

Natural transformations between braiding functors in the Fukaya category

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Abstract

We study the space of A_∞ -natural transformations between braiding functors acting on the Fukaya category associated to the Coulomb branch $\mathcal{M}(\bullet, 1)$ of the $\mathfrak{sl}_2$ quiver gauge theory. We compute all cohomologically distinct A_∞ -natural transformations $\text{Nat}(\text{id}, \text{id})$ and $\text{Nat}(\text{id}, \beta_i^-)$, where β_i^- denotes the negative braiding functor. Our computation is carried out in a diagrammatic framework compatible with the established embedding of the KLRW category into this Fukaya category. We then compute the Hochschild cohomology of the Fukaya category using an explicit projective resolution of the diagonal bimodule obtained via the Chouhy–Solotar reduction system, and use this to classify all cohomologically distinct natural transformations. These results determine the higher A_∞ -data encoded in the braiding functors and their natural transformations, and provide the first step toward a categorical formulation of braid cobordism actions on Fukaya categories.

1 Introduction

Braid group actions on triangulated categories are nontrivial and reveal a subtle interplay between topology and algebra. One interesting example is the action on Fukaya categories.

Conceptually, one considers a symplectic manifold $\mathcal{Y}_{\mathbf{a}}$ fibering over $\mathbb{C}$, determined by the data of $|\mathbf{a}|$ unordered points $\mathbf{a} = \{a_1, \dots, a_{|\mathbf{a}|}\}$ on $\mathbb{C}$, referred to as *punctures*, whose fibers exhibit certain singular behaviors.

Fix a base point $\mathbf{x}$ in the configuration space of punctures $\text{Conf}_{|\mathbf{x}|}(\mathbb{C})/S_{|\mathbf{x}|}$. For a based loop in this configuration space, we obtain an S^1 -family of symplectic manifolds equipped with fiberwise symplectic structures. The induced symplectic connection yields a monodromy symplectomorphism on the reference fiber $\mathcal{Y}_{\mathbf{x}}$. Viewing the braid group as the fundamental group of the configuration space of $|\mathbf{x}|$ unordered points in $\mathbb{C}$, we obtain a natural homomorphism

$$\text{Br}_{|\mathbf{x}|} = \pi_1(\text{Conf}_{|\mathbf{x}|}(\mathbb{C})/S_{|\mathbf{x}|}) \longrightarrow \pi_0(\text{Aut}(\mathcal{Y}_{\mathbf{x}}, \omega)). \quad (1)$$

Moreover, each symplectic automorphism of $\mathcal{Y}_{\mathbf{x}}$ induces corresponding endofunctors of its Fukaya category by transporting Lagrangian submanifolds and their Floer complexes. This gives rise to a representation

$$\rho: \text{Br}_{|\mathbf{x}|} = \pi_1(\text{Conf}_{|\mathbf{x}|}(\mathbb{C})/S_{|\mathbf{x}|}) \longrightarrow \pi_0(\text{Aut}(\mathcal{Y}_{\mathbf{x}})) \longrightarrow \text{Aut}(\text{Fuk}(\mathcal{Y}_{\mathbf{x}})), \quad (2)$$

whose image consists of the *braiding functors* on $\text{Fuk}(\mathcal{Y}_{\mathbf{x}})$. For any two braiding functors β_1 and β_2 , we can consider the space of A_∞ -natural transformations $\text{Nat}(\beta_1, \beta_2)$ in the Fukaya category.

Definition 1.1. For the standard generators σ_i of $\text{Br}_{|\mathbf{x}|}$, we denote

$$\beta_{i+} := \rho(\sigma_i), \quad \beta_{i-} := \rho(\sigma_i^{-1}),$$

and refer to them respectively as the positive and negative braiding functors.

The punctures may arise, for instance, from singular fibers or from the critical values of a superpotential. Instances of the former appear in [9], where fibrations with singular fibers are analyzed. In the present paper, we focus on the latter situation, in which the punctures specify a superpotential. More specifically, we consider the symplectic fibration

$$\pi: \mathbb{C}_x \times \mathbb{C}_y^\times \rightarrow \mathbb{C}, \quad \pi(x, y) = x.$$

For a collection of punctures $\mathbf{x} = \{x_1, \dots, x_{|\mathbf{x}|}\} \in \text{Conf}_{|\mathbf{x}|}(\mathbb{C}_x)/S_{|\mathbf{x}|}$, the associated Landau–Ginzburg superpotential $\mathcal{W}_{\mathbf{x}}$ is given by

$$\mathcal{W}_{\mathbf{x}} = y \prod_{i=1}^n (x - x_i), \quad (3)$$

which determines the fiberwise stops at $\mathcal{W}_{\mathbf{x}} \rightarrow +\infty$. We will consider the manifold with two stops in the base.

This space has a natural origin as the Coulomb branch $\mathcal{M}(\bullet, 1)$ of the $\mathfrak{sl}_2$ quiver gauge theory with dimension vector $\vec{d} = 1$, which plays a central role in Aganagic’s categorification of Khovanov cohomology [1]. We will consider a full subcategory $\text{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}}) \subset \text{Fuk}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}})$ generated by objects associated to curves without closed components [1, 2, 6], which contains all the interested objects.

The braid group itself extends naturally to the *braid cobordism category* $\text{BrCob}_{|\mathbf{x}|}$, whose objects are braid group elements (tangles from $|\mathbf{x}|$ points to themselves) and whose morphisms are braid cobordisms between them. We expect the correspondence ρ to lift to a functor

$$\rho^\sharp: \text{BrCob}_{|\mathbf{x}|} \longrightarrow \text{End}(\text{Fuk}(\mathcal{V}_{\mathbf{x}})), \quad (4)$$

sending each braid cobordism to a natural transformation between the corresponding braiding functors.

In this paper, however, we will not determine the functor $\rho^\sharp$. Instead, we compute all (cohomologically distinct) natural transformations $\text{Nat}(\text{id}, \text{id})$ and $\text{Nat}(\text{id}, \beta_{i-})$. This problem is interesting in its own right and reveals several elegant structures along the way. Once the natural transformations from the identity functor to the generating braiding functors are understood, their compositions yield much of the information about natural transformations between arbitrary braiding functors.

Remark 1.2. There are several reasons to focus on negative rather than positive braiding functors. While ordinary natural transformations from id to β_{i+} can be constructed easily, no such transformation exists from id to β_{i-} .

For the Lagrangian I_{i-1} introduced in Section 2.4, after applying the positive braiding, there is a degree 0 intersection point in $\text{Hom}(I_{i-1}, \beta_{i+} I_{i-1})$, and its corresponding natural transformation is easy to determine. However, for the negative braiding, one finds a degree 2 morphism $a \in$

$\text{Hom}^2(I_{i-1}, \beta_i - I_{i-1})$. If this morphism is to arise from a natural transformation, then there should exist a nontrivial $\eta : \text{id} \Rightarrow \beta_i -$ of degree $|\eta| = 2$.

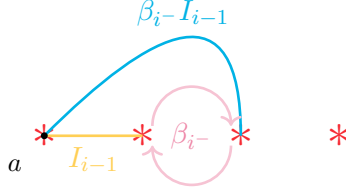


Figure 1: A degree-2 element in $\text{Hom}(I_{i-1}, \beta_i - I_{i-1})$

These indicates that higher structures must be taken into account, and that one should work with A_∞ -natural transformations. We will be able to show that this is indeed the case.

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Contents

1	Introduction	1
2	Structure of the Fukaya category	4
2.1	Generating objects	4
2.2	Morphisms between generators	5
2.3	Structure maps	6
2.3.1	Composition of morphisms	6
2.3.2	KLRW category $\mathcal{C}$ inside the Fukaya category	7
2.4	General objects from generators	8
3	Structure of the braiding functors	10
3.1	Identity functor	11
3.2	Negative braiding functor	12
3.2.1	Action on objects	12
3.2.2	Action on morphisms	12
3.2.3	Higher components	14
3.2.4	Extension to the full category	16
3.2.5	Realization as a bimodule	16
4	A_∞-natural transformations and Hochschild cohomology	19
5	The Chouhy–Solotar resolution	20
5.1	Reduction system	20
5.2	Ambiguities	21
5.3	The projective resolution	22

6	Projective resolution of the diagonal bimodule	23
6.1	Category $\mathcal{C}$ as a quiver with relations	23
6.2	Result from the Chouhy–Solotar resolution	26
7	Hochschild cohomology of the braiding functors	28
7.1	$H^\bullet \text{Hom}(\text{id}, \text{id})$	28
7.1.1	The cochain complex	28
7.1.2	The cohomology	29
7.2	$H^\bullet \text{Hom}(\text{id}, \beta_i -)$	35
8	Homotopy deformation retract from bar resolution	36
9	Components of the A_∞-natural transformations	41
9.1	$\text{Nat}(\text{id}, \text{id})$	41
9.1.1	Degree 0	41
9.1.2	Degree 1	42
9.1.3	Degree 2	43
9.2	$\text{Nat}(\text{id}, \beta_i -)$	46
9.2.1	Degree 0	46
9.2.2	Degree 1	48
9.2.3	Degree 2	49

2 Structure of the Fukaya category

2.1 Generating objects

Our interested subcategory $\text{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}})$, proved in [6], is generated by the T_i objects, whose shapes in the base are the straight vertical lines between punctures drawn in Fig. 2a, and whose shapes in the fibers are the straight lines from $y = 0$ to $y = \infty$ drawn in Fig. 2b. There are $|\mathbf{x}| + 1$ generators, we label them as $T_0, T_1, \dots, T_{|\mathbf{x}|}$ according to Fig. 2.

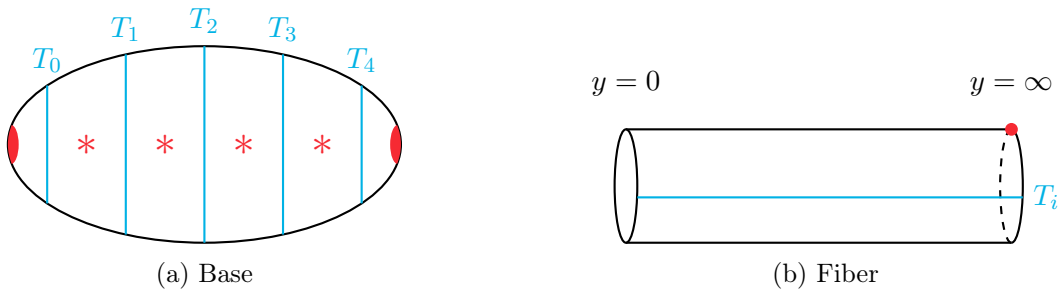


Figure 2: The generators T_i , the red dots and stars denote the stops and punctures

Remark 2.1. *Lagrangians with such shape in the fibers will be called T -branes, in the future we will simply draw the base and label the fiber type.*

2.2 Morphisms between generators

For generating objects T_i, T_j , the morphism space is given by their Floer complex

$$\mathrm{Hom}^\bullet(T_i, T_j) := CF^\bullet(T_i, T_j) \quad (5)$$

To compute this, we need to positively (counterclockwise) wrap the first input. The result for $\mathrm{Hom}(T_3, T_1)$ is shown in Fig. 3, where there is a unique intersection in the base, and infinitely many intersection points, indexed by $\mathbb{Z}_{\geq 0}$, above it in the fiber.

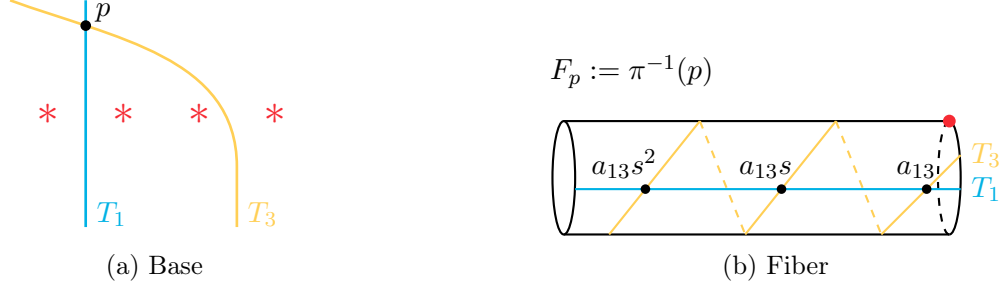


Figure 3: Morphisms between two generating T_i objects.

The same pattern holds for arbitrary T_i and T_j , and all these intersection points have degree 0.

Definition 2.2. We label the $(\alpha + 1)$ st intersection point, counted from the $y \rightarrow \infty$ side of the cylindrical fiber, by $a_{ji}s^\alpha$.

As we will see in Section 2.3.2, this corresponds to a KLRW strand diagram from j to i with α dots.

Remark 2.3. We write a_{ji} for morphisms in $\mathrm{Hom}(T_i, T_j)$, following the common convention that morphisms are composed from right to left. With this notation, an element of $\mathrm{Hom}(T_j, T_k) \otimes \mathrm{Hom}(T_i, T_j)$ appears as $a_{kj} \otimes a_{ji}$, which is notationally more convenient.

Theorem 2.4. For any i, j , the morphism space is given by

$$\mathrm{Hom}(T_i, T_j) \cong \mathbb{k}^{\mathbb{Z}_{\geq 0}} = \bigoplus_{\alpha=0}^{\infty} \mathbb{k} a_{ji}s^\alpha. \quad (6)$$

Definition 2.5. We will use the following shorthand notations:

- $a_{ji} := a_{ji}s^0$, for $i \neq j$
- $s_i^\alpha := a_{ii}s^\alpha$, $s_i := a_{ii}s^1$, $e_i := a_{ii}s^0$,
- $p_i := a_{i,i-1}$, $q_i := a_{i,i+1}$.

2.3 Structure maps

2.3.1 Composition of morphisms

In the Fukaya category, the structure maps

$$\mu^d: \text{Hom}(X_{d-1}, X_d) \otimes \cdots \otimes \text{Hom}(X_0, X_1) \rightarrow \text{Hom}(X_0, X_d)[2-d] \quad (7)$$

are defined by counting pseudoholomorphic polygons bounded by the objects $X_0, \dots, X_d$. They satisfy the A_∞ -relations

$$\sum_{m,n} (-1)^{\mathfrak{K}_n} \mu^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) = 0 \quad (8)$$

for all composable morphisms $a_i \in \text{Hom}(X_{i-1}, X_i)$, where $\mathfrak{K}_n = |a_1| + \cdots + |a_n| - n$ is the Koszul sign convention.

It is known that when restricted to the generating objects $\{T_i\}$, the compositions μ^d satisfy relations that coincide with those of the Khovanov–Lauda–Rouquier–Webster (KLRW) algebra [7, 8, 12, 14]. Namely,

Theorem 2.6. [2] *For the generators T_i , one has*

$$\mu^d = 0 \quad \text{for } d \neq 2,$$

while the binary product μ^2 is given by

$$\mu^2(a_{kj}s^\beta, a_{ji}s^\alpha) = a_{ki}s^{\beta+\alpha+\delta(i,j,k)} \quad (9)$$

where

$$\delta(i, j, k) = \begin{cases} 0, & (i-j)(j-k) \geq 0, \\ \min(|i-j|, |j-k|), & (i-j)(j-k) < 0. \end{cases} \quad (10)$$

Proof. These can be proved by a direct count of holomorphic discs. For example,

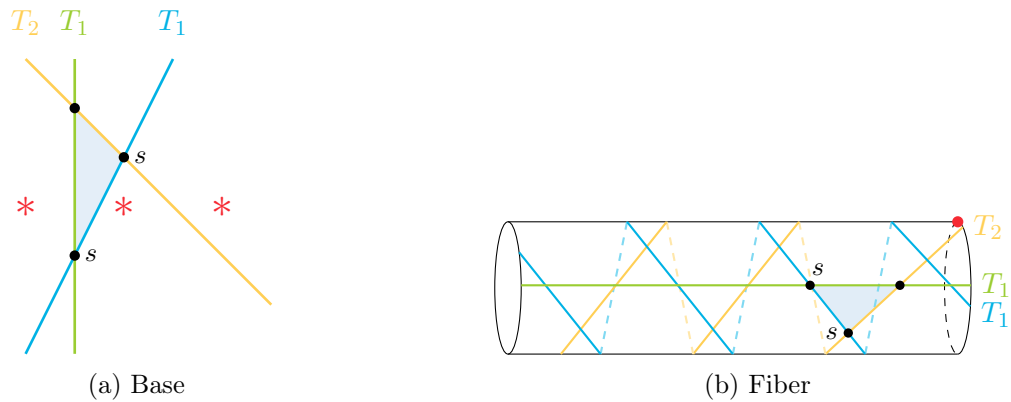


Figure 4: Proof of $\mu^2(s_1, a_{12}) = a_{12}s$.

