Computing Equilibrium Nominations in Presidential Elections

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Abstract

We study strategic candidate nomination by parties in elections decided by Plurality voting. Each party selects a nominee before the election, and the winner is chosen from the nominated candidates based on the voters' preferences. We introduce a new restriction on these preferences, which we call *party-aligned single-peakedness*: all voters agree on a common ordering of the parties along an ideological axis, but may differ in their perceptions of the positions of individual candidates within each party. The preferences of each voter are single-peaked with respect to their own axis over the candidates, which is consistent with the global ordering of the parties. We present a polynomial-time algorithm for recognizing whether a preference profile satisfies party-aligned single-peakedness. In this domain, we give polynomial-time algorithms for deciding whether a given party can become the winner under some (or all) nominations, and whether this can occur in some pure Nash equilibrium. We also prove a tight result about the guaranteed existence of pure strategy Nash equilibria for elections with up to three parties for single-peaked and party-aligned single-peaked preference profiles.

1 Introduction

Let us consider a certain computer science department that needs to select its new head. The department has three research groups, one focused on artificial intelligence, one working on theoretical computer science, and one devoted to distributed systems. In the past, each group had just one candidate, and the person who got the most votes was selected as the head, for a four-year term. However, this time more people expressed their interest in the position; the groups—worried by the possibility of splitting the vote—decided that each of them will nominate only a single one. Yet, how should they decide on whom to choose?

Faliszewski et al. (2016) studied this problem by looking for a *necessary winner*, i.e., a candidate who wins irrespective of the nominations by the other groups, or by looking for a *possible winner*, i.e., a candidate who wins, provided the other groups' nominations are favourable (formally, they referred to such candidates as *necessary* and *possible president*, as they followed a naming scheme from politics). Yet, asking for a necessary winner is too demanding—such a candidate may fail to exist—and choosing a possible winner is too optimistic—there is no reason for the other groups to cooperate. Hence, we are looking for a game-theoretic solution: we model the scenario as a game, where each group is a player, each possible nominee is a strategy, and we are looking for a pure strategy Nash equilibrium (a group's utility is 1 if its nominee wins and it is 0 otherwise), focusing on the classic Plurality rule (whoever gets the most votes wins). We are interested in two types of results: we analyse when equilibria are guaranteed to exist and, if that is not the case, what is the complexity of deciding their existence.

Henceforth, we follow the politics-based naming convention of Faliszewski et al. (2016): we speak of parties that nominate candidates, and we refer to the candidates who win under a given pure strategy Nash equilibrium as *equilibrium presidents*.

Our Contribution. We find that the guarantees for equilibria existence, as well as the complexity of recognising equilibria, are very fragile and strongly depend on the nature of the voters' preferences. In particular, we distinguish three types of preference profiles that the voters may have. First, we analyse 1D-Euclidean preferences (Enelow and Hinich, 1984, 1990), where each candidate and each voter is a point on the line, and the voters rank the candidates in the order of their distance from their points. Second, we consider the classic single-peaked setting (Black, 1958), where there is a commonly agreed order (or axis) of the candidates—such as the political left-to-right spectrum—and every prefix of every vote forms an interval within this axis. Finally, we introduce the notion of *party-aligned single-peakedness*.

In party-aligned single-peaked preferences, parties are ordered along a given axis, but each voter has their perceived axis, obtained from the party axis by replacing each party with its members, in whatever order the voter prefers. In other words, all voters agree on the positioning of the parties along the party axis, but they may disagree on the precise positioning of the parties' candidates. Such differences may stem, for example, from differing beliefs about some better known or local candidate, whom a voter perceives as more (or less) aligned with their ideals than other candidates from the same party. For example, consider parties $A = \{a_1, a_2\}$ and $B = \{b\}$ and the next three votes:

$$v_1 : a_2 \succ b \succ a_1; \quad v_2 : a_1 \succ b \succ a_2; \quad v_3 : a_1 \succ a_2 \succ b$$

based on their following perceived axes:

$$v_1, v_3 : a_1 \lhd a_2 \lhd b; \qquad v_2 : a_2 \lhd a_1 \lhd b.$$

These preferences are not single-peaked—as they rank three different candidates in the last position, which is impossible in single-peaked elections—but they are party-aligned single-peaked. The mismatch in voters' perception can give rise to non-trivial strategic decisions from the parties' side. While party B can only nominate b, party A has to make a choice between a_1 and a_2. If party A nominates a_2, voter v_2 perceives party B as preferable to party A. By contrast, this is not the case for v_1, who prefers a_2 to b. However, should party A nominate a_1 instead, they would end up gaining the vote of v_2 but losing that of v_1 to party B. Essentially, under party-aligned single-peakedness, a move to steal voters from one party can lead to losing some other ones.

We defer some of our proofs to the appendix and mark such results with the $\star$ symbol.

Remark 1. Though neither 1D-Euclidean nor single-peaked preferences need to put members of a given party on consecutive positions on the common axis (or line), all voters have to follow the same one. By contrast, party-aligned single-peaked preferences require members of each party to take consecutive positions on each axis, but voters can freely order specific party members along their individual axes. ⌟

We establish the following results:

1. Pure strategy Nash equilibria always exist, provided there are at most three parties and the preferences are both single-peaked and party-aligned single-peaked, but may fail to exist for four parties (even with at most two members in each); for 1D-Euclidean elections, equilibria may fail to exist even for two parties of size at most two.

2. Whether a pure strategy Nash equilibrium exists in a party-aligned single-peaked election can be decided in polynomial time, but the problem is NP-hard in 1D-Euclidean elections (and, hence, under single-peaked ones). Deciding if a given party can win in some pure strategy Nash equilibrium, or if it can win under some or all nominations from other parties, is also polynomial-time solvable in party-aligned single-peaked elections.

Remark 2. We focus on pure strategy Nash equilibria, which we henceforth simply refer to as Nash equilibria. Although allowing for randomisation is an important aspect of strategic decisions, we believe "pure" candidate selections are natural and often occurring events in the decision-making process of parties, e.g., through the use of primaries Borodin et al. (2024). Besides related models of strategic nominations Harrenstein et al. (2021); Sabato et al. (2017), the focus on pure strategy Nash equilibria is also widely established in other areas of computer science, such as the logical analysis of strategic reasoning in games, including, e.g., Boolean games Ågotnes et al. (2013); Gutierrez et al. (2021). ⌟

Related Literature. The line of research on party-based election models, as introduced by Faliszewski et al. (2016), has been expanded by taking into account different voting rules and investigating the related problems also from the parameterised complexity viewpoint by Misra (2019), Cechlárová et al. (2023), Schlotter et al. (2024), and Schlotter and Cechlárová (2025). In a related vein, Pierczynski and Szufa (2024) analysed a family of rules where nominations are not needed.

Finding a Nash equilibrium was also studied in a similar model describing strategic nominations over different territories by Harrenstein and Turrini (2022). Equilibrium existence and computation for nominee selection problems was further studied in the context of the Hotelling-Downs framework where both voters and candidates are located on a line, which induces a natural preference ordering of candidates Harrenstein et al. (2021); Deligkas et al. (2022). In a related paper, Sabato et al. restricts candidates' and voters' positioning to intervals, which has strong implications on equilibrium existence. Decision-making models in which strategic nomination has been studied include tournaments played by coalitions of players Lisowski et al. (2022); Lisowski (2022).

Another important related line of research is *strategic candidacy*; see Brill and Conitzer (2015); Dutta et al. (2001); Eraslan and McLennan (2004). There, candidates have preferences over their opponents and may decide not to participate in an election to achieve a more favoured outcome. Such strategic behaviour may arise, e.g., due to certain costs incurred by participation Obraztsova et al. (2015); Elkind et al. (2015). Finally, we mention that our research is directly related to the study of nominee selection in primaries; see, e.g., the work of Borodin et al. (2019).

Structured domains are surveyed in depth by Elkind et al. (2022). A variant of single-peakedness closely related to ours is the domain due to Cornaz et al. (2012, 2013) where each voter has to rank members of each party consecutively; note that this is not required for party-aligned single-peakedness.

2 Preliminaries

Strategic Nominee Selection

For an integer $k \geqslant 0$ let $[k] = \{1, \dots, k\}$ with $[0] = \varnothing$.

Consider a set C of candidates and a set $\mathcal{P} = \{P_1, \dots, P_k\}$ of *parties* where $\mathcal{P}$ is a partitioning of the candidate set C. Let $V = \{v_1, \dots, v_n\}$ be a set of voters, where each voter v has a strict preference order $\succ_v$ over the candidate set C; we call the collection $(\succ_v)_{v \in V}$ *preference profile* of the voters.

Our framework consists of two phases: a *nomination phase*, where each party selects its candidate, and an *election phase*, where voters choose among the nominated candidates based on their preferences. Thus, a *nomination scheme* is a tuple $(c_1, \dots, c_k)$ where $c_i \in P_i$ for all $i \in [k]$. Given a nomination scheme $(c_1, \dots, c_k)$, the *reduced election* takes place over the set $C' = \{c_1, \dots, c_k\}$ of *nominees*. We use the *Plurality* voting rule where each voter *votes for* their favourite candidate among C', and the candidates with the highest number of votes obtained—their *score*—are the *winners*; the party whose nominee is a winner is also called a *winner*. Some candidate or party is a winner in some nomination scheme $(c_1, \dots, c_k)$ if it is a winner in the reduced election over $\{c_1, \dots, c_k\}$.

We say that a candidate $c_i' \in P_i$ is a *Nash deviation* by party P_i for some nomination scheme $(c_1, \dots, c_k)$ if $c_i' \in P_i \setminus \{c_i\}$, and c_i is not a winner in the election resulting from $(c_1, \dots, c_k)$ but c_i'

is a winner in the election resulting from $(c_1, \dots, c_i', \dots, c_k)$. A nomination scheme is a *(pure) Nash equilibrium* if it admits no Nash deviation.

