Wightman axiomatics of the bootstrap construction of the 1+1 dimensional Sinh-Gordon model

Karol K. Kozlowski [1]

Univ Lyon, ENS de Lyon, Univ Claude Bernard Lyon 1, CNRS, Laboratoire de Physique, F-69342 Lyon, France

Alex Simon [2]

Univ Lyon, ENS de Lyon, Univ Claude Bernard Lyon 1, CNRS, Laboratoire de Physique, F-69342 Lyon, France

Abstract

The integrable bootstrap program allows one to express the tempered distributions associated with the multi-point functions of the integrable 1+1 dimensional Sinh-Gordon quantum field theory by means of explicit series. The convergence of the latter is an open problem that was only solved for the two-point case. In this work, by taking for granted the convergence of these series, we show that these expressions satisfy all of the Wightman axioms. This thus shows that, upon a yet to be proven convergence property, the integrable bootstrap based construction of correlation functions does lead to a quantum field theory.

MSC Classification : 82B23, 81Q80, 81U15, 81T99

Contents

1 Introduction **2**

2 Preliminary setup **4**
- 2.1 Wightman Axioms . . . 4
- 2.2 The correlation function built from the bootstrap program . . . 7
  - 2.2.1 The S-matrix and form factors . . . 7
  - 2.2.2 The multi-point functions . . . 10

3 Proof of the Wightman Axioms **12**
- 3.1 The covariance axiom . . . 12
- 3.2 Spectral condition . . . 13

[1]e-mail: karol.kozlowski@ens-lyon.fr
[2]e-mail: alex.simon@ens-lyon.fr

3.3 Hermiticity 16
3.4 Local Commutativity 17
3.5 Positivity 23
3.6 Cluster Decomposition 26

A Support of tempered distributions **34**

1 Introduction

On may formulate the problem of constructing rigorously a *bona fide* quantum field theory as the following requirements:

i) construct a separable Hilbert space $\mathfrak{h}$

ii) realise a continuous unitary representation of the Poincaré group on $\mathfrak{h}$

iii) provide a tempered distribution taking values in unbounded self-adjoint operators on $\mathfrak{h}$, the hermitian field operator $\boldsymbol{\Psi}$ (or multiplets thereof). The latter has to be compatible with the presence of relativistic symmetry in the model.

These objects should satisfy several requirements. First of all, there should exists a unique one dimensional subspace $\mathfrak{e}_{\mathrm{vac}} = \{\lambda \boldsymbol{v}_{\mathrm{vac}},\ \lambda \in \mathbb{C}\}$ of $\mathfrak{h}$ invariant under the action of the Poincaré group, called the "vacuum" of the model. Second, the vacuum is cyclic with respect to the action of smeared fields, what means that

$$\mathrm{Vect}\Big\{\boldsymbol{\Psi}[f_1]\cdots\boldsymbol{\Psi}[f_n]\boldsymbol{v}_{\mathrm{vac}}\ :\ n\in\mathbb{N}, f_1,\dots,f_n\in\mathcal{S}(\mathbb{R}^{1,d})\Big\} \tag{1.1}$$

is dense in $\mathfrak{h}$. Above, $\mathbb{R}^{1,d}$ refers to $\mathbb{R}^{1+d}$ endowed with the diagonal metric $(1,-1,\dots,-1)$ and $\mathcal{S}(X)$ stands for the space of Schwartz functions on X.

The search for effective ways to provide the above setting for interacting quantum field theories on rigorous mathematical grounds has been an extensive subject since the 60s. From these investigations, there emerged a mathematical construction of quantum field theories based on objects of prime physical importance: the correlation functions, see *e.g.* the monographs [3, 28]. More precisely, if one is given a collection of distributions $\mathtt{W}^{(k)}$ on $\mathcal{S}((\mathbb{R}^{1,d})^k)$, $k\in\mathbb{N}$, satisfying certain axioms, nowadays called the Wightman axioms, then one may automatically realise the aforementioned constructions, points i)-iii) above. The distributions one starts with then correspond to the constructed theory's correlation functions

$$\mathtt{W}^{(k)}[f] = \Big(\boldsymbol{v}_{\mathrm{vac}}, \boldsymbol{\Psi}[f_1]\cdots\boldsymbol{\Psi}[f_k]\boldsymbol{v}_{\mathrm{vac}}\Big) \qquad \text{with} \qquad \left\{\begin{array}{ccc} f(\boldsymbol{X}_k) & = & \prod\limits_{a=1}^{k} f_a(\boldsymbol{x}_a) \\ \boldsymbol{X}_k & = & (\boldsymbol{x}_1,\dots,\boldsymbol{x}_k) \end{array}\right. , \tag{1.2}$$

and $\boldsymbol{x}_a \in \mathbb{R}^{1,d}$. Note that it is convenient to represent these distributions in terms of the associated generalised functions which, formally speaking, correspond to their integral kernels

$$\mathtt{W}^{(k)}[f] = \int\limits_{(\mathbb{R}^{1,d})^k} \mathrm{d}^k\boldsymbol{X}\,\mathcal{W}^{(k)}(\boldsymbol{X}_k)f(\boldsymbol{X}_k) \qquad \text{with} \qquad \mathrm{d}^k\boldsymbol{X} = \prod_{a=1}^{k}\mathrm{d}\boldsymbol{x}_a\,. \tag{1.3}$$

The generalised functions $\mathcal{W}^{(k)}(\boldsymbol{X}_k)$ are referred to as Wightman functions.

However, in practice, finding distributions which verify Wightman axioms is an arduous task. First of all, due to the presence of a probabilistic setting, it is more convenient to construct Euclidean analogues of the Wightman functions, *aka* Schwinger functions. The k-point Schwinger functions are generalised functions on the Euclidean space $(\mathbb{R}^{1+d})^k$ instead of the Minkowski one $(\mathbb{R}^{1,d})^k$. In such an Euclidean setting, if one is able to construct a collection of distributions on $\mathcal{S}((\mathbb{R}^{1+d})^k)$ - the mentioned Schwinger functions- satisfying the so-called Osterwalder-Schrader axioms, then the Osterwalder-Schräder reconstruction theorem [24, 25] ensures that one may use these to construct distributions satisfying the Wightman axioms. This thus leads to the construction of the sought quantum field theory. The main advantage of working in Euclidean space, is that one may build on the probabilistic, constructive renormalisation based, rigorous construction of the associated path integral which then provides one with the Schwinger functions. This program was pioneered by [7] and achieved for many quantum field theories in various dimensions, see *e.g.* the review [29]. It provides one with the existence of the theory but does not lead to closed expressions for the Schwinger, or associated Wightman functions. The best one can do within the formalism, is to obtain their perturbative expansions in coupling constants around the free theories. More recently, there emerged various other approaches allowing one to construct Schwinger functions in an alternative way. Notably, one could cite the rigorous development of the stochastic quantisation, first proposed in [26] by the physics litterature, which presents various advantages in respect to the constructive renormalisation based approach and constituted an important progress in reaching the functional measure allowing one to build the Schwinger functions [9, 23, 13]. One should however mention that this alternative construction still provides a quite indirect access to the Schwinger functions, thus only granting a rather low control on their properties beyond a perturbative regime. The two mentioned approaches allow one to deal with quantum field theory models in various dimensions, $1 + d$, although going to higher dimensions demands increasing efforts an in various respects, going beyond $1 + d = 3$ seems quite challenging for the moment, although some partial results were obtained in $1 + d = 4$ for instance.

One should definitely mention another progress: the Gaussian multiplicative chaos approach to the construction of Euclidean path integrals for two-dimensional conformal field theories, which coupled to integrability properties of the latter allowed to access to explicit informations on the Schwinger functions for the Liouville theory in particular [11, 21]. Some progress was also achieved by this approach in respect to the massive Euclidean two-dimensional Sinh-gordon model in finite volume [10], although the construction of the infinite volume Schwinger functions through is still open. It seems fair to say that, as of today and despite important progress on various fronts, there does not exist yet an appropriate unifying framework allowing one to prove the existence of Schwinger/Wightman functions associated with an arbitrary interacting quantum field theory.

The approaches described above provide one with existence theorems for Schwinger functions associated with various classical Lagrangians. Still, ideally, one would like to have a better understanding of the Schwinger, resp. the associated Wightman, functions as generalised functions on $(\mathbb{R}^{1+d})^k$, resp. $(\mathbb{R}^{1,d})^k$, and have closed form expressions for the latter. While this could have been achieved to some extent for conformal field theories owing to their very specific feature, the matter remains widely open for any massive infinite volume quantum field theory that would not be free. Integrable quantum field theories in 1+1 dimensions constitute one instance -the sole know so-far beyond the free theories- where this seems possible. These theories are constructed within the S-matrix program [12, 30]. The idea is to take certain *formal* equations which can be argued from the *formal* Minkowski path-integral construction of these theories, as axioms which allow the theory to be built. More precisely, starting from a postulated form of a theory's S-matrix, one axiomatises a collection of coupled Riemann–Hilbert problems for unknown functions in $1, 2, \dots,$ complex variables. These functions arise as building blocks of the Wightman functions for the associated theory. This is the celebrated bootstrap program pioneered in [14, 17]. We refer to the book [27] for more details. The main point is that S-matrices of integrable quantum field theories in $1+1$ dimensions may be constructed explicitly. Indeed, integrability imples elasticity and factorisability of the scattering which, in turn, implies that the S-matrix satisfies the celebrated Yang-Baxter equation. The construction of solutions to the latter is, nowadays, understood quite well, on a deep algebraic level. In particular, this knowledge allowed to address the Bootstrap program based construction of the associated theories. Various techniques [1, 22, 27] were devised for that purpose. This approach led to closed form, explicit, expressions for various two-points arising in the considered integrable quantum field theories. The two-point functions were expressed in terms of series, whose n^{th} summand is an n-fold integral. The convergence of these series is however, generically, an open problem that could have been resolved so-far only for the space-like two-point functions of the Sinh-Gordon quantum field theory [19]. We remind that the latter refers to the quantum field theory resulting from the quantisation of the classical Sinh-Gordon field theory described by the classical Lagrangian

$$\mathcal{L}[\varphi](\boldsymbol{x}) \;=\; \big(\partial_t\varphi(\boldsymbol{x})\big)^2 \;-\; \big(\partial_x\varphi(\boldsymbol{x})\big)^2 \;-\; \frac{m^2}{g^2}\cosh\big(g\varphi(\boldsymbol{x})\big) \qquad \text{with} \qquad \boldsymbol{x}=(t,x)\in\mathbb{R}^{1,1}\,. \tag{1.4}$$

Thus, apart form the particular example of the space-like two-point functions of the Sinh-Gordon model, the bootstrap program issued expressions for the two-point functions should be considered as conjectural, in the sense that one has not established yet their well-definiteness. The construction of multi-point functions, *i.e.* Wightman functions in any number of space-time variables, was only achieved recently in full detail in [20], again in the case of the Sinh-Gordon model, and this up to the convergence conjecture of the obtained expressions. To be more precise, the multi-point correlation functions are given by series of multiple integrals in the spirit of the results for the two-point functions. The answer should be considered on the conjectural level in that the convergence of these series is an open problem so far.

The purpose of the present work is as follows. By taking for granted that the series defining the multi-point functions in the Sinh-Gordon quantum field theory are convergent, we prove that the associated tempered distributions do satisfy all the Wightman axioms. As a consequence, provided convergence holds, we do establish that the bootstrap approach does lead to a quantum field theory. This is the first time, that the complete Wightman program is achieved for an integrable quantum field theory. Some aspects of this issue were considered earlier in the literature. F. Smirnov, *c.f.* [27], considered the local commutativity axiom, taken in a non-rigorous way on the level of operator products. However, while the clustering property of the form factors was largely discussed in the physics litterarture, it turns out that the chain of heuristic ideas does not adapt sufficienly well to a proof one one has to recourse to much more involved arguemnts.

This paper is organised according to the following. Section 2 introduces all the preliminary notions and results that are necessary for obtaining to the main result of the paper. Subsection 2.1 recalls the Wightman axioms, directly in the form taken for a spinless scalar 1+1 dimensional quantum field theory, such as the Sinh-Gordon model. Subsection 2.2 reviews the results of [20]. After setting up the necessary notations in Subsection 2.2.1, closed expressions for the multi-point functions are introduced in Subsection 2.2.2. Section 3 is then devoted to the proof that the expressions for the multi-point functions obtained through the resolution of the bootstrap program do satisfy all of the Wightman axioms, this under the convergence conjecture.

2 Preliminary setup

2.1 Wightman Axioms

In order to state the Wightman axioms in the setting of interest to us, we need to introduce some preliminary notations. The Minkowski scalar product on $\mathbb{R}^{1,1}$ is defined as

$$\boldsymbol{x} * \boldsymbol{y} \;=\; x_0y_0 - x_1y_1 \qquad \text{with} \qquad \boldsymbol{x}=(x_0,x_1) \quad \text{and} \quad \boldsymbol{y}=(y_0,y_1)\,. \tag{2.1}$$

In particular, the Minkowski norm of $\boldsymbol{x}=(x_0,x_1)\in\mathbb{R}^{1,1}$ reads

$$\boldsymbol{x}^2 = x_0^2 - x_1^2\,. \tag{2.2}$$

The Minkowski scalar product is invariant under the action of the orthogonal group $O(1,1)$. The special orthogonal group

$$SO^{+}(1,1) \;=\; \Big\{\Lambda_{\theta} \;:\; \theta \in \mathbb{R}\Big\} \qquad \text{where} \qquad \Lambda_{\theta} \;=\; \begin{pmatrix} \cosh(\theta) & -\sinh(\theta) \\ -\sinh(\theta) & \cosh(\theta) \end{pmatrix} , \tag{2.3}$$

along with space-time translations $\mathtt{T}_{\boldsymbol{v}} : \boldsymbol{x} \mapsto \boldsymbol{x} + \boldsymbol{v}$, with $\boldsymbol{v} \in \mathbb{R}^{1,1}$, forms the orthochronous Poincaré group $\mathcal{P}_{\uparrow}^{+}$. It is believed that $\mathcal{P}_{\uparrow}^{+}$ should be a fundamental symmetry of a quantum relativistic theory, *viz.* that the action of $\mathcal{P}_{\uparrow}^{+}$ leaves such theories invariant. A group element $\mathtt{U}_{\theta,\boldsymbol{v}} \in \mathcal{P}_{\uparrow}^{+}$ is parameterised by $(\theta, \boldsymbol{v}) \in \mathbb{R} \times \mathbb{R}^{1,1}$ and multiplication is defined by $\mathtt{U}_{\theta,\boldsymbol{v}}\mathtt{U}_{\alpha,\boldsymbol{w}} = \mathtt{U}_{\theta+\alpha,\Lambda_{\theta}\boldsymbol{w}+\boldsymbol{v}}$. The group element $\mathtt{U}_{\theta,\boldsymbol{v}}$ acts on $f \in \mathcal{S}((\mathbb{R}^{1,1})^{k})$ as

$$(\mathtt{U}_{\theta,\boldsymbol{v}} \cdot f)(\boldsymbol{X}_k) \;=\; f(\Lambda_{\theta}^{-1}(\boldsymbol{x}_1 + \boldsymbol{v}), \dots, \Lambda_{\theta}^{-1}(\boldsymbol{x}_k + \boldsymbol{v})) \,. \tag{2.4}$$

The permutation group $\mathfrak{S}_k$ acts naturally on $\mathcal{S}((\mathbb{R}^{1,1})^{k})$ by means of coordinate permutations:

$$(\sigma \cdot G)(\boldsymbol{X}_k) \;=\; G(\sigma^{-1} \cdot \boldsymbol{X}_k) \quad \text{with} \quad \sigma^{-1} \cdot \boldsymbol{X}_k \;=\; (\boldsymbol{x}_{\sigma^{-1}(1)}, \cdots, \boldsymbol{x}_{\sigma^{-1}(k)}) \,. \tag{2.5}$$

Two kinds of permutations will be of use in the following, the inversion ι and the adjacent transpositions τ_s where it holds:

$$(\iota \cdot G)(\boldsymbol{X}_k) \;=\; G(\iota \cdot \boldsymbol{X}_k) \qquad \text{and} \qquad (\tau_s \cdot G)(\boldsymbol{X}_k) \;=\; G(\tau_s \cdot \boldsymbol{X}_k) \,, \tag{2.6}$$

with $s \in [\![\, 1 \,;\, k-1 \,]\!]$ since $\iota^{-1} = \iota$ and $\tau_s^{-1} = \tau_s$. Note that above, it is understood that

$$\iota \cdot \boldsymbol{X}_k \;=\; (\boldsymbol{x}_k, \dots, \boldsymbol{x}_1) \;=\; \overleftarrow{\boldsymbol{X}}_k \qquad \text{and} \qquad \tau_s \cdot \boldsymbol{X}_k \;=\; (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{s-1}, \boldsymbol{x}_{s+1}, \boldsymbol{x}_s, \boldsymbol{x}_{s+2}, \dots, \boldsymbol{x}_k) \,. \tag{2.7}$$

The Fourier transform $\mathscr{F}$ on $\mathcal{S}((\mathbb{R}^{1,1})^{k})$ is defined as

$$\mathscr{F}[f](\boldsymbol{q}_1, \cdots, \boldsymbol{q}_k) \;=\; \int\limits_{(\mathbb{R}^{1,1})^{k}} \mathrm{d}^k\boldsymbol{X} \cdot f(\boldsymbol{X}_k) \cdot \prod_{a=1}^{k} \mathrm{e}^{\mathrm{i}\boldsymbol{q}_a * \boldsymbol{x}_a} \,. \tag{2.8}$$

We remind that the Fourier transform of a distribution $\mathtt{D}$ is defined as

$$\mathscr{F}[\mathtt{D}][f] \;=\; \mathtt{D}[\mathscr{F}[f]] \,. \tag{2.9}$$

The spelling out of the Wightman axioms will, among other things, utilise the positive energy subspace of $(\mathbb{R}^{1,1})^{k}$

$$\mathcal{E}_k^{+} \;=\; \Big\{(\boldsymbol{q}_1, \dots, \boldsymbol{q}_k) \in (\mathbb{R}^{1,1})^{k} : \quad \boldsymbol{q}_{\ell}^{2} \geq 0 \quad \text{and} \quad q_{\ell,0} \geq 0 \quad \text{for every} \quad \ell \in [\![\, 1 \,;\, k \,]\!]\Big\} \,. \tag{2.10}$$

Definition 2.1. *A sequence $\{\mathtt{W}^{(k)}\}_{k\geq 1}$ of tempered distributions, with $\mathtt{W}^{(k)}$ being a distribution on $\mathcal{S}((\mathbb{R}^{1,1})^{k})$, is said to satisfy the Wightman axioms if the below properties hold:*

- **Covariance***: For any $(\theta, \boldsymbol{v}) \in \mathbb{R} \times \mathbb{R}^{1,1}$ and $f \in \mathcal{S}((\mathbb{R}^{1,1})^{k})$ it holds*

$$\mathtt{W}^{(k)}[\mathtt{U}_{\theta,\boldsymbol{v}} \cdot f] \;=\; \mathtt{W}^{(k)}[f] \,. \tag{2.11}$$

- **Spectral conditions**: *there exists a sequence of tempered distributions* $\{\mathtt{V}^{(k)}\}_{k\geq 1}$ *such that for every*

$$f_k(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k) = g(\boldsymbol{x}_k)\cdot G_{k-1}(\boldsymbol{x}_1-\boldsymbol{x}_2,\ldots,\boldsymbol{x}_{k-1}-\boldsymbol{x}_k) \quad with \quad (g,G_{k-1})\in\mathcal{S}(\mathbb{R}^{1,1})\times\mathcal{S}((\mathbb{R}^{1,1})^{k-1})\,, \tag{2.12}$$

it holds

$$\mathtt{W}^{(k)}[f_k] \;=\; \Big\{\int_{\mathbb{R}^{1,1}}\mathrm{d}\boldsymbol{x}\, g(x)\Big\}\cdot \mathtt{V}^{(k-1)}[G_{k-1}] \qquad with \quad k\geq 2\,. \tag{2.13}$$

Moreover, the Fourier transforms of the distributions $\mathtt{V}^{(k-1)}$ *satisfy*

$$\mathrm{supp}\Big\{\mathscr{F}[\mathtt{V}^{(k-1)}]\Big\}\subseteq\mathcal{E}^{+}_{k-1}\,, \tag{2.14}$$

with $\mathcal{E}^+_{k-1}$ *as introduced in* (2.10)*, i.e.* $\mathscr{F}[\mathtt{V}^{(k-1)}][G]=0$ *for any* $G\in\mathcal{S}((\mathbb{R}^{1,1})^{k-1})$ *such that* $\mathrm{supp}[\mathscr{F}[G]]\cap\mathcal{E}^+_{k-1}=\emptyset$.

- **Hermiticity**: *For any* $f\in\mathcal{S}((\mathbb{R}^{1,1})^k)$*, it holds*

$$\overline{\mathtt{W}^{(k)}[f]} \;=\; \mathtt{W}^{(k)}\Big[\,\overline{\iota\cdot f}\,\Big]\,, \tag{2.15}$$

where $\bar{z}$ *refers to the complex conjugate of* $z\in\mathbb{C}$*,* $\overline{f}$ *is obtained by the pointwise conjugation, and* ι *is the action of the inversion* (2.6).

- **Local commutativity**: *For any* $f\in\mathcal{S}((\mathbb{R}^{1,1})^k)$ *such that*

$$\mathrm{supp}[f]\subset\Omega_s \;=\; \Big\{(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)\in(\mathbb{R}^{1,1})^k \;:\; (\boldsymbol{x}_s-\boldsymbol{x}_{s+1})^2\;<\;0\Big\}\,, \tag{2.16}$$

it holds

$$\mathtt{W}^{(k)}[f] \;=\; \mathtt{W}^{(k)}[\tau_s\cdot f] \tag{2.17}$$

in which the transpositions act as in (2.6).

- **Positive definiteness**: *Let* $\{f^{(k)}\}_{k\geq1}$ *be any sequence of test functions with* $f^{(k)}\in\mathcal{S}((\mathbb{R}^{1,1})^k)$ *for each integer k, and such that there is only a finite number of non-zero terms. Let*

$$F_{p,q}(\boldsymbol{X}_p,\boldsymbol{Y}_q) \;=\; \overline{f^{(p)}(\overleftarrow{\boldsymbol{X}}_p)}\cdot f^{(q)}(\boldsymbol{Y}_q)\,, \tag{2.18}$$

then it holds

$$\sum_{p,q\in\mathbb{N}}\mathtt{W}^{(p+q)}[F_{p,q}]\;\geq\;0\,. \tag{2.19}$$

- **Cluster property**: *For any space-like* $\boldsymbol{v}\in\mathbb{R}^{1,1}$*, i.e.* $\boldsymbol{v}^2<0$*, and* $f\in\mathcal{S}((\mathbb{R}^{1,1})^p)$*,* $g\in\mathcal{S}((\mathbb{R}^{1,1})^q)$*, it holds*

$$\mathtt{W}^{(p+q)}[G^{(\lambda\boldsymbol{v})}_{p,q}] \;\underset{\lambda\to\pm\infty}{\longrightarrow}\; \mathtt{W}^{(p)}[f]\,\mathtt{W}^{(q)}[g] \tag{2.20}$$

where

$$G^{(\lambda\boldsymbol{v})}_{p,q}(\boldsymbol{X}_p,\boldsymbol{Y}_q) \;=\; f(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_p)\,g(\boldsymbol{y}_1+\lambda\boldsymbol{v},\ldots,\boldsymbol{y}_q+\lambda\boldsymbol{v})\,. \tag{2.21}$$

We now recall Wightman's reconstruction theorem [28], which is the central tool in re-constructing a quantum field theory from the knowledge of its correlation functions. For the needs of this work, we state it here in the case of 1+1 dimensional theories with only one scalar hermitian field.

