omega' \in H^i(X, \mathbb{K})$.

Definition 3.4. Let $\omega \in H^i(X, \mathbb{K})$ and $k \geq 1$. With the above notations, the image of $\eta \cup \alpha_k$ modulo the indeterminacy is called the $(k+1)$-*tuple Massey product associated to* η, denoted by

$$\langle [\eta], \omega \rangle_k = \langle \overbrace{[\eta], \cdots, [\eta]}^{k}, \omega \rangle.$$

If $\langle [\eta], \omega \rangle_k \neq 0$, we say the length of the higher order Massey product *on degree i* associated to η is $k+1$. Note that for $k = 2$, this definition yields a special type of classical triple Massey products $\langle [\eta], [\eta], \omega \rangle \in H^{i+1}(X, \mathbb{K})/(\eta \cup H^i(X, \mathbb{K}))$.

Proof of Theorem 1.1. By standard computations with spectral sequences (see, e.g., [2, pages 163–165]), one gets that the differential maps d_k for the above-mentioned spectral sequence are computed exactly by the $(k+1)$-tuple Massey products associated to η. Since $\widehat{R} = \mathbb{K}[[s]]$ is a principal ideal domain, any bounded complex of finitely generated $\widehat{R}$-modules is quasi-isomorphic to a finite direct sum of complexes of type $0 \to \widehat{R} \to 0$ or $0 \to \widehat{R} \xrightarrow{\cdot s^j} \widehat{R} \to 0$ for some $j \geq 0$ at various degrees. It is easy to see that the spectral sequence for the second type degenerates on the $(j+1)$-th page, while the module $\widehat{R}/(s^j)$ has the Jordan block size equal to j. So the maximal size of Jordan blocks for the eigenvalue 1 part of $H_i(X^\nu, \mathbb{K})$ is one less than the number $\min_k\{d_k^{p,q} = 0$ for all $p + q = i\}$, which by definition is exactly the length of the highest nonvanishing Massey product on degree i associated to η. Then the assertion of Theorem 1.1 follows. (One may also check [21, Proposition 9.1] for more details.) ☐

Proof of Corollary 1.2. For complex coefficients, one can use Sullivan's de Rham cdga to replace $C^*(X, \mathbb{C})$. Note that for a complex algebraic variety, Sullivan's de Rham cdga is equivalent to a cdga admitting an extra grading inducing the weight filtration on $H^*(X,\mathbb{C})$, see [4, Theorem 4.2] and [6, Theorem 8.11, Remark 8.12]. In particular, the differential map preserves the weight filtration, see [4, Theorem 4.2(ii)]. Note that the weight filtration on $H^i(X,\mathbb{C})$ has range $[0,\min\{2i,2n\}]$. For any $\alpha\in H^i(X,\mathbb{C})$, the assumption $W_0H^1(X,\mathbb{C})=0$ implies that $d_k(\alpha)$ increase the weight at least by k. Hence $d_k(\alpha)=0$ if $k>\min\{2i+2,2n\}$. Same proof applies to the case $W_1H^1(X,\mathbb{C})=0$. □

4. Prime-power cyclic covers. Homology of local systems. Betti bounds

In this section, we assume that X is a connected finite CW complex with a group epimorphism $\nu\colon\pi_1(X)\twoheadrightarrow\mathbb{Z}$. If p is a prime integer and $r>0$, by further projecting to $\mathbb{Z}_{p^r}$ we get a corresponding p^r-fold cover X_r of X. Let $\mathbb{K}=\mathbb{F}_p$ and $\eta_p\in C^1(X,\mathbb{F}_p)$ be the corresponding 1-cocycle (where the subscript is used to emphasize the characteristic of the field). Note that

$$\mathbb{F}_p\mathbb{Z}_{p^r}=\mathbb{F}_p[t]/(t^{p^r}-1)=\mathbb{F}_p[t]/((t-1)^{p^r}).$$

Then by the J-adic filtration, one gets a truncated spectral sequence induced from the one in the previous section, which computes the homology $H_*(X_r,\mathbb{F}_p)$.