Structured Preferences

Let us now introduce two types of restrictions on voters' preference profiles that may arise in elections where there is a single ideological line which strongly determines voters' behaviour such as, e.g., the left-to-right axis in politics.

Single-Peaked Profiles. We say that a preference profile $(\succ_v)_{v \in V}$ is *single-peaked* if there exists a total order $\lhd$ over the set C of candidates called an *axis* such that the preferences of each voter v are *single-peaked with respect to the axis* $\lhd$, meaning that for every three candidates $a, b, c \in C$ with $a \lhd b \lhd c$ the relation $a \succ_v b$ implies $b \succ_v c$. Roughly speaking, this means that candidates can be placed along a line such that each voter v' preferences are derived from the candidates' distances to v's "ideal point" on the line.

1-D Euclidean Profiles. A special case of single-peaked profiles appears in *1-D Euclidean elections* where voters and candidates are located on the real line, and voters' preferences over candidates stem from their distances.

Definition 1. *We say that an election $E = (C, V)$ is* 1-D Euclidean *if there is a mapping $f : C \cup V \to \mathbb{R}$, such that for every voter v in V and every pair c_i, c_j in C, $c_i \succ_v c_j$ if and only if $|f(v) - f(c_i)| < |f(v) - f(c_j)|$.*

Although 1-D Euclidean elections allow for voters to be indifferent between candidates that are equally close to them, we will ensure that the 1-D Euclidean elections used in this paper induce strict preferences.

3 Party-Aligned Single-Peakedness

We assume that parties can be ordered along a global axis recognised by all voters, with candidates belonging to the same party placed contiguously. However, candidates within a party may be perceived differently by voters, who therefore may disagree on where different candidates within a given party are located along the axis. Formally, we say that a preference profile is *party-aligned single-peaked* if there exists a *party axis* $\lhd_{\mathcal{P}}$, i.e., a total order over the set $\mathcal{P}$ of parties such that, for every voter $v \in V$, there exists a *perceived axis* $\lhd_v$ over the candidate set C for which

- v's preference order $\succ_v$ is single-peaked w.r.t. $\lhd_v$, and
- $\lhd_v$ is *compatible* with $\lhd_{\mathcal{P}}$, meaning that for each two candidates c_i and c_j belonging to different parties P_i and P_j, respectively, $c_i \lhd_v c_j$ implies $P_i \lhd_{\mathcal{P}} P_j$.

Example 1. Consider a set $\mathcal{P} = \{P_a, P_b, P_c, P_d\}$ of parties with $P_a = \{a\}$, $P_b = \{b_1, b_2\}$, $P_c = \{c_1, c_2\}$, and $P_d = \{d\}$, and three voters whose preferences are:

$$v : c_2 \succ b_1 \succ c_1 \succ d \succ b_2 \succ a;$$
$$w : b_1 \succ c_1 \succ b_2 \succ c_2 \succ a \succ d;$$
$$z : b_2 \succ c_2 \succ c_1 \succ d \succ b_1 \succ a.$$

The voters' preferences are party-aligned single-peaked with party axis $P_a \lhd_{\mathcal{P}} P_b \lhd_{\mathcal{P}} P_c \lhd_{\mathcal{P}} P_d$ and perceived axes:

$$a \lhd_v b_2 \lhd_v b_1 \lhd_v c_2 \lhd_v c_1 \lhd_v d;$$
$$a \lhd_w b_2 \lhd_w b_1 \lhd_w c_1 \lhd_w c_2 \lhd_w d;$$
$$a \lhd_z b_1 \lhd_z b_2 \lhd_z c_2 \lhd_z c_1 \lhd_z d. \qquad \lrcorner$$

Recognising Party-Aligned Single-Peaked Profiles.

We present a polynomial-time algorithm for recognising preference profiles that are party-aligned single-peaked. Formally, we deal with the following computational problem.

> RECOGNISING PARTY-ALIGNED SINGLE-PEAKEDNESS
>
> **Input:** A set C of candidates partitioned into a set $\mathcal{P}$ of parties, and a set V of voters with preference profile $(\succ_v)_{v\in V}$ over the candidate set C.
>
> **Question:** Is the preference profile $(\succ_v)_{v\in V}$ party-aligned single-peaked?

The rest of this section proves the following result.

Theorem 1. RECOGNIZING PARTY-ALIGNED SINGLE-PEAKEDNESS *can be solved, and a suitable party axis—if existent—can be computed, in* $O(|C| \cdot |V| \cdot |\mathcal{P}|)$ *time.*

Verification of Party-Aligned Single-Peakedness Under Fixed Party Ordering.

First, we formulate the necessary and sufficient conditions for a single-vote profile to be party-aligned single-peaked under a fixed ordering of parties.

Lemma 1 ($\star$). *Consider a vote $\succ_v$ over the candidate set of parties $P_1, \ldots, P_m$. For each party P_i, let h_i and l_i denote v's most- and least-preferred candidate within P_i. Let P_j be the party containing v's top candidate. Then $\succ_v$ is party-aligned single-peaked with party axis $P_1 \lhd_{\mathcal{P}} \cdots \lhd_{\mathcal{P}} P_m$ if and only if the following conditions are met.*

(a) If $P_j \notin \{P_1, P_m\}$, then $l_j \succ_v h_{j-1}$ or $l_j \succ_v h_{j+1}$.

(b) For each parties P_i, P_{i+1} with $i > j$ we have $l_i \succ_v h_{i+1}$.

(c) For each parties P_{i-1}, P_i with $i < j$ we have $l_i \succ_v h_{i-1}$.

Notice that whenever there are at most two parties, then conditions (1)–(3) of Lemma 1 are automatically true for any fixed party axis. Thus, we have the following consequence.

Corollary 1. *Every preference profile with at most two parties is party-aligned single-peaked.*

Placing the Bottom Candidates.

Following an idea by Escoffier et al. (2008), we present a lemma about the placement of a party that contains the lowest-ranked candidate in some vote.

Lemma 2 ($\star$). *Let $c \in P_i$ be the candidate ranked in the last position by a voter v. If $\succ_v$ is party-aligned single-peaked with party axis $\lhd_{\mathcal{P}}$, then P_i is either in the leftmost or the rightmost position in $\lhd_{\mathcal{P}}$.*

As a consequence of Lemma 2, in any party-aligned single-peaked consistent profile, at most two parties may have candidates that are ranked last by at least one voter.

Vote-Imposed Restrictions.

We move on to present some conditions that allow us to extend a partially determined party axis based on the given preference profile.

Suppose that we already know the position of certain parties in the party axis $\lhd_{\mathcal{P}}$ to be constructed. In particular, assume that we have an *extremal party placement* (L, R) *for* $\lhd_{\mathcal{P}}$ which consists of two sets of parties $L, R \subseteq \mathcal{P}$ such that L contains the $|L|$ leftmost parties and R contains the $|R|$ rightmost parties according to $\lhd_{\mathcal{P}}$.

Lemma 3 ($\star$). *Let (L, R) be an extremal party placement for some party axis $\lhd_{\mathcal{P}}$, and consider a vote $\succ_v$ that is party-aligned single-peaked with axis $\lhd_{\mathcal{P}}$. Assume that candidates in $\bigcup(R \cup L)$ do not form the suffix of the vote $\succ_v$. Let*

- *a be v's least favourite candidate in $\bigcup(\mathcal{P} \setminus (L \cup R))$,*
- *b be v's most favourite candidate in $\bigcup(L \cup R)$, and*
- *P_a and P_b be the parties containing a and b, respectively.*

If $P_b \in R$, then P_a directly follows the last party in L according to $\lhd_{\mathcal{P}}$, while if $P_b \in L$, then P_a directly precedes the first party in R according to $\lhd_{\mathcal{P}}$.

Reducing the Problem.

When Lemma 3 can no longer be applied, we use recursion based on the following lemma.

Lemma 4 ($\star$). *Let $\Pi = (\succ_v)_{v \in V}$ be a party-aligned single-peaked preference profile over candidate set C with a party axis $P_1 \lhd_P \cdots \lhd_{\mathcal{P}} P_m$. Assume further that the candidates in $(P_1 \cup \cdots \cup P_i) \cup (P_j \cup \cdots \cup P_m)$ form a suffix of each vote in Π, and let Q be the set of candidates not in this suffix. If the restriction of Π to Q is party-aligned single-peaked with a party axis $Q_1 \lhd_Q \cdots \lhd_Q Q_k$, then Π is party-aligned single-peaked with the party axis $P_1, \ldots, P_i, Q_1, \ldots, Q_k, P_j, \ldots, P_m$.*

Algorithm.

Armed with the aforementioned results, we present the recognition algorithm proving Theorem 1.

Placing Bottom Parties. Given a preference profile Π, we begin by identifying the set $B \subseteq \mathcal{P}$ of bottom parties, i.e., those containing a candidate appearing at the last position in some vote. Lemma 2 shows that all parties in B must appear at either the left- or the rightmost position of the hypothetical party axis $\lhd_{\mathcal{P}}$. In particular, if $|B| > 2$, then the profile is not party-aligned single-peaked, and we reject. If $|B| \leqslant 2$, then the exact placement of the parties in B is not important, because for every party axis witnessing the party-aligned single-peakedness of our profile, the reversed party axis does the same. Hence, we place the parties in B as the left- and (if $|B| = 2$) rightmost parties in $\lhd_{\mathcal{P}}$, creating an extremal party placement (L, R) for $\lhd_{\mathcal{P}}$ with $L \cup R = B \neq \varnothing$.

Vote-Imposed Restrictions. Next, we iterate over the set of voters to check whether they impose some restriction on $\lhd_{\mathcal{P}}$. More precisely, we apply Lemma 3 as long as possible, extending our extremal party placement (L, R)—together with the ordering of the parties in $L \cup R$ according to $\lhd_{\mathcal{P}}$—by fixing the placement of an additional party in the fashion determined by Lemma 3 in each step.

