q-ANALOGS OF RATIONAL NUMBERS: FROM OSTROWSKI NUMERATION SYSTEMS TO PERFECT MATCHINGS

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Abstract. We consider the q-deformation of rational numbers introduced recently by Morier-Genoud and Ovsienko. We propose three enumerative interpretations of these q-rationals: in terms of a new version of Ostrowski's numeration system for integers, in terms of order ideals of fence posets and in terms of perfect matchings of snake graphs. Contrary to previous results which are restricted to rational numbers greater than one, our interpretations work for all positive rational numbers and are based on a single combinatorial object for defining both the numerator and denominator. The proofs rest on order-preserving bijections between posets over these objects. We recover a formula for a q-analog of Markoff numbers. We also deduce a fourth interpretation given in terms of the integer points inside a polytope in $\mathbb{R}^k$ on both sides of a hyperplane where k is the length of the continued fraction expansion.

Contents

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1. Introduction

Fix a positive irrational number x with continued fraction expansion $[a_0; a_1, a_2, \dots]$ with $a_0 \geq 0$ and $a_n \geq 1$ for every $n \geq 1$. Let (p_n) and (q_n) be the sequence of numerators and denominators of the convergents p_n/q_n to x [HW08, Theorem 149]:

$$\begin{aligned} p_i &= a_i p_{i-1} + p_{i-2}, \quad \text{with } p_{-1} = 1 \text{ and } p_0 = a_0, \\ q_i &= a_i q_{i-1} + q_{i-2}, \quad \text{with } q_{-1} = 0 \text{ and } q_0 = 1. \end{aligned} \tag{1}$$

It was proved by Ostrowski a century ago [Ost21] that every positive integer n can be written uniquely as

$$n = \sum_{i=1}^{k} b_i q_i$$

where the integer coefficients satisfy $0 \leq b_i \leq a_i$ and if $b_i = a_i$ then $b_{i-1} = 0$. See also [Ber01, AS03, Arn02, Fog02] or the more recent [EFG+12].

To have uniqueness, different conditions can be imposed on the sequence $(b_i)_i$, leading to different versions of the theorem. For example, two choices are called *first* and *second* multiplicative sequences in the chapter [Arn02] describing the substitutive structure of Sturmian sequences. In this article, we propose a *third* choice leading to an alternating-sign analog of Ostrowski's theorem using the sum of the numerator and denominator of the convergents.

Our choice of sequences can be stated as follows. Let $a = [a_0; a_1, \dots, a_{k-1}]$ be the even or odd-length continued fraction expansion of a positive rational number where $a_0 \geq 0$ and $a_i \geq 1$ for every index i with $1 \leq i < k$ [HW08, Theorem 162]. We say that a finite sequence of integers $(b_i)_{0 \leq i < k}$ is *admissible for* a if it satisfies

- $0 \leq b_i \leq a_i$ for every integer i with $0 \leq i < k$,
- if $i > 0$ is odd and $b_i = a_i$, then $b_{i-1} = a_{i-1}$,
- if $i > 0$ is even and $b_i = 0$, then $b_{i-1} = 0$.

We reuse the vocabulary of "admissible" sequence from [DT89, LL24] where it was used to describe another numeration system based on a single substitution.

Also, let $(r_i)_{0 \leq i \leq k}$ be the sequence defined by

$$r_i = p_{i-1} + q_{i-1}$$

for every integer i with $0 \leq i \leq k$ where $(p_i)_{0 \leq i \leq k-1}$ and $(q_i)_{0 \leq i \leq k-1}$ are the numerators and denominators of the convergents defined as above from the finite continued fraction expansion a. Note that we have $r_0 = p_{-1} + q_{-1} = 1$ and $r_1 = p_0 + q_0 = a_0 + 1$.

Using this choice of admissible sequences, we propose the following alternating-sign version of Ostrowski's theorem. Note that it is distinct from the alternate Ostrowski numeration system proposed in [Bou16].

Theorem A. *Let $k \geq 1$ be an integer and $a = [a_0; a_1, \dots, a_{k-1}]$ be the even or odd-length continued fraction expansion of a positive rational number. Let $\mathcal{Z}(a) = \mathbb{Z} \cap [0, r_k)$ if k is odd or $\mathcal{Z}(a) = \mathbb{Z} \cap [r_{k-1} - r_k, r_{k-1})$ if k is even. Every integer $n \in \mathcal{Z}(a)$ can be written uniquely as*

$$n = \sum_{i=0}^{k-1} (-1)^i b_i r_i$$

where $(b_i)_{0 \leq i \leq k-1}$ is an admissible sequence for a.

Given a positive rational number x with continued fraction expansion $a = [a_0; a_1, \ldots, a_{k-1}]$, it is natural to define the set of admissible sequences for a:

$$\mathcal{B}(a) = \{(b_i)_{0\le i\le k-1} \mid (b_i)_{0\le i\le k-1} \text{ is admissible for } a\}.$$

Theorem A implies that $n \mapsto (b_i)_{0\le i\le k-1}$ defines a bijection $\mathrm{rep}_a : \mathcal{Z}(a) \to \mathcal{B}(a)$ representing every integer in the interval $\mathcal{Z}(a)$ as an admissible sequence for a.

When $a = [1; 1, \ldots, 1]$ is a convergent to the golden ratio, we recover similar values as in the negaFibonacci number system proposed by Donald Knuth [Knu09] in Section 7.1.3 *Bitwise tricks & techniques*. But, a difference is that admissible sequences allow using two consecutive Fibonacci numbers, whereas negaFibonacci numeration system forbids doing this; see Example 4.5.

We split the set $\mathcal{B}(a)$ into two as the disjoint union $\mathcal{B}(a) = \mathcal{B}^\bullet(a) \cup \mathcal{B}^\circ(a)$ where

$$\mathcal{B}^\bullet(a) = \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_0 = a_0 = 0 < b_1 = a_1 \text{ or } 0 < b_0\},$$
$$\mathcal{B}^\circ(a) = \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_0 = a_0 = 0 \le b_1 < a_1 \text{ or } 0 = b_0 < a_0\}.$$

When k is even, we can show that the cardinality of the two sets are equal respectively to the numerator and denominator of the rational number represented by a, that is,

$$\#\mathcal{B}^\bullet(a) = p_{k-1} \quad \text{and} \quad \#\mathcal{B}^\circ(a) = q_{k-1}.$$

This is a consequence of the following stronger result which states that the 1-norm statistics over the set of admissible sequences for a can be computed from a product of the following two matrices:

$$L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix} \qquad \text{and} \qquad R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$$

where q is some indeterminate.

Theorem B. *Let $a = [a_0; \ldots, a_{2\ell-1}]$ be the even-length continued fraction expansion of a positive rational number. The set of admissible sequences for a satisfies*

$$\begin{pmatrix} \sum_{b\in\mathcal{B}^\bullet(a)} q^{\|b\|_1} \\ \sum_{b\in\mathcal{B}^\circ(a)} q^{\|b\|_1} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

where $\|b\|_1 = b_0 + b_1 + \cdots + b_{2\ell-1}$ for every admissible sequence $(b_i)_{0\le i<2\ell} \in \mathcal{B}(a)$.

We prove in Section 11 that the set $\mathcal{B}(a)$ of admissible sequences is a convex subset in $\mathbb{Z}^{2\ell}$ whose convex hull is a polytope and that $\mathcal{B}^\bullet(a)$ and $\mathcal{B}^\circ(a)$ are the two sets of integer points in the polytope on both sides of a hyperplane.

A q-analog of all rational numbers was proposed in [MGO20]. It is given as a ratio of two polynomials in the indeterminate q. When the rational number is larger than one, they also gave a combinatorial interpretation of the coefficients of the numerator and denominator polynomials in terms of the subsets of closures of an oriented path graph. Closures can equivalently be defined as the lower order ideals of a fence poset [MSS21, KOR23]. This is the terminology that we use in this work.

We extend the definition of fence posets proposed in [MSS21, KOR23] in order to describe all of them, not only those that start with an up step. This allows to interpret simultaneously the numerator and the denominator of a q-rational number on a single fence poset. Also, this provides an enumerative interpretation of the q-analog of every positive rational number without the condition > 1 assumed in [MGO20].

To every positive rational number x, whose even-length continued fraction expansion is $a = [a_0; a_1, \ldots, a_{2\ell-1}]$, we associate a fence poset $\mathcal{F}(x)$ containing $a_0 + a_1 + \cdots + a_{2\ell-1}$ elements and $a_0 + a_1 + \cdots + a_{2\ell-1} - 1$ covering relations; please refer to the precise definition in Section 5. To each such fence poset $\mathcal{F}(x)$, we denote its set of lower order ideals by $\mathcal{J}(x)$.

The set $\mathcal{J}(x)$ of lower order ideals is naturally partitioned into a disjoint union $\mathcal{J}(x) = \mathcal{J}^\bullet(x) \cup \mathcal{J}^\circ(x)$ according to whether the left-most vertex of the fence poset is in the order ideal ($\bullet$) or not ($\circ$). We prove the following theorem about the rank polynomial of the lower order ideals.

Theorem C. *For every positive rational number $x \in \mathbb{Q}_{>0}$, whose even-length continued fraction expansion is $[a_0; a_1, \ldots, a_{2\ell-1}]$, the cardinality statistics over the set $\mathcal{J}(x)$ of order ideals of the fence poset $\mathcal{F}(x)$ satisfies*

$$\begin{pmatrix} \sum_{I \in \mathcal{J}^\bullet(x)} q^{|I|} \\ \sum_{I \in \mathcal{J}^\circ(x)} q^{|I|} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

Snake graphs have been introduced to provide combinatorial formulas for elements in cluster algebras of surface type [MS10, MSW11, CS13, MSW13], see also [Pro20]. Their relation with continued fraction expansion of rational numbers greater than one is known [NS16, ÇS20].

To every positive rational number x, whose even-length continued fraction expansion is $[a_0; a_1, \ldots, a_{2\ell-1}]$, we associate a snake graph $\mathcal{G}(x)$ embedded in the plane $\mathbb{R}^2$ containing $a_0 + a_1 + \cdots + a_{2\ell-1}$ unit squares. This is one more unit square than the usual definition [ÇS18]. This choice allows us to associate bijectively a snake graph to all positive rational numbers, not only those that are larger than one.

The bijection from positive rational numbers to the set of all snake graphs makes sense: if $x = \frac{r}{s}$ for some coprime integers $r, s \geq 1$, we show that the snake graph $\mathcal{G}(x)$ contains exactly $r + s$ perfect matchings. Moreover, the set of perfect matchings $\mathcal{M}(x)$ of the snake graph $\mathcal{G}(x)$ is naturally partitioned into a disjoint union $\mathcal{M}(x) = \mathcal{M}^\perp(x) \cup \mathcal{M}^\parallel(x)$ according to the parity of the length of the snake graph and whether the first edge of the matching is vertical or horizontal; see Definition 7.8. We prove that the cardinalities of the two sets

$$\#\mathcal{M}^\perp(x) = r \quad \text{and} \quad \#\mathcal{M}^\parallel(x) = s$$

are the numerator and denominator of the rational number x.

Again this follows from a stronger result involving Morier-Genoud–Ovsienko's q-analogs of rational numbers. We use a statistics on perfect matchings counting the area of the region enclosed by the symmetric difference $m \Delta \mathfrak{b}$ of each perfect matching m with some fixed canonical *basic* perfect matching $\mathfrak{b}$. Precise definitions are given in Section 7. The region enclosed by the symmetric difference of two perfect matchings of a snake graph was used previously to define the height monomials of a cluster algebra. This combinatorial interpretation was used to prove the positivity conjecture (nonnegativity of the coefficients of the cluster expansions of cluster variables with respect to any seed) for cluster algebras coming from a surface [MSW11, MSW13].

We prove that the area statistics of the perfect matchings of a snake graph is computed from the continued fraction expansion of the rational number defining the snake graph.

Theorem D. *For every positive rational number $x \in \mathbb{Q}_{>0}$, whose even-length continued fraction expansion is $[a_0; a_1, \ldots, a_{2\ell-1}]$, the area statistics over the set of perfect matchings of the snake graph $\mathcal{G}(x)$ satisfies*

$$\begin{pmatrix} \sum_{m\in\mathcal{M}^{\perp}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m\in\mathcal{M}^{\parallel}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

where $\mathfrak{b}$ is the basic perfect matching of $\mathcal{G}(x)$.

Note that a perfect matching m belongs to $\mathcal{M}^{\perp}(x)$ if and only if the first unit square of the snake graph is in the region enclosed by the cycles of $m\Delta\mathfrak{b}$; see Remark 7.14. In particular, $\mathrm{area}(m\Delta\mathfrak{b}) \geq 1$ when $m \in \mathcal{M}^{\perp}(x)$.

The proof of Theorem C is based on Proposition 6.4 where equivalent conditions are given for a pair of polynomial functions to be described by product of L_q and R_q matrices. Then, Theorem B and Theorem D are deduced from bijections.

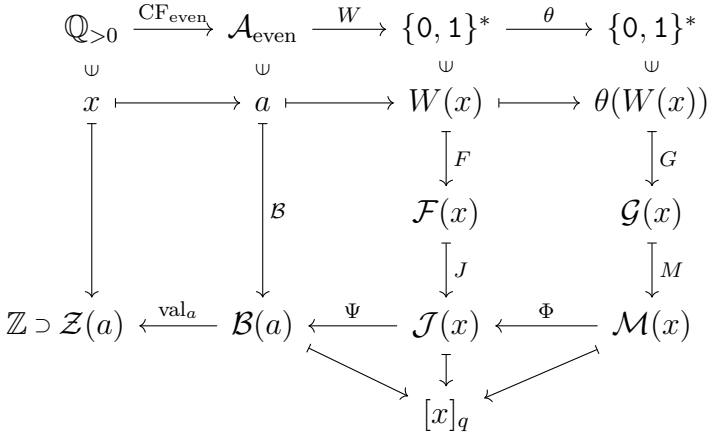


FIGURE 1. The big picture of bijections presented in this article.

There are two levels of bijections. On the first level, we represent positive rational numbers as binary words using a bijection $W \circ \mathrm{CF}_{\mathrm{even}} : \mathbb{Q}_{>0} \to \{0,1\}^*$ where $\mathrm{CF}_{\mathrm{even}}$ returns the even-length continued fraction expansion of a positive rational number. We also have an involution $\theta : \{0,1\}^* \to \{0,1\}^*$:

$$\mathbb{Q}_{>0} \xrightarrow{\mathrm{CF}_{\mathrm{even}}} \mathcal{A}_{\mathrm{even}} \xrightarrow{W} \{0,1\}^* \xrightarrow{\theta} \{0,1\}^*.$$

The second level of bijections depends on a chosen rational number $x > 0$ whose even-length continued fraction expansion is $a = [a_0; a_1, \ldots, a_{2\ell-1}]$. We prove the existence of three bijections $\Phi : \mathcal{M}(x) \to \mathcal{J}(x)$, $\Psi : \mathcal{J}(x) \to \mathcal{B}(a)$ and $\mathrm{val}_a : \mathcal{B}(a) \to \mathcal{Z}(a)$ involving the set $\mathcal{M}(x)$ of perfect matchings of the snake graph $\mathcal{G}(x)$, the set $\mathcal{J}(x)$ of order ideals of the fence poset $\mathcal{F}(x)$, the set $\mathcal{B}(a)$ of admissible sequences for a and an interval of integers $\mathcal{Z}(a) \subset \mathbb{Z}$ that depend on a:

$$\mathcal{M}(x) \xrightarrow{\Phi} \mathcal{J}(x) \xrightarrow{\Psi} \mathcal{B}(a) \xrightarrow{\mathrm{val}_a} \mathcal{Z}(a) \subset \mathbb{Z}.$$

The second level of bijections depends on the first since the snake graph $\mathcal{G}(x)$ is defined from the word $\theta \circ W \circ \mathrm{CF}_{\mathrm{even}}(x)$ and the fence poset $\mathcal{F}(x)$ is defined from the word $W \circ \mathrm{CF}_{\mathrm{even}}(x)$. The three bijections Φ, Ψ and val_a are proved in Theorem 4.4, Theorem 5.2 and Theorem 8.1. The bijections preserve statistics that are used to define polynomials in the indeterminate q. In particular, Φ and Ψ are order-preserving for some natural partial orderings defined on $\mathcal{M}(x)$, $\mathcal{J}(x)$ and $\mathcal{B}(a)$. The

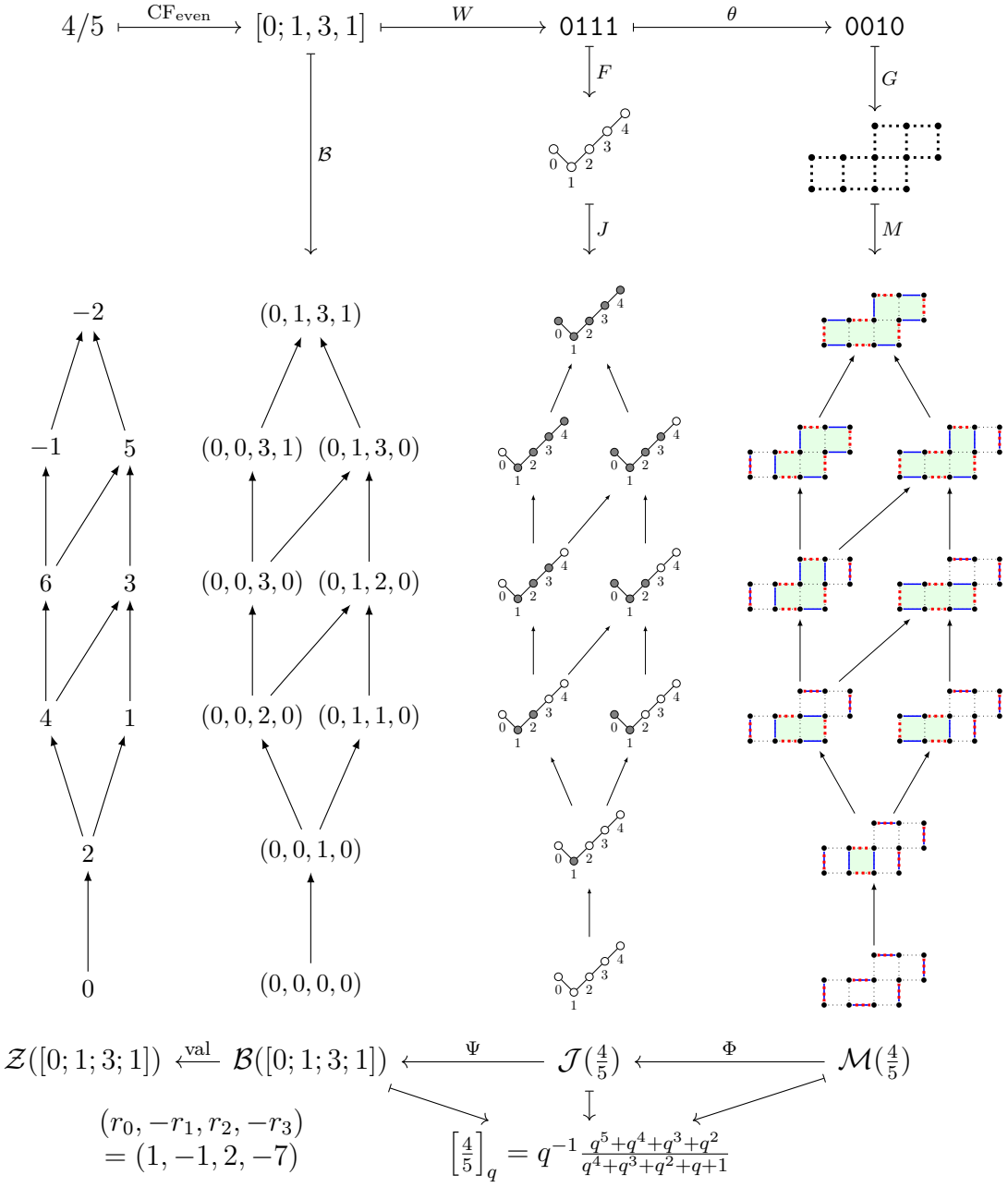


FIGURE 2. The admissible sequences $\mathcal{B}(\frac{4}{5})$ for $\frac{4}{5} = [0; 1, 3, 1]$, the order ideals $\mathcal{J}(\frac{4}{5})$ of the fence poset $\mathcal{F}(\frac{4}{5})$, and the set of perfect matchings $\mathcal{M}(\frac{4}{5})$ of the snake graph $\mathcal{G}(\frac{4}{5})$ are isomorphic posets. The poset is partitioned into two disjoint poset intervals whose rank polynomials are the numerator and denominator of $[4/5]_q$ up to a scalar q^{-1}.

big picture of all bijections of the two levels is shown in Figure 1 and are illustrated in Figure 2 for the rational number 4/5 whose even-length continued fraction expansion is $[0; 1, 3, 1]$.

As a corollary of these three bijections, we obtain three different enumerative interpretations of the Morier-Genoud and Ovsienko's q-analog of positive rational numbers [MGO20]. The definition of this q-analog is recalled Section 3.