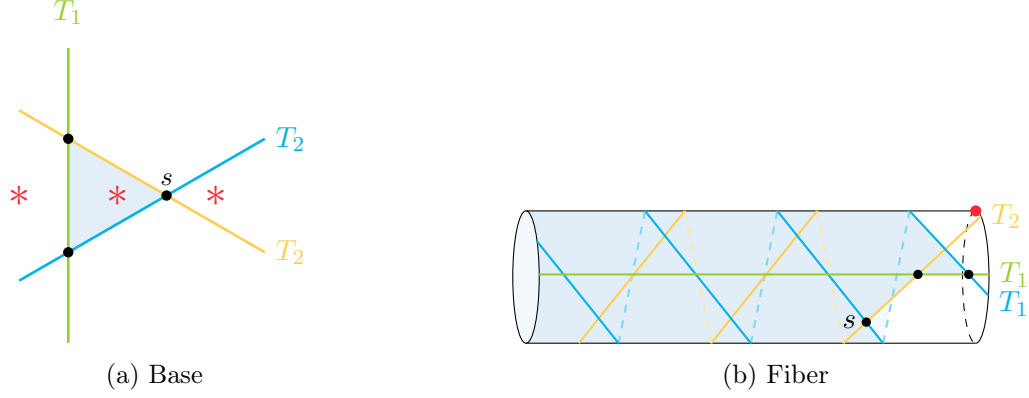


Figure 5: Proof of $\mu^2(p_2, q_1) = s_2$.

Note that in Fig. 5, the boundary of the disc in the fiber must rotate by one full turn relative to the stop. This occurs because the projection of the boundary to the base winds once around a puncture; according to the superpotential (3), moving around a puncture induces an opposite rotation of the stop in the fiber. Hence, for the boundary to close up and actually bound a disc, it must acquire an additional rotation in the fiber that cancels the rotation of the stop.

□

We will also write $a_2 \cdot a_1 := \mu^2(a_2, a_1)$ whenever no confusion can arise.

2.3.2 KLRW category $\mathcal{C}$ inside the Fukaya category

The KLRW algebra admits a convenient diagrammatic presentation, where generators and their compositions are represented by strands, dots, and crossings. In this paper, we restrict to the $\mathfrak{sl}_2$ case with $d = 1$, for which the diagrams and relations is much simplified.

Definition 2.7. Fix a collection F of $n = |\mathbf{x}|$ red points on $\mathbb{R}$. The KLRW category $\mathcal{C}$ we consider is defined as follows:

- **Objects:** configurations of a single black point on $\mathbb{R}$, distinct from the points in F .
- **Morphisms:** generated by strand diagrams in the strip $\mathbb{R} \times [0, 1]$, with no horizontal or vertical tangencies and only generic intersections. Black strands may carry dots as decorations.
- **Composition:** given by vertical stacking of diagrams. We read and compose the diagrams from the bottom to the top.
- **Relations:** diagrams are considered up to isotopy and subject to the local relations shown in Fig. 6.

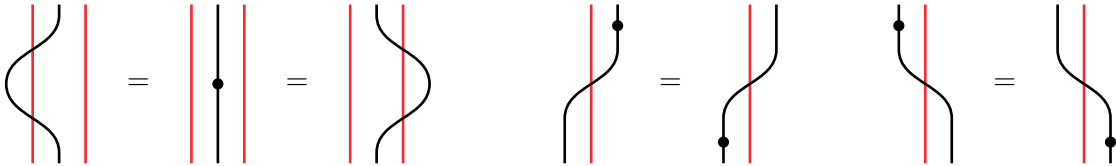


Figure 6: Nontrivial KLRW relations in $\mathcal{C}_{\bullet,1,F}$

Remark 2.8. More generally, one can define a KLRW category for any quiver Γ with dimension vector $\vec{d}$. The category $\mathcal{C}_{\Gamma, \vec{d}, F}$ corresponds precisely to the Fukaya category of the multiplicative Coulomb branch $\mathcal{M}(\Gamma, \vec{d})$ for ADE type Lie algebras. The simplified case considered above is $\mathcal{C}_{\bullet, 1, F}$, which corresponds to Aganagic's $\mathfrak{sl}_2$ theory with $d = 1$. We will recall the general definition when necessary.

The embedding $\mathcal{C} \hookrightarrow \text{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}})$ is defined as follows:

- The red points F correspond to the punctures $\mathbf{x}$ on the base of $\mathcal{M}(\bullet, 1)$. For example, we can choose $F = \text{Re}(\mathbf{x})$.
- **Objects:** the position of the black point specifies the location of the brane T_i . For instance, the KLRW object shown at the top of Fig. 7a corresponds to T_1 in the Fukaya category.
- **Morphisms:** a black strand connecting T_i to T_j , which does not cross any red strand more than once and is decorated with α dots, corresponds to the morphism $a_{ji}s^\alpha \in \text{Hom}(T_i, T_j)$ in the Fukaya category.
- **Composition:** one can directly verify that the composition in the KLRW category, under the correspondence above, reproduces the structure maps (9). Some relations follow automatically from diagram isotopy, while others arise from the defining relations shown in Fig. 6.

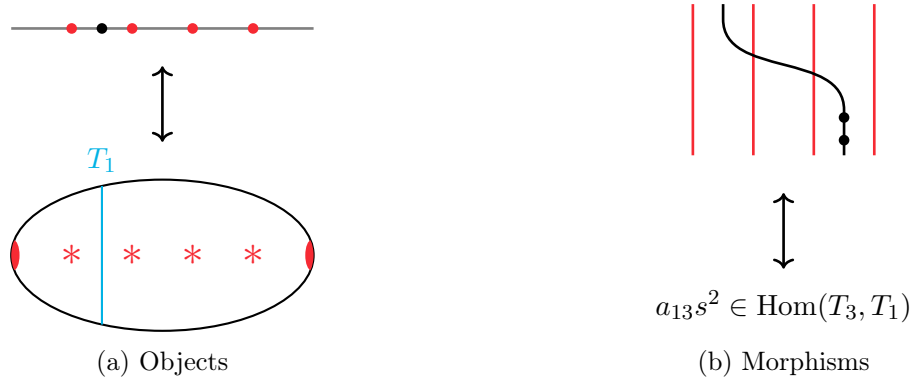


Figure 7: Illustration of the embedding of the KLRW category $\mathcal{C}_{\bullet, 1, F}$ in $\text{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}})$

2.4 General objects from generators

Besides the generating Lagrangians T_i , there exist other objects in the Fukaya category which appear as bounded twisted complexes built from the T_i . For such a twisted complex $(X^\bullet, \delta)$, the bounding cochain δ satisfies the Maurer–Cartan equation

$$\mu_{\Sigma\mathcal{C}}^1(\delta) + \mu_{\Sigma\mathcal{C}}^2(\delta, \delta) + \mu_{\Sigma\mathcal{C}}^3(\delta, \delta, \delta) + \cdots = 0. \quad (11)$$

where $\mu_{\Sigma\mathcal{C}}^d$ is the induced structure map on the additive enlargement defined in [10, §3k]. In our setting, since $\mu^d = 0$ for $d \neq 2$, this reduces to the simple condition

$$\mu^2(\delta, \delta) = \delta^2 = 0.$$

Hence $(X^\bullet, b)$ is nothing but an ordinary chain complex.

In [6], it is proved that all T -brane objects associated with curves without closed components can be generated from the basic objects T_i by a standard procedure. More precisely, it means

$$\mathrm{Tw}^b(\mathcal{C}) \cong \mathrm{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}}) \quad (12)$$

where in the latter we implicitly take all twisted complexes, which is essentially taking the pre-triangulated envelope since our category already has the idempotents e_i .

In the category $\mathrm{Tw}^b(\mathcal{C})$, the structure maps will be deformed by the bounding cochain by

$$\mu_\delta^d(a_d, \dots, a_1) = \sum_{i_0, \dots, i_d \geq 0} \mu_{\Sigma \mathcal{C}}^{d+i_0+\dots+i_d}(\delta_d^{\otimes i_d}, a_d, \delta_{d-1}^{\otimes i_{d-1}}, a_{d-1}, \dots, a_1, \delta_0^{\otimes i_0}). \quad (13)$$

since only $\mu^2 \neq 0$, the only nontrivial μ_δ^d are

$$\mu_\delta^1(a) = (-1)^{|a|} \mu^2(\delta, a) - \mu^2(a, \delta), \quad (14)$$

$$\mu_\delta^2(a_2, a_1) = (-1)^{|a_1|} \mu^2(a_2, a_1). \quad (15)$$

Here μ^2 on the RHS denotes the naïve composition in which the grading is ignored, that is, all morphisms are treated as of degree 0. We will always write $a_2 \cdot a_1 = \mu^2(a_2, a_1)$ for the naïve composition without adding any signs.

Remark 2.9. *The signs here follow strictly from Seidel's book [10, §3k], where the sign $(-1)^\triangleleft$ before each term is given by $\triangleleft = \sum_{p < q} |\phi_p^{i_p, i_{p-1}}| \cdot (|x_q^{i_q, i_{q-1}}| - 1)$ for morphisms*

$$a_q^{i_q i_{q-1}} = \phi_q^{i_q i_{q-1}} \otimes x_q^{i_q i_{q-1}} \in \mathrm{Hom}_{\mathbb{k}}(V_{q-1}^{i_{q-1}}, V_q^{i_q}) \otimes \mathrm{Hom}_{\mathcal{A}}(X_{q-1}^{i_{q-1}} \otimes X_q^{i_q}),$$

where the underlying space for each twisted complex is given by $\bigoplus_i V_i \otimes X_i$.

For our category $\mathcal{C}$, all morphisms are of degree 0, thus all $|x| = 0$, and $|\phi| = |a|$, so we have the $\triangleleft$ before each $\mu^2(x_2, x_1)$ should be $|a_1|(0 - 1) = -|a_1|$.

For the simplest examples, we can stretch and resolve them into mapping cones, i.e. Lagrangian connected sums, of T_i . For example,

$$\begin{array}{ccccc} * & * & * & * & \xleftarrow{\text{Cone}} & * & * & * & * & \longleftrightarrow & T_2 & \xrightarrow{|||} & T_1 \end{array} \quad (16)$$

$$\begin{array}{ccccc} * & * & * & * & \xleftarrow{\text{Cone}} & * & * & * & * & \longleftrightarrow & T_2 & \begin{array}{l} \xrightarrow{|||} T_1 \\ \xrightarrow{-|||} T_3 \end{array} & \oplus \end{array} \quad (17)$$

$$(18)$$

Besides T -branes, there exists another family of Lagrangian submanifolds of particular interest, whose projections to the fiber, illustrated in Fig. 8b. We refer to these objects as I -branes. Note that I -branes can end on punctures naturally, where the Lagrangian approaches $y \rightarrow \infty$, while T -branes cannot. Denote the I -brane in (21) connecting punctures x_i and x_{i+1} as I_i . These objects are *dual* generators of the brane algebra [1] to the T -branes in the sense that

$$\mathrm{Hom}(T_i, I_j) = \delta_{ij} \mathbb{k} = \mathrm{Hom}(I_i, T_j). \quad (19)$$



Figure 8: T -fiber and I -fiber

We recall some well-known results of I -branes as well.

$$(20)$$

$$(21)$$

Remark 2.10. It is an interesting fact that (16) and (20) have the same resolution. The resulting complex is an example of standard modules defined by Webster in [15, §5]. This equivalence has both a disc-counting argument in [6] and a geometric argument in [13].

3 Structure of the braiding functors

Having established the structure of the category, we now proceed to discuss functors.

In general, an A_∞ -functor between categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map $F: \text{Ob}\mathcal{A} \rightarrow \text{Ob}\mathcal{B}$ and a series of maps $\{F^d\}$ for $d \geq 1$:

$$F^d: \text{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}}(X_0, X_1) \rightarrow \text{Hom}_{\mathcal{B}}(FX_0, FX_d)[1-d], \quad (22)$$

where each $X_i \in \text{Ob}\mathcal{A}$. The maps follow the constraints [10, §1b]:

$$\begin{aligned} & \sum_{r \geq 1} \sum_{i_1 + \cdots + i_r = d} \mu_{\mathcal{B}}^r(F^{i_r}(a_d, \dots, a_{d-i_r+1}), \dots, F^{i_1}(a_{i_1}, \dots, a_1)) \\ &= \sum_{m, n} (-1)^{\mathbf{x}_n} F^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu_{\mathcal{A}}^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1). \end{aligned} \quad (23)$$

Functors are composed by

$$\begin{aligned} & (G \circ F)^d(a_d, \dots, a_1) \\ &= \sum_{r \geq 1} \sum_{i_1 + \cdots + i_r = d} G^r(F^{i_r}(a_d, \dots, a_{d-i_r+1}), \dots, F^{i_1}(a_{i_1}, \dots, a_1)) \end{aligned} \quad (24)$$

Given an A_∞ -functor $F: \mathcal{A} \rightarrow \mathcal{B}$, one associates a $(\mathcal{B}, \mathcal{A})$ -bimodule by the Yoneda embedding of $F(X)$

$$B_F(-, X) = \text{Hom}_{\mathcal{B}}(-, F(X)). \quad (25)$$

The structure maps $\mu_{B_F}^{0|1|s}$ coincide with the higher components F^s of the functor, while the abstract realization of F is given by the derived tensor functor

$$F \simeq - \otimes_{\mathcal{A}}^L B_F.$$

As mentioned in the introduction, given a set of punctures $\mathbf{x}$, the braid group $\text{Br}_{|\mathbf{x}|}$ acts on $\mathcal{M}(\bullet, n)$ by braiding these punctures, which in turn induces an action on the corresponding Fukaya category.

$$\rho: \text{Br}_{|\mathbf{x}|} \rightarrow \text{Aut}(\text{Fuk}_{|||}(\mathcal{M}(\Gamma, \vec{d}), \mathcal{W}_{\mathbf{x}})) \quad (26)$$

The most essential part of this structure is the action of the generators of $\text{Br}_{|\mathbf{x}|}$ and their inverses, and we can use (24) to get everything. In what follows, we describe these functors in detail.

3.1 Identity functor

The simplest example of a braiding functor is the identity functor $\text{id} = \rho(1)$

$$\text{id}: \text{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}}) \longrightarrow \text{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}}), \quad (27)$$

which, by definition, acts trivially on the category. Explicitly, on objects and morphisms we have

$$\text{id}(X) = X, \quad (28)$$

$$\text{id}^1(a) = a, \quad (29)$$

$$\text{id}^d = 0, \quad d \neq 1. \quad (30)$$

The A_∞ -functor equations (23) are then automatically satisfied.

