A POLYNOMIAL IMPROVEMENT FOR THE ODD CYCLE–COMPLETE RAMSEY NUMBERS

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Abstract. We give a polynomial improvement to the cycle-complete Ramsey numbers

$$r(C_\ell, K_k) \geqslant k^{1+\frac{1}{\ell-2}+\varepsilon_\ell+o(1)},$$

for all fixed odd $\ell > 7$ with $k \to \infty$, for some $\varepsilon_\ell > 0$.

1. Introduction

The *cycle-complete* Ramsey number $r(C_\ell, K_k)$ is the smallest n so that every graph on n vertices has either a cycle of length ℓ or an independent set of size k. Here our interest lies in the case when ℓ is fixed and odd and k tends to infinity. The study of these numbers goes back to the foundational work of Erdős [6] who proved

$$c_\ell k^{1+\frac{1}{2\ell}} \leqslant r(C_\ell, K_k) \leqslant k^{1+\frac{2}{\ell-1}}, \tag{1}$$

where the upper bound holds for all odd $\ell > 1$ (he proved a similar result when ℓ is even) and the lower bound was proved using an early application of the probabilistic method. The lower bound was then improved by Spencer [13] in 1977 who used the Lovász local lemma to show

$$r(C_\ell, K_k) \geqslant k^{1+\frac{1}{\ell-2}+o(1)}. \tag{2}$$

While this bound has been improved by logarithmic factors by Bohman and Keevash [1] and by Verstraëte and Mubayi [11], the exponent in (2) has stood as the best known bound on these numbers, apart from the special cases $\ell \in \{5, 7\}$, which we discuss below. In this paper we give the first polynomial improvement to this bound for all odd $\ell > 7$.

Theorem 1.1. *For all odd $\ell > 7$, we have that*

$$r(C_\ell, K_k) \geqslant k^{1+\frac{1}{\ell-2}+\varepsilon_\ell+o(1)},$$

where $\varepsilon_\ell = \Theta(\ell^{-2})$.

In fact we can take $\varepsilon_\ell = ((\ell-2)(2\ell-5))^{-1}$ in the statement of our main theorem.

What is perhaps most interesting about this result is that the bound (2) represents a natural barrier for both this and many other problems in combinatorics. Indeed, a simple

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way to construct a C_ℓ-free graph with small independence number is to sample $G \sim G(n,p)$ and then delete an edge from each instance of C_ℓ. Here one needs to choose p so that

$$p^\ell n^\ell \ll pn^2,$$

to ensure that only a small fraction of the edges of G are removed. This is known as *the deletion threshold* for the cycle C_ℓ and the bound at (2) reflects the independence number of $G(n,p)$ precisely at this deletion threshold. In many cases in combinatorics we know little about how to obtain good examples that are significantly beyond this threshold.

In this paper we show that we can push well beyond this deletion threshold for the cycle complete Ramsey numbers, using a new idea of randomly superimposing blow-ups of random graphs. This idea is due to Hefty, Horn, King and Pfender [7], which was inspired by a previous idea of Campos, Jenssen, Michelen and Sahasrabudhe [3]. In [7], this yields the lower bound on the extreme off diagonal Ramsey numbers $R(3,k)$

$$R(3,k) \geqslant \left(\frac{1}{2} + o(1)\right)\frac{k^2}{\log k},$$

which is within a factor of $2 + o(1)$ of the best upper bound due to Shearer [12], and is conjectured to be sharp [3].

It seems reasonable to guess, for odd ℓ, that Erdős's bound (1) gives the correct exponent,

$$r(C_\ell, K_k) = k^{1+\frac{2}{\ell-1}+o(1)},$$

which has been tentatively conjectured and supported by some recent conditional results [5]. It appears that the construction in this paper cannot be directly used to obtain an improvement to the lower bound beyond a factor of $n^{\varepsilon_\ell + o(1)}$ where $\varepsilon_\ell = \Theta(\ell^{-2})$.

1.1. **A brief history of the problem.** Before we turn to give the short proof of our main theorem, we briefly elaborate on the history of the problem. While the upper and lower bounds stated above are the best known for general odd ℓ, up to lower order factors, there have been several interesting advances on the lower order terms. The best known upper bound is due to Sudakov [14] and, independently, Li and Zang [9] who proved

$$r(C_\ell, K_k) \leqslant c_\ell k^{1+\frac{2}{\ell-1}} (\log k)^{-\frac{2}{\ell-1}}$$

for all odd $\ell > 2$. The lower bounds were also improved by poly-logarithmic factors, first in the influential work of Bohman and Keevash [1] on the H-free process and then, more recently, by Mubayi and Verstraëte [11] via a new method which established an important connection between lower bounds for the Ramsey numbers and spectral expanders.

