

Effective Resistance in Simplicial Complexes as Bilinear Forms: Generalizations and Properties

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Abstract

The concept of effective resistance, originally introduced in electrical circuit theory, has been extended to the setting of graphs by interpreting each edge as a resistor. In this context, the effective resistance between two vertices quantifies the total opposition to current flow when a unit current is injected at one vertex and extracted at the other. Beyond its physical interpretation, the effective resistance encodes rich structural and geometric information about the underlying graph: it defines a metric on the vertex set, relates to the topology of the graph through Foster's theorem, and determines the probability of an edge appearing in a random spanning tree. Generalizations of effective resistance to simplicial complexes have been proposed in several forms, often formulated as matrix products of standard operators associated with the complex.

In this paper, we present a twofold generalization of the effective resistance. First, we introduce a novel, basis-independent bilinear form, derived from an algebraic reinterpretation of circuit theory, that extends the classical effective resistance from graphs. Second, we extend this bilinear form to simplices, chains, and cochains within simplicial complexes. This framework subsumes and unifies all existing matrix-based formulations of effective resistance. Moreover, we establish higher-order analogues of several fundamental properties known in the graph case: (i) we prove that effective resistance induces a pseudometric on the space of chains and a metric on the space of cycles, and (ii) we provide a generalization of Foster's Theorem to simplicial complexes.

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1 Introduction

The concept of effective resistance, originating from electrical network theory, models each edge in a graph as a resistor with a given resistance. The effective resistance between two vertices reflects the total resistance to the flow of electrical current when one unit of current is injected at one vertex and extracted at the other, assuming unit resistance across all edges. This measure reflects both the direct and indirect paths between the vertices, incorporating the overall structure of the graph. It enjoys a number of important geometric and algebraic properties; see a recent review by Devriendt [15]. In particular, the effective resistance is proportional to the commute time between a pair of vertices [12] (i.e., the expected length of a random walk from one vertex to another and back); its maximum value characterizes the cover time [12] (i.e., the expected length of a random walk visiting all vertices); and it determines a metric between the vertices of a graph [29, 35]. In addition, the effective resistance between the endpoints of an edge is proportional to the probability that such edge appears in a random spanning tree [34].

Effective resistance in graphs has been widely applied in graph theory, network science, and machine learning due to its intrinsic connection with structural properties. For instance, effective resistance has been used to measure centrality [42], characterize chemical networks [4], study graph sparsification [49], node clustering (community detection) [2] and label propagation [43].

Extensions of effective resistance to simplicial complexes have been proposed in several distinct formulations. A notion of effective resistance has been introduced by Kook and Lee [36] in a simplicial complex equipped with positive resistances on its top-dimensional faces. A different formulation has been defined by Osting, Palande and Wang [43], as a means for sparsifying a simplicial complex, generalizing graph sparsification. Their approach is more general, as it applies not only to top-dimensional faces but also to simplices of arbitrary dimension (together with their associated weights). Black and Maxwell [9] discussed effective resistance of cycles in a simplicial complex. In all of these approaches, the effective resistance is defined from matrices obtained as a product of various matrix operators acting on the simplicial complexes.

In this paper, we generalize the notion of effective resistance by providing a basis-free definition, from which the matrix representations above follow, and we describe its theoretical properties. Our contributions are as follows:

- We introduce the *effective resistance bilinear form*, which extends the notion of effective resistance from edges in graphs to simplices, chains, and cochains in simplicial complexes. See **Section 5.2** and **Section 6.1**.
- We show that such a bilinear form unifies existing theory of effective resistance on simplicial complexes (i.e., [9, 36, 43]), as all formulations can be retrieved from appropriate choices of bases in the chain or cochain spaces. See **Section 6.2**.
- We provide a partial extension of the result that effective resistance is a metric on graphs, showing that effective resistance induces a pseudometric on the space of p -chains. See **Section 7**.
- We generalize Foster’s Theorem [25] from graphs to simplicial complexes, establishing a deep connection between the structure of a complex and our notion of effective resistance. See **Section 8**.

2 Related Work

Effective resistance has been extensively studied in graph theory and related fields as a tool for analysing graph connectivity, with deep connections to random walks, electrical flows, and spectral properties. In recent years, this concept has been extended to more general combinatorial structures such as hypergraphs and simplicial complexes. In what follows, we first review key properties and applications of effective resistance in graphs, and then explore how it has been generalized to these higher-dimensional settings. We omit detailed definitions here and refer the reader to the original papers for full treatment.

2.1 Properties of Effective Resistance for Graphs

Effective resistance originated in circuit theory [34], but has since become a widely-studied graph-theoretic concept due to its deep connections with the Laplacian and spanning trees. In addition to the properties mentioned in **Section 1**, it can also be characterized via current flow according to Thomson’s principle, which underlies Rayleigh’s monotonicity law: increasing edge conductance reduces the effective resistance between any pair of nodes [18]. A geometric interpretation, developed by Fiedler [23] and Sharpe and Moore [48], and later synthesized in the survey by Devriendt [15],

shows that the square root of effective resistance defines an embedding of the graph into an Euclidean simplex, offering a geometric perspective on graph structure. Related properties include the fact that the Schur complement of a Laplacian matrix yields another Laplacian and corresponds geometrically to a map between simplices. Fiedler [22] also introduced a matrix identity connecting the Laplacian and the effective resistance matrix, bringing together algebraic and geometric views of graph structure.

2.2 Applications of Effective Resistance for Graphs

Effective resistance has found broad applications across a range of domains in graph theory and network science. One important use is as a measure of network robustness [21]: it captures how well-connected a network is by accounting for both the number of alternative paths between nodes and their lengths, and it decreases as new edges are added. It also admits a natural interpretation via random walks: lower resistance between two nodes corresponds to shorter expected commute times, signaling greater resilience to node or edge failures. Effective resistance also plays a key role in graph sparsification [49]. Using effective resistance as a sampling guide enables the construction of spectral sparsifiers that improve over prior approaches by yielding subgraphs with fewer edges that preserve spectral properties with high accuracy. Effective resistance has also found applications in graph clustering [2], where a graph can be partitioned into subgraphs with bounded effective resistance diameter, discarding only a small fraction of the edges. Other related applications of effective resistance include computing maximum flows [13], finding thin trees [3], and generating random spanning trees [33, 38]. More recently, effective resistance has been used as a positional encoding technique in graph neural networks and graph transformers. A key challenge in this area is identifying which positional encodings provide the greatest expressive power. Notably, graph transformers that use effective resistance as a positional encoding have been shown to identify cut vertices, whereas those relying on shortest path encodings have not shown this capability [52]. Another recent application of effective resistance is in forms of queries to graph inference problems [8]. The latter includes graph reconstruction, verification, and property testing as special cases.

2.3 Effective Resistance Formulations for Hypergraphs

Aghdaei and Feng [1] extended the concept of effective resistance from graphs to hypergraphs using the nonlinear Laplacian framework of Chan et al. [11]. They defined a nonlinear quadratic form on the hypergraph that quantifies the weighted potential variation across its hyperedges. The effective resistance of a hyperedge is then characterized through an optimization problem comparing the potential difference between two of its vertices with this quadratic form. Building on this idea, Aghdaei and Feng developed a scalable spectral coarsening method that estimates hyperedge resistances efficiently and clusters nodes connected by low-resistance hyperedges. They further introduced a node weight propagation scheme to extend the approach to a multilevel decomposition framework.

2.4 Effective Resistance Formulations for Simplicial Complexes

Kook and Lee [36] introduced a notion of effective resistance for a simplicial network, where the network is given by a simplicial complex together with resistances assigned to its top-dimensional simplices. They considered a current generator in the form of a d -dimensional simplex attached to the complex, producing an extended cell complex. Using a combinatorial Hodge Laplacian approach, they assigned a unique harmonic class to the generator simplex. This harmonic class induces both the current and the voltage on the extended complex, and satisfies a higher-dimensional

analogue of Thomson’s principle. The effective resistance is then defined as the ratio between the voltage and current components associated with the generator simplex.

Another contribution of Kook and Lee [36] is a combinatorial interpretation of effective resistance, obtained through a connection with weighted high-dimensional tree numbers [20]. This extends the classical correspondence between effective resistance and spanning trees in one-dimensional networks. They also provided a matrix-based extension of Foster’s theorem.

Osting et al. [43] introduced effective resistance for simplicial complexes in the context of sparsification, using a modified formulation of the up Laplacian. Since the up Laplacian is not symmetric (see discussions in **Section 4.2**), they constructed a symmetric, positive semi-definite matrix to define effective resistance. In this framework, the effective resistance of a simplex is determined from the diagonal entries of a derived matrix. Another contribution of Osting et al. [43] is an extension of the Cheeger constant introduced by Gundert and Szedlák for unweighted simplicial complexes [28]. They further established that the corresponding Cheeger inequality, which links the first non-zero eigenvalue of the weighted up Laplacian to the Cheeger constant, continues to hold under sparsification.

Hansen and Ghrist [30] defined effective resistance in the setting of cellular (co)sheaves, which generalize simplicial complexes. Their formulation characterizes effective resistance between homologous cycles in terms of an optimization problem, and it applies equally to weighted sheaves since every weighted sheaf has a canonical dual weighted cosheaf. They also explored applications of this framework to sparsification.

Black and Maxwell [9] defined effective resistance for null-homologous cycles in simplicial complexes, in a way that is equivalent to the framework of Hansen and Ghrist [30]. Their formulation generalizes the graph case and is consistent with other higher-dimensional definitions [36, 43, 30], though it provides limited intuition on its own. To address this, they showed that effective resistance can be equivalently described through chains with a given boundary. They further extended classical properties, including the parallel and series laws and Rayleigh’s monotonicity principle. In addition, they proposed a quantum algorithm for testing whether a cycle is null-homologous in a simplicial complex, using a parametrization involving both effective resistance and effective capacitance.

3 Background

In this section, we review foundational definitions from linear algebra and illustrate their applications to chain and cochain spaces. This establishes the essential framework and notations used throughout this work. We assume a basic understanding of simplicial homology, see [41] for an introductory text and **Appendix A** for a summary of notations.

3.1 A Primer on Linear Algebra

Duals. Let U be a vector space over a field $\mathbb{k}$. The *dual vector space*, denoted $U^\vee$, is given by the linear forms on U , that is, $U^\vee := \text{Hom}_{\mathbb{k}}(U, \mathbb{k})$. Given a linear map $f : U \rightarrow V$ between vector spaces, the *dual map*

$$f^\vee : V^\vee \rightarrow U^\vee$$

maps each linear form φ on V to the linear form $f^\vee(\varphi)$ on U defined for all $u \in U$ as

$$f^\vee(\varphi)(u) := \varphi(f(u)).$$

Let $\mathcal{B}_U = \{u_1, \dots, u_n\}$ be a basis of a finite-dimensional vector space U . We can define the following basis on the dual space $U^\vee$, known as the *dual basis*, denoted by $\mathcal{B}_U^\vee = \{u_1^\vee, \dots, u_n^\vee\}$. Each element $u_i^\vee \in U^\vee$ is a linear form defined by its action on the basis vectors of U according to the rule:

$$u_i^\vee(u_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. This defines an isomorphism $U \cong U^\vee$ which is not canonical, i.e., it depends on the choice of the basis $\mathcal{B}_U$.

Adjoints. If we additionally assume that U and V have inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$, respectively, given a linear map $f : U \rightarrow V$ between them, we can define the *adjoint map*

$$f^* : V \rightarrow U$$

as the map satisfying

$$\langle u, f^*(v) \rangle_U = \langle f(u), v \rangle_V$$

for all $u \in U$ and $v \in V$.

Matrix representations. Given a linear map $f : U \rightarrow V$ and bases $\mathcal{B}_U$ of U and $\mathcal{B}_V$ of V , we denote by $[f]_{\mathcal{B}_U}^{\mathcal{B}_V}$ the matrix representation of f with respect to these bases, which will be a matrix of dimension $\dim(V) \times \dim(U)$. If instead we have a bilinear form $\varphi : U \times V \rightarrow \mathbb{R}$, we will denote its matrix representation as $[\varphi]_{\mathcal{B}_U, \mathcal{B}_V}$, where we omit the basis in the target space $\mathbb{R}$ as it will always be the basis given by $\{1\}$.

Matrix representation of the adjoint map. Let $f : U \rightarrow V$ be a linear map between the vector spaces U and V with inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$, respectively. Choose bases $\mathcal{B}_U$ and $\mathcal{B}_V$ in U and V respectively, and denote $[f]_{\mathcal{B}_U}^{\mathcal{B}_V} = A$, $[\langle \cdot, \cdot \rangle_U]_{\mathcal{B}_U, \mathcal{B}_U} = M$ and $[\langle \cdot, \cdot \rangle_V]_{\mathcal{B}_V, \mathcal{B}_V} = N$. It is easy to check that the matrix representation of the adjoint map $f^* : V \rightarrow U$ with respect to these bases is then given by

$$[f^*]_{\mathcal{B}_V}^{\mathcal{B}_U} = M^{-1} A^\top N. \quad (1)$$

3.2 Chain and Cochain Inner Product Spaces

Let K denote a d -dimensional finite simplicial complex, and for $0 \leq p \leq d$, let K_p be the set of p -simplices in K . We denote by $n_p := |K_p|$ the number of elements in K_p . We pick an *orientation* for K by choosing an ordering of its vertices (called the *standard orientation*).

Chain inner product space. Fixing a field $\mathbb{k}$ (typically, $\mathbb{k} = \mathbb{R}$), the *chain group* $C_p(K, \mathbb{R})$ is defined as the vector space spanned by the oriented simplices in K_p . We just write $C_p(K)$ when the field is clear from the context. Chain groups are endowed with *boundary maps* $\partial_p : C_p(K) \rightarrow C_{p-1}(K)$ (see [31] for more details on these definitions).

When, in addition, we assume that K is *weighted*, that is, we have a *weight function* $w_p : K_p \rightarrow \mathbb{R}_{>0}$ for any $p \geq 0$, we can translate this extra information to the chain spaces by endowing them with the following inner products $\langle \cdot, \cdot \rangle_{C_p} : C_p(K) \times C_p(K) \rightarrow \mathbb{R}_{\geq 0}$ defined on the p -simplices $\sigma, \tau \in K_p$ as

$$\langle \sigma, \tau \rangle_{C_p} := \begin{cases} w_p(\sigma)^{-1}, & \text{if } \sigma = \tau, \\ 0, & \text{otherwise;} \end{cases}$$

and extended linearly to general chains.

We denote by ∂_p^* the adjoint of the boundary operator ∂_p with respect to the chosen inner product. In summary, we have the chain complex:

$$\cdots \quad C_{p+1}(K) \xrightleftharpoons[\partial_{p+1}]{\partial_{p+1}^*} C_p(K) \xrightleftharpoons[\partial_p]{\partial_p^*} C_{p-1}(K) \quad \cdots \quad (2)$$

The kernel of ∂_p is called the p -th *cycle group*, and its elements are called the p -th *cycles*. The image of ∂_{p+1} is called the p -th *boundary group*, and its elements are called the p -th *boundaries*. The p -th homology group of K , denoted $H_p(K)$, is defined as the quotient of the p -th cycle group by the p -th boundary group. The dimension of $H_p(K)$ is called the p -th *Betti number*, denoted $\beta_p(K)$.

Cochain inner product space. Similarly, we can define *cochain spaces* $C^p(K, \mathbb{k}) = C_p(K, \mathbb{k})^\vee$ as the dual vector spaces of the chain spaces, also denoted $C^p(K)$ if the field is clear from the context. The *coboundary maps* $\delta_p : C^p(K) \rightarrow C^{p+1}(K)$ are then defined as the duals of the boundary maps: $\delta_p = \partial_{p+1}^\vee$. We also have an inner product in the space of cochains $\langle \cdot, \cdot \rangle_{C^p} : C^p(K) \times C^p(K) \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$\langle \sigma^\vee, \tau^\vee \rangle_{C^p} := \begin{cases} w_p(\sigma), & \text{if } \sigma = \tau, \\ 0, & \text{otherwise;} \end{cases} \quad (3)$$

for all $\sigma, \tau \in K_p$ and extended linearly.

We denote by δ_p^* the adjoint of the coboundary operator δ_p with respect to the chosen inner product. In summary, we have the cochain complex:

$$\cdots \quad C^{p+1}(K) \xrightleftharpoons[\delta_p^*]{\delta_p} C^p(K) \xrightleftharpoons[\delta_{p-1}^*]{\delta_{p-1}} C^{p-1}(K) \quad \cdots \quad (4)$$

The kernel of δ^p is called the p -th *cocycle group*, and its elements are called the p -th *cocycles*. The image of δ^{p-1} is called the p -th *coboundary group*, and its elements are called the p -th *coboundaries*. The p -th cohomology group of K , denoted $H^p(K)$, is defined as the quotient of the p -th cocycle group by the p -th coboundary group. With real coefficients, the cohomology groups are simply the dual vector spaces of the homology groups $H^p(K) = \text{Hom}_{\mathbb{R}}(H_p(K), \mathbb{R})$. Consequently, the dimension of $H^p(K)$ coincides with the p -th Betti number.

