Graphical view on linear extensions of finite posets

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Abstract

One of possible cryptomorphic definitions of a partially ordered set (= a poset) P on a non-empty finite basic set N is in terms of the set $\mathcal{L}(P)$ of all its linear extensions, that is, in terms of the set of total orders of N consonant with P. Any total order on N can be interpreted as a node of a particular graph, called the *permutohedral graph* (over N), because it is indeed the graph of a certain polytope $\Pi(N)$ in $\mathbb{R}^N$, known as the permutohedron.

It is shown in the paper that a non-empty set of total orders on N equals to $\mathcal{L}(P)$ for some poset P on N iff it is a geodetically convex set in the permutohedral graph. This result means that a purely graphical concept of geodetic convexity in this graph is a cryptomorphic definition of a finite poset. In particular, the lattice of geodetically convex sets in this graph is graded and its height function is described in graphical terms. A counter-example, however, shows that the height function does not correspond to the usual graphical diameter, relating this matter to a combinatorial concept of the *dimension of a poset*.

Two alternative cryptomorphic views on a poset P on N are also briefly commented. The geometric counterpart is its full-dimensional *braid cone* in $\mathbb{R}^N$, while a combinatorial alternative is a *topology* on N distinguishing points, often referred as a distributive lattice.

Keywords: finite poset, linear extension of a poset, geodetically convex set, permutohedral graph, graded lattice, dimension of a poset. *MSC 2020*: 06A07, 06A15, 62R01

1 Introduction

Motivation of the first author for considering different interpretations of *partially ordered sets* (= posets) on a non-empty finite basic set N has originated from his recent findings in [18]. The main message there is that non-empty faces of the cone of supermodular functions on the power set $\mathcal{P}(N)$ of N have combinatorial interpretations and the most promising approach seems to be to describe these faces by certain collections of *posets on* N, introduced already in [15, § 3.3] and called the *fans of posets*. But these mathematical objects can also alternatively be interpreted as collections of particular cones in $\mathbb{R}^N$, sometimes named *braid cones*. Yet another version of these objects are particular collections of *topologies on* N, which seem to correspond to what was discussed in [7, § 3.3(d)]. Further hiddenly equivalent mathematical objects are *rank tests* discussed in [12], which are certain partitions of the set of all total orders on N. The rank tests correspond to some statistical hypotheses tests studied in context of algebraic statistics.

In mathematics, two objects are considered to be cryptomorphic if they are equivalent, but not obviously equivalent. To put the cryptic equivalence of above mathematical objects on a solid base, one first needs to establish formally the equivalence between the components of those objects. In the given context, this means clarifying what are precisely the respective equivalent views on posets on a fixed finite set N, which the topic of the present paper. In our opinion, the most elegant way to introduce the alternative views on finite posets is to use tools and results from lattice theory on (order-reversing) *Galois connections* [2, § IV.8], as done below.

This paper is dominantly devoted to the characterization of a poset P on N in terms of the collection $\mathcal{L}(P)$ of all its *linear extensions*. The correspondence $P \leftrightarrow \mathcal{L}(P)$ has earlier been treated in the literature, both in mathematics and in computer science. Pruesse and Ruskey [16] proposed an algorithm for generating (all) linear extensions of a given poset P and discussed the computational complexity of the tasks of either generating or counting linear extensions. In their paper, they, like many previous or later authors, described total orders on N by ordered lists of (all) elements of N, which we below name *enumerations* of N. They also utilized a natural proximity relation among these when two enumerations are considered to be neighbors if one is obtained from the other by the so-called *adjacent transposition*. These neighborhood relations correspond to the edges in our *permutohedral graph*, discussed below.

Linear extensions of a finite poset P were also a topic of interest in several more recent mathematical publications. The restriction of the permutohedral graph to $\mathcal{L}(P)$, called the *linear extension graph* (of P), was studied from the point of view of graph theory in [13, 11]. A recent paper [4] offers a broad survey of inequalities for the number of linear extensions of a poset P, based on tools of enumerative combinatorics [17], which itself offers more elaborate methods to count linear extensions (see the remark concluding Section 6.2).

An important related concept is that of the *dimension of a poset* P on N, introduced by Dushnik and Miller [6] already in the 1940s. Briefly formulated, it is the smallest number of total orders on N whose intersection is P. A classic result [10] in the theory of poset dimension says that the dimension of a poset P on n-element set N, $n \geq 4$, is at most $\lfloor \frac{n}{2} \rfloor$; nonetheless, this upper bound is tight owing to a classic construction of a poset of given dimension from [6]. The theme of order dimension remains to be a topic of research interest in mathematics [20].

A kind of opposite task to the one of generating linear extensions of a given poset P on N was considered by Heath and Nema [9] in their 2013 paper. The task considered by them was the so-called *poset cover problem*, which can be re-formulated as follows. Given a set S of total orders on N and a parameter $k \in \mathbb{N}$, decide whether there exist posets $P_1, \ldots, P_k$ on N such that S is the union $\bigcup_i \mathcal{L}(P_i)$. Of course, in case $k = 1$ this reduces to the task to decide whether there exists a poset P on N with $S = \mathcal{L}(P)$. They offered an algorithm to construct a poset P with $\mathcal{L}(P) \subseteq S$ which contains a prescribed total order from S and derived a result about computational complexity of the poset cover problem. The methodological tools they used in their paper inspired the proof of the main result in the present paper.

The central concept in the present paper is that of the permutohedral graph over N, whose nodes are enumerations of N. This graph is connected and, therefore, one can interpret it as a (finite) metric space, where the distance of two enumerations ε and η is the shared length of the paths between ε and η of the least possible length, which paths are called *geodesics*. One can thus consider the concept of a *geodetically convex set*, widely studied in graph-theoretical literature [14]. It is defined as such a set of nodes in the graph which, with every two nodes, contains all the nodes on every geodesic between those two nodes. Our main result says that a non-empty set of nodes in the permutohedral graph represents the set $\mathcal{L}(P)$ of linear extensions for a poset P on N iff it is a geodetically convex set.

It looks like we are not the first researchers who came with an observation of this sort. According to Björner and Wachs [3, § 6], already in the 1970, Tits [19, Theorem 2.19] obtained a characterization of (geodetically) convex sets in the set of permutations of an ordered set of integers $[n] := \{1, \ldots, n\}$. To convey this result to non-specialists in Coxeter groups, Björner and Wachs came in [3, Propositon 6.2] with their reformulation and proof of that result by Tits. Then, in their [3, Theorem 6.6], they related these (geodetically) convex sets to the so-called *labeled posets* on n-element set, by which they meant posets on a set with a prescribed total order. Nevertheless, their presentation still remains within the framework of group theory, because their arguments rely on some additional specialized results on the so-called weak (Bruhat) order on the symmetric group $S_{[n]}$ of all permutations of the set $[n]$.

In the present paper we offer a straightforward elementary proof of the fact that linear extensions of finite posets are characterized by geodetic convexity. Our self-contained proof does not need any specialized results on the symmetric group and is, thus, conceptually simpler. Indeed, realize that the collection of total orders on a general finite set N has no distinguished element and, therefore, it does not form a group. Nonetheless, the mental transition to labeled posets and the relation to the results of [3, § 6] are also commented (Section 5.6).

The permutohedral graph can be viewed as a graph with colored edges: an equivalence on the set of its edges can be defined by introducing the so-called *combinatorial labels* on them. Specifically, a two-element set $\{u, v\} \subseteq N$ will be interpreted as a label on the edge between enumerations ε and η if these only differ in the mutual position of u and v. The same combinatorial label on different edges has geometric interpretation in terms of a certain polytope $\Pi(N) \subseteq \mathbb{R}^N$, named the *permutohedron*. This is because the edges correspond to one-dimensional faces of $\Pi(N)$ and a shared label on different edges means that the edges are parallel [18, § 2.3.2].

The collection of all geodetically convex sets in the permutohedral graph forms a lattice. This lattice is graded and its height function is characterized. The height of a non-empty geodetically convex set S of enumerations is shown to be the *number of different edge colors* within the induced subgraph for S. What is perhaps surprising for the reader is that, once $|N| \geq 6$, the value of this function for S does not coincide with the graphical *diameter* of S. An example is given, based on a poset of dimension 3; in fact, the height of $\mathcal{L}(P)$ coincides with its diameter iff the dimension of a poset P is at most two [11, Theorem 4.2].

It is also shown that, if $|N| \geq 2$, the above lattice is anti-isomorphic to the *poset-based lattice* over N, which is the collection of all posets on N extended by the full binary relation $N \times N$. In particular, its sub-collection of posets P on N forms a meet semi-lattice, which is graded and its height function is determined by the number of compared pairs of distinct elements in P.

To offer global perspective on alternative descriptions of finite posets, two other views on them are discussed in the end of the paper. In fact, these are special instances of alternative anti-isomorphic views on *preposets* on N, which are transitive and reflexive binary relations on N. The geometric approach is to associate a poset on N with a particular full-dimensional polyhedral cone in $\mathbb{R}^N$, named the *braid cone*; this relates to [15, § 3.4]. The other interpretation is combinatorial, in terms of certain rings of subsets of N, we name (finite) *topologies*; this relates to well-known Birkhoff's representation theorem for finite distributive lattices [2, § III.3]. Both interpretations are presented by means of Galois connections.

The structure of the paper is as follows. In Section 2 particular lattice-theoretical tools are recalled for reader's convenience, while, in Section 3, (most of the) specific concepts of the present paper are introduced. In Section 4 we establish formally the correspondence between posets on N and their linear extensions. Main results are presented in Section 5. Section 6 contains remarks on two other cryptomorphic views of finite posets and conclusions.

2 Lattice-theoretical tutorial

As mentioned in the Introduction, we are going to use tools from lattice theory, in particular the Galois connections methodology, to present alternative views on finite posets. This means that the reader is assumed to be familiar with the relevant concepts from lattice theory [2]. Nonetheless, being aware of the fact that a general reader, to which this paper is addressed, may not be acquaint with some of these tools, we have decided to recall these notions and results in this first preliminary section. Experts in lattice theory may skip it.

2.1 Posets, tosets, preposets, and lattices

A partially ordered set $(\mathcal{L}, \preceq)$, briefly a *poset*, is a non-empty set $\mathcal{L}$ endowed with a partial order (= ordering), that is, a binary relation $\preceq$ on $\mathcal{L}$ which is reflexive, anti-symmetric, and transitive. The symbol $l \prec u$ for $l, u \in \mathcal{L}$ will denote $l \preceq u$ with $l \neq u$. A total order is a partial order in which every two elements are comparable: $\forall\, l, u \in \mathcal{L}$ either $u \preceq l$ or $l \preceq u$. Although this is not widespread practice in the literature, we are going to use the abbreviation *toset* for a totally ordered set in the present paper.

A *preposet*, also *preorder* or *quasi-order*, is a non-empty set $\mathcal{L}$ endowed with a reflexive and transitive relation $R \subseteq \mathcal{L} \times \mathcal{L}$. The *opposite* of R is then $R^{op} := \{\, (u,l) \,:\, (l,u) \in R\}$, being also a preposet. A preposet which is additionally symmetric is an *equivalence* (on $\mathcal{L}$). An elementary fact is that, given a preposet R, one can consider its "inner" equivalence $E := R \cap R^{op}$ and interpret R as a partial order $\preceq$ on the set $\mathcal{E}$ of equivalence classes of E, defined by

$$A \preceq B \quad \text{for } A, B \in \mathcal{E} \quad := \quad [\, \exists\, l \in A \;\; \exists\, u \in B \,:\, (l,u) \in R\,]\,.$$

Indeed, then $(l,u) \in R$ iff $A \preceq B$ for $A, B \in \mathcal{E}$ determined by $l \in A$ and $u \in B$.

A partially ordered set $(\mathcal{L}, \preceq)$ is called a *lattice* [2, § I.4] if every two-element subset of $\mathcal{L}$ has both the least upper bound, also named the *supremum*, and the greatest lower bound, also named the *infimum*. A join/meet *semi-lattice* is a poset in which every two-element subset has the supremum/infimum. A finite lattice, that is, a lattice $(\mathcal{L}, \preceq)$ with $\mathcal{L}$ finite, is necessarily *complete*, which means that the requirement above holds for any subset of $\mathcal{L}$. Every complete lattice has the *least element*, denoted by $\mathbf{0}$, which satisfies $\mathbf{0} \preceq u$ for any $u \in \mathcal{L}$ and the *greatest elements*, that is, an element $\mathbf{1} \in \mathcal{L}$ satisfying $l \preceq \mathbf{1}$ for any $l \in \mathcal{L}$.

A poset $(\mathcal{L}, \preceq)$ is (order) *isomorphic* to a poset $(\mathcal{L}', \preceq')$ if there is a bijective mapping ι from $\mathcal{L}$ onto $\mathcal{L}'$ which preserves the order, that is, $l \preceq u$ iff $\iota(l) \preceq' \iota(u)$ for $l, u \in \mathcal{L}$. If this is the case for a lattice $(\mathcal{L}, \preceq)$ then both suprema and infima are preserved by the isomorphism ι and the lattices are considered to be identical mathematical structures.

On the other hand, $(\mathcal{L}, \preceq)$ is *anti-isomorphic* to a poset $(\mathcal{L}', \preceq')$ if there is a bijective mapping ϱ from $\mathcal{L}$ onto $\mathcal{L}'$ which reverses the order: for $l, u \in \mathcal{L}$, one has $l \preceq u$ iff $\varrho(u) \preceq' \varrho(l)$. In case of a lattice, the suprema are transformed to infima and conversely. We say then that the posets/lattices $(\mathcal{L}, \preceq)$ and $(\mathcal{L}', \preceq')$ are (order) *dual* each other.

2.2 Covering relation in a poset and graded lattices

Given a poset $(\mathcal{L}, \preceq)$ and $l, u \in \mathcal{L}$ such that $l \prec u$ and there is no $e \in \mathcal{L}$ with $l \prec e \prec u$, we will write $l \prec\!\cdot\, u$ and say that l is *covered* by u or, alternatively, that u *covers* l.

The covering relation is the basis for a common method to visualize finite posets, that is, posets $(\mathcal{L}, \preceq)$ with $\mathcal{L}$ finite. In the so-called *Hasse diagram* of a finite poset $(\mathcal{L}, \preceq)$, elements of $\mathcal{L}$ are represented by circles, possibly with identifiers of the elements attached. If $l \prec u$ then the circle for u is placed higher than the circle for l. If, moreover, $l \prec\!\cdot\, u$ then a segment is drawn between the circles. Clearly, any finite poset is defined up to isomorphism by its diagram.

The *atoms* of a lattice $(\mathcal{L}, \preceq)$ are its elements that cover the least element in the lattice, that is, $a \in \mathcal{L}$ satisfying $\mathbf{0} \prec\!\cdot\, a$. Analogously, *coatoms* are the elements of the lattice covered by the greatest element, that is, elements $c \in \mathcal{L}$ satisfying $c \prec\!\cdot\, \mathbf{1}$. We will call a lattice $(\mathcal{L}, \preceq)$ *atomistic* if every element of $\mathcal{L}$ is the supremum of a set of atoms (in $\mathcal{L}$). Note that some authors use the term "atomic" instead of "atomistic", see [2, § VIII.9] or [21, § 2.2], but other authors use the term "atomic" to name a weaker condition on $(\mathcal{L}, \preceq)$, namely that, for every non-least element $e \neq \mathbf{0}$ of the lattice $\mathcal{L}$, an atom a exists such that $a \preceq e$. Analogously, a lattice $(\mathcal{L}, \preceq)$ is *coatomistic* if every element of $\mathcal{L}$ is the infimum of a set of coatoms. It is evident that a lattice anti-isomorphic to an atomistic lattice is coatomistic and conversely.