From the bimodule point of view, this functor corresponds to the *diagonal bimodule*

$$\Delta(-, X) = \text{Hom}(-, X), \quad (31)$$

whose structure maps are the same as those of the category itself.

3.2 Negative braiding functor

The *negative braiding functor* $\beta_{i-} = \rho(\sigma_i^{-1})$ is the action of $\sigma_i^{-1} \in \text{Br}_n$ under (26), which clockwise exchanges the punctures x_i and x_{i+1} . We first specify its action on $\mathcal{C}$.

3.2.1 Action on objects

On the object level, as shown in Fig. 9, negative braiding β_{i-} twists T_i into the object in (17), while leaving all other generators unchanged.

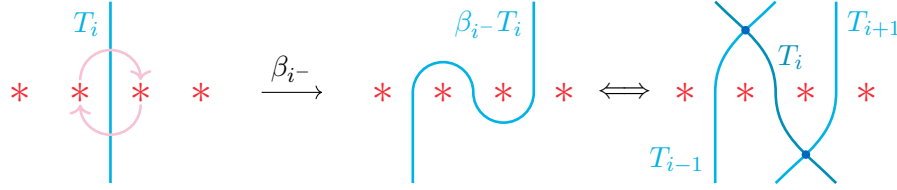


Figure 9: Negative braiding functor β_{i-} acting on T_i

Theorem 3.1. *The negative braiding functor β_{i-} acts on generators T_j by*

$$\beta_{i-}T_j = \begin{cases} T_j, & i \neq j, \\ T_i \xrightarrow{\left[\begin{array}{c} |||| \\ - |||| \end{array} \right]} T_{i-1} \oplus T_{i+1} & i = j. \end{cases} \quad (32)$$

3.2.2 Action on morphisms

On the morphism level, β_{i-}^1 acts trivially on $\text{Hom}(T_j, T_k)$ unless $j = i$ or $k = i$,

- For $a = a_{kj}s^\alpha \in \text{Hom}(T_j, T_k)$ with $j, k \neq i$:

$$\beta_{i-}^1(T_j \xrightarrow{s^\alpha} T_k) = T_j \xrightarrow{s^\alpha} T_k. \quad (33)$$

where we write s^α on an arrow $T_i \rightarrow T_j$ to indicate the morphism contains α dots, i.e. $a_{ji}s^\alpha \in \text{Hom}(T_i, T_j)$.

- For $a = a_{ij}s^\alpha \in \text{Hom}(T_j, T_i)$ with $j \neq i$:

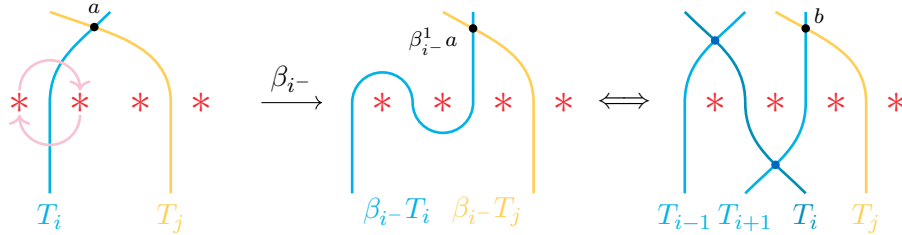


Figure 10: β_{i-}^1 acting on $\text{Hom}(T_j, T_i)$ with $j > i$

The morphism b in Fig. 10 have the same number of dots with a . Thus, we have $\beta_{i-}^1(a) = b$ for $j > i$, which is

$$\beta_{i-}^1(T_j \xrightarrow{s^\alpha} T_i) = T_j \xrightarrow{b = s^\alpha} \left[\begin{array}{c} T_i \xrightarrow{\text{red } \text{||}} T_{i-1} \\ \oplus \\ T_i \xrightarrow{\text{red } - \text{||}} T_{i+1} \end{array} \right] \quad (34)$$

Similarly, we get for $j < i$,

$$\beta_{i-}^1(T_j \xrightarrow{s^\alpha} T_i) = T_j \xrightarrow{s^\alpha} \left[\begin{array}{c} T_i \xrightarrow{\text{red } \text{||}} T_{i-1} \\ \oplus \\ T_i \xrightarrow{\text{red } - \text{||}} T_{i+1} \end{array} \right] \quad (35)$$

- For $a = a_{ki}s^\alpha \in \text{Hom}(T_i, T_k)$ with $k \neq i$:

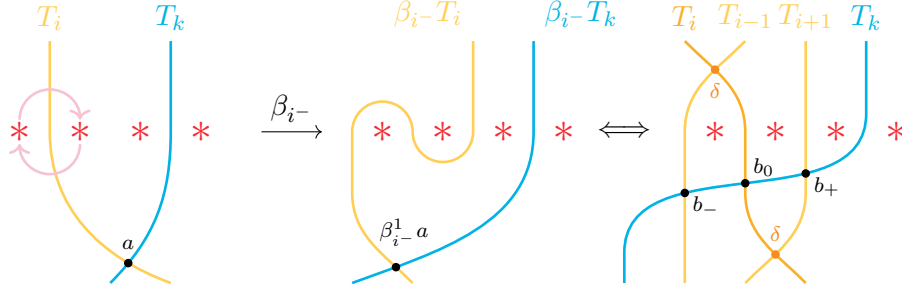


Figure 11: β_{i-}^1 acting on $\text{Hom}(T_i, T_k)$ with $k > i$

We have $\beta_{i-}^1(a) = [b_-, b_+]$. Here, b_- is directly inherited from a , thus $b_- = a_{k,i-1}s^\alpha$. b_+ and b_0 are given by the pseudoholomorphic discs $b_0 = b_- \cdot \delta = b_+ \cdot \delta$, thus $b_+ = a_{k,i+1}s^{\alpha+1}$ since one of the discs encloses a puncture.

For $k > i$,

$$\beta_{i-}^1(T_i \xrightarrow{s^\alpha} T_k) = \left[\begin{array}{c} T_i \xrightarrow{\text{red } \text{||}} T_{i-1} \\ \oplus \\ T_i \xrightarrow{\text{red } - \text{||}} T_{i+1} \end{array} \right] \xrightarrow{\begin{array}{c} s^\alpha \\ s^{\alpha+1} \end{array}} T_k \quad (36)$$

Correspondingly, for $k < i$

$$\beta_{i-}^1(T_i \xrightarrow{s^\alpha} T_k) = \left[\begin{array}{ccc} & & T_{i-1} \\ & \nearrow \textcolor{red}{\S} & \\ T_i & & \\ & \searrow -\textcolor{red}{\S} & \\ & & T_{i+1} \end{array} \oplus \begin{array}{ccc} & & T_k \\ & \nearrow s^{\alpha+1} & \\ & & \\ & \searrow s^\alpha & \\ & & \end{array} \right] \quad (37)$$

The only morphisms left are the ones in $\text{Hom}(T_i, T_i)$, which can be computed using the same procedure above. To be concise, we write out the results directly.

- For $a = s_i^\alpha \in \text{Hom}(T_i, T_i)$:

$$\beta_{i-}^1(T_i \xrightarrow{s^\alpha} T_i) = \left[\begin{array}{ccc} & & T_{i-1} \\ & \nearrow \textcolor{red}{\S} & \\ T_i & & \\ & \searrow -\textcolor{red}{\S} & \\ & & T_{i+1} \end{array} \oplus \begin{array}{ccc} & & T_{i-1} \\ & \nearrow s^\alpha & \\ & & \\ & \searrow s^\alpha & \\ & & T_{i+1} \end{array} \oplus \begin{array}{ccc} & & T_{i-1} \\ & \nearrow \textcolor{red}{\S} & \\ T_i & & \\ & \searrow -\textcolor{red}{\S} & \\ & & T_{i+1} \end{array} \right] \quad (38)$$

Theorem 3.2. (33)–(38) gives the action of β_{i-}^1 on all morphisms $\text{Hom}(T_j, T_k)$.

Remark 3.3. We can already verify the condition for a functor (23) for $d = 1$, which is

$$\mu_\delta^1 \beta_{i-}^1(a) = \beta_{i-}^1 \mu_\delta^1(a). \quad (39)$$

Since for morphisms between generators we have $\mu_\delta^1(a) = 0$ and $|\beta_{i-}^1(a)| = 0$, this is

$$\mu_\delta^1 \beta_{i-}^1(a) = \delta \beta_{i-}^1(a) - \beta_{i-}^1(a) \delta = 0, \quad (40)$$

which is followed by direct computation.

3.2.3 Higher components

There remains one piece of the puzzle to be clarified. According to (23), the components β_{i-}^d carry a cohomological degree shift of $[1 - d]$. Theorem 3.1 shows that the bimodule $\beta_{i-} T_j$ is a complex concentrated in degree 0 for $j \neq i$, while for $j = i$ it has nonzero terms in degrees 0 and -1 . Consequently, the only potentially nontrivial components β_{i-}^d on $\mathcal{C}$ occur for $d = 1$ and $d = 2$,

$$\beta_{i-}^d = 0, \quad d \geq 3. \quad (41)$$

In particular, since β_{i-}^2 has degree -1 , it can be nonzero only on composable morphisms of the form

$$T_j \longrightarrow T_k \longrightarrow T_i.$$

where β_{i-}^2 gets a morphism in $\text{Hom}^{-1}(\beta_{i-} T_j, \beta_{i-} T_i)$.

The component β_{i-}^2 can be solved directly from (23) with $d = 2$, which is

$$\mu_\delta^2(\beta_{i-}^1(a_2), \beta_{i-}^1(a_1)) + \mu_\delta^1(\beta_{i-}^2(a_2, a_1)) = \beta_{i-}^1(\mu^2(a_2, a_1)) \quad (42)$$

$$\implies \delta \beta_{i-}^2(a_2, a_1) + \beta_{i-}^2(a_2, a_1) \delta = \beta_{i-}^1(a_2) \cdot \beta_{i-}^1(a_1) - \beta_{i-}^1(a_2 \cdot a_1) \quad (43)$$

Theorem 3.4. *This was first computed by Marco David. For $a_{lk}s^\beta \otimes a_{kj}s^\alpha \in \text{Hom}(T_k, T_l) \otimes \text{Hom}(T_j, T_k)$, $\beta_{i-}^2(a_{lk}s^\beta, a_{kj}s^\alpha)$ is given by:*

- if $l \neq i$,

$$\beta_{i-}^2(a_{lk}s^\beta, a_{kj}s^\alpha) = 0; \quad (44)$$

- if $l = i$ and $j \neq i$,

$$- \text{ if } (i-j)(i-k) \geq 0,$$

$$\beta_{i-}^2(a_{ik}s^\beta, a_{kj}s^\alpha) = 0; \quad (45)$$

$$- \text{ if } (i-j)(i-k) < 0,$$

$$\beta_{i-}^2(a_{ik}s^\beta, a_{kj}s^\alpha) = \text{sgn}(i-k)a_{ij}s^{\beta+\alpha+|i-k|-1}, \quad (46)$$

which is a map from $\beta_{i-}T_j = T_j$ to the degree -1 component of $\beta_{i-}T_i$;

- if $l = j = i$,

$$\beta_{i-}^2(a_{ik}s^\beta, a_{ki}s^\alpha) = \text{sgn}(i-k)a_{i,i+\text{sgn}(i-k)}s^{\beta+\alpha+|i-k|-1}, \quad (47)$$

which is a degree -1 map from $(\beta_{i-}T_i)^0 = T_{i-1} \oplus T_{i+1}$ to $(\beta_{i-}T_i)^{-1} = T_i$.

Proof. These can be checked directly to satisfy (43). For example, for $T_j \xrightarrow{s^\alpha} T_k \xrightarrow{s^\beta} T_i$ with $k < i < j$,

$$\begin{array}{c} \begin{array}{c} T_j \xrightarrow{s^\alpha} T_k \xrightarrow{s^\beta} T_i \\ \searrow s^{\alpha+\beta+|i-k|} \end{array} \left[\begin{array}{c} \xrightarrow{s^\beta} T_{i-1} \\ \xrightarrow{s^{\alpha+\beta+|i-k|-1}} T_i \\ \xrightarrow{s^{\alpha+\beta+|i-k|}} T_{i+1} \end{array} \right] \end{array} \quad \begin{array}{l} \beta_{i-}^1 \beta_{i-}^1 \\ \beta_{i-}^2 \\ \beta_{i-}^1 \mu^2 \end{array} \quad (48)$$

for $T_i \xrightarrow{s^\alpha} T_k \xrightarrow{s^\beta} T_i$ with $k < i$,

$$\left[\begin{array}{c} T_i \xrightarrow{s^\alpha} T_k \xrightarrow{s^\beta} T_i \\ \searrow s^{\alpha+\beta+|i-k|} \end{array} \right] \left[\begin{array}{c} \xrightarrow{s^\beta} T_{i-1} \\ \xrightarrow{s^{\alpha+\beta+|i-k|-1}} T_i \\ \xrightarrow{s^{\alpha+\beta+|i-k|}} T_{i+1} \end{array} \right] \quad \begin{array}{l} \beta_{i-}^1 \beta_{i-}^1 \\ \beta_{i-}^2 \\ \beta_{i-}^1 \mu^2 \end{array} \quad (49)$$

□

Let us note that β_{i-}^2 also has direct geometrical meaning. As an illustrative example, consider $a_2 \otimes a_1 \in \text{Hom}(T_{i-1}, T_i) \otimes \text{Hom}(T_{i+1}, T_{i-1})$ under braiding,

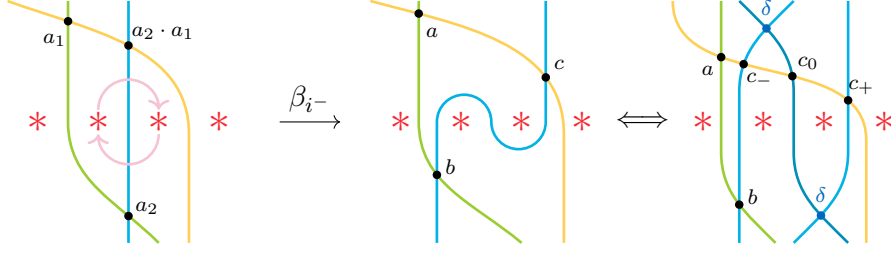


Figure 12: Action of β_{i-} on $\text{Hom}(T_{i-1}, T_i) \otimes \text{Hom}(T_{i+1}, T_{i-1})$

Before resolving $\beta_{i-}T_i$, we have a disc $c = b \cdot a$. After resolving $\beta_{i-}T_i$, the disc $c = ba$ breaks into three smaller discs, we have $c_- = b \cdot a = \beta_{i-}^1(a_2) \cdot \beta_{i-}^1(a_1)$ and $c_+ = \beta_{i-}^1(a_2 \cdot a_1)$, thus $\beta_{i-}^1(a_2) \cdot \beta_{i-}^1(a_1) \neq \beta_{i-}^1(a_2 \cdot a_1)$. Nevertheless, c_- and c_+ are cohomologous since $\delta c_0 = c_- - c_+$ is a coboundary. Thus $c_0 = \beta_{i-}^2(a_2, a_1)$ gives the information of chain homotopy.

3.2.4 Extension to the full category

One can check that the $d > 2$ equations of (23) are satisfied. Thus, we have constructed a functor

$$\beta_{i-} : \mathcal{C} \rightarrow \text{Tw}^b(\mathcal{C}). \quad (50)$$

where $\text{Tw}^b(\mathcal{C}) \cong \text{Fuk}_{|||}(\mathcal{M}(\bullet, 1), \mathcal{W}_{\mathbf{x}})$. We are then able to extend it canonically to a functor

$$\beta_{i-} : \text{Tw}^b(\mathcal{C}) \rightarrow \text{Tw}^b(\mathcal{C}). \quad (51)$$

using the construction in [10, §3m].