Corollary E. *Let $x > 0$ be a positive rational number whose even-length continued fraction expansion is $a = [a_0; \dots, a_{2\ell-1}]$. Then, the set $\mathcal{B}(a)$ of admissible sequences for a, the set $\mathcal{M}(x)$ of perfect matchings of the snake graph $\mathcal{G}(x)$ and the set $\mathcal{J}(x)$ of order ideals of the fence poset $\mathcal{F}(x)$ give three enumerative interpretations of the Morier-Genoud–Ovsienko q-analog of the rational number x:*

$$[x]_q = \frac{q^{-1}\sum_{b\in\mathcal{B}^\bullet(a)} q^{\|b\|_1}}{\sum_{b\in\mathcal{B}^\circ(a)} q^{\|b\|_1}} = \frac{q^{-1}\sum_{I\in\mathcal{J}^\bullet(x)} q^{|I|}}{\sum_{I\in\mathcal{J}^\circ(x)} q^{|I|}} = \frac{q^{-1}\sum_{m\in\mathcal{M}^\perp(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})}}{\sum_{m\in\mathcal{M}^\parallel(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})}}$$

where $\mathfrak{b}$ is the basic perfect matching of the snake graph $\mathcal{G}(x)$ and the fractions on the right-hand sides are reduced.

Our interpretation in terms of order ideals of fence posets is more general than the one proposed in [MGO20] as it is not restricted to rational numbers larger than one. Similarly, our interpretation in terms of perfect matchings of snake graphs works for all positive rational numbers and, contrary to [ÇS18, ÇS20], it is not restricted to rational numbers larger than one. Also, our interpretation of the q-analog of rational numbers in terms of admissible sequences is new.

An interesting aspect of Corollary E, compared to previous combinatorial interpretations provided in [ÇS18, ÇS20, MGO20, MOSZ23], is that the numerator polynomial and denominator polynomial are obtained from a single fence poset and a single snake graph. This seems to be an original result even in the case of $q = 1$ providing three combinatorial interpretations of the numerator and denominator of every positive rational number. The usual interpretation involving different combinatorial objects for the numerator and denominator can be deduced as a consequence of our main results; see Corollary 9.2.

Structure of the article. We recall the continued fraction expansion of rational numbers and their encodings into binary words in Section 2. Then, we recall Morier-Genoud–Ovsienko's q-analog of rational numbers in Section 3. In Section 4, we prove the bijection between admissible sequences and integers in an interval (Theorem A). In Section 5, we prove the order-preserving bijection from fence posets order ideals to admissible sequences. In Section 6, we prove Theorem C and deduce Theorem B. In Section 7, we recall the definition of snake graphs, their perfect matchings and the area statistics. In Section 8, we prove the order-preserving bijection from the set of perfect matchings of a snake graph to the set of order ideals of a fence poset. In Section 9, we prove Theorem D and Corollary E. In Section 10, we express how to recover Markoff numbers and a q-analog of them from these statistics. Finally, we show in Section 11 that the set of admissible sequences is the set of integer points inside of a polytope.

2. Continued fractions

In this section, we recall well-known notions about the continued fraction expansion of real numbers; see [HW08] for more details. We also present an encoding of rational numbers as binary words which will be useful thereafter.

2.1. Continued fraction expansions of rational numbers.

Theorem 162 from [HW08] says that a rational number can be expressed as a finite continued fraction

expansion in just two ways: one of odd-length and the other of even-length. In one form, the last partial quotient is 1, in the other, it is greater than 1. For example,

$$\frac{22}{7} = 3 + \frac{1}{7} = 3 + \frac{1}{6 + \frac{1}{1}} \qquad \text{and} \qquad \frac{7}{22} = 0 + \frac{1}{3 + \frac{1}{7}} = 0 + \frac{1}{3 + \frac{1}{6 + \frac{1}{1}}}.$$

The above is often more compactly written with square brackets as

$$\frac{22}{7} = [3; 7] = [3; 6, 1] \qquad \text{and} \qquad \frac{7}{22} = [0; 3, 7] = [0; 3, 6, 1].$$

In this work, we have a preference for using the even-length expansions:

$$\mathcal{A}_{\text{even}} = \{[a_0; a_1, \ldots, a_{2\ell-1}] \in \mathbb{N}^{2\ell} \colon \ell \geq 1, a_0 \geq 0 \text{ and } a_i > 0 \text{ when } 0 < i < 2\ell\}$$

where the a_i are called *partial quotients*. Thus, it leads to the following bijection

$$\begin{array}{rccc} \mathrm{CF}_{\text{even}} : & \mathbb{Q}_{>0} & \to & \mathcal{A}_{\text{even}} \\ & x & \mapsto & [a_0; a_1, \ldots, a_{2\ell-1}] \end{array}$$

where $[a_0; a_1, \ldots, a_{2\ell-1}]$, for some integer $\ell \geq 1$, is the unique even-length continued fraction expansion of the positive rational number x.

2.2. **Continued fraction convergents in matrix form.** Let $x \in \mathbb{Q}_{>0}$ and

$$\mathrm{CF}_{\text{even}}(x) = [a_0; a_1, \ldots, a_{2\ell-1}]$$

be its even-length continued fraction expansion for some integer $\ell \geq 1$. If $x = r/s$ where $r, s \geq 1$ are coprime integers, then it can be deduced [HW08, Theorem 149] that

$$\begin{aligned} \begin{pmatrix} r \\ s \end{pmatrix} &= R^{a_0} L^{a_1} \cdots R^{a_{2\ell-2}} L^{a_{2\ell-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= R^{a_0} L^{a_1} \cdots R^{a_{2\ell-2}} L^{a_{2\ell-1}-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{2}$$

where L and R are the elementary matrices

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

2.3. **Binary encodings of rational numbers.** The free monoid $\{\mathtt{0}, \mathtt{1}\}^*$ is the set of all finite words over the alphabet $\{\mathtt{0}, \mathtt{1}\}$ together with a binary operation of concatenation. The neutral element for the concatenation is the word of length zero, and denoted ε. If $w = ps \in \{\mathtt{0}, \mathtt{1}\}^*$ is the concatenation of two other words $p, s \in \{\mathtt{0}, \mathtt{1}\}^*$, then we say that p is a *prefix* of w and s is a *suffix* of w.

A rational number can be represented as a binary word over the alphabet $\{\mathtt{0}, \mathtt{1}\}$ using its even-length continued fraction expansion as follows:

$$\begin{array}{rccc} W : & \mathcal{A}_{\text{even}} & \to & \{\mathtt{0}, \mathtt{1}\}^* \\ & [a_0; a_1, \ldots, a_{2\ell-1}] & \mapsto & \mathtt{1}^{a_0}\mathtt{0}^{a_1} \ldots \mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1}. \end{array}$$

Here is the image of W for some rational numbers:

x	$\mathrm{CF}_{\mathrm{even}}(x)$	$W(\mathrm{CF}_{\mathrm{even}}(x))$	
1	$[0;1]$	$\mathtt{1}^0\mathtt{0}^{1-1}$	$=\varepsilon$
2	$[1;1]$	$\mathtt{1}^1\mathtt{0}^{1-1}$	$=\mathtt{1}$
1/2	$[0;2]$	$\mathtt{1}^0\mathtt{0}^{2-1}$	$=\mathtt{0}$
1/3	$[0;3]$	$\mathtt{1}^0\mathtt{0}^{3-1}$	$=\mathtt{00}$
2/3	$[0;1,1,1]$	$\mathtt{1}^0\mathtt{0}^1\mathtt{1}^1\mathtt{0}^{1-1}$	$=\mathtt{01}$
3/2	$[1;2]$	$\mathtt{1}^1\mathtt{0}^{2-1}$	$=\mathtt{10}$
3	$[2;1]$	$\mathtt{1}^2\mathtt{0}^{1-1}$	$=\mathtt{11}$

Notice that $W([a_0;\ldots,a_{2\ell-1}])$ is of length $a_0+\cdots+a_{2\ell-1}-1$.

In fact, every binary word can be expressed as the image of an even-length continued fraction expansion under the map W.

Lemma 2.1. *$W:\mathcal{A}_{even}\to\{\mathtt{0},\mathtt{1}\}^*$ is a bijection.*

Proof. Consider the following map

$$\begin{array}{rccc} Y: & \{\mathtt{0},\mathtt{1}\}^* & \to & \mathcal{A}_{\mathrm{even}} \\ & w & \mapsto & [a_0;\ldots,a_{2\ell-1}] \end{array}$$

where $(a_i)_{0\le i<2\ell}$ is the unique sequence of integers of even length such that

$$w=\mathtt{1}^{a_0}\mathtt{0}^{a_1}\ldots\mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1}$$

with $a_0\ge 0$ and $a_i\ge 1$ for every $i\neq 0$. We observe that $Y\circ W$ and $W\circ Y$ are identity mappings. Therefore, Y is the inverse map of W and we conclude that $W:\mathcal{A}_{\mathrm{even}}\to\{\mathtt{0},\mathtt{1}\}^*$ is invertible, and thus is a bijection. □

Since $\mathrm{CF}_{\mathrm{even}}:\mathbb{Q}_{>0}\to\mathcal{A}_{\mathrm{even}}$ is a well-known bijection often kept implicit, we also denote the composition $W\circ\mathrm{CF}_{\mathrm{even}}:\mathbb{Q}_{>0}\to\{\mathtt{0},\mathtt{1}\}^*$ as W in this article. We hope the reader accepts this slight abuse of notation. Thus, we have

Lemma 2.2. *$W:\mathbb{Q}_{>0}\to\{\mathtt{0},\mathtt{1}\}^*$ is a bijection.*

Proof. Follows from Lemma 2.1 since $\mathrm{CF}_{\mathrm{even}}:\mathbb{Q}_{>0}\to\mathcal{A}_{\mathrm{even}}$ is also a bijection. □

2.4. **Two involutions on rational numbers.** The inverse of rational numbers corresponds, under the map W, to a natural involution on the free monoid $\{\mathtt{0},\mathtt{1}\}^*$ which swaps the letters. On $\{\mathtt{0},\mathtt{1}\}^*$, let $w\mapsto\overline{w}$ be defined as $\overline{w}=\overline{w_1}\cdots\overline{w_k}$ if $w=w_1\cdots w_k$ where $\overline{\mathtt{0}}=\mathtt{1}$ and $\overline{\mathtt{1}}=\mathtt{0}$.

Lemma 2.3. *For every $x\in\mathbb{Q}_{>0}$, $\overline{W(x)}=W(x^{-1})$. In other words, the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{Q}_{>0} & \xrightarrow{W} & \{\mathtt{0},\mathtt{1}\}^* \\ {\scriptstyle x\mapsto x^{-1}}\downarrow & & \downarrow{\scriptstyle w\mapsto\overline{w}} \\ \mathbb{Q}_{>0} & \xrightarrow{W} & \{\mathtt{0},\mathtt{1}\}^* \end{array}$$

Proof. Let $[a_0; \dots, a_{2\ell-1}] \in \mathbb{Q}_{>0}$. If $a_0 > 0$ and $a_{2\ell-1} > 1$, we have

$$\begin{aligned}
\overline{W[a_0; \dots, a_{2\ell-1}]} &= \overline{\mathtt{1}^{a_0}\mathtt{0}^{a_1} \dots \mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1}} \\
&= \mathtt{0}^{a_0}\mathtt{1}^{a_1} \dots \mathtt{0}^{a_{2\ell-2}}\mathtt{1}^{a_{2\ell-1}-1} \\
&= \mathtt{1}^{0}\mathtt{0}^{a_0}\mathtt{1}^{a_1} \dots \mathtt{0}^{a_{2\ell-2}}\mathtt{1}^{a_{2\ell-1}-1}\mathtt{0}^{1-1} \\
&= W[0; a_0, \dots, a_{2\ell-1} - 1, 1] \\
&= W[0; a_0, \dots, a_{2\ell-1}] \\
&= W[a_0; \dots, a_{2\ell-1}]^{-1}.
\end{aligned}$$

Other cases ($a_0 = 0$ or $a_{2\ell-1} = 1$) are dealt similarly and are left to the reader. □

Also, we define the reversal $w \mapsto \widetilde{w}$ on $\{\mathtt{0}, \mathtt{1}\}^*$ as $\widetilde{w} = w_k \cdots w_1$ if $w = w_1 \cdots w_k$. The reversal is an involution whose fixed points are called palindromes. Finally, another involution on $\{\mathtt{0}, \mathtt{1}\}^*$ is $w \mapsto \widehat{w}$ defined as $\widehat{w} = \overline{w_k} \cdots \overline{w_1}$ if $w = w_1 \cdots w_k$. It satisfies $\widehat{w} = \widetilde{\overline{w}} = \overline{\widetilde{w}}$. It turns out that the map $w \mapsto \widehat{w}$ is conjugate to the involution $\tau : \mathcal{A}_{\text{even}} \to \mathcal{A}_{\text{even}}$ defined as

$$\tau([a_0; \dots, a_{2\ell-1}]) = [a_{2\ell-1} - 1, a_{2\ell-2}, \dots, a_1, a_0 + 1].$$

Lemma 2.4. *For every $a \in \mathcal{A}_{even}$, $\widehat{W(a)} = W(\tau(a))$. In other words, the following diagram is commutative:*

$$\begin{array}{ccc}
\mathcal{A}_{even} & \xrightarrow{W} & \{\mathtt{0}, \mathtt{1}\}^* \\
{\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle w \mapsto \widehat{w}} \\
\mathcal{A}_{even} & \xrightarrow{W} & \{\mathtt{0}, \mathtt{1}\}^*
\end{array}$$

Proof. Let $[a_0; a_1, \dots, a_{2\ell-1}] \in \mathcal{A}_{\text{even}}$. We have

$$\begin{aligned}
W[a_0; \widehat{a_1, \dots, a_{2\ell-1}}] &= \mathtt{1}^{a_0}\mathtt{0}^{a_1} \dots \widehat{\mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1}} \\
&= \overline{\mathtt{0}^{a_{2\ell-1}-1}\mathtt{1}^{a_{2\ell-2}} \dots \mathtt{0}^{a_1}\mathtt{1}^{a_0}} \\
&= \mathtt{1}^{a_{2\ell-1}-1}\mathtt{0}^{a_{2\ell-2}} \dots \mathtt{1}^{a_1}\mathtt{0}^{a_0} \\
&= W[a_{2\ell-1} - 1; a_{2\ell-2} \dots, a_1, a_0 + 1].
\end{aligned}$$

□

3. Morier-Genoud–Ovsienko's q-analog of rational numbers

Recently, Morier-Genoud–Ovsienko defined a q-analog for the rational numbers using the matrices

$$L_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix} \qquad \text{and} \qquad R_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix}$$

where q is some indeterminate [MGO20]. The Morier-Genoud–Ovsienko q-analog of the rational number $\frac{r}{s}$ is the rational function

$$\left[\frac{r}{s}\right]_q := \frac{R(q)}{S(q)}$$

where the numerator and denominator polynomials are obtained by replacing matrices L and R in (2) by their q-analogs L_q and R_q and computing the following vector (see Proposition 4.3 and Proposition 4.5 in [MGO20]):

$$\begin{aligned}\begin{pmatrix} R(q) \\ S(q) \end{pmatrix} &= q^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.\end{aligned} \tag{3}$$

In (3), it can be shown that $R(q)$ and $S(q)$ are polynomials and the constant coefficient of the denominator is $S(0) = 1$.

Example 3.1. *For example,*

$$\frac{7}{2} = 3 + \frac{1}{2} = [3;2], \qquad R_q^3 L_q^{2-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} q^4+q^3+2q^2+2q+1 \\ q+1 \end{pmatrix}$$

and

$$\left[\frac{7}{2}\right]_q = \frac{q^4+q^3+2q^2+2q+1}{q+1}.$$

Theorem 4 in [MGO20] gives a combinatorial interpretation of the coefficients of $R(q)$ and $S(q)$ in terms of order ideals within two distinct fence posets constructed from the partial quotients of the continued fraction expansion of $\frac{r}{s}$. In their theorem, it is implicitly assumed that $a_0 \neq 0$, that is, $\frac{r}{s} > 1$. However, Equation (3) works when $a_0 = 0$. Thus, it provides a definition of q-analog for all positive rational numbers including those in the interval $0 < \frac{r}{s} < 1$.

Example 3.2. *For example,*

$$\frac{2}{7} = 0 + \frac{1}{3 + \frac{1}{1+\frac{1}{1}}} = [0;3,1,1], \qquad R_q^0 L_q^3 R_q^1 L_q^{1-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} q^4+q^3 \\ q^4+2q^3+2q^2+q+1 \end{pmatrix}$$

and

$$\left[\frac{2}{7}\right]_q = \frac{q^4+q^3}{q^4+2q^3+2q^2+q+1}.$$

As observed by [MGO19] and [MGO20], the identity

$$\left[\frac{r}{s}+1\right]_q = q\left[\frac{r}{s}\right]_q + 1. \tag{4}$$

allows to extend the definition to all rational numbers including negative ones.

Remark 3.3. *In Example 3.2, the constant term of the numerator $R(q) = q^4 + q^3$ is 0. This is not a contradiction with Corollary 1.7 (iii) of [MGO20] that states that the constant terms of both $R(q)$ and $S(q)$ are equal to 1, because [MGO20] implicitly assumes that $\frac{r}{s} > 1$.*

An active subject. Since the discovery of q-analog of rational numbers by Morier-Genoud and Ovsienko, the subject has sparkled in many research directions. During the preparation of this article, contributions were made that are close to this article. The use of double dimer cover of graphs was considered in [Mus25]. Also, perfect matchings with weighted edges and q-deformed Markov numbers were considered in [EJMGO25]. Then, an interpretation of q-deformed rationals in terms of hyperbinary partitions was proposed in [MPS25]. Equivariant modular functions are considered in [TU25]. Finally, as observed by James Propp[1], there are three rules

[1] `https://mathenchant.wordpress.com/2025/07/17/when-999-isnt-1/`

that allow to compute $[x]_q$ for any rational number $x \in \mathbb{Q}$: $[0]_q = 0$, $[x+1]_q = q[x]_q + 1$ and $[x]_q[-x^{-1}]_q = -q^{-1}$.

4. A bijection from admissible sequences to integers in an interval

Let x be a positive irrational number with continued fraction expansion $a = [a_0; a_1, a_2, \dots]$. Let (p_n) and (q_n) be the sequence of numerators and denominators of the convergents p_n/q_n to x satisfying (1). Let $(r_i)_{i\geq 0}$ be the sequence defined as the sum of the numerator and denominator of the convergents:

$$r_i = p_{i-1} + q_{i-1}.$$

It satisfies the following recurrence:

$$r_i = a_{i-1}r_{i-1} + r_{i-2} \quad \text{for every } i > 0, \qquad \text{with } r_{-1} = r_0 = 1. \tag{5}$$

Let

$$\mathcal{B}_k(a) = \{(b_i)_{0\leq i<k} \mid (b_i)_{0\leq i<k} \text{ is admissible for } [a_0; \dots, a_{k-1}]\}$$

be the set of admissible sequences of length k.

Lemma 4.1. *The set of admissible sequences of length k has cardinality*

$$\#\mathcal{B}_k(a) = r_k.$$

Proof. We proceed by induction. We have $\#\mathcal{B}_0(a) = 1 = r_0$ and $\#\mathcal{B}_1(a) = a_1 + 1 = r_1$. Let $M = 0$ if k is odd or $M = a_{k-1}$ if k is even. The following union is disjoint:

$$\mathcal{B}_k(a) = \{(b_i)_{0\leq i<k} \in \mathcal{B}_k(a) \mid b_{k-2} = b_{k-1} = M\} \cup \{(b_i)_{0\leq i<k} \in \mathcal{B}_k(a) \mid b_{k-1} \neq M\}.$$

Therefore,

$$\begin{aligned}\#\mathcal{B}_k(a) &= \#\{(b_i)_{0\leq i<k} \in \mathcal{B}_k(a) \mid b_{k-2} = b_{k-1} = M\} \\ &\quad + \#\{(b_i)_{0\leq i<k} \in \mathcal{B}_k(a) \mid b_{k-1} \neq M\} \\ &= \#\mathcal{B}_{k-2}(a) + a_{k-1}\#\mathcal{B}_{k-1}(a) \\ &= r_{k-2} + a_{k-1}r_{k-1} = r_k.\end{aligned}$$

This concludes the proof. □

Lemma 4.2. *Let $a = [a_0; a_1, \dots, a_{k-1}]$ be the continued fraction expansion of a positive rational number. Let $n = \sum_{i=0}^{k-1}(-1)^i b_i r_i$ where $(b_i)_{0\leq i<k}$ is an admissible sequence for a. Then,*

$$n < \begin{cases} r_{k-1} & \text{if } k \text{ is even,} \\ r_k & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The upper bounds can be proved with the following telescoping sum. We have

$$n \leq \sum_{0\leq 2j<k} a_{2j}r_{2j} = \sum_{0\leq 2j<k} (r_{2j+1} - r_{2j-1}) = \begin{cases} r_{k-1} - r_{-1} & \text{if } k \text{ is even,} \\ r_k - r_{-1} & \text{if } k \text{ is odd.} \end{cases}$$

The strict inequality follows from the fact that $r_{-1} = 1$. □

Lemma 4.3. *Let $[a_0; a_1, \dots, a_{k-1}]$ be the continued fraction expansion of a positive rational number.*

(1) If k is odd, then every integer n in the interval $0 \leq n < r_k$ can be written as $n = \sum_{i=0}^{k-1}(-1)^i b_i r_i$ where $(b_i)_{0\leq i\leq k-1}$ is an admissible sequence.

