An upper bound for union-closed family size

Christopher Bouchard

Abstract

Let $\mathcal{A}$ be a union-closed family of sets with universe $\bigcup_{A\in\mathcal{A}} A = [n] = \{1, \cdots, n\}$ and length ℓ. We prove that $|\mathcal{A}| \leq \sum_{i=0}^{\ell} \binom{n}{i}$, with equality if and only if $\mathcal{A} = \bigcup_{i=0}^{\ell} \binom{[n]}{n-i}$. Additionally, by showing that $|\mathcal{A}| \leq \frac{\ell^p-1}{\ell-1} + 2^n(1-2^{-\ell})^p$ for any nonnegative integer p, we establish for all integers $1 \leq k \leq n$ that $\sum_{i=0}^{k} \binom{n}{i} \leq \frac{k^{\hat{p}}-1}{k-1} + 2^n(1-2^{-k})^{\hat{p}}$, where $\hat{p} = \lfloor (n-k)/\log_2(\frac{k}{1-2^{-k}}) \rfloor + 1$.

1. Introduction

Let $\mathcal{A}$ be a finite family of distinct finite sets (at least one of which is nonempty) with *universe* $U(\mathcal{A}) := \bigcup_{A\in\mathcal{A}} A$ denoted by $[n] = \{1, \cdots, n\}$. $\mathcal{A}$ is called *union-closed* if $X_1, X_2 \in \mathcal{A}$ implies that $X_1 \cup X_2 \in \mathcal{A}$. Such families have gained popularity by way of the union-closed sets conjecture, also known as Frankl's conjecture, which states that in any union-closed $\mathcal{A}$, there must be an element from $[n]$ that appears in at least half of its member sets (see [1–2] for an overview of many related results). One result, proved by Reimer in [4], can be expressed as an implicit least upper bound for the size of $\mathcal{A}$, namely $|\mathcal{A}| \leq (2/\log_2 |\mathcal{A}|) \sum_{A\in\mathcal{A}} |A|$ (or $|\mathcal{A}| \leq 4^{\sum_{A\in\mathcal{A}} |A|/|\mathcal{A}|}$).

A *chain* $\mathcal{C}$ in $\mathcal{A}$ is a subfamily of $\mathcal{A}$ such that $X_1, X_2 \in \mathcal{C}$ implies that $(X_1 \subseteq X_2) \vee (X_2 \subseteq X_1)$, and the *length* of $\mathcal{A}$, denoted by $\ell := \ell(\mathcal{A})$, is one less than the maximum size of a chain in $\mathcal{A}$. A result of Erdős (see Theorem 5 of [3]) states that any family of sets $\mathcal{A}$ (not necessarily union-closed) has size less than or equal to the sum of the largest $\ell+1$ binomial coefficients of n. In the present work, we prove that imposing the union-closed constraint on $\mathcal{A}$ tightens this upper bound to be the sum of the first, rather than largest, $\ell+1$ binomial coefficients. In a similar vein, we also provide an upper bound for the sum of the first k binomial coefficients of any positive integer n. Denote by $\binom{S}{r}$ the family of r-subsets of a set S. The two main theorems are stated as follows:

Theorem 1. *For any union-closed family $\mathcal{A}$ with universe $[n]$ and length ℓ,*

$$|\mathcal{A}| \leq \sum_{i=0}^{\ell} \binom{n}{i},$$

with equality and if and only if

$$\mathcal{A} = \bigcup_{i=0}^{\ell} \binom{[n]}{n-i}.$$

Theorem 2. *For all integers $1 \leq k \leq n$,*

$$\sum_{i=0}^{k} \binom{n}{i} \leq \frac{k^{\hat{p}}-1}{k-1} + 2^n(1-2^{-k})^{\hat{p}},$$