Theorem 2.2. *Let* $\{\mathtt{W}^{(k)}\}_{k\geq 1}$ *be a sequence of tempered distributions,* $\mathtt{W}^{(k)}$ *acting on* $\mathcal{S}((\mathbb{R}^{1,1})^k)$*, that satisfy the Wightman axioms. Then, there exist a separable Hilbert space* $\mathfrak{h}$*, a continuous unitary representation of the orthochronous Poincaré group* $\mathcal{P}^+_\uparrow$ *on* $\mathfrak{h}$*, a unique one dimensional subspace* $\mathfrak{e}_{\mathrm{vac}} = \{\lambda \boldsymbol{v}_{\mathrm{vac}},\ \lambda \in \mathbb{C}\}$ *of* $\mathfrak{h}$ *invariant under the action of* $\mathcal{P}^+_\uparrow$*, and an unbounded operator valued tempered distribution* $\boldsymbol{\Psi}$ *such that*

$$\mathtt{W}^{(k)}[f] = \Big(\boldsymbol{v}_{\mathrm{vac}}, \boldsymbol{\Psi}[f_1]\cdots\boldsymbol{\Psi}[f_k]\boldsymbol{v}_{\mathrm{vac}}\Big) \qquad \text{with} \qquad \left\{ \begin{array}{ccc} f(\boldsymbol{X}_k) & = & \prod\limits_{a=1}^{k} f_a(\boldsymbol{x}_a) \\ \boldsymbol{X}_k & = & (\boldsymbol{x}_1,\dots,\boldsymbol{x}_k) \in \Big(\mathbb{R}^{1,1}\Big)^k \end{array} \right. . \tag{2.22}$$

The vacuum correlation functions are exactly the Wightman functions. Moreover, this vacuum is cyclic with respect to the action of the field $\boldsymbol{\Psi}$*, and any other quantum field theory with the same Wightman functions is unitarily equivalent to this one.*

2.2 The correlation function built from the bootstrap program

We now provide the expressions for the multi-point correlation functions of the Sinh-Gordon 1+1 dimensional integrable quantum field theory which were obtained in [20] within the so-called bootstrap program. In particular, we shall discuss which of their aspects are rigorous and which still remain to be established. The description goes in several steps. We start by introducing the various building blocks of the formula for the multi-point functions, and then describe the formula *per se*.

2.2.1 The S-matrix and form factors

The fundamental building block of the Sinh-Gordon model's correlation functions is its $\mathtt{S}$-matrix, first introduced in [8]. This matrix acts diagonally, with the below scalar factor describing the scattering at rapidity β:

$$\mathtt{S}(\beta) = \frac{\tanh\big[\frac{1}{2}\beta - \mathrm{i}\pi\mathfrak{b}\big]}{\tanh\big[\frac{1}{2}\beta + \mathrm{i}\pi\mathfrak{b}\big]} = \frac{\sinh(\beta) - \mathrm{i}\sin[2\pi\mathfrak{b}]}{\sinh(\beta) + \mathrm{i}\sin[2\pi\mathfrak{b}]} \qquad \text{with} \qquad \mathfrak{b} = \frac{1}{2}\frac{g^2}{8\pi + g^2} \in [0\,;\tfrac{1}{2}] \,. \tag{2.23}$$

This $\mathtt{S}$-matrix satisfies the unitarity $\mathtt{S}(\beta)\mathtt{S}(-\beta) = 1$ and crossing $\mathtt{S}(\beta) = \mathtt{S}(\mathrm{i}\pi - \beta)$ relations. Its poles are located at $2\mathrm{i}\pi\mathfrak{b} + 2\mathrm{i}\pi\mathbb{Z}$. In particular, it has no poles in the physical strip

$$\mathscr{S} = \{0 < \Im(\beta) < \pi\} \;, \tag{2.24}$$

which translates the absence of bound states in this model. The $\mathtt{S}$-matrix is then used to construct the so-called two-body form factor $\mathtt{F}$ which can be expressed in closed form in terms of the Barnes G- and Euler Γ-functions:

$$\mathtt{F}(\beta) = \frac{1}{\Gamma\Big(1+\mathfrak{z}, -\mathfrak{z}\Big)} G\left(\begin{array}{cccc} 1-\mathfrak{b}-\mathfrak{z}, & 2-\mathfrak{b}+\mathfrak{z}, & 1-\hat{\mathfrak{b}}-\mathfrak{z}, & 2-\hat{\mathfrak{b}}+\mathfrak{z} \\ \mathfrak{b}-\mathfrak{z}, & 1+\mathfrak{b}+\mathfrak{z}, & \hat{\mathfrak{b}}-\mathfrak{z}, & 1+\hat{\mathfrak{b}}+\mathfrak{z} \end{array} \right) \quad \text{with} \quad \mathfrak{z} = \frac{\mathrm{i}\beta}{2\pi} \,, \quad \hat{\mathfrak{b}} = \frac{1}{2} - \mathfrak{b} \,. \tag{2.25}$$

Above, we made use of hypergeometric-like product conventions

$$\Gamma\left(\begin{array}{c} a_1,\dots,a_n \\ b_1,\dots,b_\ell \end{array} \right) = \frac{\prod\limits_{k=1}^{n}\Gamma(a_k)}{\prod\limits_{k=1}^{\ell}\Gamma(b_k)} \qquad \text{and} \qquad G\left(\begin{array}{c} a_1,\dots,a_n \\ b_1,\dots,b_\ell \end{array} \right) = \frac{\prod\limits_{k=1}^{n}G(a_k)}{\prod\limits_{k=1}^{\ell}G(b_k)} \,. \tag{2.26}$$

$\mathtt{F}$ is the first building block of so-called form factors. The second building block is provided by the so-called $\mathcal{K}$-transform [4, 2] which takes as input a function p_n on $\mathbb{C}^n \times \{0,1\}^n$ depending on n complex variables

$$\boldsymbol{\beta}_n = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n \tag{2.27}$$

and n discrete variables $\boldsymbol{\ell}_n \in \{0,1\}^n$. As an output, the $\mathcal{K}$-transform generates the below function of n complex variables:

$$\mathcal{K}_n[p_n](\boldsymbol{\beta}_n) = \sum_{\boldsymbol{\ell}_n \in \{0,1\}^n} (-1)^{\overline{\boldsymbol{\ell}}_n} \prod_{k<s}^{n} \left\{ 1 - \mathrm{i} \frac{\ell_{ks} \cdot \sin[2\pi \mathfrak{b}]}{\sinh(\beta_{ks})} \right\} \cdot p_n(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n) , \tag{2.28}$$

in which $\overline{\boldsymbol{\ell}}_n = \sum_{a=1}^{n} \ell_k$.

Within the bootstrap program setting, the operators, *aka* quantum fields, of the Sinh-Gordon quantum field theory are built from a specific choice of p_n functions, namely those solving the equations

a) $\boldsymbol{\beta}_n \mapsto p_n(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$ is a collection of $2\mathrm{i}\pi$ periodic holomorphic functions on $\mathbb{C}$, $p_n(\boldsymbol{\beta}_n + 2\mathrm{i}\pi \boldsymbol{e}_1 \mid \boldsymbol{\ell}_n) = p_n(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$ with $\boldsymbol{e}_1$ the first basis vector, that are symmetric with respect to simultaneous permutations of the coordinates of the vector $\boldsymbol{\beta}_n$ and $\boldsymbol{\ell}_n$;

b) $p_n(\beta_2 + \mathrm{i}\pi, \boldsymbol{\beta}'_n \mid \boldsymbol{\ell}_n) = g(\ell_1, \ell_2) p_{n-2}(\boldsymbol{\beta}''_n \mid \boldsymbol{\ell}''_n) + h(\ell_1, \ell_2 \mid \boldsymbol{\beta}'_n)$ for some function h not depending on the remaining set of variables $\boldsymbol{\ell}''_n = (\ell_3, \dots, \ell_n)$, with

$$g(0,1) = g(1,0) = \frac{-1}{\sin(2\pi\mathfrak{b})\, \mathtt{F}(\mathrm{i}\pi)} , \tag{2.29}$$

and where $\boldsymbol{\beta}'_n = (\beta_2, \dots, \beta_n)$ while $\boldsymbol{\beta}''_n = (\beta_3, \dots, \beta_n)$;

c) $p_n(\boldsymbol{\beta}_n + \theta \overline{\boldsymbol{e}}_n \mid \boldsymbol{\ell}_n) = \mathrm{e}^{\theta \mathsf{s}} \cdot p_n(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$ with $\overline{\boldsymbol{e}}_n = (1, \dots, 1) \in \mathbb{C}^n$;

d) $\left| p_n(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n) \right| \leq C \cdot \prod_{a=1}^{n} \left| \cosh\left[\Re(\beta_a) \right] \right|^{\mathfrak{w}}$.

The system of equations is labelled by two numbers characteristic of the class of operators one wants to build:

- the quantity s called the spin;
- the real number $\mathfrak{w}$ called growth index.

We do stress that these two quantities, s and $\mathfrak{w}$ are n-independent.

The system has numerous solutions, although their full classification is still an open problem. We shall henceforth label given solutions with a superscript α referring to the operator O_α that they represent, *viz.* $p_n^{(\alpha)}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$. Likewise, the spin or growth index of a given operator will be denoted s_α, $\mathfrak{w}_\alpha$. Below, we only provide the expression for the p_n function associated with the field operator $\boldsymbol{\Psi}$

$$p_n^{(\boldsymbol{\Psi})}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n) = \left(\frac{-\mathrm{i}}{\sqrt{2 \sin(\pi \mathfrak{b})}} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi\mathfrak{b}} \frac{t \mathrm{d}t}{\sin t} \right\} \right)^n \cdot \frac{2\pi \mathrm{i} \mathfrak{b}}{g} \cdot \sum_{p=1}^{n} (-1)^{\ell_p} . \tag{2.30}$$

We stress that the spin and the growth index in the case of the field $\boldsymbol{\Psi}$ p-function read

$$\mathsf{s}_{\boldsymbol{\Psi}} = \mathfrak{w}_{\boldsymbol{\Psi}} = 0 . \tag{2.31}$$

There are many other examples of p_n functions representing other operators of interest to the theory, such as the exponential of the field, its descendents, or the energy-momentum tensor. see *e.g.* [2, 5].

We are finally in position to introduce the fundamental building block of the expressions for the multi-point functions. Given a solution $p_n^{(\alpha)}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n)$ to the above mentioned system of equations, the associated form factor takes the form [4, 2]:

$$\mathcal{F}^{(\alpha)}(\boldsymbol{\beta}_n) \;=\; \prod_{a<b}^{n} \mathtt{F}(\beta_{ab}) \cdot \mathcal{K}_n[p_n^{(\alpha)}](\boldsymbol{\beta}_n) \qquad \text{with} \qquad \beta_{ab} = \beta_a - \beta_b \;, \tag{2.32}$$

where $\mathtt{F}$ is as given in (2.25), $\mathcal{K}_n$ is as introduced through (2.28) while $p_n^{(\alpha)}$ is the solution to $a) - d)$ associated with the operator indexed by α.

We stress that $\mathcal{F}^{(\alpha)}$ does depend on the dimensionality n of its argument $\boldsymbol{\beta}_n \in \mathbb{C}^n$, however to keep notations simple in what will follow, we choose to keep this dependence on the dimension implicit. This does not lead to ambiguity since it should simply be read-off from the dimensionality of the evaluated vector.

Given an operator with index α, we shall denote by $\alpha^\dagger$ the index of its adjoint operator (which may or may not coincide with α). The form factors of the adjoint are built exactly as in (2.32) with the function

$$p_n^{(\alpha^\dagger)}(\boldsymbol{\beta}_n \mid \boldsymbol{\ell}_n) \;=\; \overline{p_n^{(\alpha)}\Big(\overline{\boldsymbol{\beta}_n} + \mathrm{i}\pi\overline{\boldsymbol{e}} \mid \overleftarrow{\boldsymbol{\ell}}_n\Big)} \;. \tag{2.33}$$

Thus, whether the operator is self-adjoint or not is thus a property of its p-function. In particular, the field operator, as follows from (2.30), is self-adjoint. (2.33) implies the relation

$$\mathcal{F}^{(\alpha^\dagger)}(\boldsymbol{\beta}_n) \;=\; \overline{\mathcal{F}^{(\alpha)}\Big(\overline{\overleftarrow{\boldsymbol{\beta}_n}} + \mathrm{i}\pi\overline{\boldsymbol{e}}\Big)} \;. \tag{2.34}$$

We now list all of the properties of the form factors playing a role in the proofs to come:

- **Exchange property**:

$$\mathcal{F}^{(\alpha)}(\boldsymbol{\beta}_n) \;=\; \mathtt{S}(\beta_{a\,a+1}) \cdot \mathcal{F}^{(\alpha)}(\tau_a \cdot \boldsymbol{\beta}_n) \;. \tag{2.35}$$

- **Pole structure**: For $\{\beta_s\}_{1,\neq a}^{n}$, the maps $\beta_a \mapsto \mathcal{F}^{(\alpha)}(\boldsymbol{\beta}_n)$ are meromorphic functions on $\mathbb{C}$. For pairwise distinct coordinates, *i.e.* $\beta_b \neq \beta_c$ for any $b \neq c \in [\![\, 1\,;\, n \,]\!] \setminus \{a\}$, there are only simple poles. These are located in

$$\Big\{\beta_b + \mathrm{i}\pi + 2\mathrm{i}\pi\mathbb{Z}\Big\}_{\substack{b=1\\ \neq a}}^{n} \quad \text{and} \quad \Big\{\beta_b - 2\mathrm{i}\pi\mathfrak{b} + 2\mathrm{i}\pi\mathbb{N}^*\Big\}_{b=1}^{a-1} \quad \text{and} \quad \Big\{\beta_b - 2\mathrm{i}\pi\hat{\mathfrak{b}} + 2\mathrm{i}\pi\mathbb{N}^*\Big\}_{b=1}^{a-1} \tag{2.36}$$

  as well as in

$$\Big\{\beta_b + 2\mathrm{i}\pi\mathfrak{b} + 2\mathrm{i}\pi\mathbb{N}^*\Big\}_{b=a+1}^{n} \quad \text{and} \quad \Big\{\beta_b + 2\mathrm{i}\pi\hat{\mathfrak{b}} + 2\mathrm{i}\pi\mathbb{N}^*\Big\}_{b=a+1}^{n} \;. \tag{2.37}$$

- **Lorentz boost**: $\mathcal{F}^{(\alpha)}(\boldsymbol{\beta}_n + \theta\,\overline{\boldsymbol{e}}_n) \;=\; \mathrm{e}^{\theta \mathtt{s}_\alpha} \cdot \mathcal{F}^{(\alpha)}(\boldsymbol{\beta}_n)$ with $\overline{\boldsymbol{e}}_n = (1, \ldots, 1)$ and $\mathtt{s}_\alpha$ being the spin of the operator indexed by α.

- **Growth at infinity**: For every $1 \leq k \leq n$, $\beta_k \mapsto \mathcal{F}^{(\alpha)}(\boldsymbol{\beta}_n)$ is bounded at infinity by $C \cdot \cosh\left(\mathfrak{w}_\alpha \Re(\beta_k)\right)$, with $\mathfrak{w}_\alpha$ being the growth index of the operator labelled by α.

We stress that the properties of the form factors are completely transparent to the operator α-label. Some, weak influence of the operator manifests itself on the level of the Lorentz boost and growth at infinity property.

As we shall demonstrate in the following, the Wightman axioms hold independently on the value taken by the growth index $\mathfrak{w}_\alpha$.

The elementary exchange property (2.35) of the form factors allows one to define S-matrices subordinate to the exchange of two complex valued vectors $\boldsymbol{A} = (a_1, \ldots, a_n)$ and $\boldsymbol{B} = (b_1, \ldots, b_m)$ of arbitrary sizes n, m. For that purpose, we introduce the concatenation of two vectors

$$\boldsymbol{A} \cup \boldsymbol{B} = (a_1, \ldots, a_n, b_1, \ldots, b_m) . \tag{2.38}$$

Then,

$$\mathcal{F}(\boldsymbol{A} \cup \boldsymbol{B}) = \mathsf{S}(\boldsymbol{A} \cup \boldsymbol{B} \,|\, \boldsymbol{B} \cup \boldsymbol{A}) \cdot \mathcal{F}(\boldsymbol{B} \cup \boldsymbol{A}) . \tag{2.39}$$

In fact, these can be computed explicitly and read

$$\mathsf{S}(\boldsymbol{A} \cup \boldsymbol{B} \,|\, \boldsymbol{B} \cup \boldsymbol{A}) = \prod_{j=1}^{n} \prod_{\ell=1}^{m} \mathsf{S}(a_j - b_\ell) . \tag{2.40}$$

It will also be of use to introduce the operation of reversing a vector's coordinates: given $\boldsymbol{A} = (a_1, \ldots, a_n)$, we set $\overleftarrow{\boldsymbol{A}} = (a_n, \ldots, a_1)$.

2.2.2 The multi-point functions

The work [20] computed, on rigorous grounds, the multi-point correlation functions of appropriately regularised field operators in the Sinh-Gordon model from the bootstrap axiom formulation of the theory. These were shown to be well-defined tempered distributions. We first describe this construction and then explain how, conjecturally, these will yield the *per se* expression for the multi-point correlation functions in the Sinh-Gordon quantum field theory.

First of all, given an integer $\boldsymbol{n} = (n_{21}, n_{31}, n_{32}, n_{41}, \ldots, n_{kk-1}) \in \mathbb{N}^{\frac{k(k-1)}{2}}$, and $G \in \mathcal{S}((\mathbb{R}^{1,1})^k)$, the momentum transform of G reads

$$\mathcal{R}[G](\boldsymbol{\gamma}) = \int_{(\mathbb{R}^{1,1})^k} \mathrm{d}^k \boldsymbol{X} \cdot G(\boldsymbol{X}_k) \cdot \prod_{b>a}^{k} \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) * \boldsymbol{x}_{ba}} , \qquad \text{with} \qquad \boldsymbol{x}_{ba} = \boldsymbol{x}_b - \boldsymbol{x}_a . \tag{2.41}$$

In this expression,

$$\boldsymbol{\gamma} = (\boldsymbol{\gamma}^{(21)}, \boldsymbol{\gamma}^{(31)}, \boldsymbol{\gamma}^{(32)}, \ldots, \boldsymbol{\gamma}^{(kk-1)}) \in \mathbb{C}^{|\boldsymbol{n}|} \qquad \text{with} \qquad |\boldsymbol{n}| = \sum_{b>a}^{k} n_{ba} \tag{2.42}$$

while $\boldsymbol{\gamma}^{(ba)} \in \mathbb{C}^{n_{ba}}$. Furthermore, we agree upon

$$\overline{\boldsymbol{p}}(\boldsymbol{\beta}_m) = \sum_{a=1}^{m} \boldsymbol{p}(\beta_a) \qquad \text{with} \qquad \boldsymbol{p}(\beta) = m\,(\cosh(\beta), \sinh(\beta)) \tag{2.43}$$

being the relativistic 2-momentum of a particle of mass m.

Pick solutions $p_n^{(\alpha_1)}, \dots, p_n^{(\alpha_k)}$ to the equations $a) - d)$ associated with operator labels $\alpha_1, \dots, \alpha_k$ and consider the associated form factors $\mathcal{F}^{(\alpha_1)}, \dots, \mathcal{F}^{(\alpha_k)}$ as defined through (2.32). Given $\boldsymbol{n} \in \mathbb{N}^{\frac{k(k-1)}{2}}$ and agreeing upon $\boldsymbol{\alpha}_k = (\alpha_1, \dots, \alpha_k)$ this allows one to introduce the integral transform

$$\mathcal{I}_{\boldsymbol{\alpha}_k}^{(\boldsymbol{n})}[G] \;=\; \lim_{\boldsymbol{\varepsilon}_k \to \boldsymbol{0}^+} \int\limits_{\mathbb{R}^{|\boldsymbol{n}|}} \frac{\mathrm{d}^{|\boldsymbol{n}|}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \Big(\mathcal{S} \cdot \mathcal{R}[G] \cdot \mathcal{F}_{\boldsymbol{\alpha}_k;\boldsymbol{\varepsilon}_k}\Big)(\boldsymbol{\gamma}) \,. \tag{2.44}$$

First of all, the $\boldsymbol{\varepsilon}_k = (\varepsilon_1, \dots, \varepsilon_k) \to \boldsymbol{0}^+$ limits are to be understood as $\varepsilon_a \to 0^+$ for $a = 1, \dots, k$. These can be taken in any order and correspond to computing the + boundary values of multi-dimensional Cauchy transforms. Its definition involves two additional building blocks. $\mathcal{S}$ is built from elementary vector exchange products of S-matrices

$$\mathcal{S}(\boldsymbol{\gamma}) \;=\; \prod_{\substack{v>p \\ p\geq 3}}^{k} \prod_{u>s}^{p-1} \mathsf{S}(\boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(ps)} \mid \boldsymbol{\gamma}^{(ps)} \cup \boldsymbol{\gamma}^{(vu)}) \,. \tag{2.45}$$

The second factor is built from a product of form factors:

$$\mathcal{F}_{\boldsymbol{\alpha}_k;\boldsymbol{\varepsilon}_k}(\boldsymbol{\gamma}) \;=\; \prod_{p=1}^{k} \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \dots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\gamma}^{(kp)} \cup \dots \cup \boldsymbol{\gamma}^{(p+1p)}\Big) \,, \tag{2.46}$$

with $\boldsymbol{\varepsilon}_k = (\varepsilon_1, \dots, \varepsilon_k)$, $\overleftarrow{\boldsymbol{X}}_k$ as defined in (2.7) and where $\overline{\boldsymbol{e}}_\varepsilon$ takes the form

$$\overline{\boldsymbol{e}}_\varepsilon \;=\; \Big(1 - \frac{\varepsilon}{\pi}\Big) \cdot (1, \dots, 1) \,. \tag{2.47}$$

Note that the dimensionality of the vector $\overline{\boldsymbol{e}}_\varepsilon$ changes from one form factor to another and is undercurrent by the dimensionality of the vector to which it is added.

The well-definiteness of (2.44) was established in [20].

Proposition 2.3. *[20] For any $\boldsymbol{n} \in \mathbb{N}^{\frac{k(k-1)}{2}}$, $\mathcal{I}_{\boldsymbol{\alpha}_k}^{(\boldsymbol{n})}$ is a well-defined tempered distribution. In particular, for any $\boldsymbol{\varepsilon}_k$ small enough and such that $\varepsilon_a > 0$, the integrals are strongly convergent.*

We are finally in position to introduce the $\boldsymbol{r} \in \mathbb{N}^{k-1}$-truncated correlation functions. Given a $k-1$ dimensional integer valued vector $\boldsymbol{r} = (r_1, \dots, r_{k-1}) \in \mathbb{N}^{k-1}$, one associates with it a subset of $\mathbb{N}^{\frac{k(k-1)}{2}}$:

$$\mathcal{N}_{\boldsymbol{r}} \;=\; \Big\{\boldsymbol{n} \;=\; (n_{21}, n_{31}, n_{32}, n_{41}, \dots, n_{kk-1}) \;:\; \sum_{u=p+1}^{k} \sum_{s=1}^{p} n_{us} \;=\; r_p \quad p = 1, \dots, k-1\Big\} \,. \tag{2.48}$$

Definition 2.4. *Given $G \in \mathcal{S}((\mathbb{R}^{1,1})^k)$, the $\boldsymbol{r}$-truncated correlation functions corresponds to the below sum*

$$\mathcal{W}_{\boldsymbol{\alpha}_k}^{(\boldsymbol{r})}[G] \;=\; \sum_{\boldsymbol{n} \in \mathcal{N}_{\boldsymbol{r}}} \mathcal{I}_{\boldsymbol{\alpha}_k}^{(\boldsymbol{n})}[G] \,. \tag{2.49}$$

Above one sums over integer valued vectors $\boldsymbol{n}$ belonging to $\mathcal{N}_{\boldsymbol{r}} \subset \mathbb{N}^{\frac{k(k-1)}{2}}$ defined in (2.48).

By virtue of Proposition 2.3, being defined as finite linear combinations of the $\mathcal{I}^{(\boldsymbol{n})}_{\boldsymbol{\alpha}_k}$, the $\mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\alpha}_k}$ are well-defined tempered distributions. On this level, the role played by multi-index truncation parameter $\boldsymbol{r}$ becomes apparent in that it imposes dealing only with *finite* sums. Thus well-definiteness of the expression follows from the the one of the individual building blocks $\mathcal{I}^{(\boldsymbol{n})}_{\boldsymbol{\alpha}_k}$. In the bootstrap formalism, $\mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\alpha}_k}$ corresponds to a correlation function of certain "regularised" field operators which only connect a finite portion of the Fock Hilbert space. We refer to [20] for more details. The *per se* correlation functions of the Sinh-Gordon quantum field theory are only obtained by summing up the contributions of each individual $\boldsymbol{r}$-truncated terms $\mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\alpha}_k}$, *viz.*

$$\mathcal{W}_{\boldsymbol{\alpha}_k}[G] \;=\; \sum_{\boldsymbol{r}\in\mathbb{N}^{k-1}} \mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\alpha}_k}[G] \;. \tag{2.50}$$

The convergence problem of such series is, in general, a hard open problem that will *not* be addressed here. So far, convergence was only proven in the case of two-point functions [19], *viz.* for $k = 2$, and for test functions G having mutually purely space-like support (*i.e.* $G(\boldsymbol{x},\boldsymbol{y}) \neq 0$ only if $(\boldsymbol{x}-\boldsymbol{y})^2 < 0$)

This work established that the collection of tempered distributions $\{\mathcal{W}_{\boldsymbol{\alpha}_k}\}_{\boldsymbol{\alpha}_k}$ defined above fulfills the system of Wightman axioms, under their well-definiteness conjecture which we state below:

Conjecture 2.5. *For any $k \in \mathbb{N}$, any $G \in \mathcal{S}((\mathbb{R}^{1,1})^k)$ and any labels $\alpha_1, \dots, \alpha_k$ of solutions to the system a)-d), the k-point correlation function*

$$\mathcal{W}_{\boldsymbol{\alpha}_k}[G] = \sum_{\boldsymbol{n}\in\mathbb{N}^{\frac{k(k-1)}{2}}} \mathcal{I}^{(\boldsymbol{n})}_{\boldsymbol{\alpha}_k}[G] \;, \tag{2.51}$$

is given by an absolutely converging series and defines a tempered distribution.