(2) If k is even, then every integer n in the interval $r_{k-1} - r_k \leq n < r_{k-1}$ can be written as $n = \sum_{i=0}^{k-1}(-1)^i b_i r_i$ where $(b_i)_{0\leq i\leq k-1}$ is an admissible sequence.

Proof. The proof of existence of the admissible sequence is done by induction. We prove that if (1) holds for some odd k, then (2) hold for $k+1$, and that if (2) holds for some even k, then (1) hold for $k+1$.

First, we prove that (1) holds for $k = 1$. Recall that we have $r_{-1} = r_0 = 1$. Let $k = 1$ and n be an integer in the interval $0 \leq n < r_k = r_1 = a_0 r_0 + r_{-1} = a_0 + 1$. Let $b_0 = n$. We have $0 \leq b_0 \leq a_0$. Thus, the sequence (b_0) is admissible and satisfies

$$n = b_0 = (-1)^0 b_0 r_0.$$

Suppose that (1) holds for some odd integer $k \geq 1$. We want to show that (2) holds for $k+1$. Let n be an integer in the interval $r_k - r_{k+1} \leq n < r_k$. Let Q be the quotient and R be the remainder of the division of n by r_k. We have

$$\begin{aligned} 0 = \left\lfloor \frac{r_k - 1}{r_k} \right\rfloor \geq Q = \left\lfloor \frac{n}{r_k} \right\rfloor \geq \left\lfloor \frac{r_k - r_{k+1}}{r_k} \right\rfloor \\ = \left\lfloor \frac{r_k - (a_k r_k + r_{k-1})}{r_k} \right\rfloor = -a_k + \left\lfloor \frac{r_k - r_{k-1}}{r_k} \right\rfloor = -a_k. \end{aligned}$$

Let $b_k = -Q$ so that $0 \leq b_k \leq a_k$. By definition of the division, we have $n = Qr_k + R$ where the remainder satisfies $0 \leq R < r_k$. From (1), the integer R can be written uniquely as $\sum_{i=0}^{k-1}(-1)^i b_i r_i$ where $(b_i)_{0\leq i\leq k-1}$ is an admissible sequence. Thus,

$$n = Qr_k + R = -b_k r_k + \sum_{i=0}^{k-1}(-1)^i b_i r_i = \sum_{i=0}^{k}(-1)^i b_i r_i.$$

It remains to show that $(b_i)_{0\leq i\leq k}$ is an admissible sequence, that is, that $b_k = a_k$ implies that $b_{k-1} = a_{k-1}$. Suppose that $b_k = a_k$. Thus, $Q = -a_k$. We have

$$r_k > R = n - Qr_k = n + a_k r_k \geq r_k - r_{k+1} + a_k r_k = r_k - r_{k-1}.$$

Let $R' = \sum_{i=0}^{k-2}(-1)^i b_i r_i$. From Lemma 4.2, we have $R' < r_{k-2}$ since $k-1$ is even. Therefore,

$$b_{k-1} r_{k-1} = R - R' > r_k - r_{k-1} - r_{k-2} = a_{k-1} r_{k-1} + r_{k-2} - r_{k-1} - r_{k-2} = (a_{k-1} - 1) r_{k-1}.$$

This implies that $b_{k-1} = a_{k-1}$ in the sum $R = \sum_{i=0}^{k-1}(-1)^i b_i r_i$. We conclude that (2) holds for $k+1$.

Suppose that (2) holds for some even integer $k \geq 1$. We want to show that (1) holds for $k+1$. Let n be an integer in the interval $0 \leq n < r_{k+1}$. Let b_k be the quotient and R be the remainder of the division of $n - (r_{k-1} - r_k)$ by r_k. We have

$$\begin{aligned} 0 \leq b_k = \left\lfloor \frac{n - (r_{k-1} - r_k)}{r_k} \right\rfloor \leq \left\lfloor \frac{r_{k+1} - 1 - (r_{k-1} - r_k)}{r_k} \right\rfloor \\ = \left\lfloor \frac{a_k r_k + r_{k-1} - (r_{k-1} - r_k) - 1}{r_k} \right\rfloor = \left\lfloor \frac{a_k r_k + r_k - 1}{r_k} \right\rfloor = a_k. \end{aligned}$$

By definition of the division, we have $n - (r_{k-1} - r_k) = b_k r_k + R$ where the remainder satisfies $0 \leq R < r_k$. Thus,

$$r_{k-1} - r_k \leq R + r_{k-1} - r_k < r_{k-1}.$$

From (2), the integer $R + r_{k-1} - r_k$ can be written uniquely as $\sum_{i=0}^{k-1}(-1)^i b_i r_i$ where $(b_i)_{0\leq i\leq k-1}$ is an admissible sequence. Therefore,

$$n = b_k r_k + R + (r_{k-1} - r_k) = b_k r_k + \sum_{i=0}^{k-1}(-1)^i b_i r_i = \sum_{i=0}^{k}(-1)^i b_i r_i.$$

It remains to show that $(b_i)_{0\leq i\leq k}$ is an admissible sequence, that is, that $b_k = 0$ implies that $b_{k-1} = 0$. Suppose that $b_k = 0$. We have $n = -b_{k-1}r_{k-1} + R'$ where $R' = \sum_{i=0}^{k-2}(-1)^i b_i r_i$. From Lemma 4.2, we have $R' < r_{k-1}$ since $k-1$ is odd. Therefore, using $n \geq 0$,

$$b_{k-1}r_{k-1} = R' - n < r_{k-1} + 0 = r_{k-1}.$$

Thus, we conclude that $b_{k-1} = 0$ and $(b_i)_{0\leq i\leq k}$ is an admissible sequence. We conclude that (1) holds for $k+1$. □

Theorem 4.4. *Let $a = [a_0; a_1, \ldots, a_{k-1}]$ be a (even or odd-length) continued fraction expansion of a positive rational number. The map*

$$\begin{array}{rccc} \mathrm{val}_a : & \mathcal{B}(a) & \to & \mathcal{Z}(a) \\ & (b_i)_{0\leq i\leq k-1} & \mapsto & \sum_{i=0}^{k-1}(-1)^i b_i r_i \end{array}$$

is a bijection whose range is the following interval of integers

$$\mathcal{Z}(a) = \begin{cases} \mathbb{Z} \cap [0, r_k) & \text{if } k \text{ is odd,} \\ \mathbb{Z} \cap [r_{k-1} - r_k, r_{k-1}) & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let $\mathcal{Z}(a) = \mathbb{Z} \cap [0, r_k)$ if k is odd or $\mathcal{Z}(a) = \mathbb{Z} \cap [r_{k-1} - r_k, r_{k-1})$ if k is even. From the existence of the admissible sequence proved above in Lemma 4.3, the map $\mathrm{val}_a : \mathcal{B}(a) \to \mathcal{Z}(a)$ is onto. The cardinality of the range of val_a is $\#\mathcal{Z}(a) = r_k$. From Lemma 4.1, we also have that the set of admissible sequences for a of length k is $\#\mathcal{B}(a) = r_k$. Thus, the map $\mathrm{val}_a : \mathcal{B}(a) \to \mathcal{Z}(a)$ is also injective and it is a bijection. □

We denote the representation of integers in the interval $\mathcal{Z}(a)$ by the function $\mathrm{rep}_a : \mathcal{Z}(a) \to \mathcal{B}(a)$ defined as the inverse of the map val_a, that is, $\mathrm{rep}_a = \mathrm{val}_a^{-1}$.

Theorem A. *Let $k \geq 1$ be an integer and $a = [a_0; a_1, \ldots, a_{k-1}]$ be the even or odd-length continued fraction expansion of a positive rational number. Let $\mathcal{Z}(a) = \mathbb{Z} \cap [0, r_k)$ if k is odd or $\mathcal{Z}(a) = \mathbb{Z} \cap [r_{k-1} - r_k, r_{k-1})$ if k is even. Every integer $n \in \mathcal{Z}(a)$ can be written uniquely as*

$$n = \sum_{i=0}^{k-1}(-1)^i b_i r_i$$

where $(b_i)_{0\leq i\leq k-1}$ is an admissible sequence for a.

Proof. The statement follows from the fact that the map $\mathrm{val}_a : \mathcal{B}(a) \to \mathcal{Z}(a)$ proposed in Theorem 4.4 is a bijection. □

Theorem A is illustrated in Table 1 using the odd-length continued fraction expansion $a = [2; 2, 2]$ of $12/5$ and using the even-length continued fraction expansion $a = [2; 2, 2, 2]$ of $29/12$. The following example illustrates the case of a convergent to the golden ratio.

	a_0	a_1	a_2
	2	2	2
	p_{-1}/q_{-1}	p_0/q_0	p_1/q_1
	1/0	2/1	5/2
	r_0	$-r_1$	r_2
	1	−3	7
n	b_0	b_1	b_2
0	0	0	0
1	1	0	0
2	2	0	0
3	2	2	1
4	0	1	1
5	1	1	1
6	2	1	1
7	0	0	1
8	1	0	1
9	2	0	1
10	2	2	2
11	0	1	2
12	1	1	2
13	2	1	2
14	0	0	2
15	1	0	2
16	2	0	2

	a_0	a_1	a_2	a_3
	2	2	2	2
	p_{-1}/q_{-1}	p_0/q_0	p_1/q_1	p_2/q_2
	1/0	2/1	5/2	12/5
	r_0	$-r_1$	r_2	$-r_3$
	1	−3	7	−17
n	b_0	b_1	b_2	b_3
−24	2	2	2	2
−23	0	1	2	2
−22	1	1	2	2
−21	2	1	2	2
−20	0	0	2	2
−19	1	0	2	2
−18	2	0	2	2
−17	0	0	0	1
−16	1	0	0	1
−15	2	0	0	1
−14	2	2	1	1
−13	0	1	1	1
−12	1	1	1	1
−11	2	1	1	1
−10	0	0	1	1
−9	1	0	1	1
−8	2	0	1	1
−7	2	2	2	1
−6	0	1	2	1
−5	1	1	2	1
−4	2	1	2	1
−3	0	0	2	1
−2	1	0	2	1
−1	2	0	2	1
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	2	2	1	0
4	0	1	1	0
5	1	1	1	0
6	2	1	1	0
7	0	0	1	0
8	1	0	1	0
9	2	0	1	0
10	2	2	2	0
11	0	1	2	0
12	1	1	2	0
13	2	1	2	0
14	0	0	2	0
15	1	0	2	0
16	2	0	2	0

TABLE 1. Let $a = [2; 2, 2]$ be the continued fraction expansion of 12/5. The table on the left shows the admissible representation (b_0, b_1, b_2) of every integer n in the interval $0 \le n < r_3 = 17$. They satisfy $n = b_0r_0 - b_1r_1 + b_2r_2$. Let $a = [2; 2, 2, 2]$ be the continued fraction expansion of 29/12. The table on the right shows the admissible representation (b_0, b_1, b_2, b_3) of every integer n in the interval $-24 = r_3 - r_4 \le n < r_3 = 17$. They satisfy $n = b_0r_0 - b_1r_1 + b_2r_2 - b_3r_3$.

Example 4.5. *With $a = [1; 1, 1, 1, 1, 1]$ of even length 6, we have $(r_i)_i = (1, -2, 3, -5, 8, -13)$. Writing least-significant first, we obtain the following representations for integers in the interval $[-8, 13)$:*

n	$\mathrm{rep}_a(n)$	n	$\mathrm{rep}_a(n)$	n	$\mathrm{rep}_a(n)$
-8	111111	-1	101011	6	001110
-7	001111	0	000000	7	101110
-6	101111	1	100000	8	000010
-5	000011	2	111000	9	100010
-4	100011	3	001000	10	111010
-3	111011	4	101000	11	001010
-2	001011	5	111110	12	101010

The set of admissible sequences is naturally endowed with a partial order. Its definition is made before Theorem 5.2. Additional results can be obtained on these admissible sequences related to discrete geometry and numeration systems.

In Section 11, we prove that the set $\mathcal{B}(a)$ of admissible sequences for a is a convex set of integer points (Theorem 11.2). Thus $\mathcal{B}(a)$ is the set of integer points inside of a polytope.

Using the infinite continued fraction expansion of a positive irrational number, Theorem A can be extended to all positive integers using odd-length admissible sequences and to all integers in $\mathbb{Z}$ using even-length admissible sequences. These two alternating-sign versions of Ostrowski's theorem seem to be new and will be developed in a separate work from this article.

5. A bijection from fence poset order ideals to admissible sequences

In [MGO20], Morier-Genoud and Ovsienko proposed a q-analog of rational numbers and gave an enumerative interpretation of rational numbers larger than 1 through objects named closures of oriented path graphs. While making progress [MSS21] and solving [KOR23] a conjecture of Morier-Genoud and Ovsienko about the unimodality of the involved polynomials, the authors used the equivalent, but maybe more common, terminology of fence posets and their order ideals. In this section, we follow this choice.

The aim of this section is to relate the combinatorics of admissible sequences to order ideals of fence posets. As we shall see, we extend the definition of fence posets proposed in [MSS21, KOR23] in order to describe all of them, not only those that start with an up step. This allows to interpret simultaneously the numerator and the denominator of a q-rational number on a single fence poset. Also, this gives an enumerative interpretation of the q-analog of every positive rational number without the condition > 1 assumed in [MGO20].

5.1. **From rational numbers to fence posets.** A *partially ordered set* P (or *poset*, for short) is a set together with a binary relation denoted $\lhd$, satisfying three axioms: reflexivity, antisymmetry and transitivity; see [Sta12]. If $s, t \in P$ with $s \neq t$, then we say that t covers s or s is covered by t, if $s \lhd t$ and no element $u \in P \setminus \{s, t\}$ satisfies $s \lhd u \lhd t$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $s \lhd t$ then t is drawn above s (i.e., with a higher vertical coordinate).

A *fence poset* is a poset whose Hasse diagram is a path graph, that is, a connected acyclic directed graph whose degree of every vertex is at most 2. In [MSS21], a

fence poset was associated to every composition $(a_0, a_1, \ldots, a_{k-1})$ made of positive integers, thus forcing the first step of the fence poset to always be an up step. The same definition was used in [KOR23], but it was convenient for them to "*allow the first part of the composition to be zero*" in their proof of Morier-Genoud–Ovsienko's rank-unimodality conjecture in order to represent all fence posets. This suggests that it may be more convenient to represent fence posets using the continued fraction expansion of positive rational numbers. This is what we do in this section.

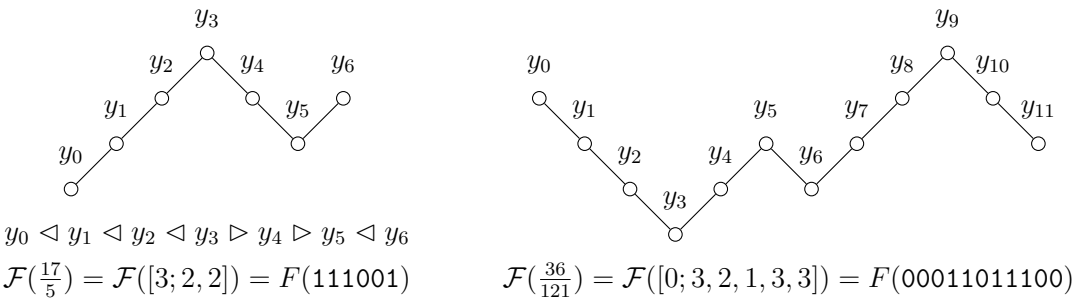


FIGURE 3. The fence posets $F(\texttt{111001})$ and $F(\texttt{00011011100})$.

For every word $w = w_1 \cdots w_n \in \{\texttt{0},\texttt{1}\}^*$, let $F(w)$ be the fence poset with elements $\{y_0, y_1, \ldots, y_n\}$ and covering relations

$$\begin{cases} y_{i-1} \lhd y_i & \text{if } w_i = \texttt{0}, \\ y_{i-1} \rhd y_i & \text{if } w_i = \texttt{1}, \end{cases}$$

for every integer index i with $1 \le i \le n$, where $\lhd$ is the order relation in $F(w)$. Every fence poset can be expressed as $F(w)$ for some binary word $w \in \{\texttt{0},\texttt{1}\}^*$. Thus, F is a bijection from $\{\texttt{0},\texttt{1}\}^*$ to the set of fence posets. For example, the fence posets $F(\texttt{000110})$ and $F(\texttt{00011011100})$ are shown in Figure 3.

We let $\mathcal{F} = F \circ W \circ \mathrm{CF}_{\mathrm{even}}$ allowing to associate bijectively a fence poset to every positive rational number:

$$\mathcal{F} : \mathbb{Q}_{>0} \to \{\text{fence posets}\}.$$

Figure 3 shows a fence poset $\mathcal{F}(x)$ associated with a rational number $x > 1$ and another associated with a rational number $x < 1$.

The map $\mathcal{F}$ maps the Stern-Brocot tree of positive rational numbers [GKP94, Section 4.5] to the binary tree of fence posets where the parent relation is defined by the prefix relation; see Figure 4.

The reader may refer to Figure 1 to locate this bijection in the big picture. Notice that the Hasse diagram of the fence poset $\mathcal{F}([a_0; \ldots, a_{2\ell-1}])$ has $a_0 + \cdots + a_{2\ell-1} - 1$ edges and $a_0 + \cdots + a_{2\ell-1}$ vertices. Also, for every $x \in \mathbb{Q}_{>0}$, the poset $\mathcal{F}(x)$ starts with a down step if and only if $x < 1$.

Following Lemma 2.3, the fence posets $\mathcal{F}(x)$ and $\mathcal{F}(x^{-1})$ are mirror image of one another under the horizontal axis. Mirror image under the vertical axis can be understood as another involution on the positive rational numbers; see Lemma 2.4.

Remark 5.1. *The construction of a fence poset from the even-length continued fraction expansion of a rational number was already proposed in [Ove23], but it is restricted to rational numbers larger than one. This choice is consistent with the combinatorial interpretation of q-rational numbers from k-vertex closures of two*

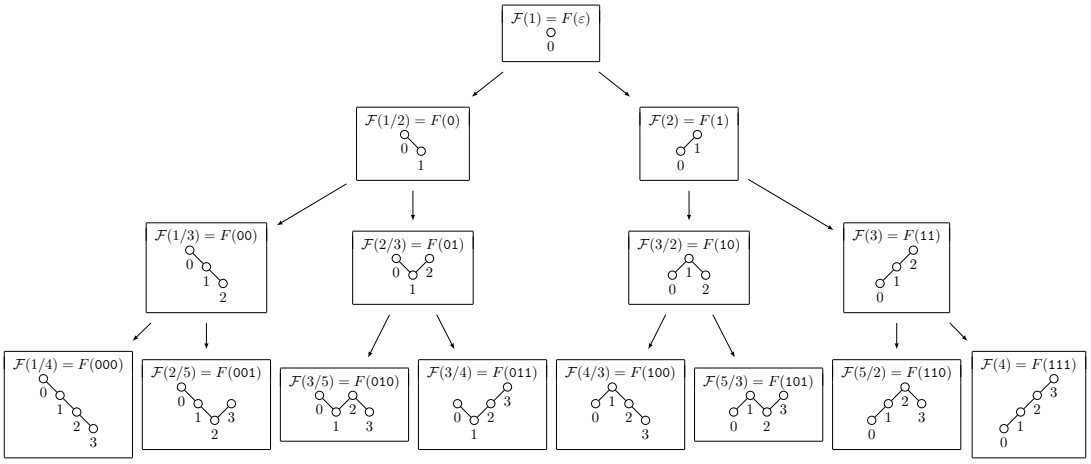


FIGURE 4. The map $\mathcal{F}$ maps the Stern-Brocot tree of positive rational numbers on the binary tree of fence posets ordered by the prefix relation.

oriented path graphs made in [MGO20]. On the contrary, the fence poset $\mathcal{F}(x)$ defined here works for all rational numbers $x > 0$.

5.2. **Fence poset order ideals.** A (lower) *order ideal* of a poset P with binary relation $\lhd$ is a subset I of P such that if $t \in I$ and $s \lhd t$, then $s \in I$ [Sta12]. The set of all order ideals of P, ordered by inclusion, forms a poset denoted $J(P)$ which is also a *distributive lattice*. The lattice $J(P)$ is graded by the cardinality, called *rank*, of the order ideals. The number of order ideals of each rank is encoded into a *rank-generating polynomial*

$$\sum_{I \in J(P)} q^{|I|}$$

where $|I|$ is the cardinality (rank) of the order ideal I.