There have also been two recent *polynomial* improvements to the cycle-complete Ramsey numbers when $\ell \in \{5,7\}$. Mubayi and Verstraëte [11] were the first to break the deletion threshold barrier for C_5 and C_7 by showing

$$r(C_5, K_k) \geqslant (1+o(1))k^{11/8} \qquad \text{and} \qquad r(C_7, K_k) \geqslant (1+o(1))k^{11/9},$$

using an ingenious construction based on incidence graphs of generalized hexagons. These bounds were then further improved by Conlon, Mattheus, Mubayi and Verstraëte [5] who showed that

$$r(C_5, K_k) \geqslant k^{10/7-o(1)} \qquad \text{and} \qquad r(C_7, K_k) \geqslant k^{5/4-o(1)}\,,$$

by leveraging ideas from the recent breakthrough of Mattheus and Verstraëte on the Ramsey numbers $R(4, k)$ [10].

In the case of even ℓ, there are no known examples that beat the deletion threshold by a polynomial factor. Again [1] and [11] supply us with poly-logarithmic improvements to the lower bound. For the upper bound, Erdős proved in the 1950s that

$$r(C_\ell, K_k) \leqslant c_\ell n^{1+\frac{2}{\ell-2}}\,,$$

which has only been improved by polylogarithmic factors by Caro, Li, Rousseau, and Zhang [4]. For even ℓ it is much less clear if Erdős's upper bound is sharp. Indeed, it is a beautiful conjecture of Erdős that

$$r(C_4, K_k) \leqslant k^{2-\varepsilon+o(1)},$$

for some $\varepsilon > 0$.

Finally we mention that when k is fixed and $\ell \to \infty$ the Ramsey numbers $r(C_\ell, K_k)$ have also received quite a bit of attention. As this regime enjoys a somewhat different behavior from what we see for fixed ℓ, we defer to the paper of Keevash, Long and Skokan [8].

2. The Construction

Let $\ell \geqslant 5$ be an odd integer. Define p_c to be the *deletion threshold* for C_ℓ in $G(n, p)$: the value for which $p_c^\ell n^\ell = p_c n^2$. That is,

$$p_c = n^{-1+\frac{1}{\ell-1}}.$$

We now define our parameters

$$\varepsilon = \varepsilon_\ell = ((2\ell-3)(\ell-1))^{-1}, \qquad p = p_c n^{\varepsilon}(\log n)^{-1}, \qquad \text{and} \qquad r = (pn)^{1/2}(\log n)^{-3/2}. \quad (3)$$

We construct a C_ℓ-free graph with density $\approx 2p$ with no independent sets of size k, where

$$k = 8p^{-1}(\log n)^3.$$

For future reference, we note that our parameters satisfy

$$p^{\ell-1}n^{\ell-2}/r \ll (\log n)^{-2} \qquad \text{and} \qquad r \leqslant 4\sqrt{n/k}. \quad (4)$$

We now let

$$[n] = V_1 \cup \cdots \cup V_{n/r} = W_1 \cup \cdots \cup W_{n/r}$$

be two independent random partitions of $[n]$ into sets of size r. We call these the *red* and *blue* partitions of $[n]$ respectively (not to be confused with the red and blue colours of a Ramsey colouring). We now define G to be a random graph defined in the following steps. We first independently sample two graphs

$$G_R^\circ, G_B^\circ \sim G(n/r, p).$$

Then let G'_R be the r-blow up of G°_R, with parts $V_1, \dots, V_{n/r}$ and let G'_B be the r-blow up of G°_B, with parts $W_1, \dots, W_{n/r}$. Then define

$$G' = G'_R \cup G'_B.$$

We refer to G'_R as the red graph and G'_B as the blue graph.

To form $G = G_R \cup G_B$ we do two steps of clean up. In a first step we delete $o(n)$ vertices to remove cycles that cross between a pair V_i, V_j or W_i, W_j more than once. Then, in a second, and more important step, we remove edges from all remaining copies of C_ℓ.

For this, we make use of two definitions. We first say that a graph $H \subset [n]^{(2)}$ is *simple* if

$$|H \cap (V_i \times V_j)| \leqslant 1 \qquad \text{and} \qquad |H \cap (W_i \times W_j)| \leqslant 1$$

for all $i < j$. Observe that if H is simple

$$\mathbb{P}(H \subset G') \leqslant (2p)^{e(H)}. \tag{5}$$

We now turn to describe the vertex deletion step.

2.1. **Vertex deletion step.** Again we need a bit of terminology. For $x_1, \dots, x_t \in [n]$, we say that $B = x_1 \dots x_t$ is a broken cycle of length t, if for each i either $x_i x_{i+1} \in G'$ or $x_i, x_{i+1} \in V_j$ or $x_i, x_{i+1} \in W_j$ for some j (subscripts to be understood modulo t). If $x_i x_{i+1} \notin G'$ we call it a broken edge. If $x_i x_{i+1} \in G'$ we call it an actual edge. We say that B is simple if the actual edges of B form a simple graph.

We can now describe our vertex deletion step: let $\widehat{G}$ denote the graph obtained by deleting all the vertices of each simple broken cycle in G' of length $\leqslant \ell - 1$ with an odd number of actual edges. We prove that this deletion removes all non-simple copies of C_ℓ.