Below, given $g_1, \dots, g_k \in \mathcal{S}(\mathbb{R}^{1,1})$, we shall often use the short-hand notation

$$G_k(\boldsymbol{X}_k) \;=\; \prod_{s=1}^{k} g_s(\boldsymbol{x}_s) \;. \tag{2.52}$$

3 Proof of the Wightman Axioms

The present section and those that will follow will be dedicated to proving that the k-point correlation functions $\mathcal{W}_{\boldsymbol{\alpha}_k}$ (2.51) do satisfy the Wightman axioms under the well-definiteness Conjecture 2.5. For that, one considers a choice of k initial $p^{(\alpha_a)}$ functions, $a = 1, \dots, k$, leading to form factors $\mathcal{F}^{(\alpha_1)}, \dots, \mathcal{F}^{(\alpha_k)}$ according to the construction (2.32) and thus satisfying the corresponding properties (exchange, pole structure, Lorentz boost, growth at infinity). The spins $\mathsf{s}_{\alpha_1}, \dots, \mathsf{s}_{\alpha_k}$ and the growth indices $\mathfrak{w}_{\alpha_1}, \dots, \mathfrak{w}_{\alpha_k}$ will thus be taken generic, what allows us to show that the Wightman axioms are satisfied by a larger class of correlation functions than those of the fields alone.

3.1 The covariance axiom

We start with the first axiom.

Proposition 3.1. *For any operator indices $\boldsymbol{\alpha}_k = (\alpha_1, \dots, \alpha_k)$, any $G \in \mathcal{S}((\mathbb{R}^{1,1})^k)$ and $(\theta, \boldsymbol{v}) \in \mathbb{R} \times \mathbb{R}^{1,1}$, it holds*

$$\mathcal{I}^{(\boldsymbol{n})}_{\boldsymbol{\alpha}_k}[U_{\theta,\boldsymbol{v}} \cdot G] \;=\; \prod_{a=1}^{k} \Big\{ \mathrm{e}^{-\theta \mathsf{s}_{\alpha_k}} \Big\} \cdot \mathcal{I}^{(\boldsymbol{n})}_{\boldsymbol{\alpha}_k}[G] \qquad \textit{with} \qquad \boldsymbol{n} \in \mathbb{N}^{k\frac{k-1}{2}} \;. \tag{3.1}$$

$\mathcal{I}^{(\boldsymbol{n})}_{\alpha_k}$ *has been introduced in* (2.44)*, while the action of the orthochronous Poincaré group on* $\mathcal{S}((\mathbb{R}^{1,1})^k)$ *is defined in* (2.4).

Moreover, provided Conjecture 2.5 holds, the covariance property is inherited by the per se correlation function

$$\mathcal{W}_{\alpha_k}[U_{\theta,\boldsymbol{v}} \cdot G] \; = \; \prod_{a=1}^{k} \Big\{ \mathrm{e}^{\theta \mathrm{s}_{\alpha_k}} \Big\} \cdot \mathcal{W}_{\alpha_k}[G] \; . \tag{3.2}$$

Proof — Starting from (2.41), one gets

$$\mathcal{R}[\mathrm{U}_{\theta,\boldsymbol{v}} \cdot G](\boldsymbol{\gamma}) \; = \int\limits_{(\mathbb{R}^{1,1})^k} \mathrm{d}^k \boldsymbol{X} \cdot G(\Lambda_\theta^{-1} \cdot (\boldsymbol{x}_1 + \boldsymbol{v}), \dots, \Lambda_\theta^{-1} \cdot (\boldsymbol{x}_k + \boldsymbol{v})) \cdot \prod_{b>a}^{k} \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) * \boldsymbol{x}_{ba}} \; . \tag{3.3}$$

At this stage, one proceeds with the change of variable whose Jacobian is 1:

$$\boldsymbol{x}_s \; = \; \Lambda_\theta \cdot \boldsymbol{y}_s - \boldsymbol{v} \qquad \text{for} \qquad s \in [\![\, 1 \, ; \, k \,]\!] \tag{3.4}$$

leading to

$$\mathcal{R}[\mathrm{U}_{\theta,\boldsymbol{v}} \cdot G](\boldsymbol{\gamma}) \; = \int\limits_{(\mathbb{R}^{1,1})^k} \mathrm{d}^k \boldsymbol{Y} \cdot G(\boldsymbol{Y}_k) \cdot \prod_{b>a}^{k} \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) * (\Lambda_\theta \cdot \boldsymbol{y}_{ba})} \; . \tag{3.5}$$

Now using the invariance of the Minkowski scalar product under the action of the $O(1,1)$ group $(\Lambda_\theta \cdot \boldsymbol{x}) * (\Lambda_\theta \cdot \boldsymbol{y}) = \boldsymbol{x} * \boldsymbol{y}$ and the fact that $\Lambda_\theta^{-1} = \Lambda_{-\theta}$, one gets

$$\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) * (\Lambda_\theta \cdot \boldsymbol{y}_{ba}) \; = \; (\Lambda_\theta^{-1} \cdot \overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)})) * \cdot \boldsymbol{y}_{ba} \; = \; \overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)} + \theta \, \overline{\boldsymbol{e}} \,) * \boldsymbol{y}_{ba} \; . \tag{3.6}$$

It then remains to translate the integration variables $\boldsymbol{\gamma} \mapsto \boldsymbol{\gamma} - \theta \overline{\boldsymbol{e}}$ and observe that

$$\mathcal{S}(\boldsymbol{\gamma} - \theta \, \overline{\boldsymbol{e}} \,) \; = \; \mathcal{S}(\boldsymbol{\gamma}) \tag{3.7}$$

since its explicit expression only depends on the difference of $\boldsymbol{\gamma}$ entries *c.f.* (2.40) and (2.45), while

$$\begin{aligned} \mathcal{F}_{\alpha_k;\boldsymbol{\varepsilon}_k}(\boldsymbol{\gamma} - \theta \overline{\boldsymbol{e}}) \; &= \; \prod_{p=1}^{k} \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \dots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi \overline{\boldsymbol{e}}_{\varepsilon_p} - \theta \overline{\boldsymbol{e}}, \boldsymbol{\gamma}^{(kp)} \cup \dots \cup \boldsymbol{\gamma}^{(p+1p)} - \theta \overline{\boldsymbol{e}} \Big) \\ &= \; \prod_{a=1}^{k} \Big\{ \mathrm{e}^{-\theta \mathrm{s}_{\alpha_k}} \Big\} \cdot \mathcal{F}_{\alpha_k;\boldsymbol{\varepsilon}_k}(\boldsymbol{\gamma}) \; , \end{aligned} \tag{3.8}$$

due to the Lorentz boost property of the form factors (axiom **IV**)). This entails the claim. ■

3.2 Spectral condition

We now establish the spectral condition axiom. For that purpose, we introduce the linear map

$$\mathcal{L} : \mathcal{S}((\mathbb{R}^{1,1})^k) \to \mathcal{S}((\mathbb{R}^{1,1})^{k-1}) \tag{3.9}$$

with

$$\mathcal{L}[G](\boldsymbol{Y}_{k-1}) \;=\; \int\limits_{\mathbb{R}^{1,1}} \mathrm{d}\boldsymbol{y}_k \; G\Big(\sum_{s=1}^{k} \boldsymbol{y}_s, \dots, \boldsymbol{y}_k\Big) \,. \tag{3.10}$$

Further, we set

$$\mathrm{P}_\ell(\boldsymbol{\gamma}) \;=\; \sum_{b=\ell+1}^{k} \sum_{a=1}^{\ell} \overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) \;=\; \Big(\mathrm{P}_\ell^{(0)}(\boldsymbol{\gamma}), \mathrm{P}_\ell^{(1)}(\boldsymbol{\gamma})\Big) \qquad \text{for} \qquad \ell \in [\![\, 1 \,;\, k-1 \,]\!] \,. \tag{3.11}$$

This allows us to introduce a reduced momentum transform on $\mathcal{S}((\mathbb{R}^{1,1})^{k-1})$

$$\widehat{\mathcal{R}}[H](\boldsymbol{\gamma}) \;=\; \int\limits_{(\mathbb{R}^{1,1})^{k-1}} \mathrm{d}^{k-1}\boldsymbol{Y} \; H(\boldsymbol{Y}_{k-1}) \prod_{\ell=1}^{k-1} \Big\{ \mathrm{e}^{-\mathrm{i}\boldsymbol{y}_\ell * \mathrm{P}_\ell(\boldsymbol{\gamma})} \Big\} \,. \tag{3.12}$$

We then introduce the tempered distribution on $\mathcal{S}((\mathbb{R}^{1,1})^{k-1})$

$$\mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}[H] \;=\; \lim_{\varepsilon_k \to 0^+} \int\limits_{\mathbb{R}^{|\boldsymbol{n}|}} \frac{\mathrm{d}^{|\boldsymbol{n}|}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \Big(\mathcal{S} \cdot \widehat{\mathcal{R}}[H] \cdot \mathcal{F}_{\alpha_k;\varepsilon_k}\Big)(\boldsymbol{\gamma}) \,. \tag{3.13}$$

The fact that $\mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}$ is well-defined will be established below.

Proposition 3.2. *For every $\boldsymbol{r} \in \mathbb{N}^{k-1}$, the distribution*

$$\mathcal{V}^{(\boldsymbol{r})}_{\alpha_k}[H] \;=\; \sum_{\boldsymbol{n} \in \mathcal{N}_{\boldsymbol{r}}} \mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}[H] \tag{3.14}$$

on $\mathcal{S}((\mathbb{R}^{1,1})^{k-1})$ is well-defined and such that it holds:

$$\mathcal{W}^{(\boldsymbol{r})}_{\alpha_k}[G] \;=\; \mathcal{V}^{(\boldsymbol{r})}_{\alpha_k}[\mathcal{L}[G]] \qquad \textit{with} \qquad G \in \mathcal{S}((\mathbb{R}^{1,1})^{k}) \,. \tag{3.15}$$

Moreover, the Fourier transform of $\mathcal{V}^{(\boldsymbol{r})}_{\alpha_k}$ satisfies the support condition

$$\mathrm{supp}\Big(\mathscr{F}[\mathcal{V}^{(\boldsymbol{r})}_{\alpha_k}]\Big) \subseteq \mathcal{E}^{+}_{k-1} \,, \tag{3.16}$$

with $\mathcal{E}^{+}_{k-1}$ as given in (2.10).

In particular, provided Conjecture 2.5 holds, the spectral condition property is inherited by the per se correlation function

$$\mathcal{W}_{\alpha_k}[G] \;=\; \mathcal{V}_{\alpha_k}[\mathcal{L}[G]] \qquad \textit{with} \qquad \mathcal{V}_{\alpha_k} \;=\; \sum_{\boldsymbol{n} \in \mathbb{N}^{\frac{1}{2}k(k-1)}} \mathcal{J}^{(\boldsymbol{n})}_{\alpha_k} \,. \tag{3.17}$$

Proof —

We first focus on the momentum transform (2.41) $\mathcal{R}[G]$ in which we implement the change of variables :

$$\boldsymbol{x}_s \;=\; \sum_{p=s}^{k} \boldsymbol{y}_p \quad \text{for} \quad s = 1, \dots, k \,, \quad \textit{viz.} \quad \left\{ \begin{array}{ccc} \boldsymbol{y}_k & = & \boldsymbol{x}_k \\ \boldsymbol{y}_s & = & \boldsymbol{x}_s - \boldsymbol{x}_{s+1} \end{array} \right. \,. \tag{3.18}$$

In particular, one has $\boldsymbol{x}_{ba} \;=\; -\sum_{p=a}^{b-1} \boldsymbol{y}_p$ for $b > a$. This yields

$$\sum_{b>a}^{k} \overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) * \boldsymbol{x}_{ba} \;=\; -\sum_{b=2}^{k}\sum_{a=1}^{b-1}\sum_{\ell=a}^{b-1} \overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) * \boldsymbol{y}_\ell \;=\; -\sum_{b=2}^{k}\sum_{\ell=1}^{b-1} \boldsymbol{y}_\ell * \sum_{a=1}^{\ell} \overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) \;=\; -\sum_{\ell=1}^{k-1} \boldsymbol{y}_\ell * \mathrm{P}_\ell(\boldsymbol{\gamma}) \,, \tag{3.19}$$

where P_ℓ is as introduced in (3.11). Since the change of variables has unit Jacobian, one gets the equality

$$\mathcal{R}[G](\boldsymbol{\gamma}) \;=\; \widehat{\mathcal{R}}[\mathcal{L}[G]](\boldsymbol{\gamma}) \,. \tag{3.20}$$

The latter immediately yields

$$\mathcal{I}^{(\boldsymbol{n})}_{\alpha_k}[G] \;=\; \mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}[\mathcal{L}[G]] \,. \tag{3.21}$$

Upon applying this identity to functions

$$G(\boldsymbol{X}_k) \;=\; g(\boldsymbol{x}_k) \cdot H(\boldsymbol{x}_1 - \boldsymbol{x}_2, \dots, \boldsymbol{x}_{k-1} - \boldsymbol{x}_k) \qquad \text{where} \qquad (g, H) \in \mathcal{S}(\mathbb{R}^{1,1}) \times \mathcal{S}((\mathbb{R}^{1,1})^{k-1}) \,, \tag{3.22}$$

and observing that in such a case $\mathcal{L}[G]$ takes the particularly simple form

$$\mathcal{L}[G](\boldsymbol{Y}_{k-1}) \;=\; H(\boldsymbol{Y}_{k-1}) \int_{\mathbb{R}^{1,1}} \mathrm{d}\boldsymbol{y}\, g(\boldsymbol{y}) \,, \tag{3.23}$$

one gets

$$\mathcal{I}^{(\boldsymbol{n})}_{\alpha_k}[G] \;=\; \mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}[H] \cdot \int_{\mathbb{R}^{1,1}} \mathrm{d}\boldsymbol{y} g(\boldsymbol{y}) \,. \tag{3.24}$$

This entails that $\mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}$ is a well-defined tempered distribution, and so is $\mathcal{V}^{(\boldsymbol{r})}_{\alpha_k}$.
Turning on to the properties of the Fourier transform, one first observes that

$$\widehat{\mathcal{R}}[\mathscr{F}[H]](\boldsymbol{\gamma}) \;=\; \int_{\left(\mathbb{R}^{1,1}\right)^{k-1}} \mathrm{d}^{k-1}\boldsymbol{Y}\; \mathscr{F}[H](\boldsymbol{Y}_{k-1}) \prod_{\ell=1}^{k-1} \Big\{ \mathrm{e}^{-\mathrm{i}\boldsymbol{y}_\ell * \mathrm{P}_\ell(\boldsymbol{\gamma})} \Big\} \;=\; (2\pi)^{2(k-1)} H(\mathrm{P}_1(\boldsymbol{\gamma}), \dots, \mathrm{P}_{k-1}(\boldsymbol{\gamma})) \,. \tag{3.25}$$

This implies that

$$\Big(\mathscr{F}[\mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}]\Big)[H] \;=\; \mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}\Big[\mathscr{F}[H]\Big] \;=\; \lim_{\varepsilon_k \to 0^+} \int_{\mathbb{R}^{\boldsymbol{n}}} \frac{\mathrm{d}^{\boldsymbol{n}}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \Big(\mathcal{S} \cdot \mathcal{F}_{\alpha_k;\varepsilon_k}\Big)(\boldsymbol{\gamma}) \cdot\; (2\pi)^{2(k-1)} H(\mathrm{P}_1(\boldsymbol{\gamma}), \dots, \mathrm{P}_{k-1}(\boldsymbol{\gamma})) \,. \tag{3.26}$$

It is obvious from (2.43) and (3.11) that $\mathrm{P}^{(0)}_\ell(\boldsymbol{\gamma}) > 0$. Further, the obvious lower bound $\cosh(\gamma^{(ba)}_k) \;>\; |\sinh(\gamma^{(ba)}_k)|$ leads to

$$\sum_{b=\ell+1}^{k}\sum_{a=1}^{\ell}\sum_{s=1}^{n_{ba}} \cosh(\gamma^{(ba)}_s) \;>\; \Big| \sum_{b=\ell+1}^{k}\sum_{a=1}^{\ell}\sum_{s=1}^{n_{ba}} \sinh(\gamma^{(ba)}_s) \Big| \tag{3.27}$$

what ensures that, for any $\ell \in [\![\, 1 \,;\, k-1 \,]\!]$, $\mathrm{P}^2_\ell(\boldsymbol{\gamma}) = (\mathrm{P}^{(0)}_\ell(\boldsymbol{\gamma}))^2 - (\mathrm{P}^{(1)}_\ell(\boldsymbol{\gamma}))^2 > 0$. This means that

$$\Big(\mathscr{F}[\mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}]\Big)[H] \;=\; 0 \qquad \text{if} \qquad \mathrm{supp}[H] \cap \mathcal{E}^+_{k-1} \;=\; \emptyset \,, \tag{3.28}$$

i.e. that $\mathrm{supp}\Big[\mathscr{F}[\mathcal{J}^{(\boldsymbol{n})}_{\alpha_k}]\Big] \subset \mathcal{E}^+_{k-1}$. ■

3.3 Hermiticity

This subsection establishes that the truncated correlation functions satisfy the hermiticity axiom.

Proposition 3.3. *Let $G \in \mathcal{S}((\mathbb{R}^{1,1})^k)$, $\boldsymbol{r} \in \mathbb{N}^{k-1}$ and pick operator indices $\alpha_1, \dots, \alpha_k$ associated with solutions to the a) − d) constrains. Let $\alpha_1^\dagger, \dots, \alpha_k^\dagger$ be the associated indices of the adjoint operators.*

Then, the truncated correlation functions enjoy the below form of hermiticity:

$$\mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\alpha}_k}[G] = \overline{\mathcal{W}^{(\overleftarrow{\boldsymbol{r}})}_{\overleftarrow{\boldsymbol{\alpha}}_k^\dagger}\big[\,\overline{\iota \cdot G}\,\big]} \,. \tag{3.29}$$

Above, $\overleftarrow{\boldsymbol{\alpha}}_k^\dagger = (\alpha_k^\dagger, \dots, \alpha_1^\dagger)$.

Proof —

The truncated correlator of interest admits the representation

$$\overline{\mathcal{W}^{(\overleftarrow{\boldsymbol{r}})}_{\overleftarrow{\boldsymbol{\alpha}}_k^\dagger}\big[\,\overline{\iota \cdot G}\,\big]} = \sum_{\boldsymbol{n} \in \mathcal{N}_{\overleftarrow{\boldsymbol{r}}}} \overline{\mathcal{I}^{(\boldsymbol{n})}_{\overleftarrow{\boldsymbol{\alpha}}_k^\dagger}\big[\,\overline{\iota \cdot G}\,\big]} \tag{3.30}$$

where the summand can be represented as

$$\overline{\mathcal{I}^{(\boldsymbol{n})}_{\overleftarrow{\boldsymbol{\alpha}}_k^\dagger}\big[\,\overline{\iota \cdot G}\,\big]} = \lim_{\boldsymbol{\varepsilon}_k \to \boldsymbol{0}^+} \int\limits_{\mathbb{R}^{|\boldsymbol{n}|}} \frac{\mathrm{d}^{|\boldsymbol{n}|}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \overline{\Big(\mathcal{S} \cdot \mathcal{R}\big[\,\overline{\iota \cdot G}\,\big] \cdot \mathcal{F}_{\overleftarrow{\boldsymbol{\alpha}}_k^\dagger ; \overleftarrow{\boldsymbol{\varepsilon}}_k}\Big)(\boldsymbol{\gamma})} \,. \tag{3.31}$$

Note that the presence of $\overleftarrow{\boldsymbol{\varepsilon}}_k$ follows from the relabelling $\varepsilon_p \hookrightarrow \varepsilon_{k+1-p}$ of the regularisation parameters in the limit. We now study the building blocks one-by-one. First of all, one has

$$\overline{\mathcal{R}\big[\,\overline{\iota \cdot G}\,\big](\boldsymbol{\gamma})} = \int\limits_{(\mathbb{R}^{1,1})^k} \mathrm{d}^k \boldsymbol{X}\, G(\overleftarrow{\boldsymbol{X}}_k) \cdot \prod_{b>a}^{k} \mathrm{e}^{-\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}) * \boldsymbol{x}_{ba}} \,. \tag{3.32}$$

There, we introduce new variables $\boldsymbol{y}_a = \boldsymbol{x}_{k+1-a}$ and denote

$$\boldsymbol{\xi}^{(ba)} = \boldsymbol{\gamma}^{(k+1-a\, k+1-b)} \quad \text{for } b > a \quad \text{as well as} \quad m_{ba} = n_{k+1-a\, k+1-b} \,. \tag{3.33}$$

This yields

$$\begin{aligned} \overline{\mathcal{R}\big[\,\overline{\iota \cdot G}\,\big](\boldsymbol{\gamma})} &= \int\limits_{(\mathbb{R}^{1,1})^k} \mathrm{d}^k \boldsymbol{Y}\, G(\boldsymbol{Y}_k) \cdot \prod_{b>a}^{k} \mathrm{e}^{-\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\xi}^{(k+1-a\, k+1-b)}) * \boldsymbol{y}_{k+1-b\, k+1-a}} & (3.34) \\ &= \int\limits_{(\mathbb{R}^{1,1})^k} \mathrm{d}^k \boldsymbol{Y}\, G(\boldsymbol{Y}_k) \cdot \prod_{b>a}^{k} \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\xi}^{(ba)}) * \boldsymbol{y}_{ba}} = \mathcal{R}[G](\boldsymbol{\xi}) \,. & (3.35) \end{aligned}$$

We now turn to the $\mathcal{S}$-factor. By using that $\mathrm{S}(\beta) = \overline{\mathrm{S}(-\beta)}$ and introducing the variables $\boldsymbol{\xi}^{(ba)}$ as in (3.33), one gets

$$\begin{aligned} \overline{\mathcal{S}(\boldsymbol{\gamma})} &= \prod_{v>p>u>s}^{k} \mathrm{S}\Big(\boldsymbol{\xi}^{(k+1-s\, k+1-p)} \cup \boldsymbol{\xi}^{(k+1-u\, k+1-v)} \mid \boldsymbol{\xi}^{(k+1-u\, k+1-v)} \cup \boldsymbol{\xi}^{(k+1-s\, k+1-p)}\Big) & (3.36) \\ &= \prod_{t>r>c>\ell}^{k} \mathrm{S}\Big(\boldsymbol{\xi}^{(tc)} \cup \boldsymbol{\xi}^{(r\ell)} \mid \boldsymbol{\xi}^{(r\ell)} \cup \boldsymbol{\xi}^{(tc)}\Big) = \mathcal{S}(\boldsymbol{\xi}) \,. & (3.37) \end{aligned}$$

Finally, we turn towards the product of form factors which we re-express in terms of the variables $\boldsymbol{\xi}^{(ba)}$:

$$\mathcal{F}_{\overleftarrow{\alpha}_k^{\dagger};\overleftarrow{\varepsilon}_k}(\boldsymbol{\gamma}) = \prod_{p=1}^{k} \mathcal{F}^{(\alpha_{k+1-p}^{\dagger})}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_{k+1-p}}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)} \Big) \tag{3.38}$$

$$= \prod_{p=1}^{k} \mathcal{F}^{(\alpha_p^{\dagger})}\Big(\overleftarrow{\boldsymbol{\gamma}^{(k+1-p\,k-p)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(k+1-p\,1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\gamma}^{(k\,k+1-p)} \cup \cdots \cup \boldsymbol{\gamma}^{(k+2-p\,k+1-p)} \Big) \tag{3.39}$$

$$= \prod_{p=1}^{k} \mathcal{F}^{(\alpha_p^{\dagger})}\Big(\overleftarrow{\boldsymbol{\xi}^{(p+1p)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\xi}^{(kp)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\xi}^{(p1)} \cup \cdots \cup \boldsymbol{\xi}^{(pp-1)} \Big)\,. \tag{3.40}$$

One then invokes the relation between the form factors of an operator with label α and its adjoint (2.34), so as to get

$$\overline{\mathcal{F}_{\overleftarrow{\alpha}_k^{\dagger};\overleftarrow{\varepsilon}_k}(\boldsymbol{\gamma})} = \prod_{p=1}^{k} \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\xi}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\xi}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}, \boldsymbol{\xi}^{(kp)} \cup \cdots \cup \boldsymbol{\xi}^{(p+1p)} - \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p} + \mathrm{i}\pi\overline{\boldsymbol{e}} \Big) \tag{3.41}$$

$$= \prod_{p=1}^{k}\Big\{ \mathrm{e}^{\mathrm{i}\varepsilon_p \mathrm{s}_{\alpha_p}} \Big\} \cdot \prod_{p=1}^{k} \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\xi}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\xi}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\xi}^{(kp)} \cup \cdots \cup \boldsymbol{\xi}^{(p+1p)} \Big) \tag{3.42}$$

$$= \prod_{p=1}^{k}\Big\{ \mathrm{e}^{\mathrm{i}\varepsilon_p \mathrm{s}_{\alpha_p}} \Big\} \cdot \mathcal{F}_{\boldsymbol{\alpha}_k;\boldsymbol{\varepsilon}_k}(\boldsymbol{\xi})\,. \tag{3.43}$$

The last line follows by invoking the Lorentz symmetry property of the form factors.