In the context of fence posets, we let $\mathcal{J} := J \circ \mathcal{F} = J \circ F \circ W \circ \mathrm{CF}_{\mathrm{even}}$. Thus, if $x > 0$ is a positive rational number, then $\mathcal{J}(x)$ is the set of lower order ideals of the fence poset $\mathcal{F}(x)$ ordered by inclusion; see these maps in the big picture in Figure 1. It was conjectured in [MGO20] and proved in [KOR23] that the rank polynomials of the distributive lattices of lower ideals of fence posets are unimodal.

For every $x \in \mathbb{Q}_{>0}$, the set $\mathcal{J}(x)$ of lower order ideals of the fence poset $\mathcal{F}(x)$ is naturally partitioned as the disjoint union $\mathcal{J}(x) = \mathcal{J}^\bullet(x) \cup \mathcal{J}^\circ(x)$ according to whether the order ideal contains the first element or not, that is,

$$\mathcal{J}^\bullet(x) = \{I \in \mathcal{J}(x) \mid y_0 \in I\},$$
$$\mathcal{J}^\circ(x) = \{I \in \mathcal{J}(x) \mid y_0 \notin I\}.$$

For example, an order ideal in the set $\mathcal{J}^\bullet(399/121)$ is shown in Figure 5.

5.3. **A bijection from fence poset order ideals to admissible sequences.** The relation between order ideals of fence posets and binary strings in $\{\mathtt{0}, \mathtt{1}\}^*$ with no two consecutive $\mathtt{1}$'s was noticed before for the particular fence poset $\mathcal{F}(\mathtt{10101}\ldots)$ whose number of order ideals is a Fibonacci number [MZS02]. In this section, we

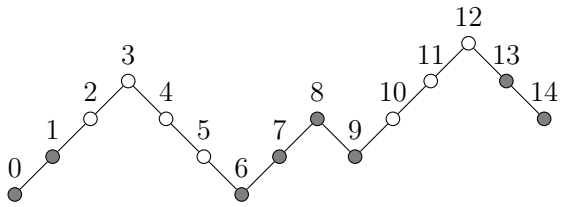


FIGURE 5. The order ideal $\{0, 1, 6, 7, 8, 9, 13, 14\}$ of cardinality 8 in the distributive lattice $\mathcal{J}^\bullet(399/121) = \mathcal{J}^\bullet([3, 3, 2, 1, 3, 3])$ is shown with dark vertices. The elements y_i of the poset are identified with their indices i.

extend this observation to all fence posets using a partition of the fence poset into chains as in [Sta12, Section 3.5].

We prove the existence of a bijection from the set of lower order ideals of a fence poset to the set of admissible sequences such that the cardinality of an order ideal correspond to the 1-norm of an admissible sequence.

Let $a = [a_0; a_1, \ldots, a_{2\ell-1}]$ be the even-length continued fraction expansion of a positive rational number x. For every i, with $0 \leq i < 2\ell$, we define the interval of integers

$$C_i(a) = \left\{ n \in \mathbb{N} \;\middle|\; \sum_{k=0}^{i-1} a_k \leq n < \sum_{k=0}^{i} a_k \right\}$$

which is a chain in the fence poset $\mathcal{F}(x)$. The family $\{C_i(a)\}_{0\leq i<2\ell}$ is made of disjoint consecutive intervals of integers and its union is the set

$$\bigcup_{0\leq i<2\ell} C_i(a) = \{0, 1, \ldots, a_0 + a_1 + \cdots + a_{2\ell-1} - 1\}.$$

To simplify the notation, we identify the elements $\{y_0, \ldots, y_{a_0+a_1+\cdots+a_{2\ell-1}-1}\}$ of the fence poset $\mathcal{F}(x)$ with their integer indices $\{0, 1, \ldots, a_0 + a_1 + \cdots + a_{2\ell-1} - 1\}$. Counting the number of elements of an order ideal of $\mathcal{F}(x)$ in each of the intervals of the family $\{C_i(a)\}_{0\leq i<2\ell}$ leads to a function $\Psi : \mathcal{J}(x) \to \mathcal{B}(a)$ defined as

$$\begin{array}{rccl} \Psi: & \mathcal{J}(x) & \to & \mathcal{B}(a) \\ & I & \mapsto & (\#I \cap C_i(a))_{0\leq i<2\ell}\,. \end{array}$$

An example of the bijection is shown in Figure 6.

Let $a = [a_0; a_1, \ldots, a_{k-1}]$ be the (even or odd-length) continued fraction expansion of a positive rational number. The set $\mathcal{B}(a)$ of admissible sequences for a is naturally equipped with a partial order $(\mathcal{B}(a), \leq)$ defined as follows:

$$(b_i)_{0\leq i\leq k-1} \leq (b'_i)_{0\leq i\leq k-1} \quad \text{if and only if} \quad b_i \leq b'_i \text{ for every } i \text{ with } 0 \leq i \leq k-1.$$

The rank function of the graded poset $(\mathcal{B}(a), \leq)$ is the 1-norm of the admissible sequences.

Theorem 5.2. *Let $x \in \mathbb{Q}_{>0}$ whose even-length continued fraction expansion is $a = [a_0; a_1, \ldots, a_{2\ell-1}]$. The map $\Psi : (\mathcal{J}(x), \subseteq) \to (\mathcal{B}(a), \leq)$ is an order-preserving bijection from the set of order ideals of the fence poset $\mathcal{F}(x)$ to the set of admissible sequences for a such that:*

- *$I \in \mathcal{J}^\bullet(x)$ if and only if $\Psi(I) \in \mathcal{B}^\bullet(a)$;*
- *$I \in \mathcal{J}^\circ(x)$ if and only if $\Psi(I) \in \mathcal{B}^\circ(a)$;*
- *for every $I \in \mathcal{J}(x)$, we have $|I| = \|\Psi(I)\|_1$.*

i	0	1	2	3	4	5
$C_i(a)$	$\{0,1,2\}$	$\{3,4,5\}$	$\{6,7\}$	$\{8\}$	$\{9,10,11\}$	$\{12,13,14\}$

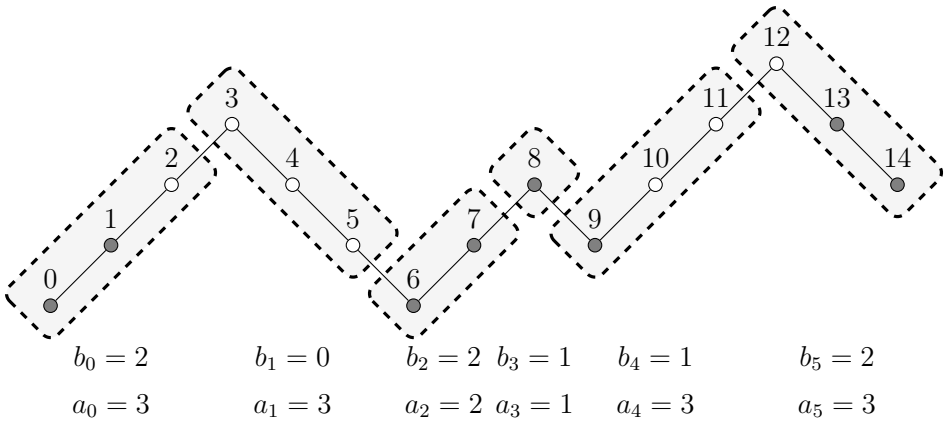


FIGURE 6. The order ideal $I = \{0, 1, 6, 7, 8, 9, 13, 14\}$ in $\mathcal{J}^\bullet(a)$ corresponding to an admissible sequence $\Psi(I) = b = (2, 0, 2, 1, 1, 2)$ for $a = [3; 3, 2, 1, 3, 3]$.

Proof. First we prove that Ψ is **well-defined**. Let $I \in \mathcal{J}(x)$ be an order ideal and $b = \Psi(I)$. For every integer i, with $0 \leq i < 2\ell$, we have

$$0 \leq b_i = \#I \cap C_i(a) \leq \#C_i(a) = a_i.$$

If i is odd and $b_i = a_i$, then $C_i(a) \subset I$. In particular, $\sum_{k=0}^{i-1} a_k \in I$. This forces the previous interval to be a subset of I as well, that is, $C_{i-1}(a) \subset I$. Thus, $b_{i-1} = \#I \cap C_{i-1}(a) = \#C_{i-1}(a) = a_{i-1}$. If i is even and $b_i = 0$, $C_i(a) \cap I = \varnothing$. In particular, $\sum_{k=0}^{i-1} a_k \notin I$. This forces all of the elements of the previous interval to not be in I as well, that is, $C_{i-1}(a) \cap I = \varnothing$. Thus, $b_{i-1} = \#I \cap C_{i-1}(a) = 0$. We conclude that $b = \Psi(I)$ is admissible for a and that the map Ψ is well-defined.

Ψ **is injective.** Let $I, I' \in \mathcal{J}^\bullet(x)$ with $I \neq I'$. From the definition of order ideal, for every integer i, the four sets $I \cap C_i(a)$, $I' \cap C_i(a)$, $C_i(a) \setminus I$ and $C_i(a) \setminus I'$ are intervals of consecutive integers. Since I and I' are distinct order ideals, there exists an integer i, with $0 \leq i < 2\ell$, such that $I \cap C_i(a) \neq I' \cap C_i(a)$. This is only possible if the cardinalities of the four sets do not match. In particular, $\#I \cap C_i(a) \neq \#I' \cap C_i(a)$. We conclude that $\Psi(I) \neq \Psi(I')$ and Ψ is injective.

Ψ **is surjective.** Let $(b_i)_{0 \leq i < 2\ell}$ be an admissible sequence for a. Let

$$I = \left(\bigcup_{0 \leq i < \ell} C_{2i+1}(a) \cap (C_{2i+1}(a) + (a_{2i+1} - b_{2i+1})) \right) \cup \left(\bigcup_{0 \leq i < \ell} C_{2i}(a) \cap (C_{2i}(a) - (a_{2i} - b_{2i})) \right)$$

be the subset made of the union of the b_i left-most (right-most, resp.) elements of $C_i(a)$ when i is even (odd, resp.). Since b is admissible, we have that I is a well-defined order ideal. By construction, we have $\Psi(I) = b$. Thus Ψ is surjective.

Ψ **is order-preserving.** Let $I, I' \in \mathcal{J}(x)$ be two order ideals. Let $(b_i)_{0 \leq i < 2\ell} = \Psi(I)$ and $(b'_i)_{0 \leq i < 2\ell} = \Psi(I')$. We have $I \subseteq I'$ if and only if $I \cap C_i(a) \subseteq I' \cap C_i(a)$ for every index i such that $0 \leq i < 2\ell$ if and only if $\#I \cap C_i(a) \leq \#I' \cap C_i(a)$ for

every index i such that $0 \leq i < 2\ell$ if and only if $b_i \leq b'_i$ for every index i such that $0 \leq i < 2\ell$ if and only if $(b_i)_{0\leq i<2\ell} \leq (b'_i)_{0\leq i<2\ell}$ if and only if $\Psi(I) \subset \Psi(I')$.

Items 1 and 2. Let $I \in \mathcal{J}^\bullet(x)$, that is $0 \in I$. We need to distinguish two cases whether $a_0 = 0$ or not. If $a_0 > 0$, $0 \in I$ implies that $b_0 > 0$, so that $\Psi(I) \in \mathcal{B}^\bullet(a)$. If $a_0 = 0$, $0 \in I$ implies that $b_0 = 0$, $C_1(a) \subset I$ and $0 < b_1 = a_1$, so that $\Psi(I) \in \mathcal{B}^\bullet(a)$.

Let $I \in \mathcal{J}^\circ(x)$, that is $0 \notin I$. We need to distinguish two cases whether $a_0 = 0$ or not. If $a_0 > 0$, $0 \notin I$ implies that $b_0 = 0$, so that $\Psi(I) \in \mathcal{B}^\circ(a)$. If $a_0 = 0$, $0 \notin I$ implies that $b_0 = 0$, $C_1(a) \not\subset I$ and $0 \leq b_1 < a_1$, so that $\Psi(I) \in \mathcal{B}^\circ(a)$.

This is sufficient to prove both Items 1 and 2 since $\mathcal{B}(a) = \mathcal{B}^\bullet(a) \cup \mathcal{B}^\circ(a)$ is a disjoint union.

Item 3. Let $I \in \mathcal{J}(x)$ and $(b_i)_{0\leq i<2\ell} = \Psi(I)$. We have

$$\|\Psi(I)\|_1 = \sum_{i=0}^{2\ell-1} b_i = \sum_{i=0}^{2\ell-1} \#I \cap C_i(a) = \#I \cap \left(\cup_{i=0}^{2\ell-1} C_i(a)\right) = \#I = |I|,$$

since the intervals $(C_i(a))_{0\leq i<2\ell}$ are disjoint and $I \subset \cup_{i=0}^{2\ell-1} C_i(a)$. □

Corollary 5.3. *Let $x \in \mathbb{Q}_{>0}$ whose even-length continued fraction expansion is $a = [a_0; a_1, \ldots, a_{2\ell-1}]$. The posets $(\mathcal{J}(x), \subseteq)$ and $(\mathcal{B}(a), \leq)$ are isomorphic.*

Proof. It follows from the order-preserving bijection proved in Theorem 5.2. □

6. Proof of Theorem B and Theorem C

In this section, we introduce an equivalent condition for a pair of polynomial functions to be counted by q-analog of continued fraction expansions. We use this condition to count the number of order ideal of a fence poset keeping track of the cardinality statistics, thus proving Theorem C. Using the bijection defined in Theorem 5.2, we deduce Theorem B.

First, we define the homomorphism

$$\begin{array}{rccl} \nu_q : & \{\mathtt{0},\mathtt{1}\}^* & \to & \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}]) \\ & \mathtt{0} & \mapsto & L_q, \\ & \mathtt{1} & \mapsto & R_q. \end{array}$$

Reusing the involution $w \mapsto \widehat{w}$ defined for binary words in Section 2.3, we have the following two results.

Lemma 6.1. *For every word $w \in \{\mathtt{0},\mathtt{1}\}^*$, the homomorphism ν_q satisfies*

$$\nu_q(w) \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) \nu_q(\widehat{w})^T \tag{6}$$

where $\widehat{w} = \widetilde{\overline{w}}$.

Proof. The matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ conjugates the matrix R_q into the transpose of L_q and conjugates the matrix L_q into the transpose of R_q:

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) = L_q^T,$$
$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} L_q \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) = R_q^T.$$

The conclusion follows. □

Lemma 6.2. *Let $a = [a_0; a_1, \ldots, a_{2\ell-1}]$ be the even-length continued fraction expansion of a positive rational number. Then*

$$\nu_q(W(a)) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) = R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

Proof. We have $W(a) = \mathtt{1}^{a_0}\mathtt{0}^{a_1}\cdots\mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1}$. Thus,

$$\begin{aligned}\nu_q(W(a))\begin{pmatrix}q\\q\end{pmatrix} &= \nu_q(\mathtt{1}^{a_0}\mathtt{0}^{a_1}\cdots\mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1})L_q\begin{pmatrix}1\\0\end{pmatrix}\\ &= R_q^{a_0}L_q^{a_1}\cdots R_q^{a_{2\ell-2}}L_q^{a_{2\ell-1}-1}L_q\begin{pmatrix}1\\0\end{pmatrix}\\ &= R_q^{a_0}L_q^{a_1}\cdots R_q^{a_{2\ell-2}}L_q^{a_{2\ell-1}}\begin{pmatrix}1\\0\end{pmatrix}.\end{aligned}$$

□

Then, we have the following two more important results.

Lemma 6.3. *Let $X : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{Z}[q]$ and $Y : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{Z}[q]$ be polynomial functions such that*

$$\begin{aligned}&X(\mathtt{1}w) = qX(w) + qY(w), \quad X(\mathtt{0}w) = qX(w),\\ &Y(\mathtt{1}w) = Y(w), \qquad\qquad Y(\mathtt{0}w) = X(w) + Y(w)\end{aligned}$$

for every $w \in \{\mathtt{0},\mathtt{1}\}^*$. Then

$$\begin{pmatrix}X(w)\\Y(w)\end{pmatrix} = \nu_q(\widehat{w})^T\begin{pmatrix}X(\varepsilon)\\Y(\varepsilon)\end{pmatrix}. \tag{7}$$

Proof. The proof is done by induction. It holds for the empty word $w = \varepsilon$. Suppose that (7) holds for every $w \in \{\mathtt{0},\mathtt{1}\}^k$ for some integer $k \geq 0$. We have

$$\begin{aligned}\begin{pmatrix}X(\mathtt{1}w)\\Y(\mathtt{1}w)\end{pmatrix} &= \begin{pmatrix}qX(w)+qY(w)\\Y(w)\end{pmatrix} = \begin{pmatrix}q & q\\0 & 1\end{pmatrix}\begin{pmatrix}X(w)\\Y(w)\end{pmatrix}\\ &= L_q^T\nu_q(\widehat{w})^T\begin{pmatrix}X(\varepsilon)\\Y(\varepsilon)\end{pmatrix} = \nu_q(\widehat{w\mathtt{0}})^T\begin{pmatrix}X(\varepsilon)\\Y(\varepsilon)\end{pmatrix} = \nu_q(\widehat{\mathtt{1}w})^T\begin{pmatrix}X(\varepsilon)\\Y(\varepsilon)\end{pmatrix}.\end{aligned}$$

Also,

$$\begin{aligned}\begin{pmatrix}X(\mathtt{0}w)\\Y(\mathtt{0}w)\end{pmatrix} &= \begin{pmatrix}qX(w)\\X(w)+Y(w)\end{pmatrix} = \begin{pmatrix}q & 0\\1 & 1\end{pmatrix}\begin{pmatrix}X(w)\\Y(w)\end{pmatrix}\\ &= R_q^T\nu_q(\widehat{w})^T\begin{pmatrix}X(\varepsilon)\\Y(\varepsilon)\end{pmatrix} = \nu_q(\widehat{w\mathtt{1}})^T\begin{pmatrix}X(\varepsilon)\\Y(\varepsilon)\end{pmatrix} = \nu_q(\widehat{\mathtt{0}w})^T\begin{pmatrix}X(\varepsilon)\\Y(\varepsilon)\end{pmatrix}.\end{aligned}$$

□

Proposition 6.4. *Let $X : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{Z}[q]$ and $Y : \{\mathtt{0},\mathtt{1}\}^* \to \mathbb{Z}[q]$ be polynomial functions. We have*

$$\begin{pmatrix}X(w)\\Y(w)\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & q\end{pmatrix}^{-1}\nu_q(w)\begin{pmatrix}q\\q\end{pmatrix}$$

for every $w \in \{\mathtt{0},\mathtt{1}\}^*$ if and only if $X(\varepsilon) = q$, $Y(\varepsilon) = 1$ and

$$\begin{aligned}&X(\mathtt{1}w) = qX(w) + qY(w), \quad X(\mathtt{0}w) = qX(w),\\ &Y(\mathtt{1}w) = Y(w), \qquad\qquad Y(\mathtt{0}w) = X(w) + Y(w)\end{aligned} \tag{8}$$

for every $w \in \{\mathtt{0},\mathtt{1}\}^*$.

Proof. Let $w \in \{\mathtt{0},\mathtt{1}\}^*$. First, we assume that $X(\varepsilon) = q$, $Y(\varepsilon) = 1$ and (8) holds. Using Lemma 6.3, we have

$$\begin{aligned}\begin{pmatrix} X(w) \\ Y(w) \end{pmatrix} &\overset{(7)}{=} \nu_q(\widehat{w})^T \begin{pmatrix} X(\varepsilon) \\ Y(\varepsilon) \end{pmatrix} \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) \nu_q(\widehat{w})^T \left(\begin{smallmatrix} q \\ 1 \end{smallmatrix}\right) \\ &\overset{(6)}{=} \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(w) \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right) \left(\begin{smallmatrix} q \\ 1 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right).\end{aligned}$$

Conversely, $\left(\begin{smallmatrix} X(\varepsilon) \\ Y(\varepsilon) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(\varepsilon) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) = \left(\begin{smallmatrix} q \\ 1 \end{smallmatrix}\right)$. Also,

$$\begin{aligned}\begin{pmatrix} X(\mathtt{1}w) \\ Y(\mathtt{1}w) \end{pmatrix} &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(\mathtt{1}w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q\nu_q(w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) \\ &= L_q^T \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) = L_q^T \begin{pmatrix} X(w) \\ Y(w) \end{pmatrix} = \begin{pmatrix} qX(w) + qY(w) \\ Y(w) \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} X(\mathtt{0}w) \\ Y(\mathtt{0}w) \end{pmatrix} &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(\mathtt{0}w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} L_q\nu_q(w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) \\ &= R_q^T \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) = R_q^T \begin{pmatrix} X(w) \\ Y(w) \end{pmatrix} = \begin{pmatrix} qX(w) \\ X(w) + Y(w) \end{pmatrix}.\end{aligned}$$

□

6.1. **A first application: counting order ideals with cardinality statistics.** We can now provide a formula computing the cardinality statistics over the set of order ideals of the fence poset $F(w)$ for every word $w \in \{\mathtt{0},\mathtt{1}\}^*$.