Lemma 2.1. *$\widehat{G}$ contains only simple copies of C_ℓ.*

We also need to show that this vertex deletion leaves most of the graph intact.

Lemma 2.2. *$\widehat{G}$ has at least $n/2$ vertices, with high probability.*

Before turning to the proofs of these lemmas, we record a useful observation. For this, note that if $C \subset G'$ is not simple, then it contains two edges e, f both contained in $V_i \times V_j$ or $W_i \times W_j$ for some $i < j$. If e and f are incident then we call the path of length two they form a *kink*.

Observation 2.3. *If B is a broken cycle with an odd number of actual edges and no kinks, then there is a simple broken cycle with an odd number of actual edges whose vertex set is contained in $V(B)$.*

Proof. We prove the claim by applying induction on the number of actual edges in B. If there is one such edge then B is simple and we are done. So assume B contains at least three actual edges. Now, if B is not simple then there are distinct $e = x_1 y_1$, $f = x_2 y_2$ so that

$$e, f \in B \cap (V_i \times V_j),$$

for some $i < j$, or similarly for $W_i \times W_j$ in place of $V_i \times V_j$. Since B contains no kinks, e, f must be disjoint and thus we may assume $x_1, x_2 \in V_i$ and $y_1, y_2 \in V_j$ for distinct vertices.

Now, let P denote the (broken) path from x_1 to x_2 in B that does not contain y_1 and let Q denote the path from y_1 to y_2 in B that does not contain x_2. Observe that one of these two paths must contain an odd number of actual edges. Without loss, we may assume that this path is P. We then consider the broken cycle $P + x_1x_2$, which has no kinks and an odd number of actual edges which is strictly fewer than B. We thus apply induction to $P + x_1x_2$. □

We now turn to prove Lemma 2.1.

Proof of Lemma 2.1. Assume $C \subset G'$ is a C_ℓ that is not simple. Now let C' be broken cycle with no kinks which is obtained by removing kinks from C, in particular, by iteratively replacing a kink x_1y, yx_2 with the broken edge x_1x_2.

Note that $V(C') \subset V(C)$ and that C' has an odd number of actual edges, since each step removes two actual edges and introduces one broken edge. Also note that C' contains no kinks. Thus by Observation 2.3 there is a simple broken cycle B on vertex set $V(C') \subset V(C)$ with an odd number of actual edges. Since we must have deleted a vertex from $V(B)$ to form $\widehat{G}$, we conclude that $C \not\subset \widehat{G}$, as desired. □

We now prove Lemma 2.2.

Proof of Lemma 2.2. We show the expected number of simple broken cycles with an odd number of actual edges is at most $o(n)$ and then apply Markov's inequality to finish. We claim that the expected number of simple broken cycles in G' of length $1 \leqslant t \leqslant \ell - 1$ with a actual edges and $t - a$ broken edges is at most

$$2^\ell (2r)^{t-a} n^a (2p)^a \,.$$

Indeed, if we write the cycle $x_1 \dots x_t$, there are $\leqslant 2^\ell$ ways of specifying which edges are broken and which are not. There are then n choices for x_1 and $\leqslant 2r$ choices for x_2 if x_1x_2 is broken and $\leqslant n$ choices for x_2 otherwise. It follows that there are $\leqslant 2^\ell(2r)^{t-a}n^a$ choices for $x_1, \dots, x_t$. Since the cycle is simple, the probability that the a actual edges appear in G' is at most $(2p)^a$ from (5). Finally we note that if a is odd, then

$$r^{t-a} n^a p^a \leqslant r(np)^{\ell-2} = o(n) \,,$$

as desired. □

2.2. **Edge deletion step.** We now turn to describe the edge deletion step which we perform on the graph $\widehat{G}$ to arrive at our final graph G. The definition of the deletion rule is somewhat subtle and might appear strange at first, but it will make our later computations much easier.

To prepare the ground for this, we order all the edges of K_n in two ways, giving them a *red* ordering and a *blue* ordering. These orderings are arbitrary, apart from the property that in the red ordering, the set of edges in $V_i \times V_j$ are consecutive, for all $i < j$. Likewise, in the blue ordering the edges of $W_i \times W_j$ are consecutive.

Now recall that G' and therefore $\widehat{G}$ has an order on the vertex set. Let $C \subset G'$ be a copy of C_ℓ. We now crucially note that since C is an odd cycle there exists a vertex v where both incident edges in C have the same colour. Call the largest such vertex $x \in V(C)$ the *apex* of C and call the edges incident with x in C *apex* edges.

We now remove cycles of length ℓ from $\widehat{G}$ as follows. For each C_ℓ in $\widehat{G}$, consider the apex vertex. If the cycle is not monochromatic and the apex is adjacent to two red edges (respectively two blue edges), then delete the largest blue edge (respectively red edge) in the cycle where we use the blue ordering (respectively red ordering). If the cycle is monochromatic, then delete the largest edge in the respective ordering.

