Ore-type condition for antidirected Hamilton cycles in oriented graphs

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Abstract

An antidirected cycle in a digraph G is a subdigraph whose underlying graph is a cycle, and in which no two consecutive arcs form a directed path in G. Let $\sigma_{+-}(G)$ be the minimum value of $d^+(x) + d^-(y)$ over all pairs of vertices x, y such that there is no arc from x to y, that is,

$$\sigma_{+-}(G) = \min\{d^+(x) + d^-(y) : \{x, y\} \subseteq V(G), xy \notin E(G)\}.$$

In 1972, Woodall extended Ore's theorem to digraphs by showing that every digraph G on n vertices with $\sigma_{+-}(G) \geqslant n$ contains a directed Hamilton cycle. Very recently, this result was generalized to oriented graphs under the condition $\sigma_{+-}(G) \geqslant (3n-3)/4$. In this paper, we give the exact Ore-type degree threshold for the existence of antidirected Hamilton cycles in oriented graphs. More precisely, we prove that for sufficiently large even integer n, every oriented graph G on n vertices with $\sigma_{+-}(G) \geqslant (3n+2)/4$ contains an antidirected Hamilton cycle. Moreover, we show that this degree condition is best possible.

Keywords: `antidirected cycle, Ore-type degree condition, oriented graph, Hamilton cycle, orientation`

1 Introduction

Notation follows [2], we only recall a few key definitions here (see also Section 2). A *digraph* is not allowed to have parallel arcs or loops and an *oriented graph* is a digraph with no cycle of length 2. For two vertices x, y in a digraph, we write xy to denote the arc from x to y. A path or cycle is called *directed* if all of its arcs are oriented in the same direction, and *antidirected* if it contains no directed path of length 2. Note that an antidirected cycle must have an even number of vertices.

The decision problem of whether a (di)graph has a Hamilton cycle is NP-complete, so a complete characterization of Hamiltonian graphs appears unlikely. It is natural, therefore, to seek degree conditions that guarantee the existence of Hamilton cycles. A classical result in this direction is Dirac's theorem [10], which states that every graph on $n \geqslant 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle. This was strengthened by Ore's theorem [18]: every graph on $n \geqslant 3$ vertices with $d(x) + d(y) \geqslant n$ for every pair of non-adjacent vertices x, y contains a Hamilton cycle. A further generalization of Dirac's theorem is Pósa's theorem [19], which allows for many vertices with small degree. Moreover, Chvátal's theorem [6] characterized

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all degree sequences that ensure the existence of Hamilton cycles in graphs. Surveys such as [5] and [13] provide extensive overviews of this topic.

While Hamiltonicity in graphs has been studied extensively for decades, its directed analogue, especially in oriented graphs poses distinct challenges and continues to attract significant interest. In particular, the problem of finding sufficient conditions for the existence of Hamilton cycles in oriented graphs has led to several deep results. Instead of minimum degree, one may consider the minimum semidegree $\delta^0(G)$, which is the minimum value over all the in-degrees and out-degrees of the vertices in G. That is, $\delta^0(G) = \min\{d^+(u), d^-(v) : u, v \in V(G)\}$. Ghouila-Houri [12] extended Dirac's theorem to digraphs by proving that every digraph on n vertices with $\delta^0(G) \geqslant n/2$ contains a directed Hamilton cycle. For oriented graphs, Keevash, Kühn and Osthus [14] showed that every large oriented graph on n vertices with $\delta^0(G) \geqslant (3n-4)/8$ contains a directed Hamilton cycle.

In 1972, Woodall [23] extended Ore's theorem to digraphs by proving the following result, where

$$\sigma_{+-}(G) = \min\{d^+(x) + d^-(y) : \{x, y\} \subseteq V(G), xy \notin E(G)\}.$$

Theorem 1.1. *[23] Let G be a digraph on n vertices. If $\sigma_{+-}(G) \geqslant n$, then G contains a directed Hamilton cycle.*

For oriented graphs, Kelly, Kühn and Osthus [15] proved that for every $\varepsilon > 0$, there exists an integer n_0 such that every oriented graph on $n \geqslant n_0$ vertices with $\sigma_{+-}(G) \geqslant (3/4 + \varepsilon)n$ contains a directed Hamilton cycle. Very recently, Chang et al. [4] gave the exact Ore-type degree threshold for a directed Hamilton cycle in oriented graphs.

Theorem 1.2. *[4] There exists an integer n_0 such that every oriented graph G on $n \geqslant n_0$ vertices with $\sigma_{+-}(G) \geqslant (3n-3)/4$ contains a directed Hamilton cycle.*

Rather than focusing solely on directed Hamilton cycles, one may also ask whether a digraph or an oriented graph contains some orientation of a Hamilton cycle under certain degree conditions. In this direction, Debiasio et al. [7] completely determined the exact semidegree threshold for every possible orientation of a Hamilton cycle in digraphs. They showed that every large digraph G with $\delta^0(G) \geqslant n/2$ contains every orientation of a Hamilton cycle, except, possibly, the antidirected one. Recently, Wang, Wang and Zhang [22] extended this result to oriented graphs. They showed that the minimum semidegree threshold for oriented graphs to contain all possible orientations of a Hamilton cycle is $\delta^0(G) \geqslant (3n-1)/8$.

Among various orientations of a Hamilton cycle in digraphs, antidirected Hamilton cycles form a particularly interesting subclass. The existence of such cycles depends delicately on both the local and global structures of the given digraph, and sufficient conditions that guarantee their existence often involve refined degree constraints. Diwan et al. [11] showed that determining whether a digraph contains an antidirected cycle-factor is NP-complete. In 2015, DeBiasio and Molla [8] proved that every large digraph of even order n with $\delta^0(G) \geqslant n$ contains an antidirected Hamilton cycle except for two counterexamples. For oriented graphs, a result of Wang et al. [22] implies that every large oriented graph G on even n vertices with $\delta^0(G) \geqslant (3n-1)/8$ contains an antidirected Hamilton cycle. In this paper, we extend this result by considering Ore-type degree condition. More precisely, we prove the following result.

Theorem 1.3. *There exists an integer n_0 such that every oriented graph G on even $n \geqslant n_0$ vertices with $\sigma_{+-}(G) \geqslant (3n+2)/4$ contains an antidirected Hamilton cycle.*

The following proposition shows that the degree bound in Theorem 1.3 is the best possible for all even $n \in \mathbb{N}$.

Proposition 1.4. *For any even integer $n \geqslant 4$, there are infinitely many oriented graphs G on n vertices with $\sigma_{+-}(G) = \lceil (3n+2)/4 \rceil - 1$ that do not contain an antidirected Hamilton cycle.*

The rest of the paper is organized as follows. Section 2 introduces additional terminology and notation. Key technical tools and preliminary results are presented in Section 3, and the proof of Theorem 1.3 is given in Section 4. In Section 5, we discuss the sharpness of the degree condition by proving Proposition 1.4. Finally, Section 6 offers conclusions of this paper and suggests some open problems.

2 Notation and Terminology

In this section, we provide some basic terminology and notation. For a digraph $G = (V, E)$, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of arcs of G, respectively. The order of G, i.e., the number of vertices, is written as $|G|$, and the size of G, i.e., the number of arcs, is denoted by $e(G)$. The subdigraph induced by a vertex subset X is written as $G[X]$, and we define $G - X = G[V \backslash X]$.

For a subdigraph $H \subseteq G$ and a vertex $v \in V(G)$, the out-neighborhood of v in H is denoted by $N_G^+(v, H)$, and the out-degree by $d_G^+(v, H) = |N_G^+(v, H)|$. Similarly, the in-neighborhood $N_G^-(v, H)$ and the in-degree $d_G^-(v, H)$ are defined. Set $d(v, H) = d^+(v, H) + d^-(v, H)$. We write $d^{\pm}(v, H) \geqslant d$ if both $d^+(v, H)$ and $d^-(v, H)$ are at least d. When the digraph G is clear from context, we often omit the subscript and simply write $N^+(v, H), d^+(v, H)$, etc. When $H = G$, we simply write $N^+(v), d^+(v)$, etc. For a path or cycle L, we say that $v \in V(L)$ is a *sink* vertex (of L) if $d^+(v, L) = 0$ and a *source* vertex if $d^-(v, L) = 0$.

For two subsets $A, B \subseteq V(G)$, we denote by $E(A, B)$ the set of arcs from A to B in G, and write $E(A)$ for $E(A, A)$. Set $e(A, B) = |E(A, B)|$ and $e(A) = |E(A)|$. For a path or a cycle $R = v_1 v_2 \cdots v_l$, the length of R is the number of its arcs. Given sets $X_1, X_2, \ldots, X_l \subseteq V(G)$, we say that $R = v_1 v_2 \cdots v_l$ has *form* $X_1 X_2 \cdots X_l$ if $v_i \in X_i$ for all $i \in [l]$. For simplicity, we will say that R has form X^l if every vertex of R belongs to X.

Let f and g be two mappings from $\mathbb{N}$ to $\mathbb{R}^+$. We write $f(n) = O(g(n))$ if there exists some constant $K > 0$ such that $f(n) \leqslant Kg(n)$ for all sufficiently large n. We adopt the notation $\alpha \leqslant \beta \leqslant \gamma$ to mean that there exist monotone functions f and g such that $\beta \leqslant f(\gamma)$ and $\alpha \leqslant g(\beta)$. For an integer n, $[n]$ will denote the set $\{1, 2, \ldots, n\}$.

3 Preparation

The first lemma is an immediate consequence of Chernoff Bound.

Lemma 3.1. *[8] For any $\varepsilon > 0$, there exists an integer n_0 such that the following holds. Suppose that G is a digraph on $n \geqslant n_0$ vertices. Let $S \subseteq V(G)$ and $|S| \geqslant m$, and define $K = m/|S|$. Then there exists a subset $T \subseteq S$ of order m such that for every $v \in V(G)$,*

$$|d^{\pm}(v, T) - Kd^{\pm}(v, S)| \leqslant \varepsilon n \text{ and } |d^{\pm}(v, S \backslash T) - (1 - K)d^{\pm}(v, S)| \leqslant \varepsilon n.$$

The following classical proposition shows that every graph contains a subgraph with minimum degree at least half of its average degree, where $\delta(H)$ is the minimum degree of H.