A poset $(\mathcal{L}, \preceq)$ is called *graded* [2, § I.3] if there exists a function $h : \mathcal{L} \to \mathbb{Z}$, called a *height function*, which satisfies, for any $l, u \in \mathcal{L}$,

- if $l \prec u$ then $h(l) < h(u)$,
- if $l \prec\!\cdot\, u$ then $h(u) = h(l) + 1$.

Note that, in case of a finite poset, the former condition is superfluous because it follows from the latter one. The graded lattices can be characterized in terms of finite *chains*, which are (non-empty) finite subsets of $\mathcal{L}$ that are totally ordered; the *length* of such a chain is the number of its elements minus 1. Note that the maximal chains (in sense of inclusion) in a finite lattice with $\mathbf{0} \neq \mathbf{1}$ have necessarily the form $\mathbf{0} \prec\!\cdot \ \ldots \prec\!\cdot \ \mathbf{1}$. A well-known fact is that a finite lattice is graded iff all maximal chains in it have the same length. A formally stronger, but equivalent, is the *Jordan-Dedekind chain condition* requiring that all maximal chains between the same endpoints have the same length [2, Lemma 1 in § I.3]. The function h which assigns to every $u \in \mathcal{L}$ the (shared) length of maximal chains between $\mathbf{0}$ and u is then a (standardized) height function for the lattice $(\mathcal{L}, \preceq)$. Clearly, the dual lattice to a graded lattice is a graded lattice.

2.3 Galois connections for representation of a complete lattice

There is a general method for generating examples of complete lattices in which method a lattice together with its dual lattice is defined on basis of a given binary relation between two sets. The method is universal in sense that every complete lattice is isomorphic to a lattice defined in this way. Moreover, every finite lattice is isomorphic to a lattice defined on basis of a given binary relation between elements of two finite sets.

This method uses the so-called *Galois connections*, also called *polarities* [2, § V.7], and allows one to represent every finite lattice in the form of a *concept lattice*, which notion has an elegant interpretation from the point of view of formal ontology [8]. To present the method we first need to recall a couple of relevant notions.

Given a non-empty set X, its power set will be denoted by $\mathcal{P}(X) := \{\, S : \ S \subseteq X \,\}$. A *Moore family* of subsets of X, also called a *closure system* (for X), is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ of subsets of X which is closed under (arbitrary) set intersection and includes the set X itself: $\mathcal{D} \subseteq \mathcal{F} \ \Rightarrow \ \bigcap \mathcal{D} \in \mathcal{F}$ with a convention $\bigcap \mathcal{D} = X$ for $\mathcal{D} = \emptyset$. An elementary observation is that any Moore family $(\mathcal{F}, \subseteq)$, equipped with the set inclusion relation as a partial order, is a complete lattice [2, Theorem 2 in § V.1].

A *closure operation* on subsets of X is a mapping $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ which is *extensive*, which means $S \subseteq \mathrm{cl}\,(S)$ for $S \subseteq X$, *isotone*, which means $S \subseteq T \subseteq X \Rightarrow \mathrm{cl}\,(S) \subseteq \mathrm{cl}\,(T)$, and *idempotent*, which means $\mathrm{cl}\,(\mathrm{cl}\,(S)) = \mathrm{cl}\,(S)$ for $S \subseteq X$. A set $S \subseteq X$ is then called *closed with respect to* cl if $S = \mathrm{cl}\,(S)$. The basic fact is that the collection of subsets of X closed with respect to cl is a Moore family and any Moore family can be defined in this way [2, Theorem 1 in § V.1]. Specifically, given a Moore family $\mathcal{F}$ of subsets of X, the formula

$$\mathrm{cl}_{\mathcal{F}}(S) := \bigcap \{\, F \subseteq X : S \subseteq F \in \mathcal{F} \,\} \quad \text{for } S \subseteq X\,,$$

defines a closure operation on subsets of X having $\mathcal{F}$ as the collection of closed sets with respect to $\mathrm{cl}_{\mathcal{F}}$; see [8, Theorem 1].

In other words, there is a one-to-one correspondence between closure operations on subsets of X and Moore families of subsets of X, which are examples of complete lattices relative to set inclusion relation $\subseteq$. Thus, any closure operation cl on (subsets of) X yields an example of a complete lattice $(\mathcal{F}, \subseteq)$, where $\mathcal{F}$ is the collection of sets closed with respect to cl.

We now present the method of Galois connections itself. Let X and Y be non-empty sets and $\mathfrak{r}$ a binary relation between elements of X and Y: $\mathfrak{r} \subseteq X \times Y$. We are going to write $x\,\mathfrak{r}\,y$ to denote that $x \in X$ and $y \in Y$ are in this *incidence* relation: $(x, y) \in \mathfrak{r}$. Then every subset S of X can be assigned a subset of Y as follows (the *forward* direction):

$$S \subseteq X \quad \mapsto \quad S^{\vartriangleright} := \{\, y \in Y \;:\; x\,\mathfrak{r}\,y \quad \text{for every } x \in S \,\} \;\subseteq Y\,.$$

Analogously, every subset T of Y can be assigned a subset of X (the *backward* direction):

$$T \subseteq Y \quad \mapsto \quad T^{\vartriangleleft} := \{\, x \in X \;:\; x\,\mathfrak{r}\,y \quad \text{for every } y \in T \,\} \;\subseteq X\,.$$

The mappings $S \mapsto S^{\vartriangleright}$ and $T \mapsto T^{\vartriangleleft}$ are then *Galois connections* between the lattices $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$, also called *polarities* [2, § V.7]. These are *order-reversing* connections, because $S_1 \subseteq S_2 \subseteq X \;\Rightarrow\; S_1^{\vartriangleright} \supseteq S_2^{\vartriangleright}$ and $T_1 \subseteq T_2 \subseteq Y \;\Rightarrow\; T_1^{\vartriangleleft} \supseteq T_2^{\vartriangleleft}$. The point is that the mapping $S \mapsto S^{\vartriangleright\vartriangleleft} := (S^{\vartriangleright})^{\vartriangleleft}$ is a closure operation on subsets of X and the mapping $T \mapsto T^{\vartriangleleft\vartriangleright} := (T^{\vartriangleleft})^{\vartriangleright}$ is a closure operation on subsets of Y. Thus, they define a complete lattice $(\mathcal{X}^{\mathfrak{r}}, \subseteq)$ of closed subsets of X and a complete lattice $(\mathcal{Y}^{\mathfrak{r}}, \subseteq)$ of closed subsets of Y. In addition to that,

- one has $S \in \mathcal{X}^{\mathfrak{r}}$ iff $S = T^{\vartriangleleft}$ for some $T \subseteq Y$,
- analogously $T \in \mathcal{Y}^{\mathfrak{r}}$ iff $T = S^{\vartriangleright}$ for some $S \subseteq X$, and
- the inverse of the mapping $S \in \mathcal{X}^{\mathfrak{r}} \mapsto S^{\vartriangleright}$ is the mapping $T \in \mathcal{Y}^{\mathfrak{r}} \mapsto T^{\vartriangleleft}$, defining an anti-isomorphism between lattices $(\mathcal{X}^{\mathfrak{r}}, \subseteq)$ and $(\mathcal{Y}^{\mathfrak{r}}, \subseteq)$ [2, Theorem 19 in § V.7].
  This anti-isomorphism is sometimes called the (respective) *Galois correspondence*.

In particular, the complete lattice $(\mathcal{X}^{\mathfrak{r}}, \subseteq)$ and its dual lattice $(\mathcal{Y}^{\mathfrak{r}}, \subseteq)$ have together been defined on basis of a given binary relation $\mathfrak{r} \subseteq X \times Y$. Note that in case of finite X, or of finite Y, the lattice $(\mathcal{X}^{\mathfrak{r}}, \subseteq)$ is finite, too. Useful observation is that $S^{\vartriangleright} = S^{\vartriangleright\vartriangleleft\vartriangleright}$ for any $S \subseteq X$.

The arguments why this method is universal for representing complete lattices are based on results from [2, § V.9]. More specifically, given a complete lattice $(\mathcal{L}, \preceq)$, one can put $X := \mathcal{L}$, $Y := \mathcal{L}$ and consider a binary relation $\mathfrak{r} := \{\, (x, y) \in X \times Y \;:\; x \preceq y\}$. Then, for any $S \subseteq X = \mathcal{L}$, $S^{\vartriangleright}$ is the set of upper bounds for S in $(\mathcal{L}, \preceq)$ while, for any $T \subseteq Y = \mathcal{L}$, $T^{\vartriangleleft}$ is the set of lower bounds for T. Then [2, Theorem 23 in § V.9] implies that every closed subset $S \subseteq X$, that is, every set $S \in \mathcal{X}^{\mathfrak{r}}$, is a principal ideal in $(\mathcal{L}, \preceq)$, which means it has the form $S = \{z \in \mathcal{L} \;:\; z \preceq a\,\}$ for some $a \in \mathcal{L}$. Thus, $(\mathcal{X}^{\mathfrak{r}}, \subseteq)$ coincides with the lattice of principal ideals in $(\mathcal{L}, \preceq)$, which is known to be isomorphic to the lattice $(\mathcal{L}, \preceq)$ itself by [2, Theorem 3 in § II.3]. Note that in case of a finite lattice $(\mathcal{L}, \preceq)$, the sets X and Y are both finite by the construction.

As a by-product of the above construction we observe that any complete lattice is isomorphic to a Moore family of subsets of a non-empty set X (ordered by set inclusion), and, therefore, it can be introduced by means of a closure operation on subsets of X.

3 Other preliminaries

We also assume the reader is familiar with elementary graph-theoretical concepts and has mild knowledge of polyhedral geometry [21]. Despite, we recall our basic concepts in this section.

3.1 Basic notational conventions

Throughout the (rest of the) paper, the symbol N will denote a finite non-empty *basic set* of variables, while the number of its elements will be denoted by $n := |N| \geq 1$. We intentionally regard the basic set N as an unordered set. Subtle reasons for that, not evident at first sight, are commented in Section 3.4. On contrary to that, given a natural number $n \in \mathbb{N}$, the symbol $[n] := \{1, \ldots, n\}$ will denote an ordered set of integers between 1 and n.

Recall that the power set of N is denoted by $\mathcal{P}(N) := \{A : A \subseteq N\}$. The *diagonal* in the Cartesian product $N \times N$ will be denoted by $\Delta := \{\, (u, u) \; : \; u \in N\}$. Given $u \in N$, the symbol u will also be used to denote the one-element set (= singleton) $\{u\}$. Given $A, B \subseteq N$, the symbol AB will be a shortened notation for the union $A \cup B$. The *incidence vector* χ_A for a subset $A \subseteq N$, is a vector $[x_u]_{u\in N}$ in $\mathbb{R}^N$ specified by $x_u := +1$ if $u \in A$ and $x_u := 0$ if $u \in N \setminus A$. Our (scarcely used) infix notation for composition of functions τ and ς uses a dot: $(\tau \cdot \varsigma)(x) := \tau(\varsigma(x))$ for any argument x of ς.

3.2 Transitive closure and acyclic binary relations

Because the intersection of transitive relations on N is transitive, for every binary relation $R \subseteq N \times N$, there exists the least transitive relation $T \subseteq N \times N$ with $R \subseteq T$, which is called the *transitive closure of* R and denoted by $\mathrm{tr}(R)$. It is easy to see that $(u, v) \in \mathrm{tr}(R)$ iff there exists a sequence $u = u_1, \ldots, u_k = v$, $k \geq 2$, in N with $(u_i, u_{i+1}) \in R$ for $i = 1, \ldots, k-1$; additionally, without loss of generality one can assume that $u_1, \ldots, u_{k-1}$ and $u_2, \ldots, u_k$ are distinct.

We say that a binary relation $R \subseteq N \times N$ is *acyclic* if there is no sequence $u_1, u_2, \ldots, u_k$, $k \geq 2$, of elements of N such that $u_k = u_1$ and $(u_i, u_{i+1}) \in R$ for $i = 1, \ldots k-1$. This basically means that $\Delta \cap \mathrm{tr}(R) = \emptyset$. A well-known equivalent definition of acyclicity is that all elements of N can be ordered into a sequence $u_1, \ldots, u_n$, $n = |N|$, such that $(u_i, u_j) \in R$ implies $i < j$. Indeed, the necessity can be verified by induction on n, because every acyclic relation R has a terminal variable, which is $t \in N$ such that there is no $w \in N$ with $(t, w) \in R$.

For a transitive relation $T \subseteq N \times N$, the next two conditions are equivalent to acyclicity:

- T is *irreflexive*, which means that there is no $u \in N$ with $(u, u) \in T$,
- T is *asymmetric*, which means that $(u, v) \in T$ implies $(v, u) \notin T$.

Indeed, realize that, for a transitive relation T, negations of those conditions are equivalent.

These observations allow one to verify easily the next characterization of posets on N:

Fact A relation $P \subseteq N \times N$ is a poset iff $\Delta \subseteq P$ and $P \setminus \Delta$ is both transitive and acyclic.

Given a poset P on N and $(u, v) \in N \times N$, we will sometimes apply the next symbols:

- $u \preceq v$ $[P]$ for *non-strict comparison*, that is, $(u,v) \in P$,
- $u \prec v$ $[P]$ for *strict comparison*, that is, $(u,v) \in P \setminus \Delta$,
- $u \prec\!\cdot\, v$ $[P]$ for *covering*, that is, $(u,v) \in P \setminus \Delta$, $\neg[\,\exists\, w \in N \setminus \{u,v\} \ : \ (u,w),(w,v) \in P\,]$,
- $u \| v\, [P]$ for *incomparability* of u and v in P, that is, $(u,v) \notin P$ and $(v,u) \notin P$.

When the poset P is clear from the context, the symbol $[P]$ might be omitted.

3.3 Finite posets depicted as directed acyclic graphs

There are two (different) standard ways to represent partial orders on a non-empty finite basic set N by means of directed acyclic graphs having N as the set of nodes. The point is that every acyclic binary relation R on N can be depicted in the form of a *directed acyclic graph* having N as the set of nodes, where an *arrow* (= directed edge) $u \to v$ is drawn iff $(u,v) \in R$.

The first way to represent a poset P on N is based on its *strict version*, that is, $R := P \setminus \Delta$. Owing to the characterization of posets on N from Section 3.2, it leads to a *transitive directed acyclic* graph with node set N. Formulated in other words, transitive directed acyclic graphs over N are in obvious one-to-one correspondence with posets on N.

The second way comes from the *covering relation* $C := \{\,(u,v) \in N \times N \ : \ u \prec\!\cdot\, v\ [P]\,\}$. Since $C \subseteq P \setminus \Delta$, it is also acyclic. The respective directed graph is (strongly) *anti-transitive* in sense that $(u_i, u_{i+1}) \in C$ for $i = 1, \ldots k-1$, $k \geq 3$, implies $(u_1, u_k) \notin C$. This yields the so-called *directed Hasse diagram* of the poset P, whose underlying undirected graph is called the *cover graph* of P in [20]. In fact, C is the least subset of $P \setminus \Delta$ such that $\mathrm{tr}(C) = P \setminus \Delta$.

3.4 Enumerations and permutohedron

Ordered lists of (all) elements of N (without repetition) will be called *enumerations* (of N). Formally, an enumeration of N is a one-to-one (= bijective) mapping $\varepsilon : [n] \to N$ from $[n]$ onto N. We will use the record $|\,\varepsilon(1)| \ldots |\varepsilon(n)\,|$ to specify such an enumeration. Every enumeration of N can be interpreted as a total order (= toset) on N. In fact, there are two ways to do that: one can either interpret ε in the ascending way as $\varepsilon(1) \prec \ldots \prec \varepsilon(n)$ or one may interpret it in the descending way as $\varepsilon(1) \succ \ldots \succ \varepsilon(n)$.