By Proposition 2.2, this spectral sequence has p^r columns, hence it always degenerates at most at the E_{p^r}-page. Moreover, the differential map on the E_1-page of the spectral sequence is given by the left cup product with η_p, and the differential map d_k on the E_k-page for $k\geq 2$ is computed exactly by the $(k+1)$-tuple Massey products associated to η_p, since this spectral sequence is a truncated version of the one used in the previous section. Then Proposition 1.4 follows directly. Let us next prove Proposition 1.3 and Theorem 1.5.

Proof of Proposition 1.3. By the E_2-page of the spectral sequence, we have

$$\begin{aligned}b_i(X_r,\mathbb{F}_p)\leq&\sum_{k=i-p^r+1}^{i}\dim E_2^{i-k,k}\\=&\dim\ker(d_2^{0,i})+\dim\operatorname{coker}(d_2^{p^r-1,i-p^r+1})+(p^r-2)\cdot\beta_i(X,\eta_p)\\=&b_i(X,\mathbb{F}_p)+(p^r-1)\cdot\beta_i(X,\eta_p)\end{aligned}$$

Here the second equality follows from the fact

$$\ker(d_2^{0,i})=\ker\{H^i(X,\mathbb{F}_p)\xrightarrow{\eta\cup}H^{i+1}(X,\mathbb{F}_p)\}$$

and

$$\operatorname{coker}(d_2^{p^j-1,i-p^j+1})=\operatorname{coker}\{H^{i-1}(X,\mathbb{F}_p)\xrightarrow{\eta\cup}H^i(X,\mathbb{F}_p)\}.$$

□

Proof of Theorem 1.5. Let Y denote the p-fold cover of X associated to $\nu\colon\pi_1(X)\to\mathbb{Z}_p$. Since p is a prime number, we have

$$b_i(Y,\mathbb{C})=b_i(X,\mathbb{C})+(p-1)b_i(X,L_\lambda).$$

On the other hand, we have by the universal coefficient theorem and (1) that

$$b_i(Y,\mathbb{C})\leq b_i(Y,\mathbb{F}_p)\leq b_i(X,\mathbb{F}_p)+(p-1)\cdot\beta_i(X,\eta_p).$$

Then the inequality (3) follows. The inequality (4) is obvious.

If a k-tuple Massey product on degree j associated to η_p is non-trivial for some $k \leq p$, then we have by Proposition 1.4 that

$$b_j(Y, \mathbb{F}_p) < b_j(X, \mathbb{F}_p) + (p-1) \cdot \beta_j(X, \eta_p),$$

so the claim follows. □

5. Hyperplane arrangements

In this section, we specialize to the case of hyperplane arrangement complements. For relevant background on hyperplane arrangements, we refer to Dimca's book [10].

Let $\mathcal{A}$ be an arrangement of d hyperplanes in $\mathbb{C}^n$ with complement X. Note that $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^d$ is torsion free. Then any epimorphism $\pi_1(X) \twoheadrightarrow \mathbb{Z}_{p^r}$ must factor through some group epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}$. Hence Proposition 1.3, Proposition 1.4 and Theorem 1.5 apply to the case of hyperplane arrangement complements.

It is well-known that the complement of a hyperplane arrangement is formal over $\mathbb{C}$ in the sense of Sullivan's rational homotopy theory, see, e.g., [12] for a discussion and references, but in general it is not formal over $\mathbb{F}_p$. We recall here the following example given by Matei [18].

Example 5.1. Let $p > 2$ be a prime integer. Consider the full monomial arrangement $\mathcal{A}(p, 1, 3)$ in $\mathbb{C}^3$ defined by the zero locus of

$$z_1 \cdot z_2 \cdot z_3 \cdot \prod_{1 \leq i < j \leq 3} (z_i^p - z_j^p).$$

We label the hyperplanes as follows

$$H_{i,j}^q = \{z_i - \lambda^q z_j\}, \text{ where } \lambda = \exp(2\pi i/p) \text{ and } 1 \leq i < j \leq 3 \text{ and } 0 \leq q \leq p-1,$$
$$H_{3p+i} = \{z_i = 0\} \text{ for } 1 \leq i \leq 3.$$

Matei showed [18] that the complement X of $\mathcal{A}(p, 1, 3)$ has non-trivial triple Massey products on $H^2(X, \mathbb{F}_p)$. More precisely, for

$$\nu = (\overbrace{1, \cdots, 1}^{p}, \overbrace{-1, \cdots, -1}^{p}, \overbrace{0, \cdots, 0}^{p}, 0, 0, 0) \text{ and } \omega = (\overbrace{0, \cdots, 0}^{p}, \overbrace{1, \cdots, 1}^{p}, \overbrace{-1, \cdots, -1}^{p}, 0, 0, 0)$$

he showed that $\langle \nu, \nu, \omega \rangle \neq 0$.