Proposition 3.2. *Every graph G contains a subgraph H with $\delta(H) \geqslant e(G)/|G|$.*

Robust expanders, introduced by Kühn, Osthus and Treglown [17], play a central role in many results involving Hamilton cycles. We now recall the definition and some basic properties.

Definition 1. (Robust outexpander) *Given $0 < \nu \leqslant \tau < 1$. For a digraph G on n vertices and a subset $S \subseteq V(G)$, the ν-out-neighborhood of S in G, denoted by $RN_{\nu,G}^+(S)$, is the set of vertices that have at least νn in-neighbors in S. We say that G is a robust (ν, τ)-outexpander if*

$$|RN_{\nu,G}^+(S)| \geqslant |S| + \nu n$$

for every $S \subseteq V(G)$ with $\tau n < |S| < (1 - \tau)n$.

In [16], Kühn and Osthus established a sufficient semidegree condition for any sufficiently large oriented graphs to be a robust outexpander, as detailed in the following lemma.

Lemma 3.3. *[16] Let n_0 be a positive integer and let ν, τ, ε be positive constants such that $0 < 1/n_0 \ll \nu \ll \tau \leqslant \varepsilon/2 \leqslant 1$. If G is an oriented graph on $n \geqslant n_0$ vertices satisfying $\delta^0(G) \geqslant (3/8 + \varepsilon)n$, then G is a robust (ν, τ)-outexpander.*

Taylor [21] proved that every sufficiently large robust outexpander with linear minimum semidegree contains every possible orientation of a Hamilton cycle.

Theorem 3.4. *[21] Let n_0 be a positive integer and let ν, τ, γ be positive constants such that $1/n_0 \ll \nu \leqslant \tau \ll \gamma < 1$. Suppose that G is a digraph on $n \geqslant n_0$ vertices with $\delta^0(G) \geqslant \gamma n$. If G is a robust (ν, τ)-outexpander, then G contains every possible orientation of a Hamilton cycle.*

A slight modification of the proof of Theorem 3.4 yields the following result, its proof can also be found in [9].

Theorem 3.5. *[9] Let n_0 be a positive integer and let ν, τ, γ be positive constants such that $1/n_0 \ll \nu \leqslant \tau \ll \gamma < 1$. Suppose that G is a digraph on $n \geqslant n_0$ vertices with $\delta^0(G) \geqslant \gamma n$. If G is a robust (ν, τ)-outexpander, then for any two distinct vertices $x, y \in V(G)$ and any oriented path P on n vertices, there is a copy of P in G that starts at x and ends at y.*

The next proposition shows that a large Ore-type degree indeed implies a linear minimum semidegree.

Proposition 3.6. *Let G be an oriented graph on n vertices. If $\sigma_{+-}(G) \geqslant (n + 3\gamma n)/2$, then G has minimum semidegree $\delta^0(G) \geqslant \gamma n$.*

Proof. By symmetry, it suffices to show $\delta^-(G) \geqslant \gamma n$, in other words, every vertex of G has in-degree at least γn in G. Suppose to the contrary, there exists a vertex $v \in V(G)$ with $d^-(v) < \gamma n$. Set $X = N^-(v)$ and $Y = V(G) \backslash (X \cup \{v\})$. For any vertex $y \in Y$, the Ore-type degree condition implies that $d^+(y) + d^-(v) \geqslant (n + 3\gamma n)/2$. Then $d^+(y) \geqslant (n + 3\gamma n)/2 - |X|$ and thus $e(Y, G) \geqslant |Y|(n/2 + 3\gamma n/2 - |X|)$. On the other hand, $e(Y, G) \leqslant |X||Y| + |Y|(|Y| - 1)/2$. Combining these bounds yields $|X| + (|Y| - 1)/2 \geqslant (n + 3\gamma n)/2 - |X|$, which contradicts the fact that $|X| = d^-(v) < \gamma n$. □

The following lemma is a direct consequence of Theorem 3.4 and Proposition 3.6.

Lemma 3.7. *Let $0 < 1/n_0 \ll \nu \leqslant \tau \ll \gamma < 1$ and let G be an oriented graph on $n \geqslant n_0$ vertices. If $\sigma_{+-}(G) \geqslant (n + 3\gamma n)/2$ and G is a robust (ν, τ)-outexpander, then G contains every possible orientation of a Hamilton cycle. In particular, G has an antidirected Hamilton cycle when n is even.*

Lemma 3.8. *Let ε be a real with $\varepsilon \ll 1$. Suppose that G is a digraph and X, Y are two disjoint subsets of $V(G)$.*

(i) *If $e(X, Y) \geqslant |X||Y| - O(\varepsilon n^2)$, then there are at most $\varepsilon^{1/3} n$ vertices x in X with $d^+(x, Y) \leqslant |Y| - \sqrt{\varepsilon} n$ and at most $\varepsilon^{1/3} n$ vertices y in Y with $d^-(y, X) \leqslant |X| - \sqrt{\varepsilon} n$.*

(ii) *If $e(X, Y) \leqslant O(\varepsilon n^2)$, then there are at most $\varepsilon^{1/3} n$ vertices x in X with $d^+(x, Y) \geqslant \sqrt{\varepsilon} n$ and at most $\varepsilon^{1/3} n$ vertices y in Y with $d^-(y, X) \geqslant \sqrt{\varepsilon} n$.*

Proof. Observe that if $e(X, Y) \leqslant O(\varepsilon n^2)$, then in the complement digraph $\overline{G}$ of G we have $e_{\overline{G}}(X, Y) \geqslant |X||Y| - O(\varepsilon n^2)$. So (ii) follows directly from the first statement. Moreover, by symmetry, it suffices to prove the former statement of (i). Suppose to the contrary that there are at least $\varepsilon^{1/3} n$ vertices $x \in X$ with $d^+(x, Y) \leqslant |Y| - \sqrt{\varepsilon} n$. Then each such x has at least $\sqrt{\varepsilon} n$ missing arcs to Y, so the total number of missing arcs from X to Y is at least $\varepsilon^{5/6} n^2$. This contradicts the assumption that $e(X, Y) \geqslant |X||Y| - O(\varepsilon n^2)$, and then the claim holds. □

4 Proof of Theorem 1.3

Note that the proof of Theorem 1.3 can be split into two cases, depending on whether the given oriented graph G is a robust (ν, τ)-outexpander or not. If G is a robust (ν, τ)-outexpander, then the result follows immediately from Lemma 3.7. It remains to consider the case when G is not a robust (ν, τ)-outexpander. This case is covered by the following theorem, whose proof will be given in the end of the section.

Theorem 4.1. *Let n be a positive even integer and let ν, τ be positive reals such that $0 < 1/n \ll \nu \leqslant \tau \ll 1$. Suppose that G is an oriented graph on n vertices with $\sigma_{+-}(G) \geqslant (3n+2)/4$. If G is not a robust (ν, τ)-outexpander, then it contains an antidirected Hamilton cycle.*

Let G be an oriented graph and let A, B, C, D be disjoint subsets of $V(G)$ with $V(G) = A \cup B \cup C \cup D$ (here A or C may be empty). For simplicity, we refer to (A, B, C, D) as a partition of G. As mentioned before, to complete the proof of Theorem 1.3, it suffices to analyze the non-expander case. In this case, we first show that G is structurally "close" to a well-behaved configuration. To formalize the notion of "closeness" to such a configuration, we introduce the following definition.

Definition 2. *(Nice partition) Let ε be a positive constant and let G be an oriented graph on n vertices. A partition (A, B, C, D) of G is called an ε-*nice partition *if the sets A, B, C, D satisfy:*

NP1 $|A| \leqslant |C|$ *and* $|A| + |C| = n/2 \pm O(\varepsilon n)$*;*

NP2 $|B|, |D| = n/4 \pm O(\varepsilon n)$*;*

NP3 $e(A \cup D, C \cup D) \leqslant \varepsilon^2 n^2$.

The following lemma shows that any sufficiently large oriented graph G with high $\sigma_{+-}(G)$ value that fails to be a robust outexpander must admit a nice partition.

Lemma 4.2. *Let n be a positive integer and let ν, τ be positive reals such that $0 < 1/n \ll \nu \leqslant \tau, \varepsilon \ll 1$. Suppose that G is an oriented graph on n vertices with $\sigma_{+-}(G) \geqslant 3n/4 - O(\varepsilon n)$. If G is not a robust (ν, τ)-outexpander, then it has an ε-nice partition (A, B, C, D).*

Proof. Since G is not a robust (ν, τ)-outexpander, there exists a subset $S \subseteq V(G)$ with $\tau n < |S| < (1-\tau)n$ such that $|RN^+_{\nu,G}(S)| < |S| + \nu n$. Set $A = RN^+_{\nu,G}(S) \cap S$, $B = RN^+_{\nu,G}(S) \backslash S$, $C = V(G) \backslash (RN^+_{\nu,G}(S) \cup S)$ and $D = S \backslash RN^+_{\nu,G}(S)$. Clearly, $|RN^+_{\nu,G}(S)| = |A| + |B|$, $|S| = |A| + |D|$ and (A, B, C, D) is a partition of $V(G)$.

