ON $\mathbb{A}$-GENERATORS OF THE COHOMOLOGY $H^*(V^{\oplus 5}) = \mathbb{Z}/2[u_1, \ldots, u_5]$ AND THE COHOMOLOGICAL TRANSFER OF RANK 5

ĐẶNG VÕ PHÚC*

ABSTRACT. Let denote $V^{\oplus n}$ the n-dimensional vector space over the prime field $\mathbb{Z}/2$. We write $\mathbb{A}$ as the 2-primary Steenrod algebra, which is the algebra of stable natural endomorphisms of the mod 2 cohomology functor on topological spaces. Working at the prime 2, computing the cohomology of $\mathbb{A}$ is an important problem of Algebraic topology since it is the initial page of the Adams spectral sequence (the ASS, for short) converging to stable homotopy groups of the spheres. A particularly effective technique for characterizing this cohomology is the cohomological transfer of rank n, which was first introduced by W. Singer in his seminal work [Math. Z. **202**, 493-523 (1989)]. This transfer maps from a certain subquotient of a divided power algebra to the cohomology of $\mathbb{A}$. Actually, the Singer transfer is induced over the E_2-term of the ASS by the geometrical transfer map $\Sigma^\infty(B(V^{\oplus n})_+) \longrightarrow \Sigma^\infty(\mathbb{S}^0)$ in stable homotopy theory. Singer formulated a pivotal conjecture that *the cohomological transfer is always a one-to-one homomorphism*. The Singer transfer is closely linked to the classical "hit problem" first proposed by Frank Peterson in [Abstracts Papers Presented Am. Math. Soc. **833**, 55-89 (1987)]. The hit problem involves finding a minimal generating set for the unstable $\mathbb{A}$-module $H^*(V^{\oplus n}) = \mathbb{Z}/2[u_1, \ldots, u_n]$. Despite several decades of research, this problem remains unsolved for all $n \geq 5$. In this paper, we study the hit problem for the $\mathbb{A}$-module $H^*(V^{\oplus n})$ and verify Singer's conjecture for the cases where $n = 5$ and the general degree $d = 2^{t+5} + 2^{t+2} + 2^{t+1} - 5$ for any non-negative integer t. The results of our study demonstrate that the Singer cohomological transfer is an isomorphism for $n = 5$ in degree d. This provides a positive answer to Singer's conjecture in the considered cases. The appendix provides our new algorithm implemented on the computer algebra system `OSCAR`, through which all principal results of this paper have been completely verified.

1. INTRODUCTION

It is well-known that there is a group homomorphism Sq^n for $n \geq 0$, between mod-2 cohomology groups of a topological space, called Steenrod squares of degrees n. They are stable cohomology operations, that is, they commute with suspension maps. For further information on Steenrod operations over the field $\mathbb{Z}/p$ (with p being any prime number), we refer to, for example, the works by Brunetti, Ciampella, and Lomonaco [5, 6]. All the Steenrod squares form an algebraic structure which is known as the 2-primary Steenrod algebra $\mathbb{A}$ subject to the Adem relations. It was applied to the vector fields on spheres and the Hopf invariant one problem, which asks for which n there exist maps of Hopf invariant ± 1. So, the Steenrod algebra is one of the important tools in Algebraic topology. Specifically, its cohomology $\mathrm{Ext}_{\mathbb{A}}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ is an algebraic object that serves as the input to the Adams spectral sequence (the ASS, for short) [1] and therefore, computing this cohomology is of fundamental importance to the study of the stable homotopy groups of spheres. These

2020 *Mathematics Subject Classification.* 55T15, 55S10, 55S05, 55R12.
Key words and phrases. Adams spectral sequences, Steenrod algebra, Hit problem, Algebraic transfer.
*ORCID: https://orcid.org/0000-0002-6885-3996.

subjects have also been extensively studied by numerous algebraic topologists. For instance, one can refer to the works by Palmieri [20, 21].

The cohomological transfer, defined by Singer [32], could be a useful approach to describe the mysterious structure of the cohomology algebra of $\mathbb{A}$. In order to better understand this transfer, we will use the following notations and the relevant concepts. Let denote $V^{\oplus n}$ the n-dimensional vector space over the prime field $\mathbb{Z}/2$. Then, we write $H^*(V^{\oplus n})$ and $H_*(V^{\oplus n})$ for mod-2 cohomology and homology of $BV^{\oplus n}$ (the classifying space of $V^{\oplus n}$). One should note that $BV^{\oplus n}$ is homotopy equivalent to the cartesian product of n copies of the union of the finite projective spaces. As it is known, $H^*(V^{\oplus n})$ is identified with the polynomial algebra $\mathbb{Z}/2[u_1, \ldots, u_n]$ on generators of degree 1, equipped with the canonical unstable algebra structure over the Steenrod algebra (i.e., it is a commutative, associative, graded $\mathbb{Z}/2$-algebra equipped with a structure of unstable $\mathbb{A}$-module and satisfying two relations, one called the Cartan formula, and the other called the instability relation: $Sq^{\deg(x)}(x) = x^2$.) By dualizing, $H_*(V^{\oplus n})$ has a natural basis dual to the monomial basis of $H^*(V^{\oplus n})$. We denote by $a_1, \ldots, a_n$ the basis of $H_1(V^{\oplus n})$ dual to the basis $u_1, \ldots, u_n$ of $H^1(V^{\oplus n}) = \mathrm{Hom}(V^{\oplus n}, \mathbb{Z}/2)$, so that $a_i(u_j) = 1$ if $i = j$ and is 0 if $i \neq j$. We write the dual of $u_1^{d_1} \ldots u_n^{d_n}$ as $a_1^{(d_1)} \ldots a_n^{(d_n)}$ where the parenthesized exponents are called *divided powers*, and be careful that in the corresponding situation over a field of characteristic 0 in place of $\mathbb{Z}/2$, $a_i^{(d_i)} = a_i^d/d!$, which fits with the formula $a^{(d)}a^{(e)} = \binom{d+e}{d}a^{(d+e)}$. This product gives a commutative graded algebra $\Gamma(a_1, \ldots, a_n)$ over $\mathbb{Z}/2$ called a *divided power algebra*, where $\Gamma(a_1, \ldots, a_n) = H_*(V^{\oplus n})$, and an element $a_1^{(d_1)} \ldots a_n^{(d_n)}$ in $H_*(V^{\oplus n})$ corresponding to a monomial $u_1^{d_1} \ldots u_n^{d_n}$ in $H^*(V^{\oplus n})$ is called *d-monomial*. The (right) action of the Steenrod ring on $H_*(V^{\oplus n})$ is given by

$$(a_j^{(t)})Sq^k = \binom{t-k}{k} a_j^{(t-k)},\ 1 \leq j \leq n,$$

and the standard Cartan formula.

Now, let $P_{\mathbb{A}}H_*(V^{\oplus n})$ be the subspace of $H_*(V^{\oplus n})$ consisting of all elements that are annihilated by all Steenrod squares of positive degrees. The general linear group $GL_n = GL(V^{\oplus n})$ acts regularly on the classifying space $BV^{\oplus n}$ and therefore on $H^*(V^{\oplus n})$ and $H_*(V^{\oplus n})$. This action commutes with that of the algebra $\mathbb{A}$ and so acts on $\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus n})$ and $P_{\mathbb{A}}H_*(V^{\oplus n})$. For each $n \geq 0$, Singer constructed in [32] a linear transformation from $P_{\mathbb{A}}H_*(V^{\oplus n})$ to the n-th cohomology group $\mathrm{Ext}_{\mathbb{A}}^{n,n+\bullet}(\mathbb{Z}/2, \mathbb{Z}/2)$ of $\mathbb{A}$, which commutes with two Sq^0's on $P_{\mathbb{A}}H_*(V^{\oplus n})$ and $\mathrm{Ext}_{\mathbb{A}}^{n,n+*}(\mathbb{Z}/2, \mathbb{Z}/2)$ (see Boardman [2] and Minami [16] for more about this). He shows that this map factors through the quotient of its domain's GL_n-coinvariants to give rise the so-called *cohomological transfer of rank n*

$$\phi_n^*(\mathbb{Z}/2) : (\mathbb{Z}/2 \otimes_{GL_n} P_{\mathbb{A}}H_*(V^{\oplus n}))_\bullet \longrightarrow \mathrm{Ext}_{\mathbb{A}}^{n,n+\bullet}(\mathbb{Z}/2, \mathbb{Z}/2).$$

Here $(\mathbb{Z}/2 \otimes_{GL_n} P_{\mathbb{A}}H_*(V^{\oplus n}))_\bullet$ is dual to the space of GL_n-invariants $((\mathbb{Z}/2 \otimes_{\mathbb{A}} H^*(V^{\oplus n}))_\bullet)^{GL_n}$. It is to be noted that $\phi_n^*(\mathbb{Z}/2)$ is induced over the E_2-term of the ASS by the geometrical transfer map $\Sigma^\infty(B(V^{\oplus n})_+) \longrightarrow \Sigma^\infty(\mathbb{S}^0)$ in stable homotopy theory (see also Mitchell [17]). The work of Minami [16] indicated that these transfers play a key role in finding permanent cycles in the ASS. In the second cohomology groups of $\mathbb{A}$, following Mahowald [14] and Lin-Mahowald [12], the classes h_1h_j for $j \geq 3$ and h_j^2 for $0 \leq j \leq 5$, are known to be the permanent cycles in the ASS. In 2016, Hill, Hopkins, and Ravenel [10] showed that when $j \geq 7$, the class h_j^2 is not a permanent cycle in the ASS. It is surprising that so far there is no answer for $j = 6$. The question of whether these h_j^2 are the permanent cycles in the ASS or not is called *Kervaire invariant problem* in literature [4]. This is one of the oldest unresolved issues in Differential and Algebraic topology.

Directly computing the domain and codomain of the Singer transfer $\phi_n^*(\mathbb{Z}/2)$ is a challenging task. Singer himself demonstrated that $\phi_n^*(\mathbb{Z}/2)$ is an isomorphism for $n \leq 2$, and Boardman showed this for $n = 3$. Notably, in the same paper [32], William Singer proposed a conjecture that $\phi_n^*(\mathbb{Z}/2)$ *is a one-to-one mapping for all* n. This conjecture was previously established by Singer himself for $n = 1$, 2 and by Boardman [2] for $n = 3$. We established in [27] that the conjecture is valid for $n = 4$, but it fails in general; see [30] for a counterexample of rank 6 in degree 36. The computation was fully checked in the computer algebra system `OSCAR`.

As it is known, the polynomial ring $P_n := \mathbb{Z}/2[u_1, \ldots, u_n]$ is a connected $\mathbb{Z}$-graded algebra. That is, $P_n = \mathbb{Z}/2 \bigoplus(\bigoplus_{d>0}(P_n)_d)$, where $(P_n)_d$ is the vector space of homogeneous polynomials of degree d. The Singer transfer we are discussing is closely related to the *hit problem* in literature [22] of determination of a minimal generating set for the unstable $\mathbb{A}$-module P_n. Historically, the hit problem was initially raised in 1982 by Boudriga and Zarati [3] for the $\mathbb{A}$-module $H^*(K(\mathbb{Z}/2, n))$, $n = 1$, 2, 4, in order to study the multiple points of a given codimensional 2 immersions. In this context, $K(\mathbb{Z}_2, n)$ refers to the Eilenberg-Maclane space, and note that $H^*(K(\mathbb{Z}/2, 1)) = P_1$. By Minami [15], the hit problem is also considered as a useful tool for studying permanent cycles in the ASS. A homogeneous element $f \in (P_n)_d$ is *hit* if there is a finite sum $f = \sum_{j>0} Sq^j(f_j)$, where the homogeneous elements f_j belong to $(P_n)_{d-j}$, $d > j$. The interested reader may consult the following references for further information on hit problems: Crabb-Hubbuck [8], Kameko [11], Monks [19], Moetele-Mothebe [18], Phúc and Sum [23], Phúc [24, 25, 26, 28], Sum [34, 35, 36], Repka-Selick [31], Singer [33], Walker-Wood [37], Wood [38], and others. When $\mathbb{Z}/2$ is a trivial $\mathbb{A}$-module, solving the hit problem is equivalent to determining the "cohits"

$$QP_n := \mathbb{Z}/2 \otimes_{\mathbb{A}} P_n = P_n/\overline{\mathbb{A}}P_n = \mathrm{Tor}_0^{\mathbb{A}}(\mathbb{Z}/2, P_n),$$

as a graded vector space, or more generally as a graded module over the group algebra $\mathbb{Z}/2[GL_n]$. Here $\overline{\mathbb{A}}$ denotes the augmentation ideal in $\mathbb{A}$. Denote by $(QP_n)_d$ the subspace of QP_n consisting of all the equivalence classes represented by the homogeneous polynomials in $(P_n)_d$. Frank Peterson [22] conjectured that $(QP_n)_d = 0$ if and only if $\alpha(n + d) \leq n$, for all d where $\alpha(k)$ is the number of 1's in the dyadic expansion of a positive integer k. His motivation for this was to prove that if $\mathcal{M}$ is a smooth manifold of dimension d such that all products of length n of Stiefel-Whiney classes of its nomal bundle vanish, then either $\alpha(d) \leq n$ or $\mathcal{M}$ is cobordant to zero. The conjecture was established by Wood [38]. While the general case remains open, the hit problem has been fully resolved for any $n \leq 4$ and degree d by Peterson [22], Kameko [11], and Sum [35].

Despite the efforts discussed above, knowledge of the hit problem and the Singer transfer in five variables remains limited. Accordingly, and as a continuation of our previous work in [26], this work studies the hit problem for the $\mathbb{A}$-module P_5 and the fifth cohomological transfer $\phi_5^*(\mathbb{Z}/2)$ in all degrees of the form

$$d = 2^{t+5} + 2^{t+2} + 2^{t+1} - 5 \quad (t \in \mathbb{Z}_{\geq 0}).$$

The choice of this "generic" degree d stems from the intrinsic interest and computational complexity of determining bases for $(QP_5)_d$ and its invariant subspaces. A detailed treatment is provided in Section 3, and Appendix 4 contains an explicit description of the associated admissible bases together with links to our algorithm implemented on the computer algebra system `OSCAR`, by means of which all main results of Section 3 have been fully verified.

2. Background

As preparation for the main results of the paper, we recall basic material on the hit problem in a concise form; for more context, consult [11, 37].

Definition 2.1 (Weight vector and exponent vector). We say that a sequence of non-negative integers $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ is a *weight vector*, if $\omega_i = 0$, for $i \gg 0$. Then, one defines $\deg(\omega) = \sum_{i\geq 1} 2^{i-1}\omega_i$. With a natural number d, let denote $\alpha_j(d)$ the j-th coefficients in dyadic expansion of d, then $\alpha(d) = \sum_{j\geq 0}\alpha_j(d)$, and $n = \sum_{j\geq 0}\alpha_j(d)2^j$, where $\alpha_j(d) \in \{0,1\}$ for all $j\geq 0$. For a monomial $u = u_1^{a_1}u_2^{a_2}\ldots u_n^{a_n} \in P_n$, let us consider two sequences $\omega(u) := (\omega_1(u), \ldots, \omega_i(u), \ldots)$ and $(a_1, a_2, \ldots, a_n)$, where $\omega_i(u) = \sum_{1\leq j\leq n}\alpha_{i-1}(a_j)$, for every i. They are respectively called the *weight vector* and the *exponent vector* of u. By convention, the sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Definition 2.2 (Linear order on P_n). Let $u = u_1^{a_1}u_2^{a_2}\ldots u_n^{a_n}$ and $y = u_1^{b_1}u_2^{b_2}\ldots u_n^{b_n}$ be monomials in P_n. We write a, b for the exponent vectors of u and y, respectively. We say that $u < y$ if and only if one of the following holds:

(i) $\omega(u) < \omega(y)$;
(ii) $\omega(u) = \omega(y)$ and $a < b$.

Definition 2.3 (Equivalence relations on P_n). For a weight vector ω of degree d, let us denote two subspaces associated with ω by

$$\begin{aligned}(P_n)_d(\omega) &= \langle\{u \in (P_n)_d \,|\, \omega(u) \leq \omega\}\rangle,\\ (P_n)_d(<\omega) &= \langle\{u \in (P_n)_d \,|\, \omega(u) < \omega\}\rangle.\end{aligned}$$

Let f and g be two homogeneous polynomials in $(P_n)_d$. Equivalence relations "$\sim$" and "$\sim_\omega$" can be defined on P_n in the following manner:

(i) $f \sim g$ if and only if $(f-g) \in \overline{\mathbb{A}}P_n$. (Note that since working mod 2, $f - g = f + g$.)
(ii) $f \sim_\omega g$ if and only if $f, g \in (P_n)_d(\omega)$ and $(f-g) \in (\overline{\mathbb{A}}P_n \cap (P_n)_d(\omega) + (P_n)_d(<\omega))$.

In particular, if $f \sim 0$ (resp. $f \sim_\omega 0$), then we say that f is *hit* (resp. ω-*hit*).

Let us denote $(QP_n)_d(\omega)$ as the quotient space of $(P_n)_d(\omega)$ via the equivalence relation "$\sim_\omega$". According to [37, 36], this $(QP_n)_d(\omega)$ is also a GL_n-module. Notably, an isomorphism exists, given by

$$(QP_n)_d \cong \bigoplus_{\deg(\omega)=d}(QP_n)_d(\omega) \text{ (see also [28]).}$$

If we put

$$\begin{aligned}(P_n)^0 &:= \langle\{u = \textstyle\prod_{1\leq j\leq n}u_j^{\gamma_j} \in P_n :\ \prod_{1\leq j\leq n}\gamma_j = 0\}\rangle,\\ (P_n)^{>0} &:= \langle\{u = \textstyle\prod_{1\leq j\leq n}u_j^{\gamma_j} \in P_n :\ \prod_{1\leq j\leq n}\gamma_j > 0\}\rangle,\end{aligned}$$

then $(P_n)^0$ and $(P_n)^{>0}$ are $\mathbb{A}$-submodules of P_n. We denote by $(QP_n)^0 := Q(P_n)^0$ and $(QP_n)^{>0} := Q(P_n)^{>0}$. Then $QP_n \cong (QP_n)^0\bigoplus(QP_n)^{>0}$. Hence, $(QP_n)_d \cong (QP_n)^0_d\bigoplus(QP_n)^{>0}_d$ for any positive degree d.

Definition 2.4 (Admissible monomial and inadmissible monomial). A monomial u in P_n is said to be *inadmissible* if there exist monomials $u_1, u_2, \ldots, u_k$ in P_n such that $u_j < u$ for all j, $1\leq j\leq k$, and $u \sim (u_1 + u_2 + \cdots + u_k)$. Then, we say that u is *admissible*, if it is not inadmissible.

Additionally, one defined strictly inadmissible monomials as follows. A monomial u in P_n is said to be *strictly inadmissible* if there exist monomials $u_1, u_2, \ldots, u_k$ in P_n such that $u_j < u$ for all j, $1 \leq j \leq k$, and $u = \sum_{1\leq j\leq k} u_j + \sum_{0\leq m\leq s-1} Sq^{2^m}(u'_m)$, where $s = \max\{i \in \mathbb{N} : \omega_i(u) > 0\}$ and suitable polynomials $u'_m \in P_n$ with $\deg(u'_m) = \deg(u) - 2^m$.

The following important theorem, which is due to Kameko [11] and Sum [35], can be viewed as a criterion in the calculation of the inadmissible monomials.

Theorem 2.5 (**Criteria on inadmissible monomials**). *The following assertions are true:*

(i) (see [11]) *Let z, w be monomials in P_n and let r be a positive integer. Suppose that $\omega_r(z) \neq 0$ and $\omega_j(z) = 0$ for all $j > r$. Consequently, if z is strictly inadmissible, then so is zw^{2^r}.*

(ii) (see [35]) *Let y and z be monomials in P_n. For an integer $r > 0$, assume that $\omega_i(y) = 0$ for all $i > r$. Consequently, if z is inadmissible, then so is yz^{2^r}.*

From the above data, we have seen that $(QP_n)_d$ is a $\mathbb{Z}/2$-vector space with a basis consisting of all the classes represented by the admissible monomials in $(P_n)_d$.

Definition 2.6 (**Spike monomial**). A monomial $u_1^{b_1}u_2^{b_2}\ldots u_n^{b_n}$ in P_n is called a *spike* if every exponent b_j is of the form $2^{c_j} - 1$. In particular, if the exponents c_j can be arranged to satisfy $c_1 > c_2 > \ldots > c_{r-1} \geq c_r \geq 1$, where only the last two smallest exponents can be equal, and $c_j = 0$ for $r + 1 \leq j \leq n$, then $u_1^{b_1}u_2^{b_2}\ldots u_n^{b_n}$ is called a *minimal spike*.

Lemma 2.7 ([23]). *All the spikes in P_n are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \ldots)$ of degree n is weakly decreasing and $\omega_1 \leq n$, then there is a spike $z \in P_n$ such that $\omega(z) = \omega$.*

Notation 2.8. It will be helpful to use the following notation: we will denote the set of all admissible monomials of degree d in the $\mathbb{A}$-module P_n by $\mathscr{C}_d^{\otimes n}$. For a monomial $u \in (P_n)_d$, write $[u]$ for the equivalence class of u in $(QP_n)_d$. If ω is a weight vector and $u \in (P_n)_d(\omega)$, then we denote by $[u]_\omega$ the equivalence class of u in $(QP_n)_d(\omega)$. Moreover if ω is a weight vector of a minimal spike, then $[u]_\omega = [u]$.

The following tools are useful for studying the hit problem. We consider the arithmetic function $\mu : \mathbb{N} \longrightarrow \mathbb{N}$ that is determined by

$$\begin{aligned}\mu(d) &= \min\Big\{n \in \mathbb{N} :\ \alpha(d+n) \leq n\Big\}\\ &= \min\{n \in \mathbb{N} :\ d = \textstyle\sum_{1\leq j\leq n}(2^{s_j} - 1),\ s_j > 0,\ 1 \leq j \leq n\}.\end{aligned}$$

In [11], Kameko defined the epimorphism

$$\begin{aligned}(\widetilde{Sq^0_*})_{2d+n} :\quad (QP_n)_{2d+n} &\longrightarrow (QP_n)_{d=\frac{(2d+n)-n}{2}}\\ [x] &\longmapsto \begin{cases} [y] & \text{if } x = \prod_{1\leq j\leq n} u_j y^2,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Theorem 2.9. *For each positive integer d, the following hold:*

(i) (see [38]). *$(QP_n)_d$ is trivial if $\mu(d) > n$. Consequently QP_n is trivial in degrees d unless d is of the form $d = 2^{t_1} + 2^{t_2} + \cdots + 2^{t_n} - n$ with $t_1 \geq t_2 \geq \cdots \geq t_n \geq 0$.*

(ii) (see [11]). *The homomorphism $(\widetilde{Sq^0_*})_{2d+n}$ is an isomorphism if and only if $\mu(2d+n) = n$.*

Thus, utilizing the aforementioned theorem and the isomorphism between the domain of the Singer transfer and $((QP_n)_d)^{GL_n}$, we can focus exclusively on studying the hit problem for the $\mathbb{A}$-module P_n and the behavior of the Singer transfer $\phi_n^*(\mathbb{Z}/2)$ in degrees d such that $\mu(d) \leq n$.

In his paper [33], Singer observed that if $\mu(d) \leq n$, then there exists a unique minimal spike of degree d in P_n. Additionally, he proved the following outcome.

Theorem 2.10 (cf. Singer [33])**.** *Suppose that u is a monomial in P_n with $\mu(\deg(u)) \leq n$. Let z be the minimal spike in P_n with $\deg(z) = \deg(u)$. Then, if $\omega(u) < \omega(z)$, then $u \in \overline{\mathbb{A}}P_n$.*

The converse of this theorem does not hold in general.

3. Key outcomes and their respective demonstrations

As previously noted, this section focuses on investigating the hit problem for the $\mathbb{A}$-module P_5 and the behavior of the Singer transfer of rank 5 in the general degree $d = 2^{t+5} + 2^{t+2} + 2^{t+1} - 5$, where $t \geq 0$, $t \in \mathbb{Z}$.

