BOURGAIN'S CONDITION, STICKY KAKEYA, AND NEW EXAMPLES

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Abstract. We prove that in all dimensions at least 3 and for any Hörmander-type oscillatory integral operator satisfying Bourgain's condition, the sticky case of the corresponding curved Kakeya conjecture reduces to the sticky case of the classical Kakeya conjecture. This supports a conjecture of Guo–Wang–Zhang, that an operator satisfies the same L^p bounds as in the restriction conjecture exactly when it satisfies Bourgain's condition.

Our result follows from a new geometric characterization of Bourgain's condition based on the structure of curved δ-tubes in a $\delta^{1/2}$-tube. We find examples in all dimensions at least 3 which show this property does not persist in a larger tube, and in particular these are the first operators satisfying Bourgain's condition for which there is no diffeomorphism taking the corresponding families of curves to lines. This suggests that a general to sticky reduction in the spirit of Wang–Zahl needs substantial new ideas. We expect these examples to provide a good starting point.

1. Introduction

We recall the definition of a Hörmander-type phase function and the corresponding Hörmander-type oscillatory integral operator. Fix $n \geqslant 2$ and let $\Sigma \subset \mathbb{R}^{n-1}$ and $M \subset \mathbb{R}^n$ be simply connected open sets, with origins $0_M \in M$ and $0_\Sigma \in \Sigma$. We reserve the right to restrict M and Σ to smaller simply connected open sets containing 0_M and 0_Σ respectively. We say that a smooth function $\phi : M \times \Sigma \to \mathbb{R}$ is a *Hörmander-type phase function* if it satisfies the following conditions:

(H1) $\operatorname{rank}(\nabla_{\mathbf{x}} \nabla_\xi \phi(\mathbf{x}, \xi)) = n-1$ on $M \times \Sigma$, and

(H2) with the Gauss map $G : M \times \Sigma \to \mathbb{R}^n \setminus \{0\}$ defined by

$$G(\mathbf{x}, \xi) = \bigwedge_{j=1}^{n-1} \partial_{\xi_j} \nabla_{\mathbf{x}} \phi(\mathbf{x}, \xi), \tag{1.1}$$

the matrix

$$(G(\mathbf{x}, \xi) \cdot \nabla_{\mathbf{x}}) \nabla_\xi^2 \phi(\mathbf{x}, \xi) \tag{1.2}$$

is nondegenerate on $M \times \Sigma$.

We say that (H2+) holds if the matrix (1.2) is further positive-definite, and we call ϕ a *positive-definite* Hörmander-type phase function in that case. Let $a : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}$ be a smooth bump function supported on $M \times \Sigma$. We define the Hörmander-type oscillatory integral operator associated to ϕ by

$$T_\phi^\lambda f(\mathbf{x}) = \int_{\mathbb{R}^{n-1}} e^{i\lambda\phi(\mathbf{x}/\lambda, \xi)} f(\xi) a(\mathbf{x}/\lambda, \xi) d\xi. \tag{1.3}$$

A central open problem is to determine the range of p for which

$$\|T_\phi^\lambda f\|_p \leqslant C_{p,\varepsilon}\lambda^\varepsilon \|f\|_\infty. \tag{1.4}$$

This is often called Hörmander's oscillatory integral problem. Writing $\mathbf{x} = (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and choosing

$$\phi_{n,\mathrm{rest}}(\mathbf{x},\xi) = x\cdot\xi + \tfrac{1}{2}t|\xi|^2, \quad (0_M, 0_\Sigma) = (0,0), \tag{1.5}$$

$T^\lambda_{\phi_{n,\mathrm{rest}}}$ becomes the Fourier extension operator for the paraboloid. In that case we obtain the Fourier restriction problem, and the conjectured range for (1.4) is $p > \frac{2n}{n-1}$. By choosing

$$\phi_{n,\mathrm{BR}}(\mathbf{x},\xi) = t^{-1}\sqrt{1+|x-t\xi|^2}, \quad (0_M, 0_\Sigma) = ((0,1),0), \tag{1.6}$$

$T^\lambda_{\phi_{n,\mathrm{BR}}}$ becomes the *reduced* Bochner–Riesz operator. This language was used in [DGGZ24] for Carleson–Sjölin operators on a manifold, where the Bochner–Riesz operator is the special case of $\mathbb{R}^n$ with the standard metric as the manifold. If (1.4) holds with $\phi = \phi_{n,\mathrm{BR}}$ for all $p > \frac{2n}{n-1}$, then the Bochner–Riesz conjecture is true (see [DGGZ24] for an exposition of this connection). These examples begin to demonstrate the importance of the following more refined question.

Question 1.1. *For which Hörmander-type phase functions ϕ does T^λ_ϕ satisfy* (1.4) *for all $p > \frac{2n}{n-1}$?*

Going forward, we will often write *phase function* to mean Hörmander-type phase function. As shown by Hörmander, the answer to Question 1.1 is all phase functions in the $n = 2$ case [Hor73]. However Bourgain proved that (1.4) fails for a *generic* phase function in the $n = 3$ case [Bou91]. This construction was later generalized to all $n \geqslant 3$ by Guo–Wang–Zhang [GWZ24]: if ϕ is a phase function such that for some $(\mathbf{x},\xi) \in M \times \Sigma$,

$$(G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})^2\nabla^2_\xi\phi(\mathbf{x},\xi) \text{ is not a multiple of } (G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})\nabla^2_\xi\phi(\mathbf{x},\xi), \tag{1.7}$$

then (1.4) fails for all

$$p < \frac{2n}{n-1} + \frac{2}{(n-1)(2n^2-n-2)}. \tag{1.8}$$

Motivated by these examples, Guo–Wang–Zhang introduced *Bourgain's condition.*

Definition 1.2 (Bourgain's condition)**.** Let ϕ be a Hörmander-type phase function. Then ϕ satisfies Bourgain's condition if there exists a (necessarily smooth) function $\lambda : M \times \Sigma \to \mathbb{R}$ such that on $M \times \Sigma$,

$$(G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})^2\nabla^2_\xi\phi(\mathbf{x},\xi) = \lambda(\mathbf{x},\xi)(G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})\nabla^2_\xi\phi(\mathbf{x},\xi). \tag{1.9}$$

The phase functions $\phi_{n,\mathrm{rest}}$ and $\phi_{n,\mathrm{BR}}$ notably satisfy Bourgain's condition, and (1.7) holding at some point $(\mathbf{x},\xi)$ is the statement that Bourgain's condition fails. This leads to the following conjecture proposed by Guo–Wang–Zhang to hopefully answer Question 1.1, and unify the Bochner–Riesz and Fourier restriction conjectures.

Conjecture 1.3 (Hörmander dichotomy conjecture)**.** *The operators T^λ_ϕ satisfy* (1.4) *for $p > \frac{2n}{n-1}$ if and only if ϕ satisfies Bourgain's condition.*

Guo–Wang–Zhang proved, via a surprising calculation, that Bourgain's condition is invariant under diffeomorphism in $\mathbf{x}$ and ξ separately—this is promising since the estimate (1.4) is also invariant under such diffeomorphisms. Finding a geometric explanation for this invariance is one of the goals of our work. In support of Conjecture 1.3, Guo–Wang–Zhang proved strong oscillatory integral estimates for phase functions satisfying Bourgain's condition, using a polynomial partitioning argument. Precisely, if ϕ satisfies Bourgain's condition and (H2+), then (1.4) holds for

$$p > 2 + \frac{2.5921}{n} + O(n^{-2}). \tag{1.10}$$

At the time, this gave the best bounds for the restriction conjecture in high dimensions. Both this positive result when Bourgain's condition is satisfied, and the negative result when Bourgain's condition fails, rely on the geometry of the wavepackets of T_ϕ^λ. In other words, they rely on the properties of certain Kakeya sets of curves.

1.1. The wavepacket geometry of T_ϕ^λ and Kakeya sets of curves.

Definition 1.4 (ϕ-curves and (δ,ϕ)-tubes)**.** Let ϕ be a Hörmander-type phase function.

- We define a ϕ-curve $\ell_{\xi,v}$ by
$$\ell_{\xi,v} = \{\mathbf{x} \in M : \nabla_\xi \phi(\mathbf{x}, \xi) = v\}. \tag{1.11}$$
After possibly shrinking M and Σ, the condition (H1) guarantees that $\ell_{\xi,v}$ is a curve for $(\xi, v) \in \Sigma \times \mathcal{V}$, where $\mathcal{V} := \nabla_\xi \phi(M, \Sigma) \subset \mathbb{R}^{n-1}$ is an open set. We also define the origin $0_\mathcal{V} := \nabla_\xi \phi(0_M, 0_\Sigma) \in \mathcal{V}$, which has the property that $0 \in \ell_{0_\Sigma, 0_\mathcal{V}}$. We define the $2(n-1)$-parameter family of ϕ-curves
$$\mathcal{C}(\phi) := \{\ell_{\xi,v} : (\xi, v) \in \Sigma \times \mathcal{V}\}. \tag{1.12}$$
- A (ϕ, δ)-tube $T^\delta_{\xi,v} \subset M$ is the δ-neighborhood of the ϕ-curve $\ell_{\xi,v}$. When the parameter δ is clear from context, we call these ϕ-tubes.

The $\phi_{n,\mathrm{rest}}$-curves are the lines

$$\mathrm{line}_{\xi,v} := (v, 0) - \mathbb{R}(\xi, 1). \tag{1.13}$$

We will always refer to the $\phi_{n,\mathrm{rest}}$-curve with parameter (ξ, v) by $\mathrm{line}_{\xi,v}$. We will often call $\phi_{n,\mathrm{rest}}$-tubes *straight tubes*. In analogy with this case, the parameter $\xi \in \Sigma$ will be called the *direction*. As a consequence of (H2), the parameter ξ is really the correct generalization of direction: if $\ell_{\xi,v}$ and $\ell_{\xi',v'}$ intersect at a point, then $\angle(\ell_{\xi,v}, \ell_{\xi',v'}) \approx |\xi - \xi'|$. Given $\ell = \ell_{\xi,v}$, we define $\mathrm{dir}(\ell) = \xi$.

The counterexamples of Bourgain and Guo–Wang–Zhang arose from special configurations of ϕ-tubes, parts of which can compress into a surface. On the other hand when ϕ satisfies Bourgain's condition, Guo–Wang–Zhang proved that ϕ-tubes satisfy a strong form of the *polynomial Wolff axioms* [GWZ24, Theorem 6.2]. This is the only part of the proof of their positive result for T_ϕ^λ [GWZ24, Theorem 1.3] that used Bourgain's condition.

More recently, Wang–Wu put forward a powerful technique to convert Kakeya-type incidence estimates to restriction type estimates, which has led to the state of the art for the restriction conjecture in all dimensions [WW24]. To reach the full restriction conjecture, one likely needs a more complete understanding of the incidence geometry of tubes, and a first step is to understand the Kakeya conjecture—a compact subset of $\mathbb{R}^n$ containing a line segment in every direction has Hausdorff dimension n. Wang–Zahl have recently made spectacular progress in this direction by solving the $n = 3$ case of the Kakeya conjecture [WZ22, WZ25a, WZ25b]. Our understanding of Hörmander-type oscillatory integrals in dimension $n \geqslant 3$ follows the same paradigm: one can often convert progress on certain Kakeya-type problems for curves associated to ϕ to progress on Hörmander's oscillatory integral problem (see [GWZ24, CGG$^+$24, Bou91, DGGZ24, GHI19] for examples).