Note that by the definition of $RN^+_{\nu,G}(S)$, every vertex not in $RN^+_{\nu,G}(S)$ has in-degree less than νn in S. Hence, $e(A \cup D, C \cup D) \leqslant \nu n^2 \leqslant \varepsilon^2 n^2$ as $\nu \ll \varepsilon$. By reversing all arcs of G and exchanging the labels of A and C if necessary, we may assume without loss of generality that $|A| \leqslant |C|$. Furthermore, observe that the inequality $|RN^+_{\nu,G}(S)| < |S| + \nu n$ implies that

$$|B| < |D| + \nu n. \tag{1}$$

Recall that $e(A \cup D, C \cup D) \leqslant \nu n^2$. By Lemma 3.8 (ii), there exist two subsets $A^L \subseteq A$, $C^L \subseteq C$ with $|A^L|, |C^L| \leqslant \nu^{1/3} n$ such that $d^+(a, C \cup D) \leqslant \sqrt{\nu} n$ for each $a \in A \backslash A^L$ and $d^-(c, A \cup D) \leqslant \sqrt{\nu} n$ for each $c \in C \backslash C^L$. Furthermore, there exists a subset $D^L \subseteq D$ with $|D^L| \leqslant 2\nu^{1/3} n$ such that $d^+(d, C \cup D) \leqslant \sqrt{\nu} n$ and $d^-(d, A \cup D) \leqslant \sqrt{\nu} n$ for each vertex $d \in D \backslash D^L$.

To complete the proof, it remains to prove **NP1** and **NP2**. We begin by estimating the size of D.

Claim 1. $\tau n/2 < |D| \leqslant n/4 + O(\varepsilon n)$.

Proof. We first prove that $|D| > \tau n/2$. Suppose to the contrary that $|D| \leqslant \tau n/2$. By (1) and the fact that $\nu \leqslant \tau$, we have $|B| \leqslant 2\tau n/3$. Moreover, it follows by $\tau n < |S| < (1-\tau)n$ that $|A| = |S| - |D| \geqslant \tau n/2$ and $|C| = n - |S| - |B| \geqslant \tau n/3$.

Let $a \in A\backslash A^L$ be a vertex with minimum out-degree in $G[A\backslash A^L]$. Clearly, $d^+(a, A\backslash A^L) \leqslant |A\backslash A^L|/2$ as G is an oriented graph. Then $d^+(a, A) \leqslant |A|/2 + |A^L|$ and thus $d^+(a) \leqslant |A|/2 + |A^L| + |B| + d^+(a, C \cup D) \leqslant |A|/2 + O(\tau n)$. Note that as $a \notin A^L$, the vertex a has at most $\sqrt{\nu}n$ out-neighbors in C. Similarly, let c be a vertex of $C\backslash(C^L \cup N^+(a, C))$ with minimum in-degree. Then $d^-(v, C) \leqslant |C|/2 + |C^L \cup N^+(a, C)|$ and thus $d^-(c) \leqslant |C|/2 + |C^L| + |N^+(a, C)| + |B| + d^-(c, A \cup D) \leqslant |C|/2 + O(\tau n)$. So we have $d^+(a) + d^-(c) \leqslant n/2 + O(\tau n)$, which contradicts the Ore-type degree condition as ac is not an arc of G. Therefore, $|D| > \tau n/2$.

Next, we prove that $|D| \leqslant n/4 + O(\varepsilon n)$. Recall that $D > \tau n/2$ and $|D^L| \leqslant 2\nu^{1/3}n$. Then $|D\backslash D^L| \geqslant \tau n/3$. Since $e(D) \leqslant e(A \cup D, C \cup D) \leqslant \nu n^2$, we have that $G[D\backslash D^L]$ is not a tournament. Hence, there are two vertices d_1, d_2 in $D\backslash D^L$ such that $d_1 d_2 \notin E(G)$ and $d_2 d_1 \notin E(G)$. By the Ore-type degree condition, we get $d(d_1) + d(d_2) = (d^+(d_1) + d^-(d_2)) + (d^+(d_2) + d^-(d_1)) \geqslant 3n/2 - O(\varepsilon n)$. On the other hand, $d(d_1) + d(d_2) \leqslant 2(|A| + |B| + |C| + 2\sqrt{\nu}n) = 2(n - |D| + 2\sqrt{\nu}n)$. Then the upper and lower bounds imply that $|D| \leqslant n/4 + O(\varepsilon n)$, which proves the claim. ☐

Claim 2. *$|B| = n/4 \pm O(\varepsilon n)$.*

Proof. By Claim 1 and the inequality (1), we have $|B| \leqslant n/4 + O(\varepsilon n)$ and thus $|A| + |C| \geqslant n/2 - O(\varepsilon n)$. Next we divide the proof into the following two cases based on the value of $|A|$.

Case 1: $|A| \leqslant \tau n/2$**.**

In this case, we have $|D| = |S| - |A| \geqslant \tau n/2$. Thus one can choose a vertex $d \in D\backslash D^L$ such that $d^+(d, C \cup D) \leqslant \sqrt{\nu}n$. Clearly, $|N^+(d, C)| \leqslant \sqrt{\nu}n$. Since $|C| \geqslant n/2 - O(\varepsilon n) - |A| \geqslant n/3$, we can choose a vertex $c \in C\backslash(C^L \cup N^+(d, C))$ with minimum in-degree in $G[C\backslash(C^L \cup N^+(d, C))]$. So $d^-(c, C) \leqslant |C|/2 + |C^L \cup N^+(d, C)|$. As $dc \notin E(G)$, we obtain

$$\begin{aligned} d^+(d) + d^-(c) &\leqslant \big(|A| + |B| + \sqrt{\nu}n\big) + (|C|/2 + O(\nu^{1/3}n) + |B| + \sqrt{\nu}n) \\ &\leqslant (|A| + |B| + |C| + |D|)/2 + |A|/2 + |B| + O(\nu^{1/3}n) \\ &\leqslant n/2 + |B| + O(\varepsilon n). \end{aligned}$$

By the condition $\sigma_{+-}(G) \geqslant 3n/4 - O(\varepsilon n)$, we have $|B| \geqslant n/4 - O(\varepsilon n)$ and thus $|B| = n/4 \pm O(\varepsilon n)$.

Case 2: $|A| > \tau n/2$**.**

In this case, one may choose a vertex $a \in A\backslash A^L$ with minimum out-degree and then $d^+(a, A) \leqslant |A|/2 + |A^L|$. As $a \notin A^L$, we also have $d^+(a, C \cup D) \leqslant \sqrt{\nu}n$. Since $|C| \geqslant |A| > \tau n/2$, there exists a vertex $c \in C\backslash(C^L \cup N^+(a, C))$ with minimum in-degree and thus $d^-(c, C) \leqslant |C|/2 + |C^L| + \sqrt{\nu}n$. By the Ore-type degree condition, we have

$$\begin{aligned} 3n/4 - O(\varepsilon n) \leqslant d^+(a) + d^-(c) &\leqslant (|A|/2 + O(\nu^{1/3}n) + |B|) + (|C|/2 + |B| + O(\nu^{1/3}n)) \\ &\leqslant (|A| + |B| + |C| + |D|)/2 + |B| + O(\nu^{1/3}n) \\ &\leqslant n/2 + |B| + O(\varepsilon n). \end{aligned}$$

This implies that $|B| \geqslant n/4 - O(\varepsilon n)$ and thus $|B| = n/4 \pm O(\varepsilon n)$. ☐

By the above two claims and (1), we have $n/4 - O(\varepsilon n) \leqslant |B| - \nu n < |D| \leqslant n/4 + O(\varepsilon n)$. Moreover, we have $|A| + |C| = n - |B| - |D| = n/2 \pm O(\varepsilon n)$, which completes the proof. ☐

Having established the existence of a nice partition, we now introduce a classification of vertices based on their degree properties relative to this partition.

Definition 3. (Good/bad vertex) *For a partition (A,B,C,D) of an oriented graph G, a vertex v is called a δ-good vertex of G if it satisfies the following conditions, according to its part:*

- **GA** *if $v \in A$: $d^{\pm}(v,A) \geqslant |A|/2 - \delta n$, $d^{+}(v,B) \geqslant |B| - \delta n$ and $d^{-}(v,D) \geqslant |D| - \delta n$;*
- **GB** *if $v \in B$: $d^{+}(v,C) \geqslant |C| - \delta n$ and $d^{-}(v,A) \geqslant |A| - \delta n$;*
- **GC** *if $v \in C$: $d^{\pm}(v,C) \geqslant |C|/2 - \delta n$, $d^{+}(v,D) \geqslant |D| - \delta n$ and $d^{-}(v,B) \geqslant |B| - \delta n$;*
- **GD** *if $v \in D$: $d^{+}(v,B) \geqslant |C|/2 - \delta n$, $d^{-}(v,B) \geqslant |A|/2 - \delta n$, $d^{+}(v,A) \geqslant |A| - \delta n$ and $d^{-}(v,C) \geqslant |C| - \delta n$.*

Moreover, a vertex is called δ-bad if it is not δ-good.

The next lemma shows that for an oriented graph G with $\sigma_{+-}(G) \geqslant 3n/4 - O(\varepsilon n)$, the number of bad vertices in any nice partition of G is necessarily small. Its proof is similar to the proof of Lemma 4.2.

Lemma 4.3. *For each $\varepsilon > 0$, there exists a constant $\delta \gg \varepsilon$ such that if an oriented graph G satisfying $\sigma_{+-}(G) \geqslant 3n/4 - O(\varepsilon n)$, then every ε-nice partition of G contains at most δn δ-bad vertices.*

Proof. Let (A,B,C,D) be an ε-nice partition of G. Applying Lemma 3.8 (ii) with $X = A \cup D$ and $Y = C \cup D$, there are three sets A^L, D^L, C^L with $A^L \subseteq A, D^L \subseteq D, C^L \subseteq C$ satisfying

- **Y1** $|A^L|, |D^L|, |C^L| \leqslant 2\varepsilon^{2/3} n$;
- **Y2** $d^{+}(u, C \cup D) \leqslant \varepsilon n$ for each $u \in (A \backslash A^L) \cup (D \backslash D^L)$;
- **Y3** $d^{-}(v, A \cup D) \leqslant \varepsilon n$ for each $v \in (C \backslash C^L) \cup (D \backslash D^L)$.

Next we finish the proof by proving the following two claims and some additional arguments.