Proposition 6.5. *For every $w \in \{\mathtt{0},\mathtt{1}\}^*$, let*

$$\begin{aligned} X(w) &= \sum_{I \in J^\bullet(F(w))} q^{|I|}, \\ Y(w) &= \sum_{I \in J^\circ(F(w))} q^{|I|}. \end{aligned}$$

Then, $X(\varepsilon) = q$, $Y(\varepsilon) = 1$ and $X(w)$ and $Y(w)$ both satisfy (8).

Proof. Let

$$\begin{array}{rccc} \text{SHIFT}: & 2^{\mathbb{N}} & \to & 2^{\mathbb{N}} \\ & I & \mapsto & \{i-1 \mid i \in I \text{ and } i \neq 0\} \end{array}$$

be a map defined on subsets of $\mathbb{N}$ which subtract 1 to every element, and ignoring the element 0. For every $w \in \{\mathtt{0},\mathtt{1}\}^*$, we have that

- SHIFT : $J^\bullet(F(\mathtt{1}w)) \to J(F(w))$ is a bijection,
- SHIFT : $J^\circ(F(\mathtt{1}w)) \to J^\circ(F(w))$ is a bijection,
- SHIFT : $J^\bullet(F(\mathtt{0}w)) \to J^\bullet(F(w))$ is a bijection,
- SHIFT : $J^\circ(F(\mathtt{0}w)) \to J(F(w))$ is a bijection.

Therefore, for every $w \in \{0,1\}^*$, we have

$$\begin{aligned}
X(1w) &= \sum_{I \in J^\bullet(F(1w))} q^{|I|} = \sum_{I \in J(F(w))} q^{|I|+1} = q(X(w)+Y(w)),\\
Y(1w) &= \sum_{I \in J^\circ(F(1w))} q^{|I|} = \sum_{I \in J^\circ(F(w))} q^{|I|} = Y(w),\\
X(0w) &= \sum_{I \in J^\bullet(F(0w))} q^{|I|} = \sum_{I \in J^\bullet(F(w))} q^{|I|+1} = qX(w),\\
Y(0w) &= \sum_{I \in J^\circ(F(0w))} q^{|I|} = \sum_{I \in J(F(w))} q^{|I|} = X(w)+Y(w).
\end{aligned}$$

Thus, $X(w)$ and $Y(w)$ both satisfy (8). Also

$$X(\varepsilon) = \sum_{I \in J^\bullet(F(\varepsilon))} q^{|I|} = q^1 = q.$$

and

$$Y(\varepsilon) = \sum_{I \in J^\circ(F(\varepsilon))} q^{|I|} = q^0 = 1.$$

□

Proposition 6.6. *Let $w \in \{0,1\}^*$. The cardinality statistics over the set of order ideals of the fence poset $F(w)$ satisfies*

$$\begin{pmatrix} \sum_{I \in J^\bullet(F(w))} q^{|I|} \\ \sum_{I \in J^\circ(F(w))} q^{|I|} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(w) \begin{pmatrix} q \\ q \end{pmatrix}.$$

Proof. For every $w \in \{0,1\}^*$, let $X(w) = \sum_{I \in J^\bullet(F(w))} q^{|I|}$ and $Y(w) = \sum_{I \in J^\circ(F(w))} q^{|I|}$. From Proposition 6.5, we have that $X(\varepsilon) = q$, and $Y(\varepsilon) = 1$ and $X(w)$ and $Y(w)$ both satisfy (8). Using Proposition 6.4, we obtain

$$\begin{pmatrix} X(w) \\ Y(w) \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(w) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right).$$

□

Theorem C. *For every positive rational number $x \in \mathbb{Q}_{>0}$, whose even-length continued fraction expansion is $[a_0; a_1, \ldots, a_{2\ell-1}]$, the cardinality statistics over the set $\mathcal{J}(x)$ of order ideals of the fence poset $\mathcal{F}(x)$ satisfies*

$$\begin{pmatrix} \sum_{I \in \mathcal{J}^\bullet(x)} q^{|I|} \\ \sum_{I \in \mathcal{J}^\circ(x)} q^{|I|} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

Proof. Let $w = W \circ \mathrm{CF}_{\mathrm{even}}(x) = W(a) = 1^{a_0}0^{a_1} \cdots 1^{a_{2\ell-2}}0^{a_{2\ell-1}-1}$. Using Proposition 6.6 and Lemma 6.2, we obtain

$$\begin{aligned}
\begin{pmatrix} \sum_{I \in \mathcal{J}^\bullet(x)} q^{|I|} \\ \sum_{I \in \mathcal{J}^\circ(x)} q^{|I|} \end{pmatrix} &= \begin{pmatrix} \sum_{I \in J^\bullet(F(W(a)))} q^{|I|} \\ \sum_{I \in J^\circ(F(W(a)))} q^{|I|} \end{pmatrix}\\
&= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(W(a)) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right)\\
&= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).
\end{aligned}$$

□

6.2. **A second application: counting admissible sequences with 1-norm statistics.** This result, together with Theorem C implies the following statement.

Theorem B. *Let $a = [a_0; \ldots, a_{2\ell-1}]$ be the even-length continued fraction expansion of a positive rational number. The set of admissible sequences for a satisfies*

$$\begin{pmatrix} \sum_{b\in\mathcal{B}^\bullet(a)} q^{\|b\|_1} \\ \sum_{b\in\mathcal{B}^\circ(a)} q^{\|b\|_1} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

where $\|b\|_1 = b_0 + b_1 + \cdots + b_{2\ell-1}$ for every admissible sequence $(b_i)_{0\le i<2\ell} \in \mathcal{B}(a)$.

Proof. Let $\Psi : \mathcal{J}(x) \to \mathcal{B}(a)$ be the bijection from the set of order ideals of the fence poset $\mathcal{F}(x)$ to the set of admissible sequences $\mathcal{B}(a)$ defined in Theorem 5.2. We have

$$\begin{aligned} \begin{pmatrix} \sum_{b\in\mathcal{B}^\bullet(a)} q^{\|b\|_1} \\ \sum_{b\in\mathcal{B}^\circ(a)} q^{\|b\|_1} \end{pmatrix} &= \begin{pmatrix} \sum_{I\in\mathcal{J}^\bullet(x)} q^{\|\Psi(I)\|_1} \\ \sum_{I\in\mathcal{J}^\circ(x)} q^{\|\Psi(I)\|_1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{I\in\mathcal{J}^\bullet(x)} q^{|I|} \\ \sum_{I\in\mathcal{J}^\circ(x)} q^{|I|} \end{pmatrix}. \end{aligned}$$

The conclusion follows from Theorem C. □

7. Snake graphs and their perfect matchings

Snake graphs have been introduced to provide combinatorial formulas for elements in cluster algebras of surface type [MS10, MSW11, CS13, MSW13], see also [Pro20]. The formulas are obtained from the set of perfect matchings of a snake graph. The enumeration of perfect matchings in a graph, or equivalently, of domino tilings [Kuo04] is part of a longer history including the study of alternating-sign matrices and the Aztec diamonds [EKLP92].

Perfect matchings in certain snake graphs also give an interpretation of Markoff numbers [Aig13] and of the numerator and denominator of rational numbers from their continued fraction expansion [ÇS18, ÇS20]. However, these results are limited to rational numbers > 1 and need different snake graphs for the numerator and denominators. In what follows, we introduce snake graphs in a way that allows us to associate them bijectively to every positive rational number > 0. The representation of snake graphs by strings was proposed in [ÇS21] using a ternary alphabet $\{\rightarrow, \leftarrow, \bullet\}$. Here, we simply use the binary alphabet $\{\mathtt{0}, \mathtt{1}\}$.

A binary word $w \in \{\mathtt{0}, \mathtt{1}\}^*$ describes a path in the North East quadrant starting from the origin with letter $\mathtt{0}$ associated with a unitary horizontal step in the East direction and letter $\mathtt{1}$ associated with a unitary vertical step in the North direction as in Figure 7. The end point of the path is $\overrightarrow{w} = (|w|_{\mathtt{0}}, |w|_{\mathtt{1}})$ where $|w|_a$ denotes the number of occurrences of the letter a in the word w.

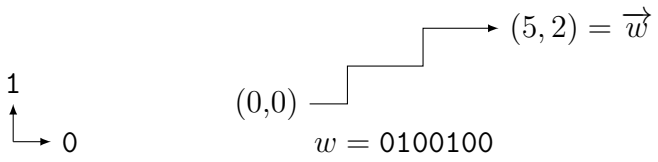


Figure 7. A path in the plane using vertical (North) and horizontal (East) unitary steps is encoded as a word over the alphabet $\{\mathtt{0}, \mathtt{1}\}$.

From a word $w \in \{0,1\}^*$ describing a path, we define the snake graph $G(w)$ as a thickening of this path. More formally, the snake graph $G(w) = (V_w, E_w)$ is defined by the following set of vertices and (undirected) edges:

$$V_w = \{\overrightarrow{p} \mid p \text{ prefix of } w\} + \{0, e_1, e_2, e_1+e_2\} \subset \mathbb{Z}^2$$
$$E_w = \bigcup_{p \text{ prefix of } w} S_{\overrightarrow{p}}$$

where $S_g = \{(g, g+e_1), (g, g+e_2), (g+e_1, g+e_1+e_2), (g+e_2, g+e_1+e_2)\}$ is the set of four edges of the unit square at point $g \in \mathbb{Z}^2$, see Figure 8. Notice that a word $w \in \{0,1\}^*$ of length $|w| = n$ has $n+1$ prefixes including itself and the empty word. Therefore, the snake graph $G(w)$ is the concatenation of $|w|+1$ unit squares.

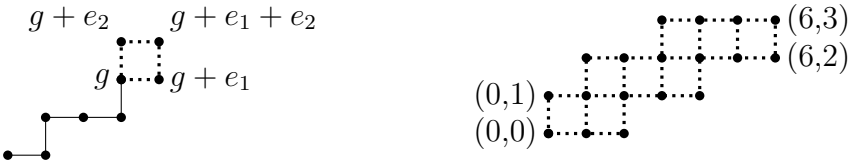


FIGURE 8. Right: the snake graph $G(0100100)$. Left: the vertices of the unit square at $g = \overrightarrow{p}$ for the prefix $p = 01001$.

7.1. **From rational numbers to snake graphs.** We denote the non-identity involution on the alphabet $\{0,1\}$ as $\overline{0} = 1$ and $\overline{1} = 0$ which extends to an involution on $\{0,1\}^*$. This involution induces a symmetry on the set of snake graphs.

Lemma 7.1. *For every $w \in \{0,1\}^*$, the snake graph $G(\overline{w})$ is the mirror image of the snake graph $G(w)$ under the map $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 : (a,b) \mapsto (b,a)$.*

Proof. Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 : (a,b) \mapsto (b,a)$. The vertices of the snake graph $\sigma(G(w))$ satisfy

$$\begin{aligned}\text{vertices of } \sigma(G(w)) &= \sigma\left(\{\overrightarrow{p} \mid p \text{ prefix of } w\} + \{0, e_1, e_2, e_1+e_2\}\right)\\ &= \{\sigma(\overrightarrow{p}) \mid p \text{ prefix of } w\} + \{0, e_1, e_2, e_1+e_2\}\\ &= \{\overrightarrow{p} \mid p \text{ prefix of } \overline{w}\} + \{0, e_1, e_2, e_1+e_2\}\\ &= \text{vertices of } G(\overline{w}).\end{aligned}$$

□

We define a length-preserving map $\theta : \{0,1\}^* \to \{0,1\}^*$ recursively by $\theta(\varepsilon) = \varepsilon$ and

$$\theta(\alpha w) = \begin{cases} \overline{\alpha}\theta(w) & \text{if } |w| \text{ is even,}\\ \alpha\theta(w) & \text{if } |w| \text{ is odd,}\end{cases}$$

for every word $w \in \{0,1\}^*$ and letter $\alpha \in \{0,1\}$.

Lemma 7.2. *$\theta : \{0,1\}^* \to \{0,1\}^*$ is an involution.*

Proof. The map θ flips the letters at even distance from the right end of the word. □

Lemma 7.3. *For every $w \in \{0,1\}^*$, $\overline{\theta(w)} = \theta(\overline{w})$.*

Proof. We proceed by induction on the length of w. The equality holds if $w = \varepsilon$ is the empty word. Let $\alpha \in \{0,1\}$ and $w \in \{0,1\}^*$. If $|w|$ is even, we have

$$\overline{\theta(\alpha w)} = \overline{\overline{\alpha}\theta(w)} = \alpha\overline{\theta(w)} = \alpha\theta(\overline{w}) = \theta(\overline{\alpha w}).$$

If $|w|$ is odd, we have

$$\overline{\theta(\alpha w)} = \overline{\alpha\theta(w)} = \overline{\alpha}\,\overline{\theta(w)} = \overline{\alpha}\theta(\overline{w}) = \theta(\overline{\alpha w}). \qquad \square$$

In other words, it follows from Lemma 2.3, Lemma 7.3 and Lemma 7.1 that the diagram shown in Figure 9 is commutative. Using the maps W and θ, we associate a snake graph $\mathcal{G}(x)$ to every rational number $x \in \mathbb{Q}_{>0}$ as follows:

$$\mathcal{G} = G \circ \theta \circ W. \tag{9}$$

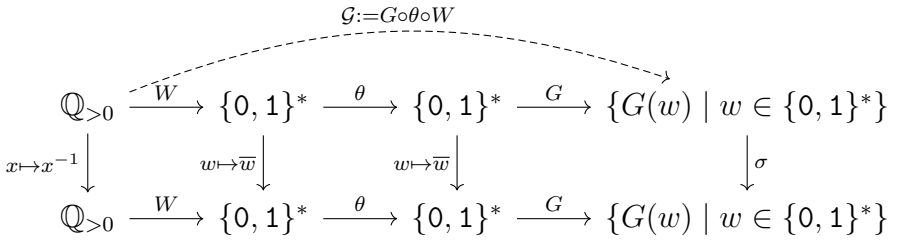


FIGURE 9. A commutative diagram involving $\mathcal{G} = G \circ \theta \circ W$.

Lemma 7.4. *The map $\mathcal{G}$ is a bijection from the set $\mathbb{Q}_{>0}$ of positive rational numbers to the set $\{G(w) \mid w \in \{0,1\}^*\}$ of snake graphs.*

Proof. It follows from the fact that intermediate maps W, θ and G are bijections. $\square$

Commutativity of the diagram shown in Figure 9 implies the following result.

Lemma 7.5. *For every $x \in \mathbb{Q}_{>0}$, the snake graph $\mathcal{G}(x^{-1})$ is the mirror image of the snake graph $\mathcal{G}(x)$ under the map $\sigma : \mathbb{R}^2 \to \mathbb{R}^2 : (a, b) \mapsto (b, a)$.*

Proof. Lemma 2.3 and Lemma 7.3 imply that $\theta \circ W(x^{-1}) = \overline{\theta \circ W(x)}$. Using Lemma 7.1, we have

$$\mathcal{G}(x^{-1}) = G(\theta \circ W(x^{-1})) = G(\overline{\theta \circ W(x)}) = \sigma(G(\theta \circ W(x))) = \sigma(\mathcal{G}(x)). \qquad \square$$

An example illustrating

$$\mathcal{G}(10/27) = \mathcal{G}([0; 2, 1, 2, 2, 1]) = G(\theta(\mathtt{0010011})) = G(\mathtt{1000110}) \text{ and}$$
$$\mathcal{G}(27/10) = \mathcal{G}([2; 1, 2, 3]) = G(\theta(\mathtt{1101100})) = G(\mathtt{0111001})$$

is shown in Figure 10. Both snake graphs contain 8 cells which is the sum of the partial quotients $8 = 2 + 1 + 2 + 3 = 0 + 2 + 1 + 2 + 2 + 1$.

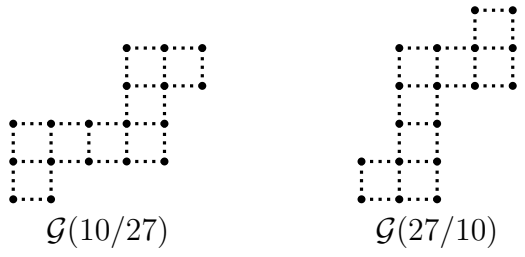


FIGURE 10. The snake graphs $\mathcal{G}(10/27)$ and $\mathcal{G}(27/10)$).

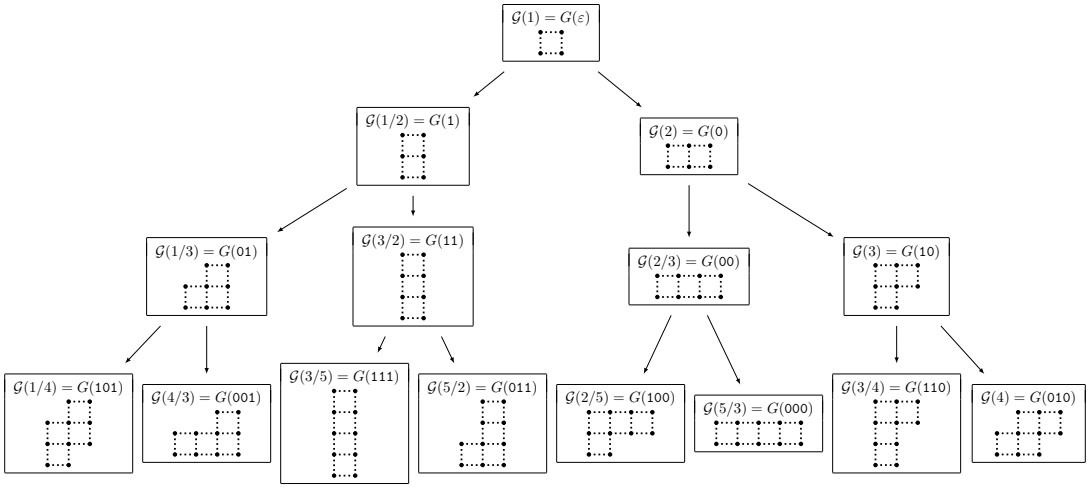


FIGURE 11. The map $\mathcal{G}$ maps the Calkin-Wilf tree of positive rational numbers onto the binary tree of snake graphs defined by the suffix relation.

The map $\mathcal{G}$ maps the Raney tree of positive rational numbers [BdL97], also known as the Calkin-Wilf tree [CW00], to the binary tree of snake graphs where the parent relation is defined by the suffix relation; see Figure 11. In general, the snake graph

$$\mathcal{G}[a_0; \dots, a_{2\ell-1}] = G(\theta(\mathtt{1}^{a_0}\mathtt{0}^{a_1} \cdots \mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1}))$$

is made of $a_0 + \cdots + a_{2\ell-1}$ unit squares. By comparison, the snake graphs defined in [LS19] contain one less unit square and are defined only for rational numbers that are larger than 1. Figure 10 can be compared with Figure 2 in [LS19] where the snake graph associated with the rational number $27/10$ contains seven unit squares instead of eight. See Corollary 9.2 and Remark 9.3 for the precise relation between these definitions of snake graphs.

7.2. **The dual map defined by Claussen.** In [Cla20] (also implicitly defined in [Pro20]), another length-preserving involution $\eta : \{\mathtt{0},\mathtt{1}\}^* \to \{\mathtt{0},\mathtt{1}\}^*$ called *dual map* that resembles θ was considered in the context of snake graphs. It is defined recursively by $\eta(\varepsilon) = \varepsilon$ and

$$\eta(w\alpha) = \begin{cases} \eta(w)\overline{\alpha} & \text{if } |w| \text{ is even,} \\ \eta(w)\alpha & \text{if } |w| \text{ is odd,} \end{cases} \tag{10}$$

for every word $w \in \{\mathtt{0},\mathtt{1}\}^*$ and letter $\alpha \in \{\mathtt{0},\mathtt{1}\}$.

It turns out that the map θ is conjugate to the dual map η through the involution $w \mapsto \widehat{w}$ on $\{\mathtt{0},\mathtt{1}\}^*$ defined in Section 6. Recall that $\widehat{w} = \overline{w_k} \cdots \overline{w_1}$ if $w = w_1 \cdots w_k$, where $\overline{\mathtt{0}} = \mathtt{1}$ and $\overline{\mathtt{1}} = \mathtt{0}$.