To avoid any artificial issues in studying a Kakeya set whose curves lie near (resp. the parameters of whose curves lie near) the boundary of M (resp. Σ or $\mathcal{V}$), we fix $M_0 \subset M$ and $\Sigma_0 \subset \Sigma$ closed balls containing 0_M and 0_Σ respectively. We also set $\mathcal{V}_0 = \nabla_\xi \phi(M_0, \Sigma_0)$.

Definition 1.5 (ϕ-Kakeya set)**.** A compact set $K \subset M_0$ is a ϕ-Kakeya set if for each $\xi \in \Sigma_0$, there is a $v \in \mathcal{V}_0$ such that $\ell_{\xi,v} \cap M_0 \subset K$.

The $\phi_{n,\mathrm{rest}}$-Kakeya sets are standard Kakeya sets in $\mathbb{R}^n$, after harmlessly taking a union with $O(1)$ many rotated copies of itself. It is a classical consequence of Khintchine's inequality that ϕ-Kakeya sets have dimension n if (1.4) hold for $p > \frac{2n}{n-1}$ (one may follow the arguments in [Dem20, Proposition 5.7] to see this). Given Conjecture 1.3, we are led to the following conjecture for ϕ-Kakeya sets.

Conjecture 1.6 (Curved Kakeya conjecture for Bourgain's condition)**.** *If ϕ satisfies Bourgain's condition, then ϕ-Kakeya sets have Hausdorff dimension n.*

Remark 1.7. One hopes that Conjecture 1.6 holds for a stronger version of dimension known as a maximal function estimate (see [Dem20, Section 5]), and possibly for further refined incidence questions analogous to those for straight tubes in [WW24]. Such estimates would likely lead to direct progress on Conjecture 1.3. If ϕ fails Bourgain's condition, the corresponding maximal function version of the ϕ-Kakeya conjecture is false, as the proof of the negative result in [GWZ24] shows, but it is unclear whether it should hold for Hausdorff dimension, of course barring certain worst-case examples (see [GHI19] for a discussion of such examples). We do not pursue these questions in this work.

So far, the best partial progress toward Conjecture 1.6 is dimension $\geqslant 2 + \frac{1}{3}$ for $n = 3$ and dimension $\geqslant (2 - \sqrt{2})n$ along an infinite sequence of n. These are both a consequence of the result that ϕ-tubes satisfy strong polynomial Wolff axioms when ϕ satisfies Bourgain's condition, and a polynomial partitioning argument from [HRZ22] (see also the discussion in [DGGZ24, Section 2]). These estimates match the state of the art in high dimensions, but fall short in low dimensions, and notably in dimension 3 we don't even have an analogue of the classical bound of 5/2 due to Wolff [Wol95]. For more discussion of adapting Wolff's hairbrush argument to curves in a slightly different context, see [Nad25].

1.2. **Geometric characterization of Bourgain's condition.** We find a new purely geometric characterization of Bourgain's condition, which is transparently invariant with respect to diffeomorphisms in $\mathbf{x}$ and ξ separately. It also quickly leads to a reduction from a sticky version of Conjecture 1.6 in dimension n to the classical sticky Kakeya conjecture in dimension n.

Stated informally, ϕ satisfies Bourgain's condition if and only if for each $(\delta^{1/2}, \phi)$-tube T_0, the family of (δ, ϕ)-tubes contained in T_0 is diffeomorphic to a family of straight δ-tubes in a straight $\delta^{1/2}$-tube, while preserving the directions. Thus we found an explicit connection between standard Kakeya sets and ϕ-Kakeya sets when ϕ satisfies Bourgain's condition. Given a curve $\gamma \subset \mathbb{R}^n$, we write $\gamma + O(r)$ to denote the $O(r)$ neighborhood of γ below.

Theorem 1.8 (Geometric characterization of Bourgain's condition)**.** *A Hörmander-type phase function ϕ satisfies Bourgain's condition if and only if there exist smooth maps $F_{\xi_0,v_0} : M \to \mathbb{R}^n$, $\Xi_{\xi_0,v_0} : \Sigma \to \mathbb{R}^{n-1}$, and $V_{\xi_0,v_0} : \Sigma \times \mathcal{V} \to \mathbb{R}^{n-1}$ depending smoothly on $(\xi_0, v_0) \in \Sigma \times \mathcal{V}$, satisfying the following:*

- *The maps F_{ξ_0,v_0} and Ξ_{ξ_0,v_0} are local diffeomorphisms.*
- *Fix $(\xi_0, v_0) \in \Sigma \times \mathcal{V}$ and let $F = F_{\xi_0,v_0}$, $\Xi = \Xi_{\xi_0,v_0}$, and $V = V_{\xi_0,v_0}$. Then*
$$F(\ell_{\xi,v}) \subset \mathrm{line}_{\Xi(\xi),V(\xi,v)} + O(|(\xi, v) - (\xi_0, v_0)|^2) \tag{1.14}$$
on $\Sigma \times \mathcal{V}$.

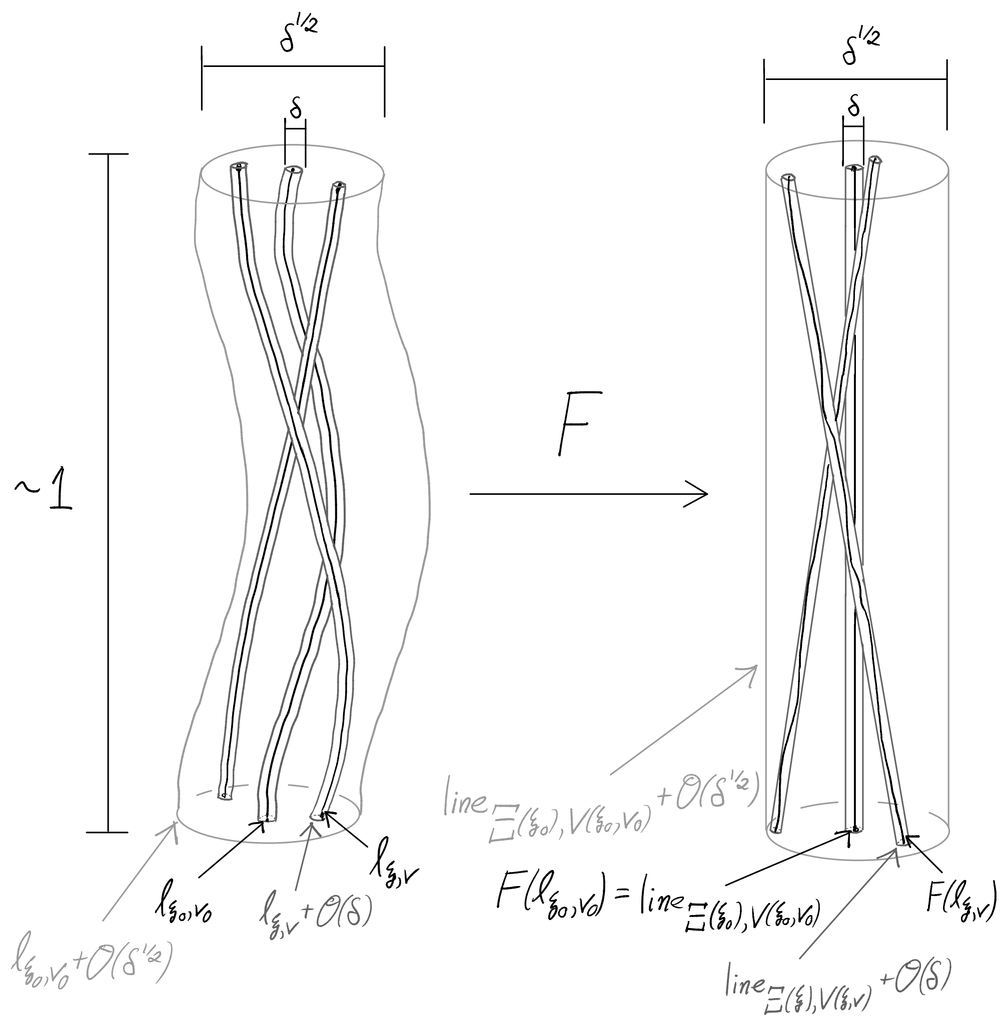

Figure 1.1. Cartoon of Theorem 1.8.

Figure 1.1 is a cartoon of Theorem 1.8. Theorem 1.8 also has the following consequence. If ϕ satisfies Bourgain's condition, then the tubular neighborhood of any ϕ-curve $\ell_{\xi,v}$ can be put into a normal form via $F = F_{\xi,v}$, in which the $2(n-1)$-parameter family of nearby ϕ-curves can be simultaneously approximately straightened. For a general phase function ϕ that doesn't satisfy Bourgain's condition, one can't expect that more than an $(n-1)$-parameter subfamily of ϕ-curves can be simultaneously straightened. For example, straightening the $(n-1)$-parameter subfamily of ϕ-curves in a fixed direction $\xi_0 \in \Sigma$ uses up essentially all the degrees of freedom of a diffeomorphism in $\mathbf{x}$, and so does straightening the $(n-1)$-parameter subfamily of ϕ-curves passing through a point $\mathbf{x}_0 \in M$.

We can see the diffeomorphism invariance in $\mathbf{x}$ and ξ of our new characterization of Bourgain's condition quickly. Let ϕ be a phase function satisfying Bourgain's condition with corresponding ϕ-curves $\ell_{\xi,v}$, and let $G : M' \to M$ and $H : \Sigma' \to \Sigma$ be diffeomorphisms. Define the transformed phase function $\tilde{\phi} : M' \times \Sigma' \to \mathbb{R}$ by $\tilde{\phi}(\mathbf{x}, \xi) = \phi(G(\mathbf{x}), H(\xi))$. Then the $\tilde{\phi}$-curves $\tilde{\ell}_{\xi,v}$ transform like

$$\tilde{\ell}_{\xi,v} = G^{-1}(\ell_{H(\xi),\nabla H(\xi)v}). \tag{1.15}$$

Thus (1.14) holds with F_{ξ_0,v_0} replaced by $F_{\xi_0,v_0} \circ G$, Ξ_{ξ_0,v_0} replaced by $\Xi_{\xi_0,v_0} \circ H$, and V_{ξ_0,v_0} replaced by $(\xi, v) \mapsto V_{\xi_0,v_0}(H(\xi), \nabla H(\xi)v)$. We will introduce examples which show that Theorem 1.8 cannot be strengthened to have an error $O(|(\xi, v)-(\xi_0, v_0)|^4)$.

Remark 1.9. The diffeomorphisms Ξ_{ξ_0,v_0} in ξ are essential in Theorem 1.8. Bourgain found the example

$$\phi_{\text{worst}}(\mathbf{x}, \xi) = x_1\xi_1 + x_2\xi_2 + t\xi_1\xi_2 + \tfrac{1}{2}t^2\xi_2^2, \quad (0_M, 0_\Sigma) = (0, 0) \in \mathbb{R}^3 \times \mathbb{R}^2, \tag{1.16}$$

which satisfies Hörmander's conditions, fails Bourgain's condition, and in fact gives the worst case bounds for Hörmander's problem [Bou91] (see [GHI19, HI22] for a study of this phenomenon). The ϕ_{worst}-curves are

$$\ell_{\xi,v} = \{(v_1 - t\xi_2, v_2 - t\xi_1 - t^2\xi_2, t) : |t| \leqslant 1\}. \tag{1.17}$$

The diffeomorphism $F(\mathbf{x}) = (x_1, x_2 - tx_1, t)$ takes the curves $\ell_{\xi,v}$ to lines,

$$F(\ell_{\xi,v}) \subset \text{line}_{(\xi_2,\xi_1+v_1),v}, \tag{1.18}$$

but crucially $(\xi, v) \mapsto (\xi_2, \xi_1 + v_1)$ is *not* a diffeomorphism in ξ alone.