where

$$\hat{p} = \Big\lfloor \frac{n-k}{\log_2(k/(1-2^{-k}))} \Big\rfloor + 1.$$

2. Proof of Theorem 1

We prove the theorem by induction on universe size. For the base case $n = 1$, the only union-closed families are $\mathcal{A} = \{1\}$ and $\mathcal{A} = \{\{1\}, \emptyset\}$. The former has $\ell = 0$ and $|\mathcal{A}| = \sum_{i=0}^{\ell} \binom{n}{i} = \binom{1}{0} = 1$, while the latter has $\ell = 1$ and $|\mathcal{A}| = \sum_{i=0}^{\ell} \binom{n}{i} = \binom{1}{0} + \binom{1}{1} = 2$. For $n > 1$, we assume for the induction hypothesis that all union-closed families $\mathcal{A}'$ with $|U(\mathcal{A}')| < n$ have $|\mathcal{A}'| \leq \sum_{i=0}^{\ell(\mathcal{A}')} \binom{|U(\mathcal{A}')|}{i}$, with equality if and only if $\mathcal{A}' = \bigcup_{i=0}^{\ell(\mathcal{A}')} \binom{U(\mathcal{A}')}{|U(\mathcal{A}')|-i}$. We let $\mathcal{A}$ be any union-closed family of sets with universe size $n > 1$ and length $\ell < n$. (If $\ell = n$, then the theorem is satisfied, as $|\mathcal{A}| \leq \sum_{i=0}^{\ell} \binom{n}{i} = 2^n$ with equality if and only if $\mathcal{A} = \bigcup_{i=0}^{\ell} \binom{[n]}{n-i} = 2^{[n]}$.) For any $x \in [n]$, we define the following families:

$$\mathcal{A}_x = \{A \in \mathcal{A} \mid x \in A\};$$
$$\hat{\mathcal{A}}_x = \{A \setminus \{x\} \mid A \in \mathcal{A}_x\};$$
$$\mathcal{A}_{\bar{x}} = \{A \in \mathcal{A} \mid x \notin A\}.$$

We observe that $\mathcal{A} = \mathcal{A}_x \cup \mathcal{A}_{\bar{x}}$, $\mathcal{A}_x \cap \mathcal{A}_{\bar{x}} = \emptyset$, and $|\mathcal{A}_x| = |\hat{\mathcal{A}}_x|$ together imply that $|\mathcal{A}| = |\hat{\mathcal{A}}_x| + |\mathcal{A}_{\bar{x}}|$. Because no member set of $\hat{\mathcal{A}}_x$ contains x, and $[n] \in \mathcal{A}_x$ implies that $[n] \setminus \{x\} \in \hat{\mathcal{A}}_x$, we have that $|U(\hat{\mathcal{A}}_x)| = n - 1$. Similarly, because no member set of $\mathcal{A}_{\bar{x}}$ contains x, we have that $|\mathcal{A}_{\bar{x}}| \leq n - 1$. Now, since $\mathcal{A}_x$ is a subfamily of $\mathcal{A}$ and $\ell(\hat{\mathcal{A}}_x) = \ell(\mathcal{A}_x)$, we have that $\ell(\hat{\mathcal{A}}_x) \leq \ell$. Let $\ell(\{\emptyset\}) = 0$. Noting that $\mathcal{A}_{\bar{x}}$ is a subfamily of $\mathcal{A}$, that $[n]$ belongs to any chain of maximum size in $\mathcal{A}$, and that $[n]$ does not belong to $\mathcal{A}_{\bar{x}}$, we also have that either $\ell = 0$ or $\ell(\mathcal{A}_{\bar{x}}) \leq \ell - 1$. Since $\hat{\mathcal{A}}_x$ is itself union-closed, we have by the induction hypothesis that

$$|\hat{\mathcal{A}}_x| \leq \sum_{i=0}^{\ell(\hat{\mathcal{A}}_x)} \binom{|U(\hat{\mathcal{A}}_x)|}{i} = \sum_{i=0}^{\ell(\hat{\mathcal{A}}_x)} \binom{n-1}{i} \leq \sum_{i=0}^{\ell} \binom{n-1}{i},$$

If $\mathcal{A}_{\bar{x}} = \emptyset$, then $|\mathcal{A}| = |\hat{\mathcal{A}}_x| \leq \sum_{i=0}^{\ell} \binom{n-1}{i} \leq \sum_{i=0}^{\ell} \binom{n}{i}$. If $\mathcal{A}_{\bar{x}} = \{\emptyset\}$, then $\ell > 0$ and $|\mathcal{A}| = |\hat{\mathcal{A}}_x| + 1 \leq 1 + \sum_{i=0}^{\ell} \binom{n-1}{i} \leq \sum_{i=0}^{\ell} \binom{n}{i}$. Else, $\mathcal{A}_{\bar{x}} \neq \emptyset$ and $\mathcal{A}_{\bar{x}} \neq \{\emptyset\}$, and $\mathcal{A}_{\bar{x}}$ is union-closed as well as $\hat{\mathcal{A}}_x$. In this case, ℓ is again positive, and we apply the induction hypothesis to $\mathcal{A}_{\bar{x}}$ in order to obtain that

$$|\mathcal{A}_{\bar{x}}| \leq \sum_{i=0}^{\ell(\mathcal{A}_{\bar{x}})} \binom{|U(\mathcal{A}_{\bar{x}})|}{i} \leq \sum_{i=0}^{\ell(\mathcal{A}_{\bar{x}})} \binom{n-1}{i} \leq \sum_{i=0}^{\ell-1} \binom{n-1}{i}.$$