Now we focus on the arrangement $\mathcal{A}(3, 1, 3)$, i.e., $p = 3$. Let L_λ be rank one $\mathbb{C}$-local system on X corresponding to

$$(\lambda, \lambda, \lambda, \lambda^{-1}, \lambda^{-1}, \lambda^{-1}, \overbrace{1, \cdots, 1}^{6}) \in (\mathbb{C}^*)^6$$

with $\lambda = \exp(2\pi i/3)$. A presentation for the fundamental group of X can be found in [18, Section 3.2]. Then a computation by Fox calculus (using a computer) yields that

$$1 = b_1(X, L_\lambda) < \beta_1(X, \eta_3) = 2.$$

This is compatible with the strict inequality (5).

On the other hand, the following result due to Cirici–Horel (stated here in the context of hyperplane arrangements) provides many examples of hyperplane arrangement complements having trivial higher order Massey products over $\mathbb{F}_p$.

Theorem 5.2. [7, 8, Theorem 7.12] *Let X be the complement of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^n$. Assume that $\mathcal{A}$ is defined over K, where K is a ℓ-adic field (i.e., a finite extension of $\mathbb{Q}_\ell$) and let $q = \ell^m$ be the cardinality of its residue field. Here ℓ is a prime number different from p. Write h for the order of q in the group $\mathbb{F}_p^*$. If $(k-2)/h \notin \mathbb{Z}$, then all k-tuple Massey products are trivial in $H^*(X, \mathbb{F}_p)$.*

Remark 5.3. Cirici-Horel use the singular cochain algebra in [7], while we work with the simplicial cochain algebra. Since every finite CW complex is homotopy equivalent to a finite simplicial complex, [7, 8, Theorem 7.12] remains valid in our setting.

Example 5.4. Let $\mathcal{A}$ be a graphic arrangement. For any prime number $p > 2$, we can choose $q = \ell = 2$. In particular, if $p = 3$, we have $h = 2$; if $p = 5$, we have $h = 4$. Hence the inequality (1) becomes an equality

$$b_i(Y, \mathbb{F}_p) = b_i(X, \mathbb{F}_p) + (p-1) \cdot \beta_i(X, \eta_p),$$

which is determined by the combinatorics with Y some p-fold cover of X for $p = 3$ or 5.

The above examples raise the question of combinatorial invariance of the higher order Massey products of hyperplane arrangement complements over $\mathbb{F}_p$. This problem seems to have been addressed to some extent in an unpublished work of Rybnikov [24].

We end this paper with the following remark.

Remark 5.5. Let X be a hyperplane arrangement complement. Using (2), Yoshinaga showed that the Betti number $b_i(Y, \mathbb{F}_2)$ for the double cover Y is determined by combinatorics [25]. He also asked the following question [25, Remark 3.8]:

Is the cohomology ring structure of $H^*(Y, \mathbb{F}_2)$ combinatorially determined?

Let Y' be a double cover of Y, hence a 4-fold cover of X. By (2), $b_i(Y', \mathbb{F}_2)$ is determined by cohomology ring structure of $H^*(Y, \mathbb{F}_2)$. On the other hand, Proposition 1.3 shows that $b_i(Y', \mathbb{F}_2)$ can be computed using the information from the higher order Massey products. This suggests that there is a subtle connection between the cohomology ring structure of $H^*(Y, \mathbb{F}_2)$ and the higher order Massey products.