Claim 3. *There are at most $\delta n/4$ δ-bad vertices in D.*

Proof. Let d^* be an arbitrary vertex in $D \backslash D^L$. By **Y2** and **Y3**, the vertex d^* has at most $2\varepsilon n$ neighbors in D. Let d^{**} be a vertex in $D \backslash (D^L \cup \{d^*\})$ which is not adjacent to d^*. By **Y1**, there are at least $|D| - 3\varepsilon^{2/3} n$ choices for d^{**}. Since $d^* d^{**} \notin E(G)$ and $d^{**} d^* \notin E(G)$, the Ore-type condition implies that $d(d^*) + d(d^{**}) \geqslant 3n/2 - O(\varepsilon n)$. On the other hand, since $d^*, d^{**} \notin D^L$, **Y2**-**Y3** yield $d(d^*) + d(d^{**}) \leqslant 4\varepsilon n + d^{-}(d^*, C) + d^{-}(d^{**}, C) + 2|B| + d^{+}(d^*, A) + d^{+}(d^{**}, A)$. Recall that $|B| = n/4 \pm O(\varepsilon n)$ and $|A| + |C| = n/2 \pm O(\varepsilon n)$. Comparing the upper and lower bounds of $d(d^*) + d(d^{**})$, we obtain $d^{-}(d^*, C) + d^{-}(d^{**}, C) + d^{+}(d^*, A) + d^{+}(d^{**}, A) \geqslant 2(|A| + |C|) - O(\varepsilon n)$. In particular, this implies

$$d^{-}(d^{**}, C) \geqslant |C| - O(\varepsilon n) \text{ and } d^{+}(d^{**}, A) \geqslant |A| - O(\varepsilon n). \tag{2}$$

Let a^* be any vertex of $N^{+}(d^{**}, A) \backslash A^L$. Since $a^* \notin A^L$ and $d^{**} \notin D^L$, we have

$$d^{+}(a^*) + d^{-}(d^{**}) \leqslant 2\varepsilon n + d^{+}(a^*, A) + d^{+}(a^*, B) + d^{-}(d^{**}, B) + d^{-}(d^{**}, C). \tag{3}$$

Next, we claim that d^{**} has at least $|A|/2 - 5\varepsilon^{2/3} n$ in-neighbors in B (see (4) below). If $|A|/2 < 5\varepsilon^{2/3} n$, the claim holds trivially. Thus, we can assume that $|A|$ (and consequently $|C|$) is large. In particular, (2) and **Y1** show that $|N^{+}(d^{**}, A) \backslash A^L|$ is large. Let a^{**} be a vertex in $N^{+}(d^{**}, A) \backslash A^L$ with minimum out-degree in $G[N^{+}(d^{**}, A) \backslash A^L]$. Since G is oriented, we have $d^{+}(a^{**}, N^{+}(d^{**}, A) \backslash A^L) \leqslant |N^{+}(d^{**}, A) \backslash A^L|/2$. By (2) and the fact that $|A^L| \leqslant 2\varepsilon^{2/3} n$, we have $|A| \leqslant |N^{+}(d^{**}, A) \backslash A^L| + 3\varepsilon^{2/3} n$ and hence $d^{+}(a^{**}, A) \leqslant |A|/2 + 3\varepsilon^{2/3} n$. Then (3) implies that $d^+(a^{**}) + d^-(d^{**}) \leqslant 2\varepsilon n + |A|/2 + 3\varepsilon^{2/3}n + |B| + d^-(d^{**}, B) + |C|$. On the other hand, since $a^{**} \in N^+(d^{**}, A)$ and G is oriented, there is no arc from a^{**} to d^{**}. Thus, the Ore-type condition gives that $d^+(a^{**}) + d^-(d^{**}) \geqslant 3n/4 - O(\varepsilon n)$. Combining the upper and lower bounds of $d^+(a^{**}) + d^-(d^{**})$, we have

$$d^-(d^{**}, B) \geqslant |A|/2 - 5\varepsilon^{2/3}n. \tag{4}$$

By symmetry, we also obtain the following inequality:

$$d^+(d^{**}, B) \geqslant |C|/2 - 5\varepsilon^{2/3}n. \tag{5}$$

To see this, let c^{**} be a vertex in $N^-(d^{**}, C)\backslash C^L$ with minimum in-degree in $G[N^-(d^{**}, C)\backslash C^L]$. A similar argument as above shows that $d^-(c^{**}, C) \leqslant |C|/2 + 3\varepsilon^{2/3}n$. Since $d^{**}c^{**} \notin E(G)$, the Ore-type condition together with **Y2**-**Y3** implies $2\varepsilon n + |C|/2 + 3\varepsilon^{2/3}n + |B| + |A| + d^+(d^{**}, B) \geqslant 3n/4 - O(\varepsilon n)$, from which (5) follows.

By (2), (4), (5) and the fact that $\varepsilon \ll \delta$, we get that d^{**} is a δ-good vertex. Since there are at least $|D| - 3\varepsilon^{2/3}n$ choices for the vertex d^{**}, we conclude that D has at most $\delta n/4$ δ-bad vertices. □

Claim 4. *A has at most $\delta n/4$ δ-bad vertices and $e(A, B) \geqslant |A||B| - O(\varepsilon^{1/6}n^2)$.*

Proof. If $|A| \leqslant \delta n/4$, the claim holds trivially. Thus, we can assume $|A| \geqslant \delta n/4$, which implies $|C| \geqslant \delta n/4$ as well. First, observe that by (5) and the fact that G is oriented, we have $d^-(d^{**}, B) \leqslant |B| - |C|/2 + 5\varepsilon^{2/3}n$. Substituting this into (3) yields $d^+(a^*) + d^-(d^{**}) \leqslant 2\varepsilon n + d^+(a^*, A) + 2|B| - |C|/2 + 5\varepsilon^{2/3}n + |C|$. Again, since there is no arc from a^* to d^{**}, the Ore-type condition gives that $d^+(a^*) + d^-(d^{**}) \geqslant 3n/4 - O(\varepsilon n)$. Combining the upper and lower bounds and the fact $|A| + |C| = n/2 \pm O(\varepsilon n)$, $|B|, |D| = n/4 \pm O(\varepsilon n)$, we have

$$d^+(a^*, A) \geqslant |A|/2 - 6\varepsilon^{2/3}n \text{ and thus } d^-(a^*, A) \leqslant |A|/2 + 6\varepsilon^{2/3}n. \tag{6}$$

Recall that a^* is an arbitrary vertex in $N^+(d^{**}, A)\backslash A^L$ and $|N^+(d^{**}, A)\backslash A^L| \geqslant |A| - O(\varepsilon^{2/3}n)$ by (2) and **Y1**. Thus there are at most $O(\varepsilon^{2/3}n)$ vertices in A having out-degree less than $|A|/2 - 6\varepsilon^{2/3}n$ in A. Moreover, it follows that $e(A) \geqslant |A|^2/2 - O(\varepsilon^{2/3}n^2)$.

Next, we claim that there are at most $\varepsilon^{1/3}n$ vertices in A having in-degree less than $|A|/2 - \varepsilon^{1/4}n$ in A. Suppose to the contrary that there are more than $\varepsilon^{1/3}n$ such vertices. Then the second statement of (6) implies that

$$\begin{aligned} e(A) &\leqslant \varepsilon^{1/3}n(|A|/2 - \varepsilon^{1/4}n) + |A^L||A| + (|A\backslash A^L| - \varepsilon^{1/3}n)(|A|/2 + 6\varepsilon^{2/3}n) \\ &\leqslant |A|^2/2 - \varepsilon^{7/12}n^2 + 8\varepsilon^{2/3}n^2, \end{aligned}$$

which contradicts the fact that $e(A) \geqslant |A|^2/2 - O(\varepsilon^{2/3}n^2)$. Therefore, there are at most $2\varepsilon^{1/3}n$ vertices in A have either out-degree or in-degree less than $|A|/2 - \varepsilon^{1/4}n$ in $G[A]$. Let A_1 be the set of these vertices. Clearly, $|A_1| \leqslant 2\varepsilon^{1/3}n$. Observe that since G is oriented, for each $a \in A\backslash A_1$, we have $d^-(a, A), d^+(a, A) = |A|/2 \pm \varepsilon^{1/4}n$.

Recall that there are at least $|D| - 3\varepsilon^{2/3}n$ choices for the vertex d^{**}. Then by (2), we get $e(D, A) \geqslant |A||D| - O(\varepsilon^{2/3}n^2)$. Due to Lemma 3.8 (i), there are at most $\varepsilon^{1/6}n$ vertices of A have in-degree less than $|D| - \varepsilon^{1/3}n$ in D. Let A_2 be the set of those vertices with small in-degree. Clearly, $|A_2| \leqslant \varepsilon^{1/6}n$. Let a be any vertex of $N^+(d^{**}, A)\backslash(A^L \cup A_1 \cup A_2)$. Recall that $d^-(d^{**}, B) \leqslant |B| - |C|/2 + 5\varepsilon^{2/3}n$ by (5). Since $a \notin A_1$, we have $d^+(a, A) = |A|/2 \pm \varepsilon^{1/4}n$. Then (3) implies that $d^+(a) + d^-(d^{**}) \leqslant 2\varepsilon n + |A|/2 + \varepsilon^{1/4}n + d^+(a, B) + |B| - |C|/2 + 5\varepsilon^{2/3}n + |C|$. On the other hand, the Ore-type condition implies $d^+(a) + d^-(d^{**}) \geqslant 3n/4 - O(\varepsilon n)$. Comparing these bounds, we obtain $d^+(a, B) \geqslant |B| - O(\varepsilon^{1/4}n)$. Hence, a is a δ-good vertex as $\varepsilon \ll \delta$ and $a \notin A_1 \cup A_2$.