Musical isomorphisms. As noted in **Section 3.1**, identifying a space with its dual via a chosen basis yields a non-canonical isomorphism. In contrast, the inner product on $C_p(K)$ defined above induces the musical (flat and sharp) maps, providing canonical isomorphisms $C_p(K) \cong C^p(K)$ (cf. [51]); these correspond to the vertical arrows in diagram (5).

Definition 3.1 (Musical isomorphisms). The *flat isomorphism* $\flat_p : C_p(K) \rightarrow C^p(K)$ sends each chain $\alpha \in C_p(K)$ to the linear form

$$\flat_p(\alpha) := \langle \alpha, \cdot \rangle_{C_p}.$$

Conversely, the *sharp isomorphism* $\sharp_p : C^p(K) \rightarrow C_p(K)$ sends each cochain $f \in C^p(K)$ to the unique chain $\alpha \in C_p(K)$ such that

$$f(\beta) = \langle \alpha, \beta \rangle_{C_p}, \quad \forall \beta \in C_p(K).$$

By construction, $\sharp_p = \flat_p^{-1}$. Indeed, since $\flat_p(\alpha)(\beta) = \langle \alpha, \beta \rangle_{C_p} \in C^p(K)$ for all $\beta \in C_p(K)$, and since $\sharp_p(\flat_p(\alpha))$ is a chain α' such that $\flat_p(\alpha)(\beta) = \langle \alpha', \beta \rangle_{C_p}$ for all $\beta \in C_p(K)$, one sees that $\alpha = \alpha'$. For notational simplicity, we will often write $\flat_p^{-1}$ instead of $\sharp_p$.

Lemma 3.2. The flat isomorphism $\flat_p: C_p(K) \rightarrow C^p(K)$ is an *orthogonal map*, that is, it preserves the inner product: for any $\alpha, \beta \in C_p(K)$,

$$\langle \alpha, \beta \rangle_{C_p} = \langle \flat_p(\alpha), \flat_p(\beta) \rangle_{C^p}.$$

Similarly, the sharp isomorphism $\sharp_p: C^p(K) \rightarrow C_p(K)$ is orthogonal, and the two are adjoint to each other: $\flat_p^* = \sharp_p$ and $\sharp_p^* = \flat_p$.

Proof. For any simplex $\sigma \in K_p$, we have $\flat_p(\sigma) = \langle \sigma, \cdot \rangle_{C_p} = w_p(\sigma)^{-1} \sigma^\vee$ by definition. Thus, for any two simplices $\sigma, \tau \in K_p$,

$$\langle \flat_p(\sigma), \flat_p(\tau) \rangle_{C^p} = w_p(\sigma)^{-1} w_p(\tau)^{-1} \langle \sigma^\vee, \tau^\vee \rangle_{C^p} = \begin{cases} w_p(\sigma)^{-1}, & \sigma = \tau, \\ 0, & \text{otherwise.} \end{cases}$$

This coincides with $\langle \sigma, \tau \rangle_{C_p}$. By bilinearity of the inner product and linearity of $\flat_p$, the equality extends to all $\alpha, \beta \in C_p(K)$, so $\flat_p$ preserves inner products. A similar argument holds for $\sharp_p$.

Since $\sharp_p = \flat_p^{-1}$ and both maps are isometries, we have $\sharp_p^* = \flat_p$ and, equivalently, $\flat_p^* = \sharp_p$. Indeed, for any $f \in C^p(K)$ and $\alpha \in C_p(K)$,

$$\langle f, \sharp_p^* \alpha \rangle_{C^p} = \langle \sharp_p f, \alpha \rangle_{C_p} = \langle \flat_p \sharp_p f, \flat_p \alpha \rangle_{C^p} = \langle f, \flat_p \alpha \rangle_{C^p},$$

where the first equality is the definition of the adjoint, the second uses that $\flat_p$ preserves inner products, and the last follows from $\flat_p = \sharp_p^{-1}$. $\square$

Lemma 3.3. The following diagram commutes:

$$\begin{array}{ccccc} \cdots & C_p(X) & \xrightarrow{\partial_p} & C_{p-1}(X) & \cdots & \cdots & C_p(X) & \xleftarrow{\partial_p^*} & C_{p-1}(X) & \cdots \\ & \flat_p \downarrow \uparrow \sharp_p & & \flat_{p-1} \downarrow \uparrow \sharp_{p-1} & & \text{and} & \flat_p \downarrow \uparrow \sharp_p & & \flat_{p-1} \downarrow \uparrow \sharp_{p-1} & \\ \cdots & C^p(X) & \xrightarrow{\delta_{p-1}^*} & C^{p-1}(X) & \cdots & \cdots & C^p(X) & \xleftarrow{\delta_{p-1}^*} & C^{p-1}(X) & \cdots \end{array} \quad (5)$$

In particular, $\flat_{p-1} \circ \partial_p = \delta_{p-1}^* \circ \flat_p$ and $\delta_{p-1} \circ \flat_{p-1} = \flat_p \circ \partial_p^*$.

Proof. It suffices to show that $\delta_{p-1} = \flat_p \circ \partial_p^* \circ \flat_{p-1}^{-1}$. Indeed, since the coboundary map is the *dual* of the boundary map, for $f \in C^{p-1}(K)$ and $\alpha \in C_p(K)$,

$$(\flat_p \partial_p^* \flat_{p-1}^{-1} f)(\alpha) = \langle \partial_p^* (\flat_{p-1}^{-1} f), \alpha \rangle_{C_p} = \langle \flat_{p-1}^{-1} f, \partial_p \alpha \rangle_{C_{p-1}} = f(\partial_p \alpha) = (\partial_p^\vee f)(\alpha) = (\delta_{p-1} f)(\alpha).$$

showing that $\delta_{p-1} = \flat_p \circ \partial_p^* \circ \flat_{p-1}^{-1}$. $\square$

Remark 3.4. In our exposition, we first equip both chain and cochain spaces with inner products and then show that the musical isomorphisms preserve them. This is equivalent to endowing only the chain space with an inner product, constructing the flat isomorphism from it, and inducing the corresponding inner product on the cochain space via the push-forward through this isomorphism.

3.3 Bases in the Spaces of Chains and Cochains

Let K be a d -dimensional simplicial complex. The set of simplices provides a standard choice of basis for the chain groups, which in turn induces a dual basis for the cochain groups.

Definition 3.5 (Standard bases). For $0 \leq p \leq \dim(K)$, the *standard basis in the space of chains* $C_p(K)$, is defined by the ordered set of p -simplices in K , that is,

$$\mathcal{B}_p := \{\sigma_1, \dots, \sigma_{n_p}\}.$$

The *standard basis in the space of cochains* is defined as the dual basis, that is,

$$\mathcal{B}^p := \{\sigma_1^\vee, \dots, \sigma_{n_p}^\vee\}.$$

Remark 3.6. We already implicitly consider this choice of basis when we define the inner products $\langle \cdot, \cdot \rangle_{C_p}$ and $\langle \cdot, \cdot \rangle_{C^p}$ in [Section 3.2](#). These inner products are precisely defined so that the standard bases are *orthogonal bases*.

By normalizing the bases above, we obtain the following orthonormal bases in the chain and cochain spaces.

Definition 3.7 (Orthonormal bases). For $0 \leq p \leq \dim(K)$, the *orthonormal basis in the space of chains* $C_p(K)$ with respect to $\langle \cdot, \cdot \rangle_{C_p}$ is given by

$$\tilde{\mathcal{B}}_p := \left\{ \sqrt{w_p(\sigma_1)} \sigma_1, \dots, \sqrt{w_p(\sigma_{n_p})} \sigma_{n_p} \right\}.$$

Similarly, the *orthonormal basis in the space of cochains* with respect to $\langle \cdot, \cdot \rangle_{C^p}$ is given by

$$\tilde{\mathcal{B}}^p := \left\{ \frac{\sigma_1^\vee}{\sqrt{w_p(\sigma_1)}}, \dots, \frac{\sigma_{n_p}^\vee}{\sqrt{w_p(\sigma_{n_p})}} \right\}.$$

Remark 3.8. The “standard” and “orthonormal” bases serve different roles. The standard bases depend only on the combinatorial structure of K , while the orthonormal bases also depend on the choice of inner products, or equivalently, on the weights w_p . Consequently, neither choice is strictly canonical once a weighting scheme is introduced; each is convenient in a different context. For this reason, we avoid referring to either as *canonical*.

3.4 Matrix Representations

We can now provide matrix representations for all the operators defined in [Section 3.2](#) with respect to the bases in [Section 3.3](#).

Inner product matrices and changes of basis. In the standard bases the inner product matrices in the spaces of chains and cochains are respectively

$$[\langle \cdot, \cdot \rangle_{C_p}]_{\mathcal{B}_p, \mathcal{B}_p} = W_p^{-1}, \quad \text{and} \quad [\langle \cdot, \cdot \rangle_{C^p}]_{\mathcal{B}^p, \mathcal{B}^p} = W_p,$$

where $W_p := \text{diag}(w_p(\sigma_1), \dots, w_p(\sigma_{n_p}))$ denotes the diagonal matrix with values given by the weights of the p -simplices. In the orthonormal bases, by definition, these matrices will be the identity. The change of basis matrix between the standard and orthonormal basis is given by

$$[\text{Id}]_{\tilde{\mathcal{B}}_p}^{\mathcal{B}_p} = W_p^{1/2}, \quad \text{and} \quad [\text{Id}]_{\tilde{\mathcal{B}}^p}^{\mathcal{B}^p} = W_p^{-1/2},$$

in the space of chains and cochains, respectively.

Boundary and coboundary operators. Let $B_p := [\partial_p]_{\mathcal{B}_p}^{\mathcal{B}_{p-1}}$ be the matrix representation of the boundary operator with respect to the standard bases, which, in some sources is called the *incidence matrix*. With respect to the orthonormal bases, taking the change of basis we have

$$[\partial_p]_{\tilde{\mathcal{B}}_p}^{\tilde{\mathcal{B}}_{p-1}} = W_{p-1}^{-1/2} B_p W_p^{1/2}.$$

Lemma 3.9. The matrix representations for the coboundary operator are given by

$$[\delta_p]_{\mathcal{B}_p}^{\mathcal{B}_{p+1}} = B_{p+1}^\top, \quad \text{and} \quad [\delta_p]_{\tilde{\mathcal{B}}_p}^{\tilde{\mathcal{B}}_{p+1}} = W_{p+1}^{1/2} B_{p+1}^\top W_p^{-1/2}.$$

Proof. Let $\sigma_i^\vee \in \mathcal{B}^p$, with $1 \leq i \leq n_p$, and $\sigma'_j \in \mathcal{B}_{p+1}$, with $1 \leq j \leq n_{p+1}$. Then, the value at position (j, i) in $[\delta_p]_{\mathcal{B}_p}^{\mathcal{B}_{p+1}}$ is precisely given by $\delta_p(\sigma_i^\vee)(\sigma'_j)$. By definition, we know that

$$\delta_p(\sigma_i^\vee)(\sigma'_j) = \sigma_i^\vee(\partial_{p+1}(\sigma'_j)),$$

the left-hand side being precisely the value of $[\partial_{p+1}]_{\mathcal{B}_{p+1}}^{\mathcal{B}_p}$ at position (i, j) , proving the first representation in the lemma. The second follows after change of basis. $\square$

Applying **Equation (1)**, we get the matrix representations of the adjoint of the boundary map:

$$[\partial_p^*]_{\mathcal{B}_{p-1}}^{\mathcal{B}_p} = W_p B_p^\top W_{p-1}^{-1}, \quad \text{and} \quad [\partial_p^*]_{\tilde{\mathcal{B}}_{p-1}}^{\tilde{\mathcal{B}}_p} = W_p^{1/2} B_p^\top W_{p-1}^{-1/2} = \left([\partial_p]_{\tilde{\mathcal{B}}_p}^{\tilde{\mathcal{B}}_{p-1}}\right)^\top, \quad (6)$$

and the matrix representations of the adjoint of the coboundary map:

$$[\delta_p^*]_{\mathcal{B}_{p+1}}^{\mathcal{B}_p} = W_p^{-1} B_{p+1} W_{p+1}, \quad \text{and} \quad [\delta_p^*]_{\tilde{\mathcal{B}}_{p+1}}^{\tilde{\mathcal{B}}_p} = W_p^{-1/2} B_{p+1} W_{p+1}^{1/2} = \left([\delta_p]_{\tilde{\mathcal{B}}_p}^{\tilde{\mathcal{B}}_{p+1}}\right)^\top, \quad (7)$$

where the second representation is to be expected, as in general the matrix representation of the adjoint of a linear map with respect to orthonormal basis equals the transpose of the matrix representation of the original map.

Musical isomorphisms. For each simplex $\sigma \in K_p$, the flat map sends σ to $w_p(\sigma)^{-1}\sigma^\vee$, and the sharp map sends $\sigma^\vee$ to $w_p(\sigma)\sigma$. Thus, with respect to the standard bases we have that the musical isomorphisms have the following matrix representations

$$[b_p]_{\mathcal{B}_p}^{\mathcal{B}_p} = W_p^{-1}, \quad \text{and} \quad [\sharp_p]_{\mathcal{B}_p}^{\mathcal{B}_p} = W_p. \quad (8)$$

We can also compute the adjoint of the flat isomorphism, using **Equations (1)** and **(8)**:

$$[b_p^*]_{\mathcal{B}_p}^{\mathcal{B}_p} = W_p W_p^{-1} W_p = W_p = [b_p^{-1}]_{\mathcal{B}_p}^{\mathcal{B}_p}.$$

4 Laplacians of Graphs and Simplicial Complexes

We review Laplacians of simplicial complexes and obtain the graph Laplacian as a particular case of this formulation. We use $\mathcal{L}$ to denote a Laplace operator and L to denote its matrix representation with respect to certain bases. Let K be a d -dimensional finite simplicial complex, and recall that K_p denotes its set of p -simplices, $0 \leq p \leq d$, and $n_p = |K_p|$.

4.1 Laplacians as Operators

In this paper, following [32], we work with the combinatorial *up Laplace operator*, *down Laplace operator*, and *Hodge Laplace operator*. As operators on the p -cochain space, they are defined as

$$\begin{aligned}\mathcal{L}_{\text{up}}^p(K) &:= \delta_p^* \delta_p; \\ \mathcal{L}_{\text{down}}^p(K) &:= \delta_{p-1} \delta_{p-1}^*; \\ \mathcal{L}_{\text{Hodge}}^p(K) &:= \mathcal{L}_{\text{up}}^p(K) + \mathcal{L}_{\text{down}}^p(K).\end{aligned}\tag{9}$$

When the underlying simplicial complex K is clear from the context, we will omit it from the notations. By construction, the Laplace operators $\mathcal{L}_{\text{up}}^p$, $\mathcal{L}_{\text{down}}^p$, and $\mathcal{L}_{\text{Hodge}}^p$ of K are self-adjoint, non-negative (therefore positive semi-definite) and compact operators [32]. As a consequence, their eigenvalues are non-negative reals and their eigenspaces are orthogonal and sum (directly) to the whole space. The up Laplace operator, $\mathcal{L}_{\text{up}}^p(K)$, in particular, gives rise to an effective resistance bilinear form (Section 5.2).

Analogously, as operators on p -chain spaces, the combinatorial *up Laplace operator*, *down Laplace operator*, and *Hodge Laplace operator* are defined as

$$\begin{aligned}\mathcal{L}_p^{\text{up}}(K) &:= \partial_{p+1} \partial_{p+1}^*; \\ \mathcal{L}_p^{\text{down}}(K) &:= \partial_p^* \partial_p; \\ \mathcal{L}_p^{\text{Hodge}}(K) &:= \mathcal{L}_p^{\text{up}}(K) + \mathcal{L}_p^{\text{down}}(K).\end{aligned}\tag{10}$$

Again, we omit K from these notations when the context is clear.

The chain and cochain Laplacians are related via the musical isomorphisms, as summarized in the following lemma. Analogous statements for the down Laplacians hold but are omitted here, since they are not needed in this paper.