References

[1] G. Bazzoni, M. Fernández, V. Muñoz, *Non-formal co-symplectic manifolds*, Trans. Amer. Math. Soc. 367 (2015), no.6, 4459–4481. 1

[2] R. Bott, L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982. 11

[3] N. Budur, Y. Liu, B. Wang, *The monodromy theorem for compact Kähler manifolds and smooth quasi-projective varieties*, Math. Ann. 371 (2018), no. 3-4, 1069–1086. 1, 8

[4] N. Budur, M. Rubió, *L-infinity pairs and applications to singularities*, Adv. Math.354 (2019), 106754, 40 pp. 2, 12

[5] N. Budur, B. Wang, *Cohomology jump loci of differential graded Lie algebras,* Compos. Math. 151 (2015), no. 8, 1499–1528. 8

[6] J. Cirici, G. Horel, *Mixed Hodge structures and formality of symmetric monoidal functors*, Ann. Sci. Éc. Norm. Supér. (4) 53 (2020), no.4, 1071–1104. 12

[7] J. Cirici, G. Horel, *Étale cohomology, purity and formality with torsion coefficients*, J. Topol. 15 (2022), no.4, 2270–2297. 4, 10, 14

[8] J. Cirici, G. Horel, *Corrigendum: Étale cohomology, purity and formality with torsion coefficients*, J. Topol. 17 (2024), no. 2, Paper No. e12348, 4 pp. 14

[9] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. 29 (1975), no. 3, 245–274. 2

[10] A. Dimca, *Hyperplane arrangements. An introduction*, Universitext. Springer, Cham, 2017. 13

[11] A. Dimca, S. Papadima, *Non-abelian cohomology jump loci from an analytic viewpoint*, Commun. Contemp. Math. 16 (2014), no. 4, 1350025. 8

[12] C. Dupont, *Purity, formality, and arrangement complements,* Int. Math. Res. Not. IMRN 2016, no. 13, 4132–4144. 13

[13] T. Ekedahl, *Two examples of smooth projective varieties with nonzero Massey products*, Algebra, algebraic topology and their interactions (Stockholm, 1983), 128–132. Lecture Notes in Math., 1183. Springer-Verlag, Berlin, 1986. 2

[14] E. Elduque, C. Geske, M. Herradón Cueto, L. Maxim, B. Wang, *Mixed Hodge structures on Alexander modules*, Mem. Amer. Math. Soc. 296 (2024), no. 1479. 2

[15] M. Fernández, A. Gray, J. Morgan, *Compact symplectic manifolds with free circle actions, and Massey products*, Michigan Math. J. 38 (1991), no.2, 271–283. 1

[16] T. Kohno, A. Pajitnov, *Novikov homology, jump loci and Massey products,* Cent. Eur. J. Math. 12 (2014), no.9, 1285-1304. 2, 11

[17] I. Limonchenko, D. Millionshchikov, *Higher order Massey products and applications*, Topology, geometry, and dynamics–V. A. Rokhlin-Memorial, 209–240. Contemp. Math., 772, American Mathematical Society, Providence, RI, 2021. 10

[18] D. Matei, *Massey products of complex hypersurface complements*, Singularity theory and its applications, Advanced Studies in Pure Mathematics, vol. 43, 205–219, Math. Soc. Japan, Tokyo, 2006. 2, 4, 13

[19] A. Pajitnov, *Massey products in mapping tori*, Eur. J. Math. 3 (2017), no.1, 34–42. 2, 11

[20] A. Pajitnov, *Twisted monodromy homomorphisms and Massey products,* Topology Appl. 255 (2019), 15–31. 2, 11

[21] S. Papadima, A. I. Suciu, *The spectral sequence of an equivariant chain complex and homology with local coefficients*, Trans. Amer. Math. Soc. 362 (2010), no. 5, 2685–2721. 1, 2, 3, 4, 9, 11

[22] S. Papadima, A. I. Suciu, *Algebraic monodromy and obstructions to formality.* Forum Math. 22 (2010), no.5, 973-983. 1

[23] A. Reznikov, *Three-manifolds class field theory (homology of coverings for a nonvirtually b_1-positive manifold)*, Selecta Math. (N.S.) 3 (1997), no. 3, 361–399. 2

[24] G. L. Rybnikov, *On the fundamental group and triple Massey's product*, arXiv:math/9805061. 14

[25] M. Yoshinaga, *Double coverings of arrangement complements and 2-torsion in Milnor fiber homology*, Eur. J. Math. 6 (2020), no. 3, 1097–1109. 3, 14

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