Thus, all-in-all, we have established that with $\boldsymbol{m}$ as defined through (3.33)

$$\overline{\mathcal{I}^{(\boldsymbol{n})}_{\overleftarrow{\alpha}_k^{\dagger}}\big[\overline{\iota\cdot G}\big]} = \lim_{\boldsymbol{\varepsilon}_k \to \boldsymbol{0}^+}\Big\{ \prod_{p=1}^{k}\Big\{ \mathrm{e}^{\mathrm{i}\varepsilon_p \mathrm{s}_{\alpha_p}} \Big\} \cdot \int_{\mathbb{R}^{|\boldsymbol{m}|}} \frac{\mathrm{d}^{|\boldsymbol{m}|}\boldsymbol{\xi}}{\boldsymbol{m}!(2\pi)^{|\boldsymbol{m}|}} \Big(\mathcal{S}\cdot\mathcal{R}[G]\cdot\mathcal{F}_{\boldsymbol{\alpha}_k;\boldsymbol{\varepsilon}_k}\Big)(\boldsymbol{\xi}) = \mathcal{I}^{(\boldsymbol{m})}_{\boldsymbol{\alpha}_k}[G]\,, \tag{3.44}$$

since both $\boldsymbol{\varepsilon}_k$ dependent factors admit $\boldsymbol{\varepsilon}_k \to \boldsymbol{0}^+$ limits.

Finally, one observes that if

$$r_{k-p} = \sum_{u=p+1}^{k}\sum_{s=1}^{p} n_{us} \quad \text{for} \quad p \in [\![\, 1\,;\, k-1 \,]\!] \qquad \text{then} \qquad r_t = \sum_{u=t+1}^{k}\sum_{s=1}^{t} m_{us} \quad \text{for} \quad t \in [\![\, 1\,;\, k-1 \,]\!]\,, \tag{3.45}$$

with m_{ba} as defined in (3.33). Thus the sets $\mathcal{N}_{\boldsymbol{r}}$ and $\mathcal{N}_{\overleftarrow{\boldsymbol{r}}}$ are in one-to-one correspondence under the map $\boldsymbol{n} \hookrightarrow \boldsymbol{m}$. The above entails that

$$\overline{\mathcal{W}^{(\overleftarrow{\boldsymbol{r}})}_{\overleftarrow{\alpha}_k^{\dagger}}\big[\overline{\iota\cdot G}\big]} = \sum_{\boldsymbol{m}\in\mathcal{N}_{\overleftarrow{\boldsymbol{r}}}} \mathcal{I}^{(\boldsymbol{m})}_{\boldsymbol{\alpha}_k}[G] = \mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\alpha}_k}[G]\,, \tag{3.46}$$

hence leading to the claim. ■

3.4 Local Commutativity

We first start by recalling an alternative representation for the truncated k-point function obtained in [20].

Proposition 3.4. *[20] Let $G \in \mathcal{S}((\mathbb{R}^{1,1})^k)$ and pick $t \in [\![1 ; k]\!]$ and $\boldsymbol{r} \in \mathbb{N}^{k-1}$. Then, one has the below, well-defined, representation for the $\boldsymbol{r}$-truncated k-point function. This alternative representation does coincide with the one given in* (2.49).

$$\mathcal{W}^{(\boldsymbol{r})}_{\alpha_k}[G] \;=\; \sum_{\boldsymbol{n}\in\mathcal{N}_{\boldsymbol{r}}} \mathcal{I}^{(\boldsymbol{n})}_{\alpha_k;t}[G] \tag{3.47}$$

The building blocks are expressed as

$$\mathcal{I}^{(\boldsymbol{n})}_{\alpha_k;t}[G] \;=\; \lim_{\boldsymbol{\varepsilon}_k\to\boldsymbol{0}^+} \int_{\mathbb{R}^{|\boldsymbol{n}|}} \frac{\mathrm{d}^{|\boldsymbol{n}|}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \Big(\mathcal{S}^{(t)}\cdot\mathcal{R}[G]\cdot\mathcal{F}^{(t)}_{\alpha_k;\boldsymbol{\varepsilon}_k}\Big)(\boldsymbol{\gamma})\;, \tag{3.48}$$

with

$$\mathcal{S}^{(t)}(\boldsymbol{\gamma}) \;=\; \mathcal{S}(\boldsymbol{\gamma})\cdot \prod_{v=t+1}^{k}\prod_{u=1}^{t-1}\Bigg\{\prod_{s=1}^{t-1} S\big(\boldsymbol{\gamma}^{(ts)}\cup\boldsymbol{\gamma}^{(vu)} \mid \boldsymbol{\gamma}^{(vu)}\cup\boldsymbol{\gamma}^{(ts)}\big)\cdot \prod_{s=t+1}^{k} S\big(\boldsymbol{\gamma}^{(vu)}\cup\boldsymbol{\gamma}^{(st)} \mid \boldsymbol{\gamma}^{(st)}\cup\boldsymbol{\gamma}^{(vu)}\big)\Bigg\}\;, \tag{3.49}$$

$\mathcal{S}$ as defined in (2.45) *and*

$$\begin{aligned}\mathcal{F}^{(t)}_{\alpha_k;\boldsymbol{\varepsilon}_k}(\boldsymbol{\gamma}) \;=\; &\prod_{\substack{p=1\\p\neq t}}^{k} \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}}\cup\cdots\cup\overleftarrow{\boldsymbol{\gamma}^{(p1)}}+\mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p},\boldsymbol{\gamma}^{(kp)}\cup\cdots\cup\boldsymbol{\gamma}^{(p+1p)}\Big)\\ &\times\mathcal{F}^{(\alpha_t)}\Big(\boldsymbol{\gamma}^{(kt)}\cup\cdots\cup\boldsymbol{\gamma}^{(t+1t)},\overleftarrow{\boldsymbol{\gamma}^{(tt-1)}}\cup\cdots\cup\overleftarrow{\boldsymbol{\gamma}^{(t1)}}-\mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t}\Big)\;.\end{aligned} \tag{3.50}$$

Finally, $\mathcal{R}[G]$ is as defined through (2.41).

We will also make use of another representation for the general term to have more freedom with the deformation of contours:

Proposition 3.5. *[20] Let $G \in \mathcal{S}((\mathbb{R}^{1,1})^k)$ and pick $t \in [\![1 ; k]\!]$ and $\boldsymbol{n} \in \mathbb{N}^{\frac{k(k-1)}{2}}$. Then, $\mathcal{I}^{(\boldsymbol{n})}_{\alpha_k;t}[G]$ as defined in* (3.48) *admit the below integral representation:*

$$\mathcal{I}^{(\boldsymbol{n})}_{\alpha_k;t}[G] \;=\; \lim_{\eta_1,\eta_2\to\boldsymbol{0}^+}\;\lim_{\delta_{\mathcal{S}},\delta_{\mathcal{R}}\to\boldsymbol{0}^+} \int_{\mathbb{R}^{|\boldsymbol{n}|}} \frac{\mathrm{d}^{|\boldsymbol{n}|}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \mathcal{S}^{(t)}(\boldsymbol{\gamma}+\mathrm{i}\boldsymbol{\delta}_{\mathcal{S}})\cdot\mathcal{R}[G](\boldsymbol{\gamma}+\mathrm{i}\boldsymbol{\delta}_{\mathcal{R}})\cdot\mathcal{F}^{(t)}_{\alpha_k}(\boldsymbol{\gamma};\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)\;, \tag{3.51}$$

where all limits can be freely interchanged and we introduced the following deformation parameters:

- *$\boldsymbol{\delta}_{\mathcal{S}} \in \mathbb{R}^{|\boldsymbol{n}|}$ is small enough so that $\boldsymbol{\gamma}\mapsto\mathcal{S}^{(t)}(\boldsymbol{\gamma}+\mathrm{i}\boldsymbol{\delta}_{\mathcal{S}})$ remains bounded on $\mathbb{R}^{|\boldsymbol{n}|}$;*
- *$\boldsymbol{\delta}_{\mathcal{R}} \in \mathbb{R}^{|\boldsymbol{n}|}$ is chosen such that $\boldsymbol{\gamma}^{(ba)} \mapsto \Re\Big[\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)}+\mathrm{i}\boldsymbol{\delta}^{(ba)}_{\mathcal{R}})*\boldsymbol{x}_{ba}\Big]$ is bounded from above for every $b > a$ uniformly in $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k)\in\operatorname{supp}[G]$, ensuring that $\mathcal{R}[G](\boldsymbol{\gamma}+\mathrm{i}\boldsymbol{\delta}_{\mathcal{R}})$ is well-defined;*
- *$\boldsymbol{\eta}_1 = (\eta_1^{(21)},\ldots,\eta_1^{(kk-1)}) \in \mathbb{R}_+^{\frac{k(k-1)}{2}}$ and $\boldsymbol{\eta}_2 = (\eta_2^{(21)},\ldots,\eta_2^{(kk-1)}) \in \mathbb{R}_+^{\frac{k(k-1)}{2}}$ are small and under the constraints that $\eta_1^{(cb)}+\eta_2^{(dc)} > 0$ for all $1\le b<c<d\le k$.*

Finally:

$$\mathcal{F}^{(t)}_{\alpha_k}(\boldsymbol{\gamma};\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) \;=\; \prod_{\substack{p=1\\p\neq t}}^{k}\Big\{\mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}}+\mathrm{i}(\pi-\eta_1^{(pp-1)})\overline{\boldsymbol{e}},\,\dots\,,\overleftarrow{\boldsymbol{\gamma}^{(p1)}}+\mathrm{i}(\pi-\eta_1^{(p1)})\overline{\boldsymbol{e}},$$

$$\boldsymbol{\gamma}^{(kp)}+\mathrm{i}\eta_2^{(kp)}\overline{\boldsymbol{e}},\dots,\boldsymbol{\gamma}^{(p+1p)}+\mathrm{i}\eta_2^{(p+1p)}\overline{\boldsymbol{e}}\Big)\Big\}$$

$$\times\,\mathcal{F}^{(\alpha_t)}\Big(\boldsymbol{\gamma}^{(kt)}-\mathrm{i}\eta_2^{(kt)}\overline{\boldsymbol{e}},\,\dots\,,\boldsymbol{\gamma}^{(t+1t)}-\mathrm{i}\eta_2^{(t+1t)}\overline{\boldsymbol{e}},\overleftarrow{\boldsymbol{\gamma}^{(tt-1)}}-\mathrm{i}(\pi-\eta_1^{(tt-1)})\overline{\boldsymbol{e}},\,\dots\,,\overleftarrow{\boldsymbol{\gamma}^{(t1)}}-\mathrm{i}(\pi-\eta_1^{(t1)})\overline{\boldsymbol{e}}\Big)\,. \tag{3.52}$$

The representation for the $\boldsymbol{r}$-truncated k-point function provided by propositions 3.4-3.5 is well-tailored for proving the local commutativity axiom of the k-point function. The proof we shall give below follows the line of ideas developed in the context of studying the weak local commutativity of two local operators in the Sinh-Gordon model: first in the context of the quantum Gelfand-Levitan-Marchenko equation construction of the model [15], second through certain identities which translate weak local commutativity into one of the bootstrap axioms [16] and third for the Thirring model in [18]. The main idea is to relate the correlation functions with operator indices α_t and α_{t+1} swapped and with the test function $\tau_t\cdot G$ to the original one by performing a contour integral deformation in (3.48). However, the technical details between considering the weak commutativity of the local operator and the one within a correlation function are different. Moreover, the mentioned work did not consider various rigorous aspects of the handlings.

Proposition 3.6. *Let* $t\in[\![\,1\,;\,k-1\,]\!]$ *and pick* $G\in\mathcal{S}((\mathbb{R}^{1,1})^k)$ *such that* $\operatorname{supp}[G]\subset\Omega_t$ *with* Ω_t *as given in* (2.16) *Then, provided that Conjecture 2.5 holds, the correlation functions satisfy*

$$\mathcal{W}_{\alpha_k}[G] \;=\; \mathcal{W}_{\tau_t\cdot\alpha_k}[\tau_t\cdot G]\,, \tag{3.53}$$

where τ_t *is the transposition* $(tt+1)$*, c.f.* (2.6)-(2.7).

Proof —

To start, observe that

$$\Omega_t \;=\; \Omega_t^+\cup\Omega_t^- \qquad\text{with}\qquad \Omega_t^\pm \;=\; \Big\{\boldsymbol{X}_k\in\Omega_t\;:\;\pm(x_{t;1}-x_{t+1;1})>0\Big\}\,. \tag{3.54}$$

$\Omega_t^\pm$ are open and connected. It is thus enough to focus on the case when $\operatorname{supp}[G]\subset\Omega_t^\pm$. We shall carry out the proof in the case when $\operatorname{supp}[G]\subset\Omega_t^+$, the other case can be dealt with quite similarly.

We start from the representation for the $\boldsymbol{r}$-truncated k point function provided by Proposition 3.4 with the $t+1^{\text{st}}$ operator singled out, which involves

$$\mathcal{S}^{(t+1)}(\boldsymbol{\gamma}) \;=\; \mathcal{S}(\boldsymbol{\gamma})\cdot\prod_{v=t+2}^{k}\prod_{u=1}^{t}\Big\{\prod_{s=1}^{t}\mathrm{S}(\boldsymbol{\gamma}^{(t+1s)}\cup\boldsymbol{\gamma}^{(vu)}\mid\boldsymbol{\gamma}^{(vu)}\cup\boldsymbol{\gamma}^{(t+1s)})\cdot\prod_{s=t+2}^{k}\mathrm{S}(\boldsymbol{\gamma}^{(vu)}\cup\boldsymbol{\gamma}^{(st+1)}\mid\boldsymbol{\gamma}^{(st+1)}\cup\boldsymbol{\gamma}^{(vu)})\Big\}\,, \tag{3.55}$$

as well as

$$\mathcal{F}^{(t+1)}_{\alpha_k;\varepsilon_k}(\boldsymbol{\gamma}) \;=\; \prod_{\substack{p=1\\\neq t,t+1}}^{k}\mathfrak{f}^{(p)}(\boldsymbol{\gamma})\cdot\mathcal{F}^{(\alpha_t)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(tt-1)}}\cup\cdots\cup\overleftarrow{\boldsymbol{\gamma}^{(t1)}}+\mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t},\boldsymbol{\gamma}^{(kt)}\cup\cdots\cup\boldsymbol{\gamma}^{(t+1t)}\Big)$$

$$\times\,\mathcal{F}^{(\alpha_{t+1})}\Big(\boldsymbol{\gamma}^{(kt+1)}\cup\cdots\cup\boldsymbol{\gamma}^{(t+2t+1)},\overleftarrow{\boldsymbol{\gamma}^{(t+1t)}}\cup\cdots\cup\overleftarrow{\boldsymbol{\gamma}^{(t+11)}}-\mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_{t+1}}\Big)\,, \tag{3.56}$$

where we agreed upon

$$\mathfrak{f}^{(p)}(\boldsymbol{\gamma}) \;=\; \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)}\Big)\,. \tag{3.57}$$

One now applies the contour deformation for the variables $\boldsymbol{\gamma}^{(t+1t)}$ from

$$\mathbb{R}^{n_{t+1t}} \qquad \text{to} \qquad \Big\{\mathbb{R} + \mathrm{i}(\pi - \varepsilon_t - \varepsilon_{t+1})\Big\}^{n_{t+1t}}. \tag{3.58}$$

We check that no poles of the integrand are crossed in the process. The pole structure of the form factors given in Subsection 2.2.1 indicates that the contour deformation crosses the simple poles of (3.56) located at

$$\gamma_p^{(vt)} - \gamma_q^{(t+1t)} \in \Big\{-2\mathrm{i}\pi\mathfrak{b}, -2\mathrm{i}\pi\hat{\mathfrak{b}}\Big\}\;, \quad v = t+2, \dots, k\;, \quad p \in [\![\, 1\,;\, n_{vt} \,]\!] \quad \text{and} \quad q \in [\![\, 1\,;\, n_{t+1t} \,]\!]\,. \tag{3.59}$$

These are compensated by zeroes of the S-matrix factors (3.55) at the corresponding points. Thus, the choice of the representation one starts with ensures that no poles are crossed in the process of the contour deformation. Further, one observes that the exponential factor present in the momentum transform part $\mathcal{R}[G]$ with $\mathcal{R}$ as introduced in (2.41) remains bounded due to the condition $x_{t;1} - x_{t+1;1} > 0$ being always fulfilled throughout the support of G. The contour deformation modifies the expression of the integrand. The $\boldsymbol{\gamma}^{(t+1t)}$ contribution to the momentum modifies as $\mathcal{R}[G](\boldsymbol{\gamma}) \hookrightarrow \widetilde{\mathcal{R}}[G](\boldsymbol{\gamma} \mid \varepsilon_t, \varepsilon_{t+1})$ with

$$\widetilde{\mathcal{R}}[G](\boldsymbol{\gamma} \mid \varepsilon_t, \varepsilon_{t+1}) \;=\; \int\limits_{(\mathbb{R}^{1,1})^k} \mathrm{d}^k\boldsymbol{X} \cdot G(\boldsymbol{X}_k) \cdot \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}(\varepsilon_t+\varepsilon_{t+1})\,\overline{\boldsymbol{e}}) * \boldsymbol{x}_{tt+1}} \cdot \prod_{\substack{d>c \\ (d,c)\neq(t+1,t)}} \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(dc)}) * \boldsymbol{x}_{dc}}\,. \tag{3.60}$$

one has that

$$\begin{aligned}\mathcal{F}^{(t+1)}_{\boldsymbol{\alpha}_k;\boldsymbol{\varepsilon}_k}(\boldsymbol{\gamma}) \;\hookrightarrow\; \prod_{\substack{p=1 \\ \neq t,t+1}}^{k} \mathfrak{f}^{(p)}(\boldsymbol{\gamma}) \cdot \mathcal{F}^{(\alpha_t)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(tt-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t}, \boldsymbol{\gamma}^{(kt)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t)}, \boldsymbol{\gamma}^{(t+1t)} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t+\varepsilon_{t+1}}\Big) \\ \times \mathcal{F}^{(\alpha_{t+1})}\Big(\boldsymbol{\gamma}^{(kt+1)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t+1)}, \overleftarrow{\boldsymbol{\gamma}^{(t+1t)}} - \mathrm{i}\varepsilon_t\overline{\boldsymbol{e}}, \overleftarrow{\boldsymbol{\gamma}^{(t+1t-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t+11)}} - \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_{t+1}}\Big)\,.\end{aligned} \tag{3.61}$$

Indeed, only the form factors labelled by α_t and α_{t+1} depend on $\boldsymbol{\gamma}^{(t+1t)}$. Finally, focusing on the transformation incurred by $\mathcal{S}^{(t+1)}(\boldsymbol{\gamma})$, one observes that the expression (2.45) for $\mathcal{S}(\boldsymbol{\gamma})$ involves pairs of variables $\boldsymbol{\gamma}^{(vu)}$ and $\boldsymbol{\gamma}^{(ps)}$ with indices ordered as $v > p > u > s$. Hence, the v, u or p, s integers are at least distant by 2: the expression for $\mathcal{S}(\boldsymbol{\gamma})$ does not involve the variables $\boldsymbol{\gamma}^{(t+1t)}$. It thus remains unaltered in the contour deformation process. Therefore, by unitarity of S, one gets that

$$\begin{aligned}\mathcal{S}^{(t+1)}(\boldsymbol{\gamma}) \;\hookrightarrow\; \mathcal{S}(\boldsymbol{\gamma}) \cdot \prod_{v=t+2}^{k}\prod_{u=1}^{t}\Big\{\mathsf{S}(\boldsymbol{\gamma}^{(vu)} \cup [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}] \mid [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}] \cup \boldsymbol{\gamma}^{(vu)}) \\ \times \prod_{s=1}^{t-1} \mathsf{S}(\boldsymbol{\gamma}^{(t+1s)} \cup \boldsymbol{\gamma}^{(vu)} \mid \boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(t+1s)}) \cdot \prod_{s=t+2}^{k} \mathsf{S}(\boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(st+1)} \mid \boldsymbol{\gamma}^{(st+1)} \cup \boldsymbol{\gamma}^{(vu)})\Big\}\,.\end{aligned} \tag{3.62}$$

Above, we have set $\overline{\varepsilon}_{tt+1} = \varepsilon_t + \varepsilon_{t+1}$.