Remark 3.5. In fact, the A_∞ -functor β_{i-} that acts on the dg-category of $\text{Perf}(\mathcal{C})$ is only uniquely defined up to contractible choices, there are other totally good choices. For instance, in (34), whenever $\alpha \geq 1$, the value of β_{i-}^1 on $a_{ij}s^\alpha$ may be chosen as

$$\beta_{i-}^1(a_{ij}s^\alpha) = [\lambda a_{i-1,j}s^{\alpha-1}, (1-\lambda)a_{i+1,j}s^\alpha] \quad (52)$$

for any $\lambda \in \mathbb{k}$. Geometrically, this corresponds to stretching T_j across T_i and T_{i-1} in Fig. 10, thereby producing the corresponding discs. Different choices of λ lead to different representatives of β_{i-}^1 , but these ambiguities are compensated by appropriate corrections in β_{i-}^2 , so that the resulting A_∞ -functor remains homotopy equivalent. Throughout, we use the simplest such representative.

3.2.5 Realization as a bimodule

The $(\mathcal{C}, \mathcal{C})$ -bimodule $\mathfrak{B}_{i-}$ corresponding to the functor β_{i-} admits a diagrammatic description as the crossing of the red strands associated with the punctures x_i and x_{i+1} [15, 16]. An element in the bimodule is a strand diagram, considered up to isotopy and the relations in Figs. 6 and 13. The bimodule action is given by vertical stacking of KLRW diagrams on the top and bottom.

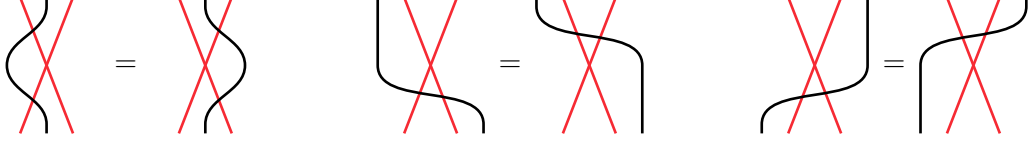


Figure 13: Additional relations for braiding bimodules

To see why this gives negative braiding, consider $\mathfrak{B}_{i-}(-, T_i)$ as a one-side module, resolving this gives

(53)

where the boxes mean all possible diagrams with the specified ends, and the maps mean appending the diagrams on the top. The resolution is precisely the Yoneda embedding $\text{Hom}(-, \beta_{i-}T_i)$. Therefore,

$$\mathfrak{B}_{i-}(-, T_i) = \text{Hom}(-, \beta_{i-}T_i) \quad (54)$$

which is expected. Here, $\mathfrak{B}_{i-}$ is treated as an object in the abelian category of $\mathcal{C}$ -bimodules. The higher component of braiding functors can be realized by resolving the corresponding one-side modules. Here in the resolution, one also finds the same choices encountered in Remark 3.5.

There is a monomorphism of bimodules

$$\iota_i : \quad \mathfrak{B}_{i-} \hookrightarrow \Delta \quad (55)$$

and it acts on other elements in the obvious way. The cokernel of this monomorphism

$$0 \longrightarrow \mathfrak{B}_{i-} \xrightarrow{\iota_i} \Delta \xrightarrow{\pi_i} \mathfrak{S}_i \longrightarrow 0 \quad (56)$$

is generated by only a single diagram e_i in Fig. 14.

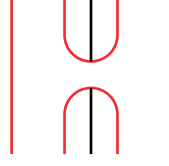


Figure 14: The only diagram e_i in $\mathfrak{S}_i = \text{coker } \iota_i$

Multiplication in $\mathfrak{S}_i$ is given by

$$a_{ki}s^\beta \cdot e_i \cdot a_{ij}s^\alpha = \begin{cases} e_i, & j = k = i, \alpha = \beta = 0, \\ 0, & \text{else.} \end{cases} \quad (57)$$

This can be summarized diagrammatically as Fig. 15 [16].

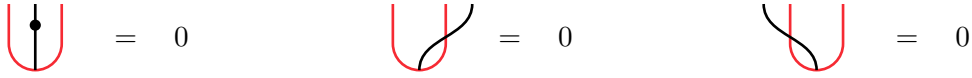


Figure 15: Relations for $\mathfrak{S}_i$

The epimorphism π_i is given by

$$\pi_i : \quad \Delta \longrightarrow \mathfrak{S}_i \quad (58)$$

and it maps everything else to 0.

Remark 3.6. *This is not a coincidence. In general, the situation can be described as follows. Suppose we have a Lefschetz fibration $f: Y \rightarrow \mathbb{C}$ with a singular value at 0. On the Fukaya category of a regular fiber $\text{Fuk}(f^{-1}(1))$, there are two twisting exact triangles [11]:*

$$S \otimes \text{Hom}(S, -) \longrightarrow \text{id} \longrightarrow T_\circ \xrightarrow{[1]}$$

and

$$T_\circ \longrightarrow \text{id} \longrightarrow S \otimes \text{Hom}(-, S)^\vee \xrightarrow{[1]}.$$

Here $T_\circ$ denotes the monodromy obtained by going around 0 counterclockwise, and $S \subset f^{-1}(1)$ is the vanishing cycle. The second triangle can be viewed as applying the clockwise monodromy $T_\circ$ to the first one, together with the usual identification $\text{Hom}(-, S)^\vee \simeq \text{Hom}(S, -)[\pm 1]$.

In Aganagic's setup, one considers a one-parameter family of spaces where the puncture positions $\mathbf{x} = \{x_i\}$ vary with a parameter t . Consider

$$(x - x_i)(x - x_{i+1}) = x^2 - t,$$

while other punctures stay fixed. The resulting family defines a generalized Lefschetz fibration $\tilde{\pi} : \tilde{Y} \rightarrow \mathbb{C}_t$, applying the twisting exact triangles gives (56), where the vanishing cycle is I_i in (21).

The construction of the positive braiding functors β_{i+} proceeds identically. For the moment, we omit it here.

4 A_∞ -natural transformations and Hochschild cohomology

In this section we recall the definition of A_∞ -natural transformations, describe their differential, explain their interpretation in terms of Hochschild cohomology and bimodules, and clarify our notation conventions.

For two A_∞ categories $\mathcal{A}$ and $\mathcal{B}$, the functors from $\mathcal{A}$ to $\mathcal{B}$ form another A_∞ -category $\text{Fun}(\mathcal{A}, \mathcal{B})$. A morphism $\eta \in \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}^g(F, G)$ is called a *pre-natural transformation*, which consists of a series of multilinear maps $\{\eta^d\}$ for $d \geq 0$:

$$\eta^d: \text{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \text{Hom}_{\mathcal{A}}(X_0, X_1) \rightarrow \text{Hom}_{\mathcal{B}}(FX_0, GX_d)[g-d], \quad (59)$$

where each $X_i \in \text{Ob}\mathcal{A}$.

There are A_∞ -structure maps $\mu_{\text{Fun}(\mathcal{A}, \mathcal{B})}^k$. In particular, the differential $\mu_{\text{Fun}(\mathcal{A}, \mathcal{B})}^1$ acts on η by

$$\begin{aligned} & \mu_{\text{Fun}(\mathcal{A}, \mathcal{B})}^1(\eta)^d(a_d, \dots, a_1) \\ &= \sum_{r,i} \sum_{s_1+\dots+s_r=d} (-1)^\dagger \mu_{\mathcal{B}}^r \left(\begin{array}{c} G^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, G^{s_{i+1}}(\dots, a_{s_i+\dots+s_{i+1}}), \\ \eta^{s_i}(a_{s_i+\dots+s_i}, \dots, a_{s_{i-1}+1}), \\ F^{s_{i-1}}(a_{s_1+\dots+s_{i-1}}, \dots), \dots, F^{s_1}(a_{s_1}, \dots, a_1) \end{array} \right) \\ & \quad - \sum_{m,n} (-1)^{\mathfrak{X}_n+|\eta|-1} \eta^{d-m+1}(a_d, \dots, a_{n+m+1}, \mu_{\mathcal{A}}^m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1), \end{aligned} \quad (60)$$

where $\dagger = (|\eta| - 1)(|a_1| + \dots + |a_{s_1+\dots+s_{i-1}}| - s_1 - \dots - s_{i-1})$.

Definition 4.1. A degree g A_∞ -natural transformation $\eta: F \Rightarrow G$ is a pre-natural transformation $\eta \in \text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{B})}^g(F, G)$ satisfying the cocycle condition

$$\mu_{\text{Fun}(\mathcal{A}, \mathcal{B})}^1(\eta) = 0. \quad (61)$$

When $\mathcal{B} = \mathcal{A}$ and $F = \text{id}_{\mathcal{A}}$, the cochain complex $(\text{Hom}_{\text{Fun}(\mathcal{A}, \mathcal{A})}^\bullet(\text{id}, G), \mu_{\text{Fun}(\mathcal{A}, \mathcal{A})}^1)$ is the Hochschild cochain $\text{CC}^\bullet(G)$, and natural transformations are the Hochschild cocycles.

The Hochschild cohomology $\text{HH}^\bullet(G) = H^\bullet \text{Hom}(\text{id}, G)$ captures the essential structure of the space of natural transformations. In the following sections, we will first determine $\text{HH}^\bullet(\text{id})$ and $\text{HH}^\bullet(\beta_{i-})$, and then write out the details of the natural transformations.

By Morita equivalence, we have

$$\text{Hom}_{\text{Fun}(\text{Tw}^b(\mathcal{C}), \text{Tw}^b(\mathcal{C}))}(F, G) \simeq \mathbf{R}\text{Hom}_{\mathcal{C}\text{-mod-}\mathcal{C}}(B_F, B_G). \quad (62)$$

Hence the Hochschild cohomology may be computed as the derived Hom complex from the diagonal bimodule to the target bimodule. To carry out this computation, one needs a projective resolution of the diagonal bimodule. The Hochschild complex is defined canonically on the bar resolution:

Definition 4.2. The bar resolution of the diagonal bimodule $\Delta_{\mathcal{C}}$ is the A_∞ -bimodule complex

$$\text{Bar}_d(\mathcal{C}) = (\mathcal{C}[1])^{\otimes d}, \quad (63)$$

whose elements are written as tuples $(a_d, \dots, a_1)$. The differential

$$\partial_{\text{Bar}}: \text{Bar}_d(\mathcal{C}) \rightarrow \text{Bar}_{d-1}(\mathcal{C}) \quad (64)$$

is defined by inserting the A_∞ -structure maps $\mu_{\mathcal{C}}^k$ into all consecutive subsequences:

$$\partial_{\text{Bar}}(a_d, \dots, a_1) = \sum_{m,n} (-1)^{\mathfrak{A}_n} (a_d, \dots, a_{m+n+1}, \mu_{\mathcal{C}}^m(a_{m+n}, \dots, a_{n+1}), a_n, \dots, a_1), \quad (65)$$

and $\text{Bar}_\bullet(\mathcal{C}) \xrightarrow{\epsilon} \Delta_{\mathcal{C}}$ is a projective resolution of the diagonal bimodule.

Computing the Hochschild cohomology is known to be a hard problem because, although the bar resolution is always available in principle, its size $\text{Bar}_n = \mathcal{C} \otimes (\mathcal{C}[1])^{\otimes n} \otimes \mathcal{C}$ grows exponentially with n , making it computationally infeasible in practice. The main drawback of the bar resolution is that it ignores the specific algebraic relations present in $\mathcal{C}$. To obtain a more tractable model for actual calculations, one seeks a projective resolution of the diagonal bimodule that makes better use of the intrinsic structure of $\mathcal{C}$.

5 The Chouhy–Solotar resolution

In 1997, M. Bardzell constructed a projective resolution for monomial algebras [3]. Later, in 2014, S. Chouhy and A. Solotar extended this construction to quiver algebras with relations [5]. As we will see momentarily, for our category $\mathcal{C}$, the main advantage of this resolution is that the size of each projective bimodule stabilizes as n increases, making the computation of Hochschild cohomology feasible in all degrees. We summarize the Chouhy–Solotar resolution below.

5.1 Reduction system

Let Q be a finite quiver with Q_0 the set of vertices and Q_1 the set of arrows. Denote by Q_n the set of paths of length n . To be consist with the notations in A_∞ -categories, our convention for a path $u = u_{k-1} \dots u_1 u_0 \in Q_k$ where $u_i \in Q_1$ is composing from the right to left, i.e. $\bullet \xleftarrow{u_{k-1}} \dots \bullet \xleftarrow{u_1} \bullet \xleftarrow{u_0} \bullet$. We write vu to denote the concatenation of paths, that is, the path obtained by connecting the target of u with the source of v . We write $v \subset u$ if v is a subpath of u .

Definition 5.1. *The path algebra $\mathbb{k}Q$ of a quiver Q over a field $\mathbb{k}$ is the tensor algebra of the vector space $\mathbb{k}Q_1$ of arrows over the semisimple algebra $\mathbb{k}Q_0$ spanned by the vertices:*

$$\mathbb{k}Q := T_{\mathbb{k}Q_0}(\mathbb{k}Q_1) = \mathbb{k}Q_0 \oplus \bigoplus_{n \geq 1} (\mathbb{k}Q_1)^{\otimes_{\mathbb{k}Q_0} n} =: \bigoplus_{n \geq 0} \mathbb{k}Q_n. \quad (66)$$

Each $\mathbb{k}Q_n$ is the $\mathbb{k}$ -span of paths of length n , and multiplication is given by concatenation of paths whenever it is defined.

Let $I \subset \mathbb{k}Q$ be a two-sided ideal, referred to as the *relations*. The quotient $A = \mathbb{k}Q/I$ is called the *path algebra of a quiver with relations*. We denote by $\pi : \mathbb{k}Q \rightarrow A$ the canonical projection and by $i : A \hookrightarrow \mathbb{k}Q$ the inclusion of a chosen section.

Definition 5.2. *A reduction system R for $\mathbb{k}Q$ consists of a collection of pairs*

$$R = \{(s, \varphi_s) \mid s \in S, \varphi_s \in \mathbb{k}Q\}, \quad (67)$$

where

- S is a subset of paths of length at least 2, and distinct elements of S share no common subpaths.
- For each $s \in S$, the elements s and φ_s are parallel, meaning they have the same source and target.
- Each φ_s is irreducible, that is, a $\mathbb{k}$ -linear combination of irreducible paths.

Here, a path is called *irreducible* if it does not contain any element of S as a subpath. We denote by

$$\text{Irr}_S(Q) = Q_\bullet \setminus Q_\bullet S Q_\bullet$$

the set of all such irreducible paths. Each element of $\text{Irr}_S(Q)$ serves as a representative of a coset in the quotient algebra $A = \mathbb{k}Q/I$. Hence, as a $\mathbb{k}$ -vector space we have the natural isomorphism

$$\mathbb{k}\text{Irr}_S(Q) \cong A = \mathbb{k}Q/I. \quad (68)$$

Given a reduction system R , we associate to it the ideal of relations

$$I = \langle s - \varphi_s \mid s \in S \rangle$$

and the corresponding path algebra with relations $A = \mathbb{k}Q/I$.

Remark 5.3. Following [4], we write $\otimes$ for $\otimes_{\mathbb{k}Q_0}$ and Hom for $\text{Hom}_{\mathbb{k}Q_0^e}$, where $\mathbb{k}Q_0^e = \mathbb{k}Q_0 \otimes_{\mathbb{k}} \mathbb{k}Q_0^{\text{op}}$.