The following mild reformulation of Bourgain's condition is important in proving Theorem 1.8. It will also be crucial in constructing Example 1.20 which we will introduce shortly.

Proposition 1.10 (Mild reformulation of Bourgain's condition)**.** *Let $\phi : M \times \Sigma \to \mathbb{R}$ be a Hörmander-type phase function satisfying Bourgain's condition. Then there exist a smooth scalar valued function $c : M \times \Sigma \to \mathbb{R}$ and two smooth symmetric-matrix-valued functions $A, B : \mathcal{V} \times \Sigma \to \mathrm{Sym}_{n-1}(\mathbb{R})$ such that*

$$\nabla_\xi^2\phi(\mathbf{x}, \xi) = A(\nabla_\xi\phi(\mathbf{x}, \xi), \xi) + c(\mathbf{x}, \xi)B(\nabla_\xi\phi(\mathbf{x}, \xi), \xi) \tag{1.19}$$

on $M \times \Sigma$. Additionally $B(v, \xi)$ is a non-degenerate matrix for each $(v, \xi) \in \mathcal{V} \times \Sigma$ and $(G(\mathbf{x}, \xi) \cdot \nabla_\mathbf{x})c(\mathbf{x}, \xi) \neq 0$ on $M \times \Sigma$.

Conversely, suppose that $\phi : M \times \Sigma \to \mathbb{R}$ is a smooth function satisfying (H1). If there are smooth $c : M \times \Sigma \to \mathbb{R}$, $A : \mathcal{V} \times \Sigma \to \mathrm{Sym}_{n-1}(\mathbb{R})$, and $B : \mathcal{V} \times \Sigma \to \mathrm{Sym}_{n-1}(\mathbb{R})$ such that

- *$B(v, \xi)$ is nondegenerate (resp. positive-definite), and*
- *$(G(\mathbf{x}, \xi) \cdot \nabla_{\mathbf{x}})c(\mathbf{x}, \xi) \neq 0$,*

then ϕ satisfies (H2) (resp. (H2+)) and Bourgain's condition.

Example 1.11 (Proposition 1.10 for the restriction and Bochner–Riesz examples)**.** The phase functions $\phi_{n,\mathrm{rest}}$ and $\phi_{n,\mathrm{BR}}$ can be written in the form (1.19) by making the following choices of A, B, c:

- $\phi_{n,\mathrm{rest}}$:

$$A(v, \xi) = 0, \tag{1.20}$$

$$B(v, \xi) = I_{n-1}, \tag{1.21}$$

$$c(\mathbf{x}, \xi) = t. \tag{1.22}$$

- $\phi_{n,\mathrm{BR}}$:

$$A(v, \xi) = 0, \tag{1.23}$$

$$B_{ij}(v, \xi) = \begin{cases} 1 - v_i^2, & i = j \\ -v_i v_j, & i \neq j \end{cases}, \tag{1.24}$$

$$c(\mathbf{x}, \xi) = \frac{t}{\sqrt{1 + |x - t\xi|^2}}. \tag{1.25}$$

1.3. **Sticky ϕ-Kakeya sets, and a reduction to classical sticky Kakeya sets under Bourgain's condition.** It was demonstrated in the series of landmark papers of Wang–Zahl [WZ22, WZ25a, WZ25b] that the sticky Kakeya conjecture is true in $\mathbb{R}^3$, and that the Kakeya conjecture in $\mathbb{R}^3$ can be reduced to this sticky case. It is expected that the sticky Kakeya conjecture is true in higher dimensions, and that a general to sticky reduction is plausible in higher dimensions (see the discussion in [Gut25a, Section 9]). Roughly speaking, a Kakeya set is sticky if it has a self-similar structure. This idea can be captured in different ways. In [WZ22], a Kakeya set $K \subset \mathbb{R}^n$ is said to be sticky if its set of (direction separated) lines has packing dimension $n - 1$. However the general to sticky reduction in [WZ25b] required one to study sets of tubes which are not necessarily direction separated, but instead only satisfy the *convex Wolff axioms*. Correspondingly, a version of the sticky Kakeya conjecture in $\mathbb{R}^3$ involving the convex Wolff axioms was needed. This required a mild adjustment to the arguments in [WZ22], which appeared in [WZ25a].

It seems plausible that the sticky case is also the enemy case for Conjecture 1.6. We choose to formalize stickiness for ϕ-Kakeya sets by generalizing the definition in [WZ22], since it is more straightforward to write in general dimension. At least in $\mathbb{R}^3$, a suitable *polynomial Wolff axiom* version, analogous to the *convex Wolff axiom* version appearing in [WZ25b] can be adapted, along with a corresponding version of Theorem 1.19 below. We very briefly discuss this in Section 1.6. To measure how closely ϕ-curves pack together, we equip $\mathcal{C}(\phi)$ with the metric

$$d(\ell_{\xi,v}, \ell_{\xi',v'}) = |\xi - \xi'| + |v - v'|. \tag{1.26}$$

In the special case of $\phi = \phi_{n,\mathrm{rest}}$, this agrees with the metric on lines used in [WZ22] up to bi-Lipschitz equivalence. We are led to the following generalization of classical sticky Kakeya sets from [WZ22] to sticky ϕ-Kakeya sets.

Definition 1.12 (Sticky ϕ-Kakeya set)**.** A compact set $K \subset M_0$ is called a sticky ϕ-Kakeya set if there is a set of ϕ-curves $L \subset \mathcal{C}(\phi)$ with packing dimension $n - 1$

that contains at least one curve for each direction $\xi \in \Sigma_0$, so that $\ell \cap M_0 \subset K$ for each $\ell \in L$. More generally, a ϕ-Kakeya set is η-close to being sticky if the packing dimension of L is at most $n-1+\eta$.

A classical sticky Kakeya set is then a sticky $\phi_{n,\mathrm{rest}}$-Kakeya set. We are led to the sticky case of Conjecture 1.6.

Conjecture 1.13 (Sticky curved Kakeya conjecture for Bourgain's condition)**.** *If ϕ satisfies Bourgain's condition, then sticky ϕ-Kakeya sets in $\mathbb{R}^n$ have Hausdorff dimension n.*

By leveraging Theorem 1.8 and a simple induction on scales argument, we reduce Conjecture 1.13 in dimension n to the classical sticky Kakeya conjecture in dimension n.

Theorem 1.14 ((Informal) Classical sticky Kakeya implies curved sticky Kakeya for Bourgain's condition)**.** *If classical sticky Kakeya sets in $\mathbb{R}^n$ have dimension n, then sticky ϕ-Kakeya sets in $\mathbb{R}^n$ have dimension n when ϕ satisfies Bourgain's condition. In particular, Conjecture 1.13 is true when $n = 3$.*

To make the implication in Theorem 1.14 formal, we need to introduce a discretization scheme analogous to [WZ22]. We first introduce some notation. Given a (δ,ϕ)-tube $T = T^{\delta}_{\xi,v}$, define $T^* = (\xi, v)$ which is the parameter of the central curve. Define $\mathrm{dir}(T) = \xi$. A shading of T is a measurable subset $Y(T) \subset T$. Write $(\mathbb{T}, Y)_\delta$ for a collection of (δ,ϕ)-tubes and their associated shadings. We say two (δ,ϕ)-tubes $T_1 = T_{\xi_1,v_1}$ and $T_2 = T_{\xi_2,v_2}$ are *essentially distinct* if $|(\xi_1, v_1) - (\xi_2, v_2)| \geqslant \delta$. We say T_1 and T_2 are *essentially parallel* if $|\mathrm{dir}(T_1) - \mathrm{dir}(T_2)| \leqslant \delta$. A collection of (ρ,ϕ)-tubes $\mathbb{T}_\rho$ *covers* a collection of (δ,ϕ)-tubes $\mathbb{T}_\delta$ if the balls $\{B_\rho(T^*) : T \in \mathbb{T}_\rho\}$ cover the balls $\{B_\delta(T^*) : T \in \mathbb{T}_\delta\}$.

We may now define the discretized version of sticky ϕ-Kakeya sets having dimension n.

Definition 1.15 (Assertion $\mathrm{SK}(\phi, \varepsilon, \eta, \delta_0)$)**.** Fix a Hörmander-type phase function ϕ. We say that $\mathrm{SK}(\phi, \varepsilon, \eta, \delta_0)$ holds if for all $\delta \in (0, \delta_0)$ and all pairs $(\mathbb{T}, Y)_\delta$ satisfying the properties

(a) the (δ,ϕ)-tubes in $\mathbb{T}$ are essentially distinct,
(b) for each $\delta \leqslant \rho \leqslant 1$, $\mathbb{T}$ can be covered by a set of (ρ,ϕ)-tubes, at most $\delta^{-\eta}$ of which are pairwise essentially parallel, and
(c) $\sum_{T\in\mathbb{T}} |Y(T)| \geqslant \delta^{\eta}$,

one has

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \geqslant \delta^{\varepsilon}. \tag{1.27}$$

Using the language of Definition 1.15, the discretized version of the classical sticky Kakeya conjecture in dimension n can be stated as follows.

Conjecture 1.16 (Discretized classical sticky Kakeya conjecture in $\mathbb{R}^n$)**.** *For every $\varepsilon > 0$ there exists $\eta = \eta_{n,\mathrm{rest}}(\varepsilon) > 0$ and $\delta_0 = \delta_{0,n,\mathrm{rest}}(\varepsilon) > 0$ such that $\mathrm{SK}(\phi_{n,\mathrm{rest}}, \varepsilon, \eta, \delta_0)$ holds.*

The main result of [WZ22] is then the following discretized statement.

Theorem 1.17 (Discretized classical sticky Kakeya in $\mathbb{R}^3$ [WZ22])**.** *Conjecture 1.16 is true in the case* $n = 3$*.*

Analogously to [WZ22], the discretized statement $\mathrm{SK}(\phi, \varepsilon, \eta, \delta_0)$ implies the following Hausdorff dimension statement. Below $\dim_{\mathrm{H}}$ denotes Hausdorff dimension.

Proposition 1.18 (Discretized to dimension statement)**.** *Suppose that for every* $\varepsilon' > 0$ *there exists* $\eta', \delta_0 > 0$ *such that* $\mathrm{SK}(\phi, \varepsilon', \eta', \delta_0)$ *holds. Then for all* $\varepsilon > 0$*, there is* $\eta > 0$ *so that the following holds. Let* K *be a* ϕ*-Kakeya set that is* η*-close to being sticky. Then* $\dim_{\mathrm{H}} K \geqslant n - \varepsilon$*. In particular, if* K *is a sticky* ϕ*-Kakeya set then* $\dim_{\mathrm{H}} K = n$*.*

We now state the formal version of Theorem 1.14.

Theorem 1.19 ((Formal) Classical sticky Kakeya implies curved sticky Kakeya for Bourgain's condition)**.** *Let* $n \geqslant 3$ *and suppose that* ϕ *is a phase function in* $\mathbb{R}^n$ *satisfying Bourgain's condition. Then*

$$(\forall \varepsilon > 0 \exists \eta, \delta_0 > 0 : \mathrm{SK}(\phi_{n,\mathrm{rest}}, \varepsilon, \eta, \delta_0)) \implies (\forall \varepsilon > 0 \exists \eta, \delta_0 > 0 : \mathrm{SK}(\phi, \varepsilon, \eta, \delta_0)). \tag{1.28}$$

Combining (1.28) *with Theorem 1.17 and Proposition 1.18, we get that Conjecture 1.13 is true for* $n = 3$*.*

Below we describe some sharpness examples for Theorem 1.19, which also demonstrate that a general to sticky reduction in the spirit of Wang–Zahl requires new ideas to attack Conjecture 1.6 in the non-sticky case.