Now we count the number of such good vertices. It follows by (2) that $|N^+(d^{**}, A)| \geqslant |A| - O(\varepsilon n)$. Moreover, since $|A_1| \leqslant 2\varepsilon^{1/3}n$, $|A_2| \leqslant \varepsilon^{1/6}n$ and $|A^L| = 2\varepsilon^{2/3}n$, there are at least $|A| - 2\varepsilon^{1/6}n$ choices for a. This means that A has at most $2\varepsilon^{1/6}n < \delta n/4$ δ-bad vertices as $\varepsilon \ll \delta$. Moreover, we have $e(A, B) \geqslant (|A| - 2\varepsilon^{1/6}n)(|B| - O(\varepsilon^{1/4}n)) \geqslant |A||B| - O(\varepsilon^{1/6}n^2)$. □

By symmetry, we can obtain that C has at most $\delta n/4$ δ-bad vertices and $e(B, C) \geqslant |B||C| - O(\varepsilon^{1/6}n^2)$. This is achieved by swapping the roles of A and C, B and D, and interchanging "+" and "−" in the proof of the preceding claim. Applying Lemma 3.8 (i) to $e(A, B)$ and $e(B, C)$ respectively, we obtain that B has at most $\delta n/8$ vertices with in-degree less than $|A| - \delta n$ in A and at most $\delta n/8$ vertices with out-degree less than $|C| - \delta n$ in C. It follows that B has at most $\delta n/4$ bad vertices in total. Therefore, the partition (A, B, C, D) contains at most δn δ-bad vertices, which completes the proof of Lemma 4.3. □

Let (A, B, C, D) be any ε-nice partition of G. An arc e is called *special* for (A, B, C, D) if it belongs to $E(A \cup D, C \cup D) \cup E(B \cup C, A \cup B)$. Special arcs are useful for embedding a long antidirected cycle in G. The following lemma shows that the degree condition $\sigma_{+-}(G) \geqslant (3n + 2)/4$ guarantees the existence of two vertex-disjoint special arcs for the nice partition (A, B, C, D).

Lemma 4.4. *Let $0 < 1/n \ll \delta \ll 1$, and let G be an oriented graph on n vertices with a δ-nice partition (A, B, C, D). If $\sigma_{+-}(G) \geqslant (3n+2)/4$, then there are at least two vertex-disjoint special arcs for the partition (A, B, C, D).*

Proof. Suppose to the contrary that there are no two vertex-disjoint special arcs for (A, B, C, D). The proof can be split into the following two cases.

Case 1: G contains exactly three special arcs and these arcs form a triangle.

Let T be the triangle induced by the three special arcs. It should be mentioned that T possibly is not directed. Observe that all arcs in $G[B]$ are special and $|B| = n/4 \pm O(\delta n)$ by **NP2**. Thus, we can choose two vertices b_1, b_2 in $B\backslash V(T)$ such that $b_1b_2 \notin E(G)$ and $b_2b_1 \notin E(G)$. By the Ore-type condition, we have

$$d(b_1) + d(b_2) = d^+(b_1) + d^-(b_1) + d^+(b_2) + d^-(b_2) \geqslant (3n + 2)/2.$$

On the other hand, since all special arcs are incident to vertices in $V(T)$ and $b_1, b_2 \notin V(T)$, both b_1 and b_2 have in-degree and out-degree zero in B. This implies that $d(b_1) + d(b_2) \leqslant 2(|A| + |C| + |D|)$. Combine the upper and lower bounds for $d(b_1) + d(b_2)$, we have $|B| = n - (|A| + |C| + |D|) \leqslant (n - 2)/4$. Similarly, by considering two non-adjacent vertices d_1, d_2 in $D\backslash V(T)$, we can get that $|D| \leqslant (n - 2)/4$, and hence $|B| + |D| \leqslant n/2 - 1$.

Since all arcs in $G[A]$ and $G[C]$ are not special, we have $|V(T) \cap A| \leqslant 1$ and $|V(T) \cap C| \leqslant 1$. It follows by **NP1** and **NP2** that both $|B\backslash V(T)|$, $|C\backslash V(T)|$ and $|D\backslash V(T)|$ are larger than one. Let $b \in B, c \in C, d \in D$ be three vertices not in $V(T)$. First we consider the case that $|A| \geqslant 1$ and assume $a \in A$. Since the arcs in $E(T)$ are the only special arcs, we have $ac, ca \notin E(G)$. The Ore-type degree condition implies that $d(a) + d(c) \geqslant (3n + 2)/2$. On the other hand, we have $d(a) + d(c) \leqslant (|A| - 1 + |B| + |D| + |V(T) \cap C|) + (|C| - 1 + |B| + |D|)$. Combining these bounds yields $|B| + |D| \geqslant n/2 + 2$, which contradicts the fact that $|B| + |D| \leqslant n/2 - 1$.

So it suffices to consider the case that $A = \emptyset$. Note that as $b, c, d \notin V(T)$, there is no arc from d to $C \cup D$ and no arc from $B \cup C$ to b. In particular, we get that $dc, cb \notin E(G)$. Again, by the Ore-type degree condition, we have $d^+(d) + d(c) + d^-(b) \geqslant (3n + 2)/2$. Meanwhile, $d^+(d) + d(c) + d^-(b) \leqslant |B| + (|B| + |D| + |C| - 1) + |D|$. Combining the two bounds again, we have $|B| + |D| \geqslant n/2 + 2$, which contradicts the fact that $|B| + |D| \leqslant n/2 - 1$.

Case 2: All special arcs are incident to the same vertex, or there are no special arcs in G.

If G has special arcs, let v be the vertex to which special arcs are incident and set $X = \{v\}$. Otherwise, set $X = \emptyset$. Remove the vertex in X from the partition and continue to denote the resulting sets as A, B, C and D. Clearly, $n = |A| + |B| + |C| + |D| + |X|$.

First, suppose $|A| \geqslant 1$. Choose one vertex from each (updated) set: $a \in A$, $b \in B$, $c \in C$, and $d \in D$. Since all special arcs are incident to the vertex in X, none of ad, dc, cb and ba is an arc of G. Then the Ore-type condition implies that

$$3n + 2 \leqslant d(a) + d(b) + d(c) + d(d) < 3(|A| + |B| + |C| + |D|) + 4|X| - 2,$$

which contradicts the fact that $|X| \leqslant 1$.

It remains to consider the case $|A| = 0$. As argued in the first paragraph of **Case 1**, we can obtain $|B| + |D| \leqslant n/2 - 1$. Choose vertices $a \in A$, $b \in B$ and $c \in C$. Again, as all special arcs are incident to the vertex in X, neither dc nor cb is an arc of G. Then the Ore-type condition implies that

$$(3n + 2)/2 \leqslant d^+(d) + d(c) + d^-(b) \leqslant |B| + (|C| - 1 + |B| + |D|) + |D| + 3|X|.$$

Then $n/2 - 1 \geqslant |B| + |D| \geqslant n/2 + 2 - 2|X|$, which contradicts the fact that $|X| \leqslant 1$.

Therefore, there are two vertex-disjoint special arcs for (A, B, C, D), which completes the proof of Lemma 4.4. □

In order to construct the desired antidirected cycle, we now introduce two key definitions that will be essential for our subsequent arguments. The first formalizes the notion of a vertex being well-connected to specific subsets of vertices, while the second defines a particular type of path that will serve as a building block in our cycle construction.

Definition 4. (Acceptable vertex) *Let $U_1, U_2 \subseteq V(G)$ and $v \in V(G)$. If $d^-(v, U_1) \geqslant n/100$ and $d^+(v, U_2) \geqslant n/100$, then v is said to be acceptable for (U_1, U_2) and denote it by $v \in \mathcal{A}(U_1, U_2)$.*

Definition 5. (Proper path) *Let $\varepsilon \ll \delta < 1$ and let (A, B, C, D) be an ε-nice partition of G. A path $P = v_1 v_2 \cdots v_d$ is called a D-proper path if it satisfies the following:*

P1 *P is an antidirected path of order at most 10;*

P2 *$v_1v_2, v_{d-1}v_d \in E(G)$;*

P3 *the end-vertices v_1, v_d are δ-good vertices of D.*

Note that every D-proper path $P = v_1 v_2 \cdots v_d$ must have even order, since it is antidirected and $v_1v_2, v_{d-1}v_d \in E(G)$.

With these definitions established, we can now state and prove the following lemma, which guarantees the existence of two vertex-disjoint proper paths under the given conditions.

Lemma 4.5. *Let n be an even positive integer and let ε be a positive constant with $0 < 1/n \ll \varepsilon \ll 1$. Suppose that G is an n-vertex oriented graph with $\sigma_{+-}(G) \geqslant (3n+2)/4$. Then for every ε-nice partition (A, B, C, D) of G, there are two vertex-disjoint D-proper paths in G.*

Proof. By Lemma 4.3, there exists a constant $\delta \gg \varepsilon$ such that G contains at most δn δ-bad vertices. Reassign each bad vertex v according to the following rules:

- Assign v to A if $v \in \mathcal{A}(A \cup D, A \cup B)$;
- Assign v to B if $v \in \mathcal{A}(A \cup D, C \cup D)$;
- Assign v to C if $v \in \mathcal{A}(B \cup C, C \cup D)$;
- Assign v to D if $v \in \mathcal{A}(B \cup C, A \cup B)$.

It is not difficult to check that the reassignment is valid as $\delta^0(G) \geqslant n/6$ by Proposition 3.6. For simplicity, we continue to denote the resulting partition as (A, B, C, D). Noted that we may assume that $|A| \leqslant |C|$ by reversing all arcs of G and swapping the labels of A and C if necessary. So the new partition remains δ-nice. Moreover, by Lemma 4.4, G contains two vertex-disjoint special arcs for (A, B, C, D).

We next show that every special arc can be extended to a D-proper path while avoiding a fixed set W with constant size. The claim then follows by constructing such paths for each of the two vertex-disjoint special arcs, ensuring that each path avoids the vertices used in the other. Note that if P is a D-proper path for the new partition (A, B, C, D), then it remains D-proper for the original ε-nice partition, since the end-vertices of P are δ-good vertices and thus belong to the original set D.