Next, one swaps the positions of $\boldsymbol{\gamma}^{(t+1p)}$ and $\boldsymbol{\gamma}^{(tp)}$ in the form factor labelled by α_p with $p < t$, by using the exchange property, *c.f.* (2.39):

$$\begin{aligned}\mathfrak{f}^{(p)}(\boldsymbol{\gamma}) \;=\; & \mathsf{S}(\boldsymbol{\gamma}^{(t+1p)} \cup \boldsymbol{\gamma}^{(tp)} \mid \boldsymbol{\gamma}^{(tp)} \cup \boldsymbol{\gamma}^{(t+1p)}) \\ & \times \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(tp)} \cup \boldsymbol{\gamma}^{(t+1p)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)}\Big)\,.\end{aligned} \tag{3.63}$$

Further, one swaps $\boldsymbol{\gamma}^{(t+1p)}$ and $\boldsymbol{\gamma}^{(tp)}$ in each form factor labelled by α_p with $p > t+1$,

$$\mathfrak{f}^{(p)}(\boldsymbol{\gamma}) \;=\; \mathrm{S}(\boldsymbol{\gamma}^{(pt+1)} \cup \boldsymbol{\gamma}^{(pt)} \mid \boldsymbol{\gamma}^{(pt)} \cup \boldsymbol{\gamma}^{(pt+1)}) \\ \times \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(pt)}} \cup \overleftarrow{\boldsymbol{\gamma}^{(pt+1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)}\Big). \quad (3.64)$$

Finally, we move $\boldsymbol{\gamma}^{(t+1t)}$ through all other variables in the form factor labelled by α_t:

$$\mathcal{F}^{(\alpha_t)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(tt-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t}, \boldsymbol{\gamma}^{(kt)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t)}, \boldsymbol{\gamma}^{(t+1t)} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\overline{\varepsilon}_{tt+1}}\Big) \\ = \prod_{a=1}^{t-1} \mathrm{S}\Big(\boldsymbol{\gamma}^{(ta)} \cup [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\varepsilon_{t+1}\overline{\boldsymbol{e}}] \mid [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\varepsilon_{t+1}\overline{\boldsymbol{e}}] \cup \boldsymbol{\gamma}^{(ta)}\Big) \prod_{a=t+2}^{k} \mathrm{S}\Big([\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}] \cup \boldsymbol{\gamma}^{(at)} \mid \boldsymbol{\gamma}^{(at)} \cup [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}]\Big) \\ \times S\Big(\boldsymbol{\gamma}^{(t+1t)} \mid \overleftarrow{\boldsymbol{\gamma}^{(t+1t)}}\Big) \cdot \mathcal{F}^{(\alpha_t)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(t+1t)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\overline{\varepsilon}_{tt+1}}, \overleftarrow{\boldsymbol{\gamma}^{(tt-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t}, \boldsymbol{\gamma}^{(kt)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t)}\Big) \quad (3.65)$$

while we reorder $\boldsymbol{\gamma}^{(tt-1)}$ in the form factor labelled by α_{t+1}

$$\mathcal{F}^{(\alpha_{t+1})}\Big(\boldsymbol{\gamma}^{(kt+1)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t+1)}, \overleftarrow{\boldsymbol{\gamma}^{(t+1t)}} - \mathrm{i}\varepsilon_t\overline{\boldsymbol{e}}, \overleftarrow{\boldsymbol{\gamma}^{(t+1t-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t+11)}} - \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_{t+1}}\Big) = \mathrm{S}\Big(\overleftarrow{\boldsymbol{\gamma}^{(t+1t)}} \mid \boldsymbol{\gamma}^{(t+1t)}\Big) \\ \times \mathcal{F}^{(\alpha_{t+1})}\Big(\boldsymbol{\gamma}^{(kt+1)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t+1)}, \boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\varepsilon_t\overline{\boldsymbol{e}}, \overleftarrow{\boldsymbol{\gamma}^{(t+1t-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t+11)}} - \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_{t+1}}\Big). \quad (3.66)$$

Gathering together all of the S-matrix terms leads us to introduce

$$\mathcal{S}_{\mathrm{mod}}(\boldsymbol{\gamma} \mid \varepsilon_t, \varepsilon_{t+1}) \;=\; \mathcal{S}(\boldsymbol{\gamma}) \cdot \prod_{v=t+2}^{k}\prod_{u=1}^{t} \Big\{\mathrm{S}(\boldsymbol{\gamma}^{(vu)} \cup [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}] \mid [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}] \cup \boldsymbol{\gamma}^{(vu)}) \\ \times \prod_{s=1}^{t-1} \mathrm{S}(\boldsymbol{\gamma}^{(t+1s)} \cup \boldsymbol{\gamma}^{(vu)} \mid \boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(t+1s)}) \cdot \prod_{s=t+2}^{k} \mathrm{S}(\boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(st+1)} \mid \boldsymbol{\gamma}^{(st+1)} \cup \boldsymbol{\gamma}^{(vu)})\Big\} \\ \times \prod_{a=1}^{t-1} \Big\{\mathrm{S}(\boldsymbol{\gamma}^{(t+1a)} \cup \boldsymbol{\gamma}^{(ta)} \mid \boldsymbol{\gamma}^{(ta)} \cup \boldsymbol{\gamma}^{(t+1a)}) \cdot \mathrm{S}(\boldsymbol{\gamma}^{(ta)} \cup [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\varepsilon_{t+1}\overline{\boldsymbol{e}}] \mid [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\varepsilon_{t+1}\overline{\boldsymbol{e}}] \cup \boldsymbol{\gamma}^{(ta)})\Big\} \\ \times \prod_{a=t+2}^{k} \Big\{\mathrm{S}(\boldsymbol{\gamma}^{(at+1)} \cup \boldsymbol{\gamma}^{(at)} \mid \boldsymbol{\gamma}^{(at)} \cup \boldsymbol{\gamma}^{(at+1)}) \cdot \mathrm{S}([\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}] \cup \boldsymbol{\gamma}^{(at)} \mid \boldsymbol{\gamma}^{(at)} \cup [\boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}])\Big\}. \quad (3.67)$$

It is also handy to set

$$\widetilde{\mathcal{F}}^{(t+1)}_{\boldsymbol{\alpha}_k;\boldsymbol{\varepsilon}_k}(\boldsymbol{\gamma} \mid \nu_t, \nu_{t+1}) \;=\; \prod_{p=1}^{t-1} \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(tp)} \cup \boldsymbol{\gamma}^{(t+1p)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)}\Big) \\ \times \prod_{p=t+2}^{k} \mathcal{F}^{(\alpha_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(pt)}} \cup \overleftarrow{\boldsymbol{\gamma}^{(pt+1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_p}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)}\Big) \\ \times \mathcal{F}^{(\alpha_t)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(t+1t)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t+\nu_{t+1}}, \overleftarrow{\boldsymbol{\gamma}^{(tt-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_t}, \boldsymbol{\gamma}^{(kt)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t)}\Big) \\ \times \mathcal{F}^{(\alpha_{t+1})}\Big(\boldsymbol{\gamma}^{(kt+1)} \cup \cdots \cup \boldsymbol{\gamma}^{(t+2t+1)}, \boldsymbol{\gamma}^{(t+1t)} - \mathrm{i}\nu_t\overline{\boldsymbol{e}}, \overleftarrow{\boldsymbol{\gamma}^{(t+1t-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(t+11)}} - \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon_{t+1}}\Big). \quad (3.68)$$

Thus, we have just shown that

$$\mathcal{W}^{(\boldsymbol{r})}_{\alpha_k}[G] \;=\; \sum_{\boldsymbol{n}\in\mathcal{N}_r} \lim_{\varepsilon_k\to\mathbf{0}^+} \int_{\mathbb{R}^{|\boldsymbol{n}|}} \frac{\mathrm{d}^{|\boldsymbol{n}|}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \Big(\widetilde{\mathcal{R}}[G]\cdot\mathcal{S}_{\mathrm{mod}}\cdot\widetilde{\mathcal{F}}^{(t+1)}_{\alpha_k;\varepsilon_k}\Big)(\boldsymbol{\gamma}\mid\varepsilon_t,\varepsilon_{t+1})\;. \tag{3.69}$$

At this stage, we change variables in the integral $\boldsymbol{\gamma}\hookrightarrow\boldsymbol{\gamma}^{\sigma_t}$ so that

$$\begin{pmatrix}\boldsymbol{\gamma}^{(pt)}\\ \boldsymbol{\gamma}^{(pt+1)}\end{pmatrix}\hookrightarrow\begin{pmatrix}\boldsymbol{\gamma}^{(pt+1)}\\ \boldsymbol{\gamma}^{(pt)}\end{pmatrix}\quad\text{for}\quad p\geq t+2 \tag{3.70}$$

and

$$\begin{pmatrix}\boldsymbol{\gamma}^{(tp)}\\ \boldsymbol{\gamma}^{(t+1p)}\end{pmatrix}\hookrightarrow\begin{pmatrix}\boldsymbol{\gamma}^{(t+1p)}\\ \boldsymbol{\gamma}^{(tp)}\end{pmatrix}\quad\text{for}\quad p\leq t-1\;, \tag{3.71}$$

all other variables being unchanged. We then swap the space-time coordinates in the integral corresponding to $\widetilde{\mathcal{R}}[G](\boldsymbol{\gamma}^{\sigma_t}\mid\varepsilon_t,\varepsilon_{t+1})$:

$$\boldsymbol{x}_t\leftrightarrow\boldsymbol{x}_{t+1} \tag{3.72}$$

leading to

$$\widetilde{\mathcal{R}}[G](\boldsymbol{\gamma}^{\sigma_t}\mid\varepsilon_t,\varepsilon_{t+1}) \;=\; \mathcal{R}[\tau_t\cdot G](\boldsymbol{\gamma}+\mathrm{i}\boldsymbol{\delta}_{\mathcal{R}})\;. \tag{3.73}$$

Here $\boldsymbol{\delta}_{\mathcal{R}}$ is chosen with $\delta^{(ba)}_{\mathcal{R},\ell} = -(\varepsilon_t+\varepsilon_{t+1})\delta_{b,t+1}\delta_{a,t}$. Further, one gets that

$$\widetilde{\mathcal{F}}^{(t+1)}_{\alpha_k;\varepsilon_k}(\boldsymbol{\gamma}^{\sigma_t}\mid\varepsilon_t,\varepsilon_{t+1}) \;=\; \mathcal{F}^{(t)}_{\tau_t\cdot\alpha_k}(\boldsymbol{\gamma};\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) \tag{3.74}$$

with $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ given by $\eta_1^{(ba)} = \varepsilon_b + (\varepsilon_{t+1}-\varepsilon_t)(\delta_{b,t}-\delta_{b,t+1}) + \varepsilon_{t+1}\delta_{b,t+1}\delta_{a,t}$ and $\eta_2^{(ba)} = \varepsilon_t\delta_{b,t+1}\delta_{a,t}$.
Dealing with the S-matrix contributions demands more care. One has that

$$\begin{aligned}
\mathcal{S}_{\mathrm{mod}}(\boldsymbol{\gamma}^{\sigma_t}\mid\varepsilon_t,\varepsilon_{t+1}) = \mathcal{S}(\boldsymbol{\gamma}^{\sigma_t})\cdot\prod_{v=t+2}^{k}\Bigg\{\prod_{u=1}^{t-1}\Big[&\mathsf{S}(\boldsymbol{\gamma}^{(vu)}\cup[\boldsymbol{\gamma}^{(t+1t)}-\mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}]\mid[\boldsymbol{\gamma}^{(t+1t)}-\mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}]\cup\boldsymbol{\gamma}^{(vu)})\\
\times\,\mathsf{S}(\boldsymbol{\gamma}^{(tu)}\cup\boldsymbol{\gamma}^{(vt+1)}\mid\boldsymbol{\gamma}^{(vt+1)}\cup\boldsymbol{\gamma}^{(tu)})\cdot\prod_{s=1}^{t-1}&\mathsf{S}(\boldsymbol{\gamma}^{(ts)}\cup\boldsymbol{\gamma}^{(vu)}\mid\boldsymbol{\gamma}^{(vu)}\cup\boldsymbol{\gamma}^{(ts)})\cdot\prod_{s=t+2}^{k}\mathsf{S}(\boldsymbol{\gamma}^{(vu)}\cup\boldsymbol{\gamma}^{(st)}\mid\boldsymbol{\gamma}^{(st)}\cup\boldsymbol{\gamma}^{(vu)})\Big]\\
\times\prod_{s=t+2}^{k}\mathsf{S}(\boldsymbol{\gamma}^{(vt+1)}\cup\boldsymbol{\gamma}^{(st)}\mid\boldsymbol{\gamma}^{(st)}\cup\boldsymbol{\gamma}^{(vt+1)})\Bigg\}&\cdot\prod_{s=t+2}^{k}\mathsf{S}(\boldsymbol{\gamma}^{(st)}\cup\boldsymbol{\gamma}^{(st+1)}\mid\boldsymbol{\gamma}^{(st+1)}\cup\boldsymbol{\gamma}^{(st)})\\
\times\prod_{u=1}^{t-1}\Big\{\mathsf{S}(\boldsymbol{\gamma}^{(tu)}\cup\boldsymbol{\gamma}^{(t+1u)}\mid\boldsymbol{\gamma}^{(t+1u)}\cup\boldsymbol{\gamma}^{(tu)})\cdot&\mathsf{S}(\boldsymbol{\gamma}^{(t+1u)}\cup[\boldsymbol{\gamma}^{(t+1t)}-\mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}]\mid[\boldsymbol{\gamma}^{(t+1t)}-\mathrm{i}\overline{\varepsilon}_{tt+1}\overline{\boldsymbol{e}}]\cup\boldsymbol{\gamma}^{(t+1u)})\Big\}\;.
\end{aligned} \tag{3.75}$$

Then, one observes that

$$\begin{aligned}
\mathcal{S}(\boldsymbol{\gamma}^{\sigma_t}) \;=\; \mathcal{S}(\boldsymbol{\gamma})\cdot\prod_{\substack{p,v=t+2\\ p\neq v}}^{k}\mathsf{S}(\boldsymbol{\gamma}^{(vt)}\cup\boldsymbol{\gamma}^{(pt+1)}\mid\boldsymbol{\gamma}^{(pt+1)}\cup\boldsymbol{\gamma}^{(vt)})\prod_{\substack{u,s=1\\ u\neq s}}^{t-1}\mathsf{S}(\boldsymbol{\gamma}^{(tu)}\cup\boldsymbol{\gamma}^{(t+1s)}\mid\boldsymbol{\gamma}^{(t+1s)}\cup\boldsymbol{\gamma}^{(tu)})\\
\prod_{v=t+2}^{k}\prod_{s=1}^{t-1}\Big\{\mathsf{S}(\boldsymbol{\gamma}^{(vt+1)}\cup\boldsymbol{\gamma}^{(ts)}\mid\boldsymbol{\gamma}^{(ts)}\cup\boldsymbol{\gamma}^{(vt+1)})\cdot\mathsf{S}(\boldsymbol{\gamma}^{(t+1s)}\cup\boldsymbol{\gamma}^{(vt)}\mid\boldsymbol{\gamma}^{(vt)}\cup\boldsymbol{\gamma}^{(t+1s)})\Big\}\;.
\end{aligned} \tag{3.76}$$

These intermediate decompositions lead to $\mathcal{S}_{\mathrm{mod}}(\boldsymbol{\gamma}^{\sigma} \mid \varepsilon_t, \varepsilon_{t+1}) = \mathcal{S}^{(t)}(\boldsymbol{\gamma} + \mathrm{i}\boldsymbol{\delta}_{\mathcal{S}}) \cdot \mathcal{S}_{\mathrm{add}}(\boldsymbol{\gamma})$ with the introduced shift $\delta^{(ba)}_{\mathcal{S},\ell} = -(\varepsilon_t + \varepsilon_{t+1})\delta_{b,t+1}\delta_{a,t}$ and

$$\mathcal{S}_{\mathrm{add}}(\boldsymbol{\gamma}) = \frac{\mathcal{S}(\boldsymbol{\gamma}^{\sigma_t})}{\mathcal{S}(\boldsymbol{\gamma})} \prod_{v=t+2}^{k} \Big\{ \prod_{u=1}^{t-1} \mathsf{S}(\boldsymbol{\gamma}^{(tu)} \cup \boldsymbol{\gamma}^{(vt+1)} \mid \boldsymbol{\gamma}^{(vt+1)} \cup \boldsymbol{\gamma}^{(tu)}) \cdot \prod_{u=t+2}^{k} \mathsf{S}(\boldsymbol{\gamma}^{(vt+1)} \cup \boldsymbol{\gamma}^{(ut)} \mid \boldsymbol{\gamma}^{(ut)} \cup \boldsymbol{\gamma}^{(vt+1)}) \Big\}$$
$$\times \prod_{u,v=1}^{t-1} \mathsf{S}(\boldsymbol{\gamma}^{(t+1u)} \cup \boldsymbol{\gamma}^{(tv)} \mid \boldsymbol{\gamma}^{(tv)} \cup \boldsymbol{\gamma}^{(t+1u)}) \prod_{v=t+1}^{k} \prod_{u=1}^{t-1} \mathsf{S}(\boldsymbol{\gamma}^{(vt)} \cup \boldsymbol{\gamma}^{(t+1u)} \mid \boldsymbol{\gamma}^{(t+1u)} \cup \boldsymbol{\gamma}^{(vt)})$$
$$\times \prod_{u=1}^{t-1} \Big\{ \mathsf{S}(\boldsymbol{\gamma}^{(t+1u)} \cup \boldsymbol{\gamma}^{(t+1t)} \mid \boldsymbol{\gamma}^{(t+1t)} \cup \boldsymbol{\gamma}^{(t+1u)}) \mathsf{S}(\boldsymbol{\gamma}^{(tu)} \cup \boldsymbol{\gamma}^{(t+1u)} \mid \boldsymbol{\gamma}^{(t+1u)} \cup \boldsymbol{\gamma}^{(tu)}) \Big\}$$
$$\times \prod_{u=t+2}^{k} \mathsf{S}(\boldsymbol{\gamma}^{(ut)} \cup \boldsymbol{\gamma}^{(ut+1)} \mid \boldsymbol{\gamma}^{(ut+1)} \cup \boldsymbol{\gamma}^{(ut)}) . \quad (3.77)$$

After simplifications, one gets that $\mathcal{S}_{\mathrm{add}}(\boldsymbol{\gamma}) = 1$.

Finally, one should accomodate for the impact of the change of variables on the dimensionality of the various integrations. This leads to introduce a new collection of integers:

$$m_{bt} = n_{bt+1} , \quad m_{bt+1} = n_{bt} , \quad b \geq t+2 \quad (3.78)$$
$$m_{ta} = n_{t+1a} , \quad m_{t+1a} = n_{ta} , \quad a \leq t-1 \quad (3.79)$$
$$m_{dc} = n_{dc} , \quad \text{if} \quad d, c \notin \{t, t+1\} . \quad (3.80)$$

We denote the inverse map by $\boldsymbol{n}(\boldsymbol{m})$. The above recasts the truncated correlation function as:

$$\mathcal{W}^{(\boldsymbol{r})}_{\alpha_k}[G] = \sum_{\boldsymbol{m}:\boldsymbol{n}(\boldsymbol{m})\in\mathcal{N}_{\boldsymbol{r}}} \lim_{\eta_1,\eta_2\to0^+} \lim_{\delta_{\mathcal{S}},\delta_{\mathcal{R}}\to0^+} \int_{\mathbb{R}^{|\boldsymbol{m}|}} \frac{\mathrm{d}^{|\boldsymbol{m}|}\boldsymbol{\gamma}}{\boldsymbol{m}!(2\pi)^{|\boldsymbol{m}|}} \mathcal{S}^{(t)}(\boldsymbol{\gamma} + \mathrm{i}\boldsymbol{\delta}_{\mathcal{S}}) \cdot \mathcal{R}[\tau_t \cdot G](\boldsymbol{\gamma} + \mathrm{i}\boldsymbol{\delta}_{\mathcal{R}}) \cdot \mathcal{F}^{(t)}_{\tau_t\cdot\alpha_k}(\boldsymbol{\gamma}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) . \quad (3.81)$$

Therefore, under the assumption that the series defining the correlation function converges, one can sum over all integers $r_1, \ldots, r_{k-1}$. This relaxes the $\boldsymbol{n}(\boldsymbol{m}) \in \mathcal{N}_{\boldsymbol{r}}$ constraint to summing up over $\boldsymbol{m} \in \mathbb{N}^{\frac{k(k-1)}{2}}$, namely that

$$\mathcal{W}_{\alpha_k}[G] = \sum_{\boldsymbol{m}\in\mathbb{N}^{\frac{k(k-1)}{2}}} \lim_{\eta_1,\eta_2\to0^+} \lim_{\delta_{\mathcal{S}},\delta_{\mathcal{R}}\to\boldsymbol{0}} \int_{\mathbb{R}^{|\boldsymbol{m}|}} \frac{\mathrm{d}^{|\boldsymbol{m}|}\boldsymbol{\gamma}}{\boldsymbol{m}!(2\pi)^{|\boldsymbol{m}|}} \mathcal{S}^{(t)}(\boldsymbol{\gamma} + \mathrm{i}\boldsymbol{\delta}_{\mathcal{S}}) \cdot \mathcal{R}[\tau_t \cdot G](\boldsymbol{\gamma} + \mathrm{i}\boldsymbol{\delta}_{\mathcal{R}}) \cdot \mathcal{F}^{(t)}_{\tau_t\cdot\alpha_k}(\boldsymbol{\gamma}; \boldsymbol{\eta}_1, \boldsymbol{\eta}_2) . \quad (3.82)$$

The above is exactly the representation for $\mathcal{W}_{\tau_t\cdot\alpha_k}[\tau_t \cdot G]$ that follows by summing up over $\boldsymbol{r} \in \mathbb{N}^{k-1}$ the result of Proposition 3.5 with the t^{th} operator being singled out and our adequate choices of shifts $\boldsymbol{\delta}_{\mathcal{S}}$, $\boldsymbol{\delta}_{\mathcal{R}}$, $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, one gets the claim. ∎

3.5 Positivity

We now pass on to proving the positivity property:

Proposition 3.7. *Fix $N \in \mathbb{N}$ and pick $f^{(p)} \in \mathcal{S}((\mathbb{R}^{1,1})^p)$, $p = 0, \ldots, N$. Let $F_{p,q} \in \mathcal{S}((\mathbb{R}^{1,1})^{p+q})$ be as introduced in* (2.18). *Further, pick labels $\{\alpha_{ab}\}^N_{a\leq b}$ corresponding to solutions to a)-d).*

If Conjecture 2.5 holds, the below linear combination of correlators is non-negative:

$$\sum_{p,q=0}^{N} \mathcal{W}_{\boldsymbol{\tau}_{p+q}}[F_{p,q}] \geq 0 \qquad \textit{with} \qquad \boldsymbol{\tau}_{p+q} = (\alpha^{\dagger}_{pp}, \ldots, \alpha^{\dagger}_{1p}, \alpha_{1q}, \ldots, \alpha_{qq}) . \quad (3.83)$$

Proof. We start by writing a convenient representation for the quantity of interest. First of all, because of absolute convergence, it holds

$$\mathcal{W}_{\tau_{p+q}}[F_{p,q}] = \lim_{R\to+\infty} \lim_{M\to+\infty} \sum_{\boldsymbol{n}_{p-1}\in\mathcal{I}_M^{p-1}} \sum_{\boldsymbol{m}_{q-1}\in\mathcal{I}_M^{q-1}} \sum_{r=0}^{R} \mathcal{W}^{(\boldsymbol{r})}_{\tau_{p+q}}[F_{p,q}] \qquad \text{where} \qquad \mathcal{I}_M = [\![\, 0 \,;\, M \,]\!] \,. \tag{3.84}$$

Moreover, $\boldsymbol{r}$ arising above is to be understood as $\boldsymbol{r} = (\overleftarrow{\boldsymbol{n}}_{p-1}, r, \boldsymbol{m}_{q-1}) \in \mathbb{N}^{p+q-1}$. It is shown in Propositions 3.1, 3.2 and 4.7 of [20], that the $\boldsymbol{r}$-truncated correlator $\mathcal{W}^{(\boldsymbol{r})}_{\tau_{p+q}}[F_{p,q}]$ admits the below, well-defined distributional integral representation -see equation (3.7) of that paper-

$$\begin{aligned}\mathcal{W}^{(\boldsymbol{r})}_{\tau_{p+q}}[F_{p,q}] = \lim_{\boldsymbol{\eta}_{p+q}\to\boldsymbol{0}^+} \prod_{a=1}^{p+q-1} \bigg\{ \int_{\mathbb{R}^{r_a}} \frac{\mathrm{d}^{r_a}\alpha_a}{r_a!(2\pi)^{r_a}} \bigg\} \mathcal{M}^{(\tau_1)}_{0;r_1}(\emptyset;\boldsymbol{\alpha}_1)_{\eta_1} \cdot \mathcal{M}^{(\tau_2)}_{r_1;r_2}(\boldsymbol{\alpha}_1;\boldsymbol{\alpha}_2)_{\eta_2} \\ \cdots \mathcal{M}^{(\tau_{p+q-1})}_{r_{p+q-2};r_{p+q-1}}(\boldsymbol{\alpha}_{p+q-2};\boldsymbol{\alpha}_{p+q-1})_{\eta_{p+q-1}} \cdot \mathcal{M}^{(\tau_{p+q})}_{r_{p+q-1};0}(\boldsymbol{\alpha}_{p+q-1};\emptyset)_{\eta_{p+q}} \cdot \mathscr{R}[F_{p,q}]\Big(\{\boldsymbol{\alpha}_a\}_{a=1}^{p+q-1}\Big) \,.\end{aligned} \tag{3.85}$$

The order in which the $\boldsymbol{\alpha}_a$ integrals are taken is irrelevant and the $\eta_k \to 0^+$ limit may be moved in front of the $\boldsymbol{\alpha}_k$ integration. The first building block of this representation corresponds to the integral transform

$$\mathscr{R}[F_{p,q}]\Big(\{\boldsymbol{\alpha}_a\}_{a=1}^{p+q-1}\Big) = \int_{\left(\mathbb{R}^{1,1}\right)^{p+q-1}} \mathrm{d}^{p+q-1}\boldsymbol{Z}\; F_{p,q}(\boldsymbol{Z}_{p+q-1}) \prod_{s=1}^{p+q-1} \Big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\alpha}_s)*z_{s+1s}} \Big\} \,. \tag{3.86}$$

The remaining building blocks are given by the below distributions

$$\mathcal{M}^{(\alpha)}_{n;m}(\boldsymbol{\lambda}_n;\boldsymbol{\beta}_m)_{\varepsilon} = \sum_{A=A_1\cup A_2} \sum_{B=B_1\overset{1}{\cup}B_2} \Delta(\boldsymbol{A}_1 \mid \boldsymbol{B}_1) \cdot \mathsf{S}\Big(\overleftarrow{\boldsymbol{A}} \mid \overleftarrow{\boldsymbol{A}_2} \cup \overleftarrow{\boldsymbol{A}_1}\Big) \cdot \mathsf{S}(\boldsymbol{B} \mid \boldsymbol{B}_1 \cup \boldsymbol{B}_2) \cdot \mathcal{F}^{(\alpha)}\Big(\overleftarrow{\boldsymbol{A}_2} + \mathrm{i}\pi\overline{\boldsymbol{e}}_{\varepsilon}, \boldsymbol{B}_2\Big) \,. \tag{3.87}$$

The expression is to be understood as follows. First of all, one associates with the vectors $\boldsymbol{\lambda}_n \in \mathbb{R}^n$, $\boldsymbol{\beta}_m \in \mathbb{R}^m$ the sets $A = \{\lambda_a\}_1^n$ and $B = \{\beta_a\}_1^m$. One then sums over all partitions $A_1 \cup A_2$ of A and all partitions $B_1 \overset{1}{\cup} B_2$ of B. Moreover, the superscript 1 in the partitions of B indicates that one should, in addition, sum-up over all permutations of the elements of B_1. These partitions are constrained by the condition $|A_1| = |B_1|$. The formula uses for an ordered set $C = \{c_{k_a}\}_{a=1}^{|C|}$, $\boldsymbol{C} = (c_{k_1}, \ldots, c_{k_{|C|}})$.