Therefore, if $\mathcal{A}_{\bar{x}} \neq \emptyset$ and $\mathcal{A}_{\bar{x}} \neq \{\emptyset\}$, then we have that

$$\begin{aligned}
|\mathcal{A}| &= |\hat{\mathcal{A}}_x| + |\mathcal{A}_{\bar{x}}| \\
&\leq \sum_{i=0}^{\ell} \binom{n-1}{i} + \sum_{i=0}^{\ell-1} \binom{n-1}{i} \\
&= \binom{n-1}{0} + \sum_{i=1}^{\ell} \left(\binom{n-1}{i} + \binom{n-1}{i-1} \right) \\
&= 1 + \sum_{i=1}^{\ell} \binom{n}{i} \\
&= \sum_{i=0}^{\ell} \binom{n}{i}.
\end{aligned}$$

(Here, the equation $\sum_{i=0}^{\ell} \binom{n}{i} = \sum_{i=0}^{\ell} \binom{n-1}{i} + \sum_{i=0}^{\ell-1} \binom{n-1}{i}$ corresponds to case $m = 1$ of the identity $\sum_{i=0}^{k} \binom{n}{i} = \sum_{i=0}^{m} \binom{m}{i} \sum_{j=0}^{k-i} \binom{n-m}{j}$ for all integers $0 \leq k \leq n$, where $0 \leq m \leq n$.) This completes the proof that $|\mathcal{A}| \leq \sum_{i=0}^{\ell} \binom{n}{i}$. Next, we show that $|\mathcal{A}| = \sum_{i=0}^{\ell} \binom{n}{i}$ if and only if $\mathcal{A} = \bigcup_{i=0}^{\ell} \binom{[n]}{n-i}$. That $|\mathcal{A}| = \sum_{i=0}^{\ell} \binom{n}{i}$ when $\mathcal{A} = \bigcup_{i=0}^{\ell} \binom{[n]}{n-i}$ is clear from symmetry of the binomial coefficient. It remains to show that the bound is only sharp if $\mathcal{A} = \bigcup_{i=0}^{\ell} \binom{[n]}{n-i}$, for which we now assume that $|\mathcal{A}| = \sum_{i=0}^{\ell} \binom{n}{i}$. If $\ell = 0$, then $|\mathcal{A}| = \sum_{i=0}^{\ell} \binom{n}{i} = \binom{n}{0} = 1$, implying that $\mathcal{A} = \{[n]\} = \binom{[n]}{n} = \bigcup_{i=0}^{\ell} \binom{[n]}{n-i}$. Else, $\ell > 0$, and we have that $\sum_{i=0}^{\ell}\binom{n}{i} = \sum_{i=0}^{\ell}\binom{n-1}{i} + \sum_{i=0}^{\ell-1}\binom{n-1}{i}$. Noting that $|\mathcal{A}| = |\hat{\mathcal{A}}_x| + |\mathcal{A}_{\bar{x}}|$ and $|\hat{\mathcal{A}}_x| \le \sum_{i=0}^{\ell}\binom{n-1}{i}$, we then have that $|\mathcal{A}_{\bar{x}}| \ge \sum_{i=0}^{\ell-1}\binom{n-1}{i}$. It again follows that $\mathcal{A}_{\bar{x}}$ is union-closed. (Otherwise, $\mathcal{A}_{\bar{x}} = \emptyset$ and $|\mathcal{A}_{\bar{x}}| = 0 \ge \sum_{i=0}^{\ell-1}\binom{n-1}{i} \ge 1$, a contradiction, or $\mathcal{A}_{\bar{x}} = \{\emptyset\}$ and $|\mathcal{A}_{\bar{x}}| = 1 \ge \sum_{i=0}^{\ell-1}\binom{n-1}{i}$ implies that $\ell = 1$, making $\mathcal{A} = \{[n], \emptyset\}$, which then implies that $n = 1$, again a contradiction). Therefore, $|\mathcal{A}_{\bar{x}}| \le \sum_{i=0}^{\ell-1}\binom{n-1}{i}$, which implies that both $|\hat{\mathcal{A}}_x| = \sum_{i=0}^{\ell}\binom{n-1}{i}$ and $|\mathcal{A}_{\bar{x}}| = \sum_{i=0}^{\ell-1}\binom{n-1}{i}$. It follows that $U(\hat{\mathcal{A}}_x) = [n] \setminus \{x\}$ with $\ell(\hat{\mathcal{A}}_x) = \ell$. (If not, then $|\hat{\mathcal{A}}_x| = \sum_{i=0}^{\ell(\hat{\mathcal{A}}_x)}\binom{|U(\hat{\mathcal{A}}_x)|}{i} \le \sum_{i=0}^{\ell}\binom{n-2}{i}$ or $|\hat{\mathcal{A}}_x| = \sum_{i=0}^{\ell(\hat{\mathcal{A}}_x)}\binom{|U(\hat{\mathcal{A}}_x)|}{i} \le \sum_{i=0}^{\ell-1}\binom{n-1}{i}$, a contradiction.) Therefore, $\hat{\mathcal{A}}_x = \bigcup_{i=0}^{\ell}\binom{[n]\setminus\{x\}}{(n-1)-i}$ by the induction hypothesis. Similarly, it also follows that $U(\mathcal{A}_{\bar{x}}) = [n] \setminus \{x\}$ with $\ell(\mathcal{A}_{\bar{x}}) = \ell - 1$. (When $\ell = 1$, having $x \notin A$ for any $A \in \mathcal{A}_{\bar{x}}$ implies that $|\mathcal{A}_{\bar{x}}| = 1$, making $\ell(\mathcal{A}_{\bar{x}}) = 0 = \ell - 1$.) Applying the induction hypothesis now to $\mathcal{A}_{\bar{x}}$, we obtain that $\mathcal{A}_{\bar{x}} = \bigcup_{i=0}^{\ell-1}\binom{[n]\setminus\{x\}}{(n-1)-i}$. We thus have that