1.4. **Sharpness examples for Theorem 1.8 and barriers to a general to sticky reduction.** In each dimension $n \geqslant 3$ we give a phase function ϕ satisfying Bourgain's condition, for which there does not exist a diffeomorphism taking the family of ϕ-curves to lines. This answers a question of Gao–Liu–Xi [GLX25] in the negative, and we first announced (but did not prove anything about) this example in the case $n = 3$ in [Nad25].

Example 1.20 (The tan-example in $\mathbb{R}^n$)**.** Fix $n \geqslant 3$ and write $\xi = (\xi', \xi_{n-1}) \in \mathbb{R}^{n-2} \times \mathbb{R}$, $\mathbf{x} = (x', x_{n-1}, t) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$. The tan-example in $\mathbb{R}^n$ is the phase function

$$\phi_{n,\tan}(x, \xi) = x' \cdot \xi' + \tfrac{1}{2} t^2 |\xi'|^2 + \log(\sec(t\xi_{n-1} + x_{n-1})), \quad (0_M, 0_\Sigma) = ((0, 0, 1), (0, 0)). \tag{1.29}$$

This example is a positive-definite Hörmander-type phase function satisfying Bourgain's condition. The corresponding $\phi_{n,\tan}$-curves are

$$\ell_{\xi,v} = \{(v' - t^2\xi', \tan^{-1}(\frac{v_{n-1}}{t}) - t\xi_{n-1}, t) : |t - 1| \leqslant 1/10\}. \tag{1.30}$$

We call $\phi_{n,\tan}$ the tan-example since the family of curves involves the function tan. In Section 6, we will construct this example using Proposition 1.10. We then prove that there is no diffeomorphism from the $(\delta, \phi_{n,\tan})$-tubes in a fixed $(\delta^{1/4}, \phi_{n,\tan})$-tube to straight δ-tubes in a straight $\delta^{1/4}$-tube. The tan-example appears to be the simplest to write example satisfying Bourgain's condition with this property.

Proposition 1.21 (tan-example is curved)**.** *For any fixed $\varepsilon_0 > 0$, there does not exist a diffeomorphism $F : B_{\varepsilon_0}(0_M) \to \mathbb{R}^n$ onto its image and smooth $\Xi, V : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that*

$$F(\ell_{\xi,v} \cap B_{\varepsilon_0}(0_M)) \subset \mathrm{line}_{\Xi(\xi,v),V(\xi,v)} + O(|(\xi,v)|^4) \tag{1.31}$$

for (ξ, v) near $(0,0)$. In particular, the family of $\phi_{n,\tan}$-curves is not diffeomorphic to a family of lines.

We will prove Proposition 1.21 by showing that the union of curves through $\ell_{0,0}$ and a point $\mathbf{p} \notin \ell_{0,0}$ is not contained in a surface, while of course a union of lines passing through a fixed line and point is contained in a plane. In [Nad25] this type of behavior is known as *coniness*, where it was exploited to give positive results for Kakeya sets of curves in a slightly different context. Figure 1.2 is a cartoon of this failure of (1.31) for $\phi_{n,\tan}$.

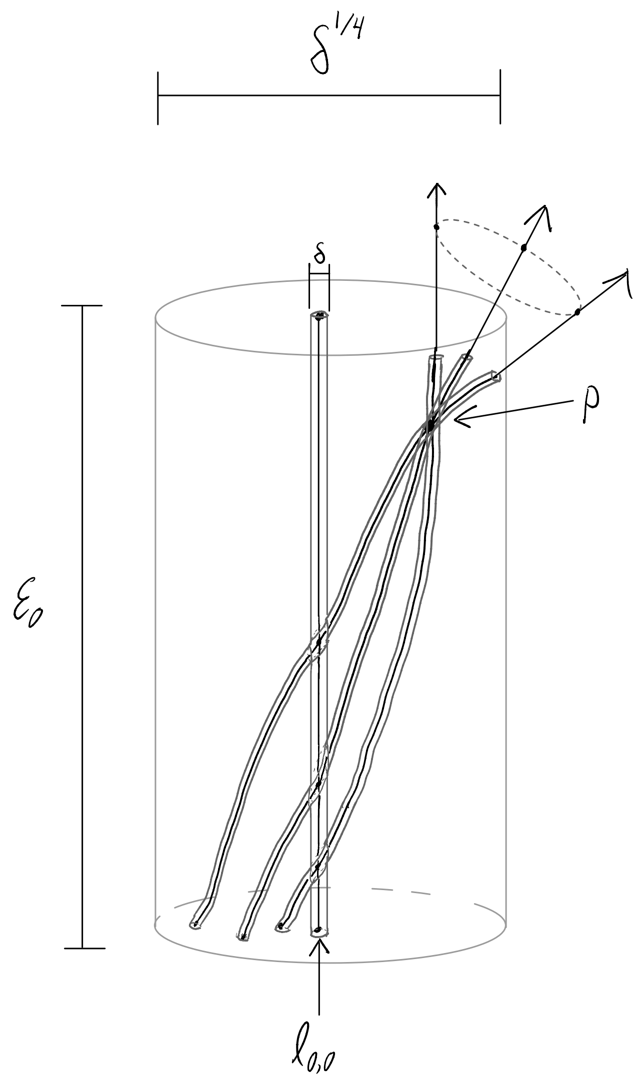

FIGURE 1.2. Cartoon of Proposition 1.21.

Proposition 1.21 shows that the error in Theorem 1.8 cannot be strengthened to $O(|(\xi,v)-(\xi_0,v_0)|^4)$, and we suspect that a version with error $O(|(\xi,v)-(\xi_0,v_0)|^3)$ is also impossible. One should contrast Example 1.20 with translation-invariant phase functions and phase functions arising from Carleson–Sjölin operators on manifolds. If the phase function ϕ in either of those cases satisfies Bourgain's condition, then $\mathcal{C}(\phi)$ is diffeomorphic to a family of lines [DGGZ24, GLX25]. Not only is $\mathcal{C}(\phi_{n,\tan})$ not diffeomorphic to a family of lines, which we believe is the first example satisfying Bourgain's condition with this property, it is not even diffeomorphic to a family of lines between scales δ and $\delta^{1/4}$.

Numerous arguments in the study of Kakeya sets ultimately use the property that lines may be efficiently organized into planes. For example, Wolff's hairbrush argument uses that the δ-tubes through a fixed stem can be organized into essentially disjoint slabs [Wol95]. The argument of Wang–Zahl uses that δ-tubes can be organized into planks [WZ25b]. At least outside a fixed $(\delta^{1/4},\phi_{n,\tan})$-tube, the $(\delta,\phi_{n,\tan})$-tubes do not have properties analogous to these. Thus to go beyond the sticky case of Conjecture 1.6, new ideas are needed.

1.5. **Organization of article.**

- In Section 2 we prove Proposition 1.10.
- In Section 3 we prove Theorem 1.8 using Proposition 1.10.
- In Section 4 we sketch how to discretize the sticky ϕ-Kakeya problem in a similar way to [WZ22, Section 2.3], and sketch the proof of Proposition 1.18.
- In Section 5 we prove Theorem 1.19.
- In Section 6 we construct Example 1.20 using Proposition 1.10, and prove Proposition 1.21.

1.6. **Discussion and future work.** Having proved the sticky case of Conjecture 1.6 in dimension 3, a natural next step is to attempt a general to sticky reduction in dimension 3 in the spirit of Wang–Zahl. As mentioned above, in light of the tan-example, this will require new ideas. The most promising route seems to be to replace the Katz-Tao and Frostman non-concentration conditions in convex sets [WZ25b], with the analogous non-concentration conditions in semi-algebraic sets with bounded complexity. Following the outline in [Gut25b], a collection of (δ,ϕ)-tubes is then sticky if it satisfies the Katz–Tao *polynomial* Wolff axioms along a sequence of nearby scales. The version of Conjecture 1.13 in dimension 3 with this notion of stickiness continues to be true by running an induction on scales argument similar to the proof of Theorem 1.19. Though one must adjust the inductive hypothesis accordingly and replace the application of Theorem 1.17 from [WZ22] with [WZ25b, Theorem 10.2]. The general to sticky reduction for Conjecture 1.6 will then require a better understanding of the grains that can arise from (δ,ϕ)-tubes. We are currently trying to understand these new phenomena for the tan-example.

By following the proof of Theorem 1.19, even when ϕ is a phase function failing Bourgain's condition, the study of sticky ϕ-Kakeya sets can be reduced to the study of sticky (δ,ϕ)-tubes in a $(\delta^{1/2},\phi)$-tube. By Taylor expanding, we need to understand sticky Kakeya sets of certain *semi-linearized* families of curves of the form

$$\ell_{\xi,v}=\{(v+A(t)\xi,t):|t|\leqslant 1\}, \tag{1.32}$$

where $A(t) = tI_{n-1} + O(t^2)$. Theorem 1.8 shows that the structure of the semi-linearized families can completely describe Bourgain's condition. It would be interesting to explore whether the semi-linearized families for other phase functions might suggest a more refined classification for Hörmander's oscillatory integral problem. Some work in this direction based instead on the polynomial method and *contact order* appears in [DGGZ24].

As part of an in-progress collaboration, we build off Proposition 1.10 to find more global information about Bourgain's condition, and study the genericity of examples like the tan-example.

1.7. **Thanks.** The author would like to thank Terence Tao for many helpful conversations about this project, and especially for suggesting the argument for Proposition 1.10. The author would also like to thank Hong Wang, Shaoming Guo, and Siddharth Mulherkar for their encouragement and helpful conversations.

2. Proof of Proposition 1.10

Recall the Gauss map $G(\mathbf{x},\xi) = \bigwedge_{j=1}^{n-1} \partial_{\xi_j} \nabla_{\mathbf{x}} \phi(\mathbf{x},\xi)$. We will make use of the fact that

$$(G(\mathbf{x},\xi) \cdot \nabla_{\mathbf{x}}) \nabla_{\xi} \phi(\mathbf{x},\xi) = 0 \tag{2.1}$$

below.

2.1. **Forward implication.**

Proof. Assume that ϕ satifies Bourgain's condition and fix $\xi_0 \in \Sigma$. Since $G(\cdot,\xi_0)$ is a nonzero vector field on M, there are local coordinates, depending smoothly on the choice of ξ_0, for which $G(\mathbf{x},\xi_0) = e_n$. In the new coordinates (2.1) becomes $\partial_{x_n} \nabla_{\xi} \phi(\mathbf{x},\xi_0) = 0$. Thus $\nabla_{\xi} \phi(\mathbf{x},\xi_0)$ depends only on $x = (x_1,\dots,x_{n-1})$, and by (H1) is therefore diffeomorphic to x (depending smoothly on the choice of ξ_0).

In the new coordinates (1.9) becomes

$$\partial_{x_n}^2 \nabla_{\xi}^2 \phi(\mathbf{x},\xi_0) = \lambda(\mathbf{x},\xi_0) \partial_{x_n} \nabla_{\xi}^2 \phi(\mathbf{x},\xi_0). \tag{2.2}$$

We may solve this differential equation in x_n to find smooth symmetric-matrix-valued functions $\tilde{A},\tilde{B} : \mathcal{V} \times \Sigma \to \mathrm{Sym}_{n-1}(\mathbb{R})$, and a smooth function $\tilde{c} : M \times \Sigma \to \mathbb{R}$ such that

$$\nabla_{\xi}^2 \phi(\mathbf{x},\xi_0) = \tilde{A}(x,\xi_0) + \tilde{c}(\mathbf{x},\xi_0) B(x,\xi_0). \tag{2.3}$$

By using that $\nabla_{\xi} \phi(\mathbf{x},\xi)$ is diffeomorphic to x and then undoing the change of variables in $\mathbf{x}$, we obtain A, B, c satisfying

$$\nabla_{\xi}^2 \phi(\mathbf{x},\xi_0) = A(\nabla_{\xi} \phi(\mathbf{x},\xi_0),\xi_0) + c(\mathbf{x},\xi_0) B(\nabla_{\xi} \phi(\mathbf{x},\xi_0),\xi_0). \tag{2.4}$$

Since the change of variables in $\mathbf{x}$ depends smoothly on the choice of ξ_0, the functions $A(v,\xi_0), B(v,\xi_0), c(\mathbf{x},\xi_0)$ are also smooth in ξ_0.