Definition 5.4. Finally, we define a relation $\preceq$ on the set

$$\mathbb{k}^\times Q_{\geq 0} := \{\lambda p \mid \lambda \in \mathbb{k}^\times, p \in Q_{\geq 0}\} \cup \{0\}.$$

We define $\preceq$ as the least reflexive and transitive relation satisfying that $\lambda p \preceq \mu q$ whenever the μq can be reduced, through a finite (including length 0) sequence of applications of the reduction system, that is, by successively replacing a subpath $s \subset q$ with φ_s for some $s \in S$, to a linear combination $\lambda p + x$ in which p does not appear among the terms of x . In addition, we write $0 \preceq \lambda p$ for all $\lambda p \in \mathbb{k}^\times Q_{\geq 0}$.

5.2 Ambiguities

Definition 5.5. A (right) n -ambiguity is a path $u = u_{n+1} \cdots u_1 u_0$ satisfying the following conditions:

- For every i , the concatenation $u_{i+1}u_i$ belongs to S ;
- For every factorization $u_{i+1} = rq$ with $r \neq 0$, one has $qu_i \notin S$.

Remark 5.6. Left n -ambiguities are defined analogously. It can be shown that left and right ambiguities are equivalent notions.

Such paths are called *ambiguities* because they admit multiple possible reductions. A reduction system is said to be *reduction unique* if all possible reduction sequences of a path yield the same irreducible result. In what follows, all quivers under consideration are assumed to be reduction unique.

For convenience, we set $S_0 = Q_0$, $S_1 = Q_1$, $S_2 = S$, and for $n > 2$, let S_n denote the set of $(n - 2)$ -ambiguities.

Definition 5.7. For each $n \geq 0$, define a $\mathbb{k}$ -linear map

$$\text{split}_n : \mathbb{k}Q \longrightarrow A \otimes \mathbb{k}S_n \otimes A$$

by the formula

$$\text{split}_n(w) := \sum_{\substack{w=urv \\ r \in S_n}} \pi(u) \otimes r \otimes \pi(v), \quad (69)$$

where the sum runs over all decompositions of w through a subpath $r \in S_n$. In addition, define

$$\text{split}^R(w) = \pi(u_R) \otimes r_R \otimes \pi(v_R) \quad (70)$$

$$\text{split}^L(w) = \pi(u_L) \otimes r_L \otimes \pi(v_L) \quad (71)$$

where $u_R r_R v_R = u_L r_L v_L = w$, and r_R (resp. r_L) is the rightmost (resp. leftmost) subpath of w which lies in S .

5.3 The projective resolution

We now describe the Chouhy–Solotar projective resolution of $A = \mathbb{k}Q/I$. It takes the form of a chain complex

$$\cdots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \rightarrow 0, \quad (72)$$

where $P_n = A \otimes \mathbb{k}S_n \otimes A$. In particular,

$$P_0 = A \otimes \mathbb{k}S_0 \otimes A = A \otimes_{\mathbb{k}Q_0} \mathbb{k}Q_0 \otimes_{\mathbb{k}Q_0} A = A \otimes_{\mathbb{k}Q_0} A.$$

Each P_n is an (A, A) -bimodule, with left and right actions given by

$$x \cdot (u \otimes w \otimes v) \cdot y := \pi(xu) \otimes w \otimes \pi(vy), \quad (73)$$

for $x, y, u, v \in A$ and $w \in \mathbb{k}S_n$.

Definition 5.8. Define maps $\delta_n : P_n \longrightarrow P_{n-1}$ by

$$\delta_0(x \otimes y) = \pi(xy), \quad (74)$$

$$\delta_1(x \otimes w \otimes y) = x \cdot (1 \otimes w - w \otimes 1) \cdot y = x \otimes \pi(wy) - \pi(xw) \otimes y, \quad (75)$$

$$\delta_n(x \otimes w \otimes y) = \begin{cases} x \cdot (\text{split}_{n-1}^L(w) - \text{split}_{n-1}^R(w)) \cdot y, & n \geq 2 \text{ odd}, \\ x \cdot \text{split}_{n-1}(w) \cdot y, & n \geq 2 \text{ even}. \end{cases} \quad (76)$$

Definition 5.9. For $\mu w \in \mathbb{k}^\times Q_{\geq 0}$, define subsets of P_n

$$\bar{\mathcal{L}}_n^<(\mu w) := \{\lambda u \otimes r \otimes v \mid u, v \in A, \lambda urv \prec \mu w\}. \quad (77)$$

One of the main theorem of [5] is,

Theorem 5.10. If $\partial_\bullet$ satisfies the following conditions

- $\partial_{i-1} \circ \partial_i = 0, \forall i \in \mathbb{Z}_{\geq 0};$

- $(\partial_i - \delta_i)(1 \otimes w \otimes 1) \in \langle \tilde{\mathcal{L}}_{i-1}^{\prec}(w) \rangle_{\mathbb{k}}, \forall i \in \mathbb{Z}_{\geq 0} \text{ and } \forall w \in S_i;$

then the complex (72) is exact.

The paper [5] further proved for a reduction system satisfying some conditions, such $\partial_{\bullet}$ always exists and gave an explicit construction, where the differentials ∂_n are constructed from the auxiliary maps δ_n and $\gamma_{n-1}, \rho_{n-1} : P_{n-1} \rightarrow P_n$, where γ_{n-1} is defined by

$$\gamma_{-1}(x) = 1 \otimes x, \quad (78)$$

$$\gamma_{n-1}(x \otimes w \otimes y) = (-1)^n \text{split}_n(xw) \cdot y. \quad (79)$$

And the differentials ∂_n and the homotopies ρ_{n-1} are then defined recursively by

$$\partial_{-1} = 0, \quad \rho_{-2} = 0, \quad (80)$$

$$\partial_n(x \otimes w \otimes y) = x \cdot ((\text{id} - \rho_{n-2}\partial_{n-1})\delta_n)(1 \otimes w \otimes 1) \cdot y, \quad (81)$$

$$\rho_{n-1} = \gamma_{n-1} + \sum_{i \geq 1} \gamma_{n-1}(\delta_n \gamma_{n-1} - \partial_n \gamma_{n-1})^i. \quad (82)$$

Proposition 5.11. *For $n = 0$, equation (81) yields*

$$\begin{aligned} \partial_0(x \otimes w \otimes y) &= x \cdot ((\text{id} - \rho_{-2}\partial_{-1})\delta_0)(1 \otimes w \otimes 1) \cdot y \\ &= x \cdot \delta_0(1 \otimes w \otimes 1) \cdot y = \pi(xy), \end{aligned}$$

where $w \in Q_0$ is a vertex of the quiver (the target of x and the source of y). Thus we have

$$\partial_0(x \otimes y) = \delta_0(x \otimes y) = \pi(xy). \quad (83)$$

For detailed proofs and further properties of this construction, we refer to [5].

Remark 5.12. *Briefly, the resolution extends the 1997 Bardzell resolution for monomial algebras [3]. The maps δ and γ coincide with those in the monomial case. Indeed, if the algebra is monomial and $S \subset Q_2$, then $\delta^2 = 0$. Specifically,*

$$\begin{aligned} &\delta_{n-1}\delta_n(1 \otimes w \otimes 1) \\ &= \text{split}_{n-2}^L \text{split}_{n-1}^L(w) \pm \text{split}_{n-2}^L \text{split}_{n-1}^R(w) \mp \text{split}_{n-2}^R \text{split}_{n-1}^L(w) + \text{split}_{n-2}^R \text{split}_{n-1}^R(w) \\ &= \text{split}_{n-2}^L \text{split}_{n-1}^L(w) + \text{split}_{n-2}^R \text{split}_{n-1}^R(w) \\ &= 0, \end{aligned}$$

since the middle two terms cancel, and both the first and last terms factor through monomials in S , which vanish in A . For non-monomial algebras, however, $\delta^2 \neq 0$, and the differentials must be corrected to ∂_n as above so that $\partial^2 = 0$.

6 Projective resolution of the diagonal bimodule

6.1 Category $\mathcal{C}$ as a quiver with relations

We can consider our KLRW category $\mathcal{C}$ of T_i branes as a quiver with relations, where the vertices are the idempotents e_i , and the arrows are taken to be s_i, p_i and q_i defined in Definition 2.5:

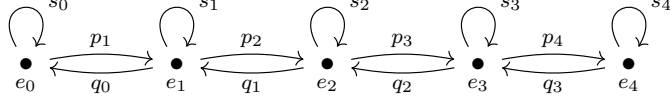


Figure 16: Category $\mathcal{C}$ as a quiver Q

$$Q_0 = \{e_i\} \tag{84}$$

$$Q_1 = \{p_i, q_i, s_i\} \tag{85}$$

The reduction system corresponding to the KLRW algebra comes from the relations in Fig. 6, where we need to specify one diagram in each relation to be irreducible:

$$R = \{(q_i p_{i+1}, s_i), (p_i q_{i-1}, s_i), (s_i p_i, p_i s_{i-1}), (s_i q_i, q_i s_{i+1})\}. \tag{86}$$

We will refer to the first two relations as *type I*, and the others as *type II*. Here we have adopted the convention that a path is considered *irreducible* if its corresponding strand diagram has all dots placed at the bottom and each black strand is monotone in the horizontal direction, that is, it never turns back after crossing a red strand. The set S is therefore

$$S = \{q_i p_{i+1}, p_i q_{i-1}, s_i p_i, s_i q_i\} \tag{87}$$

Since $S \subset Q_2$, any n -ambiguity $u \in S_{n+2}$ must arise from consecutive overlaps of minimal reducible subpaths without intermediate gaps. In other words, such a path has the form $u = u_{n+1} \dots u_1 u_0$ with $u_i \in Q_1$ and $u_{i+1} u_i \in S$, hence $S_n \subset Q_n$. Moreover, note that the dots s_i can only terminate a path in S . Consequently, a dot can appear only at the end of an n -ambiguity. There are therefore exactly two distinct families of n -ambiguities:

- **Type I ambiguities:** paths oscillating back and forth around a fixed red strand,
 - which are in the form of $p_{i+1} q_i p_{i+1} q_i p_{i+1} q_i \dots$, or $q_i p_{i+1} q_i p_{i+1} q_i p_{i+1} \dots$;
- **Type II ambiguities:** sequences ending at a dot after oscillating around a fixed red strand,
 - which are in the form of $s_{i+1} p_{i+1} q_i p_{i+1} q_i p_{i+1} \dots$, or $s_i q_i p_{i+1} q_i p_{i+1} q_i \dots$.

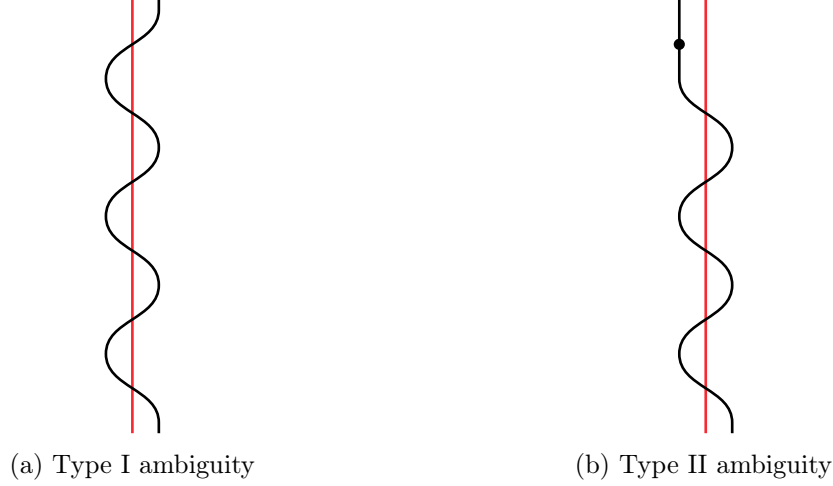


Figure 17: Two 4-ambiguities in S_6

Definition 6.1. *To streamline notation, we denote the ambiguities in S_n as*

- *Type I ambiguities:*

$$P_{i+1}^n := p_{i+1}q_i p_{i+1}q_i p_{i+1}q_i \dots \quad (88)$$

$$Q_i^n := q_i p_{i+1} q_i p_{i+1} q_i p_{i+1} \dots \quad (89)$$

- *Type II ambiguities:*

$$sP_{i+1}^n := s_{i+1}p_{i+1}q_i p_{i+1}q_i p_{i+1} \dots \quad (90)$$

$$sQ_i^n := s_i q_i p_{i+1} q_i p_{i+1} q_i \dots \quad (91)$$

This should not cause any confusion with the projective module P_n or the set of paths Q_n , as the ambiguities carry two indices.

We have for the Coulomb branch with $|\mathbf{x}|$ punctures

$$|S_n| = \begin{cases} |\mathbf{x}|, & n = 0, \\ 3|\mathbf{x}| + 1, & n = 1, \\ 4|\mathbf{x}|, & n \geq 2. \end{cases} \quad (92)$$

For example, in the 4-puncture case, we have $|S_0| = 5$, $|S_1| = 13$, and $|S_{\geq 2}| = 16$.

Remark 6.2. *Even before actually starting the computation of the projective resolution, two key advantage of the Chouhy–Solotar resolution already becomes apparent:*

- *The size of each projective bimodule $P_n = A \otimes \mathbb{k} S_n \otimes A$ remains constant for all $n \geq 2$, whereas in the bar resolution, the size of P_n grows exponentially with n . Consequently, all Hochschild cohomology groups can be computed once and for all, since the resolution size does not grow with the cohomological degree n .*

- Moreover, since all ambiguities involve only local paths between two adjacent vertices, the resolution naturally decomposes into smaller, independent components for $n \geq 3$. As a result, computations for arbitrarily many punctures $|\mathbf{x}|$ are uniformly tractable.

Definition 6.3. To be consistent with the notations in Section 2.2, we denote paths $a_{ji} \in Q_{|i-j|}$

$$a_{ji} := \begin{cases} p_{j-1}p_{j-2} \cdots p_{i+1}p_i, & i < j, \\ e_i, & i = j, \\ q_{j+1}q_{j+2} \cdots q_{i-1}q_i, & i > j. \end{cases} \quad (93)$$

Thus, all irreducible paths have the form of $a_{ji}s_i^\alpha$. The algebra $A = \mathbb{k}Q/I$, is generated by these irreducible paths, and has multiplication parallel to μ^2 in (9)

$$a_{kj}s_j^\beta \cdot a_{ji}s_i^\alpha = a_{ki}s_i^{\beta+\alpha+\delta(i,j,k)} \quad (94)$$

where $\delta(i, j, k)$ is given in (10).

6.2 Result from the Chouhy–Solotar resolution

Theorem 6.4. The result of Chouhy–Solotar resolution for our quiver algebra is

$$\cdots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \rightarrow 0 \quad (95)$$

where $P_n = A \otimes \mathbb{k}S_n \otimes A$, and ∂_n act by

- $0 \leq n \leq 1$,

$$\partial_0(1 \otimes 1) = 1 \quad (96)$$

$$\partial_1(1 \otimes w \otimes 1) = 1 \otimes w - w \otimes 1, \quad \forall w \in Q_1 \quad (97)$$

- $n \geq 2$ even,

$$\partial_n(1 \otimes Q_i^n \otimes 1) \quad (98)$$

$$= 1 \otimes Q_i^{n-1} \otimes p_{i+1} + q_i \otimes P_{i+1}^{n-1} \otimes 1 - 1 \otimes sQ_i^{n-1} \otimes 1$$

$$\partial_n(1 \otimes sP_{i+1}^n \otimes 1) \quad (99)$$

$$= 1 \otimes sP_{i+1}^{n-1} \otimes p_{i+1} + s_{i+1} \otimes P_{i+1}^{n-1} \otimes 1 - p_{i+1} \otimes sQ_i^{n-1} \otimes 1 - 1 \otimes P_{i+1}^{n-1} \otimes s_i$$

- $n \geq 2$ odd,

$$\partial_n(1 \otimes Q_i^n \otimes 1) \quad (100)$$

$$= 1 \otimes Q_i^{n-1} \otimes q_i - q_i \otimes P_{i+1}^{n-1} \otimes 1 + 1 \otimes sQ_i^{n-1} \otimes 1$$

$$\partial_n(1 \otimes sP_{i+1}^n \otimes 1) \quad (101)$$

$$= 1 \otimes sP_{i+1}^{n-1} \otimes q_i - s_{i+1} \otimes P_{i+1}^{n-1} \otimes 1 + p_{i+1} \otimes sQ_i^{n-1} \otimes 1 + 1 \otimes P_{i+1}^{n-1} \otimes s_{i+1}$$

and the equations from (98)–(101) by the replacement $P \leftrightarrow Q$, $i \leftrightarrow i+1$.