The set of (all) enumerations of N will be denoted by $\Upsilon(N)$. Every enumeration can equivalently be described by the corresponding *rank vector*, which is the vector $[\varepsilon_{-1}(u)]_{u \in N}$ in $\mathbb{R}^N$. Formally, the rank vector (for an enumeration ε) is the inverse $\varepsilon_{-1} : N \to [n]$ of the mapping $\varepsilon : [n] \to N$. We are going to denote it by ρ_ε in a geometric context (of $\mathbb{R}^N$).

The *permutohedron* in $\mathbb{R}^N$, denoted by $\Pi(N)$, is defined as the convex hull of the rank vectors for (all) enumerations of N. Formally, $\Pi(N) := \mathrm{conv}\,(\,\{\rho_\varepsilon \ : \ \varepsilon \in \Upsilon(N)\}\,)$. It is a subset of the affine space $\{\, y \in \mathbb{R}^N \ : \ \sum_{\ell \in N} y_\ell = \binom{n+1}{2}\,\}$; therefore, its (Euclidean) dimension is $n-1$.

Comment Note in this context that some authors dealing with related topics [15, 12], in their definitions of the permutohedron, have adopted a seemingly innocuous identification convention that N coincides with $[n]$. Unfortunately, such a highly specific convention is actually harmful, as it leads to misunderstandings on the side of the reader. The reason is that then the rank vectors (or enumerations) become the self-bijections of $[n]$, that is, elements of the symmetric group $S_{[n]}$, and gain many other meanings, which mislead the reader. The artificially enforced group structure on them causes later identification problems (for more explanation see [18, § 2.6]). To avoid these problems, in this paper, we assume that the set of basic variables N has no deeper mathematical structure, or, put it in another way, we intentionally regard N as an unordered set and distinguish it consistently from the ordered set $[n] = \{1, \ldots, n\}$, where $n = |N|$.

3.5 Face lattice of the permutohedron

Recall that the set of (all) faces of any polytope, ordered by inclusion, is a finite lattice, known to be both atomistic and coatomistic. In addition to that, this *face lattice* is a graded lattice, where the dimension (of faces) plays the role of a height function [21, Theorem 2.7].

In this subsection we present an elegant *combinatorial* description of (non-empty) faces of the permutohedron $\Pi(N)$, mentioned in [21, p. 18], whose proof can be found in [1, § 1]. Every non-empty face of the permutohedron $\Pi(N)$ corresponds to an ordered partition of the basic set N into non-empty blocks. We will use the symbol $|\, A_1| \ldots |A_m\,|$ to denote such a partition into m blocks, $1 \leq m \leq n$. Note that this notation is consistent with our notation for enumerations. In fact, the ordered partition serves as a compact description of the set of those enumerations of N which encode the *vertices* of the corresponding face of $\Pi(N)$: these are the respective rank vectors. Specifically, it is the set of those enumerations ε of N which are consistent with $|\, A_1| \ldots |A_m\,|$: these are the ordered lists ε of elements of N where first (all) the elements of A_1 are listed, then (all) the elements of A_2 follow and so on. Thus, any face of $\Pi(N)$ also corresponds to a special subset of $\Upsilon(N)$. The (Euclidean) dimension of the face described by $|\, A_1| \ldots |A_m\,|$ is $n - m$ then. Later, in Section 4, we interpret such (non-empty) face-associated subsets of $\Upsilon(N)$ as a special case of subsets of $\Upsilon(N)$ assigned to partial orders on N.

3.6 Some graph-theoretical concepts

We assume that the reader is familiar with elementary graph-theoretical notions and understands the next definitions, which apply to any *connected undirected graph* with a node set U.

A *geodesic* between nodes $\varepsilon, \eta \in U$ is a walk between ε and η (in the graph) which has the shortest possible length among such walks. Note that it is necessarily a path (= nodes are not repeated in it) and that several geodesics may exist between two nodes. The (graphical) *distance* $\mathrm{dist}\,(\varepsilon, \eta)$ between two nodes $\varepsilon, \eta \in U$ is the length of a geodesic between them. It makes no problem to observe that it satisfies the axioms of a metric and defines a finite metric space.

We say that a node $\sigma \in U$ *is between nodes* $\varepsilon \in U$ and $\eta \in U$ if σ belongs to some geodesic between ε and η; an equivalent condition is that $\mathrm{dist}\,(\varepsilon, \sigma) + \mathrm{dist}\,(\sigma, \eta) = \mathrm{dist}\,(\varepsilon, \eta)$. A set $S \subseteq U$ is *geodetically convex* if, for any $\varepsilon, \eta \in S$, all nodes between them belong to S. It is evident that the intersection of geodetically convex sets is a geodetically convex set.

3.7 Permutohedral graph

We say that enumerations $\varepsilon, \eta \in \Upsilon(N)$ differ by an *adjacent transposition* if

$$\exists\, 1 \leq i < n \quad \text{such that} \quad \varepsilon(i) = \eta(i+1),\ \varepsilon(i+1) = \eta(i), \qquad \text{and} \quad \varepsilon(k) = \eta(k) \qquad \text{for remaining } k \in [n] \setminus \{i, i+1\}\,. \tag{1}$$

We are going to interpret the set of enumerations of N as an undirected *permutohedral graph* in which $\Upsilon(N)$ is the set of nodes and edges are determined by adjacent transpositions. The reason for this terminology is as follows. Despite the given definition of adjacency for enumerations is fully combinatorial, it has a natural geometric interpretation. Enumerations $\varepsilon, \eta \in \Upsilon(N)$ are adjacent in this graph iff the segment $[\rho_\varepsilon, \rho_\eta] \subseteq \mathbb{R}^N$ between the respective rank vectors is a face (= a geometric edge) of the permutohedron $\Pi(N)$. This follows from the results presented in Section 3.5: the geometric edges of $\Pi(N)$ are described by ordered partitions of the form $|\, A_1| \ldots |A_{n-1}\,|$; then $|A_i| = 2$ for unique $i \in [n-1]$ and $|A_j| = 1$ for remaining $j \neq i$.

To prove our main result (Theorem 3) is instrumental to introduce particular *edge-labeling* (= edge coloring) for this graph. Specifically, we are going to label its edges by two-element subsets $\{u, v\}$ of our basic set N: if $\varepsilon, \eta \in \Upsilon(N)$ differ by an adjacent transposition (1) then we label the edge between ε and η by the set $\{u, v\} := \{\varepsilon(i), \varepsilon(i+1)\} = \{\eta(i), \eta(i+1)\}$. We will call the set $\{u, v\}$ a *combinatorial label* on the edge and denote this by $\varepsilon \xleftrightarrow{uv} \eta$.

If specifically $u = \varepsilon(i)$ and $v = \varepsilon(i+1)$ then (1) yields $\rho_\varepsilon - \rho_\eta = \chi_v - \chi_u$, that is, the rank vectors only differ in the components u and v (by one). In particular, if $\varepsilon \xleftrightarrow{uv} \eta$ and $\sigma \xleftrightarrow{uv} \pi$ then $\rho_\varepsilon - \rho_\eta = \pm(\rho_\sigma - \rho_\pi)$ meaning that the segments $[\rho_\varepsilon, \rho_\eta]$ and $[\rho_\sigma, \rho_\pi]$ are *parallel*, or worded in a different way, they are perpendicular to the same hyperplane $\mathsf{H}_{uv} := \{\, z \in \mathbb{R}^N \,:\, z_u = z_v \,\}$. Thus, the respective *combinatorial equivalence* of edges in the permutohedral graph, defined as having the same combinatorial label, has natural geometric interpretation.

Every two-element subset $\{u, v\}$ of N also defines an *automorphism* of the permutohedral graph. Specifically, the transposition $\tau_{uv} : N \to N$ of u and v, defined by $\tau_{uv}(u) = v$, $\tau_{uv}(v) = u$, and $\tau_{uv}(t) = t$ for $t \in N \setminus \{u, v\}$, is a (self-inverse) permutation of N and every enumeration $\varsigma : [n] \to N$ can be assigned its composition $\tau_{uv} \cdot \varsigma$ with $\tau_{uv} : N \to N$. Note that, thanks to our distinguishing (unordered) N from (ordered) $[n]$, there is a unique way to compose these two bijections. The mapping $\varsigma \mapsto \tau_{uv} \cdot \varsigma$ is a self-inverse transformation of $\Upsilon(N)$ preserving the edges in the permutohedral graph. In fact, it maps bijectively the set

$$S_{u \prec v} := \{\, \varsigma \in \Upsilon(N) \,:\, \varsigma_{-1}(u) < \varsigma_{-1}(v) \,\} \quad \text{of enumerations in which } u \text{ precedes } v$$

to its complement $S_{v \prec u} = \{\, \xi \in \Upsilon(N) \,:\, \xi_{-1}(v) < \xi_{-1}(u) \,\}$.

4 Linear extensions

In this section we introduce a particular finite *poset-based lattice*, extending the collection of posets on N, by the method of Galois connections, as described in Section 2.3.

To this end we introduce a binary relation $\circ$ between elements of the set $X := \Upsilon(N)$ of enumerations of N and the elements of the Cartesian product $Y := N \times N$. More specifically, if $\varepsilon : [n] \to N$ is an enumeration of N and $(u,v) \in N \times N$ then we consider ε to be in an incidence relation $\circ$ with the pair (u,v) iff u *precedes* v in ε :

$$\varepsilon \circ (u,v) \quad := \quad \varepsilon_{-1}(u) \leq \varepsilon_{-1}(v)\,, \quad \text{written also in the form } u \preceq_{\varepsilon} v.$$

We will denote Galois connections based on this precedence relation $\circ$ using larger triangles $\triangleright$ and $\triangleleft$, in order to distinguish them from later Galois connections (in Section 6). Thus, any enumeration $\varepsilon \in \Upsilon(N)$ of N is assigned (by forward Galois connection) a toset

$$T_{\varepsilon} := (\{\varepsilon\})^{\triangleright} = \{\, (u,v) \in N \times N \,:\, u \preceq_{\varepsilon} v \,\}\,.$$

We also introduce special notation for the case that u *strictly precedes* v in ε :

$$u \prec_{\varepsilon} v \qquad \text{will denote } \; \varepsilon_{-1}(u) < \varepsilon_{-1}(v), \quad \text{that is, } \; u \prec v \; [T_{\varepsilon}].$$

Analogously, in the case that u *immediately precedes* v in ε we will write

$$u \prec_{\varepsilon}\!\cdot\; v \qquad \text{to denote } \; \varepsilon_{-1}(v) - \varepsilon_{-1}(u) = 1, \quad \text{that is, } \; u \prec\!\cdot\; v \; [T_{\varepsilon}].$$

Any subset T of $Y = N \times N$ is assigned (by backward Galois connection) an enumeration set

$$\mathcal{L}(T) := T^{\triangleleft} = \{\, \varepsilon \in \Upsilon(N) \,:\, \forall\,(u,v) \in T \;\; u \preceq_{\varepsilon} v \,\}\,,$$

which is interpreted as the set of *linear extensions* for T (provided $T \setminus \Delta$ is an acyclic relation on N, as otherwise $\mathcal{L}(T)$ turns out to be empty).

Lemma 1 Let $(\mathcal{X}^{\circ}, \subseteq)$ and $(\mathcal{Y}^{\circ}, \subseteq)$ denote the finite lattices defined using the Galois connections $\triangleright$ and $\triangleleft$ based on the incidence relation $\circ$ between X and Y introduced above.

(i) If $n = |N| \geq 2$ then $[\, T \in \mathcal{Y}^{\circ}$ and $T \subset N \times N\,]$ iff T is a poset on N.

(ii) One has $[\, S \in \mathcal{X}^{\circ}$ and $S \neq \emptyset\,]$ iff a (uniquely determined) poset $(N, \preceq)$ exists such that $S = \{\, \varepsilon \in \Upsilon(N) \,:\, \varepsilon_{-1}(u) < \varepsilon_{-1}(v)$ whenever $u \prec v \,\}$, i.e. $S = \mathcal{L}(P)$ for a poset P on N.

(iii) Any $S \subseteq \Upsilon(N)$ representing a non-empty face of $\Pi(N)$ (as in Section 3.5) belongs to $\mathcal{X}^{\circ}$.

(iv) The lattice $(\mathcal{X}^{\circ}, \subseteq)$ is atomistic, coatomistic, and graded.

Coatoms in $\mathcal{X}^{\circ}$ are precisely the sets $S_{u \prec v} := \{\, \varepsilon \in \Upsilon(N) \,:\, u \prec_{\varepsilon} v \,\}$, where $u, v \in N$, $u \neq v$.

Note in this context that the class $\mathcal{X}^{\circ} \setminus \{\emptyset\}$ of poset-interpretable sets of enumerations can alternatively be introduced by means of "strict" incidence relation $\varepsilon \odot (u,v)$ defined by $\varepsilon_{-1}(u) < \varepsilon_{-1}(v)$. Except the degenerate case $|N| = 1$, the resulting lattice $(\mathcal{X}^{\odot}, \subseteq)$ coincides with $(\mathcal{X}^{\circ}, \subseteq)$ while the elements of $\mathcal{Y}^{\odot}$ are strict versions of posets on N instead.

Proof. For the necessity in (i) assume $T \in \mathcal{Y}^\circ$ and $T \neq N \times N$. Therefore,

$$T = S^{\rhd} := \{ (u,v) \in N \times N : \ \forall \varepsilon \in S \ \ \varepsilon_{-1}(u) \leq \varepsilon_{-1}(v) \}$$

for some $S \subseteq \Upsilon(N)$ (see Section 2.3). The reflexivity and transitivity of T follows directly from its definition on basis of S. Then T is anti-symmetric as otherwise $S = \emptyset$ gives $T = N \times N$.

For the sufficiency in (i) assume that $T \subset N \times N$ is a poset. As explained in Sections 3.2 and 3.3, the relation $T \setminus \Delta$ is then acyclic and coincides with the set of arrows of a transitive directed acyclic graph G over N. Hence, the set

$$\begin{aligned} S := T^{\lhd} \ &= \ \{ \varepsilon \in \Upsilon(N) \ : \ \forall (u,v) \in T \ \ \varepsilon_{-1}(u) \leq \varepsilon_{-1}(v) \} \\ &= \ \{ \varepsilon \in \Upsilon(N) \ : \ \varepsilon_{-1}(a) < \varepsilon_{-1}(b) \quad \text{whenever } a \to b \text{ in } G \} \end{aligned}$$

of enumerations of N which are consonant with G is non-empty. Evidently, $T \subseteq S^{\rhd}$ and to show $S^{\rhd} \subseteq T$ one needs, for any $(u,v) \notin T$, to find an enumeration $\varepsilon \in S$ with $\varepsilon_{-1}(u) > \varepsilon_{-1}(v)$. Since $S \neq \emptyset$, this is evident in case $v \to u$ in G. In case of a pair (u,v) of non-adjacent distinct nodes of G one can consider the set $A \subseteq N$ of $a \in N$ with $(a,v) \in T$. Then $v \in A$, $u \notin A$, and the induced subgraph G_A is acyclic. Hence, an enumeration of A exists which is consonant with G_A and this can be extended to an enumeration ε of N consonant with G. Thus, $\varepsilon \in S$ with $\varepsilon_{-1}(v) < \varepsilon_{-1}(u)$ was obtained. Altogether, we have shown $T = S^{\rhd}$, which means that $T \in \mathcal{Y}^\circ$.