Reducing the Problem. If at some point there is no voter for which Lemma 3 can be applied, then the set of candidates contained in our current extremal party placement (L, R) forms a suffix in each vote. Hence, in line with Lemma 4, we delete all parties in $L \cup R$ together with all of their candidates from the profile Π, and find a suitable party axis for the resulting profile Π' (containing at least one party less than Π) in a recursive manner. Note that if Π is party-aligned single-peaked, then so is Π', and a party axis witnessing the latter can be turned into a party axis for Π using Lemma 4 and the ordering of the parties contained in the extremal party placement (L, R) determined so far.

Checking the Resulting Profile. After obtaining a party axis $\lhd_{\mathcal{P}}$, we check if profile Π is party-aligned single-peaked under $\lhd_{\mathcal{P}}$ by checking the conditions in Lemma 1. If Π is not party-aligned single-peaked under $\lhd_{\mathcal{P}}$, then Lemmas 2–4 guarantee that Π is not party-aligned single-peaked.

Running Time. The algorithm runs in $O(|C| \cdot |V| \cdot |\mathcal{P}|)$ time. First, identifying the bottom parties can be done in $O(|V|)$ time, and such a step is performed at most $|\mathcal{P}|$ times. Second, checking the instance for vote-imposed restrictions and then applying Lemma 3 takes $O(|C| \cdot |V|)$ time; again, such a step is performed at most $|\mathcal{P}|$ times. Third, checking whether our instance can be reduced recursively by Lemma 4 takes $O(|C| \cdot |V|)$ time, not counting the time necessary to deal with the reduced instance. This yields an overall running time of $O(|C| \cdot |V| \cdot |\mathcal{P}|)$.

4 Equilibrium Existence

We move on to present the results concerning the existence of Nash equilibria in the considered model.

Theorem 2. *If voters' preference profiles are single-peaked and also party-aligned single-peaked with at most three parties, then a Nash equilibrium always exists.*

Proof. Let $(\succ_v)_{v \in V}$ be a preference profile that is both single-peaked and party-aligned single-peaked, i.e., each vote is single-peaked with respect to the same axis $\lhd$ that is compatible with some party axis. We call a nomination scheme $\vec{c}$ *centrist* if the leftmost party P_L nominates its rightmost candidate, and the rightmost party P_R nominates its leftmost candidate in $\vec{c}$. If a voter v does not vote for P_L in a centrist nomination scheme $\vec{c}$, then its favourite candidate is to the right of every candidate of P_L, and hence there is no Nash deviation for $\vec{c}$ by P_L due to the single-peakedness of $\succ_v$ w.r.t. $\lhd$. A symmetric argument shows that $\vec{c}$ admits no Nash deviation by P_R either.

We now distinguish between two cases. If there is a candidate $c \notin P_L \cup P_R$ that is a winner in some centrist nomination scheme $\vec{c}$, then neither the party containing c, nor any of the parties P_L and P_R has a Nash deviation for $\vec{c}$ which is thus a Nash equilibrium. By contrast, if in all centrist nomination schemes, no party other than P_L and P_R wins, then any such nomination scheme is a Nash equilibrium, since no party has a Nash deviation. $\square$

Non-Existence of Nash Equilibria. Let us show an example proving the tightness of Theorem 2 in the sense that party-aligned single-peakedness alone does not guarantee a Nash equilibrium, even if the number of parties is only two.

Theorem 3. *There is an election $E = (C, V)$ containing two size-two parties and three voters with a party-aligned single-peaked preference profile that admits no Nash equilibria.*

Proof. Consider the election $E = (C, V)$ with two parties $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ and voters $V = \{v_1, v_2, v_3\}$ whose preferences are

$$v_1 : a_1 \succ b_1 \succ a_2 \succ b_2;$$
$$v_2 : b_1 \succ a_2 \succ b_2 \succ a_1;$$
$$v_3 : a_2 \succ b_2 \succ a_1 \succ b_1.$$

By Corollary 1, this profile is party-aligned single-peaked. To see that there is no Nash equilibrium, it suffices to consider each possible nomination scheme and the number of voters supporting the two parties in each of them, as depicted by Figure 1. It is easy to see that each of the four possible nomination schemes admits a Nash deviation. $\square$

We proceed to show the tightness of Theorem 2 in terms of the number of parties.

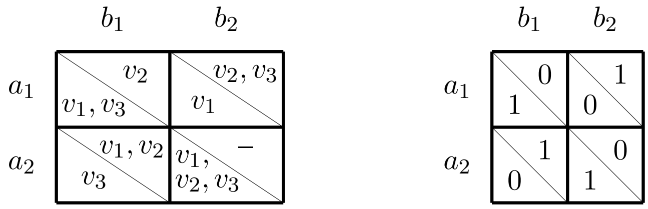

Figure 1: Illustration for Theorem 3. The left figure shows the voters supporting candidates of A and B in all nomination schemes. On the right, the corresponding winners and non-winners are indicated by "1" and "0", respectively.

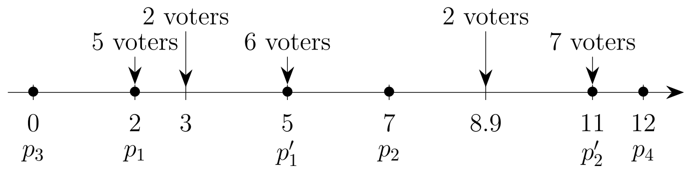

Figure 2: Location of voters and candidates on the real line in the election constructed in Theorem 4. In all figures, voters are indicated by arrows, and candidates by black circles.

Theorem 4. *There exists a 1-D Euclidean election $\mathcal{E}$ with four parties and maximum party size* 2 *with both single-peaked and party-aligned single-peaked preference profiles that does not admit any Nash equilibria.*

Proof. Let us construct an instance with four parties that is both single-peaked and party-aligned single-peaked. We do this by constructing the following 1-D Euclidean election.

Let the *distribution function* φ of voters assign to each point x on the real line the number $\varphi(x)$ of voters located at x. We set the distribution function φ of our instance such that $\varphi(2) = 5$, $\varphi(3) = 2$, $\varphi(5) = 6$, $\varphi(8.9) = 2$, $\varphi(11) = 7$, and $\varphi(x) = 0$ for all other points x on the real line. We define the set of parties $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$. Let $P_1 = \{p_1, p_1'\}$, $P_2 = \{p_2, p_2'\}$, $P_3 = \{p_3\}$ and $P_4 = \{p_4\}$. The positions of the candidates are as shown in Figure 2.

It is straightforward to verify that all preference profiles in this election are party-aligned single-peaked (note that voters' preferences can be formulated as strict total orders) and also single-peaked. Furthermore, we observe that parties P_3 and P_4 nominate the same candidate in each nomination scheme, and that their candidates cannot win in any nomination scheme. Figure 3 shows the number of votes obtained by parties P_1 and P_2, as well as the winner of the resulting election, depending on the nominated candidates. In each nomination scheme, either party P_1 or party P_2 has a Nash deviation, so the instance admits no Nash equilibria. $\square$

5 Finding a Nash Equilibrium

Let us now turn our attention to the problem of finding a Nash equilibrium whenever it exists. As we have already shown in Section 4, a Nash equilibrium may not exist even in very restricted domains involving just four parties. Formally, we define the corresponding problem as follows:

> EQUILIBRIUM EXISTENCE
>
> **Input:** A set C of candidates partitioned into a set $\mathcal{P}$ of parties and a set V of voters with preference profile $(\succ_v)_{v \in V}$ over the set C.
>
> **Question:** Does $(C, \mathcal{P}, V, (\succ_v)_{v \in V})$ admit a Nash equilibrium?

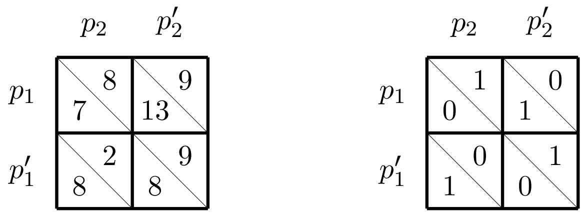

Figure 3: The left figure shows the numbers of voters supporting candidates of P_1 and P_2 in all possible nomination schemes. On the right, the corresponding winners and non-winners are indicated by "1" and "0", respectively.

It turns out that EQUILIBRIUM EXISTENCE is already NP-hard for single-peaked preference profiles, as we show in Theorem 5. In stark contrast with this, we give a polynomial-time algorithm that not only finds a Nash equilibrium, but also decides if a given party can win in some Nash equilibrium. We solve the following problem.

> EQUILIBRIUM PRESIDENT
>
> **Input:** A set C of candidates partitioned into a set $\mathcal{P}$ of parties, a distinguished party $P \in \mathcal{P}$, and a set V of voters with preference profile $(\succ_v)_{v \in V}$ over the set C.
>
> **Question:** Is there a nomination scheme that is a Nash equilibrium for the instance and where P is a winner in the resulting election?

Theorem 5 ($\star$). EQUILIBRIUM EXISTENCE *and* EQUILIBRIUM PRESIDENT *are* NP*-hard for 1-D Euclidean, and hence, for single-peaked preference profiles.*

We move on to present our main result:

Theorem 6. EQUILIBRIUM EXISTENCE *and* EQUILIBRIUM PRESIDENT *are polynomial-time solvable if the preference profile is party-aligned single-peaked.*

We present a polynomial-time algorithm for EQUILIBRIUM PRESIDENT. Note that applying this algorithm for each party as the distinguished one results in a polynomial-time algorithm for EQUILIBRIUM EXISTENCE.

Let $(C, \mathcal{P}, P, V, (\prec_v)_{v \in V})$ be our input instance for the EQUILIBRIUM PRESIDENT problem. Our algorithm uses dynamic programming and relies heavily on the structure of the preference domain.