This might seem confusing at first glance, but the structure becomes simple when written diagrammatically. For $w \in S_n$, we have $\partial_n(1 \otimes w \otimes 1) \in P_{n-1} = A \otimes \mathbb{k}S_{n-1} \otimes A$. Although the detailed construction of the resolution in Section 5.3 is rather intricate, the operations involved in defining ∂_n essentially consist of only splitting and reduction operations. Formally, we may write

$$\partial_n(1 \otimes w \otimes 1) = \sum_i \lambda_i u_i \otimes r_i \otimes v_i, \quad (102)$$

where $\lambda_i \in \mathbb{k}$, $u_i, v_i \in A$, and $r_i \in S_{n-1}$. Then we have

$$u_i r_i v_i \preceq w \quad \text{for all } i. \quad (103)$$

It is easy to find out that the possible forms of $u_i r_i v_i$ are quite limited—namely, there are only three in the case of type I ambiguities.

and four options for type II ambiguities:

The first two terms in both (104) and (105) are the naïve terms, where $u_i v_i r_i = w$. The others are nontrivial, where $u_i v_i r_i \prec w$. Writing (104) and (105) in formulae, we get precisely (98) – (101).

Proof. According to Theorem 5.10, since our resolution satisfies condition (103), the second assumption is satisfied. The only remaining point to check is that $\partial^2 = 0$, which can be verified by a straightforward computation. $\square$

7 Hochschild cohomology of the braiding functors

We now proceed to derive the Hochschild cohomology of braiding functors. For a general braiding functor β , its Hochschild cohomology is isomorphic to the complex $(\text{Hom}^{\bullet-d}(P_d, B_\beta), d_\bullet)$, where B_β is the corresponding bimodule and $d_\bullet$ is the induced differential.

7.1 $H^\bullet \text{Hom}(\text{id}, \text{id})$

For the identity functor, on $\mathcal{C}$ we can only have degree 0 morphisms, thus the cochain complex is given by

$$\cdots \rightarrow \text{Hom}(P_{n-1}, \Delta) \xrightarrow{d_{n-1}} \text{Hom}(P_n, \Delta) \xrightarrow{d_n} \text{Hom}(P_{n+1}, \Delta) \rightarrow \cdots \quad (106)$$

where $d_n \phi = \phi \circ \partial_{n+1}$.

7.1.1 The cochain complex

Since $P_n = A \otimes \mathbb{k}S_n \otimes A$ is a free bimodule, to define a homomorphism $\phi: P_n \rightarrow \Delta$, it suffices to specify the images of the generators S_n . For a generator $g \in S$ whose target T_i and source $T_j = T_i, T_{i\pm 1}$, the element $\phi(g)$ may be an arbitrary morphism in $\text{Hom}(T_j, T_i)$. Using the basis $\{a_{ij}s^\ell\}_{\ell \geq 0}$ of this morphism space, we may write

$$\phi(g) = \sum_{\ell=0}^{\infty} \phi_\ell(g) a_{ij} s^\ell \quad (107)$$

where each $\phi_\ell(g) \in \mathbb{k}$. Thus the data of a bimodule homomorphism

$$\phi: P_n \rightarrow \Delta$$

is equivalent to a family of maps

$$\phi_\ell: S_n \rightarrow \mathbb{k}, \quad \ell \geq 0. \quad (108)$$

Proposition 7.1.

$$\text{Hom}(P_n, \Delta) \cong \text{Hom}_{\text{Set}}(\mathbb{Z}_{\geq 0}, \text{Hom}_{\text{Set}}(S_n, \mathbb{k})) \quad (109)$$

$$\cong \text{Hom}_{\text{Set}}(\mathbb{Z}_{\geq 0} \times S_n, \mathbb{k}) = \mathbb{k}^{\mathbb{Z}_{\geq 0} \times S_n} \quad (110)$$

Theorem 7.2. *Let $\phi \in \text{Hom}(P_n, \Delta)$ have coefficient maps $\{\phi_\ell\}_{\ell \geq 0}$. The coefficient maps of $d_n \phi$ are given as follows:*

- $n = 0$,

$$(d_0 \phi)_\ell(p_{i+1}) = \phi_\ell(e_{i+1}) - \phi_\ell(e_i) \quad (111)$$

$$(d_0 \phi)_\ell(q_i) = \phi_\ell(e_i) - \phi_\ell(e_{i+1}) \quad (112)$$

$$(d_0 \phi)_\ell(s_i) = 0 \quad (113)$$

- $n = 1$,

$$(d_1\phi)_\ell(p_{i+1}q_i) = \phi_{\ell-1}(p_{i+1}) + \phi_{\ell-1}(q_i) - \phi_\ell(s_{i+1}) \quad (114)$$

$$(d_1\phi)_\ell(q_i p_{i+1}) = \phi_{\ell-1}(q_i) + \phi_{\ell-1}(p_{i+1}) - \phi_\ell(s_i) \quad (115)$$

$$(d_1\phi)_\ell(s_{i+1}p_{i+1}) = \phi_\ell(s_{i+1}) - \phi_\ell(s_i) \quad (116)$$

$$(d_1\phi)_\ell(s_i q_i) = \phi_\ell(s_i) - \phi_\ell(s_{i+1}) \quad (117)$$

- $n \geq 2$ even,

$$(d_n\phi)_\ell(P_{i+1}^{n+1}) = \phi_\ell(P_{i+1}^n) - \phi_\ell(Q_i^n) + \phi_\ell(sP_{i+1}^n) \quad (118)$$

$$(d_n\phi)_\ell(Q_i^{n+1}) = \phi_\ell(Q_i^n) - \phi_\ell(P_{i+1}^n) + \phi_\ell(sQ_i^n) \quad (119)$$

$$(d_n\phi)_\ell(sP_{i+1}^{n+1}) = \phi_{\ell-1}(sP_{i+1}^n) + \phi_{\ell-1}(sQ_i^n) \quad (120)$$

$$(d_n\phi)_\ell(sQ_i^{n+1}) = \phi_{\ell-1}(sQ_i^n) + \phi_{\ell-1}(sP_{i+1}^n) \quad (121)$$

- $n \geq 2$ odd,

$$(d_n\phi)_\ell(P_{i+1}^{n+1}) = \phi_{\ell-1}(P_{i+1}^n) + \phi_{\ell-1}(Q_i^n) - \phi_\ell(sP_{i+1}^n) \quad (122)$$

$$(d_n\phi)_\ell(Q_i^{n+1}) = \phi_{\ell-1}(Q_i^n) + \phi_{\ell-1}(P_{i+1}^n) - \phi_\ell(sQ_i^n) \quad (123)$$

$$(d_n\phi)_\ell(sP_{i+1}^{n+1}) = \phi_\ell(sP_{i+1}^n) - \phi_\ell(sQ_i^n) \quad (124)$$

$$(d_n\phi)_\ell(sQ_i^{n+1}) = \phi_\ell(sQ_i^n) - \phi_\ell(sP_{i+1}^n) \quad (125)$$

where we take all $\phi_{-1} := 0$.

Proof. Direct computation. □

7.1.2 The cohomology

Theorem 7.3.

$$H^n \text{Hom}(\text{id}, \text{id}) = \text{HH}^n(\Delta) = \begin{cases} \mathbb{k}^{\mathbb{Z}_{\geq 0}}, & 0 \leq x \leq 1, \\ \mathbb{k}^{|\mathbf{x}|-1}, & n = 2, \\ 0, & n \geq 3. \end{cases} \quad (126)$$

Proof. For $n \geq 3$, the resolution of the quiver algebra $A = \mathbb{k}Q/I$ decomposes into blocks of the shape

$$Q^{(i)} = \begin{array}{ccc} & \overset{s_i}{\curvearrowright} & \overset{s_{i+1}}{\curvearrowright} \\ & \bullet & \bullet \\ e_i & \xrightleftharpoons[p_i]{q_i} & e_{i+1} \end{array} \quad (127)$$

with only four ambiguities in each block $Q^{(i)}$. Written explicitly, for $n \geq 2$ we have $|Q_n^{(i)} \cap S_n| = 4$ and

$$Q_n^{(i)} \cap S_n = \{P_{i+1}^n, Q_i^n, sP_{i+1}^n, sQ_i^n\}, \quad (128)$$

$$(Q_n^{(i)} \cap S_n) \cap (Q_n^{(j)} \cap S_n) = \emptyset, \quad i \neq j. \quad (129)$$

Hence the matrix for d_n is block diagonal, we may treat all $Q^{(i)}$ separately, representing the Hochschild differential as a family of 4×4 matrices parametrized by $0 \leq i < |\mathbf{x}|$.

Case $n \geq 3$ even.

- If $\phi \in \ker d_n$, i.e. $d_n \phi = 0$, then by Theorem 7.2 for all $\ell \geq 0$ and $0 \leq i < |\mathbf{x}|$ we have

$$0 = \begin{bmatrix} (d_n \phi)_\ell (P_{i+1}^{n+1}) \\ (d_n \phi)_\ell (Q_i^{n+1}) \\ (d_n \phi)_{\ell+1} (sP_{i+1}^{n+1}) \\ (d_n \phi)_{\ell+1} (sQ_i^{n+1}) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_\ell (P_{i+1}^n) \\ \phi_\ell (Q_i^n) \\ \phi_\ell (sP_{i+1}^n) \\ \phi_\ell (sQ_i^n) \end{bmatrix}. \quad (130)$$

This matrix has rank 2, eigenvalues $\{2, 2, 0, 0\}$, and its null space is spanned by the zero-eigenvalue eigenvectors $V_{i,P}^{n,\ell} := [1, 0, -1, 1]^\top$ and $V_{i,Q}^{n,\ell} := [0, 1, 1, -1]^\top$.

- If $\phi \in \text{im } d_{n-1}$, i.e. $\phi = d_{n-1} \psi$, then for all $\ell \geq 0$ and $0 \leq i < |\mathbf{x}|$ we have

$$\begin{bmatrix} \phi_\ell (P_{i+1}^n) \\ \phi_\ell (Q_i^n) \\ \phi_\ell (sP_{i+1}^n) \\ \phi_\ell (sQ_i^n) \end{bmatrix} = \begin{bmatrix} (d_{n-1} \psi)_\ell (P_{i+1}^{n-1}) \\ (d_{n-1} \psi)_\ell (Q_i^{n-1}) \\ (d_{n-1} \psi)_\ell (sP_{i+1}^{n-1}) \\ (d_{n-1} \psi)_\ell (sQ_i^{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \psi_{\ell-1} (P_{i+1}^{n-1}) \\ \psi_{\ell-1} (Q_i^{n-1}) \\ \psi_\ell (sP_{i+1}^{n-1}) \\ \psi_\ell (sQ_i^{n-1}) \end{bmatrix}. \quad (131)$$

- For $\ell \geq 1$, all the components of the ψ vector on the RHS of (131) can be chosen freely. The matrix in (131) has rank 2, eigenvalues $\{2, 2, 0, 0\}$, with the image spanned by the nonzero-eigenvalue eigenvectors, which are also $V_{i,P}^{n,\ell} := [1, 0, -1, 1]^\top$ and $V_{i,Q}^{n,\ell} := [0, 1, 1, -1]^\top$;
- For $\ell = 0$, the components $\psi_{-1} (P_{i+1}^{n-1})$ and $\psi_{-1} (Q_i^{n-1})$ have to be fix to zero, since there are no diagrams with -1 dot. Thus (131) is reduced to

$$\begin{bmatrix} \phi_0 (P_{i+1}^n) \\ \phi_0 (Q_i^n) \\ \phi_0 (sP_{i+1}^n) \\ \phi_0 (sQ_i^n) \end{bmatrix} = \begin{bmatrix} (d_{n-1} \psi)_0 (P_{i+1}^{n-1}) \\ (d_{n-1} \psi)_0 (Q_i^{n-1}) \\ (d_{n-1} \psi)_0 (sP_{i+1}^{n-1}) \\ (d_{n-1} \psi)_0 (sQ_i^{n-1}) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \psi_0 (sP_{i+1}^{n-1}) \\ \psi_0 (sQ_i^{n-1}) \end{bmatrix}. \quad (132)$$

Still, this matrix has rank 2, with the image spanned by the column vectors $[-1, 0, 1, -1]^\top$, $[0, -1, -1, 1]^\top$, which is the same subspace as $\ell \geq 1$.

Thus for all ℓ , we have $\ker d_n = \text{im } d_{n-1}$.

$$\text{HH}^n(\Delta) = 0, \quad n \geq 3 \text{ even.} \quad (133)$$

Case $n \geq 3$ odd.

- If $\phi \in \ker d_n$, i.e. $d_n \phi = 0$, then for all $\ell \geq 0$ and $0 \leq i < |\mathbf{x}|$ we have

$$0 = \begin{bmatrix} (d_n \phi)_\ell (P_{i+1}^{n+1}) \\ (d_n \phi)_\ell (Q_i^{n+1}) \\ (d_n \phi)_\ell (sP_{i+1}^{n+1}) \\ (d_n \phi)_\ell (sQ_i^{n+1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{\ell-1} (P_{i+1}^n) \\ \phi_{\ell-1} (Q_i^n) \\ \phi_\ell (sP_{i+1}^n) \\ \phi_\ell (sQ_i^n) \end{bmatrix}. \quad (134)$$

- If $\phi \in \text{im } d_{n-1}$, i.e. $\phi = d_{n-1} \psi$, then for all $\ell \geq 0$ and $0 \leq i < |\mathbf{x}|$ we have

$$\begin{bmatrix} \phi_{\ell-1} (P_{i+1}^n) \\ \phi_{\ell-1} (Q_i^n) \\ \phi_\ell (sP_{i+1}^n) \\ \phi_\ell (sQ_i^n) \end{bmatrix} = \begin{bmatrix} (d_{n-1} \psi)_{\ell-1} (P_{i+1}^{n-1}) \\ (d_{n-1} \psi)_{\ell-1} (Q_i^{n-1}) \\ (d_{n-1} \psi)_\ell (sP_{i+1}^{n-1}) \\ (d_{n-1} \psi)_\ell (sQ_i^{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \psi_{\ell-1} (P_{i+1}^{n-1}) \\ \psi_{\ell-1} (Q_i^{n-1}) \\ \psi_{\ell-1} (sP_{i+1}^{n-1}) \\ \psi_{\ell-1} (sQ_i^{n-1}) \end{bmatrix}. \quad (135)$$

- For $\ell \geq 1$, all the components of the vectors on the RHS of both (134) and (135) can be chosen freely. The two matrices are the same as the even case with the order reversed. Thus, the kernel of the matrix in (134) and the image of the matrix in (135) are both spanned by the other eigenvectors in the $n \geq 3$ even case, which are both $\text{Span}\{V_{i,P}^{n,\ell-1} := (1, 0, 1, 1)^\top, V_{i,Q}^{n,\ell-1} := (0, 1, 1, 1)^\top\}$.
- For $\ell = 0$, (134) and (135) are reduced to

$$0 = \begin{bmatrix} (d_n \phi)_0(P_{i+1}^{n+1}) \\ (d_n \phi)_0(Q_i^{n+1}) \\ (d_n \phi)_0(sP_{i+1}^{n+1}) \\ (d_n \phi)_0(sQ_i^{n+1}) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \phi_0(sP_{i+1}^n) \\ \phi_0(sQ_i^n) \end{bmatrix} \quad (136)$$

$$\begin{bmatrix} \phi_0(sP_{i+1}^n) \\ \phi_0(sQ_i^n) \end{bmatrix} = \begin{bmatrix} (d_{n-1} \psi)_0(sP_{i+1}^{n-1}) \\ (d_{n-1} \psi)_0(sQ_i^{n-1}) \end{bmatrix} = 0 \quad (137)$$

The only solution to (136) is $[\phi_0(sP_{i+1}^n), \phi_0(sQ_i^n)]^\top = [0, 0]^\top$, thus the kernel and the image are both 0.