As concerns (ii), this holds trivially in case $n = 1$. In case $n \geq 2$ apply (i) and realize the fact that $(\mathcal{X}^\circ, \subseteq)$ and $(\mathcal{Y}^\circ, \subseteq)$ are anti-isomorphic; see Section 2.3. Specifically, the correspondence $T \in \mathcal{Y}^\circ \mapsto T^{\lhd} \in \mathcal{X}^\circ$ maps bijectively the partial orders on N onto non-empty sets in $\mathcal{X}^\circ$.

Note (iii) holds trivially if $n = 1$. If $n \geq 2$ then take a non-empty face of the permutohedron represented by an ordered partition $|\, A_1| \ldots |A_m\,|$, $m \geq 1$, of N. It defines a partial order

$$T := \Delta \cup \{ (u,v) \in N \times N \ : \ u \in A_i \text{ and } v \in A_j \ \text{ for some } i < j \ \}$$

on N. Easily, $\mathcal{L}(T)$ coincides with the set of enumerations consistent with $|\, A_1| \ldots |A_m\,|$.

As concerns (iv), this holds trivially in case $n = 1$. In case $n \geq 2$ consider $\varepsilon \in \Upsilon(N)$ with its toset T_ε and observe that $(T_\varepsilon)^{\lhd} = \{ \varepsilon \}$. Hence, the atoms of $(\mathcal{X}^\circ, \subseteq)$ are just singleton subsets of $\Upsilon(N)$. Since any $S \in \mathcal{X}^\circ$ is the union of singletons, the lattice $(\mathcal{X}^\circ, \subseteq)$ is atomistic.

The fact that $(\mathcal{X}^\circ, \subseteq)$ is coatomistic follows from the fact that $(\mathcal{Y}^\circ, \subseteq)$ is atomistic. To show the latter fact realize that, for every $(u,v) \in (N \times N) \setminus \Delta$, the set $\Delta \cup \{ (u,v) \}$ is a poset on N. Thus, using (i), the atoms of $(\mathcal{Y}^\circ, \subseteq)$ correspond to singleton subsets of $(N \times N) \setminus \Delta$. This allows one to observe, again using (i), that $(\mathcal{Y}^\circ, \subseteq)$ is atomistic.

Note that $(\mathcal{X}^\circ, \subseteq)$ is graded iff $(\mathcal{Y}^\circ, \subseteq)$ is graded. To verify the latter fact one can use the characterization of $\mathcal{Y}^\circ$ from (i). The anti-isomorphism of $(\mathcal{X}^\circ, \subseteq)$ and $(\mathcal{Y}^\circ, \subseteq)$ implies that the coatoms of $(\mathcal{Y}^\circ, \subseteq)$ are the tosets T_ε, $\varepsilon \in \Upsilon(N)$, where every $T_\varepsilon \setminus \Delta$ has the cardinality $\binom{n}{2}$. As explained in Section 2.2, to reach our goal it is enough to show that any maximal chain in $(\mathcal{Y}^\circ \setminus \{Y\}, \subseteq)$ has the length $\binom{n}{2}$, that is, $\binom{n}{2} + 1$ elements.

This follows by repeated application of the following *sandwiche principle*. Given two posets $P' \subset P$ on N there exists a poset P'' on N with $P' \subseteq P'' \subset P$ and $|P \setminus P''| = 1$. To verify the

principle realize that there exists $(u,v)\in N\times N$ with $u \prec\!\cdot\, v$ $[P]$ and $(u,v)\notin P'$.
Indeed, otherwise the covering relation $\prec\!\cdot$ for P is contained in P' and its transitive closure, which is $P\setminus\Delta$ (see Section 3.3), is also contained, yielding a contradictory conclusion $P\subseteq P'$. Take a pair (u,v) with $u \prec\!\cdot\, v$ $[P]$ and $(u,v)\notin P'$ and put $P'':=P\setminus\{\,(u,v)\,\}$. The transitive closure of P'' is contained in (a transitive relation) P. Nevertheless, (u,v) cannot be in the transitive closure of P'', because this contradicts the assumption that $u \prec\!\cdot\, v$ $[P]$. Thus, $P''\setminus\Delta$ is both transitive and irreflexive, which verifies the sandwich principle.

The claim about coatoms in $\mathcal{X}^\circ$ follows from the description of atoms in $(\mathcal{Y}^\circ,\subseteq)$: these have the form $A_{(u,v)}=\Delta\cup\{\,(u,v)\,\}$ for $u,v\in N$, $u\neq v$, and $A^{\lhd}_{(u,v)}=\{\,(u,v)\,\}^{\lhd}=S_{u\not\prec v}$. □

Note in this context what is the closure operation corresponding to the *poset-based lattice* $(\mathcal{Y}^\circ,\subseteq)$. Given $T\subseteq Y=N\times N$, one can show that

$$T^{\lhd\rhd} \;=\; \begin{cases} \mathrm{tr}(T\cup\Delta) & \text{if } T\setminus\Delta \text{ is acyclic},\\ N\times N & \text{otherwise.}\end{cases}$$

5 Main results

In this section we formulate and prove our results on linear extensions. Section 5.1 brings basic observations on geodesics in the permutohedral graph. The main theorem, which is a graphical characterization of poset-interpretable enumeration sets, is derived in Section 5.2. It is shown in Section 5.3 that basic order-theoretical relations for pairs of variables in a poset P on N can be recognized graphically on basis of the set $\mathcal{L}(P)$ of its linear extensions. Section 5.4 brings an example that the height function of the lattice of geodetically convex sets in the permutohedral graph is not given by the graphical diameter for $n\geq 6$ and discuss a relevant concept of the *dimension of a poset*. Section 5 is concluded by remarks, namely on an "algorithmic" characterization of geodetically convex sets presented in [9] (Section 5.5) and on encoding posets relative to a fixed reference total order of N applied in [3] (Section 5.6).

5.1 Geodesics in the permutohedral graph

Given $\varepsilon,\eta\in\Upsilon(N)$, a two-element subset $\{u,v\}$ of N will be called an *inversion between* $\varepsilon\in\Upsilon(N)$ and $\eta\in\Upsilon(N)$ if the mutual orders of u and v in ε and η differ:

$$\mathrm{Inv}\,[\varepsilon,\eta] \;:=\; \{\ \{u,v\}\subseteq N\ :\ \text{either}\ [\,u\prec_\varepsilon v\ \&\ v\prec_\eta u\,]\ \text{ or }\ [\,v\prec_\varepsilon u\ \&\ u\prec_\eta v\,]\ \}\ .$$

The next lemma brings basic observations on the permutohedral graph.

Lemma 2 Consider $\varepsilon,\eta\in\Upsilon(N)$. Then

(i) $\mathrm{dist}\,(\varepsilon,\eta)=|\,\mathrm{Inv}\,[\varepsilon,\eta]\,|$ is the number of inversions between ε and η. In particular, ε and η are adjacent in the permutohedral graph iff they differ by one inversion only. The label on the edge between ε and η is then the only inversion between them.

(ii) A walk in the permutohedral graph is a geodesic (between ε and η) iff no label on its edges is repeated. If this is the case then the set of labels on the geodesic coincides with the set $\mathrm{Inv}\,[\varepsilon,\eta]$ of inversions between ε and η.

(iii) An enumeration $\sigma \in \Upsilon(N)$ is between ε and η iff $\mathrm{Inv}\,[\varepsilon,\sigma] \cap \mathrm{Inv}\,[\sigma,\eta] = \emptyset$. Another equivalent condition is $T_\varepsilon \cap T_\eta \subseteq T_\sigma$, that is, $[\, u \prec_\varepsilon v \;\&\; u \prec_\eta v \,] \;\Rightarrow\; u \prec_\sigma v$.

Proof. Assume $n \geq 2$ as otherwise all claims are trivial. Let us start with the (particular) observation in (i) that $\varepsilon, \eta \in \Upsilon(N)$ are adjacent iff they differ by one inversion only. The necessity follows immediately from (1). For the sufficiency assume that $\{u,v\} \in \mathrm{Inv}\,[\varepsilon,\eta]$ is unique, meaning that any other pair of elements of N has the same mutual order in ε and in η. Assume specifically that $u \prec_\varepsilon v$ and $v \prec_\eta u$ as otherwise we exchange ε for η. Thus, one has $u = \varepsilon(i)$ and $v = \varepsilon(j)$ with $i < j$. One has $j = i+1$ as otherwise there is $t \in N$ with $u \prec_\varepsilon t \prec_\varepsilon v$ implying a contradictory conclusion $u \prec_\eta t \prec_\eta v \prec_\eta u$. Moreover, for any $t \in N \setminus \{u,v\}$, one has $t \prec_\varepsilon u \Leftrightarrow t \prec_\eta u$, and $v \prec_\varepsilon t \Leftrightarrow v \prec_\eta t$. These facts enforce $v = \eta(i)$ and $u = \eta(i+1)$. The fact that the mutual orders in ε and η coincide among $t \in N$ with $t \prec_\varepsilon u$ and also among $t \in N$ with $v \prec_\varepsilon t$ then implies $\varepsilon(k) = \eta(k)$ for $k \in [n] \setminus \{i, i+1\}$. Thus, (1) holds. Note that the label $\{u,v\}$ on the edge between enumerations ε and η is just the only inversion between them.

The latter observation implies that, for every walk between enumerations $\varepsilon, \eta \in \Upsilon(N)$ and every $\{u,v\} \in \mathrm{Inv}\,[\varepsilon,\eta]$, there is an edge of the walk labeled by $\{u,v\}$, as otherwise u and v would have the same mutual order in ε and in η. Hence, $|\,\mathrm{Inv}\,[\varepsilon,\eta]\,|$ is a lower bound for the length of a walk between ε and η.

Therefore, to verify (i) it is enough to prove, for $\varepsilon, \eta \in \Upsilon(N)$, by induction on $|\,\mathrm{Inv}\,[\varepsilon,\eta]\,|$, that there exists a walk between them which has the length $|\,\mathrm{Inv}\,[\varepsilon,\eta]\,|$. If there is no inversion between ε and η then $\varepsilon(1) \prec_\varepsilon \varepsilon(2) \ldots \prec_\varepsilon \varepsilon(n)$ gives $\varepsilon(1) \prec_\eta \varepsilon(2) \ldots \prec_\eta \varepsilon(n)$ and $\varepsilon = \eta$, that is, a walk of the length $0 = |\,\mathrm{Inv}\,[\varepsilon,\eta]\,|$ exists. If $|\,\mathrm{Inv}\,[\varepsilon,\eta]\,| \geq 1$ then necessarily $\varepsilon \neq \eta$ and $\exists\, 1 \leq i < n$ such that one has $\eta(i+1) \prec_\varepsilon \eta(i)$, for otherwise $\eta(1) \prec_\varepsilon \eta(2) \ldots \prec_\varepsilon \eta(n)$ would give a contradictory conclusion $\eta = \varepsilon$. Define $\sigma \in \Upsilon(N)$ as the enumeration obtained from η by the respective adjacent transposition: $\sigma(i) = \eta(i+1) =: u$, $\sigma(i+1) = \eta(i) =: v$, and $\sigma(k) = \eta(k)$ for $k \in [n] \setminus \{i, i+1\}$. Since $\{u,v\}$ is an inversion between ε and η, one easily observes $\mathrm{Inv}\,[\varepsilon,\sigma] = \mathrm{Inv}\,[\varepsilon,\eta] \setminus \{\,\{u,v\}\,\}$. The induction premise implies the existence of a walk between ε and σ of the length $|\,\mathrm{Inv}\,[\varepsilon,\sigma]\,|$ which can be prolonged by the edge between σ and η. That verifies the induction step.

To prove the necessity in (ii) consider a geodesic $\mathfrak{g}$ between $\varepsilon, \eta \in \Upsilon(N)$ and assume for contradiction that a label $\{u,v\} \subseteq N$ is repeated on its edges. The last claim in (i) implies that the label cannot be repeated on consecutive edges of $\mathfrak{g}$: if an edge between $\pi \in \Upsilon(N)$ and $\zeta \in \Upsilon(N)$ has the same label as the consecutive edge between ζ and $\sigma \in \Upsilon(N)$ then $\pi = \sigma$ and $\mathfrak{g}$ can be shortened, contradicting the assumption that it is a geodesic.

Therefore, one can assume without loss of generality that the first occurrence $\{u,v\}$ on the way from ε to η is between enumerations π and ζ, with π closer to ε, and the next occurrence of $\{u,v\}$ is between enumerations σ and ξ, with ξ closer to η (see Figure 1(a) for illustration).

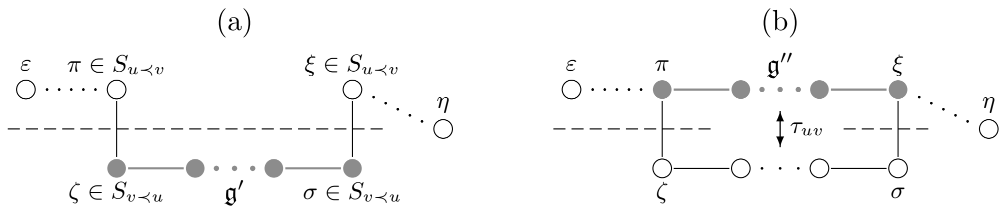

Figure 1: A picture illustrating the proof of Lemma 2.

Assume specifically $u \prec_\varepsilon v$, which enforces $\pi, \xi \in S_{u\prec v}$ and $\zeta, \sigma \in S_{v\prec u}$. Let $\mathfrak{g}'$ be the section of $\mathfrak{g}$ between ζ and σ. We know $\mathfrak{g}' \subseteq S_{v\prec u}$ and, because the map $\varsigma \mapsto \tau_{uv} \cdot \varsigma$ is an automorphism of the permutohedral graph, the section $\mathfrak{g}'$ has a copy $\mathfrak{g}''$ in $S_{u\prec v}$, which is between π and ξ (see Figure 1(b) for illustration). Therefore, the section of $\mathfrak{g}$ between π and ξ can be replaced by $\mathfrak{g}''$, yielding a walk of a shorter length and contradicting the assumption that $\mathfrak{g}$ is a geodesic.

To prove the sufficiency in (ii) assume that $\mathfrak{g}$ is a walk between ε and η in which no label is repeated. We have already observed that any inversion $\{u, v\} \in \mathrm{Inv}\,[\varepsilon, \eta]$ must occur as a label on $\mathfrak{g}$. Nevertheless, no two-element set $\{u, v\} \subseteq N$ which is *not an inversion* between ε and η can be a label on $\mathfrak{g}$: indeed, otherwise, since only one edge of $\mathfrak{g}$ is labeled by $\{u, v\}$ then, the mutual orders of u and v in ε and η must differ, which contradicts the assumption that $\{u, v\}$ is not an inversion between ε and η. Thus, we have verified both that the length of $\mathfrak{g}$ is $|\,\mathrm{Inv}\,[\varepsilon, \eta]\,| = \mathrm{dist}\,(\varepsilon, \eta)$ and the claim that $\mathrm{Inv}\,[\varepsilon, \eta]$ is the set of labels on $\mathfrak{g}$.

The first claim in (iii) then follows from (ii). If σ belongs to a geodesic between ε and η then the labels on it between ε and σ must differ from the labels between σ and η. Conversely, if $\mathrm{Inv}\,[\varepsilon, \sigma]$ and $\mathrm{Inv}\,[\sigma, \eta]$ are disjoint then the concatenation of a geodesic between ε and σ with a geodesic between σ and η yields a geodesic between ε and η.