We start by computing a party axis $\lhd_{\mathcal{P}}$ with respect to which the input profile $(\prec_v)_{v \in V}$ is party-aligned single-peaked; let $P_1, \dots, P_k$ be the parties in $\mathcal{P}$ ordered according to $\lhd_{\mathcal{P}}$, and let $P = P_\kappa$ be the distinguished party among whose candidates we are searching for a candidate that can become a winner in some Nash equilibrium.

We first prove the following lemma.

Lemma 5 ($\star$). *Suppose that the profile $(\succ_v)_{v \in V}$ is party-aligned single-peaked. Then for each voter $v \in V$, there exist at most two parties for which v might vote for in a reduced election. Moreover these two parties must be adjacent along the party axis, and can be found in $O(|C|)$ time.*

Partitioning the Voter Set. We now introduce the following sets of voters. First, for each $i \in [k]$, let V^{P_i} denote the set of those voters in V who always vote for the nominee of party P_i irrespective of the nomination scheme. Note that $v \in V^{P_i}$ if and only if the top $|P_i|$ candidates in v's preference list are exactly the candidates belonging to P_i.

Second, for each $i \in [k-1]$, let $V^{(P_i,P_{i+1})}$ denote the set of voters in $V \setminus (V^{P_i} \cup V^{P_{i+1}})$ who always vote either for the nominee of P_i or for the nominee of P_{i+1}. By Lemma 5, the sets V^{P_i} for $i \in [k]$ together with the sets $V^{(P_i,P_{i+1})}$ for $i \in [k-1]$ form a partitioning of the voter set V, and this partitioning can be computed in $O(|V| \cdot |C|)$ time.

Third, let us define the following sets of voters, crucial for our dynamic programming approach:

$$V_{\leqslant i} = \left(\bigcup_{j=1}^{i} V^{P_j} \right) \cup \left(\bigcup_{j=1}^{i-1} V^{(P_j,P_{j+1})} \right), \quad \text{and} \tag{1}$$

$$V_{\geqslant i} = \left(\bigcup_{j=i}^{k} V^{P_j} \right) \cup \left(\bigcup_{j=i}^{k-1} V^{(P_j,P_{j+1})} \right). \tag{2}$$

Note that $V_{\leqslant 1} \subseteq V_{\leqslant 2} \subseteq \cdots \subseteq V_{\leqslant k} = V$, and observe also that voters in $V_{\leqslant i}$ may only vote for nominees in $P_1 \cup \cdots \cup P_i$ in any reduced election. Similarly, $V = V_{\geqslant 1} \supseteq V_{\geqslant 2} \supseteq \cdots \supseteq V_{\geqslant k}$, and voters in $V_{\geqslant i}$ may only vote for nominees in $P_i \cup \cdots \cup P_k$ in any reduced election.

Assuming some hypothetical nomination scheme that is a Nash equilibrium and in which P_κ is a winner, we next guess the score $s^\star$ obtained by the nominee of P_κ in such a Nash equilibrium; note that $s^\star \leqslant |V|$, so there are at most $|V|$ possible values $s^\star$ can take. Henceforth, we treat $s^\star$ as fixed.

Feasible Partial Nomination Schemes. For some $i \in [k]$ at most κ, a *left-feasible partial nomination scheme for i* is a tuple $(c_1, \dots, c_i)$ where $c_j \in P_j$ for each $j \in [i]$, and

(α) for each $j \in [i-1]$, c_j obtains at most $s^\star$ votes in every nomination scheme $(c_1, \dots, c_i, c'_{i+1}, \dots, c'_k)$,[1] and

(β) for each $j \in [i-1]$, if c_j obtains less than $s^\star$ votes in some nomination scheme $(c_1, \dots, c_i, c'_{i+1}, \dots, c'_k)$, then no candidate $c'_j \in P_j \setminus \{c_j\}$ obtains at least $s^\star$ votes in $(c_1, \dots, c_{j-1}, c'_j, c_{j+1}, \dots, c_i, c'_{i+1}, \dots, c'_k)$.

Similarly, for each index $i \in [k]$ at least κ, a *right-feasible partial nomination scheme for i* is a tuple $(c_1, \dots, c_i)$ where $c_j \in P_j$ for each $j \in [i]$, and moreover,

(α') for each $j \in [k] \setminus [i]$, c_j obtains at most $s^\star$ votes in every nomination scheme $(c'_1, \dots, c'_{i-1}, c_i, \dots, c_k)$, and

(β') for each $j \in [k] \setminus [i]$, if c_j obtains less than $s^\star$ votes in some nomination scheme $(c'_1, \dots, c'_{i-1}, c_i, \dots, c_k)$, then no candidate $c'_j \in P_j \setminus \{c_j\}$ obtains at least $s^\star$ votes in $(c'_1, \dots, c'_{i-1}, c_i, \dots, c_{j-1}, c'_j, c_{j+1}, \dots, c_k)$.

Viable Scores and Score Pairs. We are now ready to define the central notions necessary for our algorithm. For some $i \in [k]$ and candidate $c_i \in P_i$, an integer s_i is a *left-* (or *right-*) *viable score for c_i* if there is a left- (or right-) feasible partial nomination scheme where c_i obtains exactly s_i votes from $V_{\leqslant i}$ (from $V_{\geqslant i}$, respectively).

Next, we define the concept of viability for *pairs of integers*. Given an index $i \in \{2, \dots, k\}$, a pair (s_{i-1}, s_i) of integers is *left-viable for a pair $(c_{i-1}, c_i) \in P_{i-1} \times P_i$ of candidates* if there is a left-feasible partial nomination scheme where c_j obtains exactly s_j votes from $V_{\leqslant i}$ for both $j \in \{i-1, i\}$. Notice that, consequently, some score s_i is left-viable for candidate $c_i \in P_i$ if and only if there exists some $s_{i-1} \in \{0, 1, \dots, s^\star\}$ and candidate $c_{i-1} \in P_{i-1}$ such that(s_{i-1}, s_i) is a left-viable pair for (c_{i-1}, c_i). The symmetric concept of *right-viability* is defined analogously.

Lemma 6 shows the importance of these notions.

[1] Note that the nominees $c'_{i+1}, \dots, c'_k$ are irrelevant here as they do not affect how voters in $V_{\leqslant i}$ vote.

Lemma 6 ($\star$). *Given a candidate $c_\kappa \in P_\kappa$, the following statements are equivalent:*

(a) there exists a nomination scheme that is a Nash equilibrium and in which c_κ is a winner with score $s^\star$;

(b) there exist non-negative integers s_l and s_r satisfying $s_l + s_r - |V^{P_\kappa}| = s^\star$ such that s_l is a left-viable score for c_κ and s_r is a right-viable score for c_κ.

Computing Viable Score Pairs. Let us now provide a dynamic programming method for computing all left-viable score pairs for each candidate pair $(c_i, c_{i+1}) \in P_i \times P_{i+1}$ increasingly for each $i \in [\kappa - 1]$; computing right-viable scores can be done in a symmetric manner. From now on, we may write viability instead of left-viability.

Notice that for a given candidate pair $(c_1, c_2) \in P_1 \times P_2$, the only viable pair can be (s_1, s_2) where

$$s_1 = f(c_1|c_2) := |V^{P_1}| + |\{v : v \in V^{(P_1,P_2)}, c_1 \succ_v c_2\}|,$$
$$s_2 = |V^{P_2}| + |\{v : v \in V^{(P_1,P_2)}, c_2 \succ_v c_1\}|. \tag{3}$$

However, such a pair (s_1, s_2) is not necessarily viable for (c_1, c_2), because it might not lead to a Nash equilibrium.

Lemma 7 ($\star$). *The pair (s_1, s_2) defined as in (3) is viable for a candidate pair $(c_1, c_2) \in P_1 \times P_2$ if and only if*
(†) $s_1 = s^\star$, or
(‡) $s_1 < s^\star$ and there is no $c'_1 \in P_1$ for which $f(c'_1|c_2) \geqslant s^\star$.

Now, we take the general case where $2 < i \leqslant k$ and aim to compute the viable score pairs for the pairs in $P_{i-1} \times P_i$.

Lemma 8 ($\star$). *A pair (s_{i-1}, s_i) of non-negative integers with $s_{i-1} \leqslant s^\star$ is viable for $(c_{i-1}, c_i) \in P_{i-1} \times P_i$ if and only if there exist integers $s'_{i-1}, s_{i-2} \in \{0, 1, \dots, s^\star\}$ and candidate $c_{i-2} \in P_{i-2}$ such that*

(i) (s_{i-2}, s'_{i-1}) is a viable pair for (c_{i-2}, c_{i-1}),

(ii) $s_{i-1} = s'_{i-1} + |\{v : v \in V^{(P_{i-1},P_i)}, c_{i-1} \succ_v c_i\}|$,

(iii) $s_i = |\{v : v \in V^{(P_{i-1},P_i)}, c_i \succ_v c_{i-1}\}| + |V^{P_i}|$, and

(iv) either (†) $s_{i-1} = s^\star$, or (‡) there is no $c'_{i-1} \in P_{i-1}$ for which $f(c'_{i-1}|c_i, c_{i-2}) \geqslant s^\star$ where

$$f(c'_{i-1}|c_i, c_{i-2}) = |\{v : v \in V^{(P_{i-1},P_i)}, c'_{i-1} \succ_v c_i\}|$$
$$+ |V^{P_{i-1}}| + \{v : v \in V^{(P_{i-2},P_{i-1})}, c'_{i-1} \succ_v c_{i-2}\}|.$$

Using dynamic programming based on Lemmas 7 and 8, we can compute all left-viable score pairs for each candidate pair $(c_{i-1}, c_i) \in P_{i-1} \times P_i$ increasingly for each $i = 2, \dots, \kappa$; note that it suffices to compute score pairs (s_{i-1}, s_i) where s_{i-1} and s_i are between 0 and $s^\star$. Hence, we need to compute a binary variable for each $s_{i-1}, s_i \in \{0, \dots, s^\star\}$ and each $c_{i-1} \in P_{i-1}$ and $c_i \in P_i$, for $i = 2, \dots, \kappa$, describing whether (s_{i-1}, s_i) is viable for (c_{i-1}, c_i) or not; we can determine these values for $i = 2$ using Lemma 7, and we can compute them for increasing values of i using Lemma 8.