By applying the vector field $G(\mathbf{x},\xi_0) \cdot \nabla_{\mathbf{x}}$ to (2.4), we see from (H2) that $(G(\mathbf{x},\xi_0) \cdot \nabla_{\mathbf{x}}) c(\mathbf{x},\xi_0) \neq 0$ and $B(\nabla_{\xi} \phi(\mathbf{x},\xi_0),\xi_0)$ is nondegenerate. When (H2+) is satisfied, $B(\nabla_{\xi} \phi(\mathbf{x},\xi_0),\xi_0)$ is furthermore positive-definite. $\square$

2.2. **Reverse implication.**

Proof. Suppose that we have data A, B, c such that (1.19) holds. Applying the operator $(G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})$ to both sides of (1.19) and using (2.1),

$$(G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})\nabla_\xi^2\phi(\mathbf{x},\xi) = (G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})c(\mathbf{x},\xi)\cdot B(\nabla_\xi\phi(\mathbf{x},\xi),\xi). \tag{2.5}$$

If $(G(\mathbf{x},\xi)\cdot\nabla_{\mathbf{x}})c(\mathbf{x},\xi)\neq 0$ and B is nondegenerate (resp. positive-definite), then (H2) (resp. (H2+)) holds. Applying the operator $G\cdot\nabla_{\mathbf{x}}$ once again,

$$(G\cdot\nabla_{\mathbf{x}})^2\nabla_\xi^2\phi = (G\cdot\nabla_{\mathbf{x}})^2 c\cdot B(\nabla_\xi\phi,\xi). \tag{2.6}$$

By combining (2.5) and (2.6), we find

$$(G\cdot\nabla_{\mathbf{x}})^2\nabla_\xi^2\phi = \frac{(G\cdot\nabla_{\mathbf{x}})^2 c}{(G\cdot\nabla_{\mathbf{x}})c}\cdot(G\cdot\nabla_x)\nabla_\xi^2\phi. \tag{2.7}$$

Therefore Bourgain's condition in the sense of Definition 1.2 holds with $\lambda = \frac{(G\cdot\nabla_{\mathbf{x}})^2 c}{(G\cdot\nabla_{\mathbf{x}})c}$. □

3. Proof of Theorem 1.8

We will need to choose a parameterization of the ϕ-curves. By a diffeomorphism in $\mathbf{x}$, we may assume that the ϕ-curves are transverse to the t-slices $\{(x,t) : x\in\mathbb{R}^{n-1}\}$. That is to say $\det(\nabla_x\nabla_\xi\phi)\neq 0$. We may further assume that $0_M = 0\in\mathbb{R}^n$ and by a constant rescaling, the ϕ-curves are defined for $t\in[-1,1]$. Finally by a translation in ξ and after replacing ϕ by $\phi(\mathbf{x},\xi)-\phi(0,\xi)$, we may assume that $0_\Sigma = 0\in\mathbb{R}^{n-1}$ and $\nabla_\xi\phi(0,0)=0$. Let $X(\xi,v,t)$ be the unique solution in x to

$$\nabla_\xi\phi(x,t,\xi) = v. \tag{3.1}$$

The curves $\ell_{\xi,v}$ are thus given by

$$\ell_{\xi,v} = \{(X(\xi,v,t),t) : |t|\leqslant 1\}, \tag{3.2}$$

and we will write $\ell_{\xi,v}(t) = (X(\xi,v,t),t)$. We record the following implicit derivative calculation of X for future use:

$$\nabla_\xi X(\xi,v,t) = -\nabla_v X(\xi,v,t)\nabla_\xi^2\phi(X(\xi,v,t),t,\xi). \tag{3.3}$$

3.1. **Forward implication.**

Proof. Let ϕ be a Hörmander-type phase function satisfying Bourgain's condition. By Proposition 1.10 there are A, B, c with B nondegenerate and $(G\cdot\nabla_{\mathbf{x}})c\neq 0$, such that (1.19) holds. Fix $(\xi_0,v_0)\in\Sigma\times\mathcal{V}$. We will find the data $F = F_{\xi_0,v_0}, \Xi = \Xi_{\xi_0,v_0}, V = V_{\xi_0,v_0}$ such that (1.14) holds. We may Taylor-expand X to first order in (ξ,v) near (ξ_0,v_0) to get

$$X(\xi,v,t) = X(\xi_0,v_0,t) + \nabla_\xi X(\xi_0,v_0,t)(\xi-\xi_0) + \nabla_v X(\xi_0,v_0,t)(v-v_0) + O, \tag{3.4}$$

where $O := O(|(\xi,v)-(\xi_0,v_0)|^2)$. We now relate the terms $\nabla_\xi X(\xi_0,v_0,t)$ and $\nabla_v X(\xi_0,v_0,t)$ to the data A, B, c. Define

$$\Xi(\xi) := B(v_0,\xi_0)(\xi-\xi_0), \tag{3.5}$$

$$V(\xi,v) := (v-v_0) - A(v_0,\xi_0)(\xi-\xi_0). \tag{3.6}$$

The map Ξ is a diffeomorphism since $B(v_0, \xi_0)$ is nondegenerate. Then by (3.3) and (1.19),

$$X(\xi, v, t) = X(\xi_0, v_0, t) + \nabla_v X(\xi_0, v_0, t)(V(\xi, v) - c(X(\xi_0, v_0, t), t)\Xi(\xi)) + O. \quad (3.7)$$

We will successively apply three diffeomorphisms to straighten the family $\ell_{\xi,v}$ into a family of lines, up to error $O(|(\xi, v) - (\xi_0, v_0)|^2)$. The first is a recentering, the second is a twist in each t-slice, and the third is a diffeomorphism in the t-component.

Step 1: Recentering.

Define the diffeomorphism $F_1(x, t) = (x - X(\xi_0, v_0, t), t)$. Then

$$\ell^1_{\xi,v} := F_1(\ell_{\xi,v}) \quad (3.8)$$
$$= \{(\nabla_v X(\xi_0, v_0, t)(V(\xi, v) - c(X(\xi_0, v_0, t), t)\Xi(\xi)) + O, t) : |t| \leqslant 1\}. \quad (3.9)$$

Step 2: Twist in each t-slice.

Define the diffeomorphism $F_2(x, t) = (\nabla_v X(\xi_0, v_0, t)^{-1}x, t)$. Then

$$\ell^2_{\xi,v} := F_2(\ell^1_{\xi,v}) \quad (3.10)$$
$$= \{(V(\xi, v) - c(X(\xi_0, v_0, t), t)\Xi(\xi) + O, t) : |t| \leqslant 1\}. \quad (3.11)$$

Step 3: Diffeomorphism in the t-component.

Define $\tilde{c}(t) = c(X(\xi_0, v_0, t), t, \xi_0)$. Since $t \mapsto (X(\xi_0, v_0, t), t)$ is an integral curve of the vector field $G(\cdot, \xi_0)$, $\frac{d}{dt}\tilde{c}(t)$ is a nonzero multiple of $(G \cdot \nabla_{\mathbf{x}})c$. Hence $\frac{d}{dt}\tilde{c}(t) \neq 0$. After a constant rescaling, we may assume that $\tilde{c}$ is a diffeomorphism when restricted to $[-1, 1]$. Thus the map $F_3(x, t) = (x, \tilde{c}(t))$ is a diffeomorphism. We compute

$$\ell^3_{\xi,v} := F_3(\ell^2_{\xi,v}) \quad (3.12)$$
$$= \{(V(\xi, v) - \tilde{c}(t)\Xi(\xi), \tilde{c}(t)) : |t| \leqslant 1\} \quad (3.13)$$
$$= \{(V(\xi, v) - s\Xi(\xi) + O, s) : |s| \lesssim 1\} \quad (3.14)$$
$$\subset \mathrm{line}_{\Xi(\xi),V(\xi,v)} + O. \quad (3.15)$$

Collecting the steps above, we can take

$$F_{\xi_0,v_0}(x, t) = (\nabla_v X(\xi_0, v_0, t)^{-1}(x - X(\xi_0, v_0, t)), c(X(\xi_0, v_0, t), t, \xi)), \quad (3.16)$$
$$\Xi_{\xi_0,v_0}(\xi) = B(v_0, \xi_0)(\xi - \xi_0), \quad (3.17)$$
$$V_{\xi_0,v_0}(\xi, v) = (v - v_0) - A(v_0, \xi_0)(\xi - \xi_0), \quad (3.18)$$

to conclude the forward direction of Theorem 1.8. □

3.2. **Reverse implication.**

Proof. Fix $(\xi_0, v_0) \in \Sigma \times \mathcal{V}$ and consider the data $F = F_{\xi_0,v_0}$, $\Xi = \Xi_{\xi_0,v_0}$, and $V = V_{\xi_0,v_0}$ such that (1.14) holds. Write $F(x, t) = (H(x, t), h(x, t)) \in \mathbb{R}^{n-1} \times \mathbb{R}$. At the level of parameterizations, (1.14) says that

$$H(X(\xi, v, t), t) = V(\xi, v) - h(X(\xi, v, t), t)\Xi(\xi) + O, \quad (3.19)$$

for a function $O(\xi, v, t) = O(|(\xi, v) - (\xi_0, v_0|^2)$. When we encounter more errors of this form, we will absorb them into O. We may replace $H(x, t)$ with $H(x, t) - (V(\xi_0, v_0) - h(X(\xi_0, v_0, t), t)$, $\Xi(\xi)$ with $\Xi(\xi) - \Xi(\xi_0)$, and $V(\xi, v)$ with $V(\xi, v) - V(\xi_0, v_0)$. This allows us to assume $H(X(\xi_0, v_0, t), t) = 0$, $\Xi(\xi_0) = 0$, and $V(\xi_0, v_0) = 0$. Compute the following Taylor expansions:

$$X(\xi, v, t) - X(\xi_0, v_0, t) = \nabla_\xi X(\xi_0, v_0, t)(\xi - \xi_0) + \nabla_v X(\xi_0, v_0, t)(v - v_0) + O, \tag{3.20}$$

$$H(x, t) = H_0(t)(x - X(\xi_0, v_0, t)) + O((|x - X(\xi_0, v_0, t)|^2), \tag{3.21}$$

$$h(x, t) = h_0(t) + O(|x - X(\xi_0, v_0, t)|), \tag{3.22}$$

$$V(\xi, v) = V_0(\xi - \xi_0) + V_1(v - v_0) + O, \tag{3.23}$$

$$\Xi(\xi, v) = \Xi_0(\xi - \xi_0) + O, \tag{3.24}$$

where $H_0(t) = \nabla_x H(X(\xi_0, v_0, t), t) : \mathbb{R} \to \mathbb{R}^{(n-1)\times(n-1)}$, $h_0(t) = h(X(\xi_0, v_0, t), t) : \mathbb{R} \to \mathbb{R}^{n-1}$, and $V_0 = \nabla_\xi V(\xi_0, v_0), V_1 = \nabla_v V(\xi_0, v_0), \Xi_0 = \nabla_\xi \Xi(\xi_0) \in \mathbb{R}^{(n-1)\times(n-1)}$. The equation (3.19) then reads

$$\begin{aligned} &H_0(t)(\nabla_\xi X(\xi_0, v_0, t)(\xi - \xi_0) + \nabla_v X(\xi_0, v_0, t)(v - v_0)) \\ &\quad = V_0(\xi - \xi_0) + V_1(v - v_0) + h_0(t)\Xi_0\xi + O. \end{aligned} \tag{3.25}$$