Assuming $\mathrm{Inv}\,[\varepsilon, \sigma] \cap \mathrm{Inv}\,[\sigma, \eta] = \emptyset$, the conditions $u \preceq_\varepsilon v$ and $u \preceq_\eta v$ together imply $u \preceq_\sigma v$ as otherwise $\{u, v\}$ belongs to the intersection of the sets of inversions. Conversely, if $T_\varepsilon \cap T_\eta \subseteq T_\sigma$ then any hypothetic simultaneous inversion $\{u, v\}$ in $\mathrm{Inv}\,[\varepsilon, \sigma] \cap \mathrm{Inv}\,[\sigma, \eta]$ must be compared equally in ε and η, and, therefore, in σ, which is a contradiction. $\square$

5.2 Characterization of poset-interpretable sets of enumerations

Two special concepts for a non-empty set $\emptyset \neq S \subseteq \Upsilon(N)$ of nodes in the permutohedral graph are needed. The set of *inversions within* S is

$$\mathrm{Inv}\,(S) \;:=\; \left\{\; \{u, v\} \subseteq N \;:\; \text{there exist } \varepsilon, \eta \in S \text{ such that } \varepsilon \overset{uv}{\longleftrightarrow} \eta \;\right\},$$

while the *covering relation* for S is the following binary relation on N:

$$\mathrm{Cov}\,(S) \;:=\; \left\{\; (u, v) \in N \times N \;:\; \text{there are } \varepsilon \in S,\, \eta \in \Upsilon(N) \setminus S \text{ with } \varepsilon \overset{uv}{\longleftrightarrow} \eta \text{ and } u \prec_\varepsilon v \;\right\}.$$

Note that the latter terminology is motivated by the fact that $\mathrm{Cov}\,(S)$ appears to coincide with the covering relation $\prec\!\cdot$ for the poset on N assigned to S (see Section 5.3). These two notions are substantial in the proof of the following result.

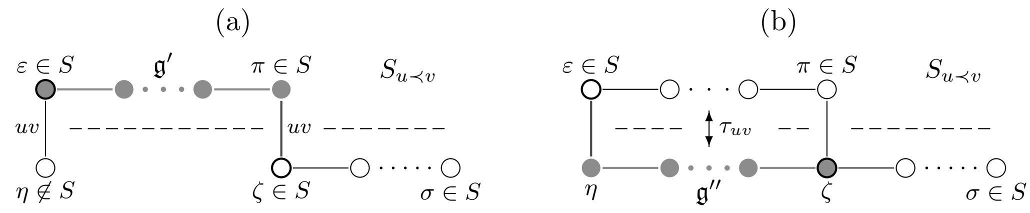

Figure 2: A picture illustrating the proof of Theorem 3.

Theorem 3 If $n = |N| \geq 2$ then, given $S \subseteq \Upsilon(N)$, $S \in \mathcal{X}^\circ$ iff S is geodetically convex.

Proof. **I.** To verify the necessity use Lemma 1(iv) saying that $(\mathcal{X}^\circ, \subseteq)$ is a coatomistic lattice. Since the intersection of geodetically convex sets is geodetically convex and so is $S = \Upsilon(N)$ it is enough to verify that the coatoms in $(\mathcal{X}^\circ, \subseteq)$ are geodetically convex. These are sets of the form $S_{u \prec v}$ for pairs $(u, v) \in N \times N$, $u \neq v$. We apply Lemma 2(ii): given a geodesic between $\varepsilon, \eta \in S_{u \prec v}$, the fact $\{u, v\} \notin \mathrm{Inv}\,[\varepsilon, \eta]$ implies that $\{u, v\}$ is not a label on any edge of the geodesic, and, thus, every node on the geodesic belongs to $S_{u \prec v}$.

II. The first step to verify the sufficiency is to observe that if S is geodetically convex then, for every $(u, v) \in \mathrm{Cov}\,(S)$ and $\sigma \in S$, one has $u \prec_\sigma v$.

Suppose specifically that an edge between enumerations $\varepsilon \in S$ and $\eta \in \Upsilon(N) \setminus S$ exists with $\varepsilon \xleftrightarrow{uv} \eta$ and $u \prec_\varepsilon v$, which implies $v \prec_\eta u$; see Figure 2(a) for illustration. Consider a geodesic $\mathfrak{g}$ between ε and σ. It belongs to S, by the assumption that S is geodetically convex. Assume for contradiction that $v \prec_\sigma u$ giving $\{u, v\} \in \mathrm{Inv}\,[\varepsilon, \sigma]$ and, by Lemma 2(ii), a unique edge of $\mathfrak{g}$ is labeled by $\{u, v\}$. Let it be the one between $\pi \in S$ and $\zeta \in S$, with π closer to ε. Therefore, the section $\mathfrak{g}'$ of $\mathfrak{g}$ between ε and π is in $S_{u \prec v}$. Because the mapping $\varsigma \mapsto \tau_{uv} \cdot \varsigma$ is an automorphism of the permutohedral graph, the section $\mathfrak{g}'$ has a copy $\mathfrak{g}''$ in $S_{v \prec u}$, which is between η and ζ; see Figure 2(b) for illustration. The walk composed of the edge $[\varepsilon, \eta]$ and $\mathfrak{g}''$ has the same length as the section of $\mathfrak{g}$ between ε and ζ then. Therefore, it is a geodesic between ε and ζ and the assumption that S is geodetically convex implies a contradictory conclusion that $\eta \in S$. Thus, one necessarily has $u \prec_\sigma v$.

III. The next step it to show that, if $\emptyset \neq S \subseteq \Upsilon(N)$ is a non-empty geodetically convex set and $\zeta \in \Upsilon(N) \setminus S$ then there is $(u, v) \in \mathrm{Cov}\,(S)$ with $v \prec_\zeta u$.

Consider a walk $\mathfrak{g}$ from ζ to some $\varepsilon \in S$ of the least possible length among walks from ζ to S. By definition, $\mathfrak{g}$ is a geodesic between ζ and ε. Let η the last node of $\mathfrak{g}$ before ε. Then necessarily $\eta \in \Upsilon(N) \setminus S$; let $\{u, v\}$ be the label of the edge between η and ε. Without loss of generality assume $u \prec_\varepsilon v$ (otherwise we exchange u for v), which gives $(u, v) \in \mathrm{Cov}\,(S)$. By Lemma 2(ii) applied to $\mathfrak{g}$, $\{u, v\}$ is an inversion between ζ and ε, which gives $v \prec_\zeta u$.

IV. As $|N| \geq 2$ is assumed, one has $\emptyset \in \mathcal{X}^\circ$ and to verify the sufficiency it is enough to show $S \in \mathcal{X}^\circ$ for a non-empty geodetically convex set $\emptyset \neq S \subseteq \Upsilon(N)$. By Step II., $S \subseteq \mathrm{Cov}\,(S)^\lhd$. By Step III., $\Upsilon(N) \setminus S \subseteq \Upsilon(N) \setminus (\mathrm{Cov}\,(S)^\lhd)$ gives $\mathrm{Cov}\,(S)^\lhd \subseteq S$. Thus, we have shown $S = \mathrm{Cov}\,(S)^\lhd$, which means $S \in \mathcal{X}^\circ$ (see Section 2.3). $\square$

5.3 Graphical recognition of incomparable and comparable pairs

Here we give a direct proof of observations that were implicitly behind of [9, § 4].

Theorem 4 Let P be a poset over N and $S=\mathcal{L}(P)$. Then, for any pair $u,v\in N$, one has

(i) $u\|v\,[P]$ iff $\{u,v\}\in \mathrm{Inv}\,(S)$,

(ii) $u \prec\cdot\; v\ [P]$ iff $(u,v)\in \mathrm{Cov}\,(S)$,

(iii) $(u,v)\in P\setminus\Delta$ iff $(u,v)\in \mathrm{tr}(\mathrm{Cov}\,(S))$,

(iv) $(u,v)\in P$ iff $(u,v)\in \mathrm{tr}(\Delta\cup \mathrm{Cov}\,(S))$.

Proof. We first verify a series of preparatory observations for a given pair $u,v\in N$ of variables.
I. If either $u\|v\,[P]$ or $u\prec\cdot\; v\ [P]$ then $\exists\,\varepsilon\in\mathcal{L}(P)\ :\ u\prec_{\varepsilon}\cdot\; v$.
We put $A:=\{\,a\in N\setminus v\ :\ (a,u)\in P \text{ or } (a,v)\in P\,\}$, $B:=N\setminus A$ and derive the next 3 facts.

- The variable v is initial in B, that is, $v\in B$ and $\neg\,[\,\exists\, b\in B\setminus v\ :\ (b,v)\in P\,]$.

Indeed, otherwise, by the definition of A, one has $b\in A$, contradicting the definition of B.

- The variable u is terminal in A, that is, $u\in A$ and $\neg\,[\,\exists\, a\in A\setminus u\ :\ (u,a)\in P\,]$.

Assume for contradiction that such $a\neq u$ exists. Then $a\in A$ means $a\neq v$ and either $(a,u)\in P$ or $(a,v)\in P$. As P is a poset, the first option $(a,u),(u,a)\in P$ contradicts $a\neq u$. The second option $(u,a),(a,v)\in P$ contradicts both $u\|v\,[P]$ (by transitivity of P) and $u\prec\cdot\; v\ [P]$.

- The set A precedes the set B in P, that is, $\neg\,[\,\exists\, b\in B\ \&\ a\in A\ :\ (b,a)\in P\,]$.

Assume for contradiction that such b and a exist and distinguish the cases $b\neq v$ and $b=v$. If $b\neq v$ then, by transitivity of P, $b\in A$, contradicting the definition of B. If $b=v$ then the option $(v,a),(a,u)\in P$ implies, by transitivity of P, $(v,u)\in P$ and contradicts both $u\|v\,[P]$ and $u\prec\cdot\; v\ [P]$. The other option $(v,a),(a,v)\in P$ contradicts $a\neq v$, as P is a poset.
Consider an ordered partition $|A_1|A_2|A_3|A_4|\ :=\ |A\setminus u\,|\,u\,|\,v\,|\,B\setminus v|$ of N. The above derived facts altogether say that P is consonant with it: $(c,d)\in P$, $c\in A_i$, $d\in A_j\ \Rightarrow\ i\leq j$. As any relation $P_i:=P\cap(A_i\times A_i)$ is a poset on A_i, for every $i=1,2,3,4$, there exists $\varepsilon(i)\in\mathcal{L}(P_i)$. One can compose an enumeration $\varepsilon\in\Upsilon(N)$ of these: $\varepsilon:=|\varepsilon(1)|\varepsilon(2)|\varepsilon(3)|\varepsilon(4)|$. It follows from the construction that $\varepsilon\in\mathcal{L}(P)$ and also $u\prec_{\varepsilon}\cdot\; v$.
II. If $u\|v\,[P]\ \&\ \varepsilon\in\mathcal{L}(P)\ :\ u\prec_{\varepsilon}\cdot\; v$ then $\eta\in\mathcal{L}(P)$ for $\eta\in\Upsilon(N)$ specified by $\varepsilon\xleftrightarrow{uv}\eta$.
Having $(c,d)\in P$, the assumption $u\|v\,[P]$ implies $(u,v)\neq(c,d)\neq(v,u)$ and $\varepsilon\in\mathcal{L}(P)$ implies $c\preceq_{\varepsilon} d$. Now, $\varepsilon\xleftrightarrow{uv}\eta$ says that T_{ε} and T_{η} only differ in the mutual position of u and v, implying thus $c\preceq_{\eta} d$. Hence, we have verified $\eta\in\mathcal{L}(P)$.
III. If $\exists\,\varepsilon,\eta\in\mathcal{L}(P)\ :\ \varepsilon\xleftrightarrow{uv}\eta$ then $u\|v\,[P]$.
Assume specifically $u\prec_{\varepsilon} v$ and $v\prec_{\eta} u$, which necessitates $\neg[v\preceq_{\varepsilon} u]$ and $\neg[u\preceq_{\eta} v]$. The

assumption $(v,u)\in P$ thus forces $\varepsilon\notin\mathcal{L}(P)$, yielding a contradiction. The assumption $(u,v)\in P$ enforces $\eta\notin\mathcal{L}(P)$, giving a contradiction, too. So, we have verified $u\|v\,[P]$.

IV. If $u \prec\!\cdot\; v\;[P]$ & $\varepsilon\in\mathcal{L}(P)\;:\; u\prec_{\varepsilon}\!\cdot\; v$ then $\eta\notin\mathcal{L}(P)$ for $\eta\in\Upsilon(N)$ specified by $\varepsilon\xleftrightarrow{uv}\eta$.
Assume $\eta\in\mathcal{L}(P)$ for contradiction. Then there are $\varepsilon,\eta\in\mathcal{L}(P)$ with $\varepsilon\xleftrightarrow{uv}\eta$ and, by Step III., $u\|v\,[P]$, yielding a contradiction with the assumption $u\prec\!\cdot\; v\;[P]$.

V. If $\exists\,\varepsilon\in\mathcal{L}(P)$ & $\eta\in\Upsilon(N)\setminus\mathcal{L}(P)\;:\;\varepsilon\xleftrightarrow{uv}\eta$ & $u\prec_{\varepsilon} v$ then $u\prec\!\cdot\; v\;[P]$.
Realize that $\varepsilon\xleftrightarrow{uv}\eta$ and $u\prec_{\varepsilon} v$ imply $u\prec_{\varepsilon}\!\cdot\; v$. If $u\|v\,[P]$ then Step II. implies a contradictory conclusion $\eta\in\mathcal{L}(P)$. If $(v,u)\in P$ then $\varepsilon\in\mathcal{L}(P)$ gives $v\preceq_{\varepsilon} u$ contradicting $u\prec_{\varepsilon}\!\cdot\; v$. Thus, necessarily $(u,v)\in P$. Assume for contradiction that $w\in N\setminus uv$ exists with $(u,w),(w,v)\in P$. Then $\varepsilon\in\mathcal{L}(P)$ implies $u\prec_{\varepsilon} w\prec_{\varepsilon} v$ contradicting $u\prec_{\varepsilon}\!\cdot\; v$. Therefore, $u\prec\!\cdot\; v\;[P]$.

Now, the claim (i) is that $u\|v\,[P]$ iff $\exists\,\varepsilon,\eta\in\mathcal{L}(P)\;:\;\varepsilon\xleftrightarrow{uv}\eta$. The necessity here follows from Steps I. and II., the sufficiency from Step III.

The claim (ii) means that $u\prec\!\cdot\; v\;[P]$ iff $\exists\,\varepsilon\in\mathcal{L}(P)$ & $\eta\in\Upsilon(N)\setminus\mathcal{L}(P)\;:\;\varepsilon\xleftrightarrow{uv}\eta$ & $u\prec_{\varepsilon} v$. The necessity follows from Steps I. and IV., the sufficiency from Step V. Further auxiliary observation is needed to derive the rest.

VI. The relation $\mathrm{Cov}\,(\mathcal{L}(P))$ on N is acyclic.
By Theorem 3, $\mathcal{L}(P)$ is geodetically convex. We choose $\sigma\in\mathcal{L}(P)$ and the goal is to verify that $\mathrm{Cov}\,(\mathcal{L}(P))\subseteq T_{\sigma}$. Assume for contradiction that $\exists\,(u,v)\in\mathrm{Cov}\,(\mathcal{L}(P))\;:\;v\prec_{\sigma} u$. Thus, by definition of $\mathrm{Cov}\,(S)$, $\exists\,\varepsilon\in\mathcal{L}(P)$ & $\eta\in\Upsilon(N)\setminus\mathcal{L}(P)\;:\;\varepsilon\xleftrightarrow{uv}\eta$ & $u\prec_{\varepsilon} v$ and Step V. gives $u\prec\!\cdot\; v\;[P]$. There exists a geodesic $\mathfrak{g}$ between ε and σ. As $\mathcal{L}(P)$ is geodetically convex, $\mathfrak{g}$ is in $\mathcal{L}(P)$. Because $u\prec_{\varepsilon} v$ and $v\prec_{\sigma} u$, one has $\{u,v\}\in\mathrm{Inv}\,[\varepsilon,\sigma]$ and, by Lemma 2(ii), there is an edge $[\pi,\zeta]$ of $\mathfrak{g}$ with $\pi\xleftrightarrow{uv}\zeta$. The fact $\pi,\zeta\in\mathcal{L}(P)$ implies, by Step III., that $u\|v\,[P]$, contradicting a previous conclusion $u\prec\!\cdot\; v\;[P]$. Clearly, T_{σ} is acyclic.