Lemma 4.1. The chain and cochain up Laplacians, as well as their pseudoinverses, are related via the musical isomorphisms:

1. $\mathcal{L}_p^{\text{up}} = \flat_p^{-1} \mathcal{L}_{\text{up}}^p \flat_p$;
2. $(\mathcal{L}_{\text{up}}^p)^+ = \flat_p (\mathcal{L}_p^{\text{up}})^+ \flat_p^{-1}$.

Proof. The first item follows directly from Lemma 3.3 and the definition of the up Laplacians.

The second item is a consequence of Proposition B.6, which asserts the compatibility of pseudoinverses under inner-product-preserving isomorphisms. $\square$

Hodge theory. By standard results from linear algebra, together with the fact that all our vector spaces are finite-dimensional, one obtains the following canonical identifications:

$$H^p(K; \mathbb{R}) \cong H_p(K; \mathbb{R}) \cong \ker(\mathcal{L}_{\text{Hodge}}^p(K)).\tag{11}$$

In particular, the p -th (co)homology group can be realized concretely as the space of *harmonic p -(co)chains*, i.e., those annihilated by the Hodge Laplacian.

Moreover, the cochain space admits the *Hodge decomposition*:

$$C^p(K; \mathbb{R}) = \text{im}(\delta_p^*) \oplus \ker(\mathcal{L}_{\text{Hodge}}^p(K)) \oplus \text{im}(\delta_{p-1}),\tag{12}$$

where the three summands are mutually orthogonal with respect to the inner product. In particular,

$$(\text{im} \delta_p^*)^\perp = \ker(\mathcal{L}_{\text{Hodge}}^p(K)) \oplus \text{im}(\delta_{p-1}), \quad \text{and} \quad (\text{im} \delta_{p-1})^\perp = \text{im}(\delta_p^*) \oplus \ker(\mathcal{L}_{\text{Hodge}}^p(K)).$$

4.2 Laplacians as Matrices

With respect to the standard or orthonormal bases, the operators defined in **Section 4.1** admit matrix representations.

In the standard basis (**Definition 3.5**), it follows from **Lemma 3.9** and **Equations (6)** and **(7)** that the matrix representations of the cochain Laplacians $\mathcal{L}_{\text{up}}^p$ and $\mathcal{L}_{\text{down}}^p$ and the chain Laplacians $\mathcal{L}_p^{\text{up}}$ and $\mathcal{L}_p^{\text{down}}$ are respectively given by

$$[\mathcal{L}_{\text{up}}^p]_{\mathcal{B}^p} = W_p^{-1} B_{p+1} W_{p+1} B_{p+1}^\top =: L_{p,\text{up}} \quad (13)$$

$$[\mathcal{L}_{\text{down}}^p]_{\mathcal{B}^p} = B_p^\top W_{p-1}^{-1} B_p W_p =: L_{p,\text{down}} \quad (14)$$

$$[\mathcal{L}_p^{\text{up}}]_{\mathcal{B}_p} = B_{p+1} W_{p+1} B_{p+1}^\top W_p^{-1} = W_p L_{p,\text{up}} W_p^{-1} \quad (15)$$

$$[\mathcal{L}_p^{\text{down}}]_{\mathcal{B}_p} = W_p B_p^\top W_{p-1}^{-1} B_p = W_p L_{p,\text{down}} W_p^{-1}. \quad (16)$$

These expressions recover the *combinatorial Hodge Laplacian operators* defined by Horak and Jost [32], which encompasses several previous approaches to define Laplace operators in simplicial complexes (e.g. [26, 19, 40]).

Observe that the matrices $L_{p,\text{up}}$ and $L_{p,\text{down}}$ need not be symmetric, even though they correspond to self-adjoint operators. The lack of symmetry stems from the fact that the standard basis is not orthonormal unless all weights are set to 1.

In the orthonormal basis (**Definition 3.7**), it follows from **Lemma 3.9** and **Equations (6)** and **(7)** that the matrix representations of the cochain Laplacians $\mathcal{L}_{\text{up}}^p$ and $\mathcal{L}_{\text{down}}^p$ and the chain Laplacians $\mathcal{L}_p^{\text{up}}$ and $\mathcal{L}_p^{\text{down}}$ are respectively given by

$$[\mathcal{L}_{\text{up}}^p]_{\tilde{\mathcal{B}}^p} = W_p^{-1/2} B_{p+1} W_{p+1} B_{p+1}^\top W_p^{-1/2} = W_p^{1/2} L_{p,\text{up}} W_p^{-1/2} \quad (17)$$

$$[\mathcal{L}_{\text{down}}^p]_{\tilde{\mathcal{B}}^p} = W_p^{1/2} B_p^\top W_{p-1}^{-1} B_p W_p^{1/2} = W_p^{1/2} L_{p,\text{down}} W_p^{-1/2} \quad (18)$$

$$[\mathcal{L}_p^{\text{up}}]_{\tilde{\mathcal{B}}_p} = [\mathcal{L}_{\text{up}}^p]_{\tilde{\mathcal{B}}^p} \quad (19)$$

$$[\mathcal{L}_p^{\text{down}}]_{\tilde{\mathcal{B}}_p} = [\mathcal{L}_{\text{down}}^p]_{\tilde{\mathcal{B}}^p}. \quad (20)$$

which are now symmetric matrices, as expected. These now recover the *normalized Hodge Laplacian operators*, see [46, Definition 4.1]. Interestingly, normalized and (un-normalized) Hodge Laplacians are often treated in the literature as distinct objects. With our approach, we demonstrate that they are just different manifestations of the same operator, after choosing different bases. This example illustrates the strength of our approach, which unifies seemingly different operators by revealing them as alternative representations of a single, basis-free operator.

Graph Laplacians as matrices. We now show how the graph Laplacian arises from the preceding formulation, as it plays a central role in the study of effective resistance in graphs.

Throughout, we consider finite, undirected, simple graphs (no self-loops or multiple edges), viewed as 1-dimensional simplicial complexes. The graph is not assumed to be connected unless explicitly required, in which case this assumption will be stated. When an orientation is needed, for instance to form an incidence matrix, we fix one arbitrarily by ordering the vertices.

Let $G = (V, E)$ be a graph, where V is the set of vertices and E the set of edges, with $|V| = n_0$ and $|E| = n_1$. We equip edges with positive weights $w: E \rightarrow \mathbb{R}_{>0}$, and take vertex weights to be unit. The edge weights define a diagonal weight matrix $W_1 \in \mathbb{R}^{n_1 \times n_1}$ containing the weights of the edges, with respect to the standard basis. The weight matrix for the vertices is simply the identity

$W_0 = \text{Id} \in \mathbb{R}^{n_0 \times n_0}$, so we often omit it. Assuming an arbitrary ordering of the vertices, we define the (signed) *incidence matrix* $B_1 \in \mathbb{R}^{n_0 \times n_1}$ as the matrix representation of the boundary operator $\partial_1: C_1(G) \rightarrow C_0(G)$ with respect to the standard bases.

In this setting, the classical *graph Laplacian* $L \in \mathbb{R}^{n_0 \times n_0}$ is

$$L := B_1 W_1 B_1^\top, \quad (21)$$

which coincides with the up Laplacian $L_{0,\text{up}}$ in **Equation (13)** for $p = 0$ and $W_0 = \text{Id}$. It is also the full Hodge Laplacian since $L_{0,\text{down}} = 0$. While $L_{p,\text{up}}$ is not symmetric in general, in this case it coincides with the symmetric classical graph Laplacian. The symmetry arises because $W_0 = \text{Id}$, which removes the asymmetry introduced by the weights in **Equation (13)**. The symmetry is therefore incidental rather than essential.

In higher-dimensional generalizations, several formulations in the literature impose symmetry of the Laplacian matrix by definition, even when both $(p-1)$ - and p -simplices are weighted, by omitting the $(p-1)$ -weights in the standard basis. In contrast, our formulation retains all weights and shows that symmetry should not be expected in the standard basis but arises naturally in the orthonormal basis, as shown in **Equation (17)**.

5 Effective Resistance in Graphs

We begin by reviewing the classical matrix-based definitions of effective resistance in graphs, distinguishing between edge-based and vertex-based formulations, with the latter being the classical notion in the literature. As one of the main contributions of this paper, we then reinterpret these definitions as bilinear forms on chain and cochain spaces, guided by their physical interpretation and formalized through musical isomorphisms. This operator-theoretic viewpoint reveals the underlying structure and extends naturally to higher-dimensional simplicial complexes, where effective resistance is defined for arbitrary simplices via analogous bilinear forms and operators.

5.1 Effective Resistance in Graphs as Matrices

Effective resistance originates from translating Kirchhoff's circuit equations in terms of the graph Laplacian [16], and has been studied by a number of authors since the early 2000s [7, 21, 27]. As before, let us consider a graph with n_1 edges and n_0 vertices, weights defined on the edges and collected in the weight matrix $W_1 \in \mathbb{R}^{n_1 \times n_1}$, while vertices have unit weights; and signed incidence matrix (or the boundary matrix) $B_1 \in \mathbb{R}^{n_0 \times n_1}$. Let $L = B_1 W_1 B_1^\top$ be its symmetric graph Laplacian as defined in **Equation (21)**.

Effective resistance of edges. Following [49], define the matrix $R \in \mathbb{R}^{n_1 \times n_1}$ as

$$R := B_1^\top L^+ B_1 = B_1^\top (B_1 W_1 B_1^\top)^+ B_1, \quad (22)$$

where L^+ denotes the Moore-Penrose inverse of L . We refer to R as the *edge-based effective resistance matrix*. The diagonal entries of R give the *effective resistance* of edges $e \in E$:

$$r_e := R(e, e). \quad (23)$$

Remark 5.1 (Terminology). The matrix R should not be confused with the classical *vertex-based* effective resistance matrix, whose rows and columns are indexed by vertices, and whose (u, v) -entry gives the effective resistance between vertices u and v (see **Equation (24)**). In contrast,

R is indexed by edges and will define a bilinear form on 1-chains. We will generalize this edge-based formulation to simplicial complexes (see **Section 6.1**). Subsequently, we will generalize the vertex-based formulation as a bilinear form on boundaries (see **Remark 6.4**).

Effective resistance between vertices. Let $\mathbf{1}_v \in \mathbb{R}^{n_0}$ denote the coordinate vector of vertex v with respect to the standard basis given by the vertices of the graph, i.e. the vector with a 1 in the v -th position and zeros elsewhere. The *effective resistance (distance)* between vertices u and v (see [35]) is given by

$$r_{uv} := (\mathbf{1}_u - \mathbf{1}_v)^\top L^+ (\mathbf{1}_u - \mathbf{1}_v). \quad (24)$$

When e is an edge from vertex u to v , the effective resistance r_e of e (**Equation (23)**) agrees with the effective resistance between the vertices u and v . Indeed, let $e \in E$ be such an edge, and let $b_e \in \mathbb{R}^{n_0}$ denote the column B_1 indexed by e . Then $b_e = \mathbf{1}_u - \mathbf{1}_v$, and we have

$$r_e = b_e^\top L^+ b_e = (\mathbf{1}_u - \mathbf{1}_v)^\top L^+ (\mathbf{1}_u - \mathbf{1}_v) = r_{uv}. \quad (25)$$

The *vertex-based effective resistance matrix* is defined as the matrix with elements $(r_{uv})_{u,v \in V}$.

Effective resistance in graphs admits several interpretations. Pairs of vertices with small effective resistance between them are typically connected by paths with small total weights, reflecting a high ease of current flow between them [49]. Physically, effective resistance measures how difficult it is for current to travel between two vertices, as detailed in **Section 5.2.1**.

5.2 Effective Resistance in Graphs as Bilinear Forms

We now reinterpret the effective resistance in graphs through bilinear forms on chain and cochain spaces. By interpreting voltages as coboundaries and currents as boundaries, and by encoding Ohm's law and Kirchhoff's laws algebraically, we arrive at expressions for the effective resistance that are independent of any particular basis. These expressions define symmetric, positive semi-definite bilinear forms on the spaces of 1-chains and 0-boundaries, which in turn give rise to edge-based and vertex-based effective resistance matrices under appropriate choices of bases.

5.2.1 Circuit Theory via Chain and Cochain Complexes

We review the algebraic formulation of circuit theory following [5, 6]. Let $G = (V, E)$ be a finite oriented graph with n_0 vertices and n_1 edges, representing an electrical circuit. We select subsets $V_{\text{in}} \subseteq V$ and $V_{\text{out}} \subseteq V$ as the sets of *input* and *output* vertices, where current is injected and extracted respectively. Their union

$$\partial G := V_{\text{in}} \cup V_{\text{out}}$$

is called the set of *terminals*. The remaining vertices are called *interior* or *non-terminal*.

To incorporate physical intuition into our model, we define edge weights based on the circuit's physical properties, following standard approach (e.g., [49]). Suppose each edge $e \in E$ is assigned a *resistance* $\rho_e > 0$ (not to be confused with the *effective resistance* r_e). We model the circuit G as a weighted 1-dimensional simplicial complex in which vertices have unit weight, and edges are weighted by the *inverse* of their resistances, referred to as *conductances*, and denoted as $w(e) = \rho_e^{-1}$.

We use the conductance weights to define an inner product on the space of 1-chains $C_1(G)$, as described in **Section 3.2**. With respect to this inner product, the standard edge basis is orthogonal, and for any edge e ,

$$\langle e, e \rangle_{C_1} = w(e)^{-1} = \rho_e. \quad (26)$$

For the space of 0-chains $C_0(G)$, we assume unit weights and thus use the standard inner product, under which the vertex basis is orthonormal.

Remark 5.2. In our model, resistance can be viewed as the effect of the circuit being curved by the weights, analogous to how gravity arises from the curvature of spacetime caused by mass. In both cases, the underlying geometry is described by a symmetric bilinear form, with resistance playing a role similar to that of a metric, determining how difficult it is for current or matter to move between points.

Given an electrical circuit, its governing physical quantities are the *voltage*, which can be described by a 1-cochain $f \in C^1(G)$, and the *current*, described by a 1-chain $\alpha \in C_1(G)$. Let ι denote the inclusion of ∂G into G , and by an abuse of notation, we also write $\iota: C_0(\partial G) \rightarrow C_0(G)$ for the induced map on chain spaces. These are subject to several physical laws (see [6, Chapter 12] or [5, Section 2.2]) that we summarize below.

- (1) **Kirchhoff's voltage law:** A (physically valid) voltage $f \in C^1(G)$ is a coboundary; that is,

$$f = \delta_0 \phi$$

for some (not necessarily unique) $\phi \in C^0(G)$ called the *potential*. This is equivalent to the condition that f evaluates to zero on every closed path (i.e., 1-cycle).

- (2) **Kirchhoff's current law:** A (physically valid) current $\alpha \in C_1(G)$ satisfies

$$\partial_1 \alpha = \iota \beta,$$

for some $\beta \in C_0(\partial G)$ called the *boundary current*. The above formula expresses conservation of current: net flow vanishes at interior vertices and is non-zero only at terminals. When $\partial G = \emptyset$, the circuit is closed and the above condition reduces to $\partial_1 \alpha = 0$.

- (3) **Ohm's law:** Given that resistances define the inner product on $C_1(G)$ (see **Equation (26)**), and using the flat isomorphism (see **Definition 3.1**), Ohm's law takes the form

$$f = \flat_1 \alpha,$$

relating $f \in C^1(G)$ and $\alpha \in C_1(G)$.

When Ohm's Law holds, given either f or α , the other is uniquely determined via the inner product induced by the resistances.

In what follows, we assume that all voltages and currents are cochains and chains, respectively, that satisfy Kirchhoff's laws and the Ohm law. Vice versa, in **Proposition 5.4**, we characterize when chains and cochains can be considered currents and voltages that satisfies Kirchhoff's and Ohm's laws.

Remark 5.3 (Interpretation of Kirchhoff's current law). We explain why Kirchhoff's current law, i.e. conservation of current, can be expressed as $\partial_1 \alpha = \iota \beta$ for some $\beta \in C_0(\partial G)$, where $\alpha \in C_1(G)$.