This allows us to compute all left–viable scores for each candidate in P_κ. Computing right-viable scores can be done in a symmetric manner. Finally, we Lemma 6 to either find a candidate $c_\kappa \in P_\kappa$ that wins in some Nash equilibrium with score $s^\star$ or conclude correctly that no such candidate exists.

Running Time. Computing the party axis can be done in polynomial time, according to Theorem 1. Guessing the winning score $s^\star$ of the distinguished party P_κ adds a factor of $|V|$ to the running time necessary for computing the viable scores. Checking the conditions of Lemmas 7 and 8 can clearly be done in polynomial; hence, the running time of our algorithm is polynomial in the input size. This finishes the proof of Theorem 6.

	EQUI PRES	POS PRES	NEC PRES
SP & PASP	P [Thm. 6]	P [Thm. 7]	P [F'16]
PASP	P [Thm. 6]	P [Thm. 7]	P [Thm. 7]
SP	NP-c [Thm. 5]	NP-c [F'16]	P [F'16]

Table 1: Summary of our algorithmic results. SP (and PASP) stands for single-peaked (party-aligned single-peaked) preferences. [F'16] marks results by Faliszewski et al. (2016).

6 Finding a Possible or Necessary President

Next, we show that the following problems are polynomial-time solvable in the domain of party-aligned single-peaked profiles, by an adaptation of our algorithm for Theorem 6:

> POSSIBLE (or NECESSARY) PRESIDENT
>
> **Input:** A set C of candidates partitioned into a set $\mathcal{P}$ of parties, a distinguished party $P \in \mathcal{P}$, and a set V of voters with preference profile $(\succ_v)_{v \in V}$ over the set C.
>
> **Question:** Is P a winner in the election resulting from *some* (or *all*, respectively) nomination scheme(s)?

Faliszewski et al. (2016) proved that POSSIBLE PRESIDENT is NP-complete for single-peaked preferences, while NECESSARY PRESIDENT is coNP-complete in general but becomes polynomial-time solvable for single-peaked preferences. They also showed that POSSIBLE PRESIDENT is polynomial-time solvable if the preferences are single-peaked with respect to an axis along which the candidates of each party are ordered consecutively, that is, if preferences are single-peaked and also party-aligned single-peaked.

The following theorem extends the tractability results of Faliszewski et al. (2016) to the domain of party-aligned single-peaked preference profiles.

Theorem 7 ($\star$)**.** *The* POSSIBLE *and* NECESSARY PRESIDENT *problems are polynomial-time solvable if the preference profile is party-aligned single-peaked.*

Proof sketch. The algorithm for EQUILIBRIUM PRESIDENT presented in Section 5 can be adapted in a straightforward way to solve the POSSIBLE PRESIDENT problem; in fact, this is a simplification, since we only need to ensure that the distinguished party is a winner in some nomination scheme $\vec{c}$, but we do not require $\vec{c}$ to be a Nash equilibrium.

To solve the NECESSARY PRESIDENT problem, we apply a slightly modified variant of our algorithm for POSSIBLE PRESIDENT that not only ensures that the distinguished party P^+ is a winner in some nomination scheme, but also guarantees that some other party P^- is *not* a winner; running this algorithm for each party P^+ other than P^-, we can decide if P^- wins in all reduced elections. $\triangle$

7 Discussion

We have explored computational questions about strategic candidate nomination by parties in a model where voters' preferences are party-aligned single-peaked, with a focus on the concept of Nash equilibria; see Table 1 for an overview of our algorithmic results. Studying different restrictions on the preference domain or voting rules other than Plurality are a natural direction for further research in the same model. More generally, extensions of our model that deal with possible interactions between the parties also seem interesting. For example, one could consider what a group of parties can achieve if they were to coordinate their nominations, with a coalition $\mathcal{C} \subseteq \mathcal{P}$ being *powerful* if there exists a tuple $T_{\mathcal{C}} = (c_i)_{P_i \in \mathcal{C}}$ of candidates such that for every nomination scheme extending $T_{\mathcal{C}}$, the Plurality winner is in $T_{\mathcal{C}}$. This definition generalizes the notion of a *necessary president* (i.e., a powerful singleton coalition), and we would like to know under what conditions we can determine it. Similar generalisations can be attempted for possible president, as well as a possible or necessary president in an equilibrium.

Understanding strategic nominations in district-based elections is another interesting direction, where the preference structure could improve equilibrium computation results. Finally, extensive interaction is worth exploring, where parties hold primaries at different times, with more nuanced (e.g., subgame-perfect) equilibria.

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Appendix A Missing proofs from Section 3

The proof of Lemma 1

Lemma 1 ($\star$). *Consider a vote $\succ_v$ over the candidate set of parties $P_1, \ldots, P_m$. For each party P_i, let h_i and l_i denote v's most- and least-preferred candidate within P_i. Let P_j be the party containing v's top candidate. Then $\succ_v$ is party-aligned single-peaked with party axis $P_1 \lhd_{\mathcal{P}} \cdots \lhd_{\mathcal{P}} P_m$ if and only if the following conditions are met.*

(a) If $P_j \notin \{P_1, P_m\}$, then $l_j \succ_v h_{j-1}$ or $l_j \succ_v h_{j+1}$.

(b) For each parties P_i, P_{i+1} with $i > j$ we have $l_i \succ_v h_{i+1}$.

(c) For each parties P_{i-1}, P_i with $i < j$ we have $l_i \succ_v h_{i-1}$.

Proof. First, we prove that if the vote $\succ_v$ is party-aligned single-peaked according to $\lhd_{\mathcal{P}}$, then conditions (a)–(c) are met.

To see that (a) holds, assume $P_j \notin \{P_1, P_m\}$. Then for every perceived axis $\lhd_v$ compatible with $\lhd_{\mathcal{P}}$ we have $h_{j-1} \lhd_v l_j \lhd_v h_{j+1}$. Hence, if $h_{j-1} \succ_v l_j$ (which is equivalent with $l_j \not\succ_v h_{j-1}$), then the single-peakedness of $\succ_v$ with axis $\lhd_v$ implies that $l_j \succ_v h_{j+1}$, proving (a).

To see condition (b), consider two parties P_i and P_{i+1} for $j < i$. Then for every perceived axis $\lhd_v$ compatible with $\lhd_{\mathcal{P}}$ we have $h_j \lhd_v l_i \lhd_v h_{i+1}$ for the top candidate h_j. Thus, due to the single-peakedness of $\succ_v$ with axis $\lhd_v$, we know that $h_j \succ_v l_i$ implies $l_i \succ_v h_{i+1}$. The argument for condition (c) is symmetric to that for condition (b).

We move on to prove that if conditions (a)–(c) are satisfied, then the vote $\succ_v$ is party-aligned single-peaked under $\lhd_{\mathcal{P}}$. We show how to construct a total ordering $\lhd_v$ of the set C of candidates that is compatible with the party ordering $\lhd_{\mathcal{P}}$ and for which vote $\succ_v$ is single-peaked under $\lhd_v$.

If P_j is the first party in $\lhd_{\mathcal{P}}$, i.e., $j = 1$, we identify the highest-ranked candidate of party P_2, that is, h_2. We place every candidate from party $P_j = P_1$ ranked by v lower than h_2 on the left of the top candidate h_1 in decreasing order of their rank according to $\succ_v$ (i.e., setting $c' \lhd_v c$ whenever $c \succ_v c'$ for any two such candidates c and c'). Next, we place the top candidate h_1, and then the remaining candidates from P_1 (those preferred to h_2) in increasing order of their rank according to $\succ_v$. Condition (b) guarantees that the remaining candidates from parties $P_2, P_3, \ldots, P_m$ can be placed in $\lhd_v$ in increasing order of their rank according to $\succ_v$ so that $\lhd_v$ becomes compatible with $\lhd_{\mathcal{P}}$. The case when $P_j = P_m$ is symmetric.

Suppose now that P_j is neither the first nor the last party according to $\lhd_{\mathcal{P}}$. Then condition (a) guarantees that at least one of the highest-ranked candidates of parties P_{j-1}, P_{j+1}, that is, one of h_{j-1} and h_{j+1}, is ranked lower than l_j by v. Breaking symmetry, we may assume that $l_j \succ_v h_{j-1}$. Similarly to the previous case, we place every candidate from party P_j ranked lower than h_{j+1} on the left of the top candidate h_j in decreasing order of their rank according to $\succ_v$. Next, we place the top candidate h_j, and then the remaining candidates from P_j in increasing order of their rank according to $\succ_v$. Conditions (b) and (c) guarantee that the candidates from the remaining parties can be placed in the total order $\lhd_v$ so that it is compatible with $\lhd_{\mathcal{P}}$. □

The proof of Lemma 2

Lemma 2 ($\star$). *Let $c \in P_i$ be the candidate ranked in the last position by a voter v. If $\succ_v$ is party-aligned single-peaked with party axis $\lhd_{\mathcal{P}}$, then P_i is either in the leftmost or the rightmost position in $\lhd_{\mathcal{P}}$.*

Proof. If party P_i is neither in the leftmost nor in the rightmost position in $\lhd_{\mathcal{P}}$, then for the perceived axis $\lhd_v$ of voter v, which must be compatible with $\lhd_{\mathcal{P}}$, there exist candidates a and b such that $a \lhd_v c \lhd_v b$. However, then $a \succ_v c$ and $b \succ_v c$ contradict the single-peakedness of $\succ_v$ with the perceived axis $\lhd_v$. □

The proof of Lemma 3

Lemma 3 ($\star$)**.** *Let (L, R) be an extremal party placement for some party axis $\lhd_{\mathcal{P}}$, and consider a vote $>_v$ that is party-aligned single-peaked with axis $\lhd_{\mathcal{P}}$. Assume that candidates in $\bigcup(R \cup L)$ do not form the suffix of the vote $>_v$. Let*

- *a be v's least favourite candidate in $\bigcup(\mathcal{P} \setminus (L \cup R))$,*
- *b be v's most favourite candidate in $\bigcup(L \cup R)$, and*
- *P_a and P_b be the parties containing a and b, respectively.*

If $P_b \in R$, then P_a directly follows the last party in L according to $\lhd_{\mathcal{P}}$, while if $P_b \in L$, then P_a directly precedes the first party in R according to $\lhd_{\mathcal{P}}$.