Lemma 7.6. *For every word $w \in \{\mathtt{0},\mathtt{1}\}^*$, $\widehat{\theta(w)} = \eta(\widehat{w})$. In other words, the following diagram is commutative:*

$$\begin{array}{ccc} \{0,1\}^* & \xrightarrow{\theta} & \{0,1\}^* \\ {\scriptstyle w\mapsto\widehat{w}}\downarrow & & {\scriptstyle w\mapsto\widehat{w}}\downarrow \\ \{0,1\}^* & \xrightarrow{\eta} & \{0,1\}^* \end{array}$$

Proof. Let $w \in \{0,1\}^*$ and $\alpha \in \{0,1\}$. If $|w|$ is even, then

$$\widehat{\theta(\alpha w)} = \widehat{\overline{\alpha}\theta(w)} = \widehat{\theta(w)}\alpha = \eta(\widehat{w})\alpha = \eta(\widehat{w}\overline{\alpha}) = \eta(\widehat{\alpha w}).$$

If $|w|$ is odd, then

$$\widehat{\theta(\alpha w)} = \widehat{\alpha\theta(w)} = \widehat{\theta(w)}\overline{\alpha} = \eta(\widehat{w})\overline{\alpha} = \eta(\widehat{w}\overline{\alpha}) = \eta(\widehat{\alpha w}).$$

□

Recall, from Lemma 2.4, that the map $w \mapsto \widehat{w}$ is conjugate to the involution $\tau : \mathcal{A}_{\text{even}} \to \mathcal{A}_{\text{even}}$. However, in this work, it is preferable to use θ instead of η in order to obtain Theorem D and have the start of the snake graph correspond to the first partial quotients of the continued fraction expansion.

7.3. **Perfect matchings.** In graph theory, a *matching* in an undirected graph is a subset of edges without common vertices and a *perfect matching* is a matching that matches all vertices of the graph. We denote by $M(g)$ the set of perfect matchings of a graph g. By abuse of notation, for every $w \in \{0,1\}^*$, we also denote by $M(w)$ the set of perfect matchings of the snake graph $G(w)$.

A snake graph admits perfect matchings. An example is shown in Figure 12.

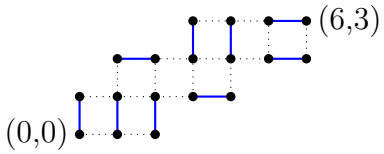


FIGURE 12. A perfect matching of the snake graph $G(w)$ with $w = 0100100$.

For every positive rational number $x \in \mathbb{Q}_{>0}$, we define $\mathcal{M}(x)$ to be the set of perfect matchings of the snake graph $\mathcal{G}(x)$. In other words, $\mathcal{M} := M \circ G \circ \theta \circ W$; see Figure 1.

A combinatorial interpretation of every rational number $\frac{r}{s} > 1$ was proposed by Çanakçi and Schiffler in terms of the quotient of cardinalities of sets of perfect matchings from two distinct snake graphs [ÇS18]. Their result can be stated using $\mathcal{G}$ and $\mathcal{M}$ as follows.

Theorem 7.7 (Theorem 3.4, [ÇS18])**.** *Let $[a_0; \ldots, a_{n-1}]$ be the continued fraction expansion of a rational number greater than 1, that is, such that $a_0 \geq 1$. Then,*

$$[a_0; \ldots, a_{n-1}] = \frac{\#\mathcal{M}[a_0 - 1, \ldots, a_{n-1}]}{\#\mathcal{M}[a_1 - 1, \ldots, a_{n-1}]}$$

and the fraction on the right-hand side is reduced.

Below, we show that the above theorem can be extended to consider the case $a_0 = 0$ while also giving a combinatorial interpretation of both the numerator and denominator of a rational number from the same snake graph using a natural dichotomy on its set of perfect matchings.

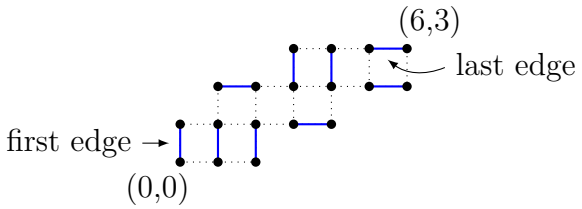


FIGURE 13. A perfect matching of the snake graph $G(w)$ with $w = \mathtt{0100100}$. The first and last edge of the matching are indicated.

The first and last edge of a matching. A notion which is important is the first and last edge of a perfect matching. The *first edge* of a perfect matching is the edge containing the vertex $(0, 0)$, that is, the bottom-most left-most vertex of the snake graph $G(w)$. The *last edge* of a perfect matching is the edge containing the vertex $\overrightarrow{w} + (1, 1)$, which is the top-most right-most vertex of the snake graph $G(w)$, see Figure 13.

A dichotomy on the set of perfect matchings. Let $w \in \{\mathtt{0}, \mathtt{1}\}^*$. Let $M(w)$ be the set of perfect matchings of the snake graph $G(w)$. The dichotomy on the set of perfect matchings is defined as follows.

Definition 7.8. *We say that a perfect matching $m \in M(w)$ is a* $\perp$-matching *if*

- $|w|$ *is even and its first edge is horizontal, or*
- $|w|$ *is odd and its first edge is vertical.*

Otherwise, we say that m is a $\|$-matching*, that is, if*

- $|w|$ *is even and its first edge is vertical, or*
- $|w|$ *is odd and its first edge is horizontal.*

The terminology $\|$- and $\perp$-matching was chosen because the first edge of a $\|$-matching is parallel to the first edge of the basic perfect matching and the first edge of a $\perp$-matching is orthogonal to the first edge of the basic perfect matching; see Remark 7.13. Let

$$M^{\perp}(w) = \{m \in M(w) \mid m \text{ is a } \perp\text{-matching}\},$$
$$M^{\|}(w) = \{m \in M(w) \mid m \text{ is a } \|\text{-matching}\}.$$

These two sets form a partition of $M(w) = M^{\|}(w) \cup M^{\perp}(w)$.

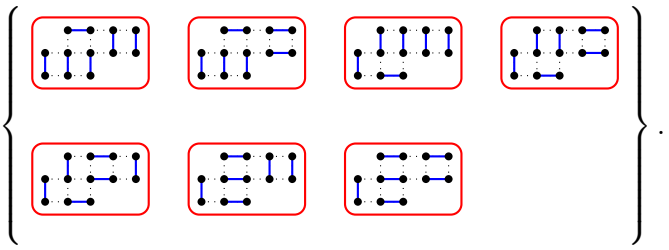
Example 7.9. *The snake graph associated with $x = \frac{2}{7}$ is $\mathcal{G}(\frac{2}{7}) = G(\mathtt{0100})$ since*

$$\theta \circ W(\tfrac{2}{7}) = \theta(W([0; 3, 1, 1])) = \theta(\mathtt{1^0 0^3 1^1 0^{1-1}}) = \theta(\mathtt{0001}) = \mathtt{0100}.$$

The perfect matchings of the snake graph $G(\mathtt{0100})$ whose first edge is horizontal are

$$M^{\perp}(\mathtt{0100}) = \left\{ \quad \right\}.$$

The perfect matchings of the snake graph $G(\mathtt{0100})$ whose first edge is vertical are

$M^{\parallel}(\mathtt{0100}) =$  .

We observe that the cardinalities $\#M^{\perp}(\mathtt{0100})$ and $\#M^{\parallel}(\mathtt{0100})$ are the numerator and denominator of the rational number x:

$$\frac{\#\mathcal{M}^{\perp}(\frac{2}{7})}{\#\mathcal{M}^{\parallel}(\frac{2}{7})} = \frac{\#M^{\perp} \circ \mathcal{G}(\frac{2}{7})}{\#M^{\parallel} \circ \mathcal{G}(\frac{2}{7})} = \frac{\#M^{\perp}(\mathtt{0100})}{\#M^{\parallel}(\mathtt{0100})} = \frac{2}{7} = x.$$

We show that this holds in general. The following result extends Theorem 7.7 to every positive rational number. It gives an interpretation of both the numerator and denominator of a rational number in terms of the perfect matchings of the same snake graph.

Theorem 7.10. *Let $x > 0$ be a positive rational number. Then,*

$$x = \frac{\#\mathcal{M}^{\perp}(x)}{\#\mathcal{M}^{\parallel}(x)}$$

and the fraction on the right-hand side is reduced.

Theorem 7.10 is proved as a consequence of Theorem D which is a q-analog of it thanks to the area statistics defined below. We prove Theorem D in Section 9. An alternative statement deduced from Theorem 7.10 is the following.

Corollary 7.11. *Let $r, s \geq 1$ be coprime integers. The snake graph $\mathcal{G}(\frac{r}{s})$ has r $\perp$-matchings and s $\parallel$-matchings. Thus,*

$$\frac{r}{s} = \frac{\#\mathcal{M}^{\perp}(\frac{r}{s})}{\#\mathcal{M}^{\parallel}(\frac{r}{s})}.$$

Example 7.12. *This example is a reproduction of Example 2.3 from [ÇS20] with our definition of snake graph. The continued fraction $[2,3,1,2,3] = \frac{84}{37}$ has convergents*

$$[2] = \frac{2}{1}, \quad [2,3] = \frac{7}{3}, \quad [2,3,1] = \frac{9}{4}, \quad [2,3,1,2] = \frac{25}{11}, \quad [2,3,1,2,3] = \frac{84}{37}.$$

These rational numbers define a path in the Stern-Brocot tree [GKP94] whose intermediate nodes are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \frac{16}{7}, \frac{25}{11}, \frac{34}{15}, \frac{59}{26}, \frac{84}{37}.$$

All these values can be interpreted in the snake graph $\mathcal{G}(\frac{84}{37}) = G(w)$ with $w = \mathtt{1001000110}$. The numbers $\#M^{\perp}(v)$ and $\#M^{\parallel}(v)$ are numerators and denominators of a convergent $\frac{84}{37}$ when v is a prefix of w. When v is a suffix of w, the numbers $\#M^{\perp}(v)$ and $\#M^{\parallel}(v)$ are consecutive values in the execution of the Euclid algorithm when computing the gcd of 84 and 37; see Figure 14.

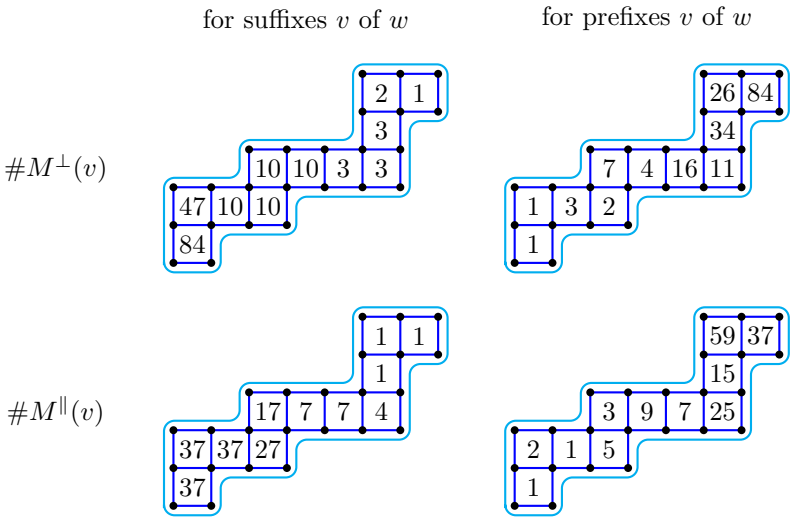


FIGURE 14. The snake graph $\mathcal{G}(\frac{84}{37}) = G(w)$ with $w = \texttt{1001000110}$. The values indicate the number of $\perp$-matchings $\parallel$-matchings in the snake graph $G(v)$ for every prefix and suffixes of w. This reproduces Example 2.3 from [ÇS20] using a single snake graph for both the numerators and denominators.

Perfect matchings with no interior edges. An edge (u, v) of a snake graph $G(w) = (V_w, E_w)$ is *interior* if the graph induced by the subset $V_w \setminus \{u, v\}$ is not connected [CS13]. Remaining edges are called *boundary* edges [MS10, MSW13]. Two of the perfect matchings use only boundary edges of a snake graph; see Figure 15.

FIGURE 15. Two perfect matchings of $G(w)$ use only boundary edges.

These two perfect matchings using only boundary edges are called *minimal* and *maximal* perfect matching in [CS13, MS10, MSW13], see also [KOY25]. The standard convention for the minimal perfect matching is that the first edge is horizontal. In this article, we need another convention. The perfect matching with only boundary edges and such that its last edge is vertical is called the *basic matching* of the snake graph. Depending on the parity of the length of the snake graph, the basic matching may correspond to the minimal or to the maximal perfect matching. We denote the basic matching as $\mathfrak{b}_w$ or as $\mathfrak{b}$ when the context is clear and we illustrate it with thick red dashed edges in illustrations; see Figure 16.

Notice that the first edge of the basic perfect matching is vertical or horizontal depending on the parity of $|w|$, where $|w|$ denotes the length of the word w; see Figure 17. In particular, the basic matching of $G(w)$ always belong to the set $M^{\parallel}(w)$.

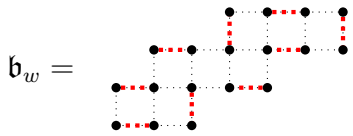
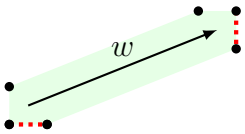
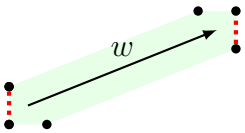


FIGURE 16. The basic matching $\mathfrak{b}$ has only boundary edges and its last edge is vertical.



first edge is horizontal when $|w|$ is odd



first edge is vertical when $|w|$ is even

FIGURE 17. The orientation of the first edge of the basic matching of $G(w)$.

Remark 7.13. *A matching $m \in G(w)$ is a* $\|$-matching *if and only if its first edge belongs to the basic perfect matching and a matching is a* $\perp$-matching *if and only if its first edge does not belong to the basic perfect matching.*

The area statistics. Given $w \in \{\mathtt{0},\mathtt{1}\}^*$ and two perfect matchings $m, m' \in M(w)$, let $m\Delta m' = (m \setminus m') \cup (m' \setminus m)$ denote their symmetric difference. We recall that the symmetric difference of sets A and B is defined as: $A\Delta B = (A \setminus B) \cup (B \setminus A)$. The symmetric difference $m\Delta m'$ is the union of disjoint cycles in the snake graph $G(w)$. Since $G(w)$ is embedded in the plane $\mathbb{R}^2$, the cycles in the symmetric difference defines a bounded region which we denote by $\text{region}(m\Delta m') \subset \mathbb{R}^2$. A statistics which takes an important role is the *area* of this region which we denote by

$$\text{area}(m\Delta m').$$

Equivalently, it is equal to the area of the region enclosed by the union $m \cup m'$.

Note that the region enclosed by $m\Delta m_-$ where m_- is the minimal perfect matching was used in the description of variables in cluster algebras from surfaces [MS10, CS13, MSW13]. In this work, for every matching $m \in M(w)$, we rather consider the region enclosed by $m\Delta\mathfrak{b}_w$ where $\mathfrak{b}_w$ is the basic perfect matching of the snake graph $G(w)$; see Figure 18. It is more convenient to use the basic perfect matching $\mathfrak{b}_w$ in the symmetric difference in order for the start of the snake graph to correspond to the initial partial quotients in the continued fraction expansion while having the first edge of the perfect matching used to obtain a dichotomy in the set of perfect matchings contributing to the numerator or denominator of the associated rational number.

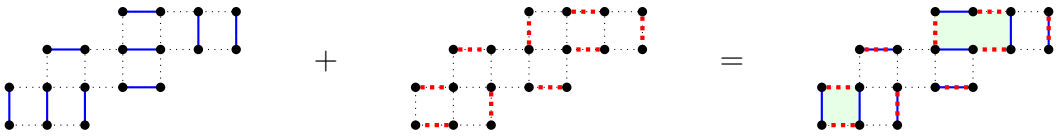


FIGURE 18. A perfect matching on top of the basic matching encloses a region of area $1 + 2 = 3$.

The area of the basic perfect matching of a snake graph is zero. The area of the other perfect matching with no interior edges is maximal, see Figure 19. Note

that the definition of area corresponds to the degree of the height monomials of a perfect matching [MS10, CS13, MSW13, KOY25], except that the height monomials are defined using the minimal perfect matching m_- instead of the basic perfect matching $\mathfrak{b}_w$.

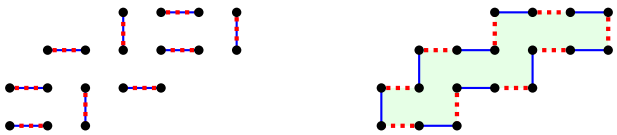


FIGURE 19. A perfect matching of minimal area 0 and maximal area.

Remark 7.14. *Let $m \in M(w)$. Observe that $m \in M^{\perp}(w)$ if and only if the unit square at the origin is inside the enclosed area. Equivalently, $m \in M^{\parallel}(w)$ if and only if the unit square at the origin is not in the enclosed area.*

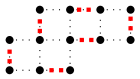
In particular, $\mathrm{area}(m\Delta\mathfrak{b}_w) \geq 1$ for every matching $m \in M^{\perp}(w)$. Note that this implies that q divides the polynomial $\sum_{m\in M^{\perp}(w)} q^{\mathrm{area}(m\Delta\mathfrak{b}_w)}$.

A graded poset. For every $w \in \{0,1\}^*$, the set of perfect matchings $M(w)$ is naturally equipped with a partial order $(M(w), \prec)$ defined as follows:

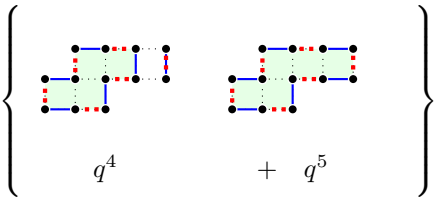
$$m \prec m' \quad \text{if and only if} \quad \mathrm{region}(m\Delta\mathfrak{b}_w) \subseteq \mathrm{region}(m'\Delta\mathfrak{b}_w)$$

for every perfect matchings $m, m' \in M(w)$. The rank function of the graded poset $(M(w), \prec)$ is the area of the perfect matchings. The basic perfect matching $\mathfrak{b}_w$ is its minimal element.

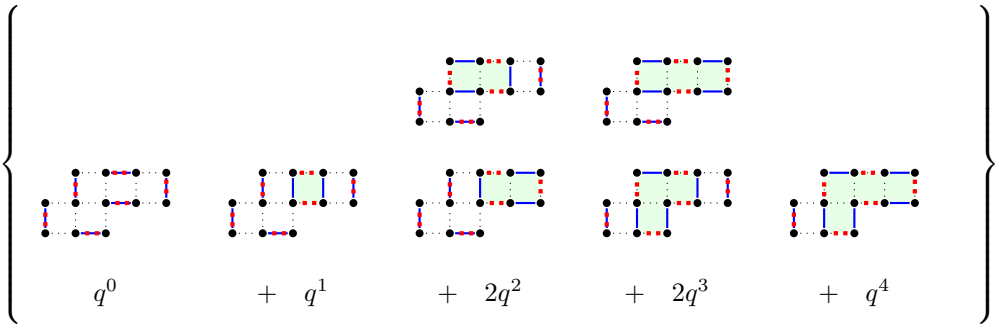
Example 7.15. *For example, let $w = 0100$. The basic perfect matching of $G(w)$ is*

$\mathfrak{b} = \mathfrak{b}_w =$  .

The region enclosed by $m\Delta\mathfrak{b}$ for every perfect matchings $m \in M^{\perp}(0100)$ is:



Thus, $\sum_{m\in M^{\perp}(0100)} q^{\mathrm{area}(m\Delta\mathfrak{b})} = q^4+q^5$. The region enclosed by $m\Delta\mathfrak{b}$ for every perfect matchings $m \in M^{\parallel}(0100)$ is:



Thus, $\sum_{m\in M^{\parallel}(\mathtt{0100})} q^{\mathrm{area}(m\Delta\mathfrak{b})} = 1+q+2q^2+2q^3+q^4$. *We observe that*

$$q^{-1}\frac{\sum_{m\in M^{\perp}(\mathtt{0100})} q^{\mathrm{area}(m\Delta\mathfrak{b})}}{\sum_{m\in M^{\parallel}(\mathtt{0100})} q^{\mathrm{area}(m\Delta\mathfrak{b})}} = q^{-1}\frac{q^4+q^5}{1+q+2q^2+2q^3+q^4} = [0;3,1,1]_q.$$

8. A bijection from perfect matchings to fence poset order ideals

The main result of this section is the existence of a bijection from the set of perfect matchings of a snake graph to the set of order ideals of a fence poset such that the area statistics of a perfect matching correspond to the size (rank) of the order ideal.

The existence of such a bijection is not new; see for instance Theorem 5.4 in [MSW13], Section 4.1 in [KOY25], Section 6 in [Cla20], Section 4 in [BOSZ24], Section 4 in [Pro20]. For completeness, but also because our definitions slightly differ from the literature, we provide a proof of the bijection which works for all snake graphs and for all fence posets and which involves the area statistics defined from our specific choice of basic perfect matching.

For every word $w \in \{\mathtt{0},\mathtt{1}\}^*$, the map is defined as:

$$\begin{array}{rccl} \Phi: & M(w) & \to & J(F(\theta(w))) \\ & m & \mapsto & \left\{|p|\colon w = ps \text{ and } \mathrm{region}(S_{\overrightarrow{p}}) \subseteq \mathrm{region}(m\Delta\mathfrak{b}_w)\right\}. \end{array}$$

where $\mathfrak{b}_w$ is the basic matching of the snake graph $G(w)$, that is, the matching using only boundary edges and whose last edge is vertical. The subset inclusion $\mathrm{region}(S_{\overrightarrow{p}}) \subseteq \mathrm{region}(m\Delta\mathfrak{b}_w)$ means that the unit square $S_{\overrightarrow{p}}$ at $\overrightarrow{p}$ is inside the region delimited by a cycle of $m\Delta\mathfrak{b}_w$. The map Φ is illustrated in Figure 20.