For the necessity in (iii) consider $(u,v)\in P\setminus\Delta$. Because N is finite, one can show by repeated "including" elements $w\in N$ between u and v, that $\exists\,u=u_1,\ldots,u_k=v$, $k\geq 2$ such that $u_i\prec\!\cdot\; u_{i+1}\;[P]$ for $i=1,\ldots,k-1$. Then the application of (ii) gives $(u_i,u_{i+1})\in\mathrm{Cov}\,(S)$ for $i=1,\ldots,k-1$, implying $(u,v)\in\mathrm{tr}(\mathrm{Cov}\,(S))$.

For the sufficiency in (iii) consider $(u,v)\in\mathrm{tr}(\mathrm{Cov}\,(S))$ and $u=u_1,\ldots,u_k=v$, $k\geq 2$ such that $(u_i,u_{i+1})\in\mathrm{Cov}\,(S)$ for $i=1,\ldots,k-1$. By (ii), one has $u_i\prec\!\cdot\; u_{i+1}\;[P]$ for these i, and transitivity of P yields $(u,v)\in P$. Assume for contradiction $u=v$, in which case one has $\exists\,u=u_1,\ldots,u_k=u$, $k\geq 2$ with $(u_i,u_{i+1})\in\mathrm{Cov}\,(S)$ for $i=1,\ldots,k-1$, meaning that $\mathrm{Cov}\,(\mathcal{L}(P))$ contains a cycle (see Section 3.2), yielding a contradiction, by Step VI.

The claim (iv) follows directly from (iii): as $P=\Delta\cup(P\setminus\Delta)$, one can apply evident equality $\Delta\cup\mathrm{tr}(R)=\mathrm{tr}(\Delta\cup R)$ valid for any $R\subseteq N\times N$. $\square$

Corollary 5 Given a poset P on N and $S=\mathcal{L}(P)$, any $(u,v)\in N\times N\setminus\Delta$ satisfies *exclusively* one of the following three conditions:

(a) $\{u,v\}\in\mathrm{Inv}\,(S)$,

(b) $(u,v) \in \mathrm{tr}(\mathrm{Cov}\,(S))$,

(c) $(v,u) \in \mathrm{tr}(\mathrm{Cov}\,(S))$.

Proof. By claims (i) and (iii) in Theorem 4, the conditions are equivalent to $u \| v\, [P]$, $(u,v) \in P$, and $(v,u) \in P$, which are clearly exclusive each other and one of them must occur. $\square$

5.4 Height function, diameter, and the dimension of a poset

This is another consequence of previous observations.

Corollary 6 If $n \geq 2$ then the function $S \in \mathcal{X}^\circ \setminus \{\emptyset\} \mapsto |\,\mathrm{Inv}\,(S)\,|$, extended by a convention $|\,\mathrm{Inv}\,(\emptyset)\,| := -1$, is a *height function* for the lattice of geodetically convex subsets of $\Upsilon(N)$.

Proof. By Theorem 3, the discussed lattice coincides with the lattice $(\mathcal{X}^\circ, \subseteq)$ introduced in Section 4. It was shown there that it is anti-isomorphic to the poset-based lattice $(\mathcal{Y}^\circ, \subseteq)$. Moreover, the lattice $(\mathcal{Y}^\circ, \subseteq)$ was shown in the proof of Lemma 1(iv) to be graded, with the height function assigning $|P \setminus \Delta|$ to a poset P on N and $\binom{n}{2}+1$ to the full relation $N \times N$. In particular, $(\mathcal{X}^\circ, \subseteq)$ is graded, with the height function assigning the value $\binom{n}{2} - |P \setminus \Delta|$ to every $S = \mathcal{L}(P)$, which is the number of incomparable pairs in P. This value is, by Theorem 4(i), the cardinality of the set $\mathrm{Inv}\,(S)$ of inversions within S. $\square$

Recall from Section 3.7 that the combinatorial equivalence of edges in the permutohedral graph can be viewed as its particular edge-labeling, with deeper geometric meaning. Therefore, Corollary 6 has the following interpretation: the value of the height function for $(\mathcal{X}^\circ, \subseteq)$ at non-empty $S \in \mathcal{X}^\circ$ is nothing but the number of different labels/colors of edges within S.

The reader might have some doubts about this claim, because the elements of $\mathcal{X}^\circ$ were characterized in Theorem 3 purely in graphical terms (= without using labels), while its height function is now described in terms of particular edge labeling/coloring of the graph. Of course, one naturally expects that the height function of the lattice $(\mathcal{X}^\circ, \subseteq)$ should be also definable in purely graphical terms. The explanation is that the discussed *combinatorial equivalence of edges* can be recognized/reconstructed solely on basis of the permutohedral graph itself (as a whole). Specifically, given an edge between $\varepsilon \in \Upsilon(N)$ and $\eta \in \Upsilon(N)$, consider the set

$$S_{\varepsilon:\eta} := \{\, \sigma \in \Upsilon(N) \;:\; \mathrm{dist}\,(\sigma, \varepsilon) < \mathrm{dist}\,(\sigma, \eta)\,\}$$

of enumerations that are closer to ε than to η. If the edge is labeled by $\{u,v\} \subseteq N$ and $u \prec_\varepsilon v$ then one can observe, using Lemma 2(ii), that $S_{u \prec v} \subseteq S_{\varepsilon:\eta}$. Analogously, $S_{v \prec u} \subseteq S_{\eta:\varepsilon}$. Since $S_{v \prec u}$ is the complement of $S_{u \prec v}$ in $\Upsilon(N)$, and similarly with $S_{\eta:\varepsilon}$ and $S_{\varepsilon:\eta}$, one has $S_{u \prec v} = S_{\varepsilon:\eta}$ and $S_{v \prec u} = S_{\eta:\varepsilon}$, meaning that $\{S_{u \prec v}, S_{v \prec u}\} = \{S_{\varepsilon:\eta}, S_{\eta:\varepsilon}\}$ is a partition of $\Upsilon(N)$, definable in purely graphical terms. In particular, two edges have the same combinatorial label $\{u,v\} \subseteq N$ iff they yield the same graphical bi-partition of $\Upsilon(N)$.

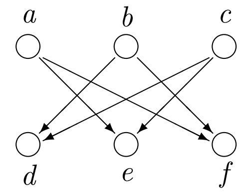

Figure 3: Directed Hasse diagram of the poset from Example 1.

The reader may tend to think that the height function for the lattice $(\mathcal{X}^\circ, \subseteq)$ coincides, for $\emptyset \neq S \in \mathcal{X}^\circ$, with the *diameter function*, which assigns the number

$$\mathrm{diam}\,(S) := \max\{\, \mathrm{dist}\,(\varepsilon,\eta) \;:\; \varepsilon,\eta \in S\,\} \qquad \text{to any } \emptyset \neq S \subseteq \Upsilon(N).$$

This is indeed the case if $|N| \leq 5$, but in general one only has $\mathrm{diam}\,(S) \leq |\,\mathrm{Inv}\,(S)\,|$ [11, Thm 4.2].

This phenomenon closely relates to the concept of *dimension* of a poset P [20, § 1.1], which can equivalently be defined by the formula $\dim(P) := \min\{\, |S| \;:\; P = S^{\triangleright}\,\}$. Note in this context that one can show that, for every pair of distinct enumerations $\varepsilon, \eta \in \Upsilon(N)$, the set of enumerations between ε and η belongs to $\mathcal{X}^\circ$ and corresponds to a poset of dimension 2, defined as the intersection of T_ε and T_η. A classic result [10] in the theory of poset dimension says that $\dim(P) \leq \lfloor \frac{n}{2} \rfloor$ for any poset P on an n-element set N. This upper bound is tight owing to a classic construction [6] of a poset of given dimension. Thus, the simplest example of a poset of dimension 3 exists in case $|N| = 6$. It also gives the simplest example of a poset-interpretable set $S \subseteq \Upsilon(N)$ of enumerations for which $\mathrm{diam}\,(S) < |\,\mathrm{Inv}\,(S)\,|$.

Example 1 Put $N := \{a,b,c,d,e,f\}$ and $P := \Delta \cup \{\,(a,e),(a,f),(b,d),(b,f),(c,d),(c,e)\,\}$; see Figure 3 for the directed Hasse diagram of this poset. One can show that the set $S := P^{\triangleleft}$ is the (disjoint) union of (face-associated) subsets of $\Upsilon(N)$:

$$|\,abc\,|\,def\,|, \quad |\,ab\,|\,f\,|\,c\,|\,de\,|, \quad |\,ac\,|\,e\,|\,b\,|\,d\,f\,|, \quad |\,bc\,|\,d\,|\,a\,|\,ef\,|\,.$$

Hence, the inversions within S are just incomparable pairs in P, that is, $|\,\mathrm{Inv}\,(S)\,| = 9$. On the other hand, for $\varepsilon \in S$, if $\varepsilon \notin |\,ab\,|\,f\,|\,c\,|\,de\,|$ then $c \prec_\varepsilon f$. This observation allows one to conclude, using Lemma 2(i), that any geodesic between elements of $S \setminus |\,ab\,|\,f\,|\,c\,|\,de\,|$ has the length at most 8. The implication $\varepsilon \in S \setminus |\,ac\,|\,e\,|\,b\,|\,d\,f\,| \;\Rightarrow\; b \prec_\varepsilon e$ and the implication $\varepsilon \in S \setminus |\,bc\,|\,d\,|\,a\,|\,ef\,| \;\Rightarrow\; a \prec_\varepsilon d$ lead to analogous conclusions. This, combined with the choice $\varepsilon := |a|c|e|b|f|d|$ and $\eta := |b|c|d|a|f|e|$, for which $\mathrm{dist}\,(\varepsilon,\eta) = |\,\mathrm{Inv}\,[\varepsilon,\eta]\,| = 8$ allows one to observe that $\mathrm{diam}\,(S) = 8$.

5.5 Remarks on an alternative characterization

Our proof of Theorem 3 has been inspired by the tools used in the proof of [9, Theorem 9], where a different characterization of poset-interpretable subsets of $\Upsilon(N)$ was presented. It can be re-phrased as follows: a (non-empty) set $S \subseteq \Upsilon(N)$ corresponds to a poset on N iff, for every $(u,v) \in N \times N$ with $u \neq v$, exclusively one of the following three conditions holds:

(a) $\{u, v\} \in \mathrm{Inv}\,(S)$,

(b) (u, v) belongs to the transitive closure of $\mathrm{Cov}\,(S)$,

(c) (v, u) belongs to the transitive closure of $\mathrm{Cov}\,(S)$.

Observe that Corollary 5 claims that the above condition is necessary for $S = \mathcal{L}(P)$, where P is a poset on N. The condition of this type is perhaps more suitable from the algorithmic point of view than the condition of geodetic convexity of S, when one wants to construct a poset P with $\varepsilon \in \mathcal{L}(P) \subseteq S$ for given $\varepsilon \in S \subseteq \Upsilon(N)$, which was the task treated in [9, § 5].

The offered proof of sufficiency of this condition from [9], however, contained three gaps. For example, the following statement was made and used (when Case 1. of the proof of [9, Thm 9] was discussed): if $\varepsilon \notin \mathcal{L}(P)$ for a poset P then there exists a pair $(b, a) \in P$ such that $a \prec_{\varepsilon} \cdot\, b$. This is indeed not the case: take, for instance, $N := \{\, a, b, c \,\}$, $P := \Delta \cup \{\, (b, a) \,\}$, and $\varepsilon := |a|c|b|$.

It was not clear (to the first author of the present paper) for a while whether [9, Theorem 9] is true as re-phrased above or whether there exists a counter-example of a disconnected set $S \subseteq \Upsilon(N)$ satisfying exclusively (a)-(c) for some large $|N|$. Now, it looks like the statement is indeed valid as formulated above but its proof is beyond the scope of the present paper.

The reason is that proving the sufficiency requires a (correct proof of a particular) deeper graph-theoretical result on the permutohedral graph. The needed result is, informally said, as follows: any walk $\mathfrak{w}$ in the permutohedral graph between enumerations ε and η can be transformed to any given geodesic $\mathfrak{g}$ between ε and η by a series of "admissible" operations, which are the removals of sections of $\mathfrak{w}$ between different occurrencies of the same node and swaps of alternative sections of $\mathfrak{w}$ of between opposite nodes on the so-called squares and hexagons.[1] Note that an algebraic analogue of such a graph-theoretical result has appeared as [12, Thm 10], accompanied with arguments that are transparent only to deep specialists on Coxeter groups. There is a plan to present a direct elementary proof of this result in the next version of [18].

Here is a hint to prove the sufficiency with the help of that result. Consider an arbitrary connectivity component C of the induced subgraph G_S, where G is the permutohedral graph. The assumption on S in combination with the discussed graphical result allow one to show that C is geodetically convex. By Theorem 3, a poset P on N exists with $C = \mathcal{L}(P)$, and Corollary 5 implies that the conditions (a)-(c) hold exclusively also for C. Finer considerations allow one to show that $\mathrm{Inv}\,(C) = \mathrm{Inv}\,(S)$ and $\mathrm{Cov}\,(C) = \mathrm{Cov}\,(S)$. The application of Theorem 4(iv) then gives $P = \mathrm{tr}(\Delta \cup \mathrm{Cov}\,(C)) = \mathrm{tr}(\Delta \cup \mathrm{Cov}\,(S))$. As the correspondence of $P \longleftrightarrow \mathcal{L}(P)$ is one-to-one, different components C of G_S have to coincide, implying that S is connected.

5.6 On encoding posets when a reference total order is given

In the present paper, we try to propagate the opinion that when dealing with posets on a finite set N, one should not assume a priori that the set N itself has a preferred total order. We believe

[1] Squares and hexagons are particular induced subgraphs of the permutohedral graph which correspond to two-dimensional faces of the permutohedron $\Pi(N)$ – see Section 3.5.

that our approach simplifies things and leads to much more transparent arguments, avoiding unnecessary technicalities. Nonetheless, habitual attitude of some authors dealing with this topic is to insist on having N of the form $[n]$, where $n \in \mathbb{N}$, that is, on having some predetermined preferred total order on N. This approach, formalized by the concept of a *labeled poset* from [3], perhaps also has some rationale, because a fixed total order on N might play the role of a kind of a coordinate system, which leads to estimates for the number of posets on a given set N.

In this section we interpret (some) results on labeled posets from [3, § 6] in our context. It is useful to realize that, when a reference enumeration $\pi \in \Upsilon(N)$ is given, which means a toset T_π on N is fixed, then posets on N can be identified with some intervals in the poset $(T_\pi \setminus \Delta, \subseteq)$.

Lemma 7 Let $\pi \in \Upsilon(N)$ be a fixed reference enumeration of N, where $|N| \geq 2$.