Expressing the current as a linear combination of the standard basis of edges $\alpha = \sum_{e \in E} \alpha_e e$, we can compute its boundary and evaluate at some vertex $x \in V$ to obtain its component over that vertex. Namely, we compute

$$\langle \partial_1 \alpha, x \rangle_{C_0} = \sum_{(u,x) \in E} \alpha_e - \sum_{(x,v) \in E} \alpha_e. \quad (27)$$

Kirchhoff's current law states that the input current must equal the output current in all internal vertices, i.e., **Equation (27)** vanishes if $x \in V$ is internal; while there is a positive excess, denoted $\beta_x > 0$ at $x \in V_{\text{in}}$ corresponding to the injected current, and a negative deficiency, $\beta_x < 0$ at $x \in V_{\text{out}}$ corresponding to the extracted current. This information can be collected in a 0-chain $\beta = \sum_{v \in V} \beta_v v$, with $\beta_v = 0$ for all internal vertices, which satisfies

$$\langle \beta, x \rangle_{C_0} = \beta_x. \quad (28)$$

Putting together **Equation (27)** and **Equation (28)**, we obtain that for all $x \in V$,

$$\langle \partial_1 \alpha, x \rangle_{C_0} = \langle \iota \beta, x \rangle_{C_0},$$

which implies $\partial_1 \alpha = \iota \beta$. We emphasize that for a different choice of the inner product on the vertices, Kirchhoff's current law would not hold in its usual formulation. Thus, the choice of unit weights on the vertices is necessary to retrieve Kirchhoff's current law in its classical form.

The following proposition extends the relationship between Kirchhoff's voltage law (KVL), Kirchhoff's current law (KCL), and Ohm's law (OL) to higher dimensions, and characterizes when chain and cochains can be interpreted as currents, voltages, or potentials. Previous work [10] developed a similar result to obtain insights related to Reidemeister torsion. However, our setting differs in that voltages are cochains rather than chains, necessitating the proposition below.

Proposition 5.4. Given $\beta \in C_{p-1}(K)$, the system of equations

$$\begin{cases} \delta_{p-1} \phi = f & \text{(KVL)} \\ \partial_p \alpha = \beta & \text{(KCL)} \\ \flat_p \alpha = f & \text{(OL)} \end{cases} \quad (29)$$

in the unknowns $\alpha \in C_p(K)$, $f \in C^p(K)$, and $\phi \in C^{p-1}(K)$ admits solutions if and only if the equation

$$\mathcal{L}_{\text{up}}^{p-1} \phi = \flat_{p-1} \beta \quad (30)$$

in the unknown ϕ does.

Proof. Let us first assume that $\alpha \in C_p(K)$, $f \in C^p(K)$, and $\phi \in C^{p-1}(K)$ exist that satisfy the system for a given β . Then, (OL) and the fact that $\flat_p$ is an isomorphism imply that $\alpha = \flat_p^{-1} f$. Hence, (KVL) implies $\alpha = \flat_p^{-1} \delta_{p-1} \phi$. By (KCL), we have $\beta = \partial_p \flat_p^{-1} \delta_{p-1} \phi$ that, by the commutativity of the diagram

$$\begin{array}{ccc} C^{p-1}(K) & \xleftarrow{\delta_{p-1}^*} & C^p(K) \\ \flat_{p-1} \uparrow & & \uparrow \flat_p \\ C_{p-1}(K) & \xleftarrow{\partial_p} & C_p(K), \end{array} \quad (31)$$

is equivalent to $\beta = \flat_{p-1}^{-1} \delta_{p-1}^* \delta_{p-1} \phi$, that is $\mathcal{L}_{\text{up}}^{p-1} \phi = \flat_{p-1} \beta$.

Vice versa, let us assume that $\phi \in C^{p-1}(K)$ exists that satisfy the equation $\mathcal{L}_{\text{up}}^{p-1} \phi = \flat_{p-1} \beta$ with β given. Then, reversing the argument above, $\beta = \partial_p \flat_p^{-1} \delta_{p-1} \phi$. So, taking $\alpha = \flat_p^{-1} \delta_{p-1} \phi$ we get a solution for equation (KCL); taking $f = \flat_p \alpha$ we get a solution for equation (OL); moreover, this choice of f and α also satisfies (KVL) because $f = \flat_p \alpha = \flat_p \flat_p^{-1} \delta_{p-1} \phi = \delta_{p-1} \phi$. $\square$

Corollary 5.5. Let $G = (V, E)$ be a connected graph. If $\beta = \sum_{u \in V} \lambda_u u \in C_0(G)$ satisfies $\sum_{u \in V} \lambda_u = 0$, then β is a valid boundary current in the sense that the linear system (**Equation (29)**) in the unknown α , ϕ , and f , with $p = 1$, admits a solution.

Proof. By **Proposition 5.4**, it is enough to prove that the equation $\mathcal{L}_{\text{up}}^0 \phi = \flat_0^G \beta$ admits a solution. To see this, we note that by connectedness $\ker \mathcal{L}_{\text{up}}^0 = \text{span}(\sum_{u \in V} u^\vee)$. Moreover, we observe that, given the unit weights in the vertices, we have

$$\langle \sum_{u \in V} u^\vee, \flat_0^G \beta \rangle_{C^0} = \sum_{u \in V} \lambda_u = 0.$$

Hence, $\flat_0^G \beta \in (\ker \mathcal{L}_{\text{up}}^0)^\perp = \text{im}(\mathcal{L}_{\text{up}}^0)^*$. Because $\mathcal{L}_{\text{up}}^0$ is self-adjoint, we have proved that $\flat_0^G \beta \in \text{im} \mathcal{L}_{\text{up}}^0$, resulting in the existence of ϕ solving $\mathcal{L}_{\text{up}}^0 \phi = \flat_0^G \beta$. $\square$

Corollary 5.6. For $p \geq 1$, let $\beta \in C_{p-1}(K)$ be a boundary current, that is, Kirchhoff's current law is satisfied: $\partial_p \alpha = \beta$ for some $\alpha \in C_p(K)$. Let $f \in C^p(K)$ be obtained by applying Ohm's law: $\flat_p \alpha = f$. Then f is a voltage, i.e., it satisfies Kirchhoff's voltage law: there exists a potential $\phi \in C^{p-1}(K)$ such that $\delta_{p-1}(\phi) = f$.

Proof. By the assumptions and the commutativity of Diagram (31), we have $\flat_{p-1} \beta = \flat_{p-1} \partial_p \alpha = \delta_{p-1}^* \flat_p \alpha = \delta_{p-1}^* f$. Hence, $\flat_{p-1} \beta \in \text{im}(\delta_{p-1}^*)$. By **Equation (57)**, $\text{im}(\delta_{p-1}^*) = \text{im}(\mathcal{L}_{\text{up}}^{p-1})$. So, there exists $\phi \in C^{p-1}(K)$ such that $\mathcal{L}_{\text{up}}^{p-1} \phi = \flat_{p-1} \beta$. The claim now follows from **Equation (29)**. $\square$

5.2.2 Vertex-Based Effective Resistance in Graphs as Bilinear Forms

We represent the effective resistance in graphs through bilinear forms. We start with the vertex-based one.

Let G be a connected graph. Given $u \in V_{\text{in}}$ and $v \in V_{\text{out}}$, let $\beta = u - v \in C_0(\partial G) \subset C_0(G)$. Since G is connected, $\beta \in \text{im} \partial_1^G$. Let $\phi \in C^0(G)$ be the solution of **Equation (30)** we get from β and the graph, assuming the electric circuit laws hold. We define

$$\tilde{r}_{uv} := \langle \phi, \flat_0^G \beta \rangle_{C^0}, \quad (32)$$

which represents the resistance created by the entire graph between u and v , measured as the potential difference we get between them when we inject a unit current in u and extract a unit current in v using β . In **Proposition 5.8**, we will show that $\tilde{r}_{uv}$ coincides with the effective resistance r_{uv} as defined in **Equation (24)**.

We first reformulate $\tilde{r}_{uv}$ in terms of chains, using the musical isomorphisms and the pseudoinverse of the 0-th Laplacian.

Lemma 5.7. Let $u \in V_{\text{in}}$, $v \in V_{\text{out}}$ so that $u - v \in C_0(\partial G)$, and let $\alpha \in C_1(G)$ satisfy $\partial_1 \alpha = \iota(u - v) \in C_0(G)$. Then $\tilde{r}_{uv}$ can be expressed as the value of an inner product in C_0 :

$$\tilde{r}_{uv} = \langle (\mathcal{L}_0^{\text{up}})^+ \partial_1 \alpha, \partial_1 \alpha \rangle_{C_0}, \quad (33)$$

and equivalently as the value of an inner product in C_1 :

$$\tilde{r}_{uv} = \langle \partial_1^* (\mathcal{L}_0^{\text{up}})^+ \partial_1 \alpha, \alpha \rangle_{C_1}. \quad (34)$$

Proof. For notational simplicity, we write $\beta = u - v \in C_0(G)$. Let $\phi \in C^0(G)$ be a potential that solves the circuit problem as described in system **Equation (29)**, whose existence is guaranteed by **Corollary 5.5**. Then by **Proposition 5.4**, we have $\mathcal{L}_{\text{up}}^0 \phi = \flat_0^G \iota \beta$. Hence, $\flat_0^G \iota \beta \in \text{im}(\mathcal{L}_{\text{up}}^0) = \ker(\mathcal{L}_{\text{up}}^0)^\perp$. From the definition of the pseudoinverse operator (cf. **Definition B.1**) we get

$$\phi = (\mathcal{L}_{\text{up}}^0)^+ \flat_0^G \iota \beta.$$

Plugging this into the expression for effective resistance (from **Equation (32)**) and applying **Lemma 4.1**, we obtain:

$$\tilde{r}_{uv} = \langle ((\mathcal{L}_{\text{up}}^0)^+ \mathfrak{b}_0^G \iota \beta, \mathfrak{b}_0^G \iota \beta)_{C_0} = \langle ((\mathfrak{b}_0^G)^{-1} (\mathcal{L}_{\text{up}}^0)^+ \mathfrak{b}_0^G \partial_1 \alpha, \partial_1 \alpha)_{C_0} = \langle (\mathcal{L}_0^{\text{up}})^+ \partial_1 \alpha, \partial_1 \alpha \rangle_{C_0}.$$

This yields the expression in **Equation (33)**. Applying the adjoint of ∂_1 , we can express the same quantity as an inner product in C_1 and obtain **Equation (34)**. $\square$

Proposition 5.8. Consider a connected graph G and vertices $u, v \in V$. Then,

$$\tilde{r}_{uv} = r_{uv},$$

where r_{uv} is the standard effective resistance between vertices u and v from **Equation (24)**.

Proof. Let $\alpha \in C_1(G)$ be any 1-chain such that $\partial_1 \alpha = u - v \in B_0(G)$. From **Equation (24)**, the effective resistance between u and v is given by

$$\begin{aligned} r_{uv} &= (\mathbf{1}_u - \mathbf{1}_v)^\top L^+ (\mathbf{1}_u - \mathbf{1}_v) && \text{(Equation (24))} \\ &= \langle (\mathcal{L}_{\text{up}}^0)^+ (\mathfrak{b}_0^G \iota u - \mathfrak{b}_0^G \iota v), (\mathfrak{b}_0^G \iota u - \mathfrak{b}_0^G \iota v) \rangle_{C_0} && (\mathbf{1}_u \text{ is the vector representing } \mathfrak{b}_0^G \iota u) \\ &= \langle (\mathcal{L}_{\text{up}}^0)^+ \mathfrak{b}_0^G \partial_1 \alpha, \mathfrak{b}_0^G \partial_1 \alpha \rangle_{C_0} && \text{(linearity of } \mathfrak{b}_0^G) \\ &= \langle (\mathcal{L}_0^{\text{up}} \partial_1 \alpha, \partial_1 \alpha)_{C_0} && \text{(Lemma 4.1)} \\ &= \tilde{r}_{uv} && \text{(Equation (33)).} \end{aligned}$$

Hence, $\tilde{r}_{uv} = r_{uv}$, as claimed. $\square$

Inspired by **Equation (33)**, we define the following bilinear form on the space of 0-boundaries of the graph G to be

$$\tilde{\mathcal{R}}_0 : B_0(G) \times B_0(G) \rightarrow \mathbb{R}_{\geq 0}, \quad \tilde{\mathcal{R}}_0(\partial_1 \alpha, \partial_1 \alpha') := \mathcal{R}_0(\alpha, \alpha'). \quad (35)$$

Remark 5.9. A limitation of the bilinear form $\tilde{\mathcal{R}}_0$ is the absence of a canonical basis in $B_0(G)$. This motivates the construction, in **Definition 5.10**, of a bilinear form defined on the full space $C_1(G)$, which admits a standard basis indexed by the edges of the graph.

5.2.3 Edge-Based Effective Resistance in Graphs as Bilinear Forms

Inspired by **Equation (34)** in **Lemma 5.7**, we introduce the following definition.

Definition 5.10. We define the *effective resistance bilinear form* on the space of 1-chains of the graph G to be

$$\begin{aligned} \mathcal{R}_1 : C_1(G) \times C_1(G) &\rightarrow \mathbb{R} \\ (\alpha, \alpha') &\mapsto \mathcal{R}_1(\alpha, \alpha') := \langle \partial_1^* (\mathcal{L}_0^{\text{up}})^+ \partial_1 \alpha, \alpha' \rangle_{C_1}. \end{aligned}$$

We note that the two bilinear forms $\mathcal{R}_1$ and $\tilde{\mathcal{R}}_0$ are related as follows: for 1-chains α, α' ,

$$\mathcal{R}_1(\alpha, \alpha') = \tilde{\mathcal{R}}_0(\partial_1 \alpha, \partial_1 \alpha').$$

The next proposition shows that, under the assumption of unit weights on the vertices and using the standard basis indexed by the edges of the graph, the matrix representation of $\mathcal{R}_1$ coincides with the edge-based effective resistance matrix R from **Equation (22)**. This justifies the name *effective resistance bilinear form*, as it extends the classical notion of effective resistance. The proof is deferred to **Proposition 6.9**, where the proposition below appears as a special case of a more general result.

Proposition 5.11. Assume unit weights on the vertices and use the standard basis indexed by the edges of the graph. Then the bilinear form $\mathcal{R}_1$ has matrix representation $B_1^\top (B_1 W_1 B_1^\top)^+ B_1$.

6 Effective Resistance in Simplicial Complexes

Effective resistance, originally defined for edges in graphs, has been extended to higher-dimensional simplices in several works, including [9, 36, 43]. These approaches rely on choosing specific bases for the chain or cochain spaces and working with the resulting matrix representations of the (co)boundary and Laplacian operators.

In **Section 6.1**, we develop a coordinate-free, operator-theoretic perspective on effective resistance by generalizing the graph-theoretic setting of **Section 5.2.3**, we reinterpret effective resistance in simplicial complexes using bilinear forms and musical isomorphisms.

Finally, in **Section 6.2**, we show that our operator-theoretic formulation recovers the matrix-based definitions found in the literature.

6.1 Effective Resistance in Simplicial Complexes as Bilinear Forms

We generalize **Definition 5.10** to the setting of simplicial complexes.

Definition 6.1. We define the *effective resistance bilinear form* on the space $C_p(K)$ of p -chains as

$$\begin{aligned} \mathcal{R}_p : C_p(K) \times C_p(K) &\rightarrow \mathbb{R} \\ (\alpha, \alpha') &\mapsto \mathcal{R}_p(\alpha, \alpha') := \langle \partial_p^*(\mathcal{L}_{p-1}^{\text{up}})^+ \partial_p \alpha, \alpha' \rangle_{C_p}. \end{aligned}$$

We define the *effective resistance bilinear form* on the space $C^p(K)$ of p -cochains as

$$\begin{aligned} \mathcal{R}^p : C^p(K) \times C^p(K) &\rightarrow \mathbb{R} \\ (f, f') &\mapsto \mathcal{R}^p(f, f') := \langle \delta_{p-1}(\mathcal{L}_{\text{up}}^{p-1})^+ \delta_{p-1}^* f, f' \rangle_{C^p}. \end{aligned}$$

We note that the effective resistance bilinear forms on chains and cochains are related to each other via the musical isomorphism. Indeed, applying **Lemma 4.1** we obtain

$$\mathcal{R}^p(f, f') = \langle \flat_{p-1} \partial_p^*(\mathcal{L}_{p-1}^{\text{up}})^+ \partial_p \flat_p^{-1} f, f' \rangle_{C^p} = \mathcal{R}_p(\flat_p^{-1} f, \flat_p^{-1} f').$$

Definition 6.2 (Effective resistance of chains and cochains). Let $\alpha \in C_p(K)$ be a p -chain in a simplicial complex K . The *effective resistance* of α is defined as

$$r_\alpha := \mathcal{R}_p(\alpha, \alpha).$$

Let $f \in C^p(K)$ be a p -cochain. The *effective resistance* of f is defined as

$$r_f := \mathcal{R}^p(f, f) = \mathcal{R}_p(\flat_p^{-1} f, \flat_p^{-1} f).$$

where $\flat_p : C_p(K) \rightarrow C^p(K)$ is the flat isomorphism (see **Definition 3.1**).