Thus for all ℓ , we have $\ker d_n = \text{im } d_{n-1}$.

$$\text{HH}^n(\Delta) = 0, \quad n \geq 3 \text{ odd.} \quad (138)$$

Case $n = 2$. This is where things become different, since

$$(Q_1^{(i)} \cap S_1) \cap (Q_1^{(i-1)} \cap S_1) = \{s_i\} \neq \emptyset, \quad (139)$$

(129) is not true for $n = 1$. Thus, $\text{im } d_1$ begins to relate different blocks $Q^{(i)}$ to each other

- If $\phi \in \ker d_2$, i.e. $d_2 \phi = 0$, the situation is the same as (130). Here, we change the order of components as

$$0 = \begin{bmatrix} (d_2 \phi)_{\ell+1}(sQ_i^3) \\ (d_2 \phi)_\ell(Q_i^3) \\ (d_2 \phi)_\ell(P_{i+1}^3) \\ (d_2 \phi)_{\ell+1}(sP_{i+1}^3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_\ell(s_i q_i) \\ \phi_\ell(q_i p_{i+1}) \\ \phi_\ell(p_{i+1} q_i) \\ \phi_\ell(s_{i+1} p_{i+1}) \end{bmatrix}. \quad (140)$$

The kernel is the same, spanned by $V_{i,Q}^{2,\ell} := [-1, 1, 0, 1]^\top$ and $V_{i,P}^{2,\ell} := [1, 0, 1, -1]^\top$, where $0 \leq i < |\mathbf{x}|$.

- If $\phi \in \text{im } d_{n-1}$, i.e. $\phi = d_{n-1} \psi$, then for all $\ell \geq 0$ and $0 \leq i < |\mathbf{x}|$ we have on block $Q^{(i)}$:

$$\begin{bmatrix} \phi_\ell(s_i q_i) \\ \phi_\ell(q_i p_{i+1}) \\ \phi_\ell(p_{i+1} q_i) \\ \phi_\ell(s_{i+1} p_{i+1}) \end{bmatrix} = \begin{bmatrix} (d_1 \psi)_\ell(s_i q_i) \\ (d_1 \psi)_\ell(q_i p_{i+1}) \\ (d_1 \psi)_\ell(p_{i+1} q_i) \\ (d_1 \psi)_\ell(s_{i+1} p_{i+1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \psi_\ell(s_i) \\ \psi_{\ell-1}(q_i) \\ \psi_{\ell-1}(p_{i+1}) \\ \psi_\ell(s_{i+1}) \end{bmatrix} \quad (141)$$

However, now s_i belongs to both blocks $Q^{(i)}$ and $Q^{(i-1)}$, the total matrix is no longer block diagonal:

$$\begin{bmatrix} \vdots \\ \phi_\ell(s_{i-1}q_{i-1}) \\ \phi_\ell(q_{i-1}p_i) \\ \phi_\ell(p_iq_{i-1}) \\ \phi_\ell(s_ip_i) \\ \phi_\ell(s_iq_i) \\ \phi_\ell(q_ip_{i+1}) \\ \phi_\ell(p_{i+1}q_i) \\ \phi_\ell(s_{i+1}p_{i+1}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & & & & \\ & 1 & 0 & 0 & -1 & & & & \\ & -1 & 1 & 1 & 0 & & & & \\ & 0 & 1 & 1 & -1 & & & & \\ & -1 & 0 & 0 & 1 & & & & \\ & & & & 1 & 0 & 0 & -1 & \\ & & & & -1 & 1 & 1 & 0 & \\ & & & & 0 & 1 & 1 & -1 & \\ & & & & -1 & 0 & 0 & 1 & \\ & & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \psi_\ell(s_{i-1}) \\ \psi_{\ell-1}(q_{i-1}) \\ \psi_{\ell-1}(p_i) \\ \psi_\ell(s_i) \\ \psi_{\ell-1}(q_i) \\ \psi_{\ell-1}(p_{i+1}) \\ \psi_\ell(s_{i+1}) \\ \vdots \end{bmatrix} \quad (142)$$

$$=: [\cdots J_\ell(s_{i-1}), J_{\ell-1}(q_{i-1}), J_{\ell-1}(p_i), J_\ell(s_i), J_{\ell-1}(q_i), J_{\ell-1}(p_{i+1}), J_\ell(s_{i+1}) \cdots] \begin{bmatrix} \vdots \\ \psi_\ell(s_{i-1}) \\ \psi_{\ell-1}(q_{i-1}) \\ \psi_{\ell-1}(p_i) \\ \psi_\ell(s_i) \\ \psi_{\ell-1}(q_i) \\ \psi_{\ell-1}(p_{i+1}) \\ \psi_\ell(s_{i+1}) \\ \vdots \end{bmatrix} \quad (143)$$

where for $w \in Q_1$ we denote by $J_\ell(w)$ the column vector in the large matrix corresponding to $\psi_\ell(w)$.

- For $\ell \geq 1$, all the components of the ψ vector on the RHS of (142) can be chosen freely. The image of d_1 is given by the linear span of all column vector of the matrix. We can see that

$$V_{0,Q}^{2,\ell} = -J_\ell(s_0) \quad (144)$$

$$V_{0,P}^{2,\ell} = -V_{0,Q}^{2,\ell} + J_{\ell-1}(q_0) = J_\ell(s_0) + J_{\ell-1}(q_0) \quad (145)$$

$$V_{1,Q}^{2,\ell} = -V_{0,P}^{2,\ell} - J_\ell(s_1) = -J_\ell(s_0) - J_{\ell-1}(q_0) - J_\ell(s_1) \quad (146)$$

$$V_{1,P}^{2,\ell} = -V_{1,Q}^{2,\ell} + J_{\ell-1}(q_1) = J_\ell(s_0) + J_{\ell-1}(q_0) + J_\ell(s_1) + J_{\ell-1}(q_1) \quad (147)$$

.....

$$V_{i,Q}^{2,\ell} = -V_{i-1,P}^{2,\ell} - J_\ell(s_i) \quad (148)$$

$$V_{i,P}^{2,\ell} = -V_{i,Q}^{2,\ell} + J_{\ell-1}(q_i) \quad (149)$$

.....

Thus, all the null eigenvectors of d_2 can be represented by a linear combination of column vectors of d_1 . Thus the $\ell \geq 1$ part is exact.

- For $\ell = 0$, the components $\psi_{-1}(q_j)$ and $\psi_{-1}(p_{j+1})$ must be set to zero. This means we can no longer use $J_{-1}(q_j)$ and $J_{-1}(p_{j+1})$ in the linear combination:

$$\begin{bmatrix} \vdots \\ \phi_0(s_{i-1}q_{i-1}) \\ \phi_0(q_{i-1}p_i) \\ \phi_0(p_iq_{i-1}) \\ \phi_0(s_ip_i) \\ \phi_0(s_iq_i) \\ \phi_0(q_ip_{i+1}) \\ \phi_0(p_{i+1}q_i) \\ \phi_0(s_{i+1}p_{i+1}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & & & & \\ & 1 & -1 & & & & & & \\ & -1 & 0 & & & & & & \\ & 0 & -1 & & & & & & \\ & -1 & 1 & & & & & & \\ & & 1 & -1 & & & & & \\ & & -1 & 0 & & & & & \\ & & 0 & -1 & & & & & \\ & & -1 & 1 & & & & & \\ & & & & \ddots & & & & \end{bmatrix} \begin{bmatrix} \vdots \\ \psi_0(s_{i-1}) \\ \psi_0(s_i) \\ \psi_0(s_{i+1}) \\ \vdots \end{bmatrix} \quad (150)$$

$$=: [\cdots J_0(s_{i-1}), J_0(s_i), J_0(s_{i+1}) \cdots] \begin{bmatrix} \vdots \\ \psi_0(s_{i-1}) \\ \psi_0(s_i) \\ \psi_0(s_{i+1}) \\ \vdots \end{bmatrix} \quad (151)$$

In this case, the term $J_{\ell-1}(q_i)$ in (149) is no longer available. Therefore, the $\ell = 0$ part is no longer exact.

Each $J_0(s_i)$ is linearly independent, since the entry corresponding to $\phi_0(q_ip_{i+1})$ is nonzero only for $J_0(s_i)$. Consequently, the matrix in (150) has rank $|\mathbf{x}| + 1$, which is the dimension of the $\ell = 0$ part of $\text{im } d_1$.

As shown earlier, the $\ell = 0$ component of $\ker d_2$ is spanned by $2|\mathbf{x}|$ linearly independent vectors $\{V_{i,P}^{2,0}, V_{i,Q}^{2,0}\}_{0 \leq i < |\mathbf{x}|}$. Hence,

$$\dim^{\ell=0} \ker d_2 = 2|\mathbf{x}|, \quad \dim^{\ell=0} \text{im } d_1 = |\mathbf{x}| + 1, \quad (152)$$

and

$$\dim \text{HH}^2(\Delta) = \dim^{\ell=0} \ker d_2 - \dim^{\ell=0} \text{im } d_1 = |\mathbf{x}| - 1. \quad (153)$$

Therefore,

$$\text{HH}^2(\Delta) = \mathbb{k}^{|\mathbf{x}|-1}. \quad (154)$$

The vectors J_ℓ span the image and the vectors $V^{2,\ell}$ span the kernel. Thus, we can take the representatives of the Hochschild cohomology as

$$\mathfrak{V}_i := V_{i-1,P}^{2,\ell} - V_{i,Q}^{2,\ell} = [0, 0, \dots, 1, 0, 1, -1, 1, -1, 0, -1, \dots, 0, 0]^\top, \quad 0 < i < |\mathbf{x}|. \quad (155)$$

Case $n = 1$.

- The condition for $\phi \in \ker d_1$ follows from (142)

$$0 = \begin{bmatrix} \vdots \\ (d_1\phi)_\ell(s_{i-1}q_{i-1}) \\ (d_1\phi)_\ell(q_{i-1}p_i) \\ (d_1\phi)_\ell(p_iq_{i-1}) \\ (d_1\phi)_\ell(s_ip_i) \\ (d_1\phi)_\ell(s_iq_i) \\ (d_1\phi)_\ell(q_ip_{i+1}) \\ (d_1\phi)_\ell(p_{i+1}q_i) \\ (d_1\phi)_\ell(s_{i+1}p_{i+1}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & & & & \\ & 1 & 0 & 0 & -1 & & & & \\ & -1 & 1 & 1 & 0 & & & & \\ & 0 & 1 & 1 & -1 & & & & \\ & -1 & 0 & 0 & 1 & & & & \\ & & & & 1 & 0 & 0 & -1 & \\ & & & & -1 & 1 & 1 & 0 & \\ & & & & 0 & 1 & 1 & -1 & \\ & & & & -1 & 0 & 0 & 1 & \\ & & & & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \phi_\ell(s_{i-1}) \\ \phi_{\ell-1}(q_{i-1}) \\ \phi_{\ell-1}(p_i) \\ \phi_\ell(s_i) \\ \phi_{\ell-1}(q_i) \\ \phi_{\ell-1}(p_{i+1}) \\ \phi_\ell(s_{i+1}) \\ \vdots \end{bmatrix} \quad (156)$$

This can be summarized concisely as

$$\begin{cases} \phi_0(s_i) = 0, \\ \phi_\ell(s_i) =: \sigma_{\ell-1}, & \ell > 0, \forall 0 \leq i \leq |\mathbf{x}|, \\ \phi_\ell(q_i) + \phi_\ell(p_{i+1}) = \sigma_\ell, & \ell \geq 0, \forall 0 \leq i < |\mathbf{x}|. \end{cases} \quad (157)$$

where the middle line comes from the blue rows of (156), while the first and third lines come from the black rows.

- The condition for $\phi = d_0\psi \in \text{im } d_0$ is, by Theorem 7.2

$$\phi_\ell(p_{i+1}) = \psi_\ell(e_{i+1}) - \psi_\ell(e_i) \quad (158)$$

$$\phi_\ell(q_i) = \psi_\ell(e_i) - \psi_\ell(e_{i+1}) \quad (159)$$

$$\phi_\ell(s_i) = 0 \quad (160)$$

which is equivalent to

$$\begin{cases} \phi_\ell(s_i) = 0, & \forall \ell \geq 0, 0 \leq i \leq |\mathbf{x}|, \\ \phi_\ell(q_i) + \phi_\ell(p_{i+1}) = 0, & \forall \ell \geq 0, 0 \leq i < |\mathbf{x}|. \end{cases} \quad (161)$$

Therefore, $\{\sigma_\ell\}_{\ell \geq 0}$ are precisely the parameters of the Hochschild cohomology:

$$\text{HH}^1(\Delta) = \mathbb{k}^{\mathbb{Z}_{\geq 0}}. \quad (162)$$

Case $n = 0$.