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_x \cup \mathcal{A}_{\bar{x}} = \{A \cup \{x\} \mid A \in \hat{\mathcal{A}}_x\} \cup \mathcal{A}_{\bar{x}} \\
&= \bigcup_{i=0}^{\ell}\Big\{A \cup \{x\} \;\Big|\; A \in \binom{[n]\setminus\{x\}}{n-i-1}\Big\} \cup \bigcup_{i=0}^{\ell-1}\binom{[n]\setminus\{x\}}{n-i-1} \\
&= \{[n]\} \cup \bigcup_{i=1}^{\ell}\Big(\Big\{A \cup \{x\} \;\Big|\; A \in \binom{[n]\setminus\{x\}}{n-i-1}\Big\} \cup \binom{[n]\setminus\{x\}}{n-i}\Big) \\
&= \binom{[n]}{n} \cup \bigcup_{i=1}^{\ell}\binom{[n]}{n-i} \\
&= \bigcup_{i=0}^{\ell}\binom{[n]}{n-i}.
\end{aligned}$$

Therefore, $|\mathcal{A}| = \sum_{i=1}^{\ell}\binom{n}{i}$ implies that $\mathcal{A} = \bigcup_{i=0}^{\ell}\binom{[n]}{n-i}$, completing the proof of Theorem 1.

Corollary 2.1. *For any union-closed family $\mathcal{A}$, there is an element from its universe $[n]$ that is in at most $\sum_{i=0}^{\ell}\binom{n-1}{i}$ of its member sets.*