Remark 6.3. When $f = \sigma^\vee$ is the dual cochain corresponding to a p -simplex σ , we have:

1. $r_{\sigma^\vee} = \mathcal{R}_p(\flat_p^{-1} \sigma^\vee, \flat_p^{-1} \sigma^\vee) = \mathcal{R}_p(w(\sigma)\sigma, w(\sigma)\sigma) = w(\sigma)^2 r_\sigma,$
2. $r_{\sqrt{w(\sigma)}\sigma} = w(\sigma) r_\sigma,$
3. $r_{\frac{1}{\sqrt{w(\sigma)}}\sigma^\vee} = w(\sigma) r_\sigma.$

In the graph case, letting $\sigma = e$ be an edge, the quantity $w(e)r_e$ has been referred to as the *relative effective resistance* in [17]. The above calculations show that the relative effective resistance of an edge e is equal to the effective resistance of either the chain $\sqrt{w(e)}e$ or the cochain $\frac{1}{\sqrt{w(e)}}e^\vee$.

Remark 6.4 (Effective resistance depends only on boundaries). The effective resistance bilinear form depends only on the boundaries of the chains. Indeed, using

$$\mathcal{R}_p(\alpha, \alpha') = \langle \partial_p^*(\mathcal{L}_{p-1}^{\text{up}})^+ \partial_p \alpha, \alpha' \rangle_{C_p} = \langle (\mathcal{L}_{p-1}^{\text{up}})^+ \partial_p \alpha, \partial_p \alpha' \rangle_{C_{p-1}},$$

if $\tilde{\alpha}$ and $\tilde{\alpha}'$ satisfy $\partial_p \tilde{\alpha} = \partial_p \alpha$ and $\partial_p \tilde{\alpha}' = \partial_p \alpha'$, then $\mathcal{R}_p(\alpha, \alpha') = \mathcal{R}_p(\tilde{\alpha}, \tilde{\alpha}')$. It follows immediately that for any $\alpha \in C_p(K)$ and $c \in Z_p(K)$, $\mathcal{R}_p(\alpha, c) = 0$.

Motivated by the above observation, one may define a bilinear form directly on the space of p -boundaries $B_p(K)$:

$$\tilde{\mathcal{R}}_p : B_p(K) \times B_p(K) \rightarrow \mathbb{R}_{\geq 0}, \quad \tilde{\mathcal{R}}_p(\beta, \beta') := \mathcal{R}_{p+1}(\alpha, \alpha'), \quad (36)$$

where α and α' are any $(p+1)$ -chains such that $\partial_{p+1} \alpha = \beta$ and $\partial_{p+1} \alpha' = \beta'$. This generalizes **Equation (35)** to arbitrary dimensions. Using the bilinear form $\tilde{\mathcal{R}}_p$, one can induce a notion of effective resistance for a p -boundary $\beta = \partial_{p+1} \alpha$ by defining it as the effective resistance of α . This perspective is adopted in [9], for instance.

In addition, when $p = 1$ and K is a connected graph, the vertex-based effective resistance arises from the bilinear form $\tilde{\mathcal{R}}_0$, where the boundaries are given by differences of vertices and the effective resistance is defined using the 1-chain connecting them.

Operators associated to effective resistance bilinear forms. To facilitate the discussion of the properties of the effective resistance bilinear forms, we introduce the following linear operators. For chains, we define the linear operator

$$\mathcal{T}_p := \partial_p^*(\mathcal{L}_{p-1}^{\text{up}})^+ \partial_p = \partial_p^*(\partial_p^*)^+. \quad (37)$$

By construction, $\mathcal{R}_p(\alpha, \alpha') = \langle \mathcal{T}_p \alpha, \alpha' \rangle_{C_p}$. Similarly, for cochains, define the linear operator

$$\mathcal{T}^p := \delta_{p-1}(\mathcal{L}_{\text{up}}^{p-1})^+ \delta_{p-1}^* = \delta_{p-1} \delta_{p-1}^+ = \flat_{p-1} \mathcal{T}_p \flat_{p-1}^{-1}. \quad (38)$$

Again, it follows immediately that $\mathcal{R}^p(f, f') = \langle \mathcal{T}^p f, f' \rangle_{C^p}$.

Because we work in finite-dimensional vector spaces, every linear operator is continuous and hence bounded. Consequently, any bilinear form of the type $\langle Ax, y \rangle$, where A is a linear operator, is also bounded, since $\langle Ax, y \rangle \leq \|A\| \|x\| \|y\|$. The Riesz representation theorem thus provides a one-to-one correspondence between bounded bilinear forms and linear operators. Therefore, the effective resistance bilinear forms $\mathcal{R}_p$ and $\mathcal{R}^p$ correspond uniquely to the operators $\mathcal{T}_p$ and $\mathcal{T}^p$, respectively, which we refer to as the *effective resistance (linear) operators*.

Remark 6.5. **Equation (37)** is based on the up Laplacian operator. However, it is also possible to consider alternative definitions of effective resistance operators based on the down Laplace or the Hodge Laplace operator. For instance, one can define the effective resistance operator on chains as

$$\partial_p^*(\mathcal{L}_{p-1}^{\text{Hodge}})^+ \partial_p = \partial_p^*(\mathcal{L}_{p-1}^{\text{up}} + \mathcal{L}_{p-1}^{\text{down}})^+ \partial_p.$$

For $p = 1$, and hence for graphs, the above formula and **Equation (37)** turn out to be the same, but for greater dimensions, they are different. Studying the theoretical properties of these Hodge-Laplace-based definitions is left for future work.

By **Equations** (37) and (38), the effective resistance operators are of the form of $\varphi\varphi^+$, it follows immediately from **Proposition B.5** that they satisfy the following properties.

Proposition 6.6. The p -th effective resistance operator $\mathcal{T}_p = \partial_p^*(\partial_p^*)^+$ on chains is the orthogonal projection of $C_p(K)$ onto $(\ker(\partial_p^*)^+)^{\perp} = (\ker \partial_p)^{\perp} = \text{im } \partial_p^*$. This means that $\mathcal{T}_p$ is self-adjoint, idempotent and has $\text{im } \partial_p^*$ as its image and $\ker(\partial_p)$ as its kernel.

Similarly, the p -th effective resistance operator $\mathcal{T}^p = \delta_{p-1}\delta_{p-1}^+$ on cochains is the orthogonal projection of $C^p(K)$ onto $\ker(\delta_{p-1}^+)^{\perp} = \ker(\delta_{p-1}^*)^{\perp} = \text{im } (\delta_{p-1})$. This means $\mathcal{T}^p$ is self-adjoint, idempotent and has $\text{im } (\delta_{p-1})$ as its image and $\ker(\delta_{p-1}^*)$ as its kernel.

Using the properties of the effective resistance operators from **Proposition 6.6**, we can now establish important properties of the effective resistance bilinear forms.

Corollary 6.7. The effective resistance bilinear forms $\mathcal{R}_p$ and $\mathcal{R}^p$ are both symmetric and positive semi-definite. Moreover, for any p -chain $\alpha \in C_p(K)$ and p -cochain $f \in C^p(K)$,

$$r_{\alpha} = \|\mathcal{T}_p(\alpha)\|_{C_p}^2 = \|\partial_p^*(\partial_p^*)^+(\alpha)\|_{C_p}^2 \quad \text{and} \quad r_f = \|\mathcal{T}^p(f)\|_{C^p}^2 = \|\delta_{p-1}\delta_{p-1}^+(f)\|_{C^p}^2. \quad (39)$$

Proof. Symmetry follows directly from the self-adjointness of the effective resistance operators. Positive semidefiniteness follows from their self-adjointness and idempotence. Indeed, for any p -chain $\alpha \in C_p(K)$, we have

$$r_{\alpha} = \langle \mathcal{T}_p(\alpha), \alpha \rangle_{C_p} = \langle (\mathcal{T}_p)^2(\alpha), \alpha \rangle_{C_p} = \langle \mathcal{T}_p(\alpha), \mathcal{T}_p(\alpha) \rangle_{C_p} = \|\mathcal{T}_p(\alpha)\|_{C_p}^2 = \|\partial_p^*(\partial_p^*)^+(\alpha)\|_{C_p}^2 \geq 0.$$

A similar argument applies to the cochain case. $\square$

Due to the analogous structure of the effective resistance operators, and their associated bilinear forms on both chains and cochains, all results apply equally in either setting. Since we state most results in terms of chains, we adopt the chain-based formulation for the remainder of the paper.

6.2 Effective Resistance in Simplicial Complexes as Matrices

In this section, we review several notions of effective resistance of simplices from [43, 36, 9] and show how they can be recovered from our notion of effective resistance bilinear form by selecting appropriate bases for the chain (or cochain) spaces.

6.2.1 Connection to the Definition of Osting, Palande, and Wang

We review the notion of effective resistance of simplices introduced by Osting et al. [43] and recover their notion using our framework in **Proposition 6.9**. Following **Section 4.2**, $[\mathcal{L}_{\text{up}}^p]_{\mathcal{B}^p}^{\mathcal{B}^p}$ is not symmetric according to **Equation** (13), despite originating from a self-adjoint operator, because the inner product in this space does not have the identity matrix as its representation. Osting et al. [43] used a standard basis but avoided working with $[\mathcal{L}_{\text{up}}^p]_{\mathcal{B}^p}^{\mathcal{B}^p}$ because of this lack of symmetry. Instead, they considered the following symmetric, positive semi-definite normalized Laplacian,

$$L_{p-1,\text{up}}^{\text{sym}} := W_{p-1}L_{p-1,\text{up}} = B_pW_pB_p^{\top}. \quad (40)$$

Based on the normalized Laplacian $L_{p-1,\text{up}}^{\text{sym}}$, they defined the matrix $R_p \in \mathbb{R}^{n_p \times n_p}$,

$$R_p := B_p^{\top}(L_{p-1,\text{up}}^{\text{sym}})^+B_p = B_p^{\top}\left(B_pW_pB_p^{\top}\right)^+B_p. \quad (41)$$

The effective resistance of a p -simplex σ is then defined as the diagonal entry of R_p corresponding to σ .

Equation (41) generalizes the edge-based effective resistance matrix R in the graph case (see **Equation (22)**). Indeed, for graphs (i.e., setting $p = 0$) with unit weights on vertices, we have:

$$R_1 = B_1^\top \left(B_1 W_1 B_1^\top \right)^+ B_1 = R.$$

Remark 6.8. The matrix defined in **Equation (40)** is indeed the matrix representation of the up Laplacian operator with respect to the standard basis $\mathcal{B}^{p-1}$ of $C^{p-1}(K)$ and the normalized basis $\hat{\mathcal{B}}^{p-1} := (w_{\sigma_1}^{-1} \sigma_1^\vee, \dots, w_{\sigma_{n_{p-1}}}^{-1} \sigma_{n_{p-1}}^\vee)$ of $C^p(K)$. Indeed, by $[\delta_{p-1}]_{\mathcal{B}^p}^{\mathcal{B}^{p-1}} = B_p^\top$ and **Equation (7)**,

$$\begin{aligned} [\delta_{p-1}^* \delta_{p-1}]_{\hat{\mathcal{B}}^{p-1}}^{\hat{\mathcal{B}}^p} &= [\delta_{p-1}^*]_{\hat{\mathcal{B}}^{p-1}}^{\hat{\mathcal{B}}^p} [\delta_{p-1}]_{\hat{\mathcal{B}}^{p-1}}^{\hat{\mathcal{B}}^p} = [\text{Id}]_{\hat{\mathcal{B}}^{p-1}}^{\hat{\mathcal{B}}^p} [\delta_{p-1}^*]_{\hat{\mathcal{B}}^{p-1}}^{\hat{\mathcal{B}}^p} [\delta_{p-1}]_{\hat{\mathcal{B}}^{p-1}}^{\hat{\mathcal{B}}^p} \\ &= W_{p-1} W_{p-1}^{-1} B_p W_p B_p^\top = B_p W_p B_p^\top = L_{p-1, \text{up}}^{\text{sym}}. \end{aligned}$$

Next, we show that the matrix representation of the effective resistance bilinear form $\mathcal{R}_p$ with respect to the standard bases and unit weights on $(p-1)$ -simplices agrees with the matrix definition in **Equation (41)**. This is achieved by generalizing **Proposition 5.11** in a straightforward way.

Proposition 6.9. Let W_{p-1} and W_p be the weight matrices on the $(p-1)$ - and p -simplices, respectively. With respect to the standard bases, the effective-resistance bilinear form $\mathcal{R}_p$ has a matrix representation

$$[\mathcal{R}_p]_{\mathcal{B}_p, \mathcal{B}_p} = B_p^\top W_{p-1}^{-1/2} \left(W_{p-1}^{-1/2} B_p W_p B_p^\top W_{p-1}^{-1/2} \right)^+ W_{p-1}^{-1/2} B_p, \quad (42)$$

which reduces to $B_p^\top (B_p W_p B_p^\top)^+ B_p$ when the $(p-1)$ -simplices have unit weights ($W_{p-1} = \text{Id}$).

Proof. Using the standard bases $\mathcal{B}_{p-1}$ and $\mathcal{B}_p$ (see **Definition 3.5**), and applying **Equations (6)** and **(15)** and **Proposition B.8**, the effective-resistance operator $\mathcal{T}_p = \partial_p^*(\mathcal{L}_{p-1}^{\text{up}})^+ \partial_p$ has matrix representation

$$\begin{aligned} [\mathcal{T}_p]_{\mathcal{B}_p}^{\mathcal{B}_p} &= [\partial_p^*]_{\mathcal{B}_{p-1}}^{\mathcal{B}_p} [(\mathcal{L}_{p-1}^{\text{up}})^+]_{\mathcal{B}_{p-1}}^{\mathcal{B}_{p-1}} [\partial_p]_{\mathcal{B}_{p-1}}^{\mathcal{B}_p} \\ &= W_p B_p^\top W_{p-1}^{-1} W_{p-1}^{1/2} \left(W_{p-1}^{-1/2} \left(B_p W_p B_p^\top W_{p-1}^{-1/2} \right)^+ W_{p-1}^{1/2} \right) W_{p-1}^{-1/2} B_p \end{aligned}$$

Since $[\mathcal{R}_p]_{\mathcal{B}_p, \mathcal{B}_p} = ([\mathcal{T}_p]_{\mathcal{B}_p}^{\mathcal{B}_p})^\top [\langle \cdot, \cdot \rangle_{C_p}]_{\mathcal{B}_p, \mathcal{B}_p} = ([\mathcal{T}_p]_{\mathcal{B}_p}^{\mathcal{B}_p})^\top W_p^{-1}$, the desired result follows after simplification. $\square$

Remark 6.10. With respect to the orthonormal bases $\tilde{\mathcal{B}}_{p-1}$ and $\tilde{\mathcal{B}}_p$, the Gram matrix $[\langle \cdot, \cdot \rangle_{C_p}]_{\tilde{\mathcal{B}}_p, \tilde{\mathcal{B}}_p}$ is the identity, so the matrix representation of the effective resistance bilinear form $\mathcal{R}_p$ coincides with that of the effective resistance operator $\mathcal{T}_p$. In particular, we have

$$[\mathcal{R}_p]_{\tilde{\mathcal{B}}_p, \tilde{\mathcal{B}}_p} = ([\mathcal{T}_p]_{\tilde{\mathcal{B}}_p}^{\tilde{\mathcal{B}}_p})^\top = [\mathcal{T}_p]_{\tilde{\mathcal{B}}_p}^{\tilde{\mathcal{B}}_p} = W_p^{-1/2} [\mathcal{T}_p]_{\mathcal{B}_p}^{\mathcal{B}_p} W_p^{1/2} = W_p^{1/2} [\mathcal{R}_p]_{\mathcal{B}_p, \mathcal{B}_p} W_p^{1/2},$$

which is equal to $W_p^{1/2} B_p^\top (B_p W_p B_p^\top)^+ B_p W_p^{1/2}$ when the $(p-1)$ -simplices have unit weights.

6.2.2 Connection to the Definition of Kook and Lee

We review the notion of effective resistance introduced by Kook and Lee [36] for simplicial networks (X, W_d) , i.e. d -dimensional simplicial complexes X with weights only on the d -simplices, gathered in the diagonal matrix W_d . They define a notion of effective resistance for d -simplices (see **Equation (47)**) on these objects. In **Proposition 6.14**, we show that their definition is recovered from our effective resistance bilinear form. Furthermore, in **Proposition 6.15**, we establish that their matrix expression **Equation (50)** is a special case of that of [43], which represents our bilinear form under suitable bases, as already shown in **Section 6.2.1**.

We first review the following notion, key to the framework by Kook and Lee.