Proof. Since the set of all candidates from parties in $R \cup L$ is not the suffix of the vote $>_v$, it holds, by definition, that $b >_v a$. Moreover, for every party in $P_i \in \mathcal{P} \setminus (L \cup R)$ different from P_a, it holds that v prefers every candidate of P_i to a.

Assume that P_a is not placed in $\lhd_{\mathcal{P}}$ as claimed. This means that there exists a party $P_i \in \mathcal{P} \setminus (L \cup R \cup \{P_a\})$ such that P_a is placed between P_i and P_b according to $\lhd_{\mathcal{P}}$. However, both P_b and P_i have at least one candidate that is ranked strictly higher than $a \in P_a$, which contradicts our assumption that $>_v$ is party-aligned single-peaked with axis $\lhd_{\mathcal{P}}$. ☐

The proof of Lemma 4

Lemma 4 ($\star$)**.** *Let $\Pi = (>_v)_{v \in V}$ be a party-aligned single-peaked preference profile over candidate set C with a party axis $P_1 \lhd_P \cdots \lhd_{\mathcal{P}} P_m$. Assume further that the candidates in $(P_1 \cup \cdots \cup P_i) \cup (P_j \cup \cdots \cup P_m)$ form a suffix of each vote in Π, and let Q be the set of candidates not in this suffix. If the restriction of Π to Q is party-aligned single-peaked with a party axis $Q_1 \lhd_Q \cdots \lhd_Q Q_k$, then Π is party-aligned single-peaked with the party axis $P_1, \dots, P_i, Q_1, \dots, Q_k, P_j, \dots, P_m$.*

Proof. Consider a voter $v \in V$, and let $\lhd_v^Q$ be a perceived axis over Q compatible with $\lhd_Q$ such that the restriction of $>_v$ to Q is single-peaked w.r.t. $\lhd_v^Q$. Let also $\lhd_v^C$ be a perceived axis compatible with $\lhd_{\mathcal{P}}$ such that $>_v$ is single-peaked w.r.t. $\lhd_v^C$. Define now a perceived axis $\lhd_v$ by combining these two such that

- for each $c, c' \in C \setminus Q$ we set $c \lhd_v c'$ exactly if $c \lhd_v^C c'$;
- for each $c, c' \in Q$ we set $c \lhd_v c'$ exactly if $c \lhd_v^Q c'$;
- for each $c \in C \setminus Q$ and $c' \in Q$, we set $c \lhd_v c'$ exactly if $c \in P_1 \cup \dots P_i$.

It is straightforward to check that $\lhd_v$ is compatible with the ordering $P_1, \dots, P_i, Q_1, \dots, Q_k, P_j, \dots, P_m$. Hence, it remains to show that $>_v$ is single-peaked w.r.t. $\lhd_v$. Consider candidates a, b, and c such that $a \lhd_v b \lhd_v c$ and $a >_v b$; we aim to show that $b >_v c$.

Now, if $\{a, b, c\}$ is a subset of $C \setminus Q$ (or of Q), then $b >_v c$ follows because the vote $>_v$ (or its restriction to Q) is single-peaked w.r.t. the perceived axis $\lhd_v^C$ (or $\lhd_v^Q$, respectively). Hence, we may assume that $|Q \cap \{a, b, c\}| \in [2]$.

First, if $b \in Q$ and $c \in C \setminus Q$, then $b >_v c$ holds, because $C \setminus Q$ forms a suffix of $>_v$. Second, if $\{a, b\} \subseteq C \setminus Q$ or $\{b, c\} \subseteq C \setminus Q$, then our definition of $\lhd_v$ yields $a \lhd_v^C b \lhd_v^C c$, so the single-peakedness of $>_v$ w.r.t. $\lhd_v^C$ means that $a >_v b$ indeed implies $b >_v c$. The only remaining possibility is that $a \in C \setminus Q$ and $b \in Q$. However, in this case $a >_v b$ cannot happen, because $C \setminus Q$ forms a suffix of $>_v$. Hence, in all possible cases, we obtain $b >_v c$, as required. ☐

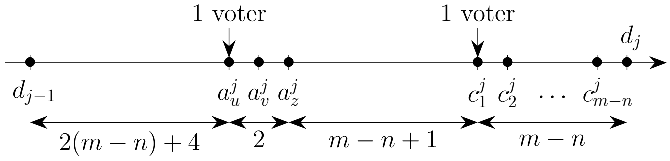

Figure 4: The location of voters and candidates within the triple segment corresponding to $S_j = \{a_u, a_v, a_z\} \in \mathcal{S}$.

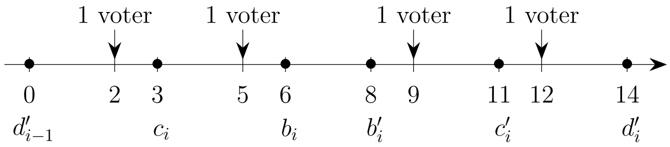

Figure 5: The location of voters and candidates within the i^{th} auxiliary segment for some $i \in [m-n]$. The numbers indicating locations are relative to the start of the segment.

Appendix B Missing proofs from Section 5

The proof of Theorem 5

Theorem 5 ($\star$)**.** Equilibrium Existence *and* Equilibrium President *are* NP*-hard for 1-D Euclidean, and hence, for single-peaked preference profiles.*

Proof. We reduce from the NP-hard Exact 3-Cover problem Garey and Johnson (1979) whose input is a set $U = \{u_1, \dots, u_{3n}\}$ and a family $\mathcal{S} = \{S_1, \dots, S_m\}$ of size-3 subsets of U, and the task is decide if there are n sets in $\mathcal{S}$ whose union is U.

Let us create an instance I of Equilibrium Existence as follows. We create m *triple segments* and $m-n$ *auxiliary segments* on the real line. Each triple segment corresponds to a set $S_j \in \mathcal{S}$ and has length $L = 4(m-n)+7$, so that the segment corresponding to S_j is located at the interval $[(j-1)L, jL]$. Each of the $m-n$ auxiliary segments has length 14, with the i^{th} located at $[K + 14(i-1), K + 14i]$ where $K = Lm + 1$ is the start of the first auxiliary segment. There will be two voters located in each triple segment, and four voters in each auxiliary segment.

We create a party $A_u = \{a_u^j : u \in S_j \in \mathcal{S}\}$ for each element $u \in U$, and two parties $B_i = \{b_i, b'_i\}$ and $C_i = \{c_i, c'_i, c_i^1, \dots, c_i^m\}$ for each index $i \in [m-n]$. We also create dummy candidates $d_1, \dots, d_m$ and $d'_0, \dots, d'_{m-n}$, each of them contained in a singleton party, and located at the beginning or the end of some segment. The locations of all candidates and all voters along the real line are given in Figures 4 and 5. It is straightforward to check that voters' preferences are strict linear orders over the candidate set, so the constructed instance is within the single-peaked preference domain.

We claim that $(U, \mathcal{S})$ is a yes-instance of Exact 3-Cover if and only if the constructed instance I admits a Nash equilibrium.

Notice that any candidate a_u^j can only obtain the vote of at most one voter, namely the "left" voter within the j-th triple segment, because the "right" voter in that segment prefers d_j to a_u^j. This will imply that no party A_u, $u \in U$, can win in any reduced election.

Assume that $\mathcal{S}' \subseteq \mathcal{S}$ contains exactly n sets whose union is U. Let us fix a bijection $\varphi : [m-n] \to \{j : S_j \in \mathcal{S} \setminus \mathcal{S}'\}$. Consider the nomination scheme $\vec{c}$ where party A_u nominates the candidate a_u^j for which $u \in S_j \in \mathcal{S}'$, party C_i nominates $c_i^{\varphi(i)}$, and all other nominations are fixed arbitrarily. Notice that each party B_i obtains two votes from the two middle voters in the i^{th} auxiliary segment, because no candidate c_i or c'_i is nominated. Also, each party C_i obtains two votes from the two voters within the triple segment corresponding to $S_{\varphi(i)}$, because no other party has a nominee within that segment (except for the parties of dummy candidates). It is also clear that each dummy candidate obtains at most two votes. Thus, all

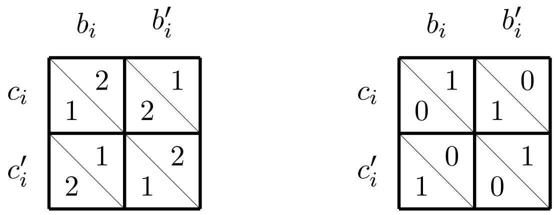

Figure 6: The left figure shows the numbers of voters voting for candidates of C_i and B_i in all nomination schemes where C_i nominates c_i or c'_i. On the right, the corresponding winners and non-winners among these parties are indicated by "1" and "0", respectively.

parties B_i and C_i, $i \in [m - n]$, are winners, and furthermore, no Nash deviation is possible for any party A_u as each such party can obtain at most one vote. Hence, $\vec{c}$ is a Nash equilibrium.