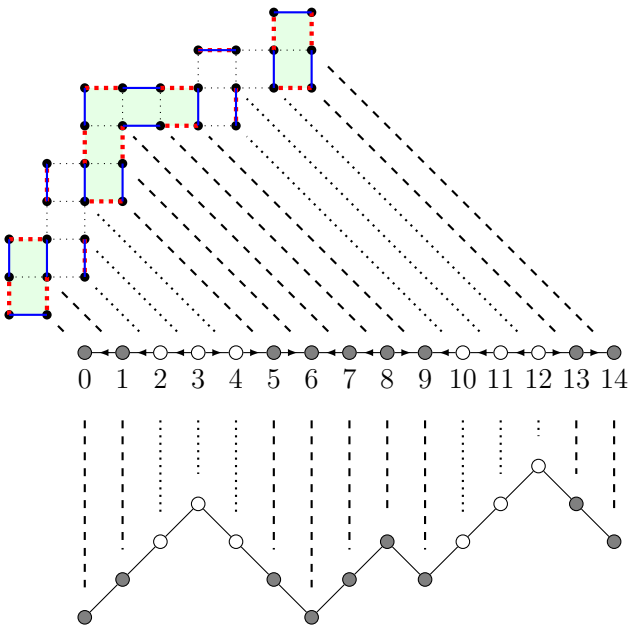


FIGURE 20. The image of a perfect matching of the snake graph $G(\mathtt{10110110001001})$ under the map Φ is the order ideal $\{0,1,5,6,7,8,9,13,14\}$ of the fence poset $F(\theta(\mathtt{10110110001001})) = F(\mathtt{11100011011100})$.

We show that the map Φ is a well-defined bijection from the set of perfect matchings $M(w)$ to the set of fence poset order ideals $J(F(\theta(w)))$.

Theorem 8.1. *Let $w \in \{0,1\}^*$. The map $\Phi : (M(w), \prec) \to (J(F(\theta(w))), \subseteq)$ is an order-preserving bijection from the set $M(w)$ of perfect matchings of the snake graph $G(w)$ to the set $J(F(\theta(w)))$ of order ideals of the fence poset $F(\theta(w))$ such that:*

- *$m \in M^{\perp}(w)$ if and only if $\Phi(m) \in J^{\bullet}(F(\theta(w)))$;*
- *$m \in M^{\parallel}(w)$ if and only if $\Phi(m) \in J^{\circ}(F(\theta(w)))$;*
- *for every $m \in M(w)$, we have $\mathrm{area}(m \Delta \mathfrak{b}_w) = |\Phi(m)|$.*

Proof. Let $u^- = (0, e_1)$ denote the horizontal bottom edge and $u^{|} = (0, e_2)$ denote the vertical left edge of the first unit square of the snake graph. Let S be the four edges $\{u^-, u^{|}, (e_1, e_1+e_2), (e_2, e_1+e_2)\}$ of the first unit square at the origin. Using S, we define a function from $M(\alpha w)$ to $M(w)$ which erases the information in the first unit square of the snake graph $G(w)$:

$$\begin{array}{rccl} \textsc{pop}: & M(\alpha w) & \to & M(w) \\ & m & \mapsto & \begin{cases} m \setminus \{u^{|}\} - e_1 & \text{if } \alpha = \mathtt{0} \text{ and } u^{|} \in m, \\ (m \Delta S) \setminus \{u^{|}\} - e_1 & \text{if } \alpha = \mathtt{0} \text{ and } u^{-} \in m, \\ m \setminus \{u^{-}\} - e_2 & \text{if } \alpha = \mathtt{1} \text{ and } u^{-} \in m, \\ (m \Delta S) \setminus \{u^{-}\} - e_2 & \text{if } \alpha = \mathtt{1} \text{ and } u^{|} \in m. \end{cases} \end{array}$$

Using POP, we can equivalently define the map Φ by induction as follows:

$$\begin{array}{rccl} \Phi: & M(w) & \to & J(F(\theta(w))) \\ & m & \mapsto & \begin{cases} \varnothing & \text{if } w = \varepsilon \text{ and } m \in M^{\parallel}(\varepsilon), \\ \{0\} & \text{if } w = \varepsilon \text{ and } m \in M^{\perp}(\varepsilon), \\ \Phi(\textsc{pop}(m)) + 1 & \text{if } w \neq \varepsilon \text{ and } m \in M^{\parallel}(w), \\ \{0\} \cup (\Phi(\textsc{pop}(m)) + 1) & \text{if } w \neq \varepsilon \text{ and } m \in M^{\perp}(w). \end{cases} \end{array}$$

The proof that Φ is a bijection is done by induction on the length of the word w. Suppose that $\Phi : M(w) \to J(F(\theta(w)))$ is a bijection for every word $w \in \{0,1\}^*$ of length n. Let $\alpha \in \{0,1\}$ be some letter. We want to show that $\Phi : M(\alpha w) \to J(F(\theta(\alpha w)))$ is also a bijection.

First we prove that Φ is **well-defined**. Let $m \in M(\alpha w)$ and $\beta \in \{0,1\}$ be the first letter of $\theta(\alpha w) = \beta\theta(w)$. From the induction hypothesis, we have that $\Phi(\textsc{pop}(m))$ is an order ideal in $J(F(\theta(w)))$. Thus, $\Phi(\textsc{pop}(m)) + 1$ is an order ideal in $J(F(\theta(\alpha w)))$. It remains to show that $\Phi(m)$ is also an order ideal. This depends on β and the subset $\{0,1\} \cap \Phi(m)$. Suppose that $\beta = \mathtt{1}$. We want to show that $1 \in \Phi(m)$ implies that $0 \in \Phi(m)$. Suppose by contradiction that $1 \in \Phi(m)$ and $0 \notin \Phi(m)$. There are two cases to consider:

- Case $|\alpha w|$ is even. From the definition of θ, we must have $\alpha = \mathtt{1}$. Since $1 \in \Phi(m)$ and $0 \notin \Phi(m)$, we have the interior edge $(e_2, e_1+e_2) \in m$. Since $|\alpha w|$ is even, we have $u^{|} \in \mathfrak{b}_{\alpha w}$. Since $0 \notin \Phi(m)$, we have $m \in M^{\parallel}(\alpha w)$. Thus, $u^{|} \in m$ as well. This is a contradiction, because vertex e_2 may not be paired with two distinct vertices in the perfect matching m.
- Case $|\alpha w|$ is odd. From the definition of θ, we must have $\alpha = \mathtt{0}$. Since $1 \in \Phi(m)$ and $0 \notin \Phi(m)$, we have the interior edge $(e_1, e_1+e_2) \in m$. Since $|\alpha w|$ is odd, we have $u^{-} \in \mathfrak{b}_{\alpha w}$. Since $0 \notin \Phi(m)$, we have $m \in M^{\parallel}(\alpha w)$. Thus, $u^{-} \in m$ as well. This is a contradiction, because vertex e_1 may not be paired with two distinct vertices in the perfect matching m.

Suppose that $\beta = \mathtt{0}$. We need to show that $0 \in \Phi(m)$ implies that $1 \in \Phi(m)$. As above, we may show that assuming $0 \in \Phi(m)$ and $1 \notin \Phi(m)$ leads to a contradiction. We conclude that the map Φ is well-defined.

Φ is surjective. Let $I \in J(F(\theta(\alpha w)))$ be an order ideal. We have $(I \setminus \{0\}) - 1 \in J(F(\theta(w)))$. By induction, there exists a perfect matching $m \in M(w)$ such that $\Phi(m) = (I \setminus \{0\}) - 1$. Let $m' \in M(\alpha w)$ be a perfect matching of $G(\alpha w)$ defined as follows:

$$m' = \begin{cases} (m + e_1) \cup \{u^{\mid}\} & \text{if } \alpha = \mathtt{0}, 0 \in I \text{ and } |w| \text{ is even} \\ & \text{or } \alpha = \mathtt{0}, 0 \notin I \text{ and } |w| \text{ is odd}, \\ ((m + e_1) \cup \{u^{\mid}\})\Delta S & \text{if } \alpha = \mathtt{0}, 0 \in I \text{ and } |w| \text{ is odd} \\ & \text{or } \alpha = \mathtt{0}, 0 \notin I \text{ and } |w| \text{ is even}, \\ ((m + e_2) \cup \{u^{-}\})\Delta S & \text{if } \alpha = \mathtt{1}, 0 \in I \text{ and } |w| \text{ is even} \\ & \text{or } \alpha = \mathtt{1}, 0 \notin I \text{ and } |w| \text{ is odd}, \\ (m + e_2) \cup \{u^{-}\} & \text{if } \alpha = \mathtt{1}, 0 \in I \text{ and } |w| \text{ is odd} \\ & \text{or } \alpha = \mathtt{1}, 0 \notin I \text{ and } |w| \text{ is even}. \end{cases}$$

This matching satisfies that $\textsc{pop}(m') = m$. If $0 \in I$, we have that $m' \in M^{\perp}(\alpha w)$ and

$$\begin{aligned} \Phi(m') &= \{0\} \cup (\Phi(\textsc{pop}(m')) + 1) = \{0\} \cup (\Phi(m) + 1) \\ &= \{0\} \cup ((I \setminus \{0\}) - 1 + 1) = \{0\} \cup (I \setminus \{0\}) = I. \end{aligned}$$

If $0 \notin I$, we have that $m' \in M^{\parallel}(\alpha w)$ and

$$\begin{aligned} \Phi(m') &= \Phi(\textsc{pop}(m')) + 1 = \Phi(m) + 1 \\ &= (I \setminus \{0\}) - 1 + 1 = I \setminus \{0\} = I. \end{aligned}$$

We conclude that Φ is surjective.

Φ is injective. Let $m_1, m_2 \in M(\alpha w)$ be two perfect matchings such that $\Phi(m_1) = \Phi(m_2)$. We have

$$\begin{aligned} \Phi(\textsc{pop}(m_1)) &= \Phi(\textsc{pop}(m_1)) + 1 - 1 = \Phi(m_1) \setminus \{0\} - 1 \\ &= \Phi(m_2) \setminus \{0\} - 1 = \Phi(\textsc{pop}(m_2)) + 1 - 1 = \Phi(\textsc{pop}(m_2)). \end{aligned}$$

We may now use the induction hypothesis. Since $\Phi : M(w) \to J(F(\theta(w)))$ is injective and $\textsc{pop}(m_1), \textsc{pop}(m_2) \in M(w)$, we must have that $\textsc{pop}(m_1) = \textsc{pop}(m_2)$. If $0 \in \Phi(m_1) = \Phi(m_2)$, then $m_1, m_2 \in M^{\perp}(w)$. If $0 \notin \Phi(m_1) = \Phi(m_2)$, then $m_1, m_2 \in M^{\parallel}(w)$. In particular, the first edge of m_1 is the same as the first edge of m_2. Assume w starts with $\mathtt{0}$. If the first edge of m_1 and m_2 is $u^{\mid}$, then $m_1 \setminus \{u^{\mid}\} - e_1 = \textsc{pop}(m_1) = \textsc{pop}(m_2) = m_2 \setminus \{u^{\mid}\} - e_1$. Thus, $m_1 = m_2$. If the first edge of m_1 and m_2 is u^{-}, then $(m_1 \Delta S) \setminus \{u^{\mid}\} - e_1 = \textsc{pop}(m_1) = \textsc{pop}(m_2) = (m_2 \Delta S) \setminus \{u^{\mid}\} - e_1$. Thus, $m_1 = m_2$. We reach the same conclusion if we assume w that starts with $\mathtt{1}$. We conclude that Φ is injective.

Φ is order-preserving. Let $m, m' \in M(w)$ be two perfect matchings. We have $m \prec m'$ if and only if $\text{region}(m \Delta \mathfrak{b}_w) \subseteq \text{region}(m' \Delta \mathfrak{b}_w)$ if and only if $\Phi(m) \subset \Phi(m')$.

Items 1 and 2. By definition of Φ, we have $m \in M^{\perp}(w)$ if and only if $0 \in \Phi(m)$ if and only if $\Phi(m) \in J^{\bullet}(F(\theta(w)))$. Since $M(w) = M^{\perp}(w) \cup M^{\parallel}(w)$ is a disjoint union, this also implies that $m \in M^{\parallel}(w)$ if and only if $0 \notin \Phi(m)$ if and only if $\Phi(m) \in J^{\circ}(F(\theta(w)))$.

Item 3. It follows from the definition of Φ that $\mathrm{area}(m\Delta\mathfrak{b}_w) = |\Phi(m)|$ for every perfect matching $m \in M(w)$. □

Since θ is an involution on $\{\mathtt{0},\mathtt{1}\}^*$, Theorem 8.1 also means that the map Φ is a bijection $M(\theta(w)) \to J(F(w))$ from the set of perfect matchings of the snake graph $G(\theta(w))$ to the set of order ideals of the fence poset $F(w)$.

Corollary 8.2. *For every $w \in \{\mathtt{0},\mathtt{1}\}^*$, the posets $(M(\theta(w)), \prec)$ and $(J(F(w)), \subseteq)$ are isomorphic.*

Proof. It follows from the order-preserving bijection proved in Theorem 8.1. □

9. Proof of Theorem D and Corollary E

Similarly to Proposition 6.6, we have the following result involving the map θ as a consequence of Theorem 8.1.

Proposition 9.1. *Let $w \in \{\mathtt{0},\mathtt{1}\}^*$. The area statistics over the set of perfect matchings of the snake graph $G(w)$ satisfies*

$$\begin{pmatrix} \sum_{m\in M^{\perp}(w)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m\in M^{\parallel}(w)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(\theta(w)) \begin{pmatrix} q \\ q \end{pmatrix}.$$

where $\mathfrak{b}$ is the basic perfect matching of the snake graph $G(w)$.

Proof. For every $w \in \{\mathtt{0},\mathtt{1}\}^*$, let $\Phi : M(w) \to J(F(\theta w))$ be the bijection from the set of perfect matchings of the snake graph $G(w)$ to the set of order ideals of the fence poset $F(\theta w)$ defined in Theorem 8.1. Using Proposition 6.6, we have

$$\begin{aligned} \begin{pmatrix} \sum_{m\in M^{\perp}(w)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m\in M^{\parallel}(w)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} &= \begin{pmatrix} \sum_{I\in J^{\bullet}(F(\theta w))} q^{\mathrm{area}(\Phi^{-1}(I)\Delta\mathfrak{b})} \\ \sum_{I\in J^{\circ}(F(\theta w))} q^{\mathrm{area}(\Phi^{-1}(I)\Delta\mathfrak{b})} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{I\in J^{\bullet}(F(\theta w))} q^{|I|} \\ \sum_{I\in J^{\circ}(F(\theta w))} q^{|I|} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(\theta w) \begin{pmatrix} q \\ q \end{pmatrix}. \end{aligned}$$

□

The following result says that the area statistics of the perfect $\parallel$-matchings and $\perp$-matchings of a snake graph is related to the q-analog of a continued fraction expansion of a rational number.

Theorem D. *For every positive rational number $x \in \mathbb{Q}_{>0}$, whose even-length continued fraction expansion is $[a_0; a_1, \ldots, a_{2\ell-1}]$, the area statistics over the set of perfect matchings of the snake graph $\mathcal{G}(x)$ satisfies*

$$\begin{pmatrix} \sum_{m\in \mathcal{M}^{\perp}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m\in \mathcal{M}^{\parallel}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

where $\mathfrak{b}$ is the basic perfect matching of $\mathcal{G}(x)$.

Proof. Let $a = [a_0; a_1, \ldots, a_{2\ell-1}]$ be the even-length continued fraction expansion of a positive rational number $x > 0$. Let $w = W(a) = \mathtt{1}^{a_0}\mathtt{0}^{a_1} \cdots \mathtt{1}^{a_{2\ell-2}}\mathtt{0}^{a_{2\ell-1}-1}$. Using

Proposition 9.1 and Lemma 6.2, we obtain

$$\begin{aligned}\begin{pmatrix} \sum_{m\in\mathcal{M}^{\perp}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m\in\mathcal{M}^{\parallel}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} &= \begin{pmatrix} \sum_{m\in M^{\perp}(\theta(W(a)))} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m\in M^{\parallel}(\theta(W(a)))} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(\theta^2(W(a))) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} \nu_q(W(a)) \left(\begin{smallmatrix} q \\ q \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).\end{aligned}$$

□

Also, in terms of the Morier-Genoud–Ovsienko q-analog of rational numbers, the following result holds in general.

Corollary E. *Let $x > 0$ be a positive rational number whose even-length continued fraction expansion is $a = [a_0; \ldots, a_{2\ell-1}]$. Then, the set $\mathcal{B}(a)$ of admissible sequences for a, the set $\mathcal{M}(x)$ of perfect matchings of the snake graph $\mathcal{G}(x)$ and the set $\mathcal{J}(x)$ of order ideals of the fence poset $\mathcal{F}(x)$ give three enumerative interpretations of the Morier-Genoud–Ovsienko q-analog of the rational number x:*

$$[x]_q = \frac{q^{-1}\sum_{b\in\mathcal{B}^{\bullet}(a)} q^{\|b\|_1}}{\sum_{b\in\mathcal{B}^{\circ}(a)} q^{\|b\|_1}} = \frac{q^{-1}\sum_{I\in\mathcal{J}^{\bullet}(x)} q^{|I|}}{\sum_{I\in\mathcal{J}^{\circ}(x)} q^{|I|}} = \frac{q^{-1}\sum_{m\in\mathcal{M}^{\perp}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})}}{\sum_{m\in\mathcal{M}^{\parallel}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})}}$$

where $\mathfrak{b}$ is the basic perfect matching of the snake graph $\mathcal{G}(x)$ and the fractions on the right-hand sides are reduced.

Proof. Let $a = [a_0; \ldots, a_{2\ell-1}]$ be the even-length continued fraction expansion of a positive rational number $\frac{r}{s} > 0$. Let $R(q)$ and $S(q)$ be the polynomials such that the Morier-Genoud–Ovsienko q-analog of the rational number $\frac{r}{s}$ is

$$\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}.$$

Using Theorem D, we have

$$\begin{aligned}\begin{pmatrix} \sum_{m\in\mathcal{M}^{\perp}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m\in\mathcal{M}^{\parallel}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)^{-1} q \cdot q^{-1} R_q^{a_0} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\overset{(3)}{=} \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right) \begin{pmatrix} R(q) \\ S(q) \end{pmatrix} = \begin{pmatrix} q\cdot R(q) \\ S(q) \end{pmatrix}.\end{aligned}$$

We obtain

$$\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)} = \frac{q^{-1}\sum_{m\in M^{\perp}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})}}{\sum_{m\in M^{\parallel}(x)} q^{\mathrm{area}(m\Delta\mathfrak{b})}}.$$

The two other equalities follow from Theorem B and Theorem C. □

A combinatorial interpretation of rational numbers [ÇS18, Theorem 3.4], stated here as Theorem 7.7, and of their q-analogs [MGO20, Theorem 4] was already provided in terms of order ideals of fence posets and snake graphs. All of these formulas are involving different combinatorial objects for the numerator and denominator and are restricted to rational numbers larger than one. We can deduce their results as a consequence of ours.

Corollary 9.2. *Let $x > 1$ be a rational number whose even-length continued fraction expansion is $a = [a_0; a_1, \ldots, a_{2\ell-1}]$. Then,*

$$[x]_q = \frac{\sum_{b\in\mathcal{B}(x')} q^{\|b\|_1}}{\sum_{b\in\mathcal{B}(x'')} q^{\|b\|_1}} = \frac{\sum_{I\in\mathcal{J}(x')} q^{|I|}}{\sum_{I\in\mathcal{J}(x'')} q^{|I|}} = \frac{\sum_{m\in\mathcal{M}(x')} q^{\mathrm{area}(m\Delta\mathfrak{b}')}}{\sum_{m\in\mathcal{M}(x'')} q^{\mathrm{area}(m\Delta\mathfrak{b}'')}} \tag{11}$$

where

$$\begin{aligned} x' &= x - 1 && = [a_0 - 1; a_1, a_2, \ldots, a_{2\ell-1}], \\ x'' &= ((x - \lfloor x \rfloor)^{-1} - 1)^{-1} && = [0; a_1 - 1, a_2, \ldots, a_{2\ell-1}], \end{aligned}$$

and $\mathfrak{b}'$ (resp. $\mathfrak{b}''$) is the basic perfect matching of the snake graph $\mathcal{G}(x')$ (resp. $\mathcal{G}(x'')$).