Definition 6.11 (*Current generator*, [36]). Let X be a simplicial complex of dimension d with n vertices. A *current generator* is a subset $\sigma \subset \{1, \dots, n\}$ with $|\sigma| = d + 1$ such that

$$\partial_d \sigma = -\partial_d^X(c)$$

for some $c \in C_d(X, \mathbb{R})$. In the formula above, ∂_d refers to the boundary operator defined on d -simplices in an abstract simplicial complex, and $\partial_d^X : C_d(X) \rightarrow C_{d-1}(X)$ is that of X .

We recall the necessary elements to define the effective resistance of a current generator σ in a simplicial network (X, W_d) according to [36], adapted to our formalism and language. We first extend the chain complex of X by summing a copy of $H_d(X) = \ker \partial_d^X$ in degree $d + 1$ and $\text{span}(\sigma)$ in degree d . This yields the complex

$$Z : H_d(X) \xrightarrow{\partial_{d+1}^Z} C_d(X \sqcup \{\sigma\}) \cong C_d(X) \oplus \text{span}(\sigma) \xrightarrow{\partial_d^Z} C_{d-1}(X) \rightarrow \dots, \quad (43)$$

where $\text{im } \partial_{d+1}^Z = \ker \partial_d^X \subset C_d(X) \subset C_d(X \sqcup \{\sigma\})$, $\partial_d^Z|_{C_d(X)} = \partial_d^X$ and $\partial_d^Z(\sigma) = -\partial_d^X(c)$, c being the chain in the definition of the current generator (**Definition 6.11**). We endow $H_d(X) = \ker \partial_d^X$ with the inner product induced from $C_d(X)$. For $C_d(X \sqcup \{\sigma\})$, we extend the inner product on $C_d(X)$ by declaring the new basis element σ to be orthogonal to the standard basis of $C_d(X)$ and have weight one. In other words, for $\alpha, \beta \in C_d(X \sqcup \{\sigma\})$ written as $\alpha = \alpha_X + \alpha_\sigma \sigma$ and $\beta = \beta_X + \beta_\sigma \sigma$ with $\alpha_X, \beta_X \in C_d(X)$ and $\alpha_\sigma, \beta_\sigma \in \mathbb{R}$, we set

$$\langle \alpha, \beta \rangle_{C_d(X \sqcup \{\sigma\})} = \langle \alpha_X, \beta_X \rangle_{C_d(X)} + \alpha_\sigma \beta_\sigma, \quad \langle \sigma, \sigma \rangle_{C_d(X \sqcup \{\sigma\})} = 1.$$

The d th Hodge Laplacian of the complex in **Equation (43)** is then

$$\mathcal{L}_d^Z := \mathcal{L}_d^{\text{Hodge}}(Z) = (\partial_d^Z)^* \partial_d^Z + \partial_{d+1}^Z (\partial_{d+1}^Z)^*,$$

where adjoints are taken with respect to these inner products.

Notice that now, the homology of Z at level d is 1-dimensional:

$$H_d(Z) = \frac{\ker \partial_d^Z}{\text{im } \partial_{d+1}^Z} = \frac{\ker \partial_d^X \oplus \text{span}(\sigma + c)}{\ker \partial_d^X} \cong \text{span}(\sigma + c).$$

Hence, the kernel of the d th Hodge Laplacian of Z is 1-dimensional.

The definition of the effective resistance by Kook and Lee [36] requires the three following steps.

1. **Current construction:** First, fixing a value $i_\sigma \in \mathbb{R} \setminus \{0\}$, Kook and Lee [36] defined the chain $I_\sigma \in C_d(X \sqcup \{\sigma\})$ as the unique vector in the kernel of the d th Hodge Laplacian of Z whose projection onto σ is precisely i_σ , i.e.,

$$I_\sigma \in \ker(\mathcal{L}_d^Z) \cong \ker(\partial_d^Z) \cap \ker((\partial_{d+1}^Z)^*), \text{ s.t. } I_\sigma := I + i_\sigma \sigma, \quad (44)$$

with $I \in C_d(X)$, also uniquely defined since $\dim \ker(\mathcal{L}_d^Z) = 1$.

Both I_σ and I satisfy Kirchhoff's current law for their respective complexes. For I_σ , this follows from it being a cycle, $\partial_d^Z I_\sigma = 0$. Similarly, for I we have that $0 = \partial_d^Z I_\sigma = \partial_d^X I + i_\sigma \partial_d^Z(\sigma)$, and hence $\partial_d^X I = -i_\sigma \partial_d^Z(\sigma) = i_\sigma \partial_d^X(c)$ with $c \in C_d(X)$, so that $\beta = i_\sigma \partial_d^X(c) \in C_{d-1}(X)$ is the boundary current in our terminology, where i_σ ampères are being injected/extracted in each simplex of the boundary. This explains the intuition behind the value i_σ set at the beginning.

We note that

$$\ker((\partial_{d+1}^Z)^*) \cong \text{im}(\partial_{d+1}^Z)^\perp = \ker(\partial_d^X)^\perp \subset C_d(X),$$

which ensures that $I \in \ker(\partial_d^X)^\perp$.

2. **Voltage construction:** The authors then defined a cochain $V_\sigma \in C^d(X \sqcup \{\sigma\})$ by applying Ohm's law to the current I constructed on X , which in our language means setting $V = \flat_d^X I$, and defining

$$V_\sigma := V + v_\sigma \sigma^\vee, \text{ s.t. } V_\sigma(I_\sigma) = 0, \quad (45)$$

or, in other words, setting v_σ to be the value that satisfies

$$V_\sigma(I_\sigma) = \langle I, I \rangle_{C_d(X)} + i_\sigma v_\sigma = 0. \quad (46)$$

Note that, from **Equation (46)** we deduce that $i_\sigma v_\sigma < 0$, i.e., they must have different signs.

In this setting, V satisfies Ohm's law in X by definition with respect to I , and I has boundary current $\beta = i_\sigma \partial_d^X(c)$. By **Corollary 5.6**, Kirchhoff's voltage law holds for V in X ; that is, there exists $\phi \in C_{d-1}(X)$ such that $\delta_{d-1}^X \phi = V$. Hence, V is indeed a voltage according to our definition.

The case of V_σ is more subtle. Proving that V_σ and I_σ are related through Ohm's law would entail

$$\flat_d^Z I_\sigma = \langle I + i_\sigma \sigma, \cdot \rangle_{C_d(X \sqcup \{\sigma\})} = \flat_d^X I + i_\sigma \langle \sigma, \cdot \rangle_{C_d(X \sqcup \{\sigma\})},$$

being equal to

$$V_\sigma = \flat_d^X I + v_\sigma \sigma^\vee, \quad \text{where} \quad \sigma^\vee = \frac{\langle \sigma, \cdot \rangle_{C_d(X \sqcup \{\sigma\})}}{\langle \sigma, \sigma \rangle_{C_d(X \sqcup \{\sigma\})}}.$$

However, with our definition of the inner product on $C_d(X \sqcup \{\sigma\})$, this equality fails in general since $i_\sigma \neq v_\sigma$. Modifying the inner product to mirror the weighting in X , i.e. setting $\langle \sigma, \sigma \rangle_{C_d(X \sqcup \{\sigma\})} = w(\sigma)^{-1}$ with σ having weight $w(\sigma)$ in the complex $X \sqcup \{\sigma\}$, would impose

$$w(\sigma) = \frac{i_\sigma}{v_\sigma}.$$

This is impossible because $i_\sigma v_\sigma < 0$, implying $w(\sigma) < 0$, whereas the inner product requires $\langle \sigma, \sigma \rangle_{C_d(X \sqcup \{\sigma\})} = w(\sigma)^{-1} \geq 0$. Thus, Ohm's law cannot hold for V_σ . Nevertheless, we will show in **Lemma 6.12** that V_σ satisfies Kirchhoff's voltage law independently of Ohm's law, so two of the three electric laws remain valid for the pair I_σ and V_σ .

3. **Effective resistance definition:** With these definitions, Kook and Lee defined the effective resistance of the current generator σ [36, Definition 3.1] precisely as the negative of its weight

$$r'_\sigma := -\frac{v_\sigma}{i_\sigma}. \quad (47)$$

We prove the following two lemmas establishing some properties of the voltage obtained for V and V_σ from Kirchhoff's voltage law. Similar properties are already stated with brief proofs in [36, Section 4.1], here we provide the statements and proofs adapted to our setting.

Lemma 6.12. A potential $\phi \in C^{d-1}(X)$ for V , i.e. $V = \delta_{d-1}^X \phi$, is also a potential for V_σ , that is

$$\delta_{d-1}^Z \phi = V_\sigma,$$

with δ_{d-1}^Z the dual map of ∂_d^Z in **Equation (43)**.

Proof. We check that the restriction of $\delta_{d-1}^Z \phi$ to X agrees with V and that $\delta_{d-1}^Z \phi(I_\sigma) = 0$. Indeed, for $\alpha \in C_d(X)$, we have

$$\delta_{d-1}^Z \phi(\alpha) = \phi(\partial_d^Z \alpha) = \phi(\partial_d^X \alpha) = \delta_{d-1}^X \phi(\alpha) = V(\alpha).$$

In addition, by definition (**Equation (44)**), we have that $I_\sigma \in \ker \partial_d^Z$, which implies

$$\delta_{d-1}^Z \phi(I_\sigma) = \phi(\partial_d^Z I_\sigma) = 0,$$

proving that $\delta_{d-1}^Z \phi = V_\sigma$ as it satisfies its two defining conditions. $\square$

Lemma 6.13. Given a potential $\phi \in C^{d-1}(X)$ for V , i.e. $V = \delta_{d-1}^X \phi$, we have

$$v_\sigma = \phi(\partial_d \sigma) \tag{48}$$

where v_σ is the component accompanying $\sigma^\vee$ in **Equation (45)**.

Proof. From **Lemma 6.12**, we know that $\delta_{d-1}^Z \phi = V_\sigma$. Let $\alpha \in C_d(X \sqcup \{\sigma\})$, and write $\alpha = \alpha_X + \alpha_\sigma \sigma$. Then, we have

$$\delta_{d-1}^Z \phi(\alpha) = \phi(\partial_d^Z(\alpha)) = \phi(\partial_d^X(\alpha_X)) + \alpha_\sigma \phi(\partial_d(\sigma)) = V(\alpha_X) + \alpha_\sigma \phi(\partial_d(\sigma)),$$

but also,

$$V_\sigma(\alpha) = V(\alpha_X) + v_\sigma \alpha_\sigma \sigma^\vee(\sigma) = V(\alpha_X) + v_\sigma \alpha_\sigma.$$

Since $\delta_{d-1}^Z \phi = V_\sigma$, and given that α_σ is arbitrary, we conclude:

$$\phi(\partial_d \sigma) = v_\sigma$$

as desired. $\square$

Finally, we are ready to prove that the definition of effective resistance by Kook and Lee coincides with **Definition 6.2** when $\sigma \in C_d(X)$.

Proposition 6.14. Let $\sigma \in C_d(X)$ be a current generator, and define the current $I \in C_d(X)$ and $i_\sigma \in \mathbb{R}$ (resp. the voltage $V \in C^d(X)$ and $v_\sigma \in \mathbb{R}$) as in **Equation (44)** (resp. **Equation (45)**). Let $\phi \in C^{d-1}(X)$ be a potential for V , i.e. $\delta_{d-1}^X \phi = V$. Then, the following equality holds:

$$\mathcal{R}_d(\sigma, \sigma) = -\frac{v_\sigma}{i_\sigma}$$

or equivalently, $r_\sigma = r'_\sigma$.

Proof. Denoting the boundary current $\beta = \partial_d^X I$, from **Proposition 5.4** we know that the potential ϕ must satisfy $\mathcal{L}_{d-1}^{\text{up}} \phi = \flat_{d-1}^X \beta$. Using the pseudo inverse, we obtain $\phi = (\mathcal{L}_{d-1}^{\text{up}})^+ \flat_{d-1}^X \beta$. Applying $(\flat_d^X)^{-1}$ to both sides, and **Lemma 4.1**,

$$(\flat_d^X)^{-1} \phi = (\mathcal{L}_{d-1}^{\text{up}})^+ \beta \in C_{d-1}(X).$$

Considering the inner product with $\partial_d \sigma$ in $C_{d-1}(X)$, we obtain the following sequence of equivalences:

$$\begin{aligned} \langle (\mathcal{L}_{d-1}^{\text{up}})^+ \beta, \partial_d \sigma \rangle_{C_{d-1}(X)} &= \langle (\flat_d^X)^{-1} \phi, \partial_d \sigma \rangle_{C_{d-1}(X)} \\ &= \phi(\partial_d \sigma) && \text{(definition of } \flat_d^{-1}) \\ &= v_\sigma. && \text{(Lemma 6.13)} \end{aligned} \quad (49)$$

We know that $\beta = \partial_d^X I = -i_\sigma \partial_d(\sigma)$ since $0 = \partial_d^Z I_\sigma = \partial_d^X I + i_\sigma \partial_d(\sigma)$. Substituting this in **Equation (49)**, we conclude

$$v_\sigma = -i_\sigma \langle (\mathcal{L}_{d-1}^{\text{up}})^+ \partial_d \sigma, \partial_d \sigma \rangle_{C_{d-1}(X)} = -i_\sigma \langle \partial_d^*(\mathcal{L}_{d-1}^{\text{up}})^+ \partial_d \sigma, \sigma \rangle_{C_d(X)} = -i_\sigma \mathcal{R}_d(\sigma, \sigma).$$

Hence, the result follows. $\square$

In [36, Theorem 4.2], the authors provided a matrix characterization of their notion of effective resistance (**Equation (47)**) in terms of the *Green's function*, i.e., the inverse of the Hodge Laplacian matrix for X in the standard bases, that is $L_{d-1} = L_{d-1, \text{up}} + L_{d-1, \text{down}}$, where $L_{d-1, \text{up}}$ and $L_{d-1, \text{down}}$ are given in **Equations (13)** and **(14)**. Assuming that the reduced homology in dimension $d-1$ is trivial (i.e., $\bar{H}_{d-1}(K, \mathbb{R}) = \{0\}$), the Hodge theorem guarantees that L_{d-1} is invertible. Under this assumption, Kook and Lee [36] defined

$$R'_d := B_d^\top L_{d-1}^{-1} B_d. \quad (50)$$

For a given d -simplex $\tau \in K_d$, the effective resistance defined in **Equation (47)** is equivalent to the diagonal entry of R'_d , that is, $r'_\tau = R'_d(\tau, \tau)$.

We now show that the matrix R'_d defined by [36] coincides with the matrix R_d (see **Equation (41)**) introduced by Osting et al. [43], where the version from [43] is formulated in a more general setting (with no assumptions on the invertibility of the Laplacian) and has already been shown to be the matrix representation of our effective resistance bilinear form in the standard bases in **Section 6.2.1**.

Proposition 6.15. When the Hodge Laplacian L_{d-1} is invertible, and only the d -dimensional simplices are weighted, the two resistance matrices agree: $R_d = R'_d$, or, equivalently,

$$B_d^\top \left(B_d W_d B_d^\top \right)^+ B_d = B_d^\top (B_d W_d D_d^\top + B_{d-1}^\top B_{d-1}) B_d.$$

To prove this proposition we first need the following result from [24]. For a subspace Ω , let P_Ω denote the matrix of the orthogonal projection onto Ω .

Lemma 6.16 (Theorem 3, [24]). Let $A, C \in \mathbb{R}^{n \times n}$ with $\text{rank}(A + C) = \text{rank}(A) + \text{rank}(C)$. Then

$$(A + C)^+ = (I - S)A^+(I - T) + SC^+T, \quad (51)$$

where I is the identity matrix, $S := \left(P_{\text{col}(C^\top)} P_{\text{col}(A^\top)} \right)^+$ and $T := \left(P_{\text{col}(A)^\perp} P_{\text{col}(C)} \right)^+$.

Proof of Proposition 6.15. Let $A = B_d W_d B_d^\top$ and $C = B_{d-1}^\top B_{d-1}$. $L_{d-1} = A + C$ is invertible, so $(A + C)^{-1} = (A + C)^+$. On the other hand, invertibility implies $\ker(A + C) = \{0\}$, and from the Hodge decomposition for simplicial complexes we get

$$C_{d-1}(K, \mathbb{R}) \simeq \text{col}(A) \oplus \text{col}(C), \quad (52)$$

ensuring that $\text{rank}(A + C) = \text{rank}(A) + \text{rank}(C)$. This means that we can apply **Lemma 6.16**.