Assume now that there is a nomination scheme $\vec{c}$ that is a Nash equilibrium. It is not hard to check that each nominee in a segment can obtain at most two votes. Moreover, no candidate c_i or c'_i can be present in $\vec{c}$, because whenever c_i or c'_i is nominated, there always exists a Nash deviation either for B_i or for C_i, as shown in Figure 6. In fact, every party C_i has to be a winner in $\vec{c}$, as otherwise it has a Nash deviation by nominating c_i or c'_i according to Figure 6. Consider the nominees c_i^j of parties C_i for $i \in [m - n]$, and notice that each such nominee c_i^j must be the only nominee located in the triple segment S_j, as otherwise c_i^j (or some other nominee $c_{i'}^j$) would not obtain the two votes necessary for becoming a winner. Hence, all $3n$ nominees of the parties A_u, $u \in U$ must be located in the n triple segments that do not contain any candidate nominated by some party C_i. This means that the n sets in $\mathcal{S}$ corresponding to these segments cover U, as required, showing the correctness of our reduction and proving that EQUILIBRIUM EXISTENCE is NP-hard.

To show the NP-hardness of EQUILIBRIUM PRESIDENT, we can modify the reduction as follows: we add a single new candidate $c^\star$ forming its own singleton party, and placed sufficiently far away from every other candidate; we further add two voters who are placed at the same place as $c^\star$ and thus always vote for $c^\star$. Then $c^\star$ is a winner in some Nash equilibrium for the modified instance I' if and only if there exists a Nash equilibrium for the original instance I which in turn happens if and only if $(U, \mathcal{S})$ is a yes-instance of EXACT 3-COVER. This proves the NP-hardness of EQUILIBRIUM PRESIDENT. □

The proof of Lemma 5

Lemma 5 ($\star$). *Suppose that the profile $(\succ_v)_{v \in V}$ is party-aligned single-peaked. Then for each voter $v \in V$, there exist at most two parties for which v might vote for in a reduced election. Moreover these two parties must be adjacent along the party axis, and can be found in $O(|C|)$ time.*

Proof. Let party A contain the top candidate for voter v, and let a and a' be the two extremal candidates of party A according to the perceived axis $\lhd_v$ for v, so that each candidate of party A appears between a and a' along the axis $\lhd_v$. Let us further break symmetry by assuming $a \succ_v a'$ and $a' \lhd_v a$ (otherwise, we may reverse our axes or rename the candidates). Define b as the candidate directly following a along $\lhd_v$ so that $a' \lhd_v a \lhd_v b$. Let B the party containing b; then party-aligned single-peakedness yields $A \lhd_{\mathcal{P}} B$, with no party placed between A and B along $\lhd_{\mathcal{P}}$.

We claim that in all nomination schemes, v's most-preferred candidate belongs to party A or party B. Indeed: since A contains v's top candidate, we know that for each party C with $A \lhd_{\mathcal{P}} B \lhd_{\mathcal{P}} C$, v prefers every candidate of B to each candidate of C; similarly, for each party C with $C \lhd_{\mathcal{P}} A \lhd_{\mathcal{P}} B$, candidate a' and, hence, all candidates of A, are preferred by v to each candidate of C. In either case, the nominee of C in any nomination scheme is less preferred by v than the nominee of either A or of B, and hence cannot obtain v's vote.

Furthermore, if there is a candidate outside A that v prefers to some candidate in A, then the most-preferred such candidate must be exactly b; this yields an efficient way to determine A and B by simply going through the top $|A| + 1$ candidates for v. □

The proof of Lemma 6

Lemma 6 ($\star$). *Given a candidate $c_\kappa \in P_\kappa$, the following statements are equivalent:*

(a) there exists a nomination scheme that is a Nash equilibrium and in which c_κ is a winner with score $s^\star$;

(b) there exist non-negative integers s_l and s_r satisfying $s_l + s_r - |V^{P_\kappa}| = s^\star$ such that s_l is a left-viable score for c_κ and s_r is a right-viable score for c_κ.

Proof. Assume first that (a) holds. Define s_l and s_r as the number of votes that c_κ obtains from voters in $V_{\leqslant\kappa}$ and in $V_{\geqslant\kappa}$, respectively, in the nomination scheme $\vec{c} = (c_1, \ldots, c_k)$. Recall that $V_{\leqslant\kappa} \cap V_{\geqslant\kappa} = V^{P_\kappa}$ and thus $s^\star = s_r + s_l - |V^{P_\kappa}|$. It is clear that $\vec{c}$ is left-feasible: condition (α) holds because no nominee obtains more than $s^\star$ votes in $\vec{c}$, and condition (β) holds since $\vec{c}$ is a Nash equilibrium and hence admits no Nash deviation.

Assume now that (b) holds. Let a left- and a right-feasible partial nomination scheme $(c_1, \ldots, c_\kappa)$ and $(c_\kappa, \ldots, c_k)$, respectively. for which c_κ obtains s_l votes from $V_{\leqslant\kappa}$ in the former and s_r votes from $V_{\geqslant\kappa}$ votes in the latter. Then c_κ is a winner in the nomination scheme $\vec{v} = (c_1, \ldots, c_\kappa, \ldots, c_k)$ with score $s^\star$ due to (α), and $\vec{v}$ is a Nash equilibrium due to (β). □

The proof of Lemma 7

Lemma 7 ($\star$). *The pair (s_1, s_2) defined as in (3) is viable for a candidate pair $(c_1, c_2) \in P_1 \times P_2$ if and only if*
(†) $s_1 = s^\star$, or
(‡) $s_1 < s^\star$ and there is no $c_1' \in P_1$ for which $f(c_1'|c_2) \geqslant s^\star$.

Proof. First, if (†) or (‡) holds, then (c_1, c_2) is a partial nomination scheme satisfying both (α) and (β) in the definition of (left-)feasibility. Second, if neither (†) nor (‡) holds, then either $s_1 > s^\star$ in which case (c_1, c_2) violates (α), or $s_1 < s^\star$ but there exists some $c_1' \in P_1$ that can obtain at least $s^\star$ votes in every nomination scheme containing c_1' and c_2 (and thus lead to a Nash deviation) in which case (c_1, c_2) violates (β). □

The proof of Lemma 8

Lemma 8 ($\star$). *A pair (s_{i-1}, s_i) of non-negative integers with $s_{i-1} \leqslant s^\star$ is viable for $(c_{i-1}, c_i) \in P_{i-1} \times P_i$ if and only if there exist integers $s_{i-1}', s_{i-2} \in \{0, 1, \ldots, s^\star\}$ and candidate $c_{i-2} \in P_{i-2}$ such that*

(i) (s_{i-2}, s_{i-1}') is a viable pair for (c_{i-2}, c_{i-1}),

(ii) $s_{i-1} = s_{i-1}' + |\{v : v \in V^{(P_{i-1}, P_i)}, c_{i-1} \succ_v c_i\}|$,

(iii) $s_i = |\{v : v \in V^{(P_{i-1}, P_i)}, c_i \succ_v c_{i-1}\}| + |V^{P_i}|$, and

(iv) either (†) $s_{i-1} = s^\star$, or (‡) there is no $c_{i-1}' \in P_{i-1}$ for which $f(c_{i-1}'|c_i, c_{i-2}) \geqslant s^\star$ where

$$f(c_{i-1}'|c_i, c_{i-2}) = |\{v : v \in V^{(P_{i-1}, P_i)}, c_{i-1}' \succ_v c_i\}| + |V^{P_{i-1}}| + \{v : v \in V^{(P_{i-2}, P_{i-1})}, c_{i-1}' \succ_v c_{i-2}\}|.$$

Proof. Let $\vec{c} = (c_1, \ldots, c_{i-2}, c_{i-1}, c_i)$ be a left-feasible partial nomination scheme for i where c_j obtains s_j votes for $j \in \{i-1, i\}$ from voters in $V_{\leqslant i}$, witnessing the viability of (s_{i-1}, s_i) for (c_{i-1}, c_i). Let s_{i-2}

and s'_{i-1} be the votes that c_{i-2} and c_{i-1}, respectively, obtain from voters in $V_{\leqslant i-1}$. It is then straightforward to verify that (i)–(iii) hold. Moreover, if $s_{i-1} < s^\star$, then (as there can be Nash deviation for $\vec{c}$ by party P_{i-1} due to condition (β) in the definition of left-feasibility) there is no $c'_{i-1} \in P_{i-1}$ that obtains at least $s^\star$ votes when nominated together with c_{i-2} and c_i—which is equivalent with the condition $f(c'_{i-1}|c_i, c_{i-2}) \geqslant s^\star$. This means that (iv) is satisfied as well.

Assume now that integers $s_{i-2}, s'_{i-1} \in \{0, 1, \dots, s^\star\}$ and candidate $c_{i-2} \in P_{i-2}$ satisfy conditions (i)–(iv), and let $\vec{c}_- = (c_1, \dots, c_{i-2}, c_{i-1})$ be a left-feasible partial nomination scheme for $i-1$ witnessing (i). Define the partial nomination scheme $\vec{c} = (c_1, \dots, c_{i-2}, c_{i-1}, c_i)$. Then $\vec{c}$ satisfies (α), due to the left-feasibility of $\vec{c}_-$ and because c_{i-1} obtains $s_{i-1} \leqslant s^\star$ votes in $\vec{c}$ from voters in $V_{\leqslant i}$ due to (ii). Additionally, $\vec{c}$ also satisfies (β) due to (iv) and the left-feasibility of $\vec{c}_-$. Thus, $\vec{c}$ is left-feasible for i, and since c_i obtains s_i votes from voters in $V_{\leqslant i}$ due to (iii), $\vec{c}$ witnesses the viability of (s_{i-1}, s_i) for (c_{i-1}, c_i). □

Appendix C The missing proof of Theorem 7

This section is dedicated to prove the following result.