Finally, given $\delta_{x,y}$ the Dirac mass centred at $x = y$, we have set

$$\Delta(\boldsymbol{A}_1 \mid \boldsymbol{B}_1) = \prod_{a=1}^{|A_1|} \Big\{ 2\pi\delta_{\lambda_{k_a},\beta_{i_a}} \Big\} \qquad \text{where} \qquad \left\{ \begin{array}{lcl} A_1 & = & \{\lambda_{k_a}\}_{a=1}^{|A_1|} \quad , \quad k_1 < \cdots < k_{|A_1|} \\ B_1 & = & \{\beta_{i_a}\}_{a=1}^{|B_1|} \quad , \quad i_1 \neq \cdots \neq i_{|B_1|} \end{array} \right. \,. \tag{3.88}$$

Upon implementing the relabelling of the integration variables

$$\boldsymbol{\alpha}_a = \boldsymbol{\lambda}_{p-a} \qquad \text{for} \quad a \in [\![\, 1 \,;\, p-1 \,]\!] \,, \quad \boldsymbol{\alpha}_p = \boldsymbol{\vartheta} \,, \quad \text{and} \quad \boldsymbol{\alpha}_{p+a} = \boldsymbol{\beta}_a \qquad \text{for} \quad a \in [\![\, 1 \,;\, q-1 \,]\!] \,, \tag{3.89}$$

changing the space-time integration variables in (3.86) to $\boldsymbol{Z}_{p+q-1} = (\overleftarrow{\boldsymbol{X}}_p, \boldsymbol{Y}_q)$ and reparameterising the regularisation parameters $\boldsymbol{\eta}_{p+q} = (\overleftarrow{\boldsymbol{\varepsilon}}_p, \boldsymbol{\delta}_q)$, the $\boldsymbol{r}$-truncated correlator is recast as

$$\mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\tau}_{p+q}}[F_{p,q}] = \int\limits_{\mathbb{R}^r} \frac{\mathrm{d}^r\boldsymbol{\vartheta}}{(2\pi)^r r!} \lim_{\boldsymbol{\varepsilon}_p\to 0^+} \prod_{s=1}^{p-1}\Bigg\{ \int\limits_{\mathbb{R}^{n_s}} \frac{\mathrm{d}^{n_s}\boldsymbol{\lambda}_s}{(2\pi)^{n_s} n_s!}\Bigg\} \cdot \lim_{\boldsymbol{\delta}_q\to 0^+} \prod_{j=1}^{q-1}\Bigg\{ \int\limits_{\mathbb{R}^{m_j}} \frac{\mathrm{d}^{m_j}\boldsymbol{\beta}_j}{(2\pi)^{m_j} m_j!}\Bigg\} \cdot \int\limits_{(\mathbb{R}^{1,1})^p} \mathrm{d}^p\boldsymbol{X} \cdot \int\limits_{(\mathbb{R}^{1,1})^q} \mathrm{d}^q\boldsymbol{Y}$$
$$\mathcal{M}^{(\alpha^\dagger_{pp})}_{0;n_{p-1}}(\emptyset;\boldsymbol{\lambda}_{p-1})_{\varepsilon_p} \cdot \mathcal{M}^{(\alpha^\dagger_{p-1p})}_{n_{p-1};n_{p-2}}(\boldsymbol{\lambda}_{p-1};\boldsymbol{\lambda}_{p-2})_{\varepsilon_{p-1}} \cdots \mathcal{M}^{(\alpha^\dagger_{2p})}_{n_2;n_1}(\boldsymbol{\lambda}_2;\boldsymbol{\lambda}_1)_{\varepsilon_2} \cdot \mathcal{M}^{(\alpha^\dagger_{1p})}_{n_1;r}(\boldsymbol{\lambda}_1;\boldsymbol{\vartheta})_{\varepsilon_1} \cdot \overline{f^{(p)}(\boldsymbol{X}_p)}$$
$$\times \mathcal{M}^{(\alpha_{1q})}_{r;m_1}(\boldsymbol{\vartheta};\boldsymbol{\beta}_1)_{\delta_1} \cdot \mathcal{M}^{(\alpha_{2q})}_{m_1;m_2}(\boldsymbol{\beta}_1;\boldsymbol{\beta}_2)_{\delta_2} \cdots \mathcal{M}^{(\alpha_{q-1q})}_{m_{q-2};m_{q-1}}(\boldsymbol{\beta}_{q-2};\boldsymbol{\beta}_{q-1})_{\delta_{q-1}} \cdot \mathcal{M}^{(\alpha_{qq})}_{m_{q-1};0}(\boldsymbol{\beta}_{q-1};\emptyset)_{\delta_q} \cdot f^{(q)}(\boldsymbol{Y}_q)$$
$$\times\, \mathrm{e}^{\mathrm{i}(\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_1)-\overline{\boldsymbol{p}}(\boldsymbol{\gamma}))*\boldsymbol{x}_1} \cdot \prod_{s=2}^{p-1} \mathrm{e}^{\mathrm{i}(\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_s)-\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_{s-1}))*\boldsymbol{x}_s} \cdot \mathrm{e}^{-\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_{p-1})*\boldsymbol{x}_p} \cdot \mathrm{e}^{-\mathrm{i}(\overline{\boldsymbol{p}}(\boldsymbol{\beta}_1)-\overline{\boldsymbol{p}}(\boldsymbol{\gamma}))*\boldsymbol{y}_1} \cdot \prod_{s=2}^{q-1} \mathrm{e}^{-\mathrm{i}(\overline{\boldsymbol{p}}(\boldsymbol{\beta}_s)-\overline{\boldsymbol{p}}(\boldsymbol{\beta}_{s-1}))*\boldsymbol{y}_s} \cdot \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\beta}_{q-1})*\boldsymbol{y}_q} . \tag{3.90}$$

By using the expression for the form factors of the adjoint operators (2.34), the elementary behaviour of the S-factors under complex conjugation

$$\overline{\mathsf{S}\big(\overleftarrow{\boldsymbol{A}} \mid \overleftarrow{\boldsymbol{A}_2} \cup \overleftarrow{\boldsymbol{A}_1}\big)} = \mathsf{S}(\boldsymbol{A} \mid \boldsymbol{A}_1 \cup \boldsymbol{A}_2) \tag{3.91}$$

the Lorentz boost property of the form factors and elementary properties of sums over partitions one infers from (3.87) that

$$\mathcal{M}^{(\alpha^\dagger)}_{n;m}(\boldsymbol{\lambda}_n;\boldsymbol{\beta}_m)_{\varepsilon} = \overline{\mathcal{M}^{(\alpha)}_{m;n}(\boldsymbol{\beta}_m;\boldsymbol{\lambda}_n)_{\varepsilon}} \cdot \mathrm{e}^{-\mathrm{i}\varepsilon \mathsf{s}_\alpha} . \tag{3.92}$$

By applying this relation and introducing

$$\mathscr{G}^{(\boldsymbol{n}_{p-1})}_{\boldsymbol{\alpha}_p}[f^{(p)}](\boldsymbol{\vartheta}) = \lim_{\boldsymbol{\varepsilon}_p\to 0^+} \prod_{s=1}^{p-1}\Bigg\{ \int\limits_{\mathbb{R}^{n_s}} \frac{\mathrm{d}^{n_s}\boldsymbol{\lambda}_s}{(2\pi)^{n_s} n_s!}\Bigg\} \int\limits_{(\mathbb{R}^{1,1})^p} \mathrm{d}^p\boldsymbol{X}\, f^{(p)}(\boldsymbol{X}_p)\, \mathrm{e}^{\mathrm{i}(\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_1)-\overline{\boldsymbol{p}}(\boldsymbol{\gamma}))*\boldsymbol{x}_1}$$
$$\times \prod_{s=2}^{p-1} \mathrm{e}^{\mathrm{i}(\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_s)-\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_{s-1}))*\boldsymbol{x}_s} \cdot \mathrm{e}^{-\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\lambda}_{p-1})*\boldsymbol{x}_p} \cdot \mathcal{M}^{(\alpha_{1p})}_{r;n_1}(\boldsymbol{\vartheta};\boldsymbol{\lambda}_1)_{\varepsilon_1} \cdots \mathcal{M}^{(\alpha_{pp})}_{n_{p-1};0}(\boldsymbol{\lambda}_{p-1};\emptyset)_{\varepsilon_q} \tag{3.93}$$

one gets that

$$\mathcal{W}^{(\boldsymbol{r})}_{\boldsymbol{\tau}_{p+q}}[F_{p,q}] = \int\limits_{\mathbb{R}^r} \frac{\mathrm{d}^r\boldsymbol{\vartheta}}{(2\pi)^r r!} \overline{\mathscr{G}^{(\boldsymbol{n}_{p-1})}_{\boldsymbol{\alpha}_p}[f^{(p)}](\boldsymbol{\vartheta})} \cdot \mathscr{G}^{(\boldsymbol{m}_{q-1})}_{\boldsymbol{\alpha}_p}[f^{(q)}](\boldsymbol{\vartheta}) . \tag{3.94}$$

Note that the factors $\overline{\mathscr{G}^{(\boldsymbol{n}_{p-1})}_{\boldsymbol{\alpha}_p}[f^{(p)}](\boldsymbol{\vartheta})}$ decay as $\boldsymbol{\vartheta} \to \infty$ faster than any polynomial in $\cosh(\vartheta_1), \dots, \cosh(\vartheta_r)$ as can be inferred from direct estimates of the momentum transform $\mathcal{R}$ and the calculation of the Dirac mass distributions as well as the taking of the $\boldsymbol{\varepsilon}_p \to \boldsymbol{0}^+$ limits. We refer to [20] for further details.

We are now in position to insert this result into (3.84), what gives

$$\sum_{p,q=0}^{N} \mathcal{W}_{\boldsymbol{\tau}_{p+q}}[F_{p,q}] = \lim_{R\to+\infty} \lim_{M\to+\infty} \int\limits_{\mathbb{R}^r} \frac{\mathrm{d}^r\boldsymbol{\vartheta}}{(2\pi)^r r!} \big|\mathscr{I}^{(M)}_r[f](\boldsymbol{\vartheta})\big|^2 \tag{3.95}$$

where

$$\mathscr{I}_r^{(M)}[f](\boldsymbol{\vartheta}) \;=\; \sum_{p=0}^{N} \sum_{\boldsymbol{n}_{p-1}\in\mathcal{I}_M^{p-1}} \mathscr{G}_{\boldsymbol{\alpha}_p}^{(\boldsymbol{n}_{p-1})}[f^{(p)}](\boldsymbol{\vartheta}) \tag{3.96}$$

and $\mathcal{I}_M$ is as defined in (3.84).

On the left hand side, the result is well-defined by assumption. On the right hand side, one takes the limit of a strictly increasing sequence. This limit is thus well-defined, and one gets that the sum in the left hand side is non-negative. ■

3.6 Cluster Decomposition

We finally check the cluster decomposition axiom which translates the loss of correlations at sufficiently large space-like interval separations. The proof relies on a few auxiliary lemmas.

Lemma 3.8. *Pick $p,q \in \mathbb{N}^*$. Let $(f,g) \in \mathcal{S}((\mathbb{R}^{1,1})^p)\times\mathcal{S}((\mathbb{R}^{1,1})^q)$, $\boldsymbol{v}\in\mathbb{R}^{1,1}$ and $G_{p,q}^{(\lambda\boldsymbol{v})}$ as introduced in* (2.21). *Let $\boldsymbol{\alpha}_p$, resp. $\boldsymbol{\beta}_q$, be p, resp. q, dimensional vectors of operators labels corresponding to solutions to a) − d) and*

$$\boldsymbol{\ell} = (\ell_{21},\ell_{31},\dots,\ell_{p+q\,p+q-1}) \in \mathbb{N}^{(p+q)\frac{p+q-1}{2}} \qquad \textit{be such that} \quad \ell_{dc} \;=\; 0 \quad \textit{for all} \quad d \;>\; p \;\geq\; c\,. \tag{3.97}$$

Then with $\boldsymbol{\tau}_{p+q} \;=\; (\boldsymbol{\alpha}_p,\boldsymbol{\beta}_q)$, one has the factorisation

$$\mathcal{I}_{\boldsymbol{\tau}_{p+q}}^{(\boldsymbol{\ell})}[G_{p,q}^{(\lambda\boldsymbol{v})}] \;=\; \mathcal{I}_{\boldsymbol{\alpha}_p}^{(\boldsymbol{n})}[f]\cdot\mathcal{I}_{\boldsymbol{\beta}_q}^{(\boldsymbol{m})}[g]\,. \tag{3.98}$$

The factorising distributions are labelled by the vector integers

$$\boldsymbol{n} \;=\; (\ell_{21},\ell_{31},\ell_{32},\dots,\ell_{pp-1}) \qquad \textit{and} \qquad \boldsymbol{m} \;=\; (\ell_{p+2\,p+1},\ell_{p+3\,p+1},\ell_{p+3\,p+2},\dots,\ell_{p+q\,p+q-1})\,. \tag{3.99}$$

Proof—

Taken that $\boldsymbol{\gamma}^{(ba)}\in\mathbb{R}^{\ell_{ba}}$, there are no integration variables $\boldsymbol{\gamma}^{(ba)}$ such that $b\geq p+1$ and $a\leq p$ in the integral representation (2.44) for $\mathcal{I}_{\boldsymbol{\tau}_{p+q}}^{(\boldsymbol{\ell})}[G_{p,q}^{(\lambda\boldsymbol{v})}]$ as soon as $\boldsymbol{\ell}$ is as in (3.97). Thus, one has that

$$\sum_{b>a}^{p+q}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)})*z_{ba} \;=\; \sum_{b>a}^{p}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)})*z_{ba} \;+\; \sum_{\substack{b>a\\ p+1}}^{p+q}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)})*z_{ba}\,. \tag{3.100}$$

Thus, by setting

$$\boldsymbol{\lambda} \;=\; (\boldsymbol{\lambda}^{(21)},\boldsymbol{\lambda}^{(31)},\dots,\boldsymbol{\lambda}^{(pp-1)}) \quad \text{with} \quad \boldsymbol{\lambda}^{(ba)} \;=\; \boldsymbol{\gamma}^{(ba)} \quad 1\leq a<b\leq p \tag{3.101}$$

and

$$\boldsymbol{\vartheta} \;=\; (\boldsymbol{\vartheta}^{(21)},\boldsymbol{\vartheta}^{(31)},\dots,\boldsymbol{\vartheta}^{(qq-1)}) \quad \text{with} \quad \boldsymbol{\vartheta}^{(ba)} \;=\; \boldsymbol{\gamma}^{(b+p\,a+p)} \quad 1\leq a<b\leq q \tag{3.102}$$

and reparameterising $\boldsymbol{Z}_{p+q} \;=\; (\boldsymbol{X}_p,\boldsymbol{Y}_q+\lambda\boldsymbol{v}_q)$ with $\boldsymbol{v}_q=(\boldsymbol{v},\dots,\boldsymbol{v})\in(\mathbb{R}^{1,1})^q$, one thus gets

$$\mathcal{R}[G_{p,q}^{(\lambda\boldsymbol{v})}](\boldsymbol{\gamma}) \;=\; \int\limits_{(\mathbb{R}^{1,1})^p}\mathrm{d}^p\boldsymbol{X}\int\limits_{(\mathbb{R}^{1,1})^q}\mathrm{d}^q\boldsymbol{Y} f(\boldsymbol{X}_p)\cdot g(\boldsymbol{Y}_q)\prod_{b>a}^{p}\Big\{\mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\lambda}^{(ba)})*\boldsymbol{x}_{ba}}\Big\}\cdot\prod_{b>a}^{q}\Big\{\mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\vartheta}^{(ba)})*\boldsymbol{y}_{ba}}\Big\} \;=\; \mathcal{R}[f](\boldsymbol{\lambda})\cdot\mathcal{R}[g](\boldsymbol{\vartheta})\,. \tag{3.103}$$

Further, the product of S-matrices given in (2.45) may be presented as

$$\mathcal{S}(\boldsymbol{\gamma}) = \prod_{v>r>u>s}^{p+q} \mathsf{S}(\boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(rs)} \mid \boldsymbol{\gamma}^{(rs)} \cup \boldsymbol{\gamma}^{(vu)}) . \tag{3.104}$$

A given S-matrix factor will be present if both $\boldsymbol{\gamma}^{(vu)}$ and $\boldsymbol{\gamma}^{(rs)}$ are non trivial. $\boldsymbol{\gamma}^{(vu)}$ is non-trivial if, either, $p \geq v$ or $u \geq p+1$. In the first case one necessarily has $p \geq r > s$. In the second case scenario, one necessarily has $r > p+1$ so that $\boldsymbol{\gamma}^{(rs)}$ will be non-trivial provided that $s \geq p+1$. This leads to the factorisation

$$\mathcal{S}(\boldsymbol{\gamma}) = \prod_{v>r>u>s}^{p} \mathsf{S}(\boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(rs)} \mid \boldsymbol{\gamma}^{(rs)} \cup \boldsymbol{\gamma}^{(vu)}) \cdot \prod_{\substack{v>r>u>s \\ \geq p+1}}^{p+q} \mathsf{S}(\boldsymbol{\gamma}^{(vu)} \cup \boldsymbol{\gamma}^{(rs)} \mid \boldsymbol{\gamma}^{(rs)} \cup \boldsymbol{\gamma}^{(vu)}) = \mathcal{S}(\boldsymbol{\lambda}) \cdot \mathcal{S}(\boldsymbol{\vartheta}) . \tag{3.105}$$

Finally, we focus on the building blocks of $\mathcal{F}_{\boldsymbol{\tau}_{p+q};\boldsymbol{\eta}_{p+q}}(\boldsymbol{\gamma})$ while reparameterising $\boldsymbol{\eta}_{p+q} = (\boldsymbol{\varepsilon}_p, \boldsymbol{\delta}_q)$. Then, for $s \leq p$, one has:

$$\begin{aligned}
\mathscr{F}_s(\boldsymbol{\gamma}) &\equiv \mathcal{F}^{(\tau_s)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(ss-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(s1)}} + \mathrm{i}\pi \overline{\boldsymbol{e}}_{\eta_s}, \boldsymbol{\gamma}^{(p+qs)} \cup \cdots \cup \boldsymbol{\gamma}^{(s+1s)} \Big) && (3.106) \\
&= \mathcal{F}^{(\alpha_s)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(ss-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(s1)}} + \mathrm{i}\pi \overline{\boldsymbol{e}}_{\varepsilon_s}, \boldsymbol{\gamma}^{(ps)} \cup \cdots \cup \boldsymbol{\gamma}^{(s+1s)} \Big) && (3.107) \\
&= \mathcal{F}^{(\alpha_s)}\Big(\overleftarrow{\boldsymbol{\lambda}^{(ss-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\lambda}^{(s1)}} + \mathrm{i}\pi \overline{\boldsymbol{e}}_{\varepsilon_s}, \boldsymbol{\lambda}^{(ps)} \cup \cdots \cup \boldsymbol{\lambda}^{(s+1s)} \Big) . && (3.108)
\end{aligned}$$

Further, for $s = u + p$ with $u \in [\![1 ; q]\!]$, one rather has

$$\begin{aligned}
\mathscr{F}_s(\boldsymbol{\gamma}) &= \mathcal{F}^{(\beta_u)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(u+p\,u+p-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(u+p\,p+1)}} + \mathrm{i}\pi \overline{\boldsymbol{e}}_{\delta_u}, \boldsymbol{\gamma}^{(p+q\,u+p)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+u+1\,p+u)} \Big) && (3.109) \\
&= \mathcal{F}^{(\beta_u)}\Big(\overleftarrow{\boldsymbol{\vartheta}^{(uu-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\vartheta}^{(u1)}} + \mathrm{i}\pi \overline{\boldsymbol{e}}_{\delta_u}, \boldsymbol{\vartheta}^{(qu)} \cup \cdots \cup \boldsymbol{\vartheta}^{(u+1u)} \Big) . && (3.110)
\end{aligned}$$

This thus leads to the factorisation of the form factor contribution

$$\mathcal{F}_{\boldsymbol{\tau}_{p+q},\boldsymbol{\eta}_{p+q}}(\boldsymbol{\gamma}) = \mathcal{F}_{\boldsymbol{\alpha}_p,\boldsymbol{\varepsilon}_p}(\boldsymbol{\lambda}) \cdot \mathcal{F}_{\boldsymbol{\beta}_q,\boldsymbol{\delta}_q}(\boldsymbol{\vartheta}) . \tag{3.111}$$

The factorisation of the integrand that we have just demonstrated ensures the claimed factorisation of the integral transform we started with. ■

Lemma 3.9. *Let $p, q \in \mathbb{N}^*$, $(f, g) \in \mathcal{S}((\mathbb{R}^{1,1})^p) \times \mathcal{S}((\mathbb{R}^{1,1})^q)$, $\boldsymbol{v} \in \mathbb{R}^{1,1}$ and set $G_{p,q}^{(\lambda\boldsymbol{v})}$ as introduced in* (2.21). *Let $\boldsymbol{\alpha}_p$, resp. $\boldsymbol{\beta}_q$, be p, resp. q, dimensional vectors of operators labels corresponding to solutions to a) − d) and $\boldsymbol{\tau}_{p+q} = (\boldsymbol{\alpha}_p, \boldsymbol{\beta}_q)$ be their concatenation. Let $\sigma \in \mathfrak{S}_{p+q}$ refer to the below permutation of the coordinates*

$$\sigma \cdot (\boldsymbol{X}_p, \boldsymbol{Y}_q) = (\boldsymbol{Y}_q, \boldsymbol{X}_p) . \tag{3.112}$$

Assume Conjecture 2.5 holds. Then, the distributions

$$\mathfrak{M}^{(\lambda\boldsymbol{v})}[f, g] = \mathcal{W}_{\boldsymbol{\tau}_{p+q}}[G_{p,q}^{(\lambda\boldsymbol{v})}] - \mathcal{W}_{\boldsymbol{\alpha}_p}[f] \cdot \mathcal{W}_{\boldsymbol{\beta}_q}[g] \tag{3.113}$$

and

$$\mathfrak{M}_{\sigma}^{(\lambda\boldsymbol{v})}[f, g] = \mathcal{W}_{\sigma\cdot\boldsymbol{\tau}_{p+q}}[\sigma \cdot G_{p,q}^{(\lambda\boldsymbol{v})}] - \mathcal{W}_{\boldsymbol{\beta}_q}[g] \cdot \mathcal{W}_{\boldsymbol{\alpha}_p}[f] , \tag{3.114}$$

where the action of the symmetric group on $\mathcal{S}((\mathbb{R}^{1,1})^{p+q})$ has been introduced in (2.5)*, can be recast as*

$$\mathfrak{M}^{(\lambda\boldsymbol{\nu})}[f,g] \;=\; \mathfrak{N}^{(\lambda\boldsymbol{\nu})}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]] \qquad \textit{and} \qquad \mathfrak{M}^{(\lambda\boldsymbol{\nu})}_{\sigma}[f,g] \;=\; \mathfrak{N}^{(\lambda\boldsymbol{\nu})}_{\sigma}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]] \;. \tag{3.115}$$