We define our final graph G to be the graph we arrive at after this edge deletion step. Our main theorem is that this graph G is the desired graph with high probability.

Theorem 2.4. *For $\ell \geqslant 5$ and odd, G is a C_ℓ-free graph on at least $n/2$ vertices, with*

$$\alpha(G) \leqslant 8p^{-1}(\log n)^3,$$

with high probability, where $p = p_\ell$ is defined at (3).

We see that Theorem 2.4 implies Theorem 1.1 by taking n to be sufficiently large and recalling the definition of p.

3. Basic Pseudo-random properties

In this section we define a pseudo-randomness property $\mathcal{A}$ for the graphs G_R°, G_B°, G' and show that $\mathcal{A}$ holds with high probability. The first property we include into $\mathcal{A}$ is basic control on the degrees. For all $x \in V(G_R^\circ)$ we have

$$d_{G_R^\circ}(x) = (1 + O(n^{-c}))p(n/r), \tag{6}$$

where $c = c_\ell > 0$ and likewise for G_B°. The second property we include into $\mathcal{A}$ is the following spectral pseudo-randomness property for the pre-blown up graphs G_R° and G_B°. We let A_R, A_B be the adjacency matrices of G_R°, G_B° respectively. We include the event

$$\|A_R - pJ_{n/r}\|_{op} \leqslant 3\sqrt{pn/r} \tag{7}$$

and likewise for A_B. Here J_t denotes the t by t matrix of all ones.

We also need a pseudo-randomness property that captures the typical interaction of the red and blue partitions $\{V_i\}_i$ and $\{W_i\}_i$. For a set $S \subset [n]$, we define the "projections" of S into the red and blue partitions by

$$\pi_R(S) = \big\{i : S \cap V_i \neq \emptyset\big\} \qquad \text{and} \qquad \pi_B(S) = \big\{i : S \cap W_i \neq \emptyset\big\}.$$

We include into the event $\mathcal{A}$ the following. For every $t \leqslant k$ and $S \in [n]^{(t)}$ we have

$$\max\big\{|\pi_R(S)|, |\pi_B(S)|\big\} \geqslant \delta t\,, \tag{8}$$

where we set

$$\delta = (\log n)^{-1}, \tag{9}$$

for use throughout the paper. On a technical note, we require that there are not too many edges incident to a vertex v that are in both G'_R and G'_B. Here we include into $\mathcal{A}$ the weak bound

$$\max_{v \in [n]} \big|N_{G'_R}(v) \cap N_{G'_B}(v)\big| \leqslant n^{1-\eta}p, \tag{10}$$

for some fixed $\eta = \eta_\ell > 0$. Finally, we need the graph G' to have good vertex expansion properties. For a graph H and set of vertices $S \subset V(H)$ define

$$\partial_H(S) = \big\{v \in V(H) \setminus S : N_H(v) \cap S \neq \emptyset\big\}$$

to be the vertex boundary of S. We include into the event $\mathcal{A}$ the property that every set $S \subset [n]$ with $|S| \leqslant (2p)^{-1}$ satisfies

$$|\partial_{G'}(S)| \geqslant \delta pn|S|/8\,. \tag{11}$$

We now turn to show that properties (6),(7),(8),(10) and (11) hold with high probability. The only elements that are not standard, or follow from well-known results are (8) and (11). We isolate these here.

Observation 3.1. *The event* (8) *holds with high probability.*

Proof. For $t \leqslant k$, fix $S \in [n]^{(t)}$ and consider the probability that $|\pi_R(S)| < \delta t$. By symmetry, this is the same as fixing the partition and choosing $S \in [n]^{(t)}$ uniformly at random. Thus

$$\mathbb{P}\big(|\pi_R(S)| < \delta t\big) \leqslant t\binom{n/r}{\delta t}\binom{r\delta t}{t}\binom{n}{t}^{-1},$$

since there are n/r parts of which we choose $< \delta t$ for the red projection. We then choose S from the union of these parts, which has size $< r(\delta t)$.

By independence of the red and blue partitions we have

$$\mathbb{P}\big(|\pi_R(S)| < \delta t \,\wedge\, |\pi_B(S)| < \delta t\big) \leqslant \left[t\binom{n/r}{\delta t}\binom{r\delta t}{t}\binom{n}{t}^{-1}\right]^2 \leqslant \left(\frac{C\delta^2 r^2}{(n/t)}\right)^t\binom{n}{t}^{-1},$$

for some constant $C > 0$. We note that, by our choices of r, k at (3), we have $r \leqslant 4(n/k)^{1/2}$. Thus the above is $\ll 2^{-t}\binom{n}{t}^{-1}$. Thus we may union bound over all S and t to obtain the desired result. $\square$