Proof. Let $x > 1$ be a rational number whose even-length continued fraction expansion is $a = [a_0; a_1, \ldots, a_{2\ell-1}]$. Since $x > 1$, we have $a_0 > 0$. Below, we use the following identity

$$q^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} R_q = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} R_q^{a_0} L_q. \tag{12}$$

For the numerator of $[x]_q$ expressed in Corollary E, we can rewrite it as follows using Theorem C and (12):

$$\begin{aligned} q^{-1} \sum_{I\in\mathcal{J}^\bullet(x)} q^{|I|} &= q^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} R_q \cdot R_q^{a_0-1} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} \cdot R_q^{a_0-1} L_q^{a_1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \sum_{I\in\mathcal{J}^\bullet[a_0-1;a_1,\ldots,a_{2\ell-1}]} q^{|I|} + \sum_{I\in\mathcal{J}^\circ[a_0-1;a_1,\ldots,a_{2\ell-1}]} q^{|I|} \\ &= \sum_{I\in\mathcal{J}[a_0-1;a_1,\ldots,a_{2\ell-1}]} q^{|I|} = \sum_{I\in\mathcal{J}(x')} q^{|I|}. \end{aligned}$$

For the denominator of $[x]_q$ expressed in Corollary E, we can rewrite it as follows using Theorem C and (12):

$$\begin{aligned} \sum_{I\in\mathcal{J}^\circ(x)} q^{|I|} &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} R_q^{a_0} L_q \cdot L_q^{a_1-1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} \cdot L_q^{a_1-1} \cdots R_q^{a_{2\ell-2}} L_q^{a_{2\ell-1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \sum_{I\in\mathcal{J}^\bullet[0;a_1-1,\ldots,a_{2\ell-1}]} q^{|I|} + \sum_{I\in\mathcal{J}^\circ[0;a_1-1,\ldots,a_{2\ell-1}]} q^{|I|} \\ &= \sum_{I\in\mathcal{J}[0;a_1-1,\ldots,a_{2\ell-1}]} q^{|I|} = \sum_{I\in\mathcal{J}(x'')} q^{|I|}. \end{aligned}$$

Notice that if $a_1 = 1$, then $x'' = [a_2; a_3, \ldots, a_{2\ell-1}]$. The other equalities are deduced from the order-preserving bijections proved in Theorem 5.2 and Theorem 8.1. □

One may observe that the fence posets $\mathcal{F}(x')$ and $\mathcal{F}(x'')$ are the two oriented path graphs used to define the numerator and denominator formulas for $[x]_q$ in Theorem 4 of [MGO20]. Here is a remark why (11) is a q-analog of Theorem 3.4 from [ÇS18].

Remark 9.3. *When $q = 1$, the perfect matchings part of Equation* (11) *becomes*

$$x = \frac{\#\mathcal{M}(x')}{\#\mathcal{M}(x'')} = \frac{\#\mathcal{M}(x')}{\#\mathcal{M}((x'')^{-1})} = \frac{\#\mathcal{M}([a_0 - 1; a_1, \ldots, a_{2\ell-1}])}{\#\mathcal{M}([a_1 - 1, a_2, \ldots, a_{2\ell-1}])}$$

where we used the fact that the number of perfect matchings of the snake graph $\mathcal{G}(x'')$ is equal to the number of perfect matchings of its mirror image snake graph $\mathcal{G}((x'')^{-1})$; see Lemma 7.5. Thus, we recover Theorem 3.4 from [ÇS18]. However, replacing x'' by its inverse does not work in general when $q \neq 1$. Thus, (11) *is a legitimate q-analog of Theorem 3.4 from [ÇS18].*

It remains open whether Theorem 1.3 from [MOSZ23] can also be obtained from the statistics of the m-dimer cover over a single snake graph.

10. Markoff numbers and their q-analogs

A Markoff triple is a positive solution of the Diophantine equation $x^2+y^2+z^2 = 3xyz$ [Mar80, Mar79]. Markoff triples can be defined recursively as follows: $(1,1,1)$, $(1,2,1)$ and $(1,5,2)$ are Markoff triples and if (x,y,z) is a Markoff triple with $y \geq x$ and $y \geq z$, then $(x, 3xy-z, y)$ and $(y, 3yz-x, z)$ are Markoff triples. A list of small Markoff numbers (elements of a Markoff triple) is

$$1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, \ldots$$

referenced as sequence A002559 in OEIS [OEI22].

Christoffel words are words over the alphabet $\{\mathtt{0},\mathtt{1}\}$ that can be defined recursively as follows: $\mathtt{0}$, $\mathtt{1}$ and $\mathtt{01}$ are Christoffel words and if $u, v, uv \in \{\mathtt{0},\mathtt{1}\}^*$ are Christoffel words then uuv and uvv are Christoffel words [BLRS09]. The shortest Christoffel words are:

$$\mathtt{0}, \mathtt{1}, \mathtt{01}, \mathtt{001}, \mathtt{011}, \mathtt{0001}, \mathtt{00101}, \mathtt{01011}, \mathtt{0111}, \mathtt{00001}, \mathtt{0001001}, \mathtt{00100101}, \mathtt{0010101}, \ldots$$

Note that these are usually named *lower* Christoffel words. A Christoffel word other than the letters $\mathtt{0}$ or $\mathtt{1}$ is called *proper*.

It is known that each Markoff number can be expressed in terms of a Christoffel word. More precisely, let μ be the monoid homomorphism $\{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z})$ defined by

$$\mu(\mathtt{0}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mu(\mathtt{1}) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Each Markoff number is equal to $\mu(w)_{12}$ for some Christoffel word w [Reu09], where M_{12} denotes the element above the diagonal in a matrix $M = \left(\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$.

For example, the Markoff number 194 is associated with the Christoffel word $\mathtt{00101}$ as it is the entry at position $(1,2)$ in the matrix

$$\mu(\mathtt{00101}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix}.$$

Whether the map $w \mapsto \mu(w)_{12}$ provides a bijection between Christoffel words and Markoff numbers is a question (stated differently in [Fro13]) that has remained open for more than 100 years [Aig13]. The conjecture, also known as the *uniqueness conjecture*, can be expressed in terms of the injectivity of the map $w \mapsto \mu(w)_{12}$ over the set of Christoffel words [Reu19, §3.3].

A q-analog of the Markoff numbers was proposed in [Kog20] which can be expressed using the following morphism of monoids $\mu_q : \{\mathtt{0},\mathtt{1}\}^* \to \mathrm{GL}_2(\mathbb{Z}[q^{\pm 1}])$

[LMG21]:

$$\mu_q(\mathtt{0}) = \nu_q(\mathtt{10}) = R_q L_q = \begin{pmatrix} q+q^2 & 1 \\ q & 1 \end{pmatrix},$$
$$\mu_q(\mathtt{1}) = \nu_q(\mathtt{1100}) = R_q R_q L_q L_q = \begin{pmatrix} q+2q^2+q^3+q^4 & 1+q \\ q+q^2 & 1 \end{pmatrix}.$$

It was proved that the map $w \mapsto \mu_q(w)_{12}$ is injective over the language of a balanced sequence [LL22] and over the set of Christoffel words [LLS23]. Note that different q-analogs of Markoff numbers were proposed in [EJMGO25] based on other pairs of Cohn matrices.

Consider the following morphism

$$\begin{array}{rccc} \gamma: & \{\mathtt{0},\mathtt{1}\}^* & \to & \{\mathtt{0},\mathtt{1}\}^* \\ & \mathtt{0} & \mapsto & \mathtt{00} \\ & \mathtt{1} & \mapsto & \mathtt{0110} \end{array}.$$

The following result provides a combinatorial interpretation of the q-analog of Markoff numbers based on the map μ_q in terms of the area statistics of the perfect matching of a snake graph. Its proof is short and is based on the matrices L_q and R_q.

Theorem 10.1. *The q-analog of the Markoff number associated with a proper Christoffel word $\mathtt{0}m\mathtt{1} \in \{\mathtt{0},\mathtt{1}\}^*$ is the area polynomial of the set of perfect matchings of the snake graph $G(\mathtt{0}\gamma(m)\mathtt{0})$. More precisely,*

$$\mu_q(\mathtt{0}m\mathtt{1})_{12} = \sum_{m \in M(\mathtt{0}\gamma(m)\mathtt{0})} q^{\mathrm{area}(m\Delta\mathfrak{b})}.$$

Proof. Note that the morphism

$$\begin{array}{rccc} \gamma': & \{\mathtt{0},\mathtt{1}\}^* & \to & \{\mathtt{0},\mathtt{1}\}^* \\ & \mathtt{0} & \mapsto & \mathtt{10} \\ & \mathtt{1} & \mapsto & \mathtt{1100} \end{array}$$

satisfies $\mu_q = \nu_q \circ \gamma'$ and

$$\theta(\mathtt{0}\gamma(w)\mathtt{0}) = \mathtt{0}\gamma'(w)\mathtt{1}$$

for every even-length word $w \in \{\mathtt{0},\mathtt{1}\}^*$. Using Proposition 9.1, we have

$$\begin{aligned} \sum_{m \in M(\mathtt{0}\gamma(m)\mathtt{0})} q^{\mathrm{area}(m\Delta\mathfrak{b})} &= \sum_{m \in M^{\perp}(\mathtt{0}\gamma(m)\mathtt{0})} q^{\mathrm{area}(m\Delta\mathfrak{b})} + \sum_{m \in M^{\parallel}(\mathtt{0}\gamma(m)\mathtt{0})} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \sum_{m \in M^{\perp}(\mathtt{0}\gamma(m)\mathtt{0})} q^{\mathrm{area}(m\Delta\mathfrak{b})} \\ \sum_{m \in M^{\parallel}(\mathtt{0}\gamma(m)\mathtt{0})} q^{\mathrm{area}(m\Delta\mathfrak{b})} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} \nu_q(\theta(\mathtt{0}\gamma(m)\mathtt{0})) \begin{pmatrix} q \\ q \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{-1} \nu_q(\mathtt{0}\gamma'(m)\mathtt{1}) \begin{pmatrix} q \\ q \end{pmatrix} \\ &= \begin{pmatrix} q & 1 \end{pmatrix} \cdot \nu_q(\mathtt{0}\gamma'(m)\mathtt{1}) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} R_q \cdot \nu_q(\mathtt{0}\gamma'(m)\mathtt{1}) \cdot R_q L_q L_q \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \nu_q(\mathtt{10}\gamma'(m)\mathtt{1100}) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \nu_q(\gamma'(\mathtt{0}m\mathtt{1})) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \mu_q(\mathtt{0}m\mathtt{1}) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$= \mu_q(\mathtt{0}m\mathtt{1})_{12}.$$

□

As a consequence, when evaluating at $q = 1$, we recover Theorem 4 from [Pro20], see also Theorem 7.12 from [Aig13] and Theorem C from [ÇS20].

Corollary 10.2. *The Markoff number associated with a proper Christoffel word* $\mathtt{0}m\mathtt{1} \in \{\mathtt{0},\mathtt{1}\}^*$ *is equal to the number of perfect matchings of the snake graph* $G(\mathtt{0}\gamma(m)\mathtt{0})$*, that is,*

$$\mu(\mathtt{0}m\mathtt{1})_{12} = \#M(\mathtt{0}\gamma(m)\mathtt{0}).$$

Proof. It follows from Theorem 10.1, after evaluating at $q = 1$. □

Our proof is different as it is based on Proposition 9.1 which is more general as it works for all snake graphs using a product over the elementary matrices L_q and R_q. The proof proposed in [Aig13] uses a decomposition of certain snake graphs (those obtained from Christoffel words) into blocks of size 2 or 4 each block being associated with one of the Cohn matrices $\left(\begin{smallmatrix}1 & 1\\ 1 & 2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}3 & 2\\ 4 & 3\end{smallmatrix}\right)$.

The bijections provided in Theorem 5.2 and Theorem 8.1 mean that Theorem 10.1 and Corollary 10.2 also admit equivalent interpretations of Markoff numbers and their q-analogs in terms of admissible sequences and order ideals of fence posets.

Example 10.3. *This reproduces Example 7.11 in [Aig13]. Consider the Christoffel word* $\mathtt{00101} = \mathtt{0}m\mathtt{1}$ *with* $m = \mathtt{010}$*. The associated Markoff number is*

$$\mu(\mathtt{0}m\mathtt{1})_{12} = \mu(\mathtt{00101})_{12} = 433 = \#M(\mathtt{0}\gamma(m)\mathtt{0}) = \#M(\mathtt{001100001100})$$

which is equal to the number of perfect matchings of the snake graph $G(\mathtt{001100001100}) = \mathcal{G}(179/254)$ *shown in Figure 21.*

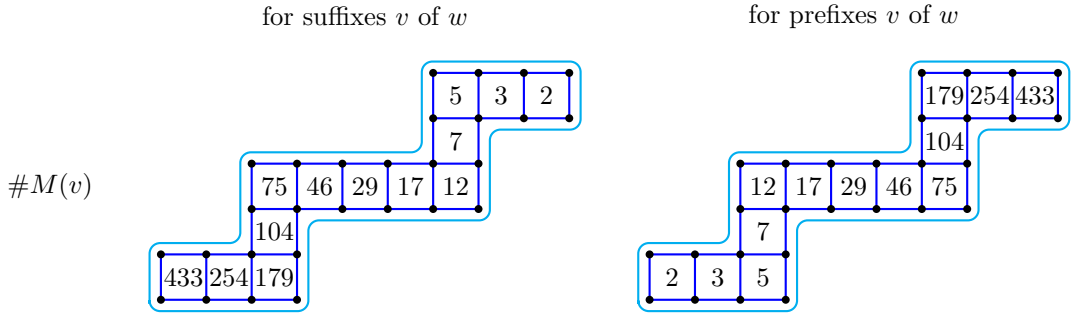


FIGURE 21. With $a = 179/254 = [0; 1, 2, 2, 1, 1, 2, 2, 2]$, we have $W(a) = 011001011001$ and $w = \theta \circ W(a) = 001100001100$. The number of perfect matchings in the snake graph $G(v)$ for all prefixes and suffixes of w are indicated in the boxes. This reproduces Example 7.11 in Aigner's book [Aig13] for the Markov number 433.

11. Convexity of the set of admissible sequences

In this section, we prove additional results about the set of admissible sequences for the even or odd-length continued fraction expansion of a positive rational number, namely it is a convex subset of $\mathbb{Z}^k$ whose convex hull is a polytope [Zie95].

Definition 11.1. *Let $k \geq 1$ be an integer. For every even or odd-length continued fraction expansion $a = [a_0; a_1, \ldots, a_{k-1}]$ of a positive rational number, we define the polytope*

$$\mathcal{P}(a) := \mathrm{conv}(\mathcal{B}(a)) \subset \mathbb{R}^k$$

where $\mathrm{conv}(X)$ denotes the convex hull of X for every subset $X \subset \mathbb{R}^k$. Also, for every positive rational number $x \in \mathbb{Q}_{>0}$, let

$$\mathcal{P}_{even}(x) = \mathcal{P}(\mathrm{CF}_{even}(x)) \qquad \text{and} \qquad \mathcal{P}_{odd}(x) = \mathcal{P}(\mathrm{CF}_{odd}(x))$$

be the two polytopes associated with $x \in \mathbb{Q}_{>0}$.

Notice that the dimension of the polytope $\mathcal{P}(a)$ is much smaller than the dimension ($= a_0 + a_1 + \cdots + a_{k-1}$) of the order polytope of the fence poset $\mathcal{F}(a)$ [Sta86].

Theorem 11.2. *Let $k \geq 1$ be an integer and $a = [a_0; a_1, \ldots, a_{k-1}]$ be the even or odd-length continued fraction expansion of a positive rational number. The set of admissible sequences for a are the integer points of the polytope $\mathcal{P}(a)$, that is,*

$$\mathcal{P}(a) \cap \mathbb{Z}^k = \mathcal{B}(a).$$

Proof. ($\supseteq$) Admissible sequences are sequences of nonnegative integers, so we have $\mathcal{B}(a) \subset \mathbb{Z}^k$. From the definition of convex hull, we have $\mathcal{B}(a) \subset \mathrm{conv}(\mathcal{B}(a)) = \mathcal{P}(a)$. The conclusion follows.

($\subseteq$) Let $(c_i)_{0\leq i\leq k-1} \in \mathcal{P}(a) \cap \mathbb{Z}^k$. There exist $m \geq 1$ admissible sequences $(b_i^{(1)})_{0\leq i\leq k-1}, \ldots, (b_i^{(m)})_{0\leq i\leq k-1} \in \mathcal{B}(a)$ such that $(c_i)_{0\leq i\leq k-1}$ is in their convex hull. In other words, there exists $(\gamma_1, \ldots, \gamma_m) \in [0,1]^m$, with $\gamma_1 + \cdots + \gamma_m = 1$, such that

$$c_i = \sum_{j=1}^{m} \gamma_j b_i^{(j)}$$

for every integer i with $0 \leq i \leq k-1$. We may assume that the number m of admissible sequences allowing to express $(c_i)_i$ is minimal. In other words, we can assume that $\gamma_j \neq 0$ for every j with $1 \leq j \leq m$. If there exists j such that $\gamma_j = 1$, then $(c_i)_{0\leq i\leq k-1} = (b_i^{(j)})_{0\leq i\leq k-1}$ is an admissible sequence. Suppose that $0 < \gamma_j < 1$ for every j with $1 \leq j \leq m$. Since every sequence $(b_i^{(j)})_{0\leq i\leq k-1}$ is admissible for a, for every integer i, we have

$$0 \leq \min_{1\leq j\leq m} (b_i^{(j)}) \leq c_i \leq \max_{1\leq j\leq m} (b_i^{(j)}) \leq a_i.$$

Assume $c_i = a_i$ for some odd integer $i > 0$. Then $c_i = b_i^{(j)} = a_i$ for every j with $1 \leq j \leq m$. Since these m sequences are admissible, we deduce that $b_{i-1}^{(j)} = a_{i-1}$ for every j with $1 \leq j \leq m$. Thus, $c_{i-1} = a_{i-1}$. Assume $c_i = 0$ for some even integer $i > 0$. Then $c_i = b_i^{(j)} = 0$ for every j with $1 \leq j \leq m$. Since these m sequences are admissible, we deduce that $b_{i-1}^{(j)} = 0$ for every j with $1 \leq j \leq m$. Thus, $c_{i-1} = 0$. Therefore, the sequence $(c_i)_{0\leq i\leq k-1}$ is admissible for a. □

Convexity also holds for the two subsets forming the partition $\mathcal{B}(a) = \mathcal{B}^\bullet(a) \cup \mathcal{B}^\circ(a)$. In fact, the two subsets are in two complementary half-spaces. For every $x \in \mathbb{R}$, let $\delta_x = 0$ if $x = 0$, and $\delta_x = 1$ if $x \neq 0$. For every integer $k \geq 2$ and every continued fraction expansion $a = [a_0; a_1, \ldots, a_{k-1}]$ of a positive real number, consider the open half-space

$$H_a^+ = \{(x_0, x_1, \ldots, x_{k-1}) \in \mathbb{R}^k \mid \delta_{a_0}(1 - x_0) + (1 - \delta_{a_0})(a_1 - x_1) > 0\} \subset \mathbb{R}^k.$$

Lemma 11.3. *We have*

$$\mathcal{B}^\bullet(a) = \mathcal{B}(a) \cap (\mathbb{R}^k \setminus H_a^+),$$
$$\mathcal{B}^\circ(a) = \mathcal{B}(a) \cap H_a^+.$$

Proof. If $a_0 = 0$, we have $\delta_{a_0} = 0$ and

$$\mathcal{B}^\bullet(a) = \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_1 = a_1\} = \mathcal{B}(a) \cap (\mathbb{R}^k \setminus H_a^+),$$
$$\mathcal{B}^\circ(a) = \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_1 < a_1\} = \mathcal{B}(a) \cap H_a^+.$$

If $a_0 \neq 0$, we have $\delta_{a_0} = 1$ and

$$\begin{aligned}\mathcal{B}^\bullet(a) &= \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_0 > 0\}\\ &= \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_0 \ge 1\} = \mathcal{B}(a) \cap (\mathbb{R}^k \setminus H_a^+),\\ \mathcal{B}^\circ(a) &= \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_0 = 0\}\\ &= \{(b_i)_{0\le i\le k-1} \in \mathcal{B}(a) \mid b_0 < 1\} = \mathcal{B}(a) \cap H_a^+.\end{aligned}$$

□

If $x \in \mathbb{Q}_{>0}$, we deduce from Corollary E (when $q = 1$) and Lemma 11.3, that the number of integer points in the polytope $\mathcal{P}_{\text{even}}(x)$ that are inside (resp., outside) of the open half-space H_a^+ is equal to the denominator (resp., numerator) of the rational number x.

Unimodality of the rank polynomials proved in [KOR23] means that the sequence of number of integers points of $\mathcal{P}(a)$ on each affine hyperplane orthogonal to the vector $(1, 1, \dots, 1)$ is also unimodal.

Given the continued fraction expansion $a = [a_0; a_1, \dots, a_{k-1}]$ of a positive rational number, the properties of the polytope $\mathcal{P}(a)$ remain to be explored. A first step would be to describe its extremal points and facets.

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