We will begin by studying the structure of the space $(QP_5)_d$.

Case $t = 0$. We have $d = 2^{0+5} + 2^{0+2} + 2^{0+1} - 5 = 33$. We observe that since the Kameko squaring operation $(\widetilde{Sq^0_*})_{33} : (QP_5)_{33} \longrightarrow (QP_5)_{14}$ is an epimorphism, we have an isomorphism

$$(QP_5)_{33} \cong \text{Ker}((\widetilde{Sq^0_*})_{33}) \bigoplus (QP_5)_{14}.$$

As it is known, there exist exactly 320 admissible monomials of degree 14 in the $\mathbb{A}$-module P_5. Consequently, $(QP_5)_{14}$ has dimension 320 (see also [26]). Thus, in order to obtain a basis for the space $(QP_5)_{33}$, we must calculate the dimension of $\text{Ker}((\widetilde{Sq^0_*})_{33})$. In this regard, the following remark may be of interest.

Remark 3.1. We claim that if g is an admissible monomial of degree 33 in P_5 such that $[g]$ belongs to $\text{Ker}((\widetilde{Sq^0_*})_{33})$, then $\omega_1(g) = 3$. Indeed, since $u_1^{31}u_2u_3$ is the minimal spike of degree 33 in the $\mathbb{A}$-module P_5 and $\omega(u_1^{31}u_2u_3) = (3, 1, 1, 1, 1)$, according to Lemma 2.7, $u_1^{31}u_2u_3$ is admissible. Because g is admissible, and $\deg(g)$ is odd, by Theorem 2.10, either $\omega_1(g) = 1$ or $\omega_1(g) = 5$. If $\omega_1(g) = 5$, then g is of the form $\prod_{1\leq j\leq 5} u_jh^2$, in which h is a monomial of degree 14 in P_5. Since $\prod_{1\leq j\leq 5} u_jh^2$ is admissible, following Theorem 2.5(i), h is also admissible. So, $(\widetilde{Sq^0_*})_{33}([\prod_{1\leq j\leq 5} u_jh^2]) = [\underline{t}] \neq [0]$, which contradicts with the fact that $[g] = [\prod_{1\leq j\leq 5} u_jh^2] \in (\widetilde{Sq^0_*})_{33}$. Thus, we must have $\omega_1(g) = 3$, and therefore by the above arguments, g can be represented as $u_iu_ju_kh^2$, where $1 \leq i < j < k \leq 5$, and h is an admissible monomial of degree 15 in P_5. According to Sum [34], $\omega(h)$ belongs to the set $\{(1, 1, 1, 1),\ (1, 1, 3),\ (3, 2, 2)\ (3, 4, 1)\}$, and therefore, we may conclude that the weight vector $\omega(g)$ of g is one of the following sequences:

$$\widetilde{\omega}_{(1)} := (3, 1, 1, 1, 1),\ \widetilde{\omega}_{(2)} := (3, 1, 1, 3),\ \widetilde{\omega}_{(3)} := (3, 3, 2, 2),\ \widetilde{\omega}_{(4)} := (3, 3, 4, 1).$$

Thus, we have an isomorphism

$$\begin{aligned}\text{Ker}((\widetilde{Sq^0_*})_{33}) &\cong (QP_5)^0_{33} \oplus \Big(\text{Ker}((\widetilde{Sq^0_*})_{33}) \bigcap (QP_5)^{>0}_{33}\Big)\\ &\cong (QP_5)^0_{33} \oplus \Big(\bigoplus_{1\leq i\leq 4}(QP_5)^{>0}_{33}(\widetilde{\omega}_{(i)})\Big).\end{aligned}$$

We now compute the subspaces $(QP_5)^0_{33}$ and $(QP_5)^{>0}_{33}(\widetilde{\omega}_{(i)})$.

Computation of $(QP_5)^0_{33}$. From [37], we have

$$\dim(QP_5)^0_{33}(\widetilde{\omega}_{(i)}) = \sum_{3\leq s\leq 4}\binom{5}{s}\dim(QP_s)^{>0}_{33}(\widetilde{\omega}_{(i)}), \tag{3.1}$$

$$\dim(QP_5)^0_{33} = \sum_{3\leq s\leq 4}\binom{5}{s}\dim(QP_s)^{>0}_{33}. \tag{3.2}$$

Since $\mu(33) = 3$, $(QP_3)_{33}$ is isomorphic to $(QP_3)_{15}$. So $\dim(QP_3)^{>0}_{33} = \dim(QP_3)_{15} = 13$ (see Kameko [11]). Following Sum [35], $\dim(QP_4)^{>0}_{33} = 84$, and so, by (3.3), we get

$$\dim(QP_3)^0_{33} = 13\binom{5}{3} + 84\binom{5}{4} = 550.$$

According to Sum [35], $(QP_4)_{33} \cong (QP_4)_{33}(\widetilde{\omega}_{(1)}) \bigoplus (QP_4)_{33}(\widetilde{\omega}_{(3)})$, which shows that $(QP_5)^0_{33}(\widetilde{\omega}_{(2)}) = (QP_5)^0_{33}(\widetilde{\omega}_{(4)}) = 0$, and

$$(QP_5)^0_{33} \cong (QP_5)^0_{33}(\widetilde{\omega}_{(1)}) \bigoplus (QP_5)^0_{33}(\widetilde{\omega}_{(3)}).$$

Moreover, by Kameko [11], we have $\dim(QP_3)^{>0}_{33}(\widetilde{\omega}_{(1)}) = 7$ and $\dim(QP_3)^{>0}_{33}(\widetilde{\omega}_{(3)}) = 6$. By Sum [35], we have $\dim(QP_4)^{>0}_{33}(\widetilde{\omega}_{(1)}) = 17$ and $\dim(QP_4)^{>0}_{33}(\widetilde{\omega}_{(3)}) = 67$. So, using (3.1), we derive:

$$\begin{aligned}\dim(QP_5)^0_{33}(\widetilde{\omega}_{(1)}) &= 7\tbinom{5}{3} + 17\tbinom{5}{4} = 155,\\ \dim(QP_5)^0_{33}(\widetilde{\omega}_{(3)}) &= 6\tbinom{5}{3} + 67\tbinom{5}{4} = 395.\end{aligned}$$

Computation of $(QP_5)^{>0}_{33}(\widetilde{\omega}_{(i)})$. We obtain

Theorem 3.2. *The following statements are true:*

$$\dim(QP_5)^{>0}_{33}(\widetilde{\omega}_{(i)}) = \begin{cases} 31 & \text{if } i = 1,\\ 0 & \text{if } i = 2,\ 4,\\ 421 & \text{if } i = 3.\end{cases}$$

Proof. We prove the theorem for the cases where $i = 1, 2, 4$. The remaining case can be obtained through analogous calculations.

• *Case* $i = 1$. According to Moetele and Mothebe [18, Proposition 1], if t is an admissible monomial in the $\mathbb{A}$-module P_n, then t must take the form $t = u_1^{2^{a_1}-1}u_2^{a_2}u_3^{a_3}\ldots u_n^{a_n}$. By this, we only need to consider the monomials of the form $t := u_iu_ju_k\underline{t}^2 = \prod_{1\leq l\leq 5}u_l^{\alpha_l}$, with $\underline{t} \in \mathscr{C}^{\otimes 5}_{15}$, $1 \leq i < j < k \leq 5$, $\alpha_1 \in \{1, 3, 7, 15\}$, and $\prod_{2\leq l\leq 5}\alpha_l > 0$. A simple computation indicates that if $t \neq \mathrm{adm}_i$, $156 \leq i \leq 186$, where

$\mathrm{adm}_{156} = u_1u_2u_3u_4^2u_5^{28}$, $\mathrm{adm}_{157} = u_1u_2u_3^2u_4u_5^{28}$, $\mathrm{adm}_{158} = u_1u_2u_3^2u_4^{28}u_5$, $\mathrm{adm}_{159} = u_1u_2^2u_3u_4u_5^{28}$,
$\mathrm{adm}_{160} = u_1u_2^2u_3u_4^{28}u_5$, $\mathrm{adm}_{161} = u_1u_2^2u_3^{28}u_4u_5$, $\mathrm{adm}_{162} = u_1u_2u_3^2u_4^4u_5^{25}$, $\mathrm{adm}_{163} = u_1u_2^2u_3u_4^4u_5^{25}$,
$\mathrm{adm}_{164} = u_1u_2^2u_3^4u_4u_5^{25}$, $\mathrm{adm}_{165} = u_1u_2^2u_3^4u_4^{25}u_5$, $\mathrm{adm}_{166} = u_1u_2u_3^2u_4^5u_5^{24}$, $\mathrm{adm}_{167} = u_1u_2^2u_3u_4^5u_5^{24}$,
$\mathrm{adm}_{168} = u_1u_2^2u_3^5u_4u_5^{24}$, $\mathrm{adm}_{169} = u_1u_2^2u_3^5u_4^{24}u_5$, $\mathrm{adm}_{170} = u_1u_2u_3^3u_4^4u_5^{24}$, $\mathrm{adm}_{171} = u_1u_2^3u_3u_4^4u_5^{24}$,
$\mathrm{adm}_{172} = u_1^3u_2u_3u_4^4u_5^{24}$, $\mathrm{adm}_{173} = u_1u_2^3u_3^4u_4u_5^{24}$, $\mathrm{adm}_{174} = u_1^3u_2u_3^4u_4u_5^{24}$, $\mathrm{adm}_{175} = u_1u_2^3u_3^4u_4^{24}u_5$,
$\mathrm{adm}_{176} = u_1^3u_2u_3^4u_4^{24}u_5$, $\mathrm{adm}_{177} = u_1u_2^2u_3^4u_4^9u_5^{17}$, $\mathrm{adm}_{178} = u_1u_2^2u_3^5u_4^8u_5^{17}$, $\mathrm{adm}_{179} = u_1u_2^3u_3^4u_4^8u_5^{17}$,
$\mathrm{adm}_{180} = u_1^3u_2u_3^4u_4^8u_5^{17}$, $\mathrm{adm}_{181} = u_1u_2^2u_3^5u_4^9u_5^{16}$, $\mathrm{adm}_{182} = u_1u_2^3u_3^4u_4^9u_5^{16}$, $\mathrm{adm}_{183} = u_1^3u_2u_3^4u_4^9u_5^{16}$,
$\mathrm{adm}_{184} = u_1u_2^3u_3^5u_4^8u_5^{16}$, $\mathrm{adm}_{185} = u_1^3u_2u_3^5u_4^8u_5^{16}$, $\mathrm{adm}_{186} = u_1^3u_2^5u_3u_4^8u_5^{16}$,

then either $t \in \{u_1^3u_2^5u_3^8u_4u_5^{16}, u_1^3u_2^5u_3^8u_4^{16}u_5\}$, or t is of the form $t'z^{16}$, where t' is one of the following inadmissible monomials: $u_1^3u_2^4u_3u_4^9$, $u_1^3u_2^4u_3^9u_4$, $u_1^3u_2^4u_3^9u_5$, $u_1^3u_2^4u_3u_4u_5^8$, $u_1^3u_2^4u_3u_4^8u_5$, $u_1^3u_2^4u_3^8u_4u_5$. Applying Cartan's formula, we obtain the following equality:

$$\begin{aligned} u_1^3u_2^5u_3^8u_4u_5^{16} &= Sq^4\Big(u_1^3u_2^3u_3^8u_4u_5^{14} + u_1^3u_2^3u_3^4u_4u_5^{18} + u_1^3u_2^3u_3^2u_4u_5^{20} + u_1^3u_2^3u_3u_4^2u_5^{20} \\ &\quad + u_1^3u_2^3u_3u_4^4u_5^{18} + u_1^3u_2^3u_3u_4^8u_5^{14}\Big) + Sq^2\Big(u_1^5u_2^3u_3^8u_4u_5^{14} + u_1^5u_2^3u_3^4u_4u_5^{18} \\ &\quad + u_1^5u_2^3u_3^2u_4u_5^{20} + u_1^5u_2^3u_3u_4^8u_5^{14} + u_1^5u_2^3u_3u_4^4u_5^{18} + u_1^5u_2^3u_3u_4^2u_5^{20} + u_1^2u_2^3u_3u_4u_5^{24}\Big) \\ &\quad + Sq^1\Big(u_1^3u_2^3u_3u_4u_5^{24}\Big) + u_1^3u_2^5u_3u_4^8u_5^{16} + u_1^3u_2^4u_3u_4u_5^{24} + u_1^2u_2^5u_3u_4u_5^{24} + \sum X, \end{aligned}$$

where $\omega(X) < \omega(u_1^3u_2^5u_3^8u_4u_5^{16}) = \widetilde{\omega}_{(1)}$. Thus the monomial $u_1^3u_2^5u_3^8u_4u_5^{16}$ is inadmissible, and so is $u_1^3u_2^5u_3^8u_4^{16}u_5$. On the other side, we observe that $\omega_4(t') = 1 \neq 0$, and $\omega_j(t') = 0$ for all $j > 4$. So, following Theorem 2.5(ii), $t = t'z^{16}$ is inadmissible. Therefore, it follows that

$$(QP_5)_{33}^{\geq 0}(\widetilde{\omega}_{(1)}) = \langle \{[\mathrm{adm}_i]_{\omega_{(1)}} = [\mathrm{adm}_i] |\, 156 \leq i \leq 186\}\rangle.$$

Let us demonstrate that the set $\Big\{[\mathrm{adm}_i] \ | \ 156 \leq i \leq 186\Big\}$ is linearly dependent in $(QP_5)_{33}^{\geq 0}(\widetilde{\omega}_{(1)})$. To do so, consider the set

$$\mathcal{N}_5 := \{(l, \mathscr{L}) \mid \mathscr{L} = (l_1, l_2, \ldots, l_r), 1 \leqslant l < l_1 < l_2 < \ldots < l_r \leq 5,\ 0 \leq r \leq 4\},$$

where by convention, $\mathscr{L} = \emptyset$, if $r = 0$. Denote by $r = \ell(\mathscr{L})$ the length of $\mathscr{L}$. For a pair $(l, \mathscr{L}) \in \mathcal{N}_5$ with $r = \ell(\mathscr{L}) = 1$, $\mathscr{L} = (l')$, and $1 \leq l < l' \leq 5$. We define an $\mathbb{A}$-homomorphism $\mathsf{p}_{(l,l')}: P_5 \to P_4$, as follows:

$$\mathsf{p}_{(l,l')}(u_j) = \begin{cases} u_j & \text{if } \ 1 \leq j \leq l-1, \\ u_{l'-1} & \text{if } j = l, \\ u_{j-1} & \text{if } l+1 \leq j \leq 5. \end{cases}$$

We assume that there is a linear relation $S = \sum_{156 \leq i \leq 186} \gamma_i \mathrm{adm}_i \sim 0$, in which $\gamma_i \in \mathbb{Z}/2$ for every i. According to Sum [35], $(QP_{33}^{\otimes 4})^{>0}(\widetilde{\omega}_{(1)}) = \langle \{[\mathrm{Adm}_j] :\ 1 \leq j \leq 17\}\rangle$, where the admissible monomials Adm_j are determined as follows:

$$\begin{array}{llll} \mathrm{Adm}_1 = u_1u_2^2u_3^5u_4^{25}, & \mathrm{Adm}_2 = u_1u_2^3u_3^4u_4^{25}, & \mathrm{Adm}_3 = u_1^3u_2u_3^4u_4^{25}, & \mathrm{Adm}_4 = u_1u_2^3u_3^5u_4^{24}, \\ \mathrm{Adm}_5 = u_1^3u_2u_3^5u_4^{24}, & \mathrm{Adm}_6 = u_1^3u_2^5u_3u_4^{24}, & \mathrm{Adm}_7 = u_1u_2u_3^2u_4^{29}, & \mathrm{Adm}_8 = u_1u_2^2u_3u_4^{29}, \\ \mathrm{Adm}_9 = u_1u_2^2u_3^{29}u_4, & \mathrm{Adm}_{10} = u_1u_2u_3u_4^{30}, & \mathrm{Adm}_{11} = u_1u_2u_3^{30}u_4, & \mathrm{Adm}_{12} = u_1u_2^{30}u_3u_4, \\ \mathrm{Adm}_{13} = u_1u_2u_3^3u_4^{28}, & \mathrm{Adm}_{14} = u_1u_2^3u_3u_4^{28}, & \mathrm{Adm}_{15} = u_1u_2^3u_3^{28}u_4, & \mathrm{Adm}_{16} = u_1^3u_2u_3u_4^{28}, \\ \mathrm{Adm}_{17} = u_1^3u_2u_3^{28}u_4. & & & \end{array}$$

From this result, we determine explicitly $\mathsf{p}_{(l,l')}(S)$ in terms Adm_j modulo $(\overline{\mathbb{A}}P_4)$. Then, a direct computation using the relations $\mathsf{p}_{(l,l')}(S) \sim 0$, for $1 \leq l \leq 3$, and $2 \leq l' \leq 4$, one gets $\gamma_i = 0$, for all i, $156 \leq i \leq 186$.

• *Case* $i = 2$. Let us recall that by Remark 3.1, the monomials of degree 33 has form $u_iu_ju_k\underline{t}^2$ with $1 \leq i < j < k \leq 5$ and $\underline{t} \in \mathscr{C}_{15}^{\otimes 5}$. Following [34], $|\mathscr{C}_{15}^{\otimes 5}| = 432$. Thanks to this result, we find that if $t \in (P_5)_{33}(\widetilde{\omega}_{(2)})$, then t is one of the following monomials: $u_1^3u_2^4u_3^8u_4^9u_5^9$, $u_1^3u_2^4u_3^9u_4^8u_5^9$, $u_1^3u_2^4u_3^9u_4^9u_5^8$, $u_1^3u_2^5u_3^8u_4^8u_5^9$, $u_1^3u_2^5u_3^8u_4^9u_5^8$, $u_1^3u_2^5u_3^9u_4^8u_5^8$. Let us consider the monomials

$u_1^3u_2^4u_3^8u_4^9u_5^9$ and $u_1^3u_2^5u_3^8u_4^8u_5^9$. Applying the Cartan formula, we get

$$\begin{aligned}
u_1^3u_2^4u_3^8u_4^9u_5^9 &= Sq^1(u_1^3u_2u_3^{10}u_4^9u_5^9)\\
&\quad + Sq^2(u_1^2u_2u_3^{10}u_4^9u_5^9 + u_1^5u_2u_3^6u_4^9u_5^{10} + u_1^5u_2u_3^6u_4^{10}u_5^9 + u_1^5u_2^2u_3^6u_4^9u_5^9)\\
&\quad + Sq^4(u_1^3u_2^2u_3^6u_4^9u_5^9 + u_1^3u_2u_3^6u_4^9u_5^{10} + u_1^3u_2u_3^6u_4^{10}u_5^9)\\
&\quad + u_1^3u_2u_3^8u_4^9u_5^{12} + u_1^3u_2u_3^8u_4^{12}u_5^9 + u_1^2u_2u_3^{12}u_4^9u_5^9 \mod ((P_5)_{33}(< \widetilde{\omega}_{(2)})),\\
u_1^3u_2^5u_3^8u_4^8u_5^9 &= Sq^1(u_1^3u_2^3u_3u_4^8u_5^{17}) + Sq^2(u_1^5u_2^3u_3u_4^8u_5^{14} + u_1^5u_2^3u_3^2u_4^4u_5^{17})\\
&\quad + Sq^4(u_1^3u_2^3u_3u_4^8u_5^{14} + u_1^3u_2^2u_3^2u_4^4u_5^{17} + u_1^3u_2^9u_3^4u_4^4u_5^9)\\
&\quad + Sq^8(u_1^3u_2^5u_3^4u_4^4u_5^9) \mod ((P_5)_{33}(< \widetilde{\omega}_{(2)})).
\end{aligned}$$

These calculations show that $u_1^3u_2^4u_3^8u_4^9u_5^9$ is strictly inadmissible and that $u_1^3u_2^5u_3^8u_4^8u_5^9$ is $\widetilde{\omega}_{(2)}$-hit. Similarly, the monomials $u_1^3u_2^4u_3^9u_4^8u_5^9$, $u_1^3u_2^4u_3^9u_4^9u_5^8$ are strictly inadmissible and $u_1^3u_2^5u_3^8u_4^9u_5^8$, $u_1^3u_2^5u_3^9u_4^8u_5^8$ are $\widetilde{\omega}_{(2)}$-hit. Thus, $(QP_5)_{33}^{>0}(\widetilde{\omega}_{(2)})$ is trivial.

- *Case* $i = 4$. Let t be a monomial in $(P_5)_{33}(\widetilde{\omega}_{(4)})$. Since $t = u_iu_ju_k\underline{t}^2$ where $1 < i < j < k \leq 5$ and $\underline{t} \in \mathscr{C}_{15}^{\otimes 5}$, from a result in Sum [34], we find that t is the permutation of each of the following monomial:

$$\begin{array}{lllll}
u_1^2u_2^4u_3^5u_4^7u_5^{15}, & u_1^2u_2^4u_3^7u_4^7u_5^{13}, & u_1^2u_2^5u_3^5u_4^6u_5^{15}, & u_1^2u_2^5u_3^5u_4^7u_5^{14}, & u_1^2u_2^5u_3^6u_4^7u_5^{13},\\
u_1^2u_2^5u_3^7u_4^7u_5^{12}, & u_1^3u_2^4u_3^4u_4^7u_5^{15}, & u_1^3u_2^4u_3^5u_4^6u_5^{15}, & u_1^3u_2^4u_3^5u_4^7u_5^{14}, & u_1^3u_2^4u_3^6u_4^7u_5^{13},\\
u_1^3u_2^4u_3^7u_4^7u_5^{12}, & u_1^3u_2^5u_3^5u_4^6u_5^{14}, & u_1^3u_2^5u_3^6u_4^6u_5^{13}, & u_1^3u_2^5u_3^6u_4^7u_5^{12}, & u_1^4u_2^4u_3^7u_4^7u_5^{11},\\
u_1^4u_2^5u_3^6u_4^7u_5^{11}, & u_1^4u_2^5u_3^7u_4^7u_5^{10}, & u_1^5u_2^5u_3^6u_4^7u_5^{10}. & &
\end{array}$$

It is straightforward to check that these monomials are inadmissible. For instance, using the Cartan formula, we find that

$$\begin{aligned}
u_1^3u_2^4u_3^4u_4^7u_5^{15} = Sq^1(u_1^3u_2u_3^2u_4^{11}u_5^{15} + u_1^3u_2u_3^2u_4^7u_5^{19}) + Sq^2(u_1^5u_2^2u_3^2u_4^7u_5^{15})\\
+ Sq^4(u_1^3u_2^2u_3^2u_4^7u_5^{15}) + \text{smaller monomials}.
\end{aligned}$$

Therefore $(QP_5)_{33}^{>0}(\widetilde{\omega}_{(4)})$ is trivial. The theorem is completely proved. □

The computational outcomes imply the following corollary.

Corollary 3.3. *There exist exactly* 1322 *admissible monomials of degree* 33 *in the* $\mathbb{A}$*-module* P_5*. Consequently,* $(QP_5)_{33}$ *is* 1322*-dimensional.*

Applying the above data, we can explicitly determine the GL_5-invariants $((QP_5)_{33})^{GL_5}$. By using Theorem 3.2, we obtain an isomorphism that can be used to compute the desired this invariant space:

$$(QP_5)_{33} \cong \mathrm{Ker}((\widetilde{Sq^0_*})_{33}) \bigoplus (QP_5)_{14} \cong (QP_5)_{33}(\widetilde{\omega}_{(1)}) \bigoplus (QP_5)_{33}(\widetilde{\omega}_{(3)}) \bigoplus (QP_5)_{14}.$$

This shows that

$$\dim((QP_5)_{33})^{GL_5} \leq \dim((QP_5)_{33}(\widetilde{\omega}_{(1)}))^{GL_5} + \dim((QP_5)_{33}(\widetilde{\omega}_{(3)}))^{GL_5} + \dim((QP_5)_{14})^{GL_5}. \tag{3.3}$$

Thus, to obtain the dimension of $((QP_5)_{33})^{GL_5}$, we need to determine the GL_5-invariants $((QP_5)_{33}(\widetilde{\omega}_{(i)}))^{GL_5}$ for $i = 1, 3$ as well as $((QP_5)_{14})^{GL_5}$. The description of $((QP_5)_{14})^{GL_5}$ can be found in our previous work [28], where the following theorem is stated.