Let uv be any special arc and let $W \subseteq V(G)\backslash\{u, v\}$ be any set with $|W| \leqslant 20$. Next we show that uv can be extended to a D-proper path avoiding all vertices of W. First, suppose $uv \in E(B \cup C, A \cup B)$. By Definition 4 and the reassignment of bad vertices, we have

$$d^+(u, C \cup D) \geqslant n/100 \quad \text{and} \quad d^-(v, A \cup D) \geqslant n/100.$$

Since there are at most δn δ-bad vertices and $|W| \leqslant 20$, we can choose a δ-good vertex u_1 in $N^+(u, C \cup D)\backslash W$ and a δ-good vertex v_1 in $N^-(v, A \cup D)\backslash W$. By **GA-GD**, u_1 has a δ-good in-neighbor $u_2 \in C\backslash W$ and u_2 has a δ-good out-neighbor $u_3 \in D\backslash W$. Similarly, if $|A| \geqslant n/200$, then v_1 has a δ-good out-neighbor in $v_2 \in A\backslash W$ and v_2 has a δ-good in-neighbor $v_3 \in D\backslash W$. Then the path $v_3v_2v_1vuu_1u_2u_3$ is the desired D-proper path. For the case that $|A| < n/200$, we have $d^-(v, D) \geqslant n/200$ as $d^-(v, A \cup D) \geqslant n/100$. Then we may assume that the vertex v_1 belongs to $D\backslash W$ and thus $v_1vuu_1u_2u_3$ is the desired D-proper path.

The case $uv \in E(A \cup D, C \cup D)$ can be handled similarly, and we omit the details. □

Building upon the previous results, we now present a more technical lemma that describes how to construct an antidirected path incorporating the proper paths while satisfying several important properties. This lemma will be instrumental in the final stage of our proof.

Lemma 4.6. *Let n be an even positive integer and let ε be a positive constant with $0 < 1/n \ll \varepsilon \ll 1$. Then there exists a constant $\delta \gg \varepsilon$ such that the following holds. Suppose that G is an n-vertex oriented graph with $\sigma_{+-}(G) \geqslant 3n/4 - O(\varepsilon n)$ and (A, B, C, D) is an ε-nice partition of G. If G contains two vertex-disjoint D-proper paths P_1 and P_2, then there exists an antidirected path P satisfying the following:*

L1 *$A \cup V(P_1) \cup V(P_2) \subseteq V(P)$ and P contains all δ-bad vertices of G;*

L2 *both end-vertices of P are δ-good vertices in D and, they are sink of P;*

L3 *$|C\backslash V(P)| > |B\backslash V(P)| + |D\backslash V(P)| + n/300$;*

L4 *$|C \cap V(P)| = O(\delta n)$.*

Proof. To ensure the condition **L3**, we first construct a long antidirected path P^* between B and D. Let H be the underlying bipartite graph with bipartition (B, D) and arc set $E(D, B)$. Since (A, B, C, D) is an ε-nice partition, the property **NP1** guarantees that $|C| \geqslant n/4 - O(\varepsilon n)$. By Lemma 4.3, G contains at most δn δ-bad vertices. Combining this with **GD**, we have

$$e(H) = e(D, B) \geqslant (|D| - \delta n)(|C|/2 - \delta n) \geqslant n^2/32 - O(\delta n^2).$$

Since $V(H) = |B \cup D| = n/2 \pm O(\varepsilon n)$, Proposition 3.2 implies that there exists a subgraph $H' \subseteq H$ with $\delta(H') \geqslant n/100$. Moreover, since G has at most δn δ-bad vertices, H' contains a path of order $n/200$ whose end-vertices are in D and all vertices are δ-good. This path is clearly corresponding to the desired antidirected path P^*.

Set $\mathcal{P} = V(P_1 \cup P_2 \cup P^*)$. Clearly, $|\mathcal{P}| \leqslant n/100$ by **P1**. Next, we reassign each δ-bad vertex v of $V(G)\backslash\mathcal{P}$ to one of the sets $\mathcal{B}_A$ and $\mathcal{B}_C$ according to the following criteria:

- Assign v to $\mathcal{B}_A$ if one of $d^+(v, A)$, $d^-(v, A)$ and $d^-(v, D)$ exceeds $n/50$;
- Assign v to $\mathcal{B}_C$ if one of $d^-(v, C)$, $d^+(v, C)$ and $d^+(v, D)$ exceeds $n/50$.

This assignment is valid. Indeed, by Proposition 3.6, we have $\delta^0(G) \geqslant n/6$. If a bad vertex v could not be assigned to $\mathcal{B}_A \cup \mathcal{B}_C$, then we would have $d(v, B) \geqslant 2(n/6 - 3n/50) > |B|$, which contradicts the fact that G is oriented. Remove all vertices of $\mathcal{B}_A \cup \mathcal{B}_C$ from the partition and continue to denote the resulting sets as A, B, C and D. Observe that now all vertices in $A \cup B \cup C \cup D$ are δ-good. Note that P_1 and P_2 are still D-proper paths since all end-vertices are δ-good vertices in D.

Let $(X, Y) \in \{(A, B), (B, C), (C, D), (D, A)\}$. It follows by **GA-GD** that $d^+(x, Y) \geqslant |Y| - \delta n$ and $d^-(y, X) \geqslant |X| - \delta n$ for any $x \in X, y \in Y$. Therefore, we have

XY For any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ with $(X, Y) \in \{(A, B), (B, C), (C, D), (D, A)\}$, we have $|N^+(x_1, Y) \cap N^+(x_2, Y)| \geqslant |Y| - 2\delta n$ and $|N^-(y_1, X) \cap N^-(y_2, X)| \geqslant |X| - 2\delta n$.

Similarly, we have the following statements by **GA-GD** and $|\mathcal{B}_A \cup \mathcal{B}_C| \leqslant \delta n$.

CD $|N^-(c, C) \cap N^-(d, C)| \geqslant |C|/2 - 4\delta n$ for any $c \in C, d \in D$.

DD $|N^+(d_1, B) \cap N^+(d_2, B)| \geqslant |C| - |B| - 4\delta n$ for any $d_1, d_2 \in D$.

AD $|N^+(a, A) \cap N^+(d, A)| \geqslant |A|/2 - 4\delta n$ and
$|N^+(a, B) \cap N^+(d, B)| \geqslant |C|/2 - 4\delta n$ for any $a \in A, d \in D$.

Now we construct a (short) path P_C in $G - \mathcal{P}$ covering all vertices of $\mathcal{B}_C$. Let $\mathcal{B}_C^1, \mathcal{B}_C^2$ and $\mathcal{B}_C^3$ be disjoint sets of vertices in $\mathcal{B}_C$ with $d^+(v, D) > n/50$, $d^+(v, C) > n/50$ and $d^-(v, C) > n/50$, respectively. For each $v \in \mathcal{B}_C^1$, pick two out-neighbors of v in $D\backslash\mathcal{P}$. Moreover, as $|\mathcal{P}| \leqslant n/100$, one may assume that those out-neighbors are pairwise distinct. Applying **XY** with $(X, Y) = (C, D)$, every pair of vertices in D has at least $|C| - 2\delta n \geqslant n/5$ common in-neighbors in $C\backslash\mathcal{P}$ as $|C| \geqslant n/4 - O(\varepsilon n)$. Repeat the above progress, one may get an antidirected path L_1 in $G - \mathcal{P}$ covering all vertices of $\mathcal{B}_C^1$ with form $d_1 C (D\mathcal{B}_C^1 DC)^{|\mathcal{B}_C^1|} d_2$, where $d_1, d_2 \in D$. Clearly d_1, d_2 are sink vertices of the path L_1.

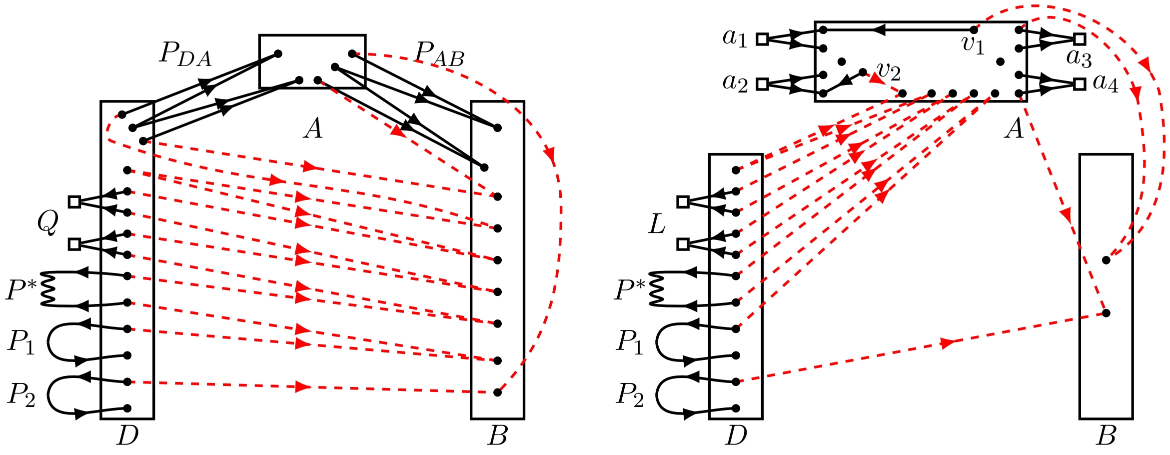
In the same way, applying **XY** with (B, C) and (C, D) respectively, $G - \mathcal{P}$ has two disjoint antidirected paths $L_2 = c_1 B (C\mathcal{B}_C^2 CB)^{|\mathcal{B}_C^2|} c_2$ and $c_3 D (C\mathcal{B}_C^3 CD)^{|\mathcal{B}_C^3|} c_4$, where $c_i \in C$ for each $i \in [4]$. Clearly, c_1, c_2 are sink vertices of L_2 and c_3, c_4 are source vertices of L_3. For c_i with $i = 3, 4$, pick an unused out-neighbor d_i of c_i in D according to **GC**. By **CD**, c_1 and d_3 (resp., c_2 and d_2) have more than $n/10$ common in-neighbors in C. Then we get the desired path $P_C = (d_1) L_1 C L_2 C D L_3 d_4$ with $|P_C| = O(\delta n)$ as $|\mathcal{B}_C| \leqslant \delta n$. Note that the end-vertices of P_C are sink vertices in D.