The operator $\mathcal{L}$ is as introduced in (3.10)*, while the auxiliary distributions $\mathfrak{N}^{(\lambda\boldsymbol{\nu})}$, $\mathfrak{N}^{(\lambda\boldsymbol{\nu})}_{\sigma}$ are defined by means of a summation over the subset of $\mathbb{N}^{\frac{(p+q)(p+q-1)}{2}}$:*

$$\mathcal{T}^{(p)} \;=\; \Big\{ \boldsymbol{n} \in \mathbb{N}^{\frac{(p+q)(p+q-1)}{2}} \;:\; \sum_{b=p+1}^{p+q} \sum_{a=1}^{p} n_{ba} \;\geq\; 1 \Big\} \,, \tag{3.116}$$

as follows

$$\mathfrak{N}^{(\lambda\boldsymbol{\nu})}[H] \;=\; \sum_{\boldsymbol{n}\in\mathcal{T}^{(p)}} \mathcal{J}^{(\boldsymbol{n};p)}_{\boldsymbol{\tau}_{p+q},\lambda\boldsymbol{\nu}}[H] \qquad \textit{and} \qquad \mathfrak{N}^{(\lambda\boldsymbol{\nu})}_{\sigma}[H] \;=\; \sum_{\boldsymbol{n}\in\mathcal{T}^{(q)}} \mathcal{J}^{(\boldsymbol{n};q)}_{\sigma\cdot\boldsymbol{\tau}_{p+q},-\lambda\boldsymbol{\nu}}[H]. \tag{3.117}$$

There, we agree upon

$$\mathcal{J}^{(\boldsymbol{n};p)}_{\boldsymbol{\tau}_{p+q},\lambda\boldsymbol{\nu}}[H] \;=\; \lim_{\boldsymbol{\varepsilon}_{p+q}\to 0^+} \int\limits_{\mathbb{R}^{|\boldsymbol{n}|}} \frac{\mathrm{d}^{|\boldsymbol{n}|}\boldsymbol{\gamma}}{\boldsymbol{n}!(2\pi)^{|\boldsymbol{n}|}} \, \mathrm{e}^{-\mathrm{i}\lambda\boldsymbol{\nu}*P_p(\boldsymbol{\gamma})} \Big(\mathcal{S}\cdot\widehat{\mathcal{R}}[H]\cdot\mathcal{F}_{\boldsymbol{\tau}_{p+q};\boldsymbol{\varepsilon}_{p+q}} \Big)(\boldsymbol{\gamma}) \;. \tag{3.118}$$

Above, P_p is as introduced in (3.11)*, the $\widehat{\mathcal{R}}$ transform has been defined in* (3.12) *while $\mathcal{S}$ and $\mathcal{F}_{\boldsymbol{\tau}_{p+q};\boldsymbol{\varepsilon}_{p+q}}$ have been, respectively, introduced in* (2.45) *and* (2.46)*.*

$\mathfrak{N}^{(\lambda\boldsymbol{\nu})}$*,* $\mathfrak{N}^{(\lambda\boldsymbol{\nu})}_{\sigma}$ *are well-defined tempered distributions on $\mathcal{S}((\mathbb{R}^{1,1})^{p+q-1})$. Their Fourier transforms are supported on forward or backward lightcones:*

$$\mathfrak{N}^{(\lambda\boldsymbol{\nu})}[\mathscr{F}[H]] \;=\; 0 \qquad \textit{if} \qquad \operatorname{supp}[H]\cap\mathcal{E}^{+,+}_{p+q-1,p} \;=\; \emptyset \,, \tag{3.119}$$

where given $\boldsymbol{Q}_{k-1} \;=\; (\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{k-1}) \in (\mathbb{R}^{1,1})^{k-1}$, one has

$$\mathcal{E}^{+,+}_{p-1,p} \;=\; \left\{ \boldsymbol{Q}_{k-1} \in (\mathbb{R}^{1,1})^{k-1} \;:\; \begin{array}{ll} q_{p,0}\geq m\,, & \boldsymbol{q}_p^2 \geq m^2 \\ q_{\ell,0}\geq 0\,, & \boldsymbol{q}_\ell^2 \geq 0 \end{array} \quad \textit{for} \quad \ell\in[\![\,1\,;\,k-1\,]\!]\setminus\{p\} \right\} \;. \tag{3.120}$$

Similarly

$$\mathfrak{N}^{(\lambda\boldsymbol{\nu})}_{\sigma}[\mathscr{F}[H]] \;=\; 0 \qquad \textit{if} \qquad \operatorname{supp}[H]\cap\mathcal{E}^{+,-}_{p+q-1,p} \;=\; \emptyset \,, \tag{3.121}$$

where we set

$$\mathcal{E}^{+,-}_{k-1,p} = \Big\{ \boldsymbol{Q}_{k-1} \in (\mathbb{R}^{1,1})^{k-1} \;:\quad q_{p,0}\leq -m\,,\quad \boldsymbol{q}_p^2 \geq m^2 \Big\} \,. \tag{3.122}$$

Proof —

The fact that $\mathcal{J}^{(\boldsymbol{n};p)}_{\boldsymbol{\tau}_{p+q},\lambda\boldsymbol{\nu}}$, $\mathfrak{N}^{(\lambda\boldsymbol{\nu})}$ and $\mathfrak{N}^{(\lambda\boldsymbol{\nu})}_{\sigma}$ are all well-defined follows from similar handlings as those outlined in Proposition 3.2.

We first focus on the distribution $\mathfrak{M}$. Starting from the expansion (2.51) applied to $\mathcal{W}_{\boldsymbol{\tau}_{p+q}}[G^{(\lambda\boldsymbol{\nu})}_{p,q}]$, $\mathcal{W}_{\boldsymbol{\alpha}_p}[f]$, $\mathcal{W}_{\boldsymbol{\beta}_q}[g]$ taking the product of the series and recalling Lemma 3.8 which ensures the factorisation (3.98) provided that condition (3.97) holds, one gets that when computing $\mathfrak{M}^{(\lambda\boldsymbol{\nu})}[f,g]$ through series expansions, only the complement of the condition (3.97) contributes, thus leading to

$$\mathfrak{M}^{(\lambda\boldsymbol{\nu})}[f,g] \;=\; \sum_{\boldsymbol{n}\in\mathcal{T}^{(p)}} \mathcal{I}^{(\boldsymbol{n})}_{\boldsymbol{\tau}_{p+q}}[G^{(\lambda\boldsymbol{\nu})}_{p,q}] \;=\; \sum_{\boldsymbol{n}\in\mathcal{T}^{(p)}} \mathcal{J}^{(\boldsymbol{n})}_{\boldsymbol{\tau}_{p+q}}[\mathcal{L}[G^{(\lambda\boldsymbol{\nu})}_{p,q}]] \tag{3.123}$$

where the second equality follows from (3.21), and $\mathcal{J}^{(\boldsymbol{n})}_{\boldsymbol{\tau}_{p+q}}$ is as defined in (3.13). Further, by implementing the change of variables $\boldsymbol{y}_s \hookrightarrow \boldsymbol{y}_s + \lambda \boldsymbol{v}(\delta_{s,p} - \delta_{s,p+q})$ for $s \in [\![1 \, ; \, p+q]\!]$, one infers that

$$\widehat{\mathcal{R}}[\mathcal{L}[G^{(\lambda\boldsymbol{v})}_{p,q}]](\boldsymbol{\gamma}) \; = \; \mathrm{e}^{-\mathrm{i}\lambda\boldsymbol{v}*\mathrm{P}_p(\boldsymbol{\gamma})} \cdot \widehat{\mathcal{R}}[\mathcal{L}[G^{(\boldsymbol{0})}_{p,q}]](\boldsymbol{\gamma}) \; . \tag{3.124}$$

This leads to the identity $\mathcal{J}^{(\boldsymbol{n})}_{\boldsymbol{\tau}_{p+q}}[\mathcal{L}[G^{(\lambda\boldsymbol{v})}_{p,q}]] \; = \; \mathcal{J}^{(\boldsymbol{n};p)}_{\boldsymbol{\tau}_{p+q},\lambda\boldsymbol{v}}[\mathcal{L}[G^{(\boldsymbol{0})}_{p,q}]]$ which, along with (3.123), yields the first equality given in (3.115).

Further, owing to (3.25), one infers that $\mathcal{J}^{(\boldsymbol{n};p)}_{\boldsymbol{\tau}_{p+q},\lambda\boldsymbol{v}}[\mathscr{F}[H]]$ will be non zero only if supp$[H]$ intersects the range of $\boldsymbol{\gamma} \mapsto (\mathrm{P}_1(\boldsymbol{\gamma}), \dots, \mathrm{P}_{p+q-1}(\boldsymbol{\gamma}))$. Now, recalling the expression for $\mathrm{P}_s(\boldsymbol{\gamma})$ (3.11), one readily infers that

$$\mathrm{P}^{(0)}_p(\boldsymbol{\gamma}) \; = \; m \sum_{b=p+1}^{p+q} \sum_{a=1}^{p} \sum_{s=1}^{n_{ba}} \cosh\big(\gamma^{(ba)}_s\big) \; \geq \; m \sum_{b=p+1}^{p+q} \sum_{a=1}^{p} n_{ba} \; , \tag{3.125}$$

as well as

$$\mathrm{P}^2_p(\boldsymbol{\gamma}) \; = \; m^2 \sum_{b=p+1}^{p+q} \sum_{a=1}^{p} n_{ba} \; + \; m^2 \sum_{\substack{b,b' \\ =p+1}}^{p+q} \sum_{a,a'=1}^{p} \sum_{s=1}^{n_{ba}} \sum_{s'=1}^{n_{b'a'}} \cosh\big(\gamma^{(ba)}_s \; - \; \gamma^{(b'a')}_{s'}\big) \geq m^2 \; , \tag{3.126}$$

for any $\boldsymbol{n} \in \mathcal{T}^{(p)}$. This entails the claim relatively to $\mathfrak{N}^{(\lambda\boldsymbol{v})}$, *viz.* (3.119).

We now move on to proving the analogous properties of the distribution $\mathfrak{M}^{(\lambda\boldsymbol{v})}_{\sigma}$. Similar reasoning ensure that

$$\mathfrak{M}^{(\lambda\boldsymbol{v})}_{\sigma}[f,g] = \sum_{\boldsymbol{n}\in\mathcal{T}^{(p)}} \mathcal{I}^{(\boldsymbol{n})}_{\sigma\cdot\boldsymbol{\tau}_{p+q}}[\sigma \cdot G^{(\lambda\boldsymbol{v})}_{p,q}] \; = \; \sum_{\boldsymbol{n}\in\mathcal{T}^{(p)}} \mathcal{J}^{(\boldsymbol{n})}_{\sigma\cdot\boldsymbol{\tau}_{p+q}}[\mathcal{L}[\sigma \cdot G^{(\lambda\boldsymbol{v})}_{p,q}]] \; . \tag{3.127}$$

At this stage, one observes that

$$\mathcal{R}[\sigma \cdot G^{(\lambda\boldsymbol{v})}_{p,q}](\boldsymbol{\gamma}) \; = \int\limits_{(\mathbb{R}^{1,1})^{p+q}} \mathrm{d}^p\boldsymbol{X} \cdot \mathrm{d}^q\boldsymbol{Y} \; G^{(\lambda\boldsymbol{v})}_{p,q}(\boldsymbol{X}_p, \boldsymbol{Y}_q) \prod_{b>a}^{q} \Big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(ba)})*\boldsymbol{y}_{ba}} \Big\} \prod_{b>a}^{p} \Big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(b+q\,a+q)})*\boldsymbol{x}_{ba}} \Big\}$$
$$\times \prod_{b=1}^{p} \prod_{a=1}^{q} \Big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(b+q\,a)})*(\boldsymbol{x}_b - \boldsymbol{y}_a)} \Big\} \; . \tag{3.128}$$

We now proceed with the relabelling $\boldsymbol{Z}_{p+q} \; = \; (\boldsymbol{X}_p, \boldsymbol{Y}_q)$, which yields

$$\mathcal{R}[\sigma \cdot G^{(\lambda\boldsymbol{v})}_{p,q}](\boldsymbol{\gamma}) \; = \int\limits_{(\mathbb{R}^{1,1})^{p+q}} \mathrm{d}^{p+q}\boldsymbol{Z} \, G^{(\lambda\boldsymbol{v})}_{p,q}(\boldsymbol{Z}_{p+q}) \prod_{\substack{b>a \\ \geq p+1}}^{p+q} \Big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(b-p\,a-p)})*\boldsymbol{z}_{ba}} \Big\} \prod_{b>a}^{p} \Big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(b+q\,a+q)})*\boldsymbol{z}_{ba}} \Big\}$$
$$\times \prod_{b=1}^{p} \prod_{a=1+p}^{q+p} \Big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\boldsymbol{\gamma}^{(b+q\,a-p)} + \mathrm{i}\pi\overline{\boldsymbol{e}})*\boldsymbol{z}_{ab}} \Big\} \; . \tag{3.129}$$

Thus, by introducing the change of variables $\boldsymbol{\gamma} \mapsto \boldsymbol{\Gamma}(\boldsymbol{\gamma})$ defined as:

$$\begin{array}{rclcl} \boldsymbol{\Gamma}^{(ba)}(\boldsymbol{\gamma}) & = & \boldsymbol{\gamma}^{(b+q\,a+q)} & , & 1 \leq a < b \leq p \\ \boldsymbol{\Gamma}^{(ba)}(\boldsymbol{\gamma}) & = & \boldsymbol{\gamma}^{(a+q\,b-p)} + \mathrm{i}\pi\overline{\boldsymbol{e}} & , & 1 \leq a < 1+p \leq b \leq p+q \\ \boldsymbol{\Gamma}^{(ba)}(\boldsymbol{\gamma}) & = & \boldsymbol{\gamma}^{(b-p\,a-p)} & , & p+1 \leq a < b \leq p+q \end{array} \tag{3.130}$$

one gets the chain of equalities

$$\mathcal{R}[\sigma \cdot G^{(\lambda \boldsymbol{v})}_{p,q}](\boldsymbol{\gamma}) \;=\; \mathcal{R}[G^{(\lambda \boldsymbol{v})}_{p,q}](\boldsymbol{\Gamma}(\boldsymbol{\gamma})) \;=\; \widehat{\mathcal{R}}[\mathcal{L}[\sigma \cdot G^{(\lambda \boldsymbol{v})}_{p,q}]](\boldsymbol{\Gamma}(\boldsymbol{\gamma})) \;=\; \mathrm{e}^{-\mathrm{i}\lambda \boldsymbol{v} * \mathrm{P}_p(\boldsymbol{\Gamma}(\boldsymbol{\gamma}))} \cdot \widehat{\mathcal{R}}[\mathcal{L} \cdot G^{(\mathbf{0})}_{p,q}](\boldsymbol{\Gamma}(\boldsymbol{\gamma}))\,, \tag{3.131}$$

where we invoked (3.124). Then, observing through direct calculations that $\mathrm{P}_p(\boldsymbol{\Gamma}(\boldsymbol{\gamma})) \;=\; -\mathrm{P}_q(\boldsymbol{\gamma})$, one eventually gets that

$$\mathcal{J}^{(\boldsymbol{n})}_{\sigma \cdot \boldsymbol{\tau}_{p+q}}[\mathcal{L}[\sigma \cdot G^{(\lambda \boldsymbol{v})}_{p,q}]] \;=\; \mathcal{J}^{(\boldsymbol{n};q)}_{\sigma \cdot \boldsymbol{\tau}_{p+q}, -\lambda \boldsymbol{v}}[\mathcal{L}[\sigma \cdot G^{(\mathbf{0})}_{p,q}]]\,. \tag{3.132}$$

This entails the second equality given in (3.115).

Again, the representation (3.25) implies that $\mathcal{J}^{(\boldsymbol{n};q)}_{\boldsymbol{\tau}_{p+q}, -\lambda \boldsymbol{v}}[\mathscr{F}[H]]$ will be non zero only if $\operatorname{supp}[H]$ intersects the range of $\boldsymbol{\gamma} \mapsto (\mathrm{P}_1(\boldsymbol{\Gamma}(\boldsymbol{\gamma})), \dots, \mathrm{P}_{p+q-1}(\boldsymbol{\Gamma}(\boldsymbol{\gamma})))$. The identity $\mathrm{P}_p(\boldsymbol{\Gamma}(\boldsymbol{\gamma})) \;=\; -\mathrm{P}_q(\boldsymbol{\gamma})$ and equations (3.125)-(3.126) thus ensure (3.121). ■

Proposition 3.10. *Let $p, q \in \mathbb{N}^*$, $(f, g) \in \mathcal{S}((\mathbb{R}^{1,1})^p) \times \mathcal{S}((\mathbb{R}^{1,1})^q)$, $\boldsymbol{v} \in \mathbb{R}^{1,1}$ and let $G^{(\lambda \boldsymbol{v})}_{p,q}$ be as introduced in* (2.21). *Let $\boldsymbol{\alpha}_p$, resp. $\boldsymbol{\beta}_q$, be p, resp. q, dimensional vectors of operators labels corresponding to solutions to a) − d). Assume that Conjecture 2.5 is valid. Given $\boldsymbol{\tau}_{p+q} \;=\; (\boldsymbol{\alpha}_p, \boldsymbol{\beta}_q)$, it holds*

$$\mathcal{W}_{\boldsymbol{\tau}_{p+q}}[G^{(\lambda \boldsymbol{v})}_{p,q}] \underset{\lambda \to \pm\infty}{\longrightarrow} \mathcal{W}_{\boldsymbol{\alpha}_p}[f] \cdot \mathcal{W}_{\boldsymbol{\beta}_q}[g]\,. \tag{3.133}$$

Proof— The proof follows closely the chain of reasoning outlined in [28]. By Conjecture 2.5, $\mathcal{W}_{\boldsymbol{\tau}_{p+q}}$ is a tempered distribution on $\mathcal{S}((\mathbb{R}^{1,1})^{p+q})$. By the structure theorem for tempered distributions (see *e.g.* Theorem 8.3.1 in [6]), there exists $\boldsymbol{\xi}_{p+q} \in \mathbb{N}^{p+q}$, a polynomially bounded continuous function Φ on $(\mathbb{R}^{1,1})^{p+q}$, *i.e.* for some $C, M > 0$

$$\Big|\Phi(\boldsymbol{Z}_{p+q})\Big| \;\le\; C\Big(1 + \|\boldsymbol{Z}_{p+q}\|\Big)^M \qquad \text{with} \qquad \|\boldsymbol{Z}_{p+q}\|^2 \;=\; \sum_{s=1}^{p+q} (z^2_{s;0} \,+\, z^2_{s;1}) \tag{3.134}$$

referring to the Euclidean norm of the vector, such that

$$\mathcal{W}_{\boldsymbol{\tau}_{p+q}}[F] \;=\! \int\limits_{(\mathbb{R}^{1,1})^{p+q}} \! \mathrm{d}^{p+q}\boldsymbol{Z}\; \Phi(\boldsymbol{Z}_{p+q})\, \partial^{\boldsymbol{\xi}_{p+q}}_{\boldsymbol{Z}_{p+q}} F(\boldsymbol{Z}_{p+q})\,. \tag{3.135}$$

Here, it is understood that $\boldsymbol{\xi}_{p+q} \;=\; (\xi_{1;0}, \xi_{1;1}, \xi_{2;0}, \dots, \xi_{p+q;1})$ and

$$\partial^{\boldsymbol{\xi}_{p+q}}_{\boldsymbol{Z}_{p+q}} \;=\; \prod_{s=1}^{p+q} \prod_{a=0}^{1} \partial^{\xi_{s;a}}_{Z_{s;a}}\,. \tag{3.136}$$

The vectors $\boldsymbol{\xi}_{p+q}$ and the function Φ along with its control (3.134) do depend, in principle, on the given choice of the operator indices $\boldsymbol{\tau}_{p+q}$. It thus follows readily from changes of variables and integrations by parts that $\delta\mathcal{W}_\sigma[F] \;=\; \mathcal{W}_{\boldsymbol{\tau}_{p+q}}[F] \,-\, \mathcal{W}_{\sigma \cdot \boldsymbol{\tau}_{p+q}}[\sigma \cdot F]$ admits the representation

$$\delta\mathcal{W}_\sigma[F] \;=\! \int\limits_{(\mathbb{R}^{1,1})^p} \! \mathrm{d}^p\boldsymbol{X} \int\limits_{(\mathbb{R}^{1,1})^q} \! \mathrm{d}^q\boldsymbol{Y}\; \Psi(\boldsymbol{X}_p, \boldsymbol{Y}_q)\, \partial^{\boldsymbol{\mu}_p}_{\boldsymbol{X}_p} \partial^{\boldsymbol{\rho}_q}_{\boldsymbol{Y}_q} F(\boldsymbol{X}_p, \boldsymbol{Y}_q)\,. \tag{3.137}$$

There $(\boldsymbol{\mu}_p, \boldsymbol{\rho}_q) \in \mathbb{N}^{2p} \times \mathbb{N}^{2q}$, Ψ is continuous on $(\mathbb{R}^{1,1})^{p+q}$ and of polynomial growth in the sense of (3.134), with possibly different values for C, M.

Let

$$\Omega_{p,q} = \Big\{(\boldsymbol{X}_p, \boldsymbol{Y}_q) \in (\mathbb{R}^{1,1})^{p+q} \ : \ (\boldsymbol{x}_s - \boldsymbol{y}_r)^2 < 0 \quad \forall (s,r) \in [\![1 \,;\, p]\!] \times [\![1 \,;\, q]\!] \Big\} . \tag{3.138}$$

It follows from Proposition 3.6 that $\delta\mathcal{W}_\sigma[F] = 0$ for any F such that $\mathrm{supp}[F] \subset \Omega_{p,q}$. By proposition A.1, one may pick Ψ arising in the representation (3.137) such that it satisfies $\mathrm{supp}[\Psi] \subset \big(\Omega_{p,q}^{\mathrm{c}}\big)_\epsilon$ with $\epsilon > 0$ and small enough and

$$\big(\Omega_{p,q}^{\mathrm{c}}\big)_\epsilon = \Big\{\boldsymbol{Z}_{p,q} \in (\mathbb{R}^{1,1})^{p+q} \ : \ \mathrm{d}(\Omega_{p,q}^{\mathrm{c}}, \boldsymbol{Z}_{p,q}) \leq \epsilon \Big\} . \tag{3.139}$$

Further, it is direct to check that

$$\delta\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] = \mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] - \mathfrak{M}_\sigma^{(\lambda\boldsymbol{v})}[f,g] \tag{3.140}$$

with $\mathfrak{M}^{(\lambda\boldsymbol{v})}, \mathfrak{M}_\sigma^{(\lambda\boldsymbol{v})}$ as introduced in (3.113), (3.114), can be recast as $\delta\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] = \delta\mathcal{W}_\sigma[G_{p,q}^{(\lambda\boldsymbol{v})}]$ with $G_{p,q}^{(\lambda\boldsymbol{v})}$ as introduced in (2.21). Thus, after shifting the $\boldsymbol{Y}_q$ variables, one gets

$$\delta\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] = \int\limits_{(\mathbb{R}^{1,1})^p} \mathrm{d}^p\boldsymbol{X} \int\limits_{(\mathbb{R}^{1,1})^q} \mathrm{d}^q\boldsymbol{Y}\, \Psi(\boldsymbol{X}_p, \boldsymbol{Y}_q - \lambda\boldsymbol{v})\, \partial_{\boldsymbol{X}_p}^{\boldsymbol{\mu}_p} f(\boldsymbol{X}_p)\, \partial_{\boldsymbol{Y}_q}^{\boldsymbol{\rho}_q} g(\boldsymbol{Y}_q) . \tag{3.141}$$

It is direct to infer from the polynomial bound on Ψ that

$$\Big|\Psi(\boldsymbol{X}_p, \boldsymbol{Y}_q - \lambda\boldsymbol{v})\Big| \leq C\,(1 + ||\boldsymbol{Z}_{p+q}||)^M (1 + |\lambda|)^M \qquad \text{with} \qquad \boldsymbol{Z}_{p+q} = (\boldsymbol{X}_p, \boldsymbol{Y}_q) . \tag{3.142}$$

Likewise, the Schwartz class of f and g ensures that for any $Q > 0$ there exist $C_Q > 0$ such that

$$\Big|\partial_{\boldsymbol{X}_p}^{\boldsymbol{\mu}_p} f(\boldsymbol{X}_p)\, \partial_{\boldsymbol{Y}_q}^{\boldsymbol{\rho}_q} g(\boldsymbol{Y}_q)\Big| \leq \frac{C_Q}{(1 + ||\boldsymbol{X}_p||)^Q \cdot (1 + ||\boldsymbol{Y}_q||)^Q} \leq \frac{C_Q}{(1 + ||\boldsymbol{Z}_{p+q}||)^Q} \tag{3.143}$$

with $\boldsymbol{Z}_{p+q}$ as introduced above. Further, observe that

$$(\boldsymbol{x}_r - \boldsymbol{y}_s + \lambda\boldsymbol{v})^2 = x_{s;0}^2 + y_{r;0}^2 - 2x_{s;0}\cdot y_{r;0} - (x_{s;1} - y_{r;1})^2 + \boldsymbol{v}^2\lambda^2 - 2\lambda\,[v_0(x_{s;0} - y_{r;0}) - v_1(x_{s;1} - y_{r;1})] . \tag{3.144}$$

Thus, for any $\boldsymbol{Z}_{p+q} = (\boldsymbol{X}_p, \boldsymbol{Y}_q)$ such that $||\boldsymbol{Z}_{p+q}|| < \alpha|\lambda|$ with α small enough, one has

$$(\boldsymbol{x}_r - \boldsymbol{y}_s - \lambda\boldsymbol{v})^2 < 4\lambda^2\Big[\alpha^2 - |\boldsymbol{v}^2| + 2\alpha(|v_0| + |v_1|)\Big] < -\frac{\lambda^2}{2}|\boldsymbol{v}^2| . \tag{3.145}$$

Thus, $\Psi(\boldsymbol{X}_p, \boldsymbol{Y}_q - \lambda\boldsymbol{v})$ is identically zero whenever $||\boldsymbol{Z}_{p+q}|| < \alpha|\lambda|$. As a consequence, it holds

$$\delta\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] = \int\limits_{(\mathbb{R}^{1,1})^{p+q}} \mathrm{d}^{p+q}\boldsymbol{Z}\, \mathbb{1}_\Upsilon(\boldsymbol{Z}_{p+q})\, \Psi(\boldsymbol{X}_p, \boldsymbol{Y}_q - \lambda\boldsymbol{v})\, \partial_{\boldsymbol{X}_p}^{\boldsymbol{\mu}_p} f(\boldsymbol{X}_p)\, \partial_{\boldsymbol{Y}_q}^{\boldsymbol{\rho}_q} g(\boldsymbol{Y}_q) . \tag{3.146}$$

where $\Upsilon = \{\boldsymbol{Z}_{p+q} \in (\mathbb{R}^{1,1})^{p+q} \ : \ ||\boldsymbol{Z}_{p+q}|| \geq \alpha|\lambda|\}$. Passing on to spherical coordinates, using the upper bounds (3.142)-(3.143) and the compactness of the angular integration one gets for some constant $\widetilde{C} > 0$

$$\begin{aligned} \Big|\delta\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g]\Big| &\leq \widetilde{C}\,(1 + |\lambda|)^M \int_{\alpha\lambda}^{+\infty} \mathrm{d}R R^{2(p+q)-1} \frac{(1+R)^M}{(1+R)^Q} \leq \widetilde{C}\,(1 + |\lambda|)^M \int_{1+\alpha\lambda}^{+\infty} \mathrm{d}R\, R^{2(p+q)+M-Q-1} \\ &\leq \widetilde{C}' \frac{(1 + |\lambda|)^M}{Q - 2(p+q) - M} (\alpha|\lambda|)^{2(p+q)+M-Q} \underset{\lambda\to\infty}{\rightarrow} 0 , \end{aligned} \tag{3.147}$$

provided that Q is taken large enough.