Observation 3.2. *The event* (11) *holds with high probability.*

Proof. By Observation 3.1 we may assume (8) holds. Let $S \subset [n]$ satisfy $|S| \leqslant (2p)^{-1}$ and let $S_0 \subset S$ be a subset with the property $|\pi_R(S_0)| = |S_0|$ so that S_0 contains exactly one element from each V_i that S intersects. Write $T = \pi_R(S_0) \subset [n/r]$. Without loss, by (8), we have $|T| = |S_0| \geqslant \delta|S|$. Now write $t = |T|$ and note

$$\big|\partial_{G_R^\circ}(T)\big| \sim \mathrm{Bin}\big(n/r - t, 1-(1-p)^t\big),$$

which has mean $\geqslant tpn/(4r)$ since $pt \leqslant 1/2$. By Chernoff and a union bound, we have

$$|\partial_{G_R^\circ}(T)| \geqslant tpn/(8r)$$

for all $T \subset [n/r]$ with $t \leqslant (2p)^{-1}$, with high probability. Thus, with high probability,

$$|\partial_{G'}(S)| \geqslant r|\partial_{G_R^\circ}(T)| \geqslant |S_0|pn/8 \geqslant \delta|S|pn/8\,,$$

where for the first inequality we observe that if $i \in \partial_{G_R^\circ}(T)$ then $V_i \subset \partial_{G'}(S)$. $\square$

We now prove that $\mathcal{A}$ holds with high probability.

Lemma 3.3. *$\mathcal{A}$ holds with high probability.*

Proof. That (6) holds with high probability is standard. A proof can be found in [2]. The fact that (7) holds follows from Theorem 1.4 in [15]. We just saw that (8) holds by Observation 3.1 and (11) holds by Observation 3.2.

To prove (10), we may assume (6) holds in which case we have $|N_{G'_R}(v)| = (1 + o(1))np$ for all v. Fix $S \subset [n]$ and note that

$$|N_{G'_B}(v) \cap S| = \sum_{i=1}^{n/r} |W_i \cap S| \cdot \xi_i$$

where the ξ_i are i.i.d. $\mathrm{Ber}(p)$ random variables. Since $|W_i \cap S| \leqslant r$ for all i, we have $\sum_i |W_i \cap S|^2 \leqslant r|S|$ and so the Azuma-Hoeffding inequality gives

$$\mathbb{P}(|N_{G'_B}(v) \cap S| \geqslant p|S| + t) \leqslant \exp\big(- 2t^2/(r|S|)\big) . \tag{12}$$

Applying (12) with a fixed realisation $S = N_{G'_R}(v)$ with $|S| = (1+o(1))np$ and recalling that $r \leqslant \sqrt{np}$ shows that $|N_{G'_B}(v) \cap N_{G'_R}(v)| \leqslant n^{1-1/(4\ell)}p$ with probability at least $1 - 1/n^2$. A union bound over v shows that (10) holds with high probability with $\eta = 1/(4\ell)$. □

4. The number of J to J walks

When calculating if a given set $I \in [n]^{(k)}$ is independent, we will make use of the event $\mathcal{A}$ and first take a set of representatives $J \subset I$, with $|J| \geqslant \delta k$ so that $|\pi_R(J)| = |J|$ or $|\pi_B(J)| = |J|$. We will then use the following lemma to show there are few pairs in $J^{(2)}$ that close a cycle of length ℓ. Recall from (9) that we set $\delta = (\log n)^{-1}$.

Lemma 4.1. *Let $J \in [n]^{(\delta k)}$. On the event $\mathcal{A}$ the number of paths of length $\ell - 1$, in G', starting and ending in J is at most*

$$\delta^{-2}2^{\ell}p^{\ell-1}n^{\ell-2}|J|^2.$$

To prove this lemma we use a spectral argument, which the reader should not confuse with the "spectral method" of [11]. Recall that A_R, A_B are the adjacency matrices of G°_R and G°_B respectively. Note that the adjacency matrices of G'_B and G'_R can be obtained from A_R, A_B by taking a tensor product with the $r \times r$ all ones matrix J_r. We then permute the coordinates of G'_R. That is if A'_R, A'_B are the adjacency matrices of G'_R, G'_B we have that $A'_R = A_R \otimes J_r$ and $A'_B = A_B \otimes J_r$. We then define

$$A = A_R \otimes J_r + P^t\big(A_B \otimes J_r\big)P,$$

where P is a permutation matrix of a random permutation on $[n]$. Note that this is not quite the adjacency matrix of G' (for instance A has some 2 entries, typically) but it entry-wise dominates the adjacency matrix of G', which is enough for us since we are only looking to upper bound the number of walks.

Observation 4.2. *On the event $\mathcal{A}$ we have*

$$\big\|A_R \otimes J_r - pJ_n\big\|_{op} \leqslant 3\sqrt{rpn}.$$

Proof. From the property (7), on the event $\mathcal{A}$ we have $\|A_R - pJ_{n/r}\|_{op} \leqslant 3\sqrt{pn/r}$. Now write

$$A_R \otimes J_r - pJ_n = (A_R - pJ_{n/r}) \otimes J_r$$

and recall that the eigenvalues of this tensor product are the pairwise products of the eigenvalues of $A_R - pJ_{n/r}$ and J_r. $\square$

We now show that the largest eigenvalue of A is essentially the degree and that its corresponding eigenvector is delocalized.