We next construct an antidirected path P_A in the remaining G to cover all vertices of $A \cup \mathcal{B}_A \cup \mathcal{P}$. In the following, we always assume that the selected vertex is unused. This is possible since $|P_1|, |P_2|, |P_C|, |\mathcal{B}_A|, |\mathcal{B}_C| = O(\delta n), |P^*| = n/200$ and the number of common neighbors are large, as stated in **XY**, **CD**, **DD** and **AD**.

First noted that **XY** shows that every pair of vertices of D has at least $|A| - 2\delta n$ common out-neighbors in A. Hence we do not have enough vertices to connect two vertices of D when $|A|$ is small. However, if $|A|$ is small, say $|A| \leqslant n/50$, then **NP1-NP2** and **DD** imply that every two vertices of D has at least $|B| - n/50 - O(\delta n)$ common out-neighbors in B. So next we construct P_A by considering the following two cases based on the size of A.

Case 1: $|A| \leqslant n/50$.

From the assignment of δ-bad vertices and the fact that $|A| \leqslant n/50$, it follows that $d^-(v, D) > n/50$ for each $v \in \mathcal{B}_A$. Similar with the construction of the path L_1, choosing two in-neighbors of each $v \in \mathcal{B}_A$ in D, we may get an antidirected path Q in the remaining G with form $(D\mathcal{B}_A DB)^{|\mathcal{B}_A|} D$ by **DD**. Moreover, we have $|Q| \leqslant O(\delta n)$ due to $|\mathcal{B}_A| \leqslant \delta n$.



(a) $|A|$ is small (b) $|A|$ is large

Figure 1: An illustration of how to find the desired path P. The white diamonds and black circles indicate the vertices in $\mathcal{B}_A \cup \mathcal{B}_C$ and δ-good vertices of G, respectively. The red dashed arcs are obtained by finding common neighbors due to **XY**, **CD**, **DD** and **AD**. For simplicity, the set C is omitted and P_1, P_2, P^* are placed outside of the partition (A, B, C, D).

Now we extend the path Q to an antidirected path P_A covering all remaining vertices of A. Applying **XY** with (A, B) and (D, A) respectively, every two vertices in A has many common out-neighbors in B and many in-neighbors in D. So we may easily construct two antidirected paths $P_{AB} = (AB)^{\lceil t/2 \rceil}A$ and $P_{DA} = D(AD)^{\lfloor t/2 \rfloor}$, where t is the number of remaining vertices of A. Finally, by **XY** with (D, A), we can connect P_{DA}, P_{AB} and Q into Q^* using common out-neighbors in B of end-vertices of P_{DA} and one end-vertex of P_{AB} and of Q, respectively. Noted that all end-vertices of Q^* and P^* are source vertices and, each of P_1, P_2 has a sink end-vertex and a source end-vertex. Again, **DD** and **AD** show that we can get the desired path P_A from P_1, P^*, Q^*, P_2 by picking common out-neighbors of these source end-vertices in B, see Figure 1 (a) for an illustration. Clearly, $|P_A| = |P^*| + O(\delta n)$, the end-vertices of P_A belong to D and they are sink vertices of P_A. Indeed, the end-vertices of P_A are the sink end-vertices of P_1 and P_2.

Case 2: $|A| > n/50$**.**

In this case, every two vertices of D has at least $|A| - 2\delta n$ common out-neighbors in A, as stated in the property **XY**. This means that the vertices of A are useful when we connect two vertices of D. In particular, they will be used to connect the paths P_1, P_2 and P^*, as shown in **Case 1**. In contrast, in this case we connect those paths first and then cover the remaining vertices of A.

Similar with the arguments when we construct the path P_C, the vertices in $\mathcal{B}_A$ can be divided into three sets $\mathcal{B}_A^1, \mathcal{B}_A^2$ and $\mathcal{B}_A^3$. More precisely, every vertex in $\mathcal{B}_A^1$ has at least $n/50$ in-neighbors in D and, every vertex in $\mathcal{B}_A^2$ and $\mathcal{B}_A^3$ has at least $n/50$ out- and in-neighbors in A, respectively. Noted that by **GA** and the fact that $|A|$ is large, we may move some vertices from A into $\mathcal{B}_A^i$ if necessary so that $|\mathcal{B}_A^i| \geqslant 2$ for each $i \in [3]$. For each $v \in \mathcal{B}_A^1$, choose two of its in-neighbors in D and similarly, pick two out-neighbors (resp., in-neighbors) of each vertex of $\mathcal{B}_A^2$ (resp., $\mathcal{B}_A^3$) in A. Since every two vertices of D has at least $|A| - 2\delta n$ common out-neighbors in A, there is an antidirected path $L = (D\mathcal{B}_A^1 DA)^{|\mathcal{B}_A^1|}D$ covering all vertices in $\mathcal{B}_A^1$. Let u_1, u_2 be the end-vertices of L. Clearly, both u_1 and u_2 are source vertices of L.

Recall that $|\mathcal{B}_A^2| \geqslant 2$ and $|\mathcal{B}_A^3| \geqslant 2$. Assume $a_1, a_2 \in \mathcal{B}_A^2$ and let $a_i', a_i'', i \in [2]$ be the chosen out-neighbors of a_i in A. Moreover, say $a_3, a_4 \in \mathcal{B}_A^3$ and let a_i', a_i'' be the chosen in-neighbors of a_i in A. Since every vertex of A has large in-degree in A, the vertex a_i' has an unused in-neighbor v_i in A for each $i \in [2]$. It follows by **AD** that u_2 and v_2 have large common out-degree in A. Again, as every two vertices of D has at least $|A| - 2\delta n$ common out-neighbors in A, we may connect the paths L, P_1 and P^* by using vertices in A. After this, one may get an antidirected path L_1' with form $P_1AP^*ALAv_2a_2'a_2a_2''$, see Figure 1 (b) for an illustration. Applying **XY** with (A, B), we have that v_1 and a_3' have an unused common out-neighbor in B. Furthermore, **AD** implies that the source end-vertex of P_2 and a_4' have an unused common out-neighbor in B. Then $L_2' := a_1''a_1a_1'v_1Ba_3'a_3a_3''$ and $L_3' := a_4''a_4a_4'BP_2$ are two antidirected paths. Applying **XY** with (D, A) and (A, B), respectively, there are two antidirected paths $L_4' := a_2''DA(\mathcal{B}_A^2A)^{|\mathcal{B}_A^2|-2}D(AD)^{\lceil s/2\rceil}a_1''$ and $L_5' := a_3''BA(\mathcal{B}_A^3A)^{|\mathcal{B}_A^3|-2}(AB)^{\lceil s/2\rceil}a_4''$ covering all remaining vertices of $A \cup \mathcal{B}_A^2 \cup \mathcal{B}_A^3$ for some $s \in \mathbb{N}$. Then $P_A = L_1'L_4'L_2'L_5'L_3'$ is the desired path.

In both cases, the end-vertices of P_A are sink vertices of P_A and they belong to D. Recall that $|P_C| = O(\delta n)$ and the end-vertices of P_C are sink vertices of D also. By **XY** with (C, D), the paths P_C and P_A can be connected by an unused vertex in C. Let P be the resulting path, that is, $P = P_CCP_A$. Now we claim that P satisfies the conditions of the lemma. By the construction of P_C and P_A, **L1** and **L2** holds clearly. Moreover, the path P satisfies **L4** as $|P_C| = O(\delta n)$ and all vertices of $V(P_A)\backslash V(P_1 \cup P_2 \cup \mathcal{B}_A)$ are not in C. Recall that $|\mathcal{B}_A \cup \mathcal{B}_C| \leqslant \delta n$, $|P^*| = n/200$ and the partition (A, B, C, D) is obtained from the original partition (A, B, C, D) by removing all vertices of $\mathcal{B}_A \cup \mathcal{B}_C$. This means that for any $X \in \{A, B, C, D\}$, the sizes of new X and original X differ by at most δn. So for the new sets A, B, C, D, we have $|B|+|D| \leqslant n/2+O(\delta n)$ and $|A|+|C| \geqslant n/2-O(\delta n)$ by **NP1** and **NP2**. Therefore, by the fact that $A \cup V(P^*) \subseteq V(P)$ and **L4**, we have

$$
\begin{aligned}
|B\backslash V(P)| + |D\backslash V(P)| &\leqslant n/2 + O(\delta n) - |P^*| - |A| \\
&\leqslant |C| + O(\delta n) - n/200 \\
&< |C\backslash V(P)| - n/300,
\end{aligned}
$$

which proves **L3**. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma 4.2, there exists a constant ε with $\varepsilon \ll 1$ such that G contains an ε-nice partition (A, B, C, D). Furthermore, by Lemmas 4.5 and 4.6, there exist a constant δ with $\varepsilon \ll \delta \ll 1$ and an antidirected path P in G satisfying **L1**-**L4**.

Next we extend P into an antidirected Hamilton cycle. Note that $|C\backslash V(P)| \geqslant n/4 - O(\varepsilon n) - |C \cap V(P)| \geqslant n/5$ by **NP1** and **L4**. Lemma 3.1 and **L3** ensure the existence of a partition C_1, C_2 of $C\backslash V(P)$ satisfying the following:

CP1 $|C_1| = |B\backslash V(P)| + |D\backslash V(P)| + 2\sqrt{\delta}n$;

CP2 $|C_2| = |C\backslash V(P)| - |C_1| \geqslant n/400$;

CP3 $|d^{\pm}(v, C_i) - \frac{|C_i|}{|C\backslash V(P)|}d^{\pm}(v, C\backslash V(P))| \leqslant \varepsilon n$ for each $i \in [2]$ and $v \in G$.