We shall now establish that this preliminary result implies that $\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] \to 0$ as $\lambda \to \infty$, which is the sought result. Recall that, by eq. (3.115) of Lemma 3.9, one has

$$\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] \;=\; \mathfrak{N}^{(\lambda\boldsymbol{v})}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]] \;=\; \mathscr{F}[\mathfrak{N}^{(\lambda\boldsymbol{v})}]\Big[\,\mathscr{F}^{-1}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]]\,\Big]\,. \tag{3.148}$$

Let $\vartheta_{\pm}$ be smooth partitions of unity on $\mathbb{R}$, $\vartheta_+ + \vartheta_- = 1$, such that

$$\vartheta_+ = 1 \quad \text{on} \quad [m\,;+\infty] \qquad \text{and} \qquad \vartheta_- = 1 \quad \text{on} \quad [-\infty\,;-m]\,, \tag{3.149}$$

m being the mass scale appearing in (3.120) and (3.122). Let

$$\chi_{\pm}(\boldsymbol{q}_1,\dots,\boldsymbol{q}_{p+q-1}) \;=\; \vartheta_{\pm}(q_{p;0})\,\mathscr{F}^{-1}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]](\boldsymbol{q}_1,\dots,\boldsymbol{q}_{p+q-1})\,. \tag{3.150}$$

One has thus $\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] \;=\; \mathfrak{N}^{(\lambda\boldsymbol{v})}[\mathscr{F}[\chi_+ + \chi_-]]$. However,

$$\operatorname{supp}[\chi_-] \cap \mathcal{E}^{+,+}_{p+q-1,p} = \emptyset \quad \text{so that} \quad \mathfrak{N}^{(\lambda\boldsymbol{v})}[\mathscr{F}[\chi_-]] = 0\,. \tag{3.151}$$

Observing further that

$$\operatorname{supp}[\chi_+] \cap \mathcal{E}^{+,-}_{p+q-1,p} = \emptyset\,, \quad \text{so that} \quad \mathfrak{N}^{(\lambda\boldsymbol{v})}_{\sigma}[\mathscr{F}[\chi_+]] = 0\,, \tag{3.152}$$

one gets that

$$\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] \;=\; \mathfrak{N}^{(\lambda\boldsymbol{v})}[\mathscr{F}[\chi_+]] \;-\; \mathfrak{N}^{(\lambda\boldsymbol{v})}_{\sigma}[\mathscr{F}[\chi_+]]\,. \tag{3.153}$$

We now compute $\mathscr{F}[\chi_+](\boldsymbol{Y}_{p+q-1})$ explicitly with $\boldsymbol{Y}_{p+q-1} = (\boldsymbol{y}_1,\dots,\boldsymbol{y}_{p+q-1})$

$$\begin{aligned}
\mathscr{F}[\chi_+](\boldsymbol{Y}_{p+q-1}) \;&=\; \int\limits_{(\mathbb{R}^{1,1})^{p+q-1}} \mathrm{d}^{p+q-1}\boldsymbol{Q}\,\mathscr{F}^{-1}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]](\boldsymbol{Q}_{p+q-1})\vartheta_+(q_{p;0}) \prod_{a=1}^{p+q-1}\Big\{\mathrm{e}^{\mathrm{i}\boldsymbol{q}_a * \boldsymbol{y}_a}\Big\} &(3.154)\\
&=\; \lim_{\epsilon\to 0^+} \int\limits_{(\mathbb{R}^{1,1})^{p+q-1}} \mathrm{d}^{p+q-1}\boldsymbol{Q}\,\mathscr{F}^{-1}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]](\boldsymbol{Q}_{p+q-1})\,\mathrm{e}^{-\epsilon q^2_{p;0}}\,\vartheta_+(q_{p;0}) \prod_{a=1}^{p+q-1}\Big\{\mathrm{e}^{\mathrm{i}\boldsymbol{q}_a * \boldsymbol{y}_a}\Big\} &(3.155)\\
&=\; \lim_{\epsilon\to 0^+} \int\limits_{\mathbb{R}} \frac{\mathrm{d}u}{2\pi}\,\mathcal{L}[G^{(\mathbf{0})}_{p,q}](\boldsymbol{Y}_{p+q-1} - u\boldsymbol{e}_{p;0})\,\mathscr{F}[r_\epsilon\vartheta_+](u)\,. &(3.156)
\end{aligned}$$

We have passed on to the second line using dominated convergence since $\mathscr{F}^{-1}[\mathcal{L}[G^{(\mathbf{0})}_{p,q}]]$ is in the Schwartz class and ϑ_+ is smooth and bounded. Further, the third line was obtained by taking explicitly the Fourier transform. There, we have denoted the canonical basis of $(\mathbb{R}^{1,1})^{p+q-1}$ by $\boldsymbol{e}_{s;a}$, $s \in [\![\,1\,;\,p+q-1\,]\!]$ and $a \in [\![\,0\,;\,1\,]\!]$, and have set $r_\epsilon(s) \;=\; \mathrm{e}^{-\epsilon s^2}$. Now, going back to the explicit expression for $\mathcal{L}[G^{(\mathbf{0})}_{p,q}]$, we get

$$\begin{aligned}
\mathscr{F}[\chi_+](\boldsymbol{Y}_{p+q-1}) \;=\; \lim_{\epsilon\to 0^+} \int\limits_{\mathbb{R}} \frac{\mathrm{d}u}{2\pi} \int\limits_{\mathbb{R}^{1,1}} \mathrm{d}\boldsymbol{y}_{p+q}\,\mathscr{F}[r_\epsilon\vartheta_+](u)\, f\Big(\sum_{s=1}^{p+q}\boldsymbol{y}_s \,-\, (u,0),\dots,\sum_{s=p}^{p+q}\boldsymbol{y}_s \,-\, (u,0)\Big)& \\
\times\, g\Big(\sum_{s=p+1}^{p+q}\boldsymbol{y}_s,\dots,\boldsymbol{y}_{p+q}\Big)\,.& \qquad (3.157)
\end{aligned}$$

The convolution term appearing above can be recast as

$$f_{\vartheta_+;\epsilon}(\boldsymbol{X}_p) = \int_{\mathbb{R}} \frac{\mathrm{d}u}{2\pi} \mathscr{F}[r_\epsilon \vartheta_+](u) f(\boldsymbol{x}_1 - (u,0), \dots, \boldsymbol{x}_p - (u,0)) \tag{3.158}$$

$$= \int_{(\mathbb{R}^{1,1})^p} \mathrm{d}^p \boldsymbol{Q}\, \mathscr{F}[f](\boldsymbol{Q}_p)\, \vartheta_+\Big(\sum_{a=1}^{p} Q_{a;0} \Big) \exp\Big\{ - \epsilon \Big(\sum_{a=1}^{p} Q_{a;0} \Big)^2 \Big\} \,. \tag{3.159}$$

For any $\epsilon \geq 0$, $f_{\vartheta_+;\epsilon}$ is thus given by the Fourier transform of a function in the Schwartz class, so that $f_{\vartheta_+;\epsilon} \in \mathcal{S}((\mathbb{R}^{1,1})^p)$. It is easy to see by integration by parts that, uniformly in $\epsilon \geq 0$ and small enough,

$$\left| f_{\vartheta_+;\epsilon}(\boldsymbol{X}_p) \right| \leq \frac{C_M}{1 + \|X_p\|^M} \tag{3.160}$$

for any $M > 0$ and an M-dependent constant $C_M > 0$. Finally, by using that $\mathscr{F}[f](\boldsymbol{Q}_p)$ is in the Schwartz class, one readily gets by dominated convergence that $f_{\vartheta_+;\epsilon}(\boldsymbol{X}_p)$ converges pointwise to $f_{\vartheta_+;0}(\boldsymbol{X}_p)$. Then, the pointwise convergence along with the domination estimates (3.160) allow one to apply dominated convergence on the level of (3.157), which yields

$$\mathscr{F}[\chi_+](\boldsymbol{Y}_{p+q-1}) = \int_{\mathbb{R}^{1,1}} \mathrm{d}\boldsymbol{y}_{p+q}\; f_{\vartheta_+;0}\Big(\sum_{s=1}^{p+q} \boldsymbol{y}_s, \dots, \sum_{s=p}^{p+q} \boldsymbol{y}_s \Big) g\Big(\sum_{s=p+1}^{p+q} \boldsymbol{y}_s, \dots, \boldsymbol{y}_{p+q} \Big) = \mathcal{L}[\widetilde{G}^{(\mathbf{0})}_{p,q}](\boldsymbol{Y}_{p+q-1}) \,, \tag{3.161}$$

where $\widetilde{G}^{(\lambda\boldsymbol{v})}_{p,q}(\boldsymbol{X}_p, \boldsymbol{Y}_q) = f_{\vartheta_+;0}(\boldsymbol{X}_p) g(\boldsymbol{Y}_q)$.

Thus, going back to the representation (3.153) and making use of the relations (3.115), we get that

$$\mathfrak{M}^{(\lambda\boldsymbol{v})}[f,g] = \mathfrak{M}^{(\lambda\boldsymbol{v})}[f_{\vartheta_+;0}, g] - \mathfrak{M}^{(\lambda\boldsymbol{v})}_{\sigma}[f_{\vartheta_+;0}, g] \underset{\lambda\to+\infty}{\longrightarrow} 0 \,, \tag{3.162}$$

by (3.147) since $f_{\vartheta_+;0} \in \mathcal{S}((R^{1,1})^p)$.

Conclusion

This work demonstrated that, under the convergence conjecture 2.5 of the defining series representation, the multi-point correlation functions from [20] and obtained by solving the integrable bootstrap equations for the Sinh-Gordon quantum field theory satisfy the Wightman axioms. The method we developed is universal to integrable quantum field theories in that we only used the structural properties of the correlation functions stemming from the bootstrap equations and did not rely on some specificities of the model. We are thus fairly confident that, upon minor modifications, it should be applicable to other, more complex quantum field theories, the Sine-Gordon integrable quantum field theory in particular.

Acknowledgment

The work of KKK and AS is supported by the ERC Project LDRAM : ERC-2019-ADG Project 884584. KKK acknowledges support from CNRS, from the joint AND-DFG TSF24 project ANR-24-CE92-0033 and the CNRS-IEA "Corrélateurs dynamiques de la chaîne XXZ en et hors équilibre thermal" grant.

A Support of tempered distributions

In this section we prove a result on the support of functions representing tempered distributions. This seems to be a very classical property. However, since we did not manage to find a reference in the literature, we include a proof here for completeness.

Proposition A.1. *Let $N \in \mathbb{N}^*$ and $\mathtt{D}$ be a tempered distribution on $\mathcal{S}(\mathbb{R}^N)$ such that* $\mathrm{supp}[\mathtt{D}] \subset \Omega^{\mathrm{c}}$*, with Ω an open subset of $\mathbb{R}^N$. Then, there exists $\boldsymbol{\mu}_N = (\mu_1, \dots, \mu_N) \in \mathbb{N}^N$ and $h \in C^0(\mathbb{R}^N)$ of at most polynomial growth*

$$\big|h(\boldsymbol{X}_p)\big| \le C \cdot (1 + \|\boldsymbol{X}_N\|^M) \quad \text{for some} \quad C, M > 0 \quad \text{with} \quad \|\boldsymbol{X}_N\|^2 = \sum_{s=1}^{N} x_s^2 \,, \tag{A.1}$$

satisfying $\mathrm{supp}[h] \subset (\Omega^{\mathrm{c}})_{\epsilon}$ *with $\epsilon > 0$ and as small as need be, $A_\epsilon = \{\boldsymbol{X}_N \in \mathbb{R}^N \ : \ \mathrm{d}(A, \boldsymbol{X}_N) \le \epsilon\}$, and such that*

$$\mathtt{D}[f] = \int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{X}\, h(\boldsymbol{X}_N)\, \partial_{\boldsymbol{X}_N}^{\boldsymbol{\mu}_N} f(\boldsymbol{X}_N) \tag{A.2}$$

for any $f \in \mathcal{S}(\mathbb{R}^N)$. There $\boldsymbol{X}_N = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\partial_{\boldsymbol{X}_N}^{\boldsymbol{\mu}_N} = \prod_{a=1}^{N} \partial_{x_a}^{\mu_a}$.

Proof —

The fact that there exists $h \in C^0(\mathbb{R}^N)$ and $\boldsymbol{\mu}_N \in \mathbb{N}^N$ such that (A.2) holds is ensured by Theorem 8.3.1 in [6]. We now show that we may reduce to a setting where $\mathrm{supp}[h] \subset (\Omega^{\mathrm{c}})_\epsilon$ with $\epsilon > 0$ and as small as need be.

Let $\epsilon > 0$ and pick Υ open in $\mathbb{R}^N$ such that $\overline{\Upsilon} \subset \Omega$ and $\Upsilon^{\mathrm{c}} \subset (\Omega^{\mathrm{c}})_\epsilon$. Let $f \in \mathcal{S}(\mathbb{R}^N)$ be such that $\mathrm{supp}[f] \subset \Upsilon$. Then, pick $\chi_\sigma \in C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ such that $\mathrm{supp}[\chi_\sigma] \subset [-\sigma\,;\sigma]^N$ for σ small enough, $\chi_\sigma \ge 0$ and χ_σ integrates to 1. Let

$$f_\sigma(\boldsymbol{X}_N) = \int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{Y} \chi_\sigma(\boldsymbol{X}_N - \boldsymbol{Y}_N) f(\boldsymbol{Y}_N) \,, \tag{A.3}$$

so that $\mathrm{supp}[f_\sigma] \subset (\mathrm{supp}[f])_\sigma$. In particular, for σ small enough, independent of f, $\mathrm{supp}[f_\sigma] \subset \Omega$. Thus, one has

$$0 = \mathtt{D}[f_\sigma] = \int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{X} \int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{Y}\, h(\boldsymbol{X}_N)\, (\partial_{\boldsymbol{X}_N}^{\boldsymbol{\mu}_N} \chi_\sigma)(\boldsymbol{X}_N - \boldsymbol{Y}_N) f(\boldsymbol{Y}_N) \tag{A.4}$$

$$= (-1)^{|\boldsymbol{\mu}_N|} \int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{Y} f(\boldsymbol{Y}_N)\, \partial_{\boldsymbol{Y}_N}^{\boldsymbol{\mu}_N} \underbrace{\int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{X} \chi_\sigma(\boldsymbol{X}_N - \boldsymbol{Y}_N) h(\boldsymbol{X}_N)}_{= h_\sigma(\boldsymbol{Y}_N)} \,. \tag{A.5}$$

h_σ is smooth and since the relation holds for any Schwartz function f with support inside of Υ, one has

$$\partial_{\boldsymbol{X}_N}^{\boldsymbol{\mu}_N} h_\sigma(\boldsymbol{X}_N) = 0 \qquad \text{with} \qquad \boldsymbol{X}_N \in \Upsilon \,. \tag{A.6}$$

Thus, there exists a multivariate polynomial $P_\sigma \in \mathbb{C}[X_1, \dots, X_N]$ with upper bounded partial degree in respect to X_s $\mathrm{d}_s[P_\sigma] < \mu_s$ uniformly in σ small enough, and such that $h_\sigma(\boldsymbol{X}_N) = P_\sigma(\boldsymbol{X}_N)$ for any $\boldsymbol{X}_N \in \Upsilon$. One has that $h_\sigma \to h$ pointwise on Υ, so that $P_\sigma(\boldsymbol{X}_N) \to P_0(\boldsymbol{X}_N)$. Obviously, P_0 is a polynomial and $\mathrm{d}_s[P_0] < \mu_s$. Therefore, one has $h = P_0$ on Υ. Now, obviously, for any $f \in \mathcal{S}(\mathbb{R}^N)$, it holds

$$\int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{Y} P_0(\boldsymbol{Y}_N)\, \partial_{\boldsymbol{Y}_N}^{\boldsymbol{\mu}_N} f(\boldsymbol{Y}_N) = 0 \,. \tag{A.7}$$

One then sets $\widetilde{h} = h - P_0$ and $\mathrm{supp}[\widetilde{h}] \subset \Upsilon^{\mathrm{c}} \subset (\Omega^{\mathrm{c}})_{\epsilon}$, and it obviously holds

$$\mathtt{D}[f_{\epsilon}] \;=\; \int\limits_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{Y} \widetilde{h}(\boldsymbol{Y}_N)\, \partial^{\boldsymbol{\mu}_N}_{\boldsymbol{Y}_N} f(\boldsymbol{Y}_N)\ . \tag{A.8}$$

The polynomial growth of $\widetilde{h}$ follows from the one of h. ■

References

[1] H. Babujian, A. Fring, M. Karowski, and A. Zapletal, *"Exact form factors in integrable quantum field theories: the sine-Gordon model."*, Nucl. Phys. B **538** (1999), 535–586.

[2] H. Babujian and M. Karowski, *"Sine-Gordon breather form factors and quantum field equations."*, J.Phys.A **35** (2002), 9081–9104.

[3] N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, *"Introduction to Axiomatic Quantum Field Theory."*, Mathematical physics monographs series, W. A. Benjamin, Inc., Advanced Book Program, Reading, Massachusetts, 1975.

[4] V. Brazhnikov and S. Lukyanov, *"Angular quantization and form factors in massive integrable models."*, Nucl. Phys. B **512** (1998), 616–636.

[5] B. Feigin and M. Lashkevich, *"Form factors of descendant operators: free field construction and reflection relations."*, J. Phys. A: Math. Theor. **42** (2009), 304014.

[6] F.G. Friedlander, *"introduction to the theory of distributions"*, Cambridge University Press, 1982.

[7] J. Glimm, A. Jaffe, and T. Spencer, *"The Wightman axioms and particle structure in the $P(\phi)_2$ quantum field model."*, Ann. Math. **100** (1974), 585–632.

[8] V.M. Gryanik and S.N. Vergeles, *"Two-dimensional quantum field theories having exact solutions."*, J. Nucl. Phys. **23** (1976), 1324–1334.

[9] M. Gubinelli and M. Hofmanová, *" Global Solutions to Elliptic and Parabolic Φ^4 Models in Euclidean Space."*, Comm Math. Phys. **368** (2019), 1201–1266.

[10] C. Guillarmou, T.S. Gunaratnam, and V. Vargas, *"2d Sinh-Gordon model on the infinite cylinder."*, math-pr:2405.04076.

[11] C. Guillarmou, A. Kupiainen, R. Rhodes, and V. Vargas, *"Conformal bootstrap in Liouville theory."*, Acta. Math. **233** (2024), 33–194.

[12] W. Heisenberg, *"Der mathematische Rahmen der Quantentheorie der Wellenfelder."*, Zeit. für Naturforschung **1** (1946), 608–622.

[13] G. Jona-Lasinio and P.K. Mitter, *"On the stochastic quantization of field theory."*, Comm. Math. Phys. **101** (1985), 409–436.

[14] M. Karowski and P. Weisz, *"Exact form factors in (1+1)-dimensional field theoretic models with soliton behaviour."*, Nucl. Phys. B **139** (1978), 455–476.

[15] I. M. Khamitov, *"A constructive approach to the quantum (coshϕ)2 model. I. The method of the Gel'fand-Levitan-Marchenko equations."*, J. Sov. Math. **40** (1988), 115–148.

[16] A.N. Kirillov, *"T-invariance, CPT-invariance, and local commutativity for the quantum (cosh ϕ)2-model."*, J. Sov. Math. **40** (1988), 6–13.

[17] A.N. Kirillov and F.A. Smirnov, *"A representation of the current algebra connected with the SU (2)-invariant Thirring model."*, Phys. Rev. B **198** (1987), 506–510.

[18] ______, *"Form-factors in the SU(2)-invariant Thirring model."*, J. Soviet. Math. **47** (1989), 2423–2450.

[19] K.K. Kozlowski, *"On convergence of form factor expansions in the infinite volume quantum Sinh-Gordon model in 1+1 dimensions."*, Inventiones mathematicae **233** (2023), 725–827.

[20] K.K. Kozlowski and A. Simon, *"Multi-point correlation functions in the infinite volume quantum Sinh-Gordon model in 1+1 dimensions."*, math-ph: 2502.03894.

[21] A. Kupiainen, R. Rhodes, and V. Vargas, *"Integrability of Liouville theory: proof of the DOZZ Formula."*, Ann. of Math. **191** (2020), 81–166.

[22] S. Lukyanov, *"Free field representation for massive integrable models."*, Comm. Math. Phys. **167** (1995), 183–226.

[23] M.Hairer, *"Regularity structures and the dynamical Φ^4_3 model."*, Current Developments in Mathematics, (2014), 49pp.

[24] K. Osterwalder and R. Schrader, *"Axioms for Euclidean Green's functions."*, Comm. Math. Phys. **31** (1973), 83–112.

[25] ______, *"On the equivalence of the Euclidean and Wightman formulation of field theory."*, Comm. Math. Phys. **37** (1974), 257–272.

[26] G. Parisi and Y. Wu, *"Perturbation theory without gauge fixing."*, Scientia Sinica **24** (1981), 483–496.

[27] F.A. Smirnov, *"Form factors in completely integrable models of quantum field theory."*, Advanced Series in Mathematical Physics, vol. 14, World Scientific, 1992.

[28] R. Streater and A. Wightman, *"PCT, Spin and statistics, and all that."*, W.A. Benjamin inc., New York New York, vol. 1, The mathematical physics monograph series, 1964.

[29] S.J. Summers, *"A Perspective on Constructive Quantum Field Theory."*, math-ph: 1203.3991 (2012), 60pp.

[30] J.A. Wheeler, *"On the mathematical description of light nuclei by the method of resonating group structure."*, Phys. Rev. B **52** (1937), 1107–1106.