Proof. Because $|\mathcal{A}| \le \sum_{i=0}^{\ell}\binom{n}{i}$, it holds that $|\mathcal{A} \setminus \bigcup_{i=0}^{\ell}\binom{[n]}{n-i}| \le |\bigcup_{i=0}^{\ell}\binom{[n]}{n-i} \setminus \mathcal{A}|$. We also have that $|X| < |Y|$ for any $X \in \mathcal{A} \setminus \bigcup_{i=0}^{\ell}\binom{[n]}{n-i}$ and $Y \in \bigcup_{i=0}^{\ell}\binom{[n]}{n-i} \setminus \mathcal{A}$. Together these observations imply that $\sum_{A\in\mathcal{A}}|A| \le \sum_{A\in\bigcup_{i=0}^{\ell}\binom{[n]}{n-i}}|A| = \sum_{i=0}^{\ell}(n-i)\binom{n}{i}$. Because $\sum_{A\in\mathcal{A}}|A| = \sum_{x\in[n]}|\mathcal{A}_x|$, we also have that $\sum_{x\in[n]}|\mathcal{A}_x| \le \sum_{i=0}^{\ell}(n-i)\binom{n}{i}$. It follows that $|\mathcal{A}_y| \le \sum_{i=0}^{\ell}(1-\frac{i}{n})\binom{n}{i}$ for some $y \in [n]$. (If not, then $|\mathcal{A}_x| > \sum_{i=0}^{\ell}(1-\frac{i}{n})\binom{n}{i}$ for all $x \in [n]$, implying that $\sum_{x\in[n]}|\mathcal{A}_x| > n(\sum_{i=0}^{\ell}(1-\frac{i}{n})\binom{n}{i}) = \sum_{i=0}^{\ell}(n-i)\binom{n}{i}$, a contradiction.) Finally, considering that $\sum_{i=0}^{\ell}(1-\frac{i}{n})\binom{n}{i}$ is equal to the size of $\mathcal{A}_x$ for $\mathcal{A} = \bigcup_{i=0}^{\ell}\binom{[n]}{n-i}$ and any $x \in [n]$, we have by double counting that $\sum_{i=0}^{\ell}(1-\frac{i}{n})\binom{n}{i} = \sum_{i=0}^{\ell}\binom{n-1}{n-i-1}$, making $|\mathcal{A}_y| \le \sum_{i=0}^{\ell}\binom{n-1}{i}$.

3. Proof of Theorem 2

Let $\mathcal{A}$ be any union-closed family with universe $[n]$ and length ℓ, and consider any chain $\mathcal{C} = \{C_1, \cdots, C_{\ell+1}\}$ in $\mathcal{A}$ of maximum size, where $1 \le i < j \le \ell+1$ implies that $C_j \subsetneq C_i$ without loss of generality. Because every member set of $\mathcal{A}$ is a subset of $[n]$, and $[n]$ must belong to $\mathcal{A}$, we have that $C_1 = [n]$. Let $[0] = \emptyset$. For each $i \in [\ell]$, we define

$$\mathcal{D}_i = \Big\{X \setminus (C_i \setminus C_{i+1}) \;\Big|\; X \in \mathcal{C}_i\Big\},$$

where
$$\mathcal{C}_i = \Big\{X \in \mathcal{A} \;\Big|\; (C_i \setminus C_{i+1} \subseteq X) \wedge (X \cap \bigcup_{j\in[i-1]} C_j \setminus C_{j+1} = \emptyset)\Big\}.$$

Theorem 3.1. *For any $i \in [\ell]$, if $C_{i+1} \neq \emptyset$, then $\mathcal{D}_i$ is union-closed.*

Proof. Let i be any element of $[\ell]$ such that $C_{i+1} \neq \emptyset$. Assuming any $X_1, X_2 \in \mathcal{C}_i$, we first have that $(C_i \setminus C_{i+1} \subseteq X_1) \wedge (C_i \setminus C_{i+1} \subseteq X_2) \implies C_i \setminus C_{i+1} \subseteq X_1 \cup X_2$ and $(X_1 \cap \bigcup_{j\in[i-1]} C_j \setminus C_{j+1} = \emptyset) \wedge (X_2 \cap \bigcup_{j\in[i-1]} C_j \setminus C_{j+1} = \emptyset) \implies (X_1 \cup X_2) \cap \bigcup_{j\in[i-1]} C_j \setminus C_{j+1} = \emptyset$. Hence, $X_1 \cup X_2$ satisfies both conditions for membership in $\mathcal{C}_i$. Because $C_i \in \mathcal{C}_i$ and $C_i \neq \emptyset$, it then follows that $\mathcal{C}_i$ is union-closed. Now, assuming any $Y_1, Y_2 \in \mathcal{D}_i$, we have that $Y_1 \cup (C_i \setminus C_{i+1}) \in \mathcal{C}_i$ and $Y_2 \cup (C_i \setminus C_{i+1}) \in \mathcal{C}_i$. Since $\mathcal{C}_i$ is union-closed, we then also have that $(Y_1 \cup (C_i \setminus C_{i+1})) \cup (Y_2 \cup (C_i \setminus C_{i+1})) = (Y_1 \cup Y_2) \cup (C_i \setminus C_{i+1}) \in \mathcal{C}_i$. It follows that $((Y_1 \cup Y_2) \cup (C_i \setminus C_{i+1})) \setminus (C_i \setminus C_{i+1}) = Y_1 \cup Y_2 \in \mathcal{D}_i$. Because $C_{i+1} \in \mathcal{D}_i$ and $C_{i+1} \neq \emptyset$, it then follows that $\mathcal{D}_i$ is also union-closed, proving Theorem 3.1.