Let $b \in B\backslash V(P)$ and $d \in D\backslash V(P)$ be two arbitrary vertices. Note that both b and d are δ-good. It follows by **GB** and **GD** that all but at most δn vertices of C are out-neighbors of b and in-neighbors of d. So $d^+(b, C_1), d^-(d, C_1) \geqslant |C_1| - \delta n$. Hence, for each $b_1, b_2 \in B\backslash V(P)$, **CP1** implies that $|N^+(b_1, C_1) \cap N^+(b_2, C_1)| \geqslant |C_1| - 2\delta n > |B\backslash V(P)| + |D\backslash V(P)| + \sqrt{\delta}n$. Similarly, we have $|N^-(d_1, C_1) \cap N^-(d_2, C_1)| > |B\backslash V(P)| + |D\backslash V(P)| + \sqrt{\delta}n$ for each $d_1, d_2 \in D\backslash V(P)$. Thus, we can extend P to P' by using vertices in C_1 to cover all vertices in $(B \cup D)\backslash V(P)$ with form

$$P' = C_1PC_1(DC_1)^{|D\backslash V(P)|}c(BC_1)^{|B\backslash V(P)|} \text{ with } c \in C_1.$$

Note that the vertex c exists as each $b \in B$ and each $c_1 \in C_1$ have many common out-neighbors in C_1. Indeed, we have $|N^+(b, C_1) \cap N^+(c_1, C_1)| \geqslant |C_1|/2 - O(\delta n)$ by **GB** and **CP3**.

Let x and y be the end-vertices of P' and let $C_0 = C_2 \cup (C_1 \backslash V(P')) \cup \{x, y\}$. Noted that all vertices of C_0 are δ-good as all bad vertices are covered by P'. A simple calculation shows that for each $c \in C_0$, we have

$$\begin{aligned} d^{\pm}(c, C_0) \geqslant d^{\pm}(c, C_2) &\geqslant \frac{|C_2|}{|C\backslash V(P)|} d^{\pm}(c, C\backslash V(P)) - \varepsilon n \\ &\geqslant \frac{|C_2|}{|C\backslash V(P)|} \left(d^{\pm}(c, C) - |C \cap V(P)|\right) - \varepsilon n \\ &\geqslant \frac{|C_2|}{|C\backslash V(P)|} (|C\backslash V(P)|/2 - O(\delta n)) - \varepsilon n \\ &\geqslant |C_2|/2 - \sqrt{\delta} n \geqslant 4|C_0|/9, \end{aligned}$$

where the above inequalities follow from **GC**, **L4**, **CP2**, and the bound $|C_0 \backslash C_2| \leqslant 2\sqrt{\delta} n$.

By Lemma 3.3 and Theorem 3.5, there exists an antidirected Hamilton path P'' in $G[C_0]$ with end-vertices x, y. Then $P'P''$ is the desired antidirected Hamilton cycle of G, which completes the proof. □

5 Proof of Proposition 1.4

In this section, we will give a proof of Proposition 1.4. First let us recall the statement.

Proposition 1.4 For any even integer $n \geqslant 4$, there are infinitely many oriented graphs G on n vertices with $\sigma_{+-}(G) = \lceil (3n+2)/4 \rceil - 1$ which does not contain antidirected Hamilton cycles.

Proof. Let G be any oriented graph shown in Figure 2. Next we claim that G has no antidirected Hamilton cycles. Suppose to the contrary that G has an antidirected Hamilton cycle $L = v_1 v_2 \cdots v_n v_1$.

First we consider the case that G is isomorphic to the graph shown in Figure 2 (a) or (b). Assume that v_1 is embedded onto a vertex $a \in A$. Suppose first that v_1 is a source vertex of L, that is, $v_1 v_2, v_1 v_n \in E(G)$. Since a has at most one out-neighbor in $V(G) \backslash (A \cup B)$, one of v_2 and v_n, say v_2, should be embedded onto a vertex in $A \cup B$. Again, as v_2 is a sink vertex of L and it is embedded into $A \cup B$, the vertex v_3 should be embedded into $A \cup D$. Continue this procedure until all vertices are embedded, we get that all vertices of L with odd indices must belong to $A \cup D$ and vertices with even indices are in $A \cup B$. This implies that the antidirected Hamilton cycle $L = v_1 v_2 \cdots v_n v_1$ only uses the vertices in $A \cup B \cup D$, a contradiction. For the case that v_1 is a sink vertex of L, one may verify similarly that L only uses the vertices in $A \cup B \cup D$, a contradiction again.

So it suffices to consider the case that G is isomorphic to the digraph shown in Figure 2 (c). In this case, observe that all arcs between B and D are useless when we embedded the antidirected cycle L. Indeed, suppose $v_1 v_2 \in E(L)$ and it is embedded onto an arc from D to B, then v_3 should in D as v_2 is a sink vertex of L. Similarly, we get that all vertices of L with odd indices must belong to D and vertices with even indices are in B, a contradiction. Therefore, the cycle L must be embedded with form $(CD)^s C \cdots C (BC)^t C \cdots C$, where the second "$C \cdots C$" may be omitted. This means that the size of C should be at least $|B| + |D| + 2$. However, the digraph in (c) has order $4s + 2$ with $|C| = 2s, |B| = |D| = s + 1$, a contradiction. Hence G has no antidirected Hamilton cycles and this completes the proof. □

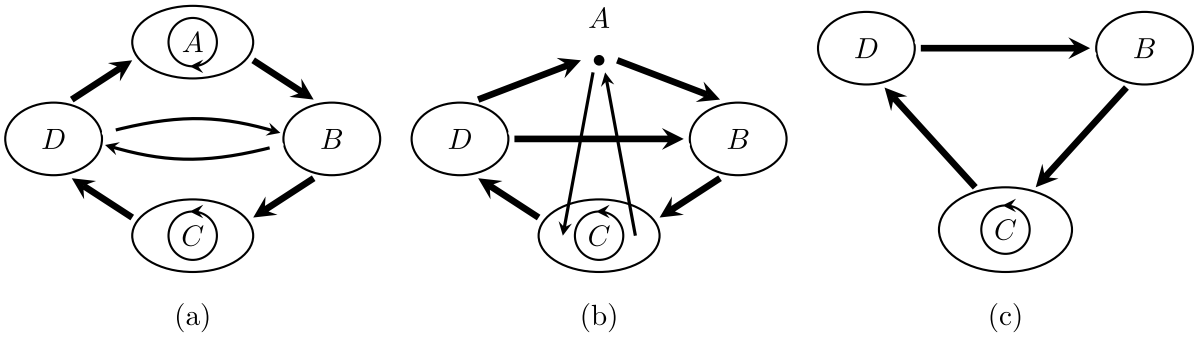


Figure 2: The oriented graphs in Proposition 1.4 with order $8s+6, 4s$ and $4s+2$, respectively. The size of each sets is given in Table 1. The bold arcs indicate that all possible arcs are present and have the directed shown. Each of A and C spans an almost regular tournament, that is, the in-degree and out-degree of every vertex differ by at most one. Both B and D are empty sets and, in (a) the oriented graph induced by B and D is an almost regular bipartite tournament. In (b), A has order one and the vertex in A has exactly one in-neighbor and out-neighbor in C.

n	$\sigma_{+-}(G)$	$\|A\|$	$\|B\| = \|D\|$	$\|C\|$
$8s+6$	$6s+4$	$2s+1$	$2s+2$	$2s+1$
$4s$	$3s$	1	s	$2s-1$
$4s+2$	$3s+1$	0	$s+1$	$2s$

Table 1: The orders of A, B, C, D and the Ore-type condition of G.

6 Conclusion

In this paper, we have established an Ore-type degree condition for the existence of antidirected Hamilton cycles in oriented graphs. More precisely, we prove that for sufficiently large even integer n, every oriented graph G on n vertices with $\sigma_{+-}(G) \geqslant (3n+2)/4$ contains an antidirected Hamilton cycle. Furthermore, we construct three surprising counterexamples showing that the degree condition is best possible. Our result contributes to the broader program of extending classical undirected Hamiltonian results to directed and oriented settings, particularly for non-standard cycle orientations. It also refines earlier work on degree conditions for Hamiltonicity in digraphs and oriented graphs, and enhances our understanding of how local degree conditions can enforce global structural properties.

An oriented graph G is said to be *vertex-pancyclic* if for every vertex $v \in V(G)$ and every integer $3 \leqslant l \leqslant n$, there is a directed cycle of length l containing v. Bondy [3] proposed the meta-conjecture: almost any nontrivial condition on graphs implying a graph is Hamiltonian also implies that it is pancyclic (with possibly a few exceptional families of graphs). This meta-conjecture has been verified for several sufficient conditions, this motivates us to examine these sufficient conditions for vertex-pancyclicity, since vertex-pancyclicity implies pancyclicity. Recall that Chang et al. proved that there exists an integer n_0 such that every oriented graph G on $n \geqslant n_0$ vertices with $\sigma_{+-}(G) \geqslant (3n-3)/4$ contains a directed Hamilton cycle (Theorem 1.2). We conjecture that the same condition also implies vertex-pancyclicity.

Conjecture 6.1. *There exists an integer n_0 such that every oriented graph G on $n \geqslant n_0$ vertices with $\sigma_{+-}(G) \geqslant (3n-3)/4$ is vertex-pancyclic.*

One direction is to investigate whether the bound $(3n-3)/4$ can be improved under additional assumptions or for specific classes of oriented graphs. The following result provides some support for this problem, as well as to the above conjecture.

Theorem 6.1. *[1] Let G be an oriented graph on $n \geqslant 9$ vertices. If G is of minimum degree $n-2$ and $\sigma_{+-}(G) \geqslant n-3$, then G is vertex-pancyclic.*

The degree condition in Theorem 6.1 is best possible in the sense that the conclusion fails if any one of the three conditions $n \geqslant 9$, $\delta(G) \geqslant n-2$ or $\sigma_{+-}(G) \geqslant n-3$ is relaxed. For further details, see [1] and [20].

Another natural and challenging problem is to determine whether a similar Ore-type degree condition can guarantees the existence of cycles with all possible orientations in oriented graphs. In this direction, we propose the following conjecture:

Conjecture 6.2. *There exists an integer n_0 such that every oriented graph G on $n \geqslant n_0$ vertices with $\sigma_{+-}(G) \geqslant (3n+2)/4$ contains all possible orientations of a (undirected) Hamilton cycle.*

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