(i) Given a poset P on N, one has $\mathcal{L}(P) = \mathcal{L}^\pi[A, B]$, where

$$\mathcal{L}^\pi[A, B] := \{ \sigma \in \Upsilon(N) : A \subseteq T_\pi \setminus T_\sigma \subseteq B \}$$

with $A := (T_\pi \cap P^{op}) \setminus \Delta$ and $B := T_\pi \setminus P$, clearly satisfying $A \subseteq B \subseteq T_\pi \setminus \Delta$.

(ii) A set $S \subseteq \Upsilon(N)$ of enumerations of N has the form $\mathcal{L}^\pi[A, B]$ for a pair $[A, B]$ of subsets of $T_\pi \setminus \Delta$ iff it is geodetically convex in the permutohedral graph.

(iii) Given a non-empty geodetically convex set $S \subseteq \Upsilon(N)$, the pair $[A, B]$ with $S = \mathcal{L}^\pi[A, B]$ and minimal difference $B \setminus A$ is unique. It has the forms $[\, \bigcap_{\sigma \in S} T_\pi \setminus T_\sigma, \bigcup_{\sigma \in S} T_\pi \setminus T_\sigma \,]$ and $[\, T_\pi \cap P^{op}, T_\pi \setminus P \,]$, where P is the unique poset on N with $S = \mathcal{L}(P)$.

The claim (i) above means that any poset P on N can be represented by an interval $[A, B]$ in $(T_\pi \setminus \Delta, \subseteq)$. By (iii), a minimal such interval is unique and can perhaps be interpreted as *encoding* of P, that is, yet another its cryptic description. The claim (ii) is our reformulation of [3, Proposition 6.2], attributed by Björner and Wachs to Tits, this time derived as a consequence of Theorem 3. The claim (iii) is a kind of our re-interpretation of [3, Proposition 6.3].

Proof. Throughout the proof we denote $T := T_\pi \setminus \Delta$.

As for (i), realize that $\mathcal{L}(P) = \{ \sigma \in \Upsilon(N) : P \subseteq T_\sigma \}$. Clearly, $P \subseteq T_\sigma$ is equivalent to the inclusion $(N \times N) \setminus T_\sigma \subseteq (N \times N) \setminus P$. Because $N \times N$ disjunctively decomposes as $T \cup \Delta \cup T^{op}$, this inclusion breaks into two conditions, namely into $T \setminus T_\sigma \subseteq T \setminus P = B$ and $T^{op} \setminus T_\sigma \subseteq T^{op} \setminus P$. Because of one-to-one correspondence $R \longleftrightarrow R^{op}$, the latter condition has an equivalent writing $T \setminus T_\sigma^{op} \subseteq T \setminus P^{op}$. However, as $T \setminus T_\sigma^{op} = T \cap T_\sigma$, it means $T \cap T_\sigma \cap P^{op} = \emptyset$, that is, $A = T \cap P^{op} \subseteq T \setminus T_\sigma$. As P is a poset, $P \cap P^{op} = \Delta$ yields $A \subseteq B$.

For the necessity in (ii) assume $\varepsilon, \eta \in \mathcal{L}^\pi[A, B]$ and $\sigma \in \Upsilon(N)$ between them. In case $(a, b) \in A \subseteq T$ one has $(b, a) \in T_\varepsilon \cap T_\eta$ and, by Lemma 2(iii), $(b, a) \in T_\sigma$, implying $(a, b) \in T \setminus T_\sigma$. The inclusion $T \setminus T_\sigma \subseteq B$ can be shown in the form $T \setminus B \subseteq T_\sigma$: if $(a, b) \in T \setminus B$ then $(a, b) \in T_\varepsilon \cap T_\eta$, and, by Lemma 2(iii), $(a, b) \in T_\sigma$. Therefore, $\sigma \in \mathcal{L}^\pi[A, B]$, which was the goal.

For the sufficiency in (ii) assume $S \neq \emptyset$, as otherwise $S = \emptyset = \mathcal{L}^\pi[T, \emptyset]$. Apply Theorem 3 and Lemma 1(ii) to S to get the unique poset P on N with $S = \mathcal{L}(P)$ and then use (i).

As concerns (iii), let us put $A(S) := \bigcap_{\sigma\in S} T\setminus T_\sigma$ and $B(S) := \bigcup_{\sigma\in S} T\setminus T_\sigma$. We known already from the above arguments concerning (ii) that $S = \mathcal{L}(P)$ for a poset on N and $S = \mathcal{L}^\pi[A,B]$ for some $[A,B]$. The inclusions $A \subseteq A(S) \subseteq T \setminus T_\sigma \subseteq B(S) \subseteq B$ valid for every $\sigma \in S$ allow one to show $S = \mathcal{L}^\pi[A(S),B(S)]$, which yields the first formula. To get the second formula one needs to show both $A(S) \subseteq T \cap P^{op}$ and $(T \setminus P) \subseteq B(S)$ and combine these inclusions with (i).

In case of $(u,v) \in A(S) = \bigcap_{\sigma\in S} T \setminus T_\sigma$ one cannot have $(u,v) \in P$ because it leads to a contradictory conclusion $(u,v) \in T_\sigma$ for arbitrary $\sigma \in \mathcal{L}(P) = S$. Additionally, $u\|v\,[P]$ implies, by Theorem 4(i), that there are $\varepsilon,\eta \in S = \mathcal{L}(P)$ with $\varepsilon \xleftrightarrow{uv} \eta$, implying that either $(u,v) \in T_\varepsilon$ or $(u,v) \in T_\eta$, both contradicting $(u,v) \in A(S)$. Hence, necessarily $(u,v) \in P^{op}$.

In case $(u,v) \in T \setminus P$ one either has $(v,u) \in P$ or $u\|v\,[P]$. In the former case one has $(v,u) \in T_\sigma$ for arbitrary $\sigma \in \mathcal{L}(P) = S$, yielding $(u,v) \notin T_\sigma$ and $(u,v) \in B(S)$. In the latter case, Theorem 4(i) implies that there are $\varepsilon,\eta \in S = \mathcal{L}(P)$ with $\varepsilon \xleftrightarrow{uv} \eta$, which gives either $(u,v) \notin T_\varepsilon$ or $(u,v) \notin T_\eta$, with both options yielding $(u,v) \in B(S)$. □

Surely, not every interval $[A,B]$ in $(T_\pi \setminus \Delta, \subseteq)$ can be interpreted as a code of a poset on N. Nonetheless, Lemma 7 at least gives the next rough estimate for the number of posets on N.

Corollary 8 The value $3^{\binom{n}{2}}$ is an upper bound for the number of posets on N with $|N| = n$.

Proof. The set $T := T_\pi \setminus \Delta$ has $\binom{n}{2}$ elements. Every interval $[A,B]$ in $(T,\subseteq)$ corresponds to a partition of T into three sets, namely A, $B \setminus A$, and $T \setminus B$. □

6 Other cryptomorphic views on finite posets and conclusions

Another important lattice we deal with in this paper is the lattice of *preposets* on N. There are two possible interpretations of these preposets. One of them is geometric, in terms of special cones in $\mathbb{R}^N$, as presented in [15, § 3.4]. The other interpretation is combinatorial, in terms of certain rings of subsets of N, and this relates to well-known Birkhoff's representation theorem for finite distributive lattices [2, § III.3]. Both interpretations can be introduced in terms Galois connections, as described in Section 2.3. Posets are special cases of preposets, for which reason the alternative views on preposets apply to posets as well.

6.1 Geometric view: braid cones

To this end we define a binary relation $\bullet$ between vectors in $X := \mathbb{R}^N$ and the elements of the Cartesian product $Y := N \times N$. Specifically, given a vector $x \in \mathbb{R}^N$ and $(u,v) \in N \times N$, we define their incidence relation $\bullet$ through the *comparison of respective vector components*:

$$x \bullet (u,v) \quad := \quad x_u \le x_v\,.$$

We will use larger black triangles $\blacktriangleright$ and $\blacktriangleleft$ to denote Galois connections based on this relation $\bullet$. Any subset T of $Y = N \times N$ is assigned (by backward Galois connection) a polyhedral cone

$$\mathsf{N}_T := T^{\blacktriangleleft} = \{\, x \in \mathbb{R}^N \;:\; \forall\,(u,v) \in T \;\; x_u \le x_v \,\}\,,$$

called the *braid cone* for T. In particular, any enumeration $\varepsilon \in \Upsilon(N)$ of N is assigned a particular full-dimensional cone in $\mathbb{R}^N$ as the braid cone for the toset T_ε :

$$\begin{aligned}\mathsf{N}^{\varepsilon} &:= \{\, x \in \mathbb{R}^N \,:\, x_{\varepsilon(1)} \le x_{\varepsilon(2)} \le \ldots \le x_{\varepsilon(n)} \,\} \\ &= \{\, x \in \mathbb{R}^N \,:\, \forall\,(u,v) \in N \times N \quad u \preceq_{\varepsilon} v \;\Rightarrow\; x_u \le x_v \,\};\end{aligned} \tag{2}$$

these cones are called *Weyl chambers* by some authors [12, 15]. The next lemma claims that the method of Galois connections yields the lattice of preposets.

Lemma 9 Let $(\mathcal{X}^{\bullet}, \subseteq)$ and $(\mathcal{Y}^{\bullet}, \subseteq)$ denote the lattices defined using the Galois connections $\blacktriangleright$ and $\blacktriangleleft$ based on the incidence relation $\bullet$ introduced above.

(i) Then $T \in \mathcal{Y}^{\bullet}$ iff T is a preposet on N.

(ii) A preposet $T \subseteq N \times N$ is a poset iff its braid cone N_T has the full dimension n.

(iii) Given a poset P on N, one has $\mathcal{L}(P) = \{\, \varepsilon \in \Upsilon(N) \,:\, \mathsf{N}^{\varepsilon} \subseteq \mathsf{N}_P \,\}$ and $\mathsf{N}_P = \bigcup_{\varepsilon \in \mathcal{L}(P)} \mathsf{N}^{\varepsilon}$.

The formulas in (iii) allow one to reconstruct $\mathcal{L}(P)$ on basis of N_P. In fact, because Weyl chambers are isometric each other, one can perhaps interpret the ratio $\frac{|\mathcal{L}(P)|}{|\Upsilon(N)|}$ geometrically as the relative volume of $\mathsf{N}_P \cap B$ within B, where B is the unit ball around the origin in $\mathbb{R}^N$.

Proof. For the necessity in (i) assume $T = \mathsf{N}^{\blacktriangleright} = \{\, (u,v) \in N \times N \,:\, \forall\, x \in \mathsf{N} \quad x_u \le x_v \,\}$ for some $\mathsf{N} \subseteq \mathbb{R}^N$ (see Section 2.3). Evidently, T is both reflexive and transitive.

For the sufficiency in (i) assume that T is a preposet on N and denote by $\mathcal{E}$ the set of equivalence classes of $E := T \cap T^{op}$. Put $\mathsf{N} := \mathsf{N}_T = T^{\blacktriangleleft}$; it is immediate that $T \subseteq \mathsf{N}^{\blacktriangleright}$. We will show $(a,b) \in (N \times N) \setminus T \;\Rightarrow\; (a,b) \notin \mathsf{N}^{\blacktriangleright}$ to verify the other inclusion $\mathsf{N}^{\blacktriangleright} \subseteq T$.
For this purpose, we interpret T in the form of a poset $\preceq$ on $\mathcal{E}$ (see Section 2.1). This poset can additionally be viewed as a transitive directed acyclic graph G over $\mathcal{E}$ (Section 3.3). The assumption $(a,b) \notin T$ thus implies that there are $A, B \in \mathcal{E}$ with $a \in A$, $b \in B$ and $\neg[A \preceq B]$, that is, both $A \neq B$ and $\neg[\, A \to B$ in $G\,]$. Therefore, an ordering $U_1, \ldots, U_m$, $m \ge 1$, of elements of $\mathcal{E}$ exists, which is consonant with G and in which B strictly precedes A. Define $x \in \mathbb{R}^N$ as the "rank vector" of this block-enumeration: put, for any $u \in N$, $x_u := i$ where $u \in U_i$. Then $(u,v) \in T$ implies either $[\exists\, i \,:\, u, v \in U_i]$ or $[\exists\, i < j \,:\, u \in U_i \,\&\, v \in U_j]$, yielding $x_u \le x_v$. Hence, $x \in \mathsf{N}_T = \mathsf{N}$ but $x_b < x_a$, and $(a,b) \notin \mathsf{N}^{\blacktriangleright}$, which was the aim.

For the necessity in (ii) realize that $\mathcal{L}(T) \neq \emptyset$. Thus, $T \subseteq T_\varepsilon$ for some $\varepsilon \in \mathcal{L}(T) \subseteq \Upsilon(N)$ and the anti-isomorphism of $(\mathcal{X}^{\bullet}, \subseteq)$ and $(\mathcal{Y}^{\bullet}, \subseteq)$ yields $\mathsf{N}^{\varepsilon} \subseteq \mathsf{N}_T$.

For the sufficiency in (ii) observe that there exists $y \in \mathsf{N}_T$ whose all components are distinct. Because of $(u,v) \in T \;\Rightarrow\; y_u \le y_v$ one has $(u,v), (v,u) \in T \;\Rightarrow\; y_u = y_v$, enforcing $u = v$.

As for (iii) realize that $\varepsilon \in \mathcal{L}(P) \;\Leftrightarrow\; P \subseteq T_\varepsilon \;\Leftrightarrow\; \mathsf{N}^{\varepsilon} \subseteq \mathsf{N}_P$, owing to the properties of Galois connections. Hence, $\bigcup_{\varepsilon \in \mathcal{L}(P)} \mathsf{N}^{\varepsilon} \subseteq \mathsf{N}_P$. For the other inclusion realize that, in case $y \in \mathsf{N}_P$ has all components distinct, unique $\eta \in \Upsilon(N)$ exists with $y \in \mathsf{N}^{\eta}$. The fact $y_u < y_v \;\Leftrightarrow\; u \prec_\eta v$ then implies $\eta \in \mathcal{L}(P)$. The observation that vectors $y \in \mathbb{R}^N$ with distinct components are dense in N_P allows one to derive the other inclusion by applying the closure in Euclidean topology. $\square$

Note that the permutohedral graph can also be recognized in this geometric context. One can identify enumerations with the Weyl chambers and the adjacency of $\varepsilon, \eta \in \Upsilon(N)$ is geometrically characterized by the condition that the dimension of the intersection $\mathsf{N}^{\varepsilon} \cap \mathsf{N}^{\eta}$ is $n-1$.

6.2 Combinatorial view: finite topologies

Define a binary relation $\bullet$ between elements of the power set $X := \mathcal{P}(N)$ and elements of $Y := N \times N$. Specifically, if D is a subset of N and $(u,v) \in N \times N$ then we consider D to be in the incidence relation $\bullet$ with the pair (u,v) if the set D *respects coming* u *before* v:

$$D \bullet (u,v) \quad := \quad [\, u \in D \;\vee\; v \notin D \,], \;\; \text{interpreted as } [\, v \in D \;\Rightarrow\; u \in D \,].$$

We will use larger gray triangles $\blacktriangleright$ and $\blacktriangleleft$ to denote Galois connections based on this relation $\bullet$. Any subset T of $Y = N \times N$ is assigned (by backward Galois connection) a set system

$$\mathcal{D}_T := T^{\blacktriangleleft} = \{\, D \subseteq N \,:\, \forall\, (u,v) \in T \;\text{ if } v \in D \text{ then } u \in D \,\} \quad \subseteq \mathcal{P}(N)\,,$$

known as the class of *down sets* for T, called alternatively *order ideals* in [17, § 3.1]. Note that any such a class $\mathcal{D}_T$ satisfies $\emptyset, N \in \mathcal{D}_T$ and is closed under set intersection and union, which means $D, E \in \mathcal{D}_T \;\Rightarrow\; D \cap E, D \cup E \in \mathcal{D}_T$. Therefore, it forms a ring of sets and can be interpreted as a (finite) *topology on* N, because in case of a finite set N, definition of a topology on N reduces to the above requirements.