Grouping terms by $\xi - \xi_0$ and $v - v_0$ in 3.25,

$$\begin{aligned} &(H_0(t)\nabla_\xi X(\xi_0, v_0, t) - V_0 - h_0(t)\Xi_0)(\xi - \xi_0) \\ &+ (H_0(t)\nabla_v X(\xi_0, v_0, t) - V_1)(v - v_0) = O. \end{aligned} \tag{3.26}$$

By taking a ξ derivative of (3.26) and setting $\xi = \xi_0$, and similarly taking a derivative of (3.26) and setting $v = v_0$, we find

$$H_0(t)\nabla_\xi X(\xi_0, v_0, t) - V_0 - h_0(t)\Xi_0 = 0, \tag{3.27}$$

$$H_0(t)\nabla_v X(\xi_0, v_0, t) - V_1 = 0. \tag{3.28}$$

By (3.28), $H_0(t) = V_1 \nabla_v X(\xi_0, v_0, t)^{-1}$. Plugging this into (3.27) and rearranging, we find

$$\nabla_\xi X(\xi_0, v_0, t) = \nabla_v X(\xi_0, v_0, t)(V_1^{-1}V_0 + h_0(t)V_1^{-1}\Xi_0). \tag{3.29}$$

Substituting (3.3) into (3.29), we get

$$\nabla_\xi^2 \phi(X(\xi_0, v_0, t), t, \xi_0) = -V_1^{-1}V_0 - h_0(t)V_1^{-1}\Xi_0. \tag{3.30}$$

After substituting back the definitions of the various quantities,

$$\begin{aligned} \nabla_\xi^2 \phi(X(\xi_0, v_0, t), t, \xi_0) &= -\nabla_v V_{\xi_0,v_0}(\xi_0, v_0)^{-1}\nabla_\xi V_{\xi_0,v_0}(\xi_0, v_0) \\ &\quad - h_{\xi_0,v_0}(X(\xi_0, v_0, t), t)\nabla_v V_{\xi_0,v_0}(\xi_0, v_0)^{-1}\nabla_\xi \Xi_{\xi_0,v_0}(\xi_0), \end{aligned} \tag{3.31}$$

and this formula holds for any choice of $(\xi_0, v_0) \in \Sigma \times \mathcal{V}$. Define

$$A(v_0, \xi_0) = -\nabla_v V_{\xi_0,v_0}(\xi_0, v_0)^{-1}\nabla_\xi V_{\xi_0,v_0}(\xi_0, v_0), \tag{3.32}$$

$$B(v, \xi) = -\nabla_v V_{\xi_0,v_0}(\xi_0, v_0)^{-1}\nabla_\xi \Xi_{\xi_0,v_0}(\xi_0), \tag{3.33}$$

$$c(\mathbf{x}) = h_{\xi_0,\nabla_\xi \phi(\mathbf{x},\xi_0)}(\mathbf{x}). \tag{3.34}$$

Thus by (3.31) we finally find that for every (ξ_0, v_0),

$$\nabla_\xi^2 \phi(\mathbf{x}, \xi_0) = A(\nabla_\xi \phi(\mathbf{x}, \xi_0), \xi_0) + c(\mathbf{x}, \xi_0)B(\mathbf{x}, \xi_0). \tag{3.35}$$

By Proposition 1.10, ϕ satisfies Bourgain's condition. $\square$

4. Discretizing sticky ϕ-Kakeya sets and Proposition 1.18

In this section, we only describe the mild adjustments to [WZ22, Section 2.3] which are needed for the standard discretization arguments to carry over to the curved case.

To analyze families of ϕ-curves with packing dimension close to $n-1$, similarly to [WZ22], define for each $\rho > 0$ and $\varepsilon > 0$ the class of "Quantitatively Sticky" sets $\mathrm{QStick}(t,\rho)$ to be the collection of sets $L \subset \mathcal{C}(\phi)$ that satisfy

$$|\{v \in \mathcal{V} : (v,\xi) \in N_\delta(L)\}| \leqslant \delta^{n-1-t} \text{ for all } \xi \in \Sigma, \delta \in (0,\rho). \tag{4.1}$$

The next lemma is the analogue of [WZ22, Lemma 2.2].

Lemma 4.1. *Let $L \subset \mathcal{C}(\phi)$ with $|\mathrm{dir}(L)| > 0$. Then for all $t > \dim_{\mathrm{P}}(L) - (n-1)$, there exists $L' \subset L$ and $\rho > 0$ with $|\mathrm{dir}(L')| > 0$ and $L' \in \mathrm{QStick}(t,\rho)$.*

The proof of Lemma 4.1 is exactly the same as the proof of [WZ22, Lemma 2.2], after replacing $\underline{p}$ with v, replacing v with ξ, and replacing the metric space $\mathcal{L}_n$ with the metric space $\mathcal{C}(\phi)$. Then Lemma 4.1 is used to prove Proposition 1.18 exactly as [WZ22, Lemma 2.2] is used to prove [WZ22, Proposition 2.1]. The necessary adjustments are, again, to replace $\mathcal{L}_n$ with $\mathcal{C}(\phi)$. One should also use that the statement

$$\forall \varepsilon' > 0 \exists \eta', \delta_0 > 0 : \mathrm{SK}(\phi, \varepsilon', \eta', \delta_0) \tag{4.2}$$

translates to the statement $\sigma_n = 0$ in the case of lines in [WZ22].

5. Proof of Theorem 1.19

It will be useful to consider a version of Definition 1.15 with a fixed δ.

Definition 5.1. The assertion $\mathrm{SK}'(\phi,\varepsilon,\eta,\delta)$ holds if for all pairs $(\mathbb{T}, Y)_\delta$ satisfying

(a) the tubes in $\mathbb{T}$ are essentially distinct,
(b) for each $\delta \leqslant \rho \leqslant 1$, $\mathbb{T}$ can be covered by a set of ρ-tubes, at most $\delta^{-\eta}$ of which are pairwise essentially parallel, and
(c) $\sum_{T\in\mathbb{T}} |Y(T)| \geqslant \delta^\eta$,

one has

$$|\bigcup_{T\in\mathbb{T}} Y(T)| \geqslant \delta^\varepsilon. \tag{5.1}$$

By extracting a single tube one gets the following trivial bound. This will serve as the base case for our iteration.

Lemma 5.2. *Fix $\varepsilon > 0$. There exists $\delta_0 = \delta_{0,\mathrm{base}}(\varepsilon)$ such that for all $\delta < \delta_0(\varepsilon)$, we have*

$$\mathrm{SK}'(\phi,\varepsilon,1,\delta^{\varepsilon/(10n)}). \tag{5.2}$$

We will pass from scale $\delta^{\varepsilon/(10n)}$ to scale δ by iterating the following lemma.

Lemma 5.3. *Fix $\varepsilon, \eta, \tilde\eta > 0$ satisfying $\tilde\eta \leqslant \eta/10$. There exists $\delta_0 = \delta_{0,\mathrm{ind}}(\varepsilon,\eta,\tilde\eta) > 0$ such that for all $\delta < \delta_0$,*

$$\mathrm{SK}'(\phi,\varepsilon,\eta,\delta^{1/2}) \wedge \mathrm{SK}'(\phi_{n,\mathrm{rest}},\eta/10,\tilde\eta,\delta^{1/2}) \implies \mathrm{SK}'(\phi,\varepsilon,\tilde\eta/10,\delta). \tag{5.3}$$

Proof. Let $(\mathbb{T}, Y)_\delta$ be a collection of (δ,ϕ)-tubes and their associated shadings satisfying (a),(b),(c) in the assertion $\mathrm{SK}'(\phi,\varepsilon,\tilde{\eta}/10,\delta)$. By (b), we may cover $\mathbb{T}$ by a set of ρ-tubes $\mathbb{T}_\rho$, where $\rho=\delta^{1/2}$. At most $\delta^{-\tilde{\eta}/10}$ of the tubes in $\mathbb{T}_\rho$ are pairwise parallel. For each $T_\rho\in\mathbb{T}_\rho$, we will study the collection of tubes covered by T_ρ, which we denote by $\mathbb{T}[T_\rho]$, and apply Theorem 1.8 to convert $\mathbb{T}[T_\rho]$ to a set of straight tubes.

For a fixed $\delta\leqslant\tau\leqslant\rho$, cover $\mathbb{T}$ by a set of τ-tubes $\mathbb{T}_\tau$, at most $\delta^{-\tilde{\eta}/10}$ of which are pairwise essentially parallel. We can slightly refine $\mathbb{T}_\tau$ so that $\mathbb{T}_\tau$ is covered by $\mathbb{T}_\rho$. So in particular for each $T_\rho\subset\mathbb{T}_\rho$, $\mathbb{T}[T_\rho]$ can be covered by a set of τ-tubes, at most $\delta^{-\tilde{\eta}/10}$ of which are pairwise essentially parallel.

For a fixed $T_\rho\in\mathbb{T}_\rho$, let (ξ_0,v_0) be the parameter of the central ϕ-curve of T_ρ. Apply Theorem 1.8 to produce the maps F,Ξ,V. By (1.14), $F(T)$ is a straight δ-tube with parameter $(\Xi(\xi),V(\xi,v))$. Since Ξ is a diffeomorphism, the family of tubes $\{F(T):T\in\mathbb{T}[T_\rho]\}$ are at most $\delta^{-\tilde{\eta}/10}$ pairwise essentially parallel.