Theorem 3.4. *The space* $((QP_5)_{14})^{GL_5}$ *has dimension one. Furthermore,*
$$((QP_5)_{14})^{GL_5} = \langle[\zeta(u_1,\ldots,u_5)]\rangle,$$
wherein
$$\begin{aligned}\zeta(u_1,\ldots,u_5) &= u_2u_3u_4^6u_5^6 + u_1u_3u_4^6u_5^6 + u_1u_2u_4^6u_5^6 + u_1u_2u_3^6u_5^6 + u_1u_2u_3^6u_4^6\\ &+u_2^3u_3^3u_4^4u_5^4 + u_1^3u_3^3u_4^4u_5^4 + u_1^3u_2^3u_4^4u_5^4 + u_1^3u_2^3u_3^4u_5^4 + u_1^3u_2^3u_3^4u_4^4\\ &+u_1u_2u_3^2u_4^4u_5^6 + u_1u_2u_3^2u_4^6u_5^4 + u_1u_2u_3^6u_4^2u_5^4 + u_1u_2^2u_3u_4^4u_5^6\\ &+u_1u_2^2u_3^4u_4u_5^6 + u_1u_2^6u_3u_4^2u_5^4 + u_1u_2^2u_3^3u_4^4u_5^4 + u_1u_2^3u_3^2u_4^4u_5^4\\ &+u_1^3u_2u_3^2u_4^4u_5^4 + u_1^3u_2u_3^4u_4^2u_5^4 + u_1u_2^2u_3^4u_4^3u_5^4 + u_1^3u_2^4u_3u_4^2u_5^4.\end{aligned}$$

Theorem 3.5. *The invariants* $((QP_5)_{33}(\widetilde{\omega}_{(1)}))^{GL_5}$ *and* $((QP_5)_{33}(\widetilde{\omega}_{(3)}))^{GL_5}$ *vanish.*

Proof. Before we proceed, we must define the action of the general linear group GL_5 on the polynomial algebra P_5. It is well-established that $GL_5 = GL(5,\mathbb{Z}/2)$ is generated by a specific set of operators. We define these generators as the set of $\mathbb{A}$-algebra homomorphisms $\{\rho_1,\ldots,\rho_5\}$, where their actions on the generators u_i of P_5 are given as follows:

(1) **The Adjacent Transpositions (ρ_j for $1 \le j \le 4$):** These 4 operators generate the symmetric group $\Sigma_5 \subset GL_5$. Each ρ_j swaps the adjacent variables u_j and u_{j+1} and fixes all others:
$$\rho_j(u_i) = \begin{cases} u_{j+1} & \text{if } i = j,\\ u_j & \text{if } i = j+1.\\ u_i & \text{otherwise.}\end{cases}$$
This set generates Σ_5 because any permutation can be decomposed into a product of adjacent transpositions.

(2) **The Transvection (ρ_5):** The final generator needed to obtain the full GL_5 group is the transvection ρ_5. This operator adds the variable u_4 to u_5 and fixes all other variables:
$$\rho_5(u_i) = \begin{cases} u_5 + u_4 & \text{if } i = 5,\\ u_i & \text{if } i < 5.\end{cases}$$

This set $\{\rho_1,\ldots,\rho_5\}$ successfully generates GL_5. The set $\{\rho_1,\ldots,\rho_4\}$ generates all permutation matrices (Type I elementary operations). The operator ρ_5 is an elementary transvection (a Type III operation, $C_5 \to C_5 + C_4$). All other transvections $C_i \to C_i + C_j$ can be generated by conjugating ρ_5 with permutation matrices, which are themselves generated by $\{\rho_1,\ldots,\rho_4\}$.

Let $f = u_1^{a_1}u_2^{a_2}\ldots u_5^{a_5}$ be a monomial in $(P_5)_{33}$. The weight vector $\omega(f)$ is invariant under the permutation of the generators $u_j,\ j = 1,2,\ldots,5$; hence $(QP_5)_{33}(\widetilde{\omega}_{(i)})$ also has a Σ_5-module structure. The linear maps ρ_j induce homomorphisms of $\mathbb{A}$-algebras, also denoted by $\rho_j : P_5 \longrightarrow P_5$.

Therefore, a class $[f]_{\widetilde{\omega}_{(i)}} \in (QP_5)_{33}(\widetilde{\omega}_{(i)})$ is a GL_5**-invariant** if and only if it is invariant under all generators:
$$\rho_j(f) \sim_{\widetilde{\omega}_{(i)}} f \quad \text{for all } 1 \le j \le 5.$$
If this condition holds only for $1 \le j \le 4$, then $[f]_{\widetilde{\omega}_{(i)}}$ is an Σ_5**-invariant**.

In what follows, let $f_1, f_2, \ldots, f_s$ be monomials in $(P_5)_{33}(\widetilde{\omega}_{(i)})$. For a subgroup G of GL_5 denote by $G(f_1; f_2; \ldots, f_s)$ the G-submodule of $(QP_5)_{33}(\widetilde{\omega}_{(i)})$ generated by the set $\{[f_i]_{\widetilde{\omega}_{(i)}} : 1 \le i \le s\}$.

Computation of the invariant space $((QP_5)_{33}(\widetilde{\omega}_{(1)}))^{GL_5}$. We have

$$(QP_5)_{33}(\widetilde{\omega}_{(1)}) \cong (QP_5)^0_{33}(\widetilde{\omega}_{(1)}) \bigoplus (QP_5)^{>0}_{33}(\widetilde{\omega}_{(1)}).$$

By the above calculations and Theorem 3.2,

$$\dim(QP_5)^0_{33}(\widetilde{\omega}_{(1)}) = 155, \quad \text{and} \quad \dim(QP_5)^{>0}_{33}(\widetilde{\omega}_{(1)}) = 31.$$

We can easily observe that the basis for $(QP_5)^0_{33}(\widetilde{\omega}_{(1)})$ is given by the set of equivalence classes represented by the following admissible monomials:

$\mathrm{adm}_1 = u_3u_4u_5^{31}$,	$\mathrm{adm}_2 = u_3u_4^{31}u_5$,	$\mathrm{adm}_3 = u_3^{31}u_4u_5$,	$\mathrm{adm}_4 = u_2u_4u_5^{31}$,
$\mathrm{adm}_5 = u_2u_4^{31}u_5$,	$\mathrm{adm}_6 = u_2u_3u_5^{31}$,	$\mathrm{adm}_7 = u_2u_3u_4^{31}$,	$\mathrm{adm}_8 = u_2u_3^{31}u_5$,
$\mathrm{adm}_9 = u_2u_3^{31}u_4$,	$\mathrm{adm}_{10} = u_2^{31}u_4u_5$,	$\mathrm{adm}_{11} = u_2^{31}u_3u_5$,	$\mathrm{adm}_{12} = u_2^{31}u_3u_4$,
$\mathrm{adm}_{13} = u_1u_4u_5^{31}$,	$\mathrm{adm}_{14} = u_1u_4^{31}u_5$,	$\mathrm{adm}_{15} = u_1u_3u_5^{31}$,	$\mathrm{adm}_{16} = u_1u_3u_4^{31}$,
$\mathrm{adm}_{17} = u_1u_3^{31}u_5$,	$\mathrm{adm}_{18} = u_1u_3^{31}u_4$,	$\mathrm{adm}_{19} = u_1^{31}u_4u_5$,	$\mathrm{adm}_{20} = u_1^{31}u_3u_5$,
$\mathrm{adm}_{21} = u_1^{31}u_3u_4$,	$\mathrm{adm}_{22} = u_1u_2u_5^{31}$,	$\mathrm{adm}_{23} = u_1u_2u_4^{31}$,	$\mathrm{adm}_{24} = u_1u_2^{31}u_5$,
$\mathrm{adm}_{25} = u_1u_2^{31}u_4$,	$\mathrm{adm}_{26} = u_1^{31}u_2u_5$,	$\mathrm{adm}_{27} = u_1^{31}u_2u_4$,	$\mathrm{adm}_{28} = u_1u_2u_3^{31}$,
$\mathrm{adm}_{29} = u_1u_2^{31}u_3$,	$\mathrm{adm}_{30} = u_1^{31}u_2u_3$,	$\mathrm{adm}_{31} = u_3u_4^3u_5^{29}$,	$\mathrm{adm}_{32} = u_3^3u_4u_5^{29}$,
$\mathrm{adm}_{33} = u_3^3u_4^{29}u_5$,	$\mathrm{adm}_{34} = u_2u_4^3u_5^{29}$,	$\mathrm{adm}_{35} = u_2u_3^3u_5^{29}$,	$\mathrm{adm}_{36} = u_2u_3^3u_4^{29}$,
$\mathrm{adm}_{37} = u_2^3u_4u_5^{29}$,	$\mathrm{adm}_{38} = u_2^3u_4^{29}u_5$,	$\mathrm{adm}_{39} = u_2^3u_3u_5^{29}$,	$\mathrm{adm}_{40} = u_2^3u_3u_4^{29}$,
$\mathrm{adm}_{41} = u_2^3u_3^{29}u_5$,	$\mathrm{adm}_{42} = u_2^3u_3^{29}u_4$,	$\mathrm{adm}_{43} = u_1u_4^3u_5^{29}$,	$\mathrm{adm}_{44} = u_1u_3^3u_5^{29}$,
$\mathrm{adm}_{45} = u_1u_3^3u_4^{29}$,	$\mathrm{adm}_{46} = u_1^3u_4u_5^{29}$,	$\mathrm{adm}_{47} = u_1^3u_4^{29}u_5$,	$\mathrm{adm}_{48} = u_1^3u_3u_5^{29}$,
$\mathrm{adm}_{49} = u_1^3u_3u_4^{29}$,	$\mathrm{adm}_{50} = u_1^3u_3^{29}u_5$,	$\mathrm{adm}_{51} = u_1^3u_3^{29}u_4$,	$\mathrm{adm}_{52} = u_1u_2^3u_5^{29}$,
$\mathrm{adm}_{53} = u_1u_2^3u_4^{29}$,	$\mathrm{adm}_{54} = u_1^3u_2u_5^{29}$,	$\mathrm{adm}_{55} = u_1^3u_2u_4^{29}$,	$\mathrm{adm}_{56} = u_1^3u_2^{29}u_5$,
$\mathrm{adm}_{57} = u_1^3u_2^{29}u_4$,	$\mathrm{adm}_{58} = u_1u_2^3u_3^{29}$,	$\mathrm{adm}_{59} = u_1^3u_2u_3^{29}$,	$\mathrm{adm}_{60} = u_1^3u_2^{29}u_3$,
$\mathrm{adm}_{61} = u_3^3u_4^5u_5^{25}$,	$\mathrm{adm}_{62} = u_2^3u_4^5u_5^{25}$,	$\mathrm{adm}_{63} = u_2^3u_3^5u_5^{25}$,	$\mathrm{adm}_{64} = u_2^3u_3^5u_4^{25}$,
$\mathrm{adm}_{65} = u_1^3u_4^5u_5^{25}$,	$\mathrm{adm}_{66} = u_1^3u_3^5u_5^{25}$,	$\mathrm{adm}_{67} = u_1^3u_3^5u_4^{25}$,	$\mathrm{adm}_{68} = u_1^3u_2^5u_5^{25}$,
$\mathrm{adm}_{69} = u_1^3u_2^5u_4^{25}$,	$\mathrm{adm}_{70} = u_1^3u_2^5u_3^{25}$,	$\mathrm{adm}_{71} = u_2u_3u_4u_5^{30}$,	$\mathrm{adm}_{72} = u_1u_3u_4u_5^{30}$,
$\mathrm{adm}_{73} = u_1u_3^{30}u_4u_5$,	$\mathrm{adm}_{74} = u_1u_2^{30}u_4u_5$,	$\mathrm{adm}_{75} = u_1u_2u_3u_4^{30}$,	$\mathrm{adm}_{76} = u_1u_2u_3u_5^{30}$,
$\mathrm{adm}_{77} = u_1u_2^{30}u_3u_5$,	$\mathrm{adm}_{78} = u_2u_3^2u_4^{29}u_5$,	$\mathrm{adm}_{79} = u_1u_3^2u_4u_5^{29}$,	$\mathrm{adm}_{80} = u_1u_2^2u_4u_5^{29}$,
$\mathrm{adm}_{81} = u_1u_2^2u_3u_4^{29}$,	$\mathrm{adm}_{82} = u_1u_2^2u_3^{29}u_4$,	$\mathrm{adm}_{83} = u_1u_2^2u_3^{29}u_5$,	$\mathrm{adm}_{84} = u_2u_3^3u_4^{28}u_5$,
$\mathrm{adm}_{85} = u_1u_3^3u_4^{28}u_5$,	$\mathrm{adm}_{86} = u_1^3u_3u_4^{28}u_5$,	$\mathrm{adm}_{87} = u_1^3u_2u_4^{28}u_5$,	$\mathrm{adm}_{88} = u_1u_2^3u_3^{28}u_4$,
$\mathrm{adm}_{89} = u_1u_2^3u_3^{28}u_5$,	$\mathrm{adm}_{90} = u_1^3u_2u_3^{28}u_5$,	$\mathrm{adm}_{91} = u_1u_2^2u_4^5u_5^{25}$,	$\mathrm{adm}_{92} = u_1u_2^2u_3^5u_5^{25}$,
$\mathrm{adm}_{93} = u_1u_3^3u_4^4u_5^{25}$,	$\mathrm{adm}_{94} = u_1u_2^3u_4^4u_5^{25}$,	$\mathrm{adm}_{95} = u_1^3u_2u_3^4u_4^{25}$,	$\mathrm{adm}_{96} = u_1^3u_2u_3^4u_5^{25}$,
$\mathrm{adm}_{97} = u_2^3u_3^5u_4u_5^{24}$,	$\mathrm{adm}_{98} = u_1^3u_3u_4^5u_5^{24}$,	$\mathrm{adm}_{99} = u_1^3u_2u_4^5u_5^{24}$,	$\mathrm{adm}_{100} = u_1u_2^3u_3^5u_4^{24}$,
$\mathrm{adm}_{101} = u_1^3u_2u_3^5u_4^{24}$,	$\mathrm{adm}_{102} = u_1^3u_2^5u_3u_4^{24}$,	$\mathrm{adm}_{103} = u_1u_2^3u_3^5u_5^{24}$,	$\mathrm{adm}_{104} = u_1^3u_2u_3^5u_5^{24}$,
$\mathrm{adm}_{105} = u_1^3u_2^5u_3u_5^{24}$,	$\mathrm{adm}_{106} = u_2u_3u_4^{30}u_5$,	$\mathrm{adm}_{107} = u_1u_3u_4^{30}u_5$,	$\mathrm{adm}_{108} = u_1u_2u_4^{30}u_5$,
$\mathrm{adm}_{109} = u_1u_2u_3^{30}u_4$,	$\mathrm{adm}_{110} = u_1u_2u_3^{30}u_5$,	$\mathrm{adm}_{111} = u_2u_3u_4^2u_5^{29}$,	$\mathrm{adm}_{112} = u_1u_3u_4^2u_5^{29}$,
$\mathrm{adm}_{113} = u_1u_2u_4^2u_5^{29}$,	$\mathrm{adm}_{114} = u_1u_2u_3^2u_4^{29}$,	$\mathrm{adm}_{115} = u_1u_2u_3^2u_5^{29}$,	$\mathrm{adm}_{116} = u_2u_3u_4^3u_5^{28}$,
$\mathrm{adm}_{117} = u_2u_3^3u_4u_5^{28}$,	$\mathrm{adm}_{118} = u_2^3u_3u_4u_5^{28}$,	$\mathrm{adm}_{119} = u_1u_3u_4^3u_5^{28}$,	$\mathrm{adm}_{120} = u_1u_3^3u_4u_5^{28}$,
$\mathrm{adm}_{121} = u_1^3u_3u_4u_5^{28}$,	$\mathrm{adm}_{122} = u_1u_2u_4^3u_5^{28}$,	$\mathrm{adm}_{123} = u_1u_2^3u_4u_5^{28}$,	$\mathrm{adm}_{124} = u_1^3u_2u_4u_5^{28}$,
$\mathrm{adm}_{125} = u_1u_2u_3^3u_4^{28}$,	$\mathrm{adm}_{126} = u_1u_2^3u_3u_4^{28}$,	$\mathrm{adm}_{127} = u_1^3u_2u_3u_4^{28}$,	$\mathrm{adm}_{128} = u_1u_2u_3^3u_5^{28}$,
$\mathrm{adm}_{129} = u_1u_2^3u_3u_5^{28}$,	$\mathrm{adm}_{130} = u_1^3u_2u_3u_5^{28}$,	$\mathrm{adm}_{131} = u_2u_3^{30}u_4u_5$,	$\mathrm{adm}_{132} = u_1u_2u_4u_5^{30}$,
$\mathrm{adm}_{133} = u_1u_2^{30}u_3u_4$,	$\mathrm{adm}_{134} = u_2u_3^2u_4u_5^{29}$,	$\mathrm{adm}_{135} = u_1u_3^2u_4^{29}u_5$,	$\mathrm{adm}_{136} = u_1u_2^2u_4^{29}u_5$,
$\mathrm{adm}_{137} = u_1u_2^2u_3u_5^{29}$,	$\mathrm{adm}_{138} = u_2^3u_3u_4^{28}u_5$,	$\mathrm{adm}_{139} = u_1u_2^3u_4^{28}u_5$,	$\mathrm{adm}_{140} = u_1^3u_2u_3^{28}u_4$,
$\mathrm{adm}_{141} = u_2u_3^2u_4^5u_5^{25}$,	$\mathrm{adm}_{142} = u_1u_3^2u_4^5u_5^{25}$,	$\mathrm{adm}_{143} = u_1u_2^2u_3^5u_4^{25}$,	$\mathrm{adm}_{144} = u_2u_3^3u_4^4u_5^{25}$,
$\mathrm{adm}_{145} = u_2^3u_3u_4^4u_5^{25}$,	$\mathrm{adm}_{146} = u_1^3u_3u_4^4u_5^{25}$,	$\mathrm{adm}_{147} = u_1^3u_2u_4^4u_5^{25}$,	$\mathrm{adm}_{148} = u_1u_2^3u_3^4u_4^{25}$,
$\mathrm{adm}_{149} = u_1u_2^3u_3^4u_5^{25}$,	$\mathrm{adm}_{150} = u_2u_3^3u_4^5u_5^{24}$,	$\mathrm{adm}_{151} = u_2^3u_3u_4^5u_5^{24}$,	$\mathrm{adm}_{152} = u_1u_3^3u_4^5u_5^{24}$,

$\mathrm{adm}_{153} = u_1^3u_3^5u_4u_5^{24},\quad \mathrm{adm}_{154} = u_1u_2^3u_4^5u_5^{24},\quad \mathrm{adm}_{155} = u_1^3u_2^5u_4u_5^{24}.$

From this, an easy computation shows that

$$\begin{aligned}
\langle[\Sigma_5(\mathrm{adm}_1)]\rangle &= \langle\{[\mathrm{adm}_j:\ 1\leq j\leq 30]\}\rangle,\\
\langle[\Sigma_5(\mathrm{adm}_{31})]\rangle &= \langle\{[\mathrm{adm}_j:\ 31\leq j\leq 60]\}\rangle,\\
\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle &= \langle\{[\mathrm{adm}_j:\ 61\leq j\leq 70]\}\rangle,\\
\langle[\Sigma_5(\mathrm{adm}_{71},\mathrm{adm}_{78},\mathrm{adm}_{91})]\rangle &= \langle\{[\mathrm{adm}_j:\ 71\leq j\leq 155]\}\rangle.
\end{aligned}$$

So, we have an isomorphism

$$\begin{aligned}
(QP_5)^0_{33}(\widetilde{\omega}_{(1)}) &\cong \langle[\Sigma_5(\mathrm{adm}_1)]\rangle\bigoplus\langle[\Sigma_5(\mathrm{adm}_{31})]\rangle\bigoplus\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle\\
&\quad\bigoplus\langle[\Sigma_5(\mathrm{adm}_{71},\mathrm{adm}_{78},\mathrm{adm}_{91})]\rangle.
\end{aligned}$$

Lemma 3.6. *The following statement hold:*

$$\begin{aligned}
\langle[\Sigma_5(\mathrm{adm}_1)]\rangle^{\Sigma_5} &= \langle[\widehat{q}_1 := \textstyle\sum_{1\leq j\leq 30}]\rangle,\\
\langle[\Sigma_5(\mathrm{adm}_{31})]\rangle^{\Sigma_5} &= \langle[\widehat{q}_2 := \textstyle\sum_{31\leq j\leq 60}]\rangle,\\
\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle^{\Sigma_5} &= \langle[\widehat{q}_3 := \textstyle\sum_{61\leq j\leq 70}]\rangle,\\
\langle[\Sigma_5(\mathrm{adm}_{71},\mathrm{adm}_{78},\mathrm{adm}_{91})]\rangle^{\Sigma_5} &= \langle[\widehat{q}_4]\rangle,\ \textit{with}\ \widehat{q}_4 = \textstyle\sum_{71\leq j\leq 130}.
\end{aligned}$$

Consequently, $((QP_5)^0_{33}(\widetilde{\omega}_{(1)}))^{\Sigma_5} = \Big\langle\Big\{[\widehat{q}_1],\ [\widehat{q}_2],\ [\widehat{q}_3],\ [\widehat{q}_4]\Big\}\Big\rangle.$

Proof. For simplicity, we only provide a description for the Σ_5-invariant $\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle^{\Sigma_5}$, and similar calculations can be used to obtain descriptions for the others.