Theorem 7 ($\star$)**.** *The* Possible *and* Necessary President *problems are polynomial-time solvable if the preference profile is party-aligned single-peaked.*

Solving the Possible President problem

We first deal with the Possible President problem. Let $(C, \mathcal{P}, P, V, (\prec_v)_{v \in V})$ be our input instance for the Possible President problem. Our algorithm is essentially a simplification of the algorithm described in Section 5 for the Equilibrium President problem: we simply need to remove the constraint that ensures that P is a winner in a Nash equilibrium. For completeness, we give a detailed proof here.

Again, we start by computing a party axis $\lhd_{\mathcal{P}}$, and let $P_1, \dots, P_k$ be the parties in $\mathcal{P}$ ordered according to $\lhd_{\mathcal{P}}$, with $P_\kappa = P$ being the distinguished party. We will re-use Lemma 5 and the consequent partitioning of the voters introduced in Section 5; in particular, we will rely on the notation for voter sets $V_{\leqslant i}$ and $V_{\geqslant i}$, as defined in (1)–(2).

After computing the party axis, our algorithm next guesses the score $s^\star$ obtained by the nominee of the distinguished party P_κ in some (hypothetical) reduced election where P_κ is a winner; we fix $s^\star$ for the remainder.

PP-feasible partial nomination schemes. We now adapt the feasibility concepts introduced in Section 5 to fit our current purposes. Given some index $i \in [k]$ at most κ, a *left-PP-feasible partial nomination scheme for* i is an i-tuple $(c_1, \dots, c_i)$ of candidates such that $c_j \in P_j$ for each $j \in [i]$, and moreover, for each $j \in [i-1]$ party P_j obtains at most $s^\star$ votes in every nomination scheme $(c_1, \dots, c_i, c'_{i+1}, \dots, c'_k)$ from voters in $V_{\leqslant i}$. The analogous notion of *right-PP-feasible partial nomination scheme* is defined symmetrically.

PP-viable scores. We next adapt our notion of viable scores from Section 5 in a straightforward manner. For some index $i \in [k]$ and candidate $c_i \in P_i$, we let s_i be a *left- (*or *right-) PP-viable score for* c_i if there is a left- (or right-) PP-feasible partial nomination scheme where c_i obtains exactly s_i votes from $V_{\leqslant i}$ (from $V_{\geqslant i}$, respectively).

The next lemma is a direct analog of Lemma 6, formulating the solvability of our instance for Possible President based on the concepts of left- and right-PP-viability.

Lemma 9. *Given a candidate $c_\kappa \in P_\kappa$, the following statements are equivalent:*

(a) *there exists a nomination scheme* $(c_1, \dots, c_\kappa, \dots, c_k)$ *in which* c_κ *is a winner with score* $s^\star$*;*

(b) *there exist non-negative integers* s_l *and* s_r *satisfying* $s_l + s_r - |V^{P_\kappa}| = s^\star$ *such that* s_l *is a left-PP-viable score for* c_κ *and* s_r *is a right-PP-viable score for* c_κ*.*

Proof. Assume first that (a) holds, and let $\mathcal{E}$ be the resulting election. Note that $V_{\leqslant i} \cap V_{\geqslant i} = V^{P_i}$. Let s_l and s_r be the number of votes that c_κ obtains in $\mathcal{E}$ from voters in $V_{\leqslant \kappa}$ and from voters in $V_{\geqslant \kappa}$, respectively. The voters counted in both of these sets are exactly those in V^{P_κ}, who all vote for c_κ in $\mathcal{E}$. Hence, the score obtained by c_κ is $s_l + s_r - |V^{P_\kappa}| = s^\star$. Moreover, as c_κ is a winner, no candidate may obtain a score higher than s, and thus $(c_1, \dots, c_\kappa)$ and $(c_\kappa, \dots, c_k)$ is a left- and a right-PP-feasible partial nomination scheme for c_κ, respectively, in which c_κ obtains s_l votes from $V_{\leqslant \kappa}$ and s_r votes from $V_{\geqslant \kappa}$, proving (b).

Assume now that (b) holds. Then there exists a left- and a right-PP-feasible partial nomination scheme $(c_1, \dots, c_\kappa)$ and $(c_\kappa, \dots, c_k)$, respectively, for c_κ, with c_κ obtaining s_l votes from $V_{\leqslant \kappa}$ in the former and s_r votes from $V_{\geqslant \kappa}$ in the latter. Thus, in the nomination scheme $(c_1, \dots, c_\kappa, \dots, c_k)$, every nominee obtains at most $s^\star$ votes, and c_κ obtains exactly $s_l + s_r - |V^{P_i}| = s^\star$ votes, and hence becomes a winner. □

Computing viable scores. Let us now provide a dynamic programming method for computing all left-PP-viable scores for each candidate in $P_1 \cup \dots \cup P_\kappa$; computing right-PP-viable scores for all candidates in $P_\kappa \cup \dots \cup P_k$ can be done in a symmetric manner. Henceforth, we may write PP-viable instead of left-PP-viable.

First, it is clear that for each $c_1 \in P_1$, the only PP-viable score is $|V^{P_1}|$, because all voters in $V_{\leqslant 1} = V^{P_1}$ vote for the nominee of P_1 irrespective of the nomination scheme.

For candidates belonging to some party P_i with $1 < i \leqslant \kappa$, we use Lemma 10 for computing the PP-viable scores.

Lemma 10. *Let* $1 < i \leqslant \kappa$*, and consider some candidate* $c_i \in P_i$*. A non-negative integer* s_i *is a PP-viable score for* c_i *if and only if there exist a candidate* $c_{i-1} \in P_{i-1}$ *and an integer* s_{i-1} *between* 0 *and* $s^\star$ *such that*

(i) s_{i-1} *is PP-viable for* c_{i-1}*,*

(ii) c_i *obtains exactly* $s'_i = s_i - |V^{P_i}|$ *votes from* $V^{(P_{i-1}, P_i)}$ *in any nomination scheme where* c_{i-1} *and* c_i *are nominated, and*

(iii) $s_{i-1} + |V^{(P_{i-1}, P_i)}| - s'_i \leqslant s^\star$.

Proof. Given a left-PP-feasible partial nomination scheme $(c_1, \dots, c_{i-1}, c_i)$ for i, observe that the votes obtained by c_i from voters in $V_{\leqslant i}$ can be partitioned into those coming from voters in V^{P_i} and those coming from voters in $V^{(P_{i-1}, P_i)}$. Hence, if c_i obtains altogether s_i votes from $V_{\leqslant i}$, then it must obtain exactly $s'_i = s_i - |V^{P_i}|$ votes from voters in $V^{(P_{i-1}, P_i)}$, proving (ii). Similarly, the votes obtained by c_{i-1} from voters in $V_{\leqslant i}$ can be partitioned into those coming from voters in $V_{\leqslant i-1}$ and those coming from voters in $V^{(P_{i-1}, P_i)}$. Let us denote the number of votes in the former category as s_{i-1}; the number of votes in the latter category is exactly $|V^{(P_{i-1}, P_i)}| - s'_i$. Since c_{i-1} cannot win more than $s^\star$ votes, we obtain that (iii) holds. Finally, it is also clear from the definition of s_{i-1} and from the left-feasibility of $(c_1, \dots, c_{i-1}, c_i)$ that s_i is a PP-viable score for c_{i-1}, with the partial nomination scheme $(c_1, \dots, c_{i-1})$ as its witness.

For the other direction, let $(c_1, \dots, c_{i-1})$ be a left-PP-feasible partial nomination scheme for $i - 1$ that witnesses (i), i.e., the PP-viability of s_{i-1} for c_{i-1}. Then $(c_1, \dots, c_{i-1}, c_i)$ is a left-PP-feasible partial nomination scheme for i, because due to (ii), c_{i-1} obtains $|V^{(P_{i-1}, P_i)}| - s'_i$ votes from voters in $V^{(P_{i-1}, P_i)}$, and thus obtains at most $s^\star$ votes in total from voters in $V_{\leqslant i}$ due to (iii). Finally, (ii) also implies that c_i obtains exactly $s'_i + |V^{P_i}| = s_i$ votes from $V_{\leqslant i}$ in every nomination scheme containing candidates $c_1, \dots, c_{i-1}, c_i$, which shows that s_i is a PP-viable score for c_i. □

Lemma 10 yields a way to compute all PP-viable scores between 0 and $s^\star$ for each candidate in $P_1 \cup \dots \cup P_\kappa$ using dynamic programming: we need to compute and store a binary variable for each $s \in \{0, \dots, s^\star\}$ and each $c_i \in P_i$, $i \in [\kappa]$, describing whether s is PP-viable for c_i or not; we already know these values for $i = 1$, and we can compute them for increasing values of i using Lemma 10. Having computed all left-PP-viable scores and all right-PP-viable scores (in a symmetric manner) for each candidate of P_κ, we can use Lemma 9 to find a nominee for P_κ that can win in some reduced election, or conclude correctly that no such candidate exists.

It is straightforward to see that the running time of the presented algorithm for the POSSIBLE PRESIDENT problem is polynomial.

Solving the NECESSARY PRESIDENT problem

By slightly modifying our algorithm for POSSIBLE PRESIDENT, we can also decide for each pair (P^+, P^-) of parties whether there is a nomination scheme that results in P^+ becoming a winner but P^- not: during the computation of PP-viable scores for each candidate, we simply need to exclude $s^\star$, the winning score, from the set of PP-viable scores for all candidates of P^-. Therefore, to NECESSARY PRESIDENT in polynomial time under party-aligned single-peaked preferences, we simply need to decide for each party $P' \in \mathcal{P} \setminus \{P\}$ whether P' can become a winner in some reduced election where P is not a winner. If so, then our instance for NECESSARY WINNER is a no-instance; otherwise, if no party P' can become a winner while also preventing P from winning, then we have a yes-instance.