Lemma 4.3. *Let $v \in \mathbb{R}^n$ with $\|v\|_2 = 1$ be the eigenvector of A corresponding to the largest eigenvalue μ. On event $\mathcal{A}$ we have $\mu = (1 + o(1))2pn$ and $\|v\|_\infty \leqslant \delta^{-1}n^{-1/2}$.*

Proof. By considering the maximum row sum we have that $\mu \leqslant (1 + O(n^{-c}))2pn$ where c is as in (6). On the other hand, by considering the Rayleigh quotient we have

$$\mu \geqslant n^{-1}\langle \mathbb{1}, A\mathbb{1}\rangle = (1 + O(n^{-c}))2pn.$$

We now show $\|v\|_\infty \leqslant \delta^{-1}n^{-1/2}$. Recall that by the Perron-Frobenius theorem we may assume v has non-negative entries. Without loss of generality, assume v_1 is the largest entry of v and assume for a contradiction that $v_1 \geqslant \delta^{-1}n^{-1/2}$.

Let γ_0 be such that $\gamma_0^{1/2^{\ell+1}} = \delta/100$. We claim that for all $\gamma \geqslant \gamma_0$ we have

$$v_x \geqslant (1-\gamma)v_1 \implies \left|\left\{y \in N_{G'}(x) : v_y \geqslant (1-\gamma^{1/2})v_1\right\}\right| \geqslant (1 - 2\gamma^{1/2})d_{G'}(x)\,. \tag{13}$$

To see this note $Av = \mu v$ which implies

$$\mu v_x = \sum_{y \sim x} v_y + O\big(pn^{1-\eta}v_1\big), \tag{14}$$

where the error term is from the contribution of edges coloured both red and blue and we use (10) to control this term. Note that by (6) and (10) we have $d_{G'}(x) = (1 + O(n^{-\rho}))2pn$ for all x where $\rho = \min\{c, \eta\}$. We then see that (14) implies (13) after noting that $\gamma_0 \gg n^{-\rho}$. Now for $t \geqslant 1$, define

$$S_t = \big\{x : v_x \geqslant (1 - \gamma_0^{1/2^t})v_1\big\}.$$

Now assume that $|S_t| \leqslant (2p)^{-1}$ for some $t \leqslant \ell$. By Observation 3.2, we then have

$$|N_{G'}(S_t)| \geqslant \delta np|S_t|/8\,. \tag{15}$$

Moreover, for each $x \in S_t$, the degree of x into $N_{G'}(S_t)\backslash S_{t+1}$ is at most $2\gamma_0^{1/2^{t+1}}d_{G'}(x)$ by (13). It follows that

$$|N_{G'}(S_t)\backslash S_{t+1}| \leqslant (1 + o(1))4np\gamma_0^{1/2^{t+1}}|S_t| \leqslant \delta np|S_t|/16,$$

recalling that $t \leqslant \ell$, and therefore

$$|S_{t+1}| \geqslant \delta np|S_t|/16 \tag{16}$$

by (15). Let τ be the least integer such that $|S_\tau| \geqslant (2p)^{-1}$. By (16) and the fact that $np \geqslant n^{1/(\ell-1)}$ we have that $\tau \leqslant \ell$. Let $S'_\tau \subset S_\tau$ with $|S'_\tau| = (2p)^{-1}$. Then arguing as above we see that $|S_{\tau+1}| \geqslant \delta np|S'_\tau|/16 = \delta n/32$. It follows that

$$\|v\|_2^2 \geqslant (\delta n/32)(1 - \gamma_0^{1/2^{\tau+1}})v_1^2 > 1,$$

a contradiction. $\square$

Lemma 4.4. *On the event $\mathcal{A}$, we can express*

$$A = \mu\big(v \otimes v\big) + M,$$

where $\mu = (1+o(1))2pn$, $\|v\|_\infty \leqslant \delta^{-1}n^{-1/2}$, $v \in \ker(M)$ and $\|M\|_{op} \leqslant 6\sqrt{rpn}$.