It is observed that $\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle$ is a $\mathbb{Z}/2$-vector space of dimension 10, with a basis consisting of all the classes represented by the 10 admissible monomials: adm_k, $61\leq k\leq 70$. Given an arbitrary $[f]$ in $\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle^{\Sigma_5}$, it should be noted that since $\widetilde{\omega}_{(1)}$ is a weight vector of a minimal spike, $[f]_{\widetilde{\omega}_{(1)}} = [f]$ for all $[f]\in\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle^{\Sigma_5}$. Then we can write $f\sim\sum_{61\leq k\leq 70}\gamma_k\mathrm{adm}_k$ wherein $\gamma_j\in\mathbb{Z}/2$ for every j. By applying the homomorphism $\rho_j: P_5\longrightarrow P_5$ for $1\leq j\leq 4$, to both sides of this equality, we obtain:

$$\begin{aligned}
\rho_1(f) &\sim \gamma_{61}\mathrm{adm}_{61}+\gamma_{62}\mathrm{adm}_{65}+\gamma_{63}\mathrm{adm}_{66}+\gamma_{64}\mathrm{adm}_{67}+\gamma_{65}\mathrm{adm}_{62}\\
&\quad+\gamma_{66}\mathrm{adm}_{63}+\gamma_{67}\mathrm{adm}_{64}+\gamma_{68}u_1^5u_2^3u_5^{25}+\gamma_{69}u_1^5u_2^3u_4^{25}+\gamma_{70}u_1^5u_2^3u_3^{25},\\
\rho_2(f) &\sim \gamma_{61}\mathrm{adm}_{62}+\gamma_{62}\mathrm{adm}_{61}+\gamma_{63}u_2^5u_3^3u_5^{25}+\gamma_{64}u_2^5u_3^3u_4^{25}+\gamma_{65}\mathrm{adm}_{65}\\
&\quad+\gamma_{66}\mathrm{adm}_{68}+\gamma_{67}\mathrm{adm}_{69}+\gamma_{68}\mathrm{adm}_{66}+\gamma_{69}\mathrm{adm}_{67}+\gamma_{70}u_1^3u_2^{25}u_3^5,\\
\rho_3(f) &\sim \gamma_{61}u_3^5u_4^3u_5^{25}+\gamma_{62}\mathrm{adm}_{63}+\gamma_{63}\mathrm{adm}_{62}+\gamma_{64}u_2^3u_3^{25}u_4^5+\gamma_{65}\mathrm{adm}_{66}\\
&\quad+\gamma_{66}\mathrm{adm}_{65}+\gamma_{67}u_1^3u_3^{25}u_4^5+\gamma_{68}\mathrm{adm}_{68}+\gamma_{69}\mathrm{adm}_{70}+\gamma_{70}\mathrm{adm}_{69},\\
\rho_4(f) &\sim \gamma_{61}u_3^3u_4^{25}u_5^5+\gamma_{62}u_2^3u_4^{25}u_5^5+\gamma_{63}\mathrm{adm}_{64}+\gamma_{64}\mathrm{adm}_{63}+\gamma_{65}u_1^3u_4^{25}u_5^5\\
&\quad+\gamma_{66}\mathrm{adm}_{67}+\gamma_{67}\mathrm{adm}_{66}+\gamma_{68}\mathrm{adm}_{69}+\gamma_{69}\mathrm{adm}_{68}+\gamma_{70}\mathrm{adm}_{70}.
\end{aligned}$$

Using the Cartan formula, we can express the following 11 inadmissible monomials in terms of the basis of $\langle[\Sigma_5(\mathrm{adm}_{61})]\rangle$:

$$\begin{gathered} u_1^5u_2^3u_5^{25},\ u_1^5u_2^3u_4^{25},\ u_1^5u_2^3u_3^{25},\ u_2^5u_3^3u_5^{25},\ u_2^5u_3^3u_4^{25},\\ u_3^5u_4^3u_5^{25},\ u_1^3u_2^{25}u_3^5,\ u_2^3u_3^{25}u_4^5,\ u_1^3u_3^{25}u_4^5,\ u_3^3u_4^{25}u_5^5,\\ u_1^3u_4^{25}u_5^5,\ u_2^3u_4^{25}u_5^5. \end{gathered}$$

Verily, a straightforward computation demonstrates:

$$\begin{aligned}
u_1^5u_2^3u_5^{25} &= Sq^2(u_1^3u_2^3u_5^{25}) + \mathrm{adm}_{68} \ \Rightarrow u_1^5u_2^3u_5^{25} \sim \mathrm{adm}_{68},\\
u_1^5u_2^3u_4^{25} &= Sq^2(u_1^3u_2^3u_4^{25}) + \mathrm{adm}_{69} \ \Rightarrow u_1^5u_2^3u_4^{25} \sim \mathrm{adm}_{69},\\
u_1^5u_2^3u_3^{25} &= Sq^2(u_1^3u_2^3u_3^{25}) + \mathrm{adm}_{70} \ \Rightarrow u_1^5u_2^3u_3^{25} \sim \mathrm{adm}_{70},\\
u_2^5u_3^3u_5^{25} &= Sq^2(u_2^3u_3^3u_5^{25}) + \mathrm{adm}_{63} \ \Rightarrow u_2^5u_3^3u_5^{25} \sim \mathrm{adm}_{63},\\
u_2^5u_3^3u_4^{25} &= Sq^2(u_2^3u_3^3u_4^{25}) + \mathrm{adm}_{64} \ \Rightarrow u_2^5u_3^3u_4^{25} \sim \mathrm{adm}_{64},\\
u_3^5u_4^3u_5^{25} &= Sq^2(u_3^3u_4^3u_5^{25}) + \mathrm{adm}_{61} \ \Rightarrow u_3^5u_4^3u_5^{25} \sim \mathrm{adm}_{61},\\
u_1^3u_2^{25}u_3^5 &= Sq^4(u_1^3u_2^{21}u_3^5 + u_1^3u_2^{13}u_3^{13} + u_1^3u_2^5u_3^{21}) + Sq^8(u_1^3u_2^{13}u_3^9 + u_1^3u_2^9u_3^{13}) + \mathrm{adm}_{70}\\
&\Rightarrow u_1^3u_2^{25}u_3^5 \sim \mathrm{adm}_{70},\\
u_2^3u_3^{25}u_4^5 &= Sq^4(u_2^3u_3^{21}u_4^5 + u_2^3u_3^{13}u_4^{13} + u_2^3u_3^5u_4^{21}) + Sq^8(u_2^3u_3^{13}u_4^9 + u_2^3u_3^9u_4^{13}) + \mathrm{adm}_{64}\\
&\Rightarrow u_2^3u_3^{25}u_4^5 \sim \mathrm{adm}_{64},\\
u_1^3u_3^{25}u_4^5 &= Sq^4(u_1^3u_3^{21}u_4^5 + u_1^3u_3^{13}u_4^{13} + u_1^3u_3^5u_4^{21}) + Sq^8(u_1^3u_3^{13}u_4^9 + u_1^3u_3^9u_4^{13}) + \mathrm{adm}_{67}\\
&\Rightarrow u_1^3u_3^{25}u_4^5 \sim \mathrm{adm}_{67},\\
u_3^3u_4^{25}u_5^5 &= Sq^4(u_3^3u_4^{21}u_5^5 + u_3^3u_4^{13}u_5^{13} + u_3^3u_4^5u_5^{21}) + Sq^8(u_3^3u_4^{13}u_5^9 + u_3^3u_4^9u_5^{13}) + \mathrm{adm}_{61}\\
&\Rightarrow u_3^3u_4^{25}u_5^5 \sim \mathrm{adm}_{61},\\
u_1^3u_4^{25}u_5^5 &= Sq^4(u_1^3u_4^{21}u_5^5 + u_1^3u_4^{13}u_5^{13} + u_1^3u_4^5u_5^{21}) + Sq^8(u_1^3u_4^{13}u_5^9 + u_1^3u_4^9u_5^{13}) + \mathrm{adm}_{65}\\
&\Rightarrow u_1^3u_4^{25}u_5^5 \sim \mathrm{adm}_{65},\\
u_2^3u_4^{25}u_5^5 &= Sq^4(u_2^3u_4^{21}u_5^5 + u_2^3u_4^{13}u_5^{13} + u_2^3u_4^5u_5^{21}) + Sq^8(u_2^3u_4^{13}u_5^9 + u_2^3u_4^9u_5^{13}) + \mathrm{adm}_{62}\\
&\Rightarrow u_2^3u_4^{25}u_5^5 \sim \mathrm{adm}_{62}.
\end{aligned}$$

From the above calculations and the relations $\rho_j(f) \sim f$ for $1 \leq j \leq 4$, we obtain $\gamma_{61} = \gamma_{62} = \cdots = \gamma_{70}$. The lemma follows. □

In the proof of Theorem 3.2, it was shown that $(QP_5)_{33}^{>0}(\widetilde{\omega}_{(1)}) = \langle\{[\mathrm{adm}_j] :\ 156 \leq j \leq 186\}\rangle$ and $\dim(QP_5)_{33}^{>0}(\widetilde{\omega}_{(1)}) = 31$. Using this result, we can obtain a direct summand decomposition of the Σ_5-submodules:

$$(QP_5)_{33}^{>0}(\widetilde{\omega}_{(1)}) = \langle[\Sigma_5(\mathrm{adm}_{156})]\rangle \bigoplus \langle[\Sigma_5(\mathrm{adm}_{162}, \mathrm{adm}_{166})]\rangle \bigoplus \langle[\Sigma_5(\mathrm{adm}_{177}, \mathrm{adm}_{181})]\rangle,$$

where $\langle[\Sigma_5(\mathrm{adm}_{156})]\rangle = \langle\{[\mathrm{adm}_j :\ 156 \leq j \leq 161]\}\rangle$, $\langle[\Sigma_5(\mathrm{adm}_{162}, \mathrm{adm}_{166})]\rangle = \langle\{[\mathrm{adm}_j :\ 162 \leq j \leq 176]\}\rangle$, and $\langle[\Sigma_5(\mathrm{adm}_{177}, \mathrm{adm}_{181})]\rangle = \langle\{[\mathrm{adm}_j :\ 177 \leq j \leq 186]\}\rangle$.

Lemma 3.7. *The following holds:*

$$((QP_5)_{33}^{>0}(\widetilde{\omega}_{(1)}))^{\Sigma_5} = \langle\{[\widehat{q}_5],\ [\widehat{q}_6],\ [\widehat{q}_7]\}\rangle,$$

where

$$\begin{aligned}\widehat{q}_5 &:= h_1,\\ \widehat{q}_6 &:= \mathrm{adm}_{156} + h_2 + h_3 + h_4,\\ \widehat{q}_7 &:= \mathrm{adm}_{156} + h_4 + h_5,\end{aligned}$$

and the polynomials h_i, $1\leq i\leq 5$ are described as follows:

$$\begin{aligned} h_1 &:= \textstyle\sum_{156\leq j\leq 161}\mathrm{adm}_j, \quad h_2 := \mathrm{adm}_{157}+\mathrm{adm}_{158},\\ h_3 &:= \textstyle\sum_{163\leq j\leq 165}\mathrm{adm}_j + \sum_{167\leq j\leq 169}\mathrm{adm}_j + \sum_{173\leq j\leq 176}\mathrm{adm}_j,\\ h_4 :&= \mathrm{adm}_{162} + \mathrm{adm}_{166} + \textstyle\sum_{170\leq j\leq 172}\mathrm{adm}_j,\\ h_5 &:= \textstyle\sum_{177\leq j\leq 186}\mathrm{adm}_j.\end{aligned}$$

Proof. The proof of the lemma follows the same line of reasoning as the proof of Lemma 3.6. ☐

Now, suppose that $[t]\in ((QP_5)_{33}(\widetilde{\omega}_{(1)}))^{GL_5}$. Then, by Lemmata 3.6 and 3.7, we have $t\sim \sum_{1\leq j\leq 7}\beta_j\widehat{q}_j$, in which $\beta_j\in\mathbb{Z}/2$, $1\leq j\leq 7$. Based on the homomorphism $\rho_5: P_5\to P_5$ and the relation $\rho_5(t)+t\sim 0$, we get $\beta_1=\beta_2=\cdots=\beta_7=0$.

Computation of the invariant space $((QP_5)_{33}(\widetilde{\omega}_{(3)}))^{GL_5}$. Recall that $(QP_5)_{33}(\widetilde{\omega}_{(3)}) = (QP_5)^0_{33}(\widetilde{\omega}_{(3)}) \oplus (QP_5)^{>0}_{33}(\widetilde{\omega}_{(3)})$, with $\dim(QP_5)^0_{33}(\widetilde{\omega}_{(3)}) = 395$ and $\dim(QP_5)^{>0}_{33}(\widetilde{\omega}_{(3)}) = 421$. Explicit bases for the spaces $(QP_5)^0_{33}(\widetilde{\omega}_{(3)})$ and $(QP_5)^0_{33}(\widetilde{\omega}_{(3)})$ are given by the output of our algorithm implemented in `OSCAR`; see Appendix 4. Also, for a more comprehensive understanding of our algorithmic approach, the reader may also refer to our `SageMath` implementation in [29].

Using these results together with the $\mathbb{A}$-homomorphisms $\rho_i: P_5\longrightarrow P_5$ for $1\leq i\leq 4$, and by hand calculations analogous to those used to determine the Σ_5-invariants of $(QP_5)_{33}\big(\widetilde{\omega}_{(1)}\big)$, we obtain the following:

$$\dim((QP_5)_{33}(\widetilde{\omega}_{(3)}))^{\Sigma_5} = 18, \quad\text{and}\quad ((QP_5)_{33}(\widetilde{\omega}_{(3)}))^{\Sigma_5} = \langle\{[\mathsf{Sigma5}[i]]_{\widetilde{\omega}_{(3)}} : 8\leq i\leq 25\}\rangle,$$

where

$$\begin{aligned}
\mathsf{Sigma5}[8] &= u_1^3u_2^5u_3^9u_4^6u_5^{10} + u_1^3u_2^5u_3^6u_4^9u_5^{10}\\
\mathsf{Sigma5}[9] &= u_1^3u_2^{13}u_3^{14}u_4u_5^2 + u_1^3u_2^3u_3^{13}u_4^{12}u_5^2 + u_1^3u_2^{13}u_3u_4^{14}u_5^2 + u_1^3u_2^5u_3^9u_4^{14}u_5^2\\
&+ u_1^3u_2u_3^{13}u_4^{14}u_5^2 + u_1u_2^3u_3^{13}u_4^{14}u_5^2 + u_1^3u_2^{13}u_3^2u_4^{12}u_5^3 + u_1^3u_2u_3^{14}u_4^{12}u_5^3\\
&+ u_1u_2^3u_3^{14}u_4^{12}u_5^3 + u_1^3u_2u_3^{12}u_4^{14}u_5^3 + u_1u_2^3u_3^{12}u_4^{14}u_5^3 + u_1u_2^2u_3^{13}u_4^{14}u_5^3\\
&+ u_1^3u_2^3u_3^7u_4^{12}u_5^8 + u_1^3u_2^5u_3^3u_4^{14}u_5^8 + u_1^3u_2^3u_3^5u_4^{14}u_5^8 + u_1^3u_2^4u_3^7u_4^9u_5^{10}\\
&+ u_1u_2^6u_3^7u_4^9u_5^{10} + u_1^3u_2u_3^7u_4^{12}u_5^{10} + u_1^3u_2^{13}u_3^3u_4^2u_5^{12} + u_1^3u_2^{13}u_3^2u_4^3u_5^{12}\\
&+ u_1^3u_2u_3^{14}u_4^3u_5^{12} + u_1u_2^3u_3^{14}u_4^3u_5^{12} + u_1^3u_2^3u_3^7u_4^8u_5^{12} + u_1^3u_2u_3^7u_4^{10}u_5^{12}\\
&+ u_1u_2^3u_3^7u_4^{10}u_5^{12} + u_1^3u_2^3u_3^3u_4^{12}u_5^{12} + u_1^3u_2^{13}u_3u_4^2u_5^{14} + u_1^3u_2^5u_3^9u_4^2u_5^{14}\\
&+ u_1^3u_2u_3^{13}u_4^2u_5^{14} + u_1u_2^3u_3^{13}u_4^2u_5^{14} + u_1^3u_2^5u_3^8u_4^3u_5^{14} + u_1u_2^2u_3^{13}u_4^3u_5^{14}\\
&+ u_1u_2u_3^{14}u_4^3u_5^{14} + u_1^3u_2^5u_3^3u_4^8u_5^{14} + u_1^3u_2^3u_3^5u_4^8u_5^{14} + u_1^3u_2^5u_3^2u_4^9u_5^{14}\\
&+ u_1^3u_2^3u_3^4u_4^9u_5^{14} + u_1^3u_2^3u_3u_4^{12}u_5^{14} + u_1^3u_2u_3^2u_4^{13}u_5^{14} + u_1u_2^3u_3^2u_4^{13}u_5^{14}
\end{aligned}$$

$$
\begin{aligned}
&+ u_1u_2^2u_3^3u_4^{13}u_5^{14} + u_1u_2u_3^3u_4^{14}u_5^{14}\\
\mathsf{Sigma5}[10] = {}& u_1^3u_2^{13}u_3^{14}u_4^3 + u_1^3u_2^{13}u_3^3u_4^{14} + u_1^3u_2^3u_3^{13}u_4^{14} + u_1^3u_2^{13}u_3^{14}u_5^3\\
&+ u_1^3u_2^{13}u_4^{14}u_5^3 + u_1^3u_3^{13}u_4^{14}u_5^3 + u_2^3u_3^{13}u_4^{14}u_5^3 + u_1^3u_2^{13}u_3^3u_5^{14}\\
&+ u_1^3u_2^3u_3^{13}u_5^{14} + u_1^3u_2^{13}u_4^3u_5^{14} + u_1^3u_3^{13}u_4^3u_5^{14} + u_2^3u_3^{13}u_4^3u_5^{14}\\
&+ u_1^3u_2^3u_4^{13}u_5^{14} + u_1^3u_3^3u_4^{13}u_5^{14} + u_2^3u_3^3u_4^{13}u_5^{14}\\
\mathsf{Sigma5}[11] = {}& u_1^7u_2^7u_3^9u_4^8u_5^2 + u_1^7u_2^7u_3^8u_4^9u_5^2 + u_1^7u_2^3u_3^{12}u_4^9u_5^2 + u_1^3u_2^7u_3^{12}u_4^9u_5^2\\
&+ u_1^7u_2^3u_3^9u_4^{12}u_5^2 + u_1^3u_2^7u_3^9u_4^{12}u_5^2 + u_1^3u_2^3u_3^{13}u_4^{12}u_5^2 + u_1^3u_2^3u_3^{12}u_4^{13}u_5^2\\
&+ u_1^3u_2u_3^{14}u_4^{13}u_5^2 + u_1u_2^3u_3^{14}u_4^{13}u_5^2 + u_1^3u_2u_3^{13}u_4^{14}u_5^2 + u_1u_2^3u_3^{13}u_4^{14}u_5^2\\
&+ u_1^7u_2^7u_3^9u_4^2u_5^8 + u_1^7u_2^3u_3^9u_4^6u_5^8 + u_1^3u_2^7u_3^9u_4^6u_5^8 + u_1^7u_2^7u_3u_4^{10}u_5^8\\
&+ u_1^7u_2^3u_3^5u_4^{10}u_5^8 + u_1^3u_2^7u_3^5u_4^{10}u_5^8 + u_1^3u_2^3u_3^7u_4^{12}u_5^8 + u_1^7u_2^7u_3^8u_4u_5^{10}\\
&+ u_1^7u_2^3u_3^{12}u_4u_5^{10} + u_1^3u_2^7u_3^{12}u_4u_5^{10} + u_1^7u_2^3u_3^9u_4^4u_5^{10} + u_1^3u_2^7u_3^9u_4^4u_5^{10}\\
&+ u_1^7u_2u_3^9u_4^6u_5^{10} + u_1u_2^7u_3^9u_4^6u_5^{10} + u_1^7u_2^7u_3u_4^8u_5^{10} + u_1^7u_2^3u_3^4u_4^9u_5^{10}\\
&+ u_1^3u_2^7u_3^4u_4^9u_5^{10} + u_1^7u_2u_3^6u_4^9u_5^{10} + u_1u_2^7u_3^6u_4^9u_5^{10} + u_1^3u_2^4u_3^7u_4^9u_5^{10}\\
&+ u_1u_2^6u_3^7u_4^9u_5^{10} + u_1^7u_2^3u_3u_4^{12}u_5^{10} + u_1^3u_2^7u_3u_4^{12}u_5^{10} + u_1^3u_2u_3^7u_4^{12}u_5^{10}\\
&+ u_1^7u_2^3u_3^9u_4^2u_5^{12} + u_1^3u_2^7u_3^9u_4^2u_5^{12} + u_1^3u_2^3u_3^{13}u_4^2u_5^{12} + u_1^3u_2^3u_3^7u_4^8u_5^{12}\\
&+ u_1^7u_2^3u_3u_4^{10}u_5^{12} + u_1^3u_2^7u_3u_4^{10}u_5^{12} + u_1^3u_2u_3^7u_4^{10}u_5^{12} + u_1u_2^3u_3^7u_4^{10}u_5^{12}\\
&+ u_1^3u_2^3u_3^3u_4^{12}u_5^{12} + u_1^3u_2^3u_3u_4^{14}u_5^{12} + u_1^3u_2^3u_3^{12}u_4u_5^{14} + u_1^3u_2u_3^{14}u_4u_5^{14}\\
&+ u_1u_2^3u_3^{14}u_4u_5^{14} + u_1^3u_2^3u_3u_4^{12}u_5^{14} + u_1^3u_2u_3u_4^{14}u_5^{14} + u_1u_2^3u_3u_4^{14}u_5^{14}\\
&+ u_1u_2u_3^3u_4^{14}u_5^{14}\\
\mathsf{Sigma5}[12] = {}& u_1^7u_2^7u_3^{11}u_4^8 + u_1^7u_2^7u_3^9u_4^{10} + u_1^7u_2^7u_3^8u_4^{11} + u_1^7u_2u_3^{14}u_4^{11}\\
&+ u_1^3u_2^5u_3^{14}u_4^{11} + u_1u_2^7u_3^{14}u_4^{11} + u_1^7u_2u_3^{11}u_4^{14} + u_1^3u_2^5u_3^{11}u_4^{14}\\
&+ u_1u_2^7u_3^{11}u_4^{14} + u_1^7u_2^7u_3^{11}u_5^8 + u_1^7u_2^7u_4^{11}u_5^8 + u_1^7u_3^7u_4^{11}u_5^8\\
&+ u_2^7u_3^7u_4^{11}u_5^8 + u_1^7u_2^7u_3^9u_5^{10} + u_1^7u_2^7u_4^9u_5^{10} + u_1^7u_3^7u_4^9u_5^{10}\\
&+ u_2^7u_3^7u_4^9u_5^{10} + u_1^7u_2^7u_3^8u_5^{11} + u_1^7u_2u_3^{14}u_5^{11} + u_1^3u_2^5u_3^{14}u_5^{11}\\
&+ u_1u_2^7u_3^{14}u_5^{11} + u_1^7u_2^7u_4^8u_5^{11} + u_1^7u_3^7u_4^8u_5^{11} + u_2^7u_3^7u_4^8u_5^{11}\\
&+ u_1^7u_2u_4^{14}u_5^{11} + u_1^3u_2^5u_4^{14}u_5^{11} + u_1u_2^7u_4^{14}u_5^{11} + u_1^7u_3u_4^{14}u_5^{11}\\
&+ u_2^7u_3u_4^{14}u_5^{11} + u_1^3u_3^5u_4^{14}u_5^{11} + u_2^3u_3^5u_4^{14}u_5^{11} + u_1u_3^7u_4^{14}u_5^{11}\\
&+ u_2u_3^7u_4^{14}u_5^{11} + u_1^7u_2u_3^{11}u_5^{14} + u_1^3u_2^5u_3^{11}u_5^{14} + u_1u_2^7u_3^{11}u_5^{14}\\
&+ u_1^7u_2u_4^{11}u_5^{14} + u_1^3u_2^5u_4^{11}u_5^{14} + u_1u_2^7u_4^{11}u_5^{14} + u_1^7u_3u_4^{11}u_5^{14}\\
&+ u_2^7u_3u_4^{11}u_5^{14} + u_1^3u_3^5u_4^{11}u_5^{14} + u_2^3u_3^5u_4^{11}u_5^{14} + u_1u_3^7u_4^{11}u_5^{14}\\
&+ u_2u_3^7u_4^{11}u_5^{14}\\
\mathsf{Sigma5}[13] = {}& u_1^7u_2^7u_3^8u_4^3u_5^8 + u_1^7u_2^3u_3^{11}u_4^4u_5^8 + u_1^3u_2^7u_3^{11}u_4^4u_5^8 + u_1^7u_2^9u_3^3u_4^6u_5^8\\
&+ u_1^3u_2^{13}u_3^3u_4^6u_5^8 + u_1^7u_2^3u_3^9u_4^6u_5^8 + u_1^3u_2^3u_3^{13}u_4^6u_5^8 + u_1^3u_2^5u_3^{10}u_4^7u_5^8\\
&+ u_1^7u_2^7u_3^3u_4^8u_5^8 + u_1^7u_2^3u_3^5u_4^{10}u_5^8 + u_1^3u_2^7u_3^5u_4^{10}u_5^8 + u_1^3u_2^5u_3^7u_4^{10}u_5^8
\end{aligned}
$$