We now show that $\text{col}(A)^\perp = \text{col}(C)$. Note that, from the vanishing of the composition of consecutive coboundary maps, we have $CA = B_{d-1}^\top B_{d-1} B_d W_d B_d^\top = 0$. Let $x = Cx' \in \text{col}(C)$ and $y = Ay' \in \text{col}(A)$. Then, using the symmetry of A and C , and the inner product on $C_{d-1}(X)$, which is the standard ℓ^2 product since W_{d-1} is the identity matrix, we compute:

$$\langle x, y \rangle_{C_{d-1}} = x^\top y = (x')^\top CAy' = 0.$$

Hence, $\text{col}(C) \subseteq \text{col}(A)^\perp$. **Equation (52)** implies that both $\text{col}(C)$ and $\text{col}(A)^\perp$ have the same dimension, and thus $\text{col}(C) = \text{col}(A)^\perp$.

Therefore, we have

$$\begin{aligned} \left(P_{\text{col}(C)^\perp} P_{\text{col}(A)^\perp} \right)^+ &= \left(P_{\text{col}(C)} P_{\text{col}(A)} \right)^+ && (C = C^\top, A = A^\top) \\ &= \left(P_{\text{col}(C)}^2 \right)^+ && (\text{col}(C) = \text{col}(A)^\perp) \\ &= \left(P_{\text{col}(C)} \right)^+ && (\text{projection matrix is idempotent}) \\ &= (CC^+)^+ && (\textbf{Proposition B.5}) \\ &= CC^+ && (\text{properties of pseudo-inverses}) \end{aligned}$$

and similarly, $\left(P_{\text{col}(A)^\perp} P_{\text{col}(C)} \right)^+ = CC^+$. Applying **Equation (51)**, we obtain:

$$\begin{aligned} (A + C)^+ &= (I - CC^+)A^+(I - CC^+) + CC^+C^+CC^+ \\ &= A^+ - A^+CC^+ - CC^+A^+ + CC^+A^+CC^+ + CC^+C^+. \end{aligned} \quad (53)$$

Next, we observe that $B_d^\top C = B_d^\top B_{d-1}^\top B_{d-1} = 0$ due to the composition of consecutive coboundary maps vanishing. In addition, using **Proposition B.5**, we get:

$$C^+ B_d = (C^\top C)^+ C^\top B_d = (C^\top C)^+ B_{d-1}^\top B_{d-1} B_d^\top = 0.$$

Therefore, when multiplying both sides of **Equation (53)** on the left by $B_d^\top$ and on the right by B_d , all terms vanish except the first, and we conclude the desired result:

$$R'_d = B_d^\top (A + C)^{-1} B_d = B_d^\top (A + C)^+ B_d = B_d^\top A^+ B_d = R_d. \quad \square$$

6.2.3 Connection to the Definition of Black and Maxwell

Black and Maxwell [9] worked with weighted simplicial complexes, but in their definition of effective resistance they assigned weights only to the p -simplices, assuming unit weights on the $(p - 1)$ -simplices. This assumption is consistent with the settings in [43, 36], but it is more restrictive than our formulation, which accommodates weights on both p -simplices and $(p - 1)$ -simplices.

The up Laplacian matrix considered in [9] in defining effective resistance is $L_{p-1, \text{up}}^{\text{sym}} = B_p W_p B_p^\top$ (see **Equation (40)**). Specifically, they defined the effective resistance of any $(p - 1)$ -cycle γ to be

$$r''_\gamma := \begin{cases} [\gamma]_{\mathcal{B}_{p-1}}^\top (L_{p-1, \text{up}}^{\text{sym}})^+ [\gamma]_{\mathcal{B}_{p-1}}, & \text{if } \gamma \text{ is a boundary} \\ \infty, & \text{otherwise} \end{cases},$$

where $[\gamma]_{\mathcal{B}_{p-1}}$ is the vector representation of γ in the standard basis $\mathcal{B}_{p-1}$ of $(p-1)$ -chains.

We now relate their definition to that of Osting et al. [43], and hence to ours. If γ is a boundary, then $\gamma = \partial_p \beta$ for some p -chain β . In this case,

$$\begin{aligned}
r''_\gamma &= [\partial_p \beta]_{\mathcal{B}_{p-1}}^\top (L_{p-1, \text{up}}^{\text{sym}})^+ [\partial_p \beta]_{\mathcal{B}_{p-1}} && \text{(definition of } r''_\gamma) \\
&= [\beta]_{\mathcal{B}_p}^\top ([\partial_p]_{\mathcal{B}_{p-1}}^{\mathcal{B}_p})^\top (L_{p, \text{up}}^{\text{sym}})^+ [\partial_p]_{\mathcal{B}_{p-1}}^{\mathcal{B}_p} [\beta]_{\mathcal{B}_p} \\
&= [\beta]_{\mathcal{B}_p}^\top B_p^\top (L_{p, \text{up}}^{\text{sym}})^+ B_p [\beta]_{\mathcal{B}_p} && \text{(matrix form of } \partial_p) \\
&= [\beta]_{\mathcal{B}_p}^\top R_p [\beta]_{\mathcal{B}_p} && \text{(Equation (41))} \\
&= r_\beta.
\end{aligned}$$

Thus, Black and Maxwell's effective resistance agrees with ours on boundaries: what they call the effective resistance of the boundary $\partial_p \beta$ is exactly what we call the effective resistance of the p -chain β . This parallels the graph case, where the effective resistance of an edge coincides with the effective resistance of its boundary, i.e., the two vertices it connects (see [Section 5.1](#)).

7 Metric Properties of Effective Resistance

In this section, we study the metric structures induced by the effective resistance bilinear forms introduced in [Section 6.1](#). We show that the p -th effective resistance bilinear form induces a pseudometric on the space of p -chains, and that the $(p+1)$ -th form induces a genuine metric on the space of p -cycles, thereby generalizing the classical vertex-based effective resistance metric in graphs [35, 47].

Recall from [Corollary 6.7](#) that the effective resistance bilinear form $\mathcal{R}_p$ is symmetric and positive semi-definite. For any $\alpha, \alpha' \in C_p(K)$, we define the *chain pseudometric*

$$d_p(\alpha, \alpha') := \sqrt{\mathcal{R}_p(\alpha - \alpha', \alpha - \alpha')} = \|\mathcal{T}_p(\alpha - \alpha')\|_{C_p}. \quad (54)$$

Because $\mathcal{R}_p$ is positive semi-definite, d_p satisfies non-negativity, symmetry, and the triangle inequality, but it may vanish for distinct elements when $\alpha - \alpha' \in \ker(\mathcal{T}_p) = \ker(\partial_p)$ (i.e., when $\alpha - \alpha'$ is a cycle). Thus, d_p defines only a *pseudometric* on $C_p(K)$.

Next, we consider the $(p+1)$ -th effective resistance bilinear form $\mathcal{R}_{p+1}$, which acts naturally on $(p+1)$ -chains and induces a metric on the cycle space $Z_p(K)$ as follows. For any pair of p -cycles $c, c' \in Z_p(K)$, choose a $(p+1)$ -chain β satisfying $\partial_{p+1} \beta = c - c'$, and define

$$\tilde{d}_p(c, c') := \sqrt{\mathcal{R}_{p+1}(\beta, \beta)} = \|\mathcal{T}_{p+1}(\beta)\|_{C_{p+1}}. \quad (55)$$

Note that the quantity $\mathcal{T}_{p+1}(\beta) = \partial_{p+1}^* (\mathcal{L}_p^{\text{up}})^+ \partial_{p+1}(\beta)$ depends only on the boundary $\partial_{p+1} \beta = c - c'$, so $\tilde{d}_p$ is well-defined.

Theorem 7.1. Let K be a finite simplicial complex and $p \geq 0$.

1. The function d_p in [Equation \(54\)](#) defines a pseudometric on $C_p(K)$.
2. If $\tilde{H}_p(K) = 0$, then the function $\tilde{d}_p$ in [Equation \(55\)](#) defines a metric on $Z_p(K)$.

Proof. The first statement follows directly from the positive semi-definiteness of $\mathcal{R}_p$. For the second, assume $\tilde{H}_p(K) = 0$. Non-negativity and symmetry of $\tilde{d}_p$ are immediate from (55). For the triangle

inequality, let $c, c', c'' \in Z_p(K)$ and choose $(p+1)$ -chains β, γ such that $\partial_{p+1}\beta = c - c'$ and $\partial_{p+1}\gamma = c' - c''$. The existence of such β, γ follows from the assumption that $\tilde{H}_p(K) = 0$. Then $\partial_{p+1}(\beta + \gamma) = c - c''$, and hence

$$\tilde{d}_p(c, c'') = \|\mathcal{T}_{p+1}(\beta + \gamma)\|_{C_{p+1}} \leq \|\mathcal{T}_{p+1}(\beta)\|_{C_{p+1}} + \|\mathcal{T}_{p+1}(\gamma)\|_{C_{p+1}} = \tilde{d}_p(c, c') + \tilde{d}_p(c', c''),$$

by the triangle inequality in the Hilbert space $C_{p+1}(K)$.

Finally, we verify the identity of indiscernibles. If $c = c'$, we may take $\beta = 0$, so $\tilde{d}_p(c, c) = \|\mathcal{T}_{p+1}(0)\|_{C_{p+1}} = 0$. Conversely, suppose $\tilde{d}_p(c, c') = 0$. Then $\|\mathcal{T}_{p+1}(\beta)\|_{C_{p+1}} = 0$ for any β satisfying $\partial_{p+1}\beta = c - c'$. This implies $\beta \in \ker(\mathcal{T}_{p+1}) = \ker(\partial_{p+1})$, so that $c - c' = \partial_{p+1}\beta = 0$, yielding $c = c'$. Therefore, $\tilde{d}_p$ satisfies all properties of a metric. $\square$

Remark 7.2. In the graph case ($p = 0$), let $K = G = (V, E)$ be connected. Then $\tilde{H}_0(G) = 0$ and $Z_0(G) = C_0(G)$, so we identify $V \subset Z_0(G)$. The restriction of $\tilde{d}_0$ from **Equation (55)** to V is therefore a metric on the vertices. For $u, v \in V$, choosing any 1-chain β with $\partial_1\beta = u - v$ yields $\tilde{d}_0(u, v) = \sqrt{\mathcal{R}_1(\beta, \beta)}$, which is the classical effective resistance distance on graphs [35, 47].

8 Foster's Theorem: From Graphs to Simplicial Complexes

In this section we extend Foster's Theorem, a property relating relative effective resistances in a graph with its 0-dimensional Betti number, to the context of simplicial complexes. We first recall Foster's Theorem for graphs.

Theorem 8.1 (Foster's Theorem in graphs, [17]). The sum of all relative effective resistances of edges in a graph $G = (V, E)$ satisfies the following property:

$$\sum_{e \in E} w(e)r_e = |V| - \beta_0(G), \quad (56)$$

where $\beta_0(G) = \dim \ker(\partial_1)$ is the number of connected components in the graph.

We are interested in a result similar to **Theorem 8.1** using the notion of simplicial effective resistance r_σ . We begin by proving this result in the operator form, **Theorem 8.2**, and then derive the matrix form as **Corollary 8.4**. Our generalization offers a deeper interpretation of the classical Foster's theorem by showing that the *total effective resistance* can be viewed as the *sum of the eigenvalues* (i.e., the trace) of the effective resistance operator, which is equal to its rank because the operator is an orthogonal projection. We now present the details.

In what follows, consider a finite (oriented) weighted simplicial complex K , and for any degree $p \geq 0$, let $n_p = |K_p|$. Let $\text{rank}(\cdot)$ and $\text{null}(\cdot)$ denote the rank and nullity of a linear operator (or a matrix), respectively.

Theorem 8.2 (Higher-dimensional Foster's theorem: operator form). Let K be a finite weighted simplicial complex equipped with an inner product. For any $p \geq 0$, let $\{\lambda_i(\mathcal{T}_p)\}$ be the eigenvalues (with multiplicities) of the effective resistance operator $\mathcal{T}_p$ on chain spaces. Then,

$$\sum_i \lambda_i(\mathcal{T}_p) = \text{rank}(\partial_p^*) = n_{p-1} - \text{null}(\partial_p^*).$$

Moreover, given any orthonormal basis $\{\alpha_i\}$ of $C_p(K)$,

$$\sum_i \langle \mathcal{T}_p(\alpha_i), \alpha_i \rangle = \text{rank}(\partial_p^*) = n_{p-1} - \text{null}(\partial_p^*).$$

To prove the theorem, we recall the following standard fact from linear algebra (included here for completeness). The *trace* of an operator φ , denoted by $\text{tr}(\varphi)$, is the sum of its eigenvalues and can also be obtained by taking the trace of its matrix representation on any orthonormal basis.

Lemma 8.3. For a linear operator $\varphi : X \rightarrow X$ on a finite-dimensional Hilbert space X and any orthonormal basis $\{x_i\}$ of X ,

$$\sum_i \langle \varphi x_i, x_i \rangle = \text{tr}(\varphi).$$

Proof. Let $\{x_i\}$ be any orthonormal basis of X , and A the matrix representation of φ with respect to $\{x_i\}$. In other words, for any i , $\varphi x_i = \sum_j A(i, j) x_j$. Since $\{x_i\}$ is orthonormal, it follows that $\langle \varphi x_i, x_j \rangle = A(i, j)$ for any i and j . Thus,

$$\sum_i \langle \varphi x_i, x_i \rangle = \sum_i A(i, i) = \text{tr}(A) = \text{tr}(\varphi). \quad \square$$

Proof of Theorem 8.2. Recall from **Proposition 6.6** that the effective resistance operator $\mathcal{T}_p = \partial_p^*(\partial_p^*)^+$ is the orthogonal projection of $C_p(K)$ onto $\text{im}(\partial_p^*)$. As an idempotent operator, $\mathcal{T}_p$ only has eigenvalues 0 or 1; consequently, the sum of its eigenvalues equals the rank of $\mathcal{T}_p$. From this, we deduce

$$\sum_i \lambda_i(\mathcal{T}_p) = \text{rank}(\mathcal{T}_p) = \text{rank}(\partial_p^*) = n_{p-1} - \text{null}(\partial_p^*),$$

establishing **Theorem 8.2**.

For the second statement, let $\{\alpha_i\}$ be any orthonormal basis of $C_p(K)$. By **Lemma 8.3**, the trace of an operator equals the sum of the inner products $\langle \mathcal{T}_p \alpha_i, \alpha_i \rangle$ over this basis, which coincides with the sum of its eigenvalues. Combining this with the previous argument yields the desired result. $\square$

Corollary 8.4 (Higher-dimensional Foster's theorem: matrix form). Let K be a finite weighted simplicial complex and fix a degree $p \geq 0$. Let B_p be the p -th boundary matrix. For any p -simplex σ , let $w(\sigma)$ and r_σ be the weight and effective resistance of σ , respectively. Then,

$$\sum_{\sigma \in K_p} w(\sigma) r_\sigma = \text{rank}(B_p^\top) = n_{p-1} - \text{null}(B_p^\top).$$

Proof. By **Remark 6.3**, the quantity $w(\sigma) r_\sigma$ equals the effective resistance of the normalized chain $\sqrt{w(\sigma)} \sigma$. Since the set $\{\sqrt{w(\sigma)} \sigma\}_{\sigma \in K_p}$ forms an orthonormal basis of $C_p(K)$, applying **Theorem 8.2** yields

$$\sum_{\sigma \in K_p} w(\sigma) r_\sigma = \sum_{\sigma \in K_p} \langle \mathcal{T}_p(\sqrt{w(\sigma)} \sigma), \sqrt{w(\sigma)} \sigma \rangle = n_{p-1} - \text{null}(\partial_p^*) = n_{p-1} - \text{null}(B_p^\top). \quad \square$$

The graph version of Foster's theorem can be equivalently expressed in terms of the rank and nullity of the incidence matrix B_1 via the rank-nullity theorem:

$$\sum_{e \in E} w(e) r_e = \text{rank}(B_1^\top) = |V| - \text{null}(B_1^\top).$$

Under this reinterpretation, we observe that the classical Foster's theorem arises as a special case of **Corollary 8.4** when the simplicial complex is a graph and $p = 1$.

Remark 8.5. Kook and Lee [36] also provided a higher-dimensional analogue of Foster's theorem in the setting of simplicial networks. Since their notion of effective resistance is a special case of ours (see **Section 6.2**), their result follows as a special case of **Corollary 8.4**. Notably, our version does not require any assumptions on the homology of the simplicial complex in dimension $d - 1$, and it applies to simplices in all dimensions, not just those of maximal dimension.

9 Conclusion

In this paper, we introduce a basis-free definition of effective resistance on the space of (co)chains of a simplicial complex via the effective resistance bilinear form, and prove that previous matrix formulations of effective resistance on simplicial complexes can be formulated within this framework with appropriate choices of basis. This formulation also satisfies an additional desirable property: it defines a pseudometric on the space of chains and a metric on the space of cycles, extending the classical result that effective resistance is a metric on graphs. We further establish a generalized Foster’s theorem, relating the eigenvalues of the effective resistance operator to the rank of the adjoint boundary operator, and demonstrate its equivalence to earlier matrix-based formulations.