Define the radial dilation $h(x,t)=h(\rho^{-1}x,t)$. Then $h(F(\mathbb{T}[T_\rho])):=\{h(F(T)):T\in\mathbb{T}[T_\rho]\}$ is a set of essentially distinct straight ρ-tubes. From above, for each $\rho\leqslant\tau\leqslant 1$, $h(F(\mathbb{T}[T_\rho]))$ can be covered by a set of ρ-tubes, at most $\rho^{-\tilde{\eta}/5}=\delta^{-\tilde{\eta}/10}$ of which are pairwise essentially parallel. From the assumption $\sum_{T\in\mathbb{T}}|Y(T)|\geqslant\delta^\eta$ and pigeonholing, we may extract a subset $\mathbb{T}_\rho'\subset\mathbb{T}_\rho$ with $|\mathbb{T}_\rho'|\geqslant\rho^{-(n-1)+\tilde{\eta}/5}$ such that for each $T_\rho\subset\mathbb{T}_\rho'$,

$$\sum_{T\in\mathbb{T}[T_\rho]}|Y(T)|\geqslant\log(1/\delta)^{-1}\rho^{n-1+2\tilde{\eta}/5}. \tag{5.4}$$

Applying $h\circ F$,

$$\sum_{T\in\mathbb{T}[T_\rho]}|h(F(Y(T)))|\geqslant C_\phi\log(1/\delta)^{-1}\rho^{2\tilde{\eta}/5}\geqslant\rho^{\tilde{\eta}/2}, \tag{5.5}$$

after possibly choosing δ_0 slightly smaller. Here C_ϕ is a constant depending on ϕ, which comes from the construction of the diffeomorphism F. Thus $h(F(\mathbb{T}[T_\rho]))$ satisfies (a),(b),(c) in the statement of the assertion $\mathrm{SK}'(\phi_{n,\mathrm{rest}},\eta/10,\tilde{\eta},\delta^{1/2})$. Applying the conclusion of this assertion, we find

$$\Big|\bigcup_{T\in\mathbb{T}[T_\rho]}h(F(Y(T)))\Big|\geqslant\rho^{\eta/10}. \tag{5.6}$$

Undoing the transformation $h\circ F$, we get a Jacobian factor of size $\rho^{-(n-1)}$. Thus

$$\Big|\bigcup_{T\in\mathbb{T}[T_\rho]}Y(T)\Big|\geqslant C_\phi\rho^{\eta/10}\rho^{n-1}\geqslant\rho^{\eta/9}\rho^{n-1}, \tag{5.7}$$

after possibly choosing δ_0 smaller. Our aim is now to apply the conclusion of the assertion $\mathrm{SK}'(\phi,\varepsilon,\eta,\delta^{1/2})$. We define a new shading $\tilde{Y}(T_\rho)$ on the tube $T_\rho\in\mathbb{T}_\rho'$ by

$$\tilde{Y}(T_\rho):=\bigcup_{T\in\mathbb{T}[T_\rho]}Y(T). \tag{5.8}$$

Then

$$\sum_{T_\rho\in\mathbb{T}_\rho'}|\tilde{Y}(T_\rho)|\geqslant\rho^{\tilde{\eta}/5+\eta/9}\geqslant\rho^\eta. \tag{5.9}$$

We may now apply the conclusion of $\mathrm{SK}'(\phi,\varepsilon,\eta,\delta^{1/2})$ with $(\mathbb{T}'_\rho,\tilde{Y})$ to find

$$|\bigcup_{T_\rho\in\mathbb{T}'_\rho}\tilde{Y}(T_\rho)|\geqslant\rho^\varepsilon\geqslant\delta^\varepsilon. \tag{5.10}$$

Finally,

$$|\bigcup_{T\in\mathbb{T}}Y(T)|\geqslant|\bigcup_{T_\rho\in\mathbb{T}[T_\rho]}\tilde{Y}(T_\rho)|\geqslant\delta^\varepsilon. \tag{5.11}$$

□

We are now ready to prove Theorem 1.19.

Proof. Fix $\varepsilon>0$. Define the sequences ρ_i, η_i, ε'_i, η'_i, and $\delta_{0,i}$ inductively as follows. Define $\rho_0=\delta^{\varepsilon/(10n)},\eta_0=1,\varepsilon'_0=\eta_0/10,\eta'_0=\eta_{,\mathrm{rest}}(\varepsilon'_0)$, and $\delta_{0,0}=\delta_{0,\mathrm{base}}(\varepsilon)$. Once $\eta_i,\rho_i,\varepsilon'_i,\eta'_i,\delta_{0,i}$ have been defined, we set

$$\rho_{i+1}=\rho_i^2, \tag{5.12}$$
$$\eta_{i+1}=\eta'_i/10, \tag{5.13}$$
$$\varepsilon'_{i+1}=\eta_{i+1}/10, \tag{5.14}$$
$$\eta'_{i+1}=\eta_{n,\mathrm{rest}}(\varepsilon'_{i+1}), \tag{5.15}$$
$$\delta_{0,i+1}=\min(\delta_{0,i},\delta_{0,\mathrm{ind}}(\varepsilon,\eta_i,\eta'_i),\delta_{0,n,\mathrm{rest}}(\varepsilon'_i)). \tag{5.16}$$

Define these sequence up to N such that $\rho_N\leqslant\delta$. Thus $N=O(\log(1/\varepsilon))$. Clearly η_N and $\delta_{0,N}$ depend only on the initial choice of ε. Finally define $\delta_0=\delta_{0,N}^{10n/\varepsilon}$. Now Lemma 5.2 asserts that $\mathrm{SK}'(\phi,\varepsilon,\eta_0,\rho_0)$ is true for $\delta<\delta_0$, and Lemma 5.3 says that

$$\mathrm{SK}'(\phi,\varepsilon,\eta_i,\rho_i)\wedge\mathrm{SK}'(\phi_{n,\mathrm{rest}},\varepsilon'_i,\eta'_i,\rho_i)\implies\mathrm{SK}'(\phi,\varepsilon,\eta_{i+1},\rho_{i+1}), \tag{5.17}$$

for $\delta<\delta_0$. Iterating N times, we conclude that $\mathrm{SK}'(\phi,\varepsilon,\eta_N,\rho_N)$ for $\delta<\delta_0$, which implies $\mathrm{SK}'(\phi,\varepsilon,\eta_N,\delta)$ for $\delta<\delta_0$. That is, the assertion $\mathrm{SK}(\phi,\varepsilon,\eta_N,\delta_0)$ holds and we are done. □

6. The tan-example and Proposition 1.21

6.1. **Constructing Example 1.20 from Proposition 1.10.** One can build Example 1.20 by choosing a simple ansatz for (1.19). We first choose a special form of ϕ to reduce to solving a system of decoupled ODEs. For $1\leqslant j\leqslant n-1$, let $f_j(x_j,t,\xi_j):\mathbb{R}^3\to\mathbb{R}$ be functions to be determined later, and define

$$\phi(\mathbf{x},\xi)=\sum_{j=1}^{n-1}\int f_j(x_j,t,\xi_j)d\xi_j. \tag{6.1}$$

We also choose the following ansatz for A,B, and c:

$$A(v,\xi)=\begin{pmatrix}0&0\\0&v_{n-1}^2\end{pmatrix}, \tag{6.2}$$
$$B(v,\xi)=I_{n-1}, \tag{6.3}$$
$$c(\mathbf{x},\xi)=t^2. \tag{6.4}$$

Then (1.19) becomes the following decoupled system of $n-1$ ODEs:

$$\partial_{\xi_j} f_j = t^2 \quad \text{for } 1 \leqslant j \leqslant n-2, \tag{6.5}$$

$$\partial_{\xi_{n-1}} f_{n-1} = f_{n-1}^2 + t^2. \tag{6.6}$$

The following choices of f_j solve these equations

$$f_j(x_j, t, \xi_j) = t^2\xi_{n-1} + x_{n-1}, \tag{6.7}$$

$$f_{n-1}(x_{n-1}, t, \xi_{n-1}) = t \cdot \tan(t\xi_{n-1} + x_{n-1}), \tag{6.8}$$

and furthermore one may compute $\det \nabla_{\mathbf{x}} \nabla_\xi \phi(\mathbf{x}, \xi) \neq 0$ for $(\mathbf{x}, \xi)$ in a neighborhood of $((0,1),0)$. After integrating the functions f_j, we obtain

$$\phi(\mathbf{x}, \xi) = x' \cdot \xi' + \tfrac{1}{2}t^2|\xi'|^2 + \log(\sec(t\xi_{n-1} + x_{n-1})). \tag{6.9}$$

It is also easy to check that $(G(\mathbf{x}, \xi) \cdot \nabla_{\mathbf{x}})c(\mathbf{x}, \xi) \neq 0$ for $(\mathbf{x}, \xi)$ in a neighborhood of $((0,1),0)$. Since B is also positive-definite, Proposition 1.10 shows that ϕ is a positive-definite Hörmander-type phase function satisfying Bourgain's condition.

6.2. **The family of $\phi_{n,\tan}$-curves.** With $\phi = \phi_{n,\tan}$, we compute

$$\nabla_\xi \phi(\mathbf{x}, \xi) = (x' + t^2\xi', t\tan(t\xi_{n-1} + x_{n-1})). \tag{6.10}$$

Writing $v = (v', v_{n-1}) \in \mathbb{R}^{n-2} \times \mathbb{R}$, we need to compute $X(\xi, v, t)$, the unique solution x to $\nabla_\xi \phi(x, t, \xi) = v$. That is,

$$x' + t^2\xi' = v', \tag{6.11}$$

$$t\tan(t\xi_{n-1} + x_{n-1}) = v_{n-1}. \tag{6.12}$$

Solving, we find

$$x' = v' - t^2\xi', \tag{6.13}$$

$$x_{n-1} = \tan^{-1}\Big(\frac{v_{n-1}}{t}\Big) - t\xi_{n-1}, \tag{6.14}$$

as long as $|t-1| \leqslant 1/10$ and $|\xi|, |v| \leqslant 1/2$. Thus $X(\xi, v, t) = (v' - t^2\xi', \tan^{-1}(\frac{v_{n-1}}{t}) - t\xi_{n-1})$ and

$$\ell_{\xi,v} = \{(v' - t^2\xi', \tan^{-1}\Big(\frac{v_{n-1}}{t}\Big) - t\xi_{n-1}, t) : |t-1| \leqslant 1/10\}. \tag{6.15}$$

6.3. **Illustration of Theorem 1.8 with $\phi_{n,\tan}$.** Before proving Proposition 1.21, let us illustrate the forward direction of Theorem 1.8 for $\phi_{n,\tan}$ by exhibiting a diffeomorphism to lines up to quadratic error. Fix a central curve ℓ_{ξ_0,v_0}. Write $\xi_0 = (\xi_0', \xi_{0,n-1})$ and $v_0 = (v_0', v_{0,n-1})$. Taylor-expanding $\tan^{-1}(v_{n-1}/t)$ to first order in v_{n-1} near $v_{n-1,0}$, we find

$$\begin{aligned}\ell_{\xi,v} = \{(v' - t^2\xi', \tan^{-1}\Big(\frac{v_{0,n-1}}{t}\Big) + \frac{t}{t^2 + v_{0,n-1}}(v_{n-1} - v_{0,n-1}) - t\xi_{n-1} + O, t) \\ : |t-1| \leqslant 1/10\},\end{aligned} \tag{6.16}$$

where $O = O(|v_{n-1} - v_{0,n-1}|^2) = O(|(\xi, v) - (\xi_0, v_0)|^2)$. We may first apply the shift in each t-slice $F_1(\mathbf{x}) = (x', x_{n-1} - \tan^{-1}(\frac{v_{0,n-1}}{t}), t)$, second apply the shear in each t-slice $F_2(\mathbf{x}) = (x', \frac{t^2+v_{0,n-1}}{t} x_{n-1}, t)$, and third apply the diffeomorphism in the t-direction $F_3(\mathbf{x}) = (x, t^2)$ to get

$$\begin{aligned}(F_3 \circ F_2 \circ F_1)(\ell_{\xi,v}) = \{&(v' - t^2\xi', v_{n-1} - v_{0,n-1}(1+\xi_{n-1}) - t^2\xi_{n-1} + O, t^2) \\ &: |t-1| \leqslant 1/10\} \end{aligned} \tag{6.17}$$

$$\subset \mathrm{line}_{\Xi(\xi),V(\xi,v)} + O(|(\xi,v) - (\xi_0, v_0)|^2), \tag{6.18}$$

where $\Xi(\xi) = \xi$ and $V(\xi, v) = (v', v_{n-1} - v_{0,n-1}(1 + \xi_{n-1}))$.