$$+ u_1^7u_2u_3^6u_4^{11}u_5^8 + u_1u_2^7u_3^6u_4^{11}u_5^8 + u_1^7u_2^3u_3^3u_4^{12}u_5^8 + u_1^3u_2^3u_3^7u_4^{12}u_5^8$$
$$+ u_1^7u_2u_3^3u_4^{14}u_5^8 + u_1u_2^7u_3^3u_4^{14}u_5^8 + u_1^3u_2^3u_3^5u_4^{14}u_5^8 + u_1^7u_2u_3^{14}u_4u_5^{10}$$
$$+ u_1^3u_2^5u_3^{14}u_4u_5^{10} + u_1u_2^7u_3^{14}u_4u_5^{10} + u_1^7u_2^9u_3^3u_4^4u_5^{10} + u_1^3u_2^{13}u_3^3u_4^4u_5^{10}$$
$$+ u_1^7u_2^3u_3^9u_4^4u_5^{10} + u_1^7u_2^9u_3^2u_4^5u_5^{10} + u_1^3u_2^{13}u_3^2u_4^5u_5^{10} + u_1^7u_2^8u_3^3u_4^5u_5^{10}$$
$$+ u_1u_2^{14}u_3^3u_4^5u_5^{10} + u_1^7u_2^3u_3^8u_4^5u_5^{10} + u_1^7u_2u_3^{10}u_4^5u_5^{10} + u_1^3u_2u_3^{14}u_4^5u_5^{10}$$
$$+ u_1u_2^3u_3^{14}u_4^5u_5^{10} + u_1^7u_2^9u_3u_4^6u_5^{10} + u_1^3u_2^{13}u_3u_4^6u_5^{10} + u_1^3u_2u_3^{12}u_4^7u_5^{10}$$
$$+ u_1u_2^3u_3^{12}u_4^7u_5^{10} + u_1u_2^2u_3^{13}u_4^7u_5^{10} + u_1u_2u_3^{14}u_4^7u_5^{10} + u_1^7u_2^3u_3^5u_4^8u_5^{10}$$
$$+ u_1^3u_2^7u_3^5u_4^8u_5^{10} + u_1^7u_2u_3^7u_4^8u_5^{10} + u_1u_2^7u_3^7u_4^8u_5^{10} + u_1^7u_2^3u_3^4u_4^9u_5^{10}$$
$$+ u_1^3u_2^7u_3^4u_4^9u_5^{10} + u_1^7u_2u_3^6u_4^9u_5^{10} + u_1u_2^7u_3^6u_4^9u_5^{10} + u_1^3u_2^4u_3^7u_4^9u_5^{10}$$
$$+ u_1u_2^6u_3^7u_4^9u_5^{10} + u_1^3u_2^7u_3u_4^{12}u_5^{10} + u_1^7u_2u_3^3u_4^{12}u_5^{10} + u_1^3u_2u_3^7u_4^{12}u_5^{10}$$
$$+ u_1^3u_2u_3^6u_4^{13}u_5^{10} + u_1u_2^3u_3^6u_4^{13}u_5^{10} + u_1u_2^2u_3^7u_4^{13}u_5^{10} + u_1^7u_2u_3u_4^{14}u_5^{10}$$
$$+ u_1^3u_2^5u_3u_4^{14}u_5^{10} + u_1u_2^7u_3u_4^{14}u_5^{10} + u_1^3u_2u_3^5u_4^{14}u_5^{10} + u_1u_2^3u_3^5u_4^{14}u_5^{10}$$
$$+ u_1u_2u_3^7u_4^{14}u_5^{10} + u_1^3u_2^5u_3^{10}u_4^4u_5^{11} + u_1^3u_2u_3^{12}u_4^6u_5^{11} + u_1u_2^3u_3^{12}u_4^6u_5^{11}$$
$$+ u_1u_2^2u_3^{13}u_4^6u_5^{11} + u_1u_2u_3^{14}u_4^6u_5^{11} + u_1^7u_2u_3^6u_4^8u_5^{11} + u_1^3u_2^5u_3^6u_4^8u_5^{11}$$
$$+ u_1u_2^7u_3^6u_4^8u_5^{11} + u_1^7u_2u_3^2u_4^{12}u_5^{11} + u_1^3u_2^5u_3^2u_4^{12}u_5^{11} + u_1u_2^7u_3^2u_4^{12}u_5^{11}$$
$$+ u_1^3u_2u_3^4u_4^{14}u_5^{11} + u_1u_2^3u_3^4u_4^{14}u_5^{11} + u_1u_2^2u_3^5u_4^{14}u_5^{11} + u_1^7u_2u_3^{11}u_4^2u_5^{12}$$
$$+ u_1^3u_2^5u_3^{11}u_4^2u_5^{12} + u_1u_2^7u_3^{11}u_4^2u_5^{12} + u_1^3u_2^3u_3^{12}u_4^3u_5^{12} + u_1^3u_2^3u_3^7u_4^8u_5^{12}$$
$$+ u_1^7u_2u_3^2u_4^{11}u_5^{12} + u_1^3u_2^5u_3^2u_4^{11}u_5^{12} + u_1u_2^7u_3^2u_4^{11}u_5^{12} + u_1^3u_2u_3^3u_4^{14}u_5^{12}$$
$$+ u_1u_2^3u_3^3u_4^{14}u_5^{12} + u_1^3u_2^3u_3^{12}u_4u_5^{14} + u_1^7u_2^9u_3u_4^2u_5^{14} + u_1^3u_2^{13}u_3u_4^2u_5^{14}$$
$$+ u_1^3u_2^5u_3^9u_4^2u_5^{14} + u_1^3u_2u_3^{13}u_4^2u_5^{14} + u_1u_2^3u_3^{13}u_4^2u_5^{14} + u_1^7u_2^3u_3u_4^8u_5^{14}$$
$$+ u_1^7u_2u_3^3u_4^8u_5^{14} + u_1^3u_2^5u_3^3u_4^8u_5^{14} + u_1u_2^7u_3^3u_4^8u_5^{14} + u_1^3u_2u_3^6u_4^9u_5^{14}$$
$$+ u_1u_2^3u_3^6u_4^9u_5^{14} + u_1u_2^2u_3^7u_4^9u_5^{14} + u_1^7u_2u_3u_4^{10}u_5^{14} + u_1^3u_2^5u_3u_4^{10}u_5^{14}$$
$$+ u_1u_2^7u_3u_4^{10}u_5^{14} + u_1^3u_2u_3^4u_4^{11}u_5^{14} + u_1u_2^3u_3^4u_4^{11}u_5^{14} + u_1u_2^2u_3^5u_4^{11}u_5^{14}$$
$$+ u_1^3u_2u_3^3u_4^{12}u_5^{14} + u_1u_2^3u_3^3u_4^{12}u_5^{14} + u_1u_2u_3^3u_4^{14}u_5^{14}$$

$$\mathsf{Sigma5}[14] = u_1^7u_2u_3^9u_4^{14}u_5^2 + u_1u_2^7u_3^9u_4^{14}u_5^2 + u_1^3u_2u_3^{13}u_4^{14}u_5^2 + u_1u_2^3u_3^{13}u_4^{14}u_5^2$$
$$+ u_1^3u_2^{13}u_3^2u_4^{12}u_5^3 + u_1^3u_2u_3^{14}u_4^{12}u_5^3 + u_1u_2^3u_3^{14}u_4^{12}u_5^3 + u_1^3u_2u_3^{12}u_4^{14}u_5^3$$
$$+ u_1u_2^3u_3^{12}u_4^{14}u_5^3 + u_1u_2^2u_3^{13}u_4^{14}u_5^3 + u_1^7u_2^7u_3^8u_4^3u_5^8 + u_1^7u_2^{11}u_3^3u_4^4u_5^8$$
$$+ u_1^3u_2^5u_3^6u_4^{11}u_5^8 + u_1^7u_2u_3^3u_4^{14}u_5^8 + u_1^3u_2^5u_3^3u_4^{14}u_5^8 + u_1u_2^7u_3^3u_4^{14}u_5^8$$
$$+ u_1^3u_2^3u_3^5u_4^{14}u_5^8 + u_1^7u_2u_3^{14}u_4u_5^{10} + u_1^3u_2^5u_3^{14}u_4u_5^{10} + u_1u_2^7u_3^{14}u_4u_5^{10}$$
$$+ u_1^3u_2^3u_3^{13}u_4^4u_5^{10} + u_1^3u_2^{13}u_3u_4^6u_5^{10} + u_1^7u_2u_3^9u_4^6u_5^{10} + u_1u_2^7u_3^9u_4^6u_5^{10}$$
$$+ u_1^7u_2u_3^7u_4^8u_5^{10} + u_1^3u_2^5u_3^7u_4^8u_5^{10} + u_1u_2^7u_3^7u_4^8u_5^{10} + u_1^3u_2^7u_3u_4^{12}u_5^{10}$$
$$+ u_1^7u_2u_3u_4^{14}u_5^{10} + u_1u_2^7u_3u_4^{14}u_5^{10} + u_1u_2u_3^7u_4^{14}u_5^{10} + u_1u_2^2u_3^{12}u_4^7u_5^{11}$$
$$+ u_1^3u_2^5u_3^6u_4^8u_5^{11} + u_1^7u_2u_3^2u_4^{12}u_5^{11} + u_1u_2^7u_3^2u_4^{12}u_5^{11} + u_1u_2^2u_3^7u_4^{12}u_5^{11}$$
$$+ u_1^3u_2u_3^4u_4^{14}u_5^{11} + u_1u_2^3u_3^4u_4^{14}u_5^{11} + u_1u_2u_3^6u_4^{14}u_5^{11} + u_1^7u_2u_3^{11}u_4^2u_5^{12}$$

$$
\begin{aligned}
&+ u_1^3u_2^5u_3^{11}u_4^2u_5^{12} + u_1u_2^7u_3^{11}u_4^2u_5^{12} + u_1^3u_2^{13}u_3^2u_4^3u_5^{12} + u_1^3u_2^3u_3^{12}u_4^3u_5^{12} \\
&+ u_1^3u_2u_3^{14}u_4^3u_5^{12} + u_1u_2^3u_3^{14}u_4^3u_5^{12} + u_1^3u_2^7u_3u_4^{10}u_5^{12} + u_1^7u_2u_3^3u_4^{10}u_5^{12} \\
&+ u_1u_2^7u_3^3u_4^{10}u_5^{12} + u_1^7u_2u_3^2u_4^{11}u_5^{12} + u_1u_2^7u_3^2u_4^{11}u_5^{12} + u_1^3u_2^3u_3^4u_4^{11}u_5^{12} \\
&+ u_1u_2^2u_3^7u_4^{11}u_5^{12} + u_1^3u_2^3u_3^{12}u_4u_5^{14} + u_1^3u_2^5u_3^9u_4^2u_5^{14} + u_1^3u_2u_3^{13}u_4^2u_5^{14} \\
&+ u_1u_2^3u_3^{13}u_4^2u_5^{14} + u_1^3u_2u_3^{12}u_4^3u_5^{14} + u_1u_2^3u_3^{12}u_4^3u_5^{14} + u_1u_2^2u_3^{13}u_4^3u_5^{14} \\
&+ u_1^7u_2u_3u_4^{10}u_5^{14} + u_1u_2^7u_3u_4^{10}u_5^{14} + u_1u_2u_3^7u_4^{10}u_5^{14} + u_1^3u_2u_3^4u_4^{11}u_5^{14} \\
&+ u_1u_2^3u_3^4u_4^{11}u_5^{14} + u_1u_2^2u_3^3u_4^{13}u_5^{14} + u_1^3u_2u_3u_4^{14}u_5^{14} + u_1u_2^3u_3u_4^{14}u_5^{14} \\
&+ u_1u_2u_3^3u_4^{14}u_5^{14}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Sigma5}[15] = \; & u_1^7u_2^{11}u_3^5u_4^8u_5^2 + u_1^7u_2^3u_3^{13}u_4^8u_5^2 + u_1^3u_2^7u_3^{13}u_4^8u_5^2 + u_1^7u_2^7u_3^8u_4^9u_5^2 \\
&+ u_1^7u_2^{11}u_3u_4^{12}u_5^2 + u_1^3u_2^{13}u_3^3u_4^{12}u_5^2 + u_1^7u_2u_3^{11}u_4^{12}u_5^2 + u_1u_2^7u_3^{11}u_4^{12}u_5^2 \\
&+ u_1^7u_2^3u_3^8u_4^{13}u_5^2 + u_1^3u_2^7u_3^8u_4^{13}u_5^2 + u_1^7u_2u_3^{10}u_4^{13}u_5^2 + u_1u_2^7u_3^{10}u_4^{13}u_5^2 \\
&+ u_1^3u_2^4u_3^{11}u_4^{13}u_5^2 + u_1u_2^6u_3^{11}u_4^{13}u_5^2 + u_1^3u_2u_3^{14}u_4^{13}u_5^2 + u_1^3u_2^{13}u_3u_4^{14}u_5^2 \\
&+ u_1^3u_2^5u_3^9u_4^{14}u_5^2 + u_1^3u_2u_3^{13}u_4^{14}u_5^2 + u_1u_2^3u_3^{13}u_4^{14}u_5^2 + u_1^7u_2^{11}u_3^5u_4^2u_5^8 \\
&+ u_1^7u_2^3u_3^{13}u_4^2u_5^8 + u_1^3u_2^7u_3^{13}u_4^2u_5^8 + u_1^7u_2^7u_3^3u_4^8u_5^8 + u_1^7u_2u_3^7u_4^{10}u_5^8 \\
&+ u_1u_2^7u_3^7u_4^{10}u_5^8 + u_1^3u_2^3u_3^7u_4^{12}u_5^8 + u_1^7u_2^3u_3u_4^{14}u_5^8 + u_1^3u_2^7u_3u_4^{14}u_5^8 \\
&+ u_1^7u_2u_3^3u_4^{14}u_5^8 + u_1^3u_2^5u_3^3u_4^{14}u_5^8 + u_1u_2^7u_3^3u_4^{14}u_5^8 + u_1^3u_2^3u_3^5u_4^{14}u_5^8 \\
&+ u_1^3u_2u_3^7u_4^{14}u_5^8 + u_1u_2^3u_3^7u_4^{14}u_5^8 + u_1^7u_2^7u_3^8u_4u_5^{10} + u_1^7u_2^9u_3u_4^6u_5^{10} \\
&+ u_1^3u_2u_3^{13}u_4^6u_5^{10} + u_1u_2^3u_3^{13}u_4^6u_5^{10} + u_1^7u_2u_3^7u_4^8u_5^{10} + u_1u_2^7u_3^7u_4^8u_5^{10} \\
&+ u_1^3u_2^4u_3^7u_4^9u_5^{10} + u_1u_2^6u_3^7u_4^9u_5^{10} + u_1u_2^3u_3^7u_4^{12}u_5^{10} + u_1^7u_2u_3u_4^{14}u_5^{10} \\
&+ u_1u_2^7u_3u_4^{14}u_5^{10} + u_1u_2u_3^7u_4^{14}u_5^{10} + u_1^7u_2^{11}u_3u_4^2u_5^{12} + u_1^3u_2^{13}u_3^3u_4^2u_5^{12} \\
&+ u_1^7u_2u_3^{11}u_4^2u_5^{12} + u_1u_2^7u_3^{11}u_4^2u_5^{12} + u_1^3u_2^3u_3^7u_4^8u_5^{12} + u_1^7u_2^3u_3u_4^{10}u_5^{12} \\
&+ u_1^3u_2^3u_3^3u_4^{12}u_5^{12} + u_1^7u_2^3u_3^8u_4u_5^{14} + u_1^3u_2^7u_3^8u_4u_5^{14} + u_1^7u_2u_3^{10}u_4u_5^{14} \\
&+ u_1u_2^7u_3^{10}u_4u_5^{14} + u_1^3u_2^4u_3^{11}u_4u_5^{14} + u_1u_2^6u_3^{11}u_4u_5^{14} + u_1^3u_2u_3^{14}u_4u_5^{14} \\
&+ u_1^3u_2^{13}u_3u_4^2u_5^{14} + u_1^7u_2u_3^9u_4^2u_5^{14} + u_1^3u_2^5u_3^9u_4^2u_5^{14} + u_1u_2^7u_3^9u_4^2u_5^{14} \\
&+ u_1^3u_2^5u_3^8u_4^3u_5^{14} + u_1^3u_2u_3^{12}u_4^3u_5^{14} + u_1u_2^3u_3^{12}u_4^3u_5^{14} + u_1u_2u_3^{14}u_4^3u_5^{14} \\
&+ u_1^7u_2^3u_3u_4^8u_5^{14} + u_1^3u_2^7u_3u_4^8u_5^{14} + u_1^7u_2u_3^3u_4^8u_5^{14} + u_1^3u_2^5u_3^3u_4^8u_5^{14} \\
&+ u_1u_2^7u_3^3u_4^8u_5^{14} + u_1^3u_2^3u_3^5u_4^8u_5^{14} + u_1^3u_2u_3^7u_4^8u_5^{14} + u_1^3u_2^5u_3^2u_4^9u_5^{14} \\
&+ u_1^3u_2^3u_3^4u_4^9u_5^{14} + u_1^7u_2u_3u_4^{10}u_5^{14} + u_1u_2^7u_3u_4^{10}u_5^{14} + u_1u_2u_3^7u_4^{10}u_5^{14} \\
&+ u_1^3u_2^4u_3u_4^{11}u_5^{14} + u_1u_2^6u_3u_4^{11}u_5^{14} + u_1^3u_2u_3^4u_4^{11}u_5^{14} + u_1u_2u_3^6u_4^{11}u_5^{14} \\
&+ u_1^3u_2^3u_3u_4^{12}u_5^{14} + u_1u_2^3u_3u_4^{14}u_5^{14} + u_1u_2u_3^3u_4^{14}u_5^{14}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Sigma5}[16] = \; & u_1^7u_2^{11}u_3^{13}u_4^2 + u_1^7u_2^7u_3^9u_4^{10} + u_1^7u_2^3u_3^{11}u_4^{12} + u_1^3u_2^7u_3^{11}u_4^{12} \\
&+ u_1^7u_2^{11}u_3u_4^{14} + u_1^7u_2u_3^{11}u_4^{14} + u_1u_2^7u_3^{11}u_4^{14} + u_1^3u_2^3u_3^{13}u_4^{14} \\
&+ u_1^7u_2^{11}u_3^{13}u_5^2 + u_1^7u_2^{11}u_4^{13}u_5^2 + u_1^7u_3^{11}u_4^{13}u_5^2 + u_2^7u_3^{11}u_4^{13}u_5^2 \\
&+ u_1^7u_2^7u_3^9u_5^{10} + u_1^7u_2^7u_4^9u_5^{10} + u_1^7u_3^7u_4^9u_5^{10} + u_2^7u_3^7u_4^9u_5^{10}
\end{aligned}
$$

$$
\begin{aligned}
&+ u_1^7u_2^3u_3^{11}u_5^{12} + u_1^3u_2^7u_3^{11}u_5^{12} + u_1^7u_2^3u_4^{11}u_5^{12} + u_1^3u_2^7u_4^{11}u_5^{12}\\
&+ u_1^7u_3^3u_4^{11}u_5^{12} + u_2^7u_3^3u_4^{11}u_5^{12} + u_1^3u_3^7u_4^{11}u_5^{12} + u_2^3u_3^7u_4^{11}u_5^{12}\\
&+ u_1^7u_2^{11}u_3u_5^{14} + u_1^7u_2u_3^{11}u_5^{14} + u_1u_2^7u_3^{11}u_5^{14} + u_1^3u_2^3u_3^{13}u_5^{14}\\
&+ u_1^7u_2^{11}u_4u_5^{14} + u_1^7u_3^{11}u_4u_5^{14} + u_2^7u_3^{11}u_4u_5^{14} + u_1^7u_2u_4^{11}u_5^{14}\\
&+ u_1u_2^7u_4^{11}u_5^{14} + u_1^7u_3u_4^{11}u_5^{14} + u_2^7u_3u_4^{11}u_5^{14} + u_1u_3^7u_4^{11}u_5^{14}\\
&+ u_2u_3^7u_4^{11}u_5^{14} + u_1^3u_2^3u_4^{13}u_5^{14} + u_1^3u_3^3u_4^{13}u_5^{14} + u_2^3u_3^3u_4^{13}u_5^{14}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Sigma5}[17] = {}& u_1^{15}u_2u_3^2u_4^{12}u_5^3 + u_1u_2^{15}u_3^2u_4^{12}u_5^3 + u_1u_2^2u_3^{15}u_4^{12}u_5^3 + u_1u_2^2u_3^{12}u_4^{15}u_5^3\\
&+ u_1^{15}u_2^3u_3^3u_4^4u_5^8 + u_1^3u_2^{15}u_3^3u_4^4u_5^8 + u_1^3u_2^3u_3^{15}u_4^4u_5^8 + u_1^3u_2^3u_3^4u_4^{15}u_5^8\\
&+ u_1^{15}u_2u_3u_4^6u_5^{10} + u_1u_2^{15}u_3u_4^6u_5^{10} + u_1u_2u_3^{15}u_4^6u_5^{10} + u_1u_2u_3^6u_4^{15}u_5^{10}\\
&+ u_1^{15}u_2u_3^2u_4^3u_5^{12} + u_1u_2^{15}u_3^2u_4^3u_5^{12} + u_1u_2^2u_3^{15}u_4^3u_5^{12} + u_1u_2^2u_3^3u_4^{15}u_5^{12}\\
&+ u_1u_2^2u_3^{12}u_4^3u_5^{15} + u_1^3u_2^3u_3^4u_4^8u_5^{15} + u_1u_2u_3^6u_4^{10}u_5^{15} + u_1u_2^2u_3^3u_4^{12}u_5^{15}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Sigma5}[18] = {}& u_1^{15}u_2^3u_3^5u_4^8u_5^2 + u_1^3u_2^{15}u_3^5u_4^8u_5^2 + u_1^3u_2^5u_3^{15}u_4^8u_5^2 + u_1^{15}u_2^3u_3u_4^{12}u_5^2\\
&+ u_1^3u_2^{15}u_3u_4^{12}u_5^2 + u_1^{15}u_2u_3^3u_4^{12}u_5^2 + u_1u_2^{15}u_3^3u_4^{12}u_5^2 + u_1^3u_2u_3^{15}u_4^{12}u_5^2\\
&+ u_1u_2^3u_3^{15}u_4^{12}u_5^2 + u_1^{15}u_2u_3u_4^{14}u_5^2 + u_1u_2^{15}u_3u_4^{14}u_5^2 + u_1u_2u_3^{15}u_4^{14}u_5^2\\
&+ u_1^3u_2^5u_3^8u_4^{15}u_5^2 + u_1^3u_2u_3^{12}u_4^{15}u_5^2 + u_1u_2^3u_3^{12}u_4^{15}u_5^2 + u_1u_2u_3^{14}u_4^{15}u_5^2\\
&+ u_1^{15}u_2^3u_3^5u_4^2u_5^8 + u_1^3u_2^{15}u_3^5u_4^2u_5^8 + u_1^3u_2^5u_3^{15}u_4^2u_5^8 + u_1^3u_2^5u_3^2u_4^{15}u_5^8\\
&+ u_1^{15}u_2u_3u_4^6u_5^{10} + u_1u_2^{15}u_3u_4^6u_5^{10} + u_1u_2u_3^{15}u_4^6u_5^{10} + u_1u_2u_3^6u_4^{15}u_5^{10}\\
&+ u_1^{15}u_2^3u_3u_4^2u_5^{12} + u_1^3u_2^{15}u_3u_4^2u_5^{12} + u_1^{15}u_2u_3^3u_4^2u_5^{12} + u_1u_2^{15}u_3^3u_4^2u_5^{12}\\
&+ u_1^3u_2u_3^{15}u_4^2u_5^{12} + u_1u_2^3u_3^{15}u_4^2u_5^{12} + u_1^3u_2u_3^2u_4^{15}u_5^{12} + u_1u_2^3u_3^2u_4^{15}u_5^{12}\\
&+ u_1^{15}u_2u_3u_4^2u_5^{14} + u_1u_2^{15}u_3u_4^2u_5^{14} + u_1u_2u_3^{15}u_4^2u_5^{14} + u_1u_2u_3^2u_4^{15}u_5^{14}\\
&+ u_1^3u_2^5u_3^8u_4^2u_5^{15} + u_1^3u_2u_3^{12}u_4^2u_5^{15} + u_1u_2^3u_3^{12}u_4^2u_5^{15} + u_1u_2u_3^{14}u_4^2u_5^{15}\\
&+ u_1^3u_2^5u_3^2u_4^8u_5^{15} + u_1u_2u_3^6u_4^{10}u_5^{15} + u_1^3u_2u_3^2u_4^{12}u_5^{15} + u_1u_2^3u_3^2u_4^{12}u_5^{15} + u_1u_2u_3^2u_4^{14}u_5^{15}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Sigma5}[19] = {}& u_1^{15}u_2u_3^{14}u_4^3 + u_1u_2^{15}u_3^{14}u_4^3 + u_1u_2^{14}u_3^{15}u_4^3 + u_1^{15}u_2^3u_3^5u_4^{10}\\
&+ u_1^3u_2^{15}u_3^5u_4^{10} + u_1^3u_2^5u_3^{15}u_4^{10} + u_1^{15}u_2u_3^3u_4^{14} + u_1u_2^{15}u_3^3u_4^{14}\\
&+ u_1u_2^3u_3^{15}u_4^{14} + u_1u_2^{14}u_3^3u_4^{15} + u_1^3u_2^5u_3^{10}u_4^{15} + u_1u_2^3u_3^{14}u_4^{15}\\
&+ u_1^{15}u_2u_3^{14}u_5^3 + u_1u_2^{15}u_3^{14}u_5^3 + u_1u_2^{14}u_3^{15}u_5^3 + u_1^{15}u_2u_4^{14}u_5^3\\
&+ u_1u_2^{15}u_4^{14}u_5^3 + u_1^{15}u_3u_4^{14}u_5^3 + u_2^{15}u_3u_4^{14}u_5^3 + u_1u_3^{15}u_4^{14}u_5^3\\
&+ u_2u_3^{15}u_4^{14}u_5^3 + u_1u_2^{14}u_4^{15}u_5^3 + u_1u_3^{14}u_4^{15}u_5^3 + u_2u_3^{14}u_4^{15}u_5^3\\
&+ u_1^{15}u_2^3u_3^5u_5^{10} + u_1^3u_2^{15}u_3^5u_5^{10} + u_1^3u_2^5u_3^{15}u_5^{10} + u_1^{15}u_2^3u_4^5u_5^{10}\\
&+ u_1^3u_2^{15}u_4^5u_5^{10} + u_1^{15}u_3^3u_4^5u_5^{10} + u_2^{15}u_3^3u_4^5u_5^{10} + u_1^3u_3^{15}u_4^5u_5^{10}\\
&+ u_2^3u_3^{15}u_4^5u_5^{10} + u_1^3u_2^5u_4^{15}u_5^{10} + u_1^3u_3^5u_4^{15}u_5^{10} + u_2^3u_3^5u_4^{15}u_5^{10}\\
&+ u_1^{15}u_2u_3^3u_5^{14} + u_1u_2^{15}u_3^3u_5^{14} + u_1u_2^3u_3^{15}u_5^{14} + u_1^{15}u_2u_4^3u_5^{14}\\
&+ u_1u_2^{15}u_4^3u_5^{14} + u_1^{15}u_3u_4^3u_5^{14} + u_2^{15}u_3u_4^3u_5^{14} + u_1u_3^{15}u_4^3u_5^{14}\\
&+ u_2u_3^{15}u_4^3u_5^{14} + u_1u_2^3u_4^{15}u_5^{14} + u_1u_3^3u_4^{15}u_5^{14} + u_2u_3^3u_4^{15}u_5^{14}
\end{aligned}
$$