Any enumeration $\varepsilon = |\,\varepsilon(1)| \ldots |\varepsilon(n)\,|$ of N can be represented by a particular topology $\mathcal{C}_{\varepsilon}$ on N, the one assigned to the toset T_{ε}, namely the respective *maximal chain* of subsets of N:

$$\mathcal{C}_{\varepsilon} \;:=\; \left\{ \bigcup_{i=1}^{j} \varepsilon(i) \quad : \; j = 0, \ldots, n \right\} \;=\; \{\, \emptyset, \{\varepsilon(1)\}\,, \{\varepsilon(1), \varepsilon(2)\}\,, \;\ldots\;, N \,\}\,. \tag{3}$$

Lemma 10 Let $(\mathcal{X}^{\bullet}, \subseteq)$ and $(\mathcal{Y}^{\bullet}, \subseteq)$ denote the lattices defined using the Galois connections $\blacktriangleright$ and $\blacktriangleleft$ based on the incidence relation $\bullet$ defined above.

(i) Then $T \in \mathcal{Y}^{\bullet}$ iff T is a preposet on N.

(ii) A set system $\mathcal{D} \subseteq \mathcal{P}(N)$ belongs to the lattice $\mathcal{X}^{\bullet}$ iff both $\emptyset, N \in \mathcal{D}$ and the implication $D, E \in \mathcal{D} \;\Rightarrow\; D \cap E, D \cup E \in \mathcal{D}$ holds ($=\mathcal{D}$ is a topology on N).

(iii) A preposet $T \subseteq N \times N$ is a poset iff the topology $\mathcal{D}_T$ distinguishes points: for any distinct $u, v \in N$ there is $D \in \mathcal{D}_T$ such that either $[\, u \in D \;\&\; v \notin D \,]$ or $[\, v \in D \;\&\; u \notin D \,]$. This happens exactly when there is $\varepsilon \in \Upsilon(N)$ such that $\mathcal{C}_{\varepsilon} \subseteq \mathcal{D}_T$.

(iv) Given a braid cone $\mathsf{N} \subseteq \mathbb{R}^N$, the set system $\mathcal{D}_{\mathsf{N}} := \{\, D \subseteq N \;:\; -\chi_D \in \mathsf{N} \,\}$ is the corresponding topology. Given a topology $\mathcal{D}$, the cone $\mathsf{N}_{\mathcal{D}} := \mathrm{con}\,(\{\chi_N\} \cup \{-\chi_D : D \in \mathcal{D}\})$, where $\mathrm{con}\,(\mathsf{N})$ denotes the conic hull of $\mathsf{N} \subseteq \mathbb{R}^N$, is the corresponding braid cone.

(v) Given a poset P on N, one has $\mathcal{L}(P) = \{\, \varepsilon \in \Upsilon(N) \,:\, \mathcal{C}_{\varepsilon} \subseteq \mathcal{D}_P \,\}$ and $\mathcal{D}_P = \bigcup_{\varepsilon \in \mathcal{L}(P)} \mathcal{C}_{\varepsilon}$.

The claim (iv) describes the transfer between the geometric representation of a poset P on N, which is the braid cone N_P, and the respective topology $\mathcal{D}_P$ on N. Analogously, the claim (v) describes the transfer between the graphical representation $\mathcal{L}(P)$ and the finite topology $\mathcal{D}_P$.

Note that, by the fundamental representation theorem for finite distributive lattices, rings of subsets of a finite set (ordered by inclusion) are universal examples of such lattices. What plays substantial role in the proof of this result is a particular correspondence between finite distributive lattices and finite posets, mediated by the concept of a down set; see [2, Corollary 1 in § III.3]. This is the correspondence we discuss here, but with one important technical difference. While distinguishing points is immaterial for the purpose of representing a (distributive) lattice, it matters for the purpose of representing preposets (on the same finite set).

Proof. For the necessity in (i) assume $T = \mathcal{D}^{\blacktriangleright} := \{ (u,v) \in N \times N \,:\, \forall\, D \in \mathcal{D} \;\; v \in D \Rightarrow u \in D \}$ for some $\mathcal{D} \subseteq \mathcal{P}(N)$ (see Section 2.3). Evidently, T is both reflexive and transitive.

For the sufficiency in (i) assume that T is a preposet on N; it is immediate that $T \subseteq (\mathcal{D}_T)^{\blacktriangleright}$. For any $v \in N$ we put $I(v) := \{ w \in N \,:\; (w,v) \in T \}$; the reflexivity of T implies $v \in I(v)$, while the transitivity of T allows one to observe $I(v) \in \mathcal{D}_T$. Given $(u,v) \in (\mathcal{D}_T)^{\blacktriangleright}$, these two facts about $I(v)$ and the definition of $\mathcal{D}^{\blacktriangleright}$ imply $u \in I(v)$, which means $(u,v) \in T$. Therefore, $(\mathcal{D}_T)^{\blacktriangleright} \subseteq T$ and $T = (\mathcal{D}_T)^{\blacktriangleright}$ yields $T \in \mathcal{Y}^{\bullet}$.

The necessity in (ii) is immediate: assume $T \subseteq N \times N$ and realize that the definition of $\mathcal{D}_T$ allows one easily to verify the conditions in (ii) for $\mathcal{D}_T$.

For the sufficiency in (ii) assume that $\mathcal{D} \subseteq \mathcal{P}(N)$ is a topology on N and put $T := \mathcal{D}^{\blacktriangleright}$; it is immediate that $\mathcal{D} \subseteq T^{\blacktriangleleft} = \mathcal{D}_T$. For any $v \in N$ we put $D(v) := \bigcap_{D \in \mathcal{D} \,:\, v \in D} D$. Then $N \in \mathcal{D}$ implies $v \in D(v)$ and, since $\mathcal{D}$ is closed under intersection, $D(v) \in \mathcal{D}$. Thus, $E \subseteq \bigcup_{v \in E} D(v)$ for any $E \subseteq N$. Given $E \in \mathcal{D}_T$, the goal is to verify the other inclusion $\bigcup_{v \in E} D(v) \subseteq E$. Indeed, having $w \in \bigcup_{v \in E} D(v)$, there exists $v \in E$ with $w \in D(v)$. The latter condition means $[\, \forall\, D \in \mathcal{D} \;\; v \in D \Rightarrow w \in D \,]$, which is nothing but $(w,v) \in \mathcal{D}^{\blacktriangleright} = T$. By the definition of $\mathcal{D}_T$, the facts $E \in \mathcal{D}_T$, $(w,v) \in T$, and $v \in E$ imply $w \in E$, concluding the proof of the inclusion. Therefore, since $\emptyset \in \mathcal{D}$ and $\mathcal{D}$ is closed under union, one gets $E = \bigcup_{v \in E} D(v) \in \mathcal{D}$. This concludes the proof of $E \in \mathcal{D}_T \;\Rightarrow\; E \in \mathcal{D}$ and $\mathcal{D} = \mathcal{D}_T = T^{\blacktriangleleft}$ yields $\mathcal{D} \in \mathcal{X}^{\bullet}$.

As concerns (iii), if T is poset then $T \subseteq T_{\varepsilon}$ for some enumeration $\varepsilon \in \Upsilon(N)$ and the anti-isomorphism of $(\mathcal{X}^{\bullet}, \subseteq)$ and $(\mathcal{Y}^{\bullet}, \subseteq)$ yields $\mathcal{C}_{\varepsilon} \subseteq \mathcal{D}_T$. Hence, $\mathcal{D}_T$ distinguishes points. Conversely, if $\mathcal{D}_T$ distinguishes points then there is no pair of distinct $u, v \in N$ with $(u,v), (v,u) \in T$, for otherwise $\forall\, D \in \mathcal{D}_T \;\; u \in D \Leftrightarrow v \in D$. Thus, $T \cap T^{op} = \Delta$ and T is anti-symmetric.

As concerns (iv), assume $\mathsf{N} = \mathsf{N}_T$ and $\mathcal{D} = \mathcal{D}_T$ for a preposet T on N. Given $D \subseteq N$ and $(u,v) \in N \times N$, re-write $D \bullet (u,v)$, that is, $[\, v \in D \Rightarrow u \in D \,]$ as $[\, \chi_D(v) = 1 \;\Rightarrow\; \chi_D(u) = 1 \,]$ and then as $-\chi_D(u) \leq -\chi_D(v)$, which means $(-\chi_D) \bullet (u,v)$. This allows one to observe that $D \in \mathcal{D}_T$ iff $-\chi_D \in \mathsf{N}_T$, implying the first claim in (iv).

The above observation also gives the inclusion $\mathrm{con}\,(\{\chi_N\} \cup \{-\chi_D : D \in \mathcal{D}_T\}) \subseteq \mathsf{N}_T$, because N_T is a cone. Therefore, the verification of the other inclusion

$$\mathsf{N}_T \subseteq \mathrm{con}\,(\{\chi_N\} \cup \{-\chi_D : D \in \mathcal{D}_T\})$$

is needed to complete the proof of the second claim in (iv).

For this purpose, consider the set $\mathcal{E}$ of equivalence classes of $E := T \cap T^{op}$ and view T as a poset on $\mathcal{E}$ and, thus, as a transitive directed acyclic graph G over $\mathcal{E}$ (see Sections 2.1 and 3.3). Observe that any vector $x \in \mathsf{N}_T$ complies with G in the following sense: if $u \in U \in \mathcal{E}$, $v \in V \in \mathcal{E}$ and $U = V$ or $U \to V$ in G then $x_u \leq x_v$. Hence, x is "constant" on sets from $\mathcal{E}$, that is, $x_u = x_v$ if $(u,v) \in E$. This allows one to construct an ordering $U(1), \ldots, U(m)$, $m \geq 1$, of (all) elements of $\mathcal{E}$, which is both consonant with G and respects x in the sense that

$$u \in U(i),\ v \in U(j),\ \text{and } i \leq j \quad \text{implies} \quad x_u \leq x_v\,.$$

Let us introduce $x(i)$ for $i = 1, \ldots, m$ as the shared value x_u for $u \in U(i)$ and write

$$x = x(m) \cdot \chi_N \,+\, \sum_{i=1}^{m-1} \left(x(i+1) - x(i)\right) \cdot \left(-\chi_{\,U(1)\ldots U(i)}\right),$$

which equality can be verified by substituting any $u \in U(j)$ for $j \in [m]$. Since $x(i+1) - x(i) \geq 0$ for $i \in [m-1]$, the vector x belongs to the conic hull of $\{\, -\chi_{\,U(1)\ldots U(i)} \,:\, i \in [m-1]\,\} \cup \{\pm\chi_N\}$. The consonancy of $U(1), \ldots, U(m)$ with G implies that $U(1)\ldots U(i) \in \mathcal{D}_T$ for $i \in [m]$, which thus yields the desired inclusion and completes the proof of the second claim in (iv).

As for (v) realize that $\varepsilon \in \mathcal{L}(P) \;\Leftrightarrow\; P \subseteq T_\varepsilon \;\Leftrightarrow\; \mathcal{C}_\varepsilon \subseteq \mathcal{D}_P$, owing to the properties of Galois connections. Hence, $\bigcup_{\varepsilon \in \mathcal{L}(P)} \mathcal{C}_\varepsilon \subseteq \mathcal{D}_P$. For the other inclusion consider $A \in \mathcal{D}_P$, put $B := N \setminus A$, and observe there is no $(b,a) \in P$ with $b \in B$ and $a \in A$. In case $A = \emptyset$ or $A = N$ one has $A \in \mathcal{C}_\varepsilon$ for any $\varepsilon \in \mathcal{L}(P)$. As $P_A := P \cap (A \times A)$ is a poset on A, there is an enumeration $\varepsilon(A) \in \mathcal{L}(P_A)$, analogously with B, and concatenation $\varepsilon := |\varepsilon(A)|\varepsilon(B)|$ yields $\varepsilon \in \mathcal{L}(P)$ with $A \in \mathcal{C}_\varepsilon$. □

Remark Let us discuss the assignment $P \mapsto \mathcal{D}_P$ confined to posets P on N. A benefit of this particular transformation is that the original poset P remains internally represented within the distributive lattice $(\mathcal{D}_P, \subseteq)$. The point is that the variables in N are in one-to-one correspondence with *join-irreducible* elements of this lattice, which are those sets in $\mathcal{D}_P$ which are not unions of their own proper subsets in $\mathcal{D}_P$. Additionally, the comparability of variables in N corresponds to the inclusion of the respective joint-irreducible sets in $\mathcal{D}_P$. Specifically, any $v \in N$ is assigned its principal order ideal $I(v) := \{\, w \in N \,:\, (w,v) \in P \,\}$ and one has $(u,v) \in P$ iff $I(u) \subseteq I(v)$.[2]

The transformation $P \mapsto \mathcal{D}_P$ of a poset P on N is also one of standard constructions in combinatorics because it leads to more elaborate methods to count linear extensions of a poset. Indeed, Lemma 10(v) allows one to identify *linear extensions* of P on basis of $\mathcal{D}_P$ and, thus, this two-step procedure forms the basis for common algorithms to compute the number of linear extensions of P, as indicated in [17, § 3.5]. A particular way of representing the lattice $(\mathcal{D}_P, \subseteq)$ in the form of a grid in $\mathbb{N}^k$ for some k is described there and linear extensions of P are identified with certain "increasing" paths in this grid. The problem of counting linear extensions of P is thus transformed to the problem of counting certain types of permutations.

[2] To avoid misunderstanding recall that the transform $P \mapsto \mathcal{D}_P$ itself is order-reversing when considered on the class of all posets P on N. Here we, however, have the situation when a poset P is fixed and claim that the embedding $v \mapsto I(v)$ of N into $\mathcal{D}_P$ is order-preserving.

According to the explanation in [17, p. 110], it follows from a classic result [5] by Dilworth that the (poset) dimension of $(\mathcal{D}_P, \subseteq)$ is the *width* of the poset P, which is the maximal cardinality of an anti-chain in P, being the least k such that N can be written as the union of k chains in P. Thus, the width of P seems to be the crucial complexity measure for computing $|\mathcal{L}(P)|$.

6.3 Conclusions

In the present paper, we have described some cryptomorphic views on posets on a non-empty finite set of variables. A contribution of the second author is a web page

`http://gogo.utia.cas.cz/posets/`

which allows the user to pass through different combinatorial representatives of posets on a variable set N of small cardinality. Specifically, besides the basic representation of a poset P in the form of a binary relation on N, it can be represented by the respective transitive directed acyclic graph over N, by its directed Hasse diagram, by the set of its linear extensions $\mathcal{L}(P)$, and by the respective finite topology $\mathcal{D}_P$, that is, in the form of a finite distributive lattice.

We believe that the alternative views on finite posets will appear to be useful in context of combinatorial description of faces of the supermodular cone [18] by means of collections of such posets, called the *fans of posets* [15, § 3.3].

We also hope that our main result on graphical characterization of sets of linear extensions of posets as geodetically convex sets might find future applications in alternative algorithms to deal with the poset cover problem, since it is conceptually simpler than [9, Theorem 9].

Appendum (from the first author)

I can hardly claim that the theme of posets was in the center of research activity of Henry Wynn. Nonetheless, he was open and willing to discuss this topic and its possible connections to Gröbner bases applications and we did so together on several occasions of conferences on algebraic statistics. I am also indebted to Victor Reiner for drawing my attention to the results presented in [3, 19], which are highly relevant to the topic of the present paper.

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