Theorem 3.2.
$$|\mathcal{A}| = 1 + \sum_{i=1}^{\ell} |\mathcal{D}_i|.$$

Proof. We first have that $\mathcal{A} \setminus \{C_{\ell+1}\} = \bigcup_{i\in[\ell]} \mathcal{C}_i$. (If not, then there exists some $X \in \mathcal{A} \setminus \{C_{\ell+1}\}$ such that either $X \subsetneq C_{\ell+1}$ or $\exists i \in [\ell] \mid (\emptyset \subsetneq (C_i \setminus C_{i+1}) \cap X \subsetneq C_i \setminus C_{i+1} \ \wedge\ \forall j \in [i-1] \mid (C_j \setminus C_{j+1}) \cap X = \emptyset)$, and it follows that either $\{C_1, \cdots, C_{\ell+1}, X\}$ or $\{C_1, \cdots, C_i, C_{i+1} \cup X, C_{i+1}, \cdots, C_{\ell+1}\}$ is a chain in $\mathcal{A}$ of size $\ell+2$, contradicting the definition of ℓ.) Next, we observe that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ for any distinct i and j in $[\ell]$. (Otherwise, there exist distinct i and j in $[\ell]$ with at least one member set X in $\mathcal{C}_i \cap \mathcal{C}_j$, where $j < i$ without loss of generality, and $X \in \mathcal{C}_j$ implies that $C_j \setminus C_{j+1} \subseteq X$, while $X \in \mathcal{C}_i$ and $j \in [i-1]$ together imply that $X \cap (C_j \setminus C_{j+1}) = \emptyset$, a contradiction.) It follows that $|\mathcal{A} \setminus \{C_{\ell+1}\}| = \sum_{i=1}^{\ell} |\mathcal{C}_i|$, making $|\mathcal{A}| = 1 + \sum_{i=1}^{\ell} |\mathcal{C}_i|$. Finally, because $C_i \setminus C_{i+1} \subseteq X$ for any $i \in [\ell]$ and any $X \in \mathcal{C}_i$, we have that $|\mathcal{D}_i| = |\mathcal{C}_i|$ for each i. It then follows that $|\mathcal{A}| = 1 + \sum_{i=1}^{\ell} |\mathcal{D}_i|$, proving Theorem 3.2.

Lemma 3.3. *$|U(\mathcal{D}_i)| \leq n - i$ and $\ell(\mathcal{D}_i) \leq \ell$ for all $i \in [\ell]$.*

Proof. For all $i \in [\ell]$, $|U(\mathcal{D}_i)| \leq n - i$ follows from having $X \cap \bigcup_{j\in[i]} C_j \setminus C_{j+1} = \emptyset$ for any $X \in \mathcal{D}_i$, and $\ell(\mathcal{D}_i) \leq \ell$ follows from $\ell(\mathcal{C}_i) = \ell(\mathcal{D}_i)$ and $\mathcal{C}_i$ being a subfamily of $\mathcal{A}$.

We now define $\Theta\colon \mathbb{Z}_{\geq 0}^3 \to \mathbb{Z}_{\geq 0}$ such that
$$\Theta(x, y, z) = \frac{x^z - 1}{x - 1} + 2^y (1 - 2^{-x})^z.$$