$$+ u_1u_2^{14}u_3^3u_5^{15} + u_1^3u_2^5u_3^{10}u_5^{15} + u_1u_2^3u_3^{14}u_5^{15} + u_1u_2^{14}u_4^3u_5^{15}$$
$$+ u_1u_3^{14}u_4^3u_5^{15} + u_2u_3^{14}u_4^3u_5^{15} + u_1^3u_2^5u_4^{10}u_5^{15} + u_1^3u_3^5u_4^{10}u_5^{15}$$
$$+ u_2^3u_3^5u_4^{10}u_5^{15} + u_1u_2^3u_4^{14}u_5^{15} + u_1u_3^3u_4^{14}u_5^{15} + u_2u_3^3u_4^{14}u_5^{15}$$

$$\text{Sigma5}[20] = u_1^{15}u_2^3u_3^{13}u_4^2 + u_1^3u_2^{15}u_3^{13}u_4^2 + u_1^3u_2^{13}u_3^{15}u_4^2 + u_1^{15}u_2u_3^{14}u_4^3$$
$$+ u_1u_2^{15}u_3^{14}u_4^3 + u_1u_2^{14}u_3^{15}u_4^3 + u_1^{15}u_2^3u_3^3u_4^{12} + u_1^3u_2^{15}u_3^3u_4^{12}$$
$$+ u_1^3u_2^3u_3^{15}u_4^{12} + u_1^{15}u_2^3u_3u_4^{14} + u_1^3u_2^{15}u_3u_4^{14} + u_1^3u_2u_3^{15}u_4^{14}$$
$$+ u_1^3u_2^{13}u_3^2u_4^{15} + u_1u_2^{14}u_3^3u_4^{15} + u_1^3u_2^3u_3^{12}u_4^{15} + u_1^3u_2u_3^{14}u_4^{15}$$
$$+ u_1^{15}u_2^3u_3^{13}u_5^2 + u_1^3u_2^{15}u_3^{13}u_5^2 + u_1^3u_2^{13}u_3^{15}u_5^2 + u_1^{15}u_2^3u_4^{13}u_5^2$$
$$+ u_1^3u_2^{15}u_4^{13}u_5^2 + u_1^{15}u_3^3u_4^{13}u_5^2 + u_2^{15}u_3^3u_4^{13}u_5^2 + u_1^3u_3^{15}u_4^{13}u_5^2$$
$$+ u_2^3u_3^{15}u_4^{13}u_5^2 + u_1^3u_2^{13}u_4^{15}u_5^2 + u_1^3u_3^{13}u_4^{15}u_5^2 + u_2^3u_3^{13}u_4^{15}u_5^2$$
$$+ u_1^{15}u_2u_3^{14}u_5^3 + u_1u_2^{15}u_3^{14}u_5^3 + u_1u_2^{14}u_3^{15}u_5^3 + u_1^{15}u_2u_4^{14}u_5^3$$
$$+ u_1u_2^{15}u_4^{14}u_5^3 + u_1^{15}u_3u_4^{14}u_5^3 + u_2^{15}u_3u_4^{14}u_5^3 + u_1u_3^{15}u_4^{14}u_5^3$$
$$+ u_2u_3^{15}u_4^{14}u_5^3 + u_1u_2^{14}u_4^{15}u_5^3 + u_1u_3^{14}u_4^{15}u_5^3 + u_2u_3^{14}u_4^{15}u_5^3$$
$$+ u_1^{15}u_2^3u_3^3u_5^{12} + u_1^3u_2^{15}u_3^3u_5^{12} + u_1^3u_2^3u_3^{15}u_5^{12} + u_1^{15}u_2^3u_4^3u_5^{12}$$
$$+ u_1^3u_2^{15}u_4^3u_5^{12} + u_1^{15}u_3^3u_4^3u_5^{12} + u_2^{15}u_3^3u_4^3u_5^{12} + u_1^3u_3^{15}u_4^3u_5^{12}$$
$$+ u_2^3u_3^{15}u_4^3u_5^{12} + u_1^3u_2^3u_4^{15}u_5^{12} + u_1^3u_3^3u_4^{15}u_5^{12} + u_2^3u_3^3u_4^{15}u_5^{12}$$
$$+ u_1^{15}u_2^3u_3u_5^{14} + u_1^3u_2^{15}u_3u_5^{14} + u_1^3u_2u_3^{15}u_5^{14} + u_1^{15}u_2^3u_4u_5^{14}$$
$$+ u_1^3u_2^{15}u_4u_5^{14} + u_1^{15}u_3^3u_4u_5^{14} + u_2^{15}u_3^3u_4u_5^{14} + u_1^3u_3^{15}u_4u_5^{14}$$
$$+ u_2^3u_3^{15}u_4u_5^{14} + u_1^3u_2u_4^{15}u_5^{14} + u_1^3u_3u_4^{15}u_5^{14} + u_2^3u_3u_4^{15}u_5^{14}$$
$$+ u_1^3u_2^{13}u_3^2u_5^{15} + u_1u_2^{14}u_3^3u_5^{15} + u_1^3u_2^3u_3^{12}u_5^{15} + u_1^3u_2u_3^{14}u_5^{15}$$
$$+ u_1^3u_2^{13}u_4^2u_5^{15} + u_1^3u_3^{13}u_4^2u_5^{15} + u_2^3u_3^{13}u_4^2u_5^{15} + u_1u_2^{14}u_4^3u_5^{15}$$
$$+ u_1u_3^{14}u_4^3u_5^{15} + u_2u_3^{14}u_4^3u_5^{15} + u_1^3u_2^3u_4^{12}u_5^{15} + u_1^3u_3^3u_4^{12}u_5^{15}$$
$$+ u_2^3u_3^3u_4^{12}u_5^{15} + u_1^3u_2u_4^{14}u_5^{15} + u_1^3u_3u_4^{14}u_5^{15} + u_2^3u_3u_4^{14}u_5^{15}$$

$$\text{Sigma5}[21] = u_1^{15}u_2u_3^2u_4^{12}u_5^3 + u_1u_2^{15}u_3^2u_4^{12}u_5^3 + u_1u_2^2u_3^{15}u_4^{12}u_5^3 + u_1u_2^2u_3^{12}u_4^{15}u_5^3$$
$$+ u_1^{15}u_2^7u_3u_4^2u_5^8 + u_1^7u_2^{15}u_3u_4^2u_5^8 + u_1^{15}u_2u_3^7u_4^2u_5^8 + u_1u_2^{15}u_3^7u_4^2u_5^8$$
$$+ u_1^7u_2u_3^{15}u_4^2u_5^8 + u_1u_2^7u_3^{15}u_4^2u_5^8 + u_1^{15}u_2u_3^2u_4^7u_5^8 + u_1u_2^{15}u_3^2u_4^7u_5^8$$
$$+ u_1u_2^2u_3^{15}u_4^7u_5^8 + u_1^7u_2u_3^2u_4^{15}u_5^8 + u_1u_2^7u_3^2u_4^{15}u_5^8 + u_1u_2^2u_3^7u_4^{15}u_5^8$$
$$+ u_1^{15}u_2u_3^2u_4^4u_5^{11} + u_1u_2^{15}u_3^2u_4^4u_5^{11} + u_1u_2^2u_3^{15}u_4^4u_5^{11} + u_1u_2^2u_3^4u_4^{15}u_5^{11}$$
$$+ u_1^{15}u_2^3u_3u_4^2u_5^{12} + u_1^3u_2^{15}u_3u_4^2u_5^{12} + u_1^{15}u_2u_3^3u_4^2u_5^{12} + u_1u_2^{15}u_3^3u_4^2u_5^{12}$$
$$+ u_1^3u_2u_3^{15}u_4^2u_5^{12} + u_1u_2^3u_3^{15}u_4^2u_5^{12} + u_1^{15}u_2u_3^2u_4^3u_5^{12} + u_1u_2^{15}u_3^2u_4^3u_5^{12}$$
$$+ u_1u_2^2u_3^{15}u_4^3u_5^{12} + u_1^3u_2u_3^2u_4^{15}u_5^{12} + u_1u_2^3u_3^2u_4^{15}u_5^{12} + u_1u_2^2u_3^3u_4^{15}u_5^{12}$$
$$+ u_1u_2^2u_3^{12}u_4^3u_5^{15} + u_1^7u_2u_3^2u_4^8u_5^{15} + u_1u_2^7u_3^2u_4^8u_5^{15} + u_1u_2^2u_3^7u_4^8u_5^{15}$$
$$+ u_1u_2^2u_3^4u_4^{11}u_5^{15} + u_1^3u_2u_3^2u_4^{12}u_5^{15} + u_1u_2^3u_3^2u_4^{12}u_5^{15} + u_1u_2^2u_3^3u_4^{12}u_5^{15}$$

$$\text{Sigma5}[22] = u_1^{15}u_2u_3^{14}u_4^3 + u_1u_2^{15}u_3^{14}u_4^3 + u_1u_2^{14}u_3^{15}u_4^3 + u_1^{15}u_2^7u_3u_4^{10}$$

$$
\begin{aligned}
&+ u_1^7u_2^{15}u_3u_4^{10} + u_1^{15}u_2u_3^7u_4^{10} + u_1u_2^{15}u_3^7u_4^{10} + u_1^7u_2u_3^{15}u_4^{10}\\
&+ u_1u_2^7u_3^{15}u_4^{10} + u_1^{15}u_2u_3^6u_4^{11} + u_1u_2^{15}u_3^6u_4^{11} + u_1u_2^6u_3^{15}u_4^{11}\\
&+ u_1^{15}u_2^3u_3u_4^{14} + u_1^3u_2^{15}u_3u_4^{14} + u_1^{15}u_2u_3^3u_4^{14} + u_1u_2^{15}u_3^3u_4^{14}\\
&+ u_1^3u_2u_3^{15}u_4^{14} + u_1u_2^3u_3^{15}u_4^{14} + u_1u_2^{14}u_3^3u_4^{15} + u_1^7u_2u_3^{10}u_4^{15}\\
&+ u_1u_2^7u_3^{10}u_4^{15} + u_1u_2^6u_3^{11}u_4^{15} + u_1^3u_2u_3^{14}u_4^{15} + u_1u_2^3u_3^{14}u_4^{15}\\
&+ u_1^{15}u_2u_3^{14}u_5^3 + u_1u_2^{15}u_3^{14}u_5^3 + u_1u_2^{14}u_3^{15}u_5^3 + u_1^{15}u_2u_4^{14}u_5^3\\
&+ u_1u_2^{15}u_4^{14}u_5^3 + u_1^{15}u_3u_4^{14}u_5^3 + u_2^{15}u_3u_4^{14}u_5^3 + u_1u_3^{15}u_4^{14}u_5^3\\
&+ u_2u_3^{15}u_4^{14}u_5^3 + u_1u_2^{14}u_4^{15}u_5^3 + u_1u_3^{14}u_4^{15}u_5^3 + u_2u_3^{14}u_4^{15}u_5^3\\
&+ u_1^{15}u_2^7u_3u_5^{10} + u_1^7u_2^{15}u_3u_5^{10} + u_1^{15}u_2u_3^7u_5^{10} + u_1u_2^{15}u_3^7u_5^{10}\\
&+ u_1^7u_2u_3^{15}u_5^{10} + u_1u_2^7u_3^{15}u_5^{10} + u_1^{15}u_2^7u_4u_5^{10} + u_1^7u_2^{15}u_4u_5^{10}\\
&+ u_1^{15}u_3^7u_4u_5^{10} + u_2^{15}u_3^7u_4u_5^{10} + u_1^7u_3^{15}u_4u_5^{10} + u_2^7u_3^{15}u_4u_5^{10}\\
&+ u_1^{15}u_2u_4^7u_5^{10} + u_1u_2^{15}u_4^7u_5^{10} + u_1^{15}u_3u_4^7u_5^{10} + u_2^{15}u_3u_4^7u_5^{10}\\
&+ u_1u_3^{15}u_4^7u_5^{10} + u_2u_3^{15}u_4^7u_5^{10} + u_1^7u_2u_4^{15}u_5^{10} + u_1u_2^7u_4^{15}u_5^{10}\\
&+ u_1^7u_3u_4^{15}u_5^{10} + u_2^7u_3u_4^{15}u_5^{10} + u_1u_3^7u_4^{15}u_5^{10} + u_2u_3^7u_4^{15}u_5^{10}\\
&+ u_1^{15}u_2u_3^6u_5^{11} + u_1u_2^{15}u_3^6u_5^{11} + u_1u_2^6u_3^{15}u_5^{11} + u_1^{15}u_2u_4^6u_5^{11}\\
&+ u_1u_2^{15}u_4^6u_5^{11} + u_1^{15}u_3u_4^6u_5^{11} + u_2^{15}u_3u_4^6u_5^{11} + u_1u_3^{15}u_4^6u_5^{11}\\
&+ u_2u_3^{15}u_4^6u_5^{11} + u_1u_2^6u_4^{15}u_5^{11} + u_1u_3^6u_4^{15}u_5^{11} + u_2u_3^6u_4^{15}u_5^{11}\\
&+ u_1^{15}u_2^3u_3u_5^{14} + u_1^3u_2^{15}u_3u_5^{14} + u_1^{15}u_2u_3^3u_5^{14} + u_1u_2^{15}u_3^3u_5^{14}\\
&+ u_1^3u_2u_3^{15}u_5^{14} + u_1u_2^3u_3^{15}u_5^{14} + u_1^{15}u_2^3u_4u_5^{14} + u_1^3u_2^{15}u_4u_5^{14}\\
&+ u_1^{15}u_3^3u_4u_5^{14} + u_2^{15}u_3^3u_4u_5^{14} + u_1^3u_3^{15}u_4u_5^{14} + u_2^3u_3^{15}u_4u_5^{14}\\
&+ u_1^{15}u_2u_4^3u_5^{14} + u_1u_2^{15}u_4^3u_5^{14} + u_1^{15}u_3u_4^3u_5^{14} + u_2^{15}u_3u_4^3u_5^{14}\\
&+ u_1u_3^{15}u_4^3u_5^{14} + u_2u_3^{15}u_4^3u_5^{14} + u_1^3u_2u_4^{15}u_5^{14} + u_1u_2^3u_4^{15}u_5^{14}\\
&+ u_1^3u_3u_4^{15}u_5^{14} + u_2^3u_3u_4^{15}u_5^{14} + u_1u_3^3u_4^{15}u_5^{14} + u_2u_3^3u_4^{15}u_5^{14}\\
&+ u_1u_2^{14}u_3^3u_5^{15} + u_1^7u_2u_3^{10}u_5^{15} + u_1u_2^7u_3^{10}u_5^{15} + u_1u_2^6u_3^{11}u_5^{15}\\
&+ u_1^3u_2u_3^{14}u_5^{15} + u_1u_2^3u_3^{14}u_5^{15} + u_1u_2^{14}u_4^3u_5^{15} + u_1u_3^{14}u_4^3u_5^{15}\\
&+ u_2u_3^{14}u_4^3u_5^{15} + u_1^7u_2u_4^{10}u_5^{15} + u_1u_2^7u_4^{10}u_5^{15} + u_1^7u_3u_4^{10}u_5^{15}\\
&+ u_2^7u_3u_4^{10}u_5^{15} + u_1u_3^7u_4^{10}u_5^{15} + u_2u_3^7u_4^{10}u_5^{15} + u_1u_2^6u_4^{11}u_5^{15}\\
&+ u_1u_3^6u_4^{11}u_5^{15} + u_2u_3^6u_4^{11}u_5^{15} + u_1^3u_2u_4^{14}u_5^{15} + u_1u_2^3u_4^{14}u_5^{15}\\
&+ u_1^3u_3u_4^{14}u_5^{15} + u_2^3u_3u_4^{14}u_5^{15} + u_1u_3^3u_4^{14}u_5^{15} + u_2u_3^3u_4^{14}u_5^{15}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Sigma5}[23] = &\ u_1^{15}u_2^7u_3^{11} + u_1^7u_2^{15}u_3^{11} + u_1^7u_2^{11}u_3^{15} + u_1^{15}u_2^7u_4^{11}\\
&+ u_1^7u_2^{15}u_4^{11} + u_1^{15}u_3^7u_4^{11} + u_2^{15}u_3^7u_4^{11} + u_1^7u_3^{15}u_4^{11}\\
&+ u_2^7u_3^{15}u_4^{11} + u_1^7u_2^{11}u_4^{15} + u_1^7u_3^{11}u_4^{15} + u_2^7u_3^{11}u_4^{15}\\
&+ u_1^{15}u_2^7u_5^{11} + u_1^7u_2^{15}u_5^{11} + u_1^{15}u_3^7u_5^{11} + u_2^{15}u_3^7u_5^{11}\\
&+ u_1^7u_3^{15}u_5^{11} + u_2^7u_3^{15}u_5^{11} + u_1^{15}u_4^7u_5^{11} + u_2^{15}u_4^7u_5^{11}
\end{aligned}
$$

$$\begin{aligned}
&+ u_3^{15}u_4^{7}u_5^{11} + u_1^{7}u_4^{15}u_5^{11} + u_2^{7}u_4^{15}u_5^{11} + u_3^{7}u_4^{15}u_5^{11}\\
&+ u_1^{7}u_2^{11}u_5^{15} + u_1^{7}u_3^{11}u_5^{15} + u_2^{7}u_3^{11}u_5^{15} + u_1^{7}u_4^{11}u_5^{15}\\
&+ u_2^{7}u_4^{11}u_5^{15} + u_3^{7}u_4^{11}u_5^{15}
\end{aligned}$$

$$\begin{aligned}
\mathsf{Sigma5}[24] = {}& u_1^{15}u_2^{15}u_3u_4^{2} + u_1^{15}u_2u_3^{15}u_4^{2} + u_1u_2^{15}u_3^{15}u_4^{2} + u_1^{15}u_2u_3^{2}u_4^{15}\\
&+ u_1u_2^{15}u_3^{2}u_4^{15} + u_1u_2^{2}u_3^{15}u_4^{15} + u_1^{15}u_2^{15}u_3u_5^{2} + u_1^{15}u_2u_3^{15}u_5^{2}\\
&+ u_1u_2^{15}u_3^{15}u_5^{2} + u_1^{15}u_2^{15}u_4u_5^{2} + u_1^{15}u_3^{15}u_4u_5^{2} + u_2^{15}u_3^{15}u_4u_5^{2}\\
&+ u_1^{15}u_2u_4^{15}u_5^{2} + u_1u_2^{15}u_4^{15}u_5^{2} + u_1^{15}u_3u_4^{15}u_5^{2} + u_2^{15}u_3u_4^{15}u_5^{2}\\
&+ u_1u_3^{15}u_4^{15}u_5^{2} + u_2u_3^{15}u_4^{15}u_5^{2} + u_1^{15}u_2u_3^{2}u_5^{15} + u_1u_2^{15}u_3^{2}u_5^{15}\\
&+ u_1u_2^{2}u_3^{15}u_5^{15} + u_1^{15}u_2u_4^{2}u_5^{15} + u_1u_2^{15}u_4^{2}u_5^{15} + u_1^{15}u_3u_4^{2}u_5^{15}\\
&+ u_2^{15}u_3u_4^{2}u_5^{15} + u_1u_3^{15}u_4^{2}u_5^{15} + u_2u_3^{15}u_4^{2}u_5^{15} + u_1u_2^{2}u_4^{15}u_5^{15}\\
&+ u_1u_3^{2}u_4^{15}u_5^{15} + u_2u_3^{2}u_4^{15}u_5^{15}
\end{aligned}$$

$$\begin{aligned}
\mathsf{Sigma5}[25] = {}& u_1^{15}u_2^{15}u_3^{3} + u_1^{15}u_2^{3}u_3^{15} + u_1^{3}u_2^{15}u_3^{15} + u_1^{15}u_2^{15}u_4^{3}\\
&+ u_1^{15}u_3^{15}u_4^{3} + u_2^{15}u_3^{15}u_4^{3} + u_1^{15}u_2^{3}u_4^{15} + u_1^{3}u_2^{15}u_4^{15}\\
&+ u_1^{15}u_3^{3}u_4^{15} + u_2^{15}u_3^{3}u_4^{15} + u_1^{3}u_3^{15}u_4^{15} + u_2^{3}u_3^{15}u_4^{15}\\
&+ u_1^{15}u_2^{15}u_5^{3} + u_1^{15}u_3^{15}u_5^{3} + u_2^{15}u_3^{15}u_5^{3} + u_1^{15}u_4^{15}u_5^{3}\\
&+ u_2^{15}u_4^{15}u_5^{3} + u_3^{15}u_4^{15}u_5^{3} + u_1^{15}u_2^{3}u_5^{15} + u_1^{3}u_2^{15}u_5^{15}\\
&+ u_1^{15}u_3^{3}u_5^{15} + u_2^{15}u_3^{3}u_5^{15} + u_1^{3}u_3^{15}u_5^{15} + u_2^{3}u_3^{15}u_5^{15}\\
&+ u_1^{15}u_4^{3}u_5^{15} + u_2^{15}u_4^{3}u_5^{15} + u_3^{15}u_4^{3}u_5^{15} + u_1^{3}u_4^{15}u_5^{15}\\
&+ u_2^{3}u_4^{15}u_5^{15} + u_3^{3}u_4^{15}u_5^{15}.
\end{aligned}$$

Suppose that $[g]_{\widetilde{\omega}_{(3)}} \in ((QP_5)_{33}(\widetilde{\omega}_{(3)}))^{GL_5}$. Since $\Sigma_5 \subset GL_5$, we have

$$g \sim_{\widetilde{\omega}_{(3)}} \sum_{8\leq i\leq 25} \beta_i \cdot \mathrm{Sigma5}[i],\ \beta_i \in \mathbb{Z}/2,\ i = 8,\ 9,\ \ldots,\ 25.$$

By a direct computation using the homomorphism $\rho_5 : P_5 \longrightarrow P_5$ and the relation $\rho_5(g) \sim_{\widetilde{\omega}_{(3)}} g$, we obtain $\beta_i = 0$, for all i. Hence $((QP_5)_{33}(\widetilde{\omega}_{(3)}))^{GL_5}$ vanishes.