6.4. **Proof of Proposition 1.21.** Fix $\varepsilon_0 > 0$ and assume for contradiction that there is a diffeomorphism $F : B_{\varepsilon_0}(0_M) \to \mathbb{R}^n$ onto its image and functions Ξ, V such that for (ξ, v) near $(0, 0)$,

$$F(\ell_{\xi,v} \cap B_{\varepsilon_0}(0_M)) \subset \mathrm{line}_{\Xi(\xi,v),V(\xi,v)} + O(|(\xi,v)|^4). \tag{6.19}$$

By adjusting the curve $\ell_{\xi,v}$ by an amount $O(|(\xi, v)|^4)$ in each t-slice, we obtain a curve $\tilde{\ell}_{\xi,v}$ satisfying

$$F(\tilde{\ell}_{\xi,v} \cap B_{\varepsilon_0}(0_M)) \subset \mathrm{line}_{\Xi(\xi,v),V(\xi,v)}. \tag{6.20}$$

We will obtain a contradiction as follows. We consider the 1-parameter family of curves $\tilde{\ell}_{\xi(s),v(s)}$ passing through $\tilde{\ell}_{0,0}$ and a point $\mathbf{p} \notin \tilde{\ell}_{0,0}$. We will show that the tangent directions $\gamma(s)$ of the curves $\tilde{\ell}_s$ at $\mathbf{p}$ are *not* all contained in a 2-dimensional vector space. Concretely, we will show that $\gamma(s) \wedge \dot{\gamma}(s) \wedge \ddot{\gamma}(s) \neq 0$ for a fixed s. This is impossible by (6.20), since the lines passing through a fixed line and a point not on that line must be contained in a plane. To make the parameterization of the curves $\tilde{\ell}_{\xi(s),v(s)}$ easier to compute, we will first simplify the curves $\tilde{\ell}_{\xi,v}$ via a Taylor expansion and a diffeomorphism. Then we will write down the parameterization, and finally show that the tangent directions through $\mathbf{p}$ are not contained in a 2-plane. The details are as follows.

Proof. **Step 1: Simplifying the family of curves.**

Each $\tilde{\ell}_{\xi,v}$ (restricted to $B_{\varepsilon_0}(0_M)$) takes the form

$$\tilde{\ell}_{\xi,v} = \{(X(\xi,v,t) + O(|(\xi,v)|^4), t) : |t-1| < \varepsilon_0\} \tag{6.21}$$

$$= \{(v' - t^2\xi' + O(|(\xi,v)|^4), \tan^{-1}(\frac{v_{n-1}}{t}) - t\xi_{n-1} + O(|(\xi,v)|^4), t) : |t-1| < \varepsilon_0\}. \tag{6.22}$$

By Taylor expanding in v_{n-1} to order 3 near 0,

$$\tilde{\ell}_{\xi,v} = \{(v' - t^2\xi' + O(|(\xi,v)|^4), \frac{v_{n-1}}{t} - \frac{v_{n-1}^3}{3t^3} - t\xi_{n-1} + O(|(\xi,v)|^4), t) : |t-1| < \varepsilon_0\}. \tag{6.23}$$

By composing F with the diffeomorphism $G(\mathbf{x}) = (x', t^3 x_{n-1}, t^2)$, we may replace $\tilde{\ell}_{\xi,v}$ with $\tilde{\ell}_{\xi,v} = \{\tilde{\ell}_{\xi,v}(t) : |t-1| \lesssim \varepsilon_0\}$, where

$$\tilde{\ell}_{\xi,v}(t) = (v' - t\xi' + O(|(\xi,v)|^4), tv_{n-1} - \frac{v_{n-1}^3}{3} - t^2\xi_{n-1} + O(|(\xi,v)|^4), t). \tag{6.24}$$

Step 2: The parameters of a curve $\tilde{\ell}_{\xi,v}$ through a point p and $\tilde{\ell}_{0,0}$.

Consider the point $\mathbf{p} = (p, t_0) = (p', p_{n-1}, t_0) \in B_{\varepsilon_0}(0_M)$, where $p \neq 0$ and $t_0 > 1$. We need to solve for the parameters $\xi_{\mathbf{p}}(s), v_{\mathbf{p}}(s)$ such that

$$\tilde{\ell}_{\xi_{\mathbf{p}}(s), v_{\mathbf{p}}(s)}(s) = \tilde{\ell}_{0,0}(s), \tag{6.25}$$

$$\tilde{\ell}_{\xi_{\mathbf{p}}(s), v_{\mathbf{p}}(s)}(t_0) = p. \tag{6.26}$$

We consider s near 1 and solve for $\xi_{\mathbf{p}}(s)$ and $v_{\mathbf{p}}(s)$ to third order in p near 0 (with $t_0 > 1$ fixed). We have $\xi_0(s) = v_0(s) = 0$, so $\xi_{\mathbf{p}}(s), v_{\mathbf{p}}(s) = O(|p|)$. Below we will suppress the "$\mathbf{p}$" in $\xi_{\mathbf{p}}(s)$ and $v_{\mathbf{p}}(s)$ for ease of notation. In coordinates, (6.25) and (6.26) give the following system of equations

$$v'(s) - s\xi'(s) + O(|p|^4) = 0, \tag{6.27}$$

$$v'(s) - t_0\xi'(s) + O(|p|^4) = p', \tag{6.28}$$

$$s v_{n-1}(s) - \frac{v_{n-1}(s)^3}{3} - s^2 \xi_{n-1}(s) + O(|p|^4) = 0, \tag{6.29}$$

$$t_0 v_{n-1}(s) - \frac{v_{n-1}(s)^3}{3} - t_0^2 \xi_{n-1}(s) + O(|p|^4) = p_{n-1}. \tag{6.30}$$

Solving for $\xi'(s)$ using (6.27) and (6.28), we get

$$\xi'(s) = -\frac{1}{t_0 - s} p' + O(|p|^4), \tag{6.31}$$

$$v'(s) = -\frac{s}{t_0 - s} p' + O(|p|^4). \tag{6.32}$$

Using (6.29) to solve for $\xi_{n-1}(s)$ in terms of $v_{n-1}(s)$,

$$\xi_{n-1}(s) = \frac{v_{n-1}(s)}{s} - \frac{v_{n-1}(s)^3}{3s^2} + O(|p|^4). \tag{6.33}$$

Substituting (6.33) into (6.30) and simplifying the result, we obtain the following cubic equation in $v_{n-1}(s)$:

$$\frac{1}{3}\left(\frac{t_0^2}{s^2} - 1\right) v_{n-1}(s)^3 - t_0\left(\frac{t_0}{s} - 1\right) v_{n-1}(s) - p_{n-1} + O(|p|^4) = 0. \tag{6.34}$$

Solving (6.34) iteratively in powers of p_{n-1}, we obtain

$$v_{n-1}(s) = -\frac{1}{t_0(\frac{t_0}{s} - 1)} p_{n-1} - \frac{1}{3} \frac{(\frac{t_0^2}{s^2} - 1)}{t_0^4(\frac{t_0}{s} - 1)^4} p_{n-1}^3 + O(|p|^4) \tag{6.35}$$

$$= -\frac{s}{t_0(t_0 - s)} p_{n-1} - \frac{s^2(t_0 + s)}{3t_0^4(t_0 - s)^3} p_{n-1}^3 + O(|p|^4). \tag{6.36}$$

Now we substitute (6.36) back into (6.33) to obtain

$$\xi_{n-1}(s) = -\frac{1}{t_0(t_0 - s)} p_{n-1} - \frac{s^2}{3t_0^4(t_0 - s)^3} p_{n-1}^3 + O(|p|^4). \tag{6.37}$$

Step 3: The curves through p and $\tilde{\ell}_{0,0}$ are not in a surface.

Define $\gamma(s) = \frac{d}{dt}\tilde{\ell}_{\xi(s),v(s)}(t_0)$, the tangent vector at point $\mathbf{p}$ of the unique curve passing through $\tilde{\ell}_{0,0}(s)$ and $\mathbf{p}$. We compute

$$\begin{aligned}\gamma(s) =&(-\xi'(s) + O(|p|^4), v_{n-1}(s) - 2t_0\xi_{n-1}(s) + O(|p|^4), 1) && (6.38)\\ =&\Big(\frac{1}{t_0 - s}p' + O(|p|^4), && (6.39)\\ &\frac{2t_0 - s}{t_0(t_0 - s)}p_{n-1} + \frac{s^2}{3t_0^4(t_0 - s)^2}p_{n-1}^3 + O(|p|^4), 1\Big). && (6.40)\end{aligned}$$

We may now compute $\gamma(1), \dot\gamma(1), \ddot\gamma(1)$:

$$\begin{aligned}\gamma(1) =&\Big(\frac{1}{t_0 - 1}p' + O(|p|^4), && (6.41)\\ &\frac{2t_0 - 1}{t_0(t_0 - 1)}p_{n-1} + \frac{1}{3t_0^4(t_0 - 1)^2}p_{n-1}^3 + O(|p|^4), 1\Big), && (6.42)\\ \dot\gamma(1) =&\Big(\frac{1}{(t_0 - 1)^2}p' + O(|p|^4), && (6.43)\\ &\frac{1}{(t_0 - 1)^2}p_{n-1} + \frac{2}{3t_0^3(t_0 - 1)^3}p_{n-1}^3 + O(|p|^4), 0\Big), && (6.44)\\ \ddot\gamma(1) =&\Big(\frac{2}{(t_0 - 1)^3}p' + O(|p|^4), && (6.45)\\ &\frac{2}{(t_0 - 1)^3}p_{n-1} + \frac{2(2 + t_0)}{3t_0^3(t_0 - 1)^4}p_{n-1}^3 + O(|p|^4), 0\Big). && (6.46)\end{aligned}$$

Choose $p' = (0, \dots, 0, p_{n-2})$ so that $\gamma(1), \dot\gamma(1), \ddot\gamma(1) \in \{0\}^{n-3} \times \mathbb{R}^3$, and we may view these as vectors in $\mathbb{R}^3$. We may finally compute

$$\begin{aligned}|\gamma(1) \wedge \dot\gamma(1) \wedge \ddot\gamma(1)| &= |\det(\gamma(1), \dot\gamma(1), \ddot\gamma(1))| && (6.47)\\ &= \left|\det\begin{pmatrix}\frac{1}{(t_0-1)^2}p' + O(|p|^4) & \frac{1}{(t_0-1)^2}p_{n-1} + \frac{2}{3t_0^3(t_0-1)^3}p_{n-1}^3 + O(|p|^4)\\ \frac{2}{(t_0-1)^3}p' + O(|p|^4) & \frac{2}{(t_0-1)^3}p_{n-1} + \frac{2(2+t_0)}{3t_0^3(t_0-1)^4}p_{n-1}^3 + O(|p|^4)\end{pmatrix}\right| && (6.48)\\ &= \frac{2|p_{n-1}|^3|p_{n-2}|}{3t_0^2(t_0 - 1)^6} + O(|p|^5). && (6.49)\end{aligned}$$

By choosing p_{n-1} and p_{n-2} small enough,

$$|\gamma(1) \wedge \dot\gamma(1) \wedge \ddot\gamma(1)| > 0. \tag{6.50}$$

Step 4: Reaching a contradiction with (6.20)

By (6.20), $F(\tilde{\ell}_{\xi,v})$ is a family of lines. Thus for s near 0, $F(\tilde{\ell}_{\xi(s),v(s)})$ is a family of lines passing through $F(\tilde{\ell}_{0,0})$ and $F(\mathbf{p})$, so it is contained in a fixed plane. In particular the tangent vectors of $F(\tilde{\ell}_{\xi(s),v(s)})$ at $F(\mathbf{p})$ lie in a fixed plane, so $\eta(s) := \frac{d}{dt}F(\tilde{\ell}_{\xi(s),v(s)}(t_0))$ satisfies $|\eta(1) \wedge \dot\eta(1) \wedge \ddot\eta(1)| = 0$. But by the chain rule

$$0 = |\eta(1) \wedge \dot\eta(1) \wedge \ddot\eta(1)| = |DF(\mathbf{p})\gamma(1) \wedge DF(\mathbf{p})\dot\gamma(1) \wedge DF(\mathbf{p})\ddot\gamma(1)|, \tag{6.51}$$

which contradicts (6.50). □

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