Proof. We isolate the largest singular value using the spectral theorem and write

$$A = \mu\big(v \otimes v\big) + M,$$

where μ is the largest eigenvalue of A, v is the corresponding unit eigenvector and $v \in \ker(M)$. Note that Lemma 4.3 shows that we have $\mu = (1+o(1))2pn$ and $\|v\|_\infty \leqslant \delta^{-1}n^{-1/2}$. We now claim that $\|M\|_{op} \leqslant 6\sqrt{rpn}$. Suppose not, then there exists orthogonal unit eigenvectors u, v with $\|Au\|_2, \|Av\|_2 > 6\sqrt{rpn}$, where here we use that $pn \gg \sqrt{rpn}$. Thus there is a unit vector $w \in \operatorname{span}\{u, v\}$ with $\langle w, \mathbb{1}\rangle = 0$ and $\|Aw\|_2 > 6\sqrt{rpn}$. But this contradicts that

$$\big\|A - 2pJ_n\big\|_{op} \leqslant \big\|A_R \otimes J_r - pJ_n\big\|_{op} + \big\|P^t\big(A_B \otimes J_r - pJ_n\big)P\big\|_{op} \leqslant 6\sqrt{rpn}. \qquad \square$$

Proof of Lemma 4.1. As we mentioned before, the number of J to J walks is at most $\langle \mathbb{1}_J, A^{\ell-1}\mathbb{1}_J\rangle$. Now apply Lemma 4.4 to write $A^{\ell-1} = \mu^{\ell-1}(v \otimes v) + M^{\ell-1}$ and thus

$$\langle \mathbb{1}_J, A^{\ell-1}\mathbb{1}_J\rangle = \mu^{\ell-1}\langle \mathbb{1}_J, v\rangle^2 + \langle \mathbb{1}_J, M^{\ell-1}\mathbb{1}_J\rangle.$$

For the first term, we have

$$\langle \mathbb{1}_J, v\rangle \leqslant \sum_{i\in J} v_j \leqslant \delta^{-1}|J|n^{-1/2}$$

using that $\|v\|_\infty \leqslant \delta^{-1}n^{-1/2}$. For the second we have

$$\langle \mathbb{1}_J, M^{\ell-1}\mathbb{1}_J\rangle \leqslant |J|\|M^{\ell-1}\|_{op} \leqslant |J|\|M\|_{op}^{\ell-1} \leqslant |J|(6\sqrt{rpn})^{\ell-1},$$

Putting this together we have that the number of J to J paths is at most

$$\leqslant \delta^{-2}2^\ell p^{\ell-1}n^{\ell-2}|J|^2$$

since $r \leqslant (pn)^{1-\frac{2}{\ell-1}}$ and $|J| = \delta k \geqslant 1/p$, by our choices at (3) and since $\ell \geqslant 5$. $\square$

5. Independent set calculation

For the independent set calculation, we fix $I \in [n]^{(k)}$ and estimate the probability that I is an independent set in G. For this, it will be crucial to ensure that not too many edges in $G' \cap I^{(2)}$ are removed in our edge deletion step. Here our first step is to show that the event $\mathcal{A}$ guarantees this. Say that a path in $G' = G'_R \cup G'_B$ is *alternating* if it alternately takes steps in the red and blue graphs G'_R, G'_B. We define the set of *closed* pairs by

$$C = \big\{xy \in [n]^{(2)} : \text{ there exists a non-alternating } xy\text{-path of length } \ell{-}1 \text{ from } x \text{ to } y \text{ in } G'\big\}.$$

We highlight that only closed pairs are in danger of being removed in our edge deletion step. The key point is that if xy is joined by a non-alternating path of length $\ell-1$ then xy must be joined by $\geqslant r$ non-alternating paths. We use this to bound the number of closed edges inside of an independent set.

Lemma 5.1. *On the event $\mathcal{A}$, we have $|J^{(2)} \cap C| = o(|J|^2)$ for every $J \in [n]^{(\delta k)}$.*

Proof. We note carefully that

$$\left|C \cap J^{(2)}\right| \cdot r \leqslant |\{\text{walks of length } \ell - 1 \text{ starting and ending in } J\}| \leqslant \delta^{-2} 2^{\ell} p^{\ell-1} n^{\ell-2} |J|^2,$$

where the walks are counted in the graph G' and the second inequality holds by Lemma 4.1 and the fact that $\mathcal{A}$ holds. Recalling our choice of r and p so that we satisfy (4) we have

$$\left|C \cap J^{(2)}\right| \leqslant \delta^{-2} 2^{\ell} \left(p^{\ell-1} n^{\ell-2} r^{-1}\right) |J|^2 \ll \delta^{-2} (\log n)^{-2} |J|^2 \leqslant |J|^2,$$

where we used that our choice of r, p satisfy (4). □

We are now ready to prove our main lemma on independent sets. Let $\mathcal{I}_k(G)$ be the set of independent sets of size k in G.