We prove by induction that, for any nonnegative integer p, $|\mathcal{A}| \leq \Theta(\ell, n, p)$ for all union-closed families $\mathcal{A}$ with universe $[n]$ and length ℓ. For the base case $p = 0$, we have that $|\mathcal{A}| \leq \Theta(\ell, n, p) = 2^n$ for any such $\mathcal{A}$. For $p > 0$, we again let $\mathcal{A}$ be any union-closed family with universe $[n]$ and length ℓ, and we assume for the induction hypothesis that any union-closed family $\mathcal{A}'$ with universe of size n' and length ℓ' satisfies $|\mathcal{A}'| \leq \Theta(\ell', n', p-1)$. By Theorem 3.1, we apply this hypothesis to all families $\mathcal{D}_i\colon i \in [\ell]$ that have $C_{i+1} \neq \emptyset$ in order to obtain that $|\mathcal{D}_i| \leq \Theta(\ell(\mathcal{D}_i), |U(\mathcal{D}_i)|, p-1)$ for all such i. (If $C_{i+1} = \emptyset$, then we may directly compute that $|\mathcal{D}_i| = \Theta(\ell(\mathcal{D}_i), |U(\mathcal{D}_i)|, p-1) = \Theta(0, 0, p-1) = 1$.) Noting that $\Theta(x, y, z)$ is always increasing with respect to each of the variables x and y, it then follows from Lemma 3.3 that $|\mathcal{D}_i| \leq \Theta(\ell, n-i, p-1)$ for all $i \in [\ell]$. Recall that by Theorem 3.2, $|\mathcal{A}| = 1 + \sum_{i=1}^{\ell} |\mathcal{D}_i|$. We substitute $\Theta(\ell, n-i, p-1)$ into this equation for every $|\mathcal{D}_i|$ to obtain an upper bound for the size of $\mathcal{A}$ as follows:

$$\begin{aligned}
|\mathcal{A}| &= 1 + \sum_{i=1}^{\ell} |\mathcal{D}_i| \\
&\leq 1 + \sum_{i=1}^{\ell} \Theta(\ell, n-i, p-1) \\
&= 1 + \sum_{i=1}^{\ell} \Big(\frac{\ell^{p-1} - 1}{\ell - 1} + 2^{n-i} (1 - 2^{-\ell})^{p-1} \Big) \\
&= 1 + \ell \Big(\frac{\ell^{p-1} - 1}{\ell - 1} \Big) + (1 - 2^{-\ell})^{p-1} \sum_{i=1}^{\ell} 2^{n-i} \\
&= \frac{\ell^p - 1}{\ell - 1} + (1 - 2^{-\ell})^{p-1} (2^n - 2^{n-\ell}) \\
&= \frac{\ell^p - 1}{\ell - 1} + 2^n (1 - 2^{-\ell})^p \\
&= \Theta(\ell, n, p).
\end{aligned}$$

This resolves the induction step. It follows that $|\mathcal{A}| \leq \min_{p\in\mathbb{Z}_{\geq 0}} \{\Theta(\ell, n, p)\}$ for any union-closed family $\mathcal{A}$ with universe $[n]$ and length ℓ.

For all integers $1 \le k \le n$, we set $\mathcal{A}$ equal to $\bigcup_{i=0}^{k} \binom{[n]}{n-i}$ to obtain that $\sum_{i=0}^{k} \binom{n}{i} \le \min_{p \in \mathbb{Z}_{\ge 0}} \{\Theta(k,n,p)\}$. As $\hat{p} \in \mathbb{Z}_{\ge 0}$, this completes the proof of Theorem 2.

To demonstrate that $\hat{p}$ is optimal, we compute that $\Theta(k,n,p+1) \le \Theta(k,n,p)$ if and only if

$$p \le \log_{\frac{k}{1-2^{-k}}}(2^{n-k}) = \frac{n-k}{\log_2(k/(1-2^{-k}))}.$$

Thus, Θ is decreasing with respect to p if and only if $p \le \lfloor \frac{n-k}{\log_2(k/(1-2^{-k}))} \rfloor$, and it follows that

$$\Theta\Big(k, n, \Big\lfloor \frac{n-k}{\log_2(k/(1-2^{-k}))} \Big\rfloor + 1\Big) = \Theta(k,n,\hat{p}) = \min_{p \in \mathbb{Z}_{\ge 0}} \{\Theta(k,n,p)\}.$$

References

[1] H. Bruhn and O. Schaudt, *The journey of the union-closed sets conjecture*, Graphs Combin. **31** (2015), 2043–2074.

[2] S. Cambie, *Progress on the union-closed conjecture and offsprings in winter 2022-2023*, preprint (2023), arXiv:2306.12351.

[3] P. Erdős, *On a lemma of Littlewood and Offord*, Bull. Amer. Math. Soc. **51** (1945), 898–902.

[4] D. Reimer, *An average set size theorem*, Combin. Probab. Comput. **12** (2003), 89–93.