Looking ahead, we envision that this basis-free formulation of effective resistance may serve as a foundation for future work on analysing the structural organization of simplicial complexes and developing machine learning methods defined on simplicial domains, where resistance could inform notions of similarity, diffusion, or label propagation across higher-order structures.

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A List of Notations

Adjoint and dual of linear operators	
U, V	vector spaces over a field $\mathbb{k}$ (typically, $\mathbb{k} = \mathbb{R}$)
$U^\vee$	dual vector space
$f: U \rightarrow V$	a linear operator
$f^*: V \rightarrow U$	adjoint of a linear operator
$f^\vee: V^\vee \rightarrow U^\vee$	dual of a linear operator
$\mathcal{B}_U, \mathcal{B}_V$	Bases of U and V respectively
$[f]_{\mathcal{B}_U}^{\mathcal{B}_V}$	matrix representation of f w.r.t. bases $\mathcal{B}_U$ and $\mathcal{B}_V$
$\varphi: U \times V \rightarrow \mathbb{k}$	a bilinear form
$[\varphi]_{\mathcal{B}_U, \mathcal{B}_V}$	matrix representation of φ w.r.t. bases $\mathcal{B}_U, \mathcal{B}_V$ and $\{1\}$
Simplicial homology	
K	a d -dimensional simplicial complex
K_p	p -simplices in K
n_p	number of p -simplices in K
$w_p: K_p \rightarrow \mathbb{R}_{>0}$	weight function on the p -simplices of K
W_p	diagonal weight matrix
$C_p(K; \mathbb{k})$	p -chain group of K with coefficients in $\mathbb{k}$
$C^p(K; \mathbb{k})$	p -cochain group of K with coefficients in $\mathbb{k}$
$\partial_p: C_p(K, \mathbb{k}) \rightarrow C_{p-1}(K, \mathbb{k})$	boundary operators
$\delta_p: C^p(K, \mathbb{k}) \rightarrow C^{p+1}(K, \mathbb{k})$	coboundary operators
$\mathcal{B}_p$	standard basis in the space of chains
$\mathcal{B}^p$	standard basis in the space of cochains
$\tilde{\mathcal{B}}_p$	orthonormal basis in the space chains
$\tilde{\mathcal{B}}^p$	orthonormal basis in the space of cochains
$\langle \cdot, \cdot \rangle_{C_p}$	inner product endowed on the space of chains
$\langle \cdot, \cdot \rangle_{C^p}$	inner product endowed on the space of cochains
$\flat_p: C_p(K) \rightarrow C^p(K)$	flat isomorphism
$\sharp_p: C^p(K) \rightarrow C_p(K)$	sharp isomorphism
Laplacian operators	
$\mathcal{L}_{\text{up}}^p: C^p(K) \rightarrow C^p(K)$	up Laplacian operator on cochain spaces
$\mathcal{L}_{\text{down}}^p: C^p(K) \rightarrow C^p(K)$	down Laplacian operator on cochain spaces
$\mathcal{L}_{\text{up}}^p: C_p(K) \rightarrow C_p(K)$	up Laplacian operator on chain spaces
$\mathcal{L}_{\text{down}}^p: C_p(K) \rightarrow C_p(K)$	down Laplacian operator on chain spaces
Effective resistance bilinear forms	
$\mathcal{R}_p: C_p(K) \times C_p(K) \rightarrow \mathbb{R}$	effective resistance bilinear form on chain spaces
$\mathcal{R}^p: C^p(K) \times C^p(K) \rightarrow \mathbb{R}$	effective resistance bilinear form on cochain spaces

B Pseudoinverse Operators

We review the concept of the *pseudoinverse* of a bounded linear operator between Hilbert spaces [14]. This notion first appeared in the work of Moore [39] and Penrose [44, 45], formulated in terms of matrices, thus it is called the *Moore–Penrose inverse* or *pseudoinverse matrix*. As we will see later

in this section, the latter is a particular case of the former, more general definition.

We restrict our attention to finite-dimensional Hilbert spaces, where all linear operators are bounded and have a closed range. Throughout this section, $\varphi : X \rightarrow Y$ denotes a linear operator from a Hilbert space $(X, \langle \cdot, \cdot \rangle_X)$ to a Hilbert space $(Y, \langle \cdot, \cdot \rangle_Y)$. The *adjoint operator* $\varphi^* : Y \rightarrow X$ is defined as the operator satisfying, for all $x \in X$ and $y \in Y$,

$$\langle \varphi(x), y \rangle_Y = \langle x, \varphi^*(y) \rangle_X.$$

The adjoint operator uniquely exists (see [50, page 249]). We recall the following property of the adjoint operator (see [37, Theorem 5.1]):

$$\text{im}(\varphi^*) = \text{im}(\varphi^* \varphi) \quad \text{and} \quad \ker(\varphi^*) = \ker(\varphi \varphi^*). \quad (57)$$

Definition B.1 (Pseudoinverse operator, [14]). Let $\varphi : X \rightarrow Y$ be a linear operator from a Hilbert space $(X, \langle \cdot, \cdot \rangle_X)$ to a Hilbert space $(Y, \langle \cdot, \cdot \rangle_Y)$. The map $\varphi^+ : Y \rightarrow X$ is said to be a *pseudoinverse* of φ if it satisfies the following three conditions:

1. $\varphi^+ \varphi(x) = x$ for all $x \in \ker(\varphi)^\perp = \text{im}(\varphi^*)$;
2. $\varphi^+(y) = 0$ for all $y \in \text{im}(\varphi)^\perp = \ker(\varphi^*)$;
3. If $y_1 \in \text{im}(\varphi)$ and $y_2 \in \ker(\varphi^*)$, then $\varphi^+(y_1 + y_2) = \varphi^+ y_1 + \varphi^+ y_2$.

Remark B.2 (Alternative definition). Let $\varphi : X \rightarrow Y$ be a bounded linear operator between Hilbert spaces. The *Moore–Penrose pseudoinverse* of φ , denoted $\varphi^+ : Y \rightarrow X$, is the unique bounded linear operator satisfying:

$$\varphi \varphi^+ \varphi = \varphi, \quad \varphi^+ \varphi \varphi^+ = \varphi^+, \quad (\varphi \varphi^+)^* = \varphi \varphi^+, \quad (\varphi^+ \varphi)^* = \varphi^+ \varphi.$$

Remark B.3 (Uniqueness of the pseudoinverse). According to **Definition B.1**, Condition 1 defines φ^+ on $\text{im}(\varphi)$ and Condition 2 on $\text{im}(\varphi)^\perp$. Uniqueness of the pseudoinverse of a linear operator then follows from the decomposition $Y = \overline{\text{im}(\varphi)} \oplus \overline{\text{im}(\varphi)}^\perp$ and the fact that linear operators on finite-dimensional Hilbert spaces are bounded and have a closed range.¹

Remark B.4 (Existence of the pseudoinverse). According to [14, Theorem 1], every operator φ admits a pseudoinverse given by

$$\varphi^+ = (\varphi^* \varphi)^+ \varphi^*. \quad (58)$$

This expression assumes the existence of a pseudoinverse for $\varphi^* \varphi$, which holds because $\varphi^* \varphi$ (as well as any self-adjoint operator) admits a pseudoinverse that can be constructed explicitly from the operator's spectral decomposition, as outlined below.

If ϕ is self-adjoint, it admits a spectral representation

$$\phi = \sum \lambda_i \phi_{E_i},$$

where the sum is taken over all non-zero eigenvalues λ_i of ϕ , and ϕ_{E_i} denotes the orthogonal projection onto the corresponding eigenspace E_i . Then, a direct computation shows that setting

$$\phi^+ = \sum \lambda_i^{-1} \phi_{E_i} \quad (59)$$

satisfies all the defining conditions of a pseudoinverse of ϕ .

¹These conditions also ensure the uniqueness of the pseudoinverse for linear operators on infinite-dimensional spaces; see [50] for more details.

Directly from the definition of the pseudoinverse, we derive the following properties. Recall that a linear operator ϕ is an *orthogonal projection* onto some subspace V if and only if it is self-adjoint ($\phi^* = \phi$), idempotent ($\phi^2 = \phi$) and has V as its image.

Proposition B.5 (Corollaries 1, 2, and 3 from [14]). For $\varphi : X \rightarrow Y$ a bounded linear operator and φ^+ its pseudoinverse operator, we have:

- (1) $\text{im}(\varphi^+) = \text{im}(\varphi^*) = \ker(\varphi)^\perp$;
- (2) $\ker(\varphi^+) = \ker(\varphi^*) = \text{im}(\varphi)^\perp$;
- (3) $(\varphi^+)^+ = \varphi$;
- (4) $\varphi^+ = \varphi^+ \varphi \varphi^+$ and $\varphi = \varphi \varphi^+ \varphi$;
- (5) $\varphi^+ \varphi$ is the orthogonal projection of X onto $\ker(\varphi)^\perp = \text{im}(\varphi^*)$. Similarly, $\varphi \varphi^+$ is the orthogonal projection of Y onto $\ker(\varphi^+)^\perp$. In particular, $\varphi^+ \varphi$ and $\varphi \varphi^+$ are self-adjoint and idempotent:
 - $(\varphi^+ \varphi)^* = \varphi^+ \varphi$ and $(\varphi \varphi^+)^* = \varphi \varphi^+$;
 - $(\varphi \varphi^+)^2 = \varphi \varphi^+$ and $(\varphi^+ \varphi)^2 = \varphi^+ \varphi$.

Proposition B.6. Let $\varphi : X \rightarrow X$ be a bounded linear operator and $\psi : X \rightarrow Y$ an isomorphism that preserves inner products. Then

$$(\psi \varphi \psi^{-1})^+ = \psi \varphi^+ \psi^{-1}.$$

Proof. Let $\Phi := \psi \varphi \psi^{-1}$ and define $\Phi' := \psi \varphi^+ \psi^{-1}$. We will verify that Φ' satisfies the three conditions of Definition B.1 for the pseudoinverse of Φ . First, we have

$$\Phi' \Phi = \psi \varphi^+ \psi^{-1} \psi \varphi \psi^{-1} = \psi \varphi^+ \varphi \psi^{-1}.$$

Take $x \in \ker(\varphi)^\perp = \text{im}(\varphi^*)$ and let $y = \psi(x)$. Then, because $\varphi^+ \varphi$ is an orthogonal projection onto $\ker(\varphi)^\perp \ni x$, we have

$$\Phi' \Phi(y) = \psi \varphi^+ \varphi \psi^{-1} \psi(x) = \psi(\varphi^+ \varphi(x)) = \psi(x) = y,$$

so $\Phi' \Phi(y) = y$ for all $y \in \ker(\Phi)^\perp = \text{im}(\Phi^*)$. Hence, condition (1) is satisfied.

Take $y \in \text{im}(\Phi)^\perp = \ker(\Phi^*)$ and let $x = \psi^{-1}(y)$. Then we have $y = \psi(x)$ and

$$\varphi^*(x) = \varphi^* \psi^{-1}(y) = \psi^{-1} \psi \varphi^* \psi^{-1}(y) = \psi^{-1} \Phi^*(y) = 0 \implies x \in \ker(\varphi^*) = \ker(\varphi^+).$$

Then

$$\Phi'(y) = \psi \varphi^+ \psi^{-1}(y) = \psi \varphi^+(x) = \psi(0) = 0.$$

Thus, condition (2) is satisfied.

Let $y_1 \in \text{im}(\Phi)$ and $y_2 \in \ker(\Phi^*)$. Then $y_1 = \psi(x_1)$ for $x_1 \in \text{im}(\varphi)$ and $y_2 = \psi(x_2)$ for $x_2 \in \ker(\varphi^*)$. Then

$$\Phi'(y_1 + y_2) = \psi \varphi^+ \psi^{-1}(y_1 + y_2) = \psi \varphi^+(x_1 + x_2) = \psi(\varphi^+(x_1) + \varphi^+(x_2)).$$

By the definition of pseudoinverse, $\varphi^+(x_2) = 0$, and φ^+ is linear on $\text{im}(\varphi)$. So

$$\Phi'(y_1 + y_2) = \psi \varphi^+(x_1) = \psi \varphi^+(x_1) + \psi \varphi^+(x_2) = \Phi'(y_1) + \Phi'(y_2).$$

Hence, condition (3) is satisfied.

Therefore, $\Phi' = \psi\varphi^+\psi^{-1}$ is the pseudoinverse of $\Phi = \psi\varphi\psi^{-1}$.

Alternatively, we verify that Φ' satisfies the four Moore–Penrose conditions from **Remark B.2**:

$$\begin{aligned}\Phi\Phi'\Phi &= \psi\varphi\psi^{-1}\psi\varphi^+\psi^{-1}\psi\varphi\psi^{-1} = \psi\varphi\varphi^+\varphi\psi^{-1} = \psi\varphi\psi^{-1} = \Phi, \\ \Phi'\Phi\Phi' &= \psi\varphi^+\varphi\varphi^+\psi^{-1} = \psi\varphi^+\psi^{-1} = \Phi', \\ (\Phi\Phi')^* &= (\psi\varphi\varphi^+\psi^{-1})^* = \psi\varphi\varphi^+\psi^{-1} = \Phi\Phi', \\ (\Phi'\Phi)^* &= (\psi\varphi^+\varphi\psi^{-1})^* = \psi\varphi^+\varphi\psi^{-1} = \Phi'\Phi.\end{aligned}$$

Thus, Φ' satisfies the Moore–Penrose pseudoinverse conditions as well. $\square$

The pseudoinverse of an operator generalizes the concept of the matrix pseudoinverse. Below, we review the definition of the matrix pseudoinverse and explain the connection.

Definition B.7 (Matrix pseudoinverse). For a matrix $A \in \mathbb{R}^{m \times n}$, the *Moore–Penrose inverse* of A is defined as a matrix $A^+ \in \mathbb{R}^{n \times m}$ satisfying four criteria:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (AA^+)^{\top} = AA^+, \quad (A^+A)^{\top} = A^+A, \quad (60)$$

In particular, AA^+ and A^+A are symmetric.

Criteria from **Equation (60)** are equivalent to the properties in **Proposition B.5** (4) and (5). Penrose [44] proved that **Equation (60)** has a unique solution which he defined as the pseudoinverse. The *Gram matrix* of a basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ of a finite-dimensional Hilbert space $(X, \langle \cdot, \cdot \rangle_X)$ is the matrix $G_X \in \mathbb{R}^{n \times n}$ defined by $G_X(i, j) = \langle b_i, b_j \rangle_X$.

Proposition B.8. Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be orthogonal bases of Hilbert spaces X and Y with Gram matrices G_X and G_Y , respectively. Let $\varphi : X \rightarrow Y$ be a linear operator with matrix representation A in these bases. Then the matrix representation of the pseudoinverse φ^+ is given by

$$[\varphi^+]_{\mathcal{B}_Y}^{\mathcal{B}_X} = G_X^{-1/2} \left(G_Y^{1/2} A G_X^{-1/2} \right)^+ G_Y^{1/2}. \quad (61)$$

Proof. When $\mathcal{B}_X$ and $\mathcal{B}_Y$ are orthonormal, $G_X = G_Y = I$, reducing **Equation (61)** to $[\varphi^+]_{\mathcal{B}_Y}^{\mathcal{B}_X} = A^+$, which holds since A^+ induces a linear operator satisfying the pseudoinverse conditions for φ .

For general orthogonal bases, let $\tilde{\mathcal{B}}_X = \{\tilde{x}_i\}$ and $\tilde{\mathcal{B}}_Y = \{\tilde{y}_j\}$ be the normalized (orthonormal) bases defined by $\tilde{x}_i = x_i / \|x_i\|$ and $\tilde{y}_j = y_j / \|y_j\|$. The matrix representation of φ in these normalized bases is $[\varphi]_{\tilde{\mathcal{B}}_Y}^{\tilde{\mathcal{B}}_X} = G_Y^{1/2} A G_X^{-1/2}$, and by the orthonormal case, the pseudoinverse φ^+ has the matrix representation

$$[\varphi^+]_{\tilde{\mathcal{B}}_Y}^{\tilde{\mathcal{B}}_X} = \left(G_Y^{1/2} A G_X^{-1/2} \right)^+.$$

Changing back to the original (orthogonal) bases yields

$$[\varphi^+]_{\mathcal{B}_Y}^{\mathcal{B}_X} = G_X^{-1/2} \left(G_Y^{1/2} A G_X^{-1/2} \right)^+ G_Y^{1/2},$$

as claimed. $\square$

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