The proof of the theorem is complete. □

Now, the combination of (3.3) with Theorems 3.4 and 3.5 yields the inequality

$$\dim((QP_5)_{33})^{GL_5} \leq 1. \tag{3.4}$$

On the other hand, by utilizing a result obtained by Lin [13] and Chen [7], we can infer that

$$\mathrm{Ext}_{\mathbb{A}}^{5,5+d}(\mathbb{Z}/2,\mathbb{Z}/2) = \mathrm{Ext}_{\mathbb{A}}^{5,2^{t+5}+2^{t+2}+2^{t+1}}(\mathbb{Z}/2,\mathbb{Z}/2) = \mathbb{Z}/2(h_{t+1}d_{t+1}) \cong \mathbb{Z}/2, \tag{3.5}$$

where $h_{t+1}d_{t+1} \neq 0$ for all $t \geq 0$. Since $h_{s+1} \in \mathrm{Im}(\phi_1^*(\mathbb{Z}/2))$ (see Singer [32]), $d_{s+1} \in \mathrm{Im}(\phi_4^*(\mathbb{Z}/2))$ (see Ha [9]), and the "total" transfer $\bigoplus_{n\geq 0}\phi_n^*(\mathbb{Z}/2)$ is a homomorphism of algebras (see [32]), $h_{t+1}d_{t+1} \in \mathrm{Im}(\phi_5^*(\mathbb{Z}/2))$. This means that

$$\dim((QP_5)_d)^{GL_5} = \dim((QP_5)_{2^{t+5}+2^{t+2}+2^{t+1}-5})^{GL_5} \geq 1,\ \forall t \geq 0. \tag{3.6}$$

From (3.4) and (3.6), it may be concluded that the invariant space $((QP_5)_{33})^{GL_5}$ is one-dimensional.

Let $g \in (P_5)_{33}$ such that $[g] \in ((QP_5)_{33})^{GL_5}$. By using Theorem 3.4 and the fact that the Kameko $(\widetilde{Sq^0_*})_{33} : (QP_5)_{33} \longrightarrow (QP_5)_{14}$ is an epimorphism of GL_5-modules, we get

$$(\widetilde{Sq^0_*})_{33}([g]) = \gamma[\zeta(u_1, u_2, \ldots, u_5)],\ \gamma \in \mathbb{Z}/2.$$

Consequently,

$$g \sim \gamma\varphi(\zeta(u_1, u_2, \ldots, u_5)) + h,$$

where $h \in (P_5)_{33}$ such that $[h] \in \mathrm{Ker}((\widetilde{Sq^0_*})_{33})$. Applying our algorithms from Appendix 4 yields:

$$g \sim \beta(\varphi(((\zeta(u_1, u_2, \ldots, u_5)) + \xi(u_1, u_2, \ldots, u_5)),\ \beta \in \mathbb{Z}/2,$$

where

$$\begin{aligned}
\xi(u_1, u_2, \ldots, u_5) &= u_1^7u_2^7u_3^{11}u_4^8 + u_1^7u_2^7u_3^9u_4^{10} + u_1^7u_2^7u_3^8u_4^{11} + u_1^7u_2u_3^{14}u_4^{11}\\
&+ u_1^3u_2^5u_3^{14}u_4^{11} + u_1u_2^7u_3^{14}u_4^{11} + u_1^7u_2u_3^{11}u_4^{14} + u_1^3u_2^5u_3^{11}u_4^{14}\\
&+ u_1u_2^7u_3^{11}u_4^{14} + u_1^3u_2^5u_3u_4^{24} + u_1^3u_2u_3^4u_4^{25} + u_1u_2^3u_3^4u_4^{25}\\
&+ u_1^3u_2u_3u_4^{28} + u_1u_2^3u_3u_4^{28} + u_1u_2u_3^3u_4^{28} + u_1u_2u_3u_4^{30}\\
&+ u_1^7u_2^7u_3^9u_4^8u_5^2 + u_1^7u_2^7u_3^8u_4^9u_5^2 + u_1^3u_2^3u_3^{13}u_4^{12}u_5^2 + u_1^3u_2^3u_3^{12}u_4^{13}u_5^2\\
&+ u_1^7u_2^7u_3^{11}u_5^8 + u_1^7u_2^7u_3^9u_4^2u_5^8 + u_1^7u_2^9u_3^3u_4^6u_5^8 + u_1^3u_2^{13}u_3^3u_4^6u_5^8\\
&+ u_1^7u_2^3u_3^9u_4^6u_5^8 + u_1^3u_2^3u_3^{13}u_4^6u_5^8 + u_1^3u_2^5u_3^{10}u_4^7u_5^8 + u_1^7u_2^7u_3^3u_4^8u_5^8\\
&+ u_1^7u_2^7u_3u_4^{10}u_5^8 + u_1^7u_2^3u_3^5u_4^{10}u_5^8 + u_1^3u_2^7u_3^5u_4^{10}u_5^8 + u_1^7u_2u_3^7u_4^{10}u_5^8\\
&+ u_1^3u_2^5u_3^7u_4^{10}u_5^8 + u_1u_2^7u_3^7u_4^{10}u_5^8 + u_1^7u_2^7u_4^{11}u_5^8 + u_1^7u_2u_3^6u_4^{11}u_5^8\\
&+ u_1^3u_2^5u_3^6u_4^{11}u_5^8 + u_1u_2^7u_3^6u_4^{11}u_5^8 + u_1^7u_3^7u_4^{11}u_5^8 + u_1u_2^6u_3^7u_4^{11}u_5^8\\
&+ u_2^7u_3^7u_4^{11}u_5^8 + u_1^3u_2^7u_3^3u_4^{12}u_5^8 + u_1^7u_2u_3^3u_4^{14}u_5^8 + u_1^3u_2^5u_3^3u_4^{14}u_5^8\\
&+ u_1u_2^7u_3^3u_4^{14}u_5^8 + u_1u_2^3u_3^7u_4^{14}u_5^8 + u_1^7u_2^7u_3^9u_5^{10} + u_1^7u_2^7u_3^8u_4u_5^{10}\\
&+ u_1^7u_2^9u_3^3u_4^4u_5^{10} + u_1^3u_2^{13}u_3^3u_4^4u_5^{10} + u_1^7u_2^3u_3^9u_4^4u_5^{10} + u_1^3u_2^3u_3^{13}u_4^4u_5^{10}\\
&+ u_1^7u_2^8u_3^3u_4^5u_5^{10} + u_1^3u_2^{12}u_3^3u_4^5u_5^{10} + u_1^7u_2^3u_3^8u_4^5u_5^{10} + u_1^3u_2^3u_3^{12}u_4^5u_5^{10}\\
&+ u_1^3u_2u_3^{12}u_4^7u_5^{10} + u_1u_2u_3^{14}u_4^7u_5^{10} + u_1^7u_2^7u_3u_4^8u_5^{10} + u_1^7u_2^3u_3^5u_4^8u_5^{10}\\
&+ u_1^3u_2^7u_3^5u_4^8u_5^{10} + u_1^7u_2u_3^7u_4^8u_5^{10} + u_1^3u_2^5u_3^7u_4^8u_5^{10} + u_1u_2^7u_3^7u_4^8u_5^{10}\\
&+ u_1^7u_2^7u_4^9u_5^{10} + u_1^7u_2^3u_3^4u_4^9u_5^{10} + u_1^3u_2^7u_3^4u_4^9u_5^{10} + u_1^7u_3^7u_4^9u_5^{10}\\
&+ u_1^3u_2^4u_3^7u_4^9u_5^{10} + u_2^7u_3^7u_4^9u_5^{10} + u_1^3u_2^3u_3^5u_4^{12}u_5^{10} + u_1^7u_2u_3u_4^{14}u_5^{10}\\
&+ u_1^3u_2^5u_3u_4^{14}u_5^{10} + u_1u_2^7u_3u_4^{14}u_5^{10} + u_1^7u_2^7u_3^8u_5^{11} + u_1^7u_2u_3^{14}u_5^{11}\\
&+ u_1^3u_2^5u_3^{14}u_5^{11} + u_1u_2^7u_3^{14}u_5^{11} + u_1^7u_2^7u_4^8u_5^{11} + u_1^7u_2u_3^6u_4^8u_5^{11}\\
&+ u_1u_2^7u_3^6u_4^8u_5^{11} + u_1^7u_3^7u_4^8u_5^{11} + u_1u_2^6u_3^7u_4^8u_5^{11} + u_2^7u_3^7u_4^8u_5^{11}\\
&+ u_1^7u_2u_3^2u_4^{12}u_5^{11} + u_1u_2^7u_3^2u_4^{12}u_5^{11} + u_1u_2^2u_3^7u_4^{12}u_5^{11} + u_1^7u_2u_4^{14}u_5^{11}\\
&+ u_1^3u_2^5u_4^{14}u_5^{11} + u_1u_2^7u_4^{14}u_5^{11} + u_1^7u_3u_4^{14}u_5^{11} + u_1u_2^6u_3u_4^{14}u_5^{11}\\
&+ u_2^7u_3u_4^{14}u_5^{11} + u_1^3u_2u_3^4u_4^{14}u_5^{11} + u_1u_2^3u_3^4u_4^{14}u_5^{11} + u_1^3u_3^5u_4^{14}u_5^{11}
\end{aligned}$$

$$
\begin{aligned}
&+ u_1u_2^2u_3^5u_4^{14}u_5^{11} + u_2^3u_3^5u_4^{14}u_5^{11} + u_1u_3^7u_4^{14}u_5^{11} + u_2u_3^7u_4^{14}u_5^{11}\\
&+ u_1^3u_2^3u_3^{13}u_4^2u_5^{12} + u_1^3u_2^5u_3^{10}u_4^3u_5^{12} + u_1^7u_2^3u_3^3u_4^8u_5^{12} + u_1^3u_2^7u_3^3u_4^8u_5^{12}\\
&+ u_1^3u_2^3u_3^7u_4^8u_5^{12} + u_1^3u_2^5u_3^3u_4^{10}u_5^{12} + u_1^3u_2^3u_3^5u_4^{10}u_5^{12} + u_1^7u_2u_3^2u_4^{11}u_5^{12}\\
&+ u_1^3u_2^5u_3^2u_4^{11}u_5^{12} + u_1u_2^7u_3^2u_4^{11}u_5^{12} + u_1^3u_2u_3^6u_4^{11}u_5^{12} + u_1u_2^2u_3^7u_4^{11}u_5^{12}\\
&+ u_1^3u_2^3u_3u_4^{14}u_5^{12} + u_1^3u_2u_3^3u_4^{14}u_5^{12} + u_1u_2^3u_3^3u_4^{14}u_5^{12} + u_1^7u_2u_3^{11}u_5^{14}\\
&+ u_1^3u_2^5u_3^{11}u_5^{14} + u_1u_2^7u_3^{11}u_5^{14} + u_1^3u_2^3u_3^{12}u_4u_5^{14} + u_1^3u_2u_3^{12}u_4^3u_5^{14}\\
&+ u_1u_2u_3^{14}u_4^3u_5^{14} + u_1^7u_2u_3^3u_4^8u_5^{14} + u_1^3u_2^5u_3^3u_4^8u_5^{14} + u_1u_2^7u_3^3u_4^8u_5^{14}\\
&+ u_1^3u_2^3u_3^5u_4^8u_5^{14} + u_1u_2^3u_3^7u_4^8u_5^{14} + u_1^3u_2^5u_3^2u_4^9u_5^{14} + u_1^3u_2u_3^6u_4^9u_5^{14}\\
&+ u_1^7u_2u_3u_4^{10}u_5^{14} + u_1u_2^7u_3u_4^{10}u_5^{14} + u_1^3u_2u_3^5u_4^{10}u_5^{14} + u_1u_2u_3^7u_4^{10}u_5^{14}\\
&+ u_1^7u_2u_4^{11}u_5^{14} + u_1^3u_2^5u_4^{11}u_5^{14} + u_1u_2^7u_4^{11}u_5^{14} + u_1^7u_3u_4^{11}u_5^{14}\\
&+ u_1u_2^6u_3u_4^{11}u_5^{14} + u_2^7u_3u_4^{11}u_5^{14} + u_1^3u_2u_3^4u_4^{11}u_5^{14} + u_1u_2^3u_3^4u_4^{11}u_5^{14}\\
&+ u_1^3u_3^5u_4^{11}u_5^{14} + u_1u_2^2u_3^5u_4^{11}u_5^{14} + u_2^3u_3^5u_4^{11}u_5^{14} + u_1u_3^7u_4^{11}u_5^{14}\\
&+ u_2u_3^7u_4^{11}u_5^{14} + u_1u_2^3u_3^3u_4^{12}u_5^{14} + u_1u_2^3u_3u_4^{14}u_5^{14} + u_1u_2^3u_3^5u_4^8u_5^{16}\\
&+ u_1^3u_2u_3^4u_4^9u_5^{16} + u_1^3u_2u_3^4u_4^8u_5^{17} + u_1u_2^3u_3^4u_4^8u_5^{17} + u_1u_2^2u_3^5u_4^8u_5^{17}\\
&+ u_1^3u_2^5u_3u_5^{24} + u_1^3u_2^5u_4u_5^{24} + u_1^3u_3^5u_4u_5^{24} + u_2^3u_3^5u_4u_5^{24}\\
&+ u_1u_2^3u_3u_4^4u_5^{24} + u_1u_2u_3^3u_4^4u_5^{24} + u_1u_2^2u_3u_4^5u_5^{24} + u_1^3u_2u_3^4u_5^{25}\\
&+ u_1u_2^3u_3^4u_5^{25} + u_1^3u_2u_4^4u_5^{25} + u_1u_2^3u_4^4u_5^{25} + u_1^3u_3u_4^4u_5^{25}\\
&+ u_1u_2^2u_3u_4^4u_5^{25} + u_2^3u_3u_4^4u_5^{25} + u_1u_3^3u_4^4u_5^{25} + u_2u_3^3u_4^4u_5^{25}\\
&+ u_1^3u_2u_3u_5^{28} + u_1u_2^3u_3u_5^{28} + u_1u_2u_3^3u_5^{28} + u_1^3u_2u_4u_5^{28}\\
&+ u_1u_2^3u_4u_5^{28} + u_1^3u_3u_4u_5^{28} + u_2^3u_3u_4u_5^{28} + u_1u_3^3u_4u_5^{28}\\
&+ u_2u_3^3u_4u_5^{28} + u_1u_2u_4^3u_5^{28} + u_1u_3u_4^3u_5^{28} + u_2u_3u_4^3u_5^{28}\\
&+ u_1u_2u_3u_5^{30} + u_1u_2u_4u_5^{30} + u_1u_3u_4u_5^{30} + u_2u_3u_4u_5^{30}.
\end{aligned}
$$

Thus, we obtain

$$((QP_5)_{33})^{GL_5} = \langle [\varphi((\zeta(u_1, u_2, \ldots, u_5)) + \xi(u_1, u_2, \ldots, u_5)] \rangle, \tag{3.7}$$

where φ is the up Kameko map $(P_5)_{14} \longrightarrow (P_5)_{33}$, which is given by $\varphi(t) = \prod_{1\leq j\leq 5} u_jt^2$ for all $t \in (P_5)_{14}$. Let us consider the following element in $H_{33}(V^{\oplus 5})$:

$$\begin{aligned}
\widetilde{\zeta_0} &= a_1^{(1)}a_2^{(7)}a_3^{(3)}a_4^{(11)}a_5^{(11)} + a_1^{(1)}a_2^{(7)}a_3^{(3)}a_4^{(13)}a_5^{(9)} + a_1^{(1)}a_2^{(7)}a_3^{(5)}a_4^{(11)}a_5^{(9)} + a_1^{(1)}a_2^{(7)}a_3^{(5)}a_4^{(13)}a_5^{(7)}\\
&+a_1^{(1)}a_2^{(7)}a_3^{(7)}a_4^{(5)}a_5^{(13)} + a_1^{(1)}a_2^{(7)}a_3^{(7)}a_4^{(7)}a_5^{(11)} + a_1^{(1)}a_2^{(7)}a_3^{(7)}a_4^{(9)}a_5^{(9)} + a_1^{(1)}a_2^{(7)}a_3^{(9)}a_4^{(3)}a_5^{(13)}\\
&+a_1^{(1)}a_2^{(7)}a_3^{(9)}a_4^{(5)}a_5^{(11)} + a_1^{(1)}a_2^{(7)}a_3^{(9)}a_4^{(7)}a_5^{(9)} + a_1^{(1)}a_2^{(7)}a_3^{(9)}a_4^{(9)}a_5^{(7)} + a_1^{(1)}a_2^{(7)}a_3^{(11)}a_4^{(5)}a_5^{(9)}\\
&+a_1^{(1)}a_2^{(7)}a_3^{(13)}a_4^{(3)}a_5^{(9)} + a_1^{(1)}a_2^{(7)}a_3^{(13)}a_4^{(5)}a_5^{(7)} + a_1^{(1)}a_2^{(11)}a_3^{(3)}a_4^{(7)}a_5^{(11)} + a_1^{(1)}a_2^{(11)}a_3^{(3)}a_4^{(13)}a_5^{(5)}\\
&+a_1^{(1)}a_2^{(11)}a_3^{(5)}a_4^{(3)}a_5^{(13)} + a_1^{(1)}a_2^{(11)}a_3^{(5)}a_4^{(5)}a_5^{(11)} + a_1^{(1)}a_2^{(11)}a_3^{(5)}a_4^{(9)}a_5^{(7)} + a_1^{(1)}a_2^{(11)}a_3^{(5)}a_4^{(11)}a_5^{(5)}\\
&+a_1^{(1)}a_2^{(11)}a_3^{(7)}a_4^{(3)}a_5^{(11)} + a_1^{(1)}a_2^{(11)}a_3^{(7)}a_4^{(9)}a_5^{(5)} + a_1^{(1)}a_2^{(11)}a_3^{(9)}a_4^{(7)}a_5^{(5)} + a_1^{(1)}a_2^{(11)}a_3^{(11)}a_4^{(3)}a_5^{(7)}\\
&+a_1^{(1)}a_2^{(11)}a_3^{(11)}a_4^{(5)}a_5^{(5)} + a_1^{(1)}a_2^{(11)}a_3^{(13)}a_4^{(3)}a_5^{(5)} + a_1^{(1)}a_2^{(13)}a_3^{(3)}a_4^{(3)}a_5^{(13)} + a_1^{(1)}a_2^{(13)}a_3^{(3)}a_4^{(5)}a_5^{(11)}\\
&+a_1^{(1)}a_2^{(13)}a_3^{(3)}a_4^{(9)}a_5^{(7)} + a_1^{(1)}a_2^{(13)}a_3^{(3)}a_4^{(13)}a_5^{(3)} + a_1^{(1)}a_2^{(13)}a_3^{(5)}a_4^{(7)}a_5^{(7)} + a_1^{(1)}a_2^{(13)}a_3^{(5)}a_4^{(11)}a_5^{(3)}\\
&+a_1^{(1)}a_2^{(13)}a_3^{(7)}a_4^{(9)}a_5^{(3)} + a_1^{(1)}a_2^{(13)}a_3^{(9)}a_4^{(7)}a_5^{(3)} + a_1^{(1)}a_2^{(13)}a_3^{(11)}a_4^{(5)}a_5^{(3)} + a_1^{(1)}a_2^{(13)}a_3^{(13)}a_4^{(3)}a_5^{(3)}.
\end{aligned}$$

We can easily verify that the element $\widetilde{\zeta_0}$ is $\overline{\mathbb{A}}$-annihilated. In fact, we need only to consider effects of the Steenrod operations Sq^{2^k} for $0 \leq k \leq 3$ due to the unstable condition. Furthermore, we have $< \widetilde{\zeta_0}, \varphi(\zeta(u_1, \ldots, u_5)) >= 1$, where $< -, - >$ denotes the dual the canonical non-singular pairing $H_*(V^{\oplus 5}) \times H^*(V^{\oplus 5}) \longrightarrow \mathbb{Z}/2$. Thus $[\widetilde{\zeta_0}]$ is dual to $[\varphi(\zeta(u_1, \ldots, u_5))]$. This means that

$$(\mathbb{Z}/2 \otimes_{GL_5} P_{\mathbb{A}}(P_5)^*)_{33} = \langle[\widetilde{\zeta_0}]\rangle.$$

As a direct result of this event and (3.5), we obtain the following corollary.

Corollary 3.8. *The cohomological transfer*

$$\phi_5^*(\mathbb{Z}/2) : (\mathbb{Z}/2 \otimes_{GL_5} P_{\mathbb{A}}(P_5)^*)_{33} \longrightarrow \mathrm{Ext}_{\mathbb{A}}^{5,5+33}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is an isomorphism.

Cases $t \geq 1$. We observe that

$$d = (2^{t+5} - 1) + (2^{t+2} - 1) + (2^t - 1) + (2^{t-1} - 1) + (2^{t-1} - 1),$$

which implies that $\mu(d) = 5$ for any $t > 1$. So, using Theorem 2.9(ii), the iterated Kameko homomorphism

$$(\widetilde{Sq^0_*})_d^{t-1} : (QP_5)_d = (QP_5)_{2^{t+5}+2^{t+2}+2^{t+1}-5} \longrightarrow (QP_5)_{71=2^{1+5}+2^{1+2}+2^{1+1}-5}$$

is an isomorphism for every $t \geq 1$. On the other hand, since the Kameko squaring operation $(\widetilde{Sq^0_*})_{71} : (QP_5)_{71} \longrightarrow (QP_5)_{33}$ is an epimorphism and $\dim((QP_5)_{33})^{GL_5} = 1$, one has an estimate

$$\dim((QP_5)_{d=2^{t+5}+2^{t+2}+2^{t+1}-5})^{GL_5} = \dim((QP_5)_{71})^{GL_5} \leq \dim(\mathrm{Ker}((\widetilde{Sq^0_*})_{71}))^{GL_5} + 1 \tag{3.8}$$

To determine the dimension of $((QP_5)_{71})^{GL_5}$, we require the following useful lemma.