Lemma 5.2. *For $I \in [n]^{(k)}$ we have*

$$\mathbb{P}\big(I \in \mathcal{I}_k(G) \wedge \mathcal{A}\big) \leqslant (1-p)^{(\delta k)^2/4}.$$

Proof. Fix I and assume that $\mathcal{A}$ holds. Without loss, assume that $|\pi_R(I)| \geqslant \delta k$ by (8) in the definition of $\mathcal{A}$. Let $J \subset I$ be such that $|\pi_R(J)| = |J| = \delta k$. We restrict our attention to bounding the probability that J is independent in the red graph. That is

$$\mathbb{P}\big(I \in \mathcal{I}_k(G) \wedge \mathcal{A}\big) \leqslant \mathbb{P}\big(J \in \mathcal{I}_{\delta k}(G_R) \wedge \mathcal{A}\big).$$

For this we need a few definitions. We say that e is *coupled to f* if

$$e, f \in V_i \times V_j \qquad \text{or} \qquad e, f \in W_i \times W_j,$$

for some $i < j$. We now expose the randomness of the graph in two rounds. We first expose all of the blue graph and all of the edges in the red graph that are not coupled to something in J. That is, we let our first exposure graph G_0 be

$$G_0 = G'_B \cup \left\{ f \in G'_R : f \text{ not coupled to any } e \in J^{(2)} \right\}.$$

We now say that e is *open* if adding e to G_0 does not complete a copy of C_ℓ. Define

$$O = \left\{ e \in J^{(2)} : e \text{ open} \right\}.$$

We now make the following deterministic claim that allows us to move from working with G_R to work with G'_R.

Claim 5.3. *If $J \in \mathcal{I}_{\delta k}(G_R)$ then $O \cap G'_R = \emptyset$.*

Proof of Claim 5.3. We prove the contrapositive. Assume $O \cap G'_R \neq \emptyset$ and let $e \in O \cap G'_R$ be the smallest edge in the red ordering. (Recall the definition of the ordering in Section 2.2). We will show that e is not deleted in the edge deletion step, and so $e \in G_R$, meaning that J is not an independent set.

For a contradiction assume that $e \notin G$ and let $C \subset G'$ be a cycle of length ℓ that accounts for the deletion of e. We first note that this cycle must be simple by Lemma 2.1. If it is not simple, then all of C will be deleted in the vertex deletion step so $J \not\subset V(G)$ and thus $J \notin \mathcal{I}_{\delta k}(G_R)$.

So we may assume that C is simple. Let us first consider the case of C monochromatic. We must have C is monochromatic in colour red since $e \in G'_R$. Now, since $e \in O$, e cannot close a cycle by itself with edges of G_0. Indeed there must exist $f' \in C \setminus \{e\}$ that is coupled to some $f \in O \cap G'_R$. We know $e \neq f$ since C is simple. By minimality of $e \in O \cap G'_R$, we know that e precedes f in the red ordering and so it must also precede f', due to the definition of the ordering. But then e is not the largest edge in C, in the red ordering, and so it would not have been deleted. This contradicts that C accounts for the deletion of e.

Finally, assume C is simple and not monochromatic. Note that C must have a blue apex. This is because our rule only removes red edges on the account of non-monochromatic cycles C with a blue apex. Now note that since $e \in O$, there must be a red edge $f' \in C \setminus \{e\}$, which is coupled to some $f \in O \cap G'_R$. By simplicity of C we know $e \neq f$. By minimality of e, we know e precedes f, in the red ordering. This means that e must also precede f', by the definition of the red ordering. Thus e is not the largest red edge in C and thus will not be deleted. Thus $e \in G_R$ which completes the proof. $\square$

We now can finish the proof of Lemma 5.2. We note importantly, that since $\mathcal{A}$ holds we can apply Lemma 5.1 to learn that

$$|O| \geqslant |J^{(2)} \setminus C| \geqslant |J|^2/4 \geqslant (\delta k)^2/4.$$

Thus we may condition on the worst case outcome of the G_0 which gives us that $|O| \geqslant (\delta k)^2/4$ (note that G_0 determines O). So we have

$$\mathbb{P}\big(J \in \mathcal{I}_{\delta k}(G_R) \wedge \mathcal{A}\big) \leqslant \max_{G_0} \mathbb{P}\big(J \in \mathcal{I}_{\delta k}(G_R) \,\big|\, G_0\big),$$

where the maximum is over all O, G_0 which satisfy $|O| \geqslant (\delta k)^2/4$, which holds on the event $\mathcal{A}$. We now use Claim 5.3 to see the above is

$$\max_{G_0} \mathbb{P}\big(O \cap G'_R = \emptyset \,\big|\, G_0\big) \leqslant \max_{|O| \geqslant (\delta k)^2/4} (1-p)^{|O|} \leqslant (1-p)^{\delta^2 k^2/4},$$

where the second last inequality holds by independence of the edges $O \cap G'_R$. $\square$

We may now prove our main technical theorem, Theorem 2.4.

Proof of Theorem 2.4. Given our preparations, the proof of our main technical theorem now follows quickly. The graph G has at least $n/2$ vertices with probability $1-o(1)$, by Lemma 2.2. It has no cycles of length ℓ, by construction. Finally, using Lemma 5.2, we see the expected number of independent sets of size k is at most

$$\binom{n}{k}(1-p)^{\delta^2 k^2/4} \leqslant \exp\big(k\big(\log n - p\delta^2 k/4\big)\big).$$

So if $p\delta^2 k > 8 \log n$, there is no independent set of size k, with probability $1 - o(1)$. $\square$

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