Lemma 3.9. *The invariant* $(\mathrm{Ker}((\widetilde{Sq^0_*})_{71}))^{GL_5}$ *vanishes.*

Combining this lemma with (3.6) and (3.8), we get

$$\dim((QP_5)_{d=2^{t+5}+2^{t+2}+2^{t+1}-5})^{GL_5} = \dim((QP_5)_{71})^{GL_5} = 1, \text{ for every } t \geq 1$$

Outline of the proof of Lemma 3.9. Suppose that t is an admissible monomial of degree 71 in the $\mathbb{A}$-module P_5 such that $[t] \in \mathrm{Ker}((\widetilde{Sq^0_*})_{71})$, then the weight vector $\omega(t)$ is one of the following sequences:

$$\begin{aligned} \overline{\omega}_{(1)} &:= (3,2,2,1,1,1),\ \overline{\omega}_{(2)} := (3,2,2,1,3),\ \overline{\omega}_{(3)} := (3,2,2,3,2),\ \overline{\omega}_{(4)} := (3,2,4,2,2),\\ \overline{\omega}_{(5)} &:= (3,2,4,4,1),\ \overline{\omega}_{(6)} := (3,4,1,1,1,1),\ \overline{\omega}_{(7)} := (3,4,1,1,3),\ \overline{\omega}_{(8)} := (3,4,3,2,2),\\ \overline{\omega}_{(9)} &:= (3,4,3,4,1). \end{aligned}$$

Indeed, it is straightforward to see that $z = u_1^{63}u_2^7u_3 \in \mathscr{C}^{\otimes 5}_{71}$ is the minimal spike, and $\omega(z) = (3,2,2,1,1,1)$. Since $t \in \mathscr{C}^{\otimes 5}_{71}$, and $\deg(t)$ is an odd number, by Theorem 2.10, either $\omega_1(t) = 3$ or $\omega_1(t) = 5$. If $\omega_1(t) = 5$ then $t = u_1u_2u_3u_4u_5y^2$ with y a monomial of degree 33 in $\mathbb{A}$-module P_5. Since t is admissible, by Theorem 2.5(i), one gets $y \in \mathscr{C}^{\otimes 5}_{33}$, which implies that $(\widetilde{Sq^0_*})_{71}([t]) = [y] \neq [0]$. This contradicts the fact that $[t] \in \mathrm{Ker}((\widetilde{Sq^0_*})_{71})$, and so, we must have that $\omega_1(t) = 3$. Then, using Theorem 2.5(i), t has the form $u_iu_ju_ky_1^2$ in which $1 \le i < j < k \le 5$ and $y_1 \in \mathscr{C}^{\otimes 5}_{34}$. Notice that the monomial $z' = u_1^{31}u_2^3$ is the minimal spike in $(P_5)_{34}$ and $\omega(z') = (2,2,1,1,1)$. Since y_1 is admissible and $\deg(y_1)$ is even number, by Theorem 2.10, either $\omega_1(y_1) = 2$ or $\omega_1(y_1) = 4$. If $\omega_1(y_1) = 2$ then y_1 is of the form $u_iu_jy_2^2$ with $y_2 \in (P_5)_{16}$ and $1 \le i < j \le 5$. Since $y_1 \in \mathscr{C}^{\otimes 5}_{34}$, by Theorem 2.5(i), $y_2 \in \mathscr{C}^{\otimes 5}_{16}$. Following the previous work in [36], one has $\omega(y_2) \in \{(2,1,1,1),\ (2,1,3),\ (2,3,2),\ (4,2,2),\ (4,4,1)\}$. This shows that

$$\omega(t) \in \{(3,2,2,1,1,1),\ (3,2,2,1,3),\ (3,2,2,3,2),\ (3,2,4,2,2),\ (3,2,4,4,1)\}.$$

Similarly, if $\omega_1(y_1) = 4$, then $y_1 = u_iu_ju_ku_ly_3^2$ with $1 \le i < j < k < l \le 5$ and $y_3 \in \mathscr{C}^{\otimes 5}_{15}$. According to [34], we have $\omega(y_3) \in \{(1,1,1,1),\ (1,1,3),\ (3,2,2),\ (3,4,1)\}$, and therefore,

$$\omega(t) \in \{(3,4,1,1,1,1),\ (3,4,1,1,3),\ (3,4,3,2,2),\ (3,4,3,4,1)\}.$$

This leads to an isomorphism

$$\mathrm{Ker}((\widetilde{Sq^0_*})_{71}) \cong \bigoplus_{1\le j\le 9} (QP_5)_{71}(\overline{\omega}_{(j)}).$$

Now, using the algorithms described in Appendix 4, we obtain:

$$\dim(QP_5)_{71}(\overline{\omega}_{(j)}) = \begin{cases} 1395 & \text{if } j = 1,\\ 124 & \text{if } j = 6,\\ 0 & \text{otherwise.} \end{cases}$$

Explicit bases for the spaces $(QP_5)_{71}\big(\overline{\omega}_{(j)}\big)$ are determined explicitly, as described in Appendix 4. Consequently,

$$\dim \mathrm{Ker}((\widetilde{Sq^0_*})_{71}) = \sum_{1\le j\le 9} \dim(QP_5)_{71}(\overline{\omega}_{(j)}) = 1395 + 124 = 1519.$$

Proceeding with our algorithm in Appendix 4 (noting that the manual computations are similar to those in Lemmata 3.6 and 3.7), we obtain the following:

$$\dim((QP_5)_{71}(\overline{\omega}_{(j)}))^{\Sigma_5} = \begin{cases} 27 & \text{if } j = 1,\\ 6 & \text{if } j = 6,\\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim[(QP_5)_{71}(\overline{\omega}_{(j)})]^{GL_5} = 0, \quad \forall j,$$

Consequently, $(\mathrm{Ker}((\widetilde{Sq^0_*})_{71}))^{GL_5}$ vanishes. The lemma is proved. ☐

Remark 3.10. As established above, $\dim((QP_5)_{71})^{GL_5} = 1$. Accordingly, an explicit basis (equivalently, a generator) of $((QP_5)_{71})^{GL_5}$ is as follows.

Take any $[h] \in \Big((QP_5)_{71}\Big)^{GL_5}$ with representative $h \in (P_5)_{71}$. Appealing to Lemma 3.9 and using that $(\widetilde{Sq^0_*})_{71}$ is a GL_5-epimorphism, we have

$$(\widetilde{Sq^0_*})_{71}\Big([h]\Big) = \gamma\,\Big[\varphi(\zeta(u_1,\ldots,u_5)) + \xi(u_1,\ldots,u_5)\Big], \quad \gamma \in \mathbb{Z}/2.$$

Consequently,

$$h \sim \gamma\Bigg(\varphi^2(\zeta(u_1,u_2,\ldots,u_5)) + \varphi(\xi(u_1,u_2,\ldots,u_5))\Bigg) + p,$$

where $p \in (P_5)_{71}$ such that $[p] \in \mathrm{Ker}((\widetilde{Sq^0_*})_{71})$. Based on our algorithm in Appendix 4, we get

$$h \sim \beta\Bigg(\varphi^2(\zeta(u_1,u_2,\ldots,u_5)) + \varphi(\xi(u_1,u_2,\ldots,u_5)) + \widetilde{\xi}(u_1,u_2,\ldots,u_5)\Bigg),\ \beta \in \mathbb{Z}/2,$$

where

$$\begin{aligned}
\widetilde{\xi}(u_1,u_2,\ldots,u_5) = {} & u_1^7u_2^7u_3^8u_4^{48}u_5 + u_1^{15}u_2u_3^2u_4^{52}u_5 + u_1^3u_2^{13}u_3^2u_4^{52}u_5 + u_1u_2^{15}u_3^2u_4^{52}u_5\\
&+ u_1^7u_2u_3^{10}u_4^{52}u_5 + u_1^3u_2u_3^{14}u_4^{52}u_5 + u_1u_2u_3^{14}u_4^{54}u_5 + u_1^7u_2^3u_3^4u_4^{56}u_5\\
&+ u_1^3u_2^7u_3^4u_4^{56}u_5 + u_1u_2^7u_3^6u_4^{56}u_5 + u_1u_2^6u_3^7u_4^{56}u_5 + u_1^3u_2^5u_3^2u_4^{60}u_5\\
&+ u_1u_2^7u_3^2u_4^{60}u_5 + u_1u_2^6u_3^3u_4^{60}u_5 + u_1u_2^3u_3^6u_4^{60}u_5 + u_1^3u_2u_3^4u_4^{62}u_5\\
&+ u_1^3u_2^7u_3^9u_4^{50}u_5^2 + u_1^3u_2^5u_3^{10}u_4^{51}u_5^2 + u_1u_2^3u_3^7u_4^{56}u_5^4 + u_1^3u_2^5u_3^2u_4^{57}u_5^4\\
&+ u_1u_2^7u_3^2u_4^{57}u_5^4 + u_1u_2^6u_3^3u_4^{57}u_5^4 + u_1u_2^2u_3^7u_4^{57}u_5^4 + u_1u_2^3u_3^5u_4^{58}u_5^4\\
&+ u_1^3u_2^4u_3u_4^{59}u_5^4 + u_1u_2^6u_3u_4^{59}u_5^4 + u_1^3u_2^3u_3u_4^{60}u_5^4 + u_1u_2^3u_3^2u_4^{61}u_5^4\\
&+ u_1u_2^2u_3^3u_4^{61}u_5^4 + u_1u_2^3u_3u_4^{62}u_5^4 + u_1^7u_2^7u_3^9u_4^{16}u_5^{32} + u_1^7u_2^7u_3^8u_4^{17}u_5^{32}\\
&+ u_1^3u_2^7u_3^9u_4^{18}u_5^{34} + u_1^3u_2^5u_3^{11}u_4^{18}u_5^{34} + u_1^7u_2^7u_3^8u_4u_5^{48} + u_1^{15}u_2^3u_3u_4^4u_5^{48}\\
&+ u_1^{15}u_2u_3^3u_4^4u_5^{48} + u_1^3u_2^5u_3^{11}u_4^4u_5^{48} + u_1u_2^7u_3^{11}u_4^4u_5^{48} + u_1^3u_2u_3^{15}u_4^4u_5^{48}\\
&+ u_1u_2^3u_3^{15}u_4^4u_5^{48} + u_1^{15}u_2u_3^2u_4^5u_5^{48} + u_1^7u_2u_3^{10}u_4^5u_5^{48} + u_1^3u_2^5u_3^{10}u_4^5u_5^{48}\\
&+ u_1u_2^7u_3^{10}u_4^5u_5^{48} + u_1^3u_2^3u_3^{12}u_4^5u_5^{48} + u_1^3u_2u_3^{14}u_4^5u_5^{48} + u_1^7u_2^7u_3u_4^8u_5^{48}\\
&+ u_1u_2^7u_3^7u_4^8u_5^{48} + u_1^7u_2u_3^6u_4^9u_5^{48} + u_1^3u_2^5u_3^6u_4^9u_5^{48} + u_1^7u_2u_3^3u_4^{12}u_5^{48}\\
&+ u_1u_2^7u_3^3u_4^{12}u_5^{48} + u_1^3u_2^3u_3^5u_4^{12}u_5^{48} + u_1^3u_2u_3^7u_4^{12}u_5^{48} + u_1u_2^7u_3^2u_4^{13}u_5^{48}\\
&+ u_1^3u_2u_3^4u_4^{15}u_5^{48} + u_1u_2^3u_3^4u_4^{15}u_5^{48} + u_1u_2^2u_3^5u_4^{15}u_5^{48} + u_1^{15}u_2u_3^2u_4^4u_5^{49}\\
&+ u_1^3u_2^5u_3^{10}u_4^4u_5^{49} + u_1u_2^7u_3^{10}u_4^4u_5^{49} + u_1^7u_2^3u_3^4u_4^8u_5^{49} + u_1^7u_2u_3^6u_4^8u_5^{49}\\
&+ u_1u_2^7u_3^6u_4^8u_5^{49} + u_1^7u_2u_3^2u_4^{12}u_5^{49} + u_1u_2^7u_3^2u_4^{12}u_5^{49} + u_1u_2^3u_3^6u_4^{12}u_5^{49}\\
&+ u_1u_2^2u_3^7u_4^{12}u_5^{49} + u_1^3u_2u_3^4u_4^{14}u_5^{49} + u_1u_2^2u_3^4u_4^{15}u_5^{49} + u_1^3u_2^7u_3^9u_4^2u_5^{50}\\
&+ u_1^3u_2^7u_3u_4^{10}u_5^{50} + u_1^3u_2^3u_3^5u_4^{10}u_5^{50} + u_1^3u_2u_3^7u_4^{10}u_5^{50} + u_1u_2^3u_3^7u_4^{10}u_5^{50}
\end{aligned}$$

$$
\begin{aligned}
&+ u_1^{3}u_2^{5}u_3^{10}u_4^{2}u_5^{51} + u_1^{3}u_2^{5}u_3^{2}u_4^{10}u_5^{51} + u_1^{3}u_2u_3^{6}u_4^{10}u_5^{51} + u_1u_2^{3}u_3^{6}u_4^{10}u_5^{51}\\
&+ u_1^{15}u_2u_3^{2}u_4u_5^{52} + u_1^{3}u_2^{13}u_3^{2}u_4u_5^{52} + u_1u_2^{15}u_3^{2}u_4u_5^{52} + u_1^{7}u_2^{3}u_3^{8}u_4u_5^{52}\\
&+ u_1^{7}u_2u_3^{10}u_4u_5^{52} + u_1^{3}u_2^{4}u_3^{11}u_4u_5^{52} + u_1u_2^{6}u_3^{11}u_4u_5^{52} + u_1^{3}u_2u_3^{14}u_4u_5^{52}\\
&+ u_1u_2^{3}u_3^{14}u_4u_5^{52} + u_1^{3}u_2^{13}u_3u_4^{2}u_5^{52} + u_1u_2^{15}u_3u_4^{2}u_5^{52} + u_1^{3}u_2u_3^{13}u_4^{2}u_5^{52}\\
&+ u_1^{3}u_2u_3^{12}u_4^{3}u_5^{52} + u_1u_2^{3}u_3^{12}u_4^{3}u_5^{52} + u_1u_2^{2}u_3^{13}u_4^{3}u_5^{52} + u_1u_2u_3^{14}u_4^{3}u_5^{52}\\
&+ u_1^{7}u_2^{3}u_3u_4^{8}u_5^{52} + u_1^{7}u_2u_3^{3}u_4^{8}u_5^{52} + u_1u_2^{7}u_3^{3}u_4^{8}u_5^{52} + u_1^{3}u_2u_3^{7}u_4^{8}u_5^{52}\\
&+ u_1^{7}u_2u_3^{2}u_4^{9}u_5^{52} + u_1^{3}u_2^{5}u_3^{2}u_4^{9}u_5^{52} + u_1u_2^{7}u_3^{2}u_4^{9}u_5^{52} + u_1^{3}u_2u_3^{6}u_4^{9}u_5^{52}\\
&+ u_1u_2^{2}u_3^{7}u_4^{9}u_5^{52} + u_1^{7}u_2u_3u_4^{10}u_5^{52} + u_1u_2^{7}u_3u_4^{10}u_5^{52} + u_1^{3}u_2u_3^{4}u_4^{11}u_5^{52}\\
&+ u_1^{3}u_2^{3}u_3u_4^{12}u_5^{52} + u_1u_2^{3}u_3^{2}u_4^{13}u_5^{52} + u_1u_2^{2}u_3^{3}u_4^{13}u_5^{52} + u_1^{3}u_2u_3u_4^{14}u_5^{52}\\
&+ u_1u_2u_3^{2}u_4^{15}u_5^{52} + u_1u_2^{3}u_3^{12}u_4^{2}u_5^{53} + u_1u_2^{2}u_3^{13}u_4^{2}u_5^{53} + u_1u_2^{2}u_3^{12}u_4^{3}u_5^{53}\\
&+ u_1^{3}u_2^{5}u_3^{2}u_4^{8}u_5^{53} + u_1^{3}u_2^{3}u_3^{4}u_4^{8}u_5^{53} + u_1^{3}u_2u_3^{6}u_4^{8}u_5^{53} + u_1u_2^{3}u_3^{6}u_4^{8}u_5^{53}\\
&+ u_1^{3}u_2u_3^{4}u_4^{10}u_5^{53} + u_1u_2^{3}u_3^{4}u_4^{10}u_5^{53} + u_1u_2^{2}u_3^{5}u_4^{10}u_5^{53} + u_1u_2u_3^{6}u_4^{10}u_5^{53}\\
&+ u_1^{3}u_2u_3^{2}u_4^{12}u_5^{53} + u_1u_2^{2}u_3u_4^{14}u_5^{53} + u_1^{3}u_2u_3^{12}u_4u_5^{54} + u_1^{3}u_2u_3u_4^{12}u_5^{54}\\
&+ u_1u_2u_3^{3}u_4^{12}u_5^{54} + u_1u_2^{2}u_3u_4^{13}u_5^{54} + u_1u_2u_3u_4^{14}u_5^{54} + u_1^{3}u_2^{7}u_3^{4}u_4u_5^{56}\\
&+ u_1^{7}u_2u_3^{6}u_4u_5^{56} + u_1u_2^{7}u_3^{6}u_4u_5^{56} + u_1u_2^{6}u_3^{7}u_4u_5^{56} + u_1^{7}u_2^{3}u_3u_4^{4}u_5^{56}\\
&+ u_1^{3}u_2^{7}u_3u_4^{4}u_5^{56} + u_1^{7}u_2u_3^{3}u_4^{4}u_5^{56} + u_1^{3}u_2^{5}u_3^{3}u_4^{4}u_5^{56} + u_1u_2^{7}u_3^{3}u_4^{4}u_5^{56}\\
&+ u_1^{3}u_2^{3}u_3^{5}u_4^{4}u_5^{56} + u_1^{3}u_2u_3^{7}u_4^{4}u_5^{56} + u_1u_2^{3}u_3^{7}u_4^{4}u_5^{56} + u_1^{3}u_2^{4}u_3^{3}u_4^{5}u_5^{56}\\
&+ u_1u_2^{6}u_3^{3}u_4^{5}u_5^{56} + u_1^{3}u_2u_3^{6}u_4^{5}u_5^{56} + u_1u_2^{3}u_3^{6}u_4^{5}u_5^{56} + u_1u_2^{2}u_3^{7}u_4^{5}u_5^{56}\\
&+ u_1^{7}u_2u_3u_4^{6}u_5^{56} + u_1u_2^{7}u_3u_4^{6}u_5^{56} + u_1^{3}u_2u_3^{5}u_4^{6}u_5^{56} + u_1u_2^{3}u_3^{5}u_4^{6}u_5^{56}\\
&+ u_1u_2^{6}u_3u_4^{7}u_5^{56} + u_1^{7}u_2u_3^{2}u_4^{4}u_5^{57} + u_1u_2^{6}u_3^{3}u_4^{4}u_5^{57} + u_1^{3}u_2^{3}u_3^{4}u_4^{4}u_5^{57}\\
&+ u_1^{3}u_2u_3^{6}u_4^{4}u_5^{57} + u_1u_2^{3}u_3^{6}u_4^{4}u_5^{57} + u_1^{3}u_2^{4}u_3u_4^{6}u_5^{57} + u_1^{3}u_2u_3^{4}u_4^{6}u_5^{57}\\
&+ u_1u_2^{3}u_3^{4}u_4^{6}u_5^{57} + u_1u_2^{2}u_3^{5}u_4^{6}u_5^{57} + u_1u_2u_3^{6}u_4^{6}u_5^{57} + u_1u_2^{2}u_3^{4}u_4^{7}u_5^{57}\\
&+ u_1^{7}u_2u_3^{2}u_4u_5^{60} + u_1^{3}u_2^{5}u_3^{2}u_4u_5^{60} + u_1u_2^{7}u_3^{2}u_4u_5^{60} + u_1u_2^{6}u_3^{3}u_4u_5^{60}\\
&+ u_1^{3}u_2u_3^{6}u_4u_5^{60} + u_1u_2^{3}u_3^{6}u_4u_5^{60} + u_1^{3}u_2^{5}u_3u_4^{2}u_5^{60} + u_1u_2^{7}u_3u_4^{2}u_5^{60}\\
&+ u_1^{3}u_2^{4}u_3u_4^{3}u_5^{60} + u_1u_2^{6}u_3u_4^{3}u_5^{60} + u_1u_2u_3^{6}u_4^{3}u_5^{60} + u_1^{3}u_2u_3^{3}u_4^{4}u_5^{60}\\
&+ u_1^{3}u_2u_3^{2}u_4^{5}u_5^{60} + u_1u_2^{3}u_3u_4^{6}u_5^{60} + u_1u_2u_3^{2}u_4^{7}u_5^{60} + u_1^{3}u_2u_3^{4}u_4^{2}u_5^{61}\\
&+ u_1u_2^{3}u_3^{4}u_4^{2}u_5^{61} + u_1u_2^{2}u_3^{5}u_4^{2}u_5^{61} + u_1u_2u_3^{6}u_4^{2}u_5^{61} + u_1^{3}u_2u_3^{2}u_4^{4}u_5^{61}\\
&+ u_1u_2^{3}u_3^{2}u_4^{4}u_5^{61} + u_1u_2^{2}u_3u_4^{6}u_5^{61} + u_1^{3}u_2^{4}u_3u_4u_5^{62} + u_1u_2^{6}u_3u_4u_5^{62}\\
&+ u_1^{3}u_2u_3^{4}u_4u_5^{62} + u_1^{3}u_2u_3u_4^{4}u_5^{62} + u_1u_2u_3^{3}u_4^{4}u_5^{62} + u_1u_2^{2}u_3u_4^{5}u_5^{62} + u_1u_2u_3u_4^{6}u_5^{62}.
\end{aligned}
$$

By direct computational verification, we also obtain

$$
\begin{aligned}
&\rho_i\Bigg(\varphi^2(\zeta(u_1,u_2,\ldots,u_5)) + \varphi(\xi(u_1,u_2,\ldots,u_5)) + \widetilde{\xi}(u_1,u_2,\ldots,u_5)\Bigg)\\
&\sim \varphi^2(\zeta(u_1,u_2,\ldots,u_5)) + \varphi(\xi(u_1,u_2,\ldots,u_5)) + \widetilde{\xi}(u_1,u_2,\ldots,u_5),\ \forall i,\ 1\leq i\leq 5.
\end{aligned}
$$

Thus, we get

$$((QP_5)_{71})^{GL_5} = \langle[\varphi^2(\zeta(u_1, u_2, \ldots, u_5)) + \varphi(\xi(u_1, u_2, \ldots, u_5)) + \widetilde{\xi}(u_1, u_2, \ldots, u_5)]\rangle.$$

By combining this result with (3.5) and taking into account the fact that $h_{t+1}d_{t+1} \in \mathrm{Im}(\phi_5^*(\mathbb{Z}/2))$, we can immediately deduce

Corollary 3.11. *The cohomological transfer*

$$\phi_5^*(\mathbb{Z}/2) : (\mathbb{Z}/2 \otimes_{GL_5} P_{\mathbb{A}}(P_5)^*)_{2^{t+5}+2^{t+2}+2^{t+1}-5} \longrightarrow \mathrm{Ext}_{\mathbb{A}}^{5,2^{t+5}+2^{t+2}+2^{t+1}}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is an isomorphism for all $t \geq 1$.

Corollaries 3.8 and 3.11 indicate that

Corollary 3.12. *Singer's conjecture is confirmed for the cohomological transfer* $\phi_n^*(\mathbb{Z}/2)$ *in the case when* $n = 5$ *and in the general degree* $2^{t+5} + 2^{t+2} + 2^{t+1} - 5$ *for arbitrary non-negative integer* t.

4. Appendix

This appendix provides links to our algorithm implemented on the computer algebra system `OSCAR`, as well as detailed computational output data for $(QP_5)_d$ with $d = 33$ and 71, together with their invariants.

• The `OSCAR` implementation of our algorithm for computing $(QP_5)_d$ for $d = 33$ and 71 together with their invariants is available at GitHub: https://github.com/phucdv2018/Code-OSCAR-33-and-71/releases/tag/v1.0.0 (archived at DOI: https://doi.org/10.5281/zenodo.17719793) and included as a supplementary file `Code_OSCAR_33_and_71`

Note that in this `OSCAR` code, within the section

```
-------- RUN (example) --------
ultra_sparse_bitpacked_main(5, 33)
```

to obtain results for $d = 71$, one simply replaces the pair $(5, 33)$ by $(5, 71)$.

• Computational data for $(QP_5)_{d=33}$ and its invariants are available at Zenodo: https://doi.org/10.5281/zenodo.17601723.

• Computational data for $(QP_5)_{d=71}$ and its invariants are available at Zenodo: https://doi.org/10.5281/zenodo.17601864.

Declaration of Competing Interests

The author declares that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data Availability

The data supporting the findings of this study are openly available at the GitHub repository and Zenodo archive referenced in Appendix 4.

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Department of Mathematics, FPT University, Quy Nhon AI Campus, An Phu Thinh New Urban Area, Quy Nhon City, Binh Dinh, Vietnam

Email address: dangphuc150488@gmail.com