- The condition for $\phi \in \ker d_0$ is simply

$$\epsilon_\ell = \phi_\ell(e_i), \quad \ell \geq 0, \forall 0 \leq i \leq |\mathbf{x}|. \quad (163)$$

Therefore, $\{\epsilon_\ell\}_{\ell \geq 0}$ parametrize the Hochschild cohomology:

$$\text{HH}^0(\Delta) = \mathbb{k}^{\mathbb{Z}_{\geq 0}}. \quad (164)$$

□

7.2 $H^\bullet \text{Hom}(\text{id}, \beta_{i-})$

Theorem 7.4.

$$H^n \text{Hom}(\text{id}, \beta_{i-}) = \text{HH}^n(\mathfrak{B}_{i-}) = \begin{cases} \mathbb{k}^{\mathbb{Z}_{\geq 1}}, & n = 0 \\ \mathbb{k}^{\mathbb{Z}_{\geq 0}}, & n = 1 \\ \mathbb{k}^{|\mathbf{x}|-2}, & n = 2 \\ 0, & n \geq 3. \end{cases} \quad (165)$$

Proof. From the short exact sequence (56), we obtain the long exact sequence of Hochschild cohomology groups:

$$\cdots \rightarrow \text{HH}^n(\mathfrak{B}_{i-}) \rightarrow \text{HH}^n(\Delta) \rightarrow \text{HH}^n(\mathfrak{S}_i) \rightarrow \text{HH}^{n+1}(\mathfrak{B}_{i-}) \rightarrow \cdots \quad (166)$$

The Hochschild cohomology of $\mathfrak{S}_i$ can be computed directly. Indeed, since $\mathfrak{S}_i$ is generated by a single diagram e_i , which goes from T_i to T_i , to define a morphism $\phi \in \text{Hom}(P_n, \mathfrak{S}_i)$, it suffices to assign a scalar in $\mathbb{k}$ to each element in S_n whose source and target are both e_i . For $n = 0, 1$, there is only one such element, namely $e_i \in S_0$ and $s_i \in S_1$; for $n \geq 2$, there are two such elements, which are P_i^n and Q_i^n for even n , and sP_i^n and sQ_i^n for odd n . Therefore,

$$\text{Hom}(P_n, \mathfrak{S}_i) = \begin{cases} \mathbb{k}, & 0 \leq n \leq 1, \\ \mathbb{k}^2, & n \geq 2. \end{cases} \quad (167)$$

Using the resolution in Theorem 7.2, the corresponding complex is easily seen to be

$$\mathbb{k} \xrightarrow{0} \mathbb{k} \xrightarrow{[-1, -1]} \mathbb{k}^2 \xrightarrow{0} \mathbb{k}^2 \xrightarrow{-\text{id}} \mathbb{k}^2 \xrightarrow{0} \mathbb{k}^2 \xrightarrow{-\text{id}} \mathbb{k}^2 \longrightarrow \cdots \quad (168)$$

Hence

$$\text{HH}^\bullet(\mathfrak{S}_i) = H^\bullet \text{Hom}(\text{id}, \mathfrak{S}_i) = \{\mathbb{k}, 0, \mathbb{k}, 0, 0, 0, \dots\} \quad (169)$$

Therefore (166) is for $n \geq 4$,

$$0 \longrightarrow \text{HH}^n(\mathfrak{B}_{i-}) \xrightarrow{\cong} \text{HH}^n(\Delta) \longrightarrow 0 \quad (170)$$

$$\text{HH}^n(\mathfrak{B}_{i-}) \cong \text{HH}^n(\Delta) = 0, \quad n \geq 4. \quad (171)$$

For the lower part of (166), we have

$$0 \longrightarrow \text{HH}^0(\mathfrak{B}_{i-}) \longrightarrow \text{HH}^0(\Delta) \longrightarrow \mathbb{k} \longrightarrow \text{HH}^1(\mathfrak{B}_{i-}) \longrightarrow \text{HH}^1(\Delta) \longrightarrow 0 \quad (172)$$

$$0 \longrightarrow \text{HH}^2(\mathfrak{B}_{i-}) \longrightarrow \text{HH}^2(\Delta) \longrightarrow \mathbb{k} \longrightarrow \text{HH}^3(\mathfrak{B}_{i-}) \longrightarrow 0 \quad (173)$$

It is easy to show that the maps $\text{HH}^0(\Delta) \rightarrow \text{HH}^0(\mathfrak{S}_i)$ and $\text{HH}^2(\Delta) \rightarrow \text{HH}^2(\mathfrak{S}_i)$ are surjections. Therefore,

$$\mathrm{HH}^0(\mathfrak{B}_{i-}) \cong \mathrm{HH}^0(\Delta)/\mathbb{k} \cong \mathbb{k}^{\mathbb{Z}_{\geq 1}} \quad (174)$$

$$\mathrm{HH}^1(\mathfrak{B}_{i-}) \cong \mathrm{HH}^1(\Delta) \cong \mathbb{k}^{\mathbb{Z}_{\geq 0}} \quad (175)$$

$$\mathrm{HH}^2(\mathfrak{B}_{i-}) \cong \mathrm{HH}^2(\Delta)/\mathbb{k} \cong \mathbb{k}^{|\mathbf{x}|-2} \quad (176)$$

$$\mathrm{HH}^3(\mathfrak{B}_{i-}) = 0 \quad (177)$$

□

Remark 7.5. One can also repeat the procedure in Section 7.1.2 to find this. More concretely, $\mathrm{HH}^0(\mathfrak{B}_{i-})$ demands $\epsilon_0 = 0$ and $\mathrm{HH}^2(\mathfrak{B}_{i-})$ demands $\vartheta_i = 0$.

8 Homotopy deformation retract from bar resolution

To obtain explicit expressions for the natural transformations, we translate our resolution $P_\bullet$ back to the bar resolution. In our A_∞ -category $\mathcal{C}$, the only nontrivial structure map is μ^2 . Moreover, defining $\bar{A} := A/(\mathbb{k}Q_0 \cdot 1_A)$, the bar resolution from Definition 4.2 can, in the language of quiver algebras, be reduced to the *normalized bar resolution*:

Definition 8.1 (normalized bar resolution).

$$\mathrm{Bar}_n = A \otimes \bar{A}^{\otimes n} \otimes A \quad (178)$$

$$\bar{\partial}_n: \mathrm{Bar}_n \rightarrow \mathrm{Bar}_{n-1} \quad (179)$$

$$\begin{aligned} \bar{\partial}_n(a_{n+1} \otimes [a_n | \cdots | a_1] \otimes a_0) = & a_{n+1} \otimes [a_n | \cdots | a_2] \otimes a_1 a_0 \\ & + \sum_{i=1}^{n-1} (-1)^i a_{n+1} \otimes [a_n | \cdots | a_{i+1} a_i | a_{i-1} | \cdots | a_1] \otimes a_0 \\ & + (-1)^n a_{n+1} a_n \otimes [a_{n-1} | \cdots | a_1] \otimes a_0 \end{aligned} \quad (180)$$

The convention in the RHS of (180) is consistent with [4, 5], which differs by an overall $\pm$ sign with $\mu_{\mathrm{Fun}(\mathcal{A}, \mathcal{B})}^1$ in [10].

A homotopy deformation retract from the normalized bar resolution onto Chouhy–Solotar resolution is constructed in [4].

Theorem 8.2. *There is a special homotopy deformation retract*

$$(P_\bullet, \partial_\bullet) \xrightleftharpoons[G_\bullet]{F_\bullet} (\mathrm{Bar}_\bullet, \bar{\partial}_\bullet) \xrightarrow{h_\bullet} (P_\bullet, \partial_\bullet) \quad (181)$$

such that

$$G_n F_n = \mathrm{id}, \quad F_n G_n = \mathrm{id} + h_{n-1} \bar{\partial}_n + \bar{\partial}_{n+1} h_n \quad (182)$$

$$h_n F_n = 0, \quad G_{n+1} h_n = 0, \quad h_{n+1} h_n = 0 \quad (183)$$

The map that we need is $G_\bullet$, which is given by

Proposition 8.3.

$$\begin{aligned} G_{-1} &= \text{id}_A \\ G_n(a \otimes y \otimes b) &= a \cdot \rho_{n-1} G_{n-1} \bar{\partial}_n (1 \otimes y \otimes b) \end{aligned}$$

where $y = [y_n | \cdots | y_1]$, and $\bar{\partial}_0: A \otimes A \rightarrow A$ is given by the multiplication in A .

Although we have proved that our cohomologically nontrivial natural transformations all concentrate in degree ≤ 2 , for completeness and potential application in the future, we here present the result for arbitrary n .

Theorem 8.4. *The result of $G_\bullet$ in our quiver is*

$$G_n(a \otimes [y_n | \cdots | y_1] \otimes b) = a \cdot \sum_{w \preceq y_n \dots y_1} \text{split}_n(w) \cdot b, \quad (184)$$

where in the sum, different w with the same result of $\text{split}_n(w)$ are counted only once.

This description can be further simplified into a straightforward rule. So before proceeding to the proof, we first define a notion and consider an example.

Definition 8.5. *For a path $w \in Q_n$, we define the turning number $\wp(w)$ of w as the number of subpaths in w that travel from a vertex e_i to its adjacent vertex $e_{i\pm 1}$ and then return to e_i .*

For example, we have $\wp(a_{ji}s_i^\alpha) = 0$, $\wp(q_{i+1}p_{i+2}p_{i+1}s_i^\alpha q_i) = 2$, $\wp(sQ_i^n) = n - 2$, $\wp(Q_i^n) = n - 1$.

Proposition 8.6. *If $v \preceq u$, then $\wp(v) \leq \wp(u)$.*

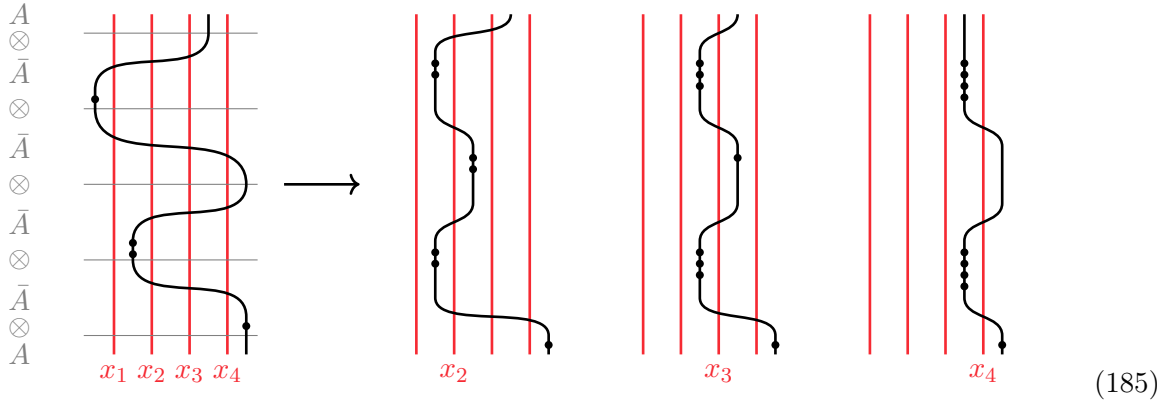
Proof. Under a type I reduction $pq \mapsto s$, the value of $\wp$ may decrease by 1, whereas under a type II reduction $sq \mapsto qs$, it always remains unchanged. Therefore, under any sequence of reductions, $\wp$ can only decrease monotonically. $\square$

Proposition 8.7. *For $y = y_n \dots y_1$, $y_i \in \bar{A}$, $\wp(y) \leq n - 1$. For $w \in S_n$, $\wp(w) = n - 1$ (type I) or $n - 2$ (type II).*

Example 8.8. *Take*

$$G_4(1 \otimes [a_{30}s_0|a_{04}|a_{41}s_1^2|a_{14}s_4] \otimes 1)$$

as an example. Since all ambiguities in S_n only bounces between two adjacent vertices, to obtain a path w with $s \subset w \preceq y_n \dots y_1$ where $s \in S_n$, we first need to restrict the middle of the path $a_{30}s_0a_{04}a_{41}s_1^2a_{14}s_4$ between some two vertices, i.e. around some red strand. In this example, this leads to 3 cases:



The next step is to perform type II reductions to split off a subpath in S_n . Since reductions only move dots downward, and ambiguities in S_n cannot contain a dot in the middle, all dots except those at the top (i.e. at the position corresponding to the source of y_n) must be moved to the bottom. Each time a top dot is dropped, a new term appears in the resulting expression. Thus, the first diagram on the RHS of (185) contributes three terms:

(186)

while the second and the third diagrams each produce four terms:

(187)

(188)

Therefore, the result contains 11 terms:

$$\begin{aligned}
G_4 (1 \otimes [a_{30}s_0|a_{04}|a_{41}s_1^2|a_{14}s_4] \otimes 1) \\
= a_{31}s_1 \otimes sQ_1^4 \otimes a_{24}s_4^5 + a_{31} \otimes sQ_1^4 \otimes a_{24}s_4^6 + a_{31} \otimes P_2^4 \otimes a_{24}s_4^7 \\
+ a_{32}s_2^2 \otimes sQ_2^4 \otimes a_{34}s_4^5 + a_{32}s_2 \otimes sQ_2^4 \otimes a_{34}s_4^6 + a_{32} \otimes sQ_2^4 \otimes a_{34}s_4^7 + 1 \otimes P_3^4 \otimes a_{34}s_4^8 \\
+ s_3^3 \otimes sQ_3^4 \otimes s_4^5 + s_3^2 \otimes sQ_3^4 \otimes s_4^6 + s_3 \otimes sQ_3^4 \otimes s_4^7 + 1 \otimes sQ_3^4 \otimes s_4^8
\end{aligned} \tag{189}$$

This example should give a comprehensive recipe for evaluating G_n on arbitrary input.

Now we write down the proof for Theorem 8.4

Proof. By induction, the base case is clear

$$G_0(a \otimes b) = a \cdot \rho_{-1} G_{-1} d_0(1 \otimes b) = a \otimes b \tag{190}$$

$$\implies G_0 = \text{id}_{A \otimes A} \tag{191}$$

which is expected. If this is too abstract, we can go a step further,

$$G_1(a \otimes y_1 \otimes b) = a \cdot \rho_0 G_0 \bar{\partial}_1(1 \otimes y_1 \otimes b) \tag{192}$$

$$= a \cdot \rho_0(1 \otimes \pi(y_1 b) - y_1 \otimes b) \tag{193}$$

$$= a \cdot \text{split}_1(y_1) \cdot b \tag{194}$$

Again, this is expected.

Now, suppose the result holds for G_{n-1} , then

$$G_n(a \otimes [y_n | \cdots | y_1] \otimes b) = a \cdot \rho_{n-1} G_{n-1} \bar{\partial}_n(1 \otimes [y_n | \cdots | y_1] \otimes b) \cdot b \tag{195}$$

$$= a \cdot \rho_{n-1} \left(\begin{aligned} & \sum_{x \preceq y_n \dots y_2} \text{split}_{n-1}(x) \cdot \pi(y_1 b) \\ & + \sum_{i=1}^{n-1} \sum_{x \preceq y_n \dots \pi(y_{i+1} y_i) \dots y_1} (-1)^i \text{split}_{n-1}(x) \cdot b \\ & + (-1)^n \sum_{x \preceq y_{n-1} \dots y_1} y_n \cdot \text{split}_{n-1}(x) \cdot y_n \end{aligned} \right) \tag{196}$$

$$= a \cdot \left(1 + \sum_{i \geq 1} (\gamma_{n-1}(\delta_n - \partial_n))^i \right) \tag{197}$$

$$(-1)^n \left(\begin{aligned} & \sum_{\substack{qwr \preceq y_n \dots y_2 \\ q \otimes w \otimes r \in P_{n-1}}} \text{split}_n(qw) \cdot \pi(ry_1 b) \\ & + \sum_{i=1}^{n-1} \sum_{\substack{qwr \preceq y_n \dots \pi(y_{i+1} y_i) \dots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} (-1)^i \text{split}_n(qw) \cdot \pi(rb) \\ & + (-1)^n \sum_{\substack{qwr \preceq y_{n-1} \dots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} \text{split}_n(\pi(y_n q)w) \cdot \pi(rb) \end{aligned} \right)$$

All but the last term in the large parenthesis in (197) vanish. The reason is as follows. For any other term to be nonvanishing, it must contain a subpath in S_n inside qw with $q \in A$ and $w \in S_{n-1}$. Since ambiguities can contain a dot only at the end of a path, w must be of type I. Moreover, because qwr is reduced from a path obtained by concatenating $n-1$ irreducible paths, we have $\wp(qwr) \leq n-2$. However, since $w \in S_{n-1}$ is a type I ambiguity, $\wp(w) = n-2$, which means that the endpoint of w must already lie in the middle of y_n . Consequently, the beginning of r can neither turn nor carry a dot, and therefore no S_n element can be split off.

So we have for now,

$$G_n(a \otimes [y_n | \cdots | y_1] \otimes b) = \cdots \quad (198)$$

$$= a \cdot \left(1 + \sum_{i \geq 1} (\gamma_{n-1} (\delta_n - \partial_n))^i \right) \sum_{\substack{qwr \preceq y_{n-1} \cdots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} \text{split}_n(\pi(y_n q)w) \cdot \pi(rb) \quad (199)$$

$$= a \cdot \sum_{\substack{qwr \preceq y_{n-1} \cdots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} (\text{split}_n(\pi(y_n q)w) \cdot r) \cdot b \quad (200)$$

$$+ a \cdot \left(1 + \sum_{i \geq 1} (\gamma_{n-1} (\delta_n - \partial_n))^i \right) \gamma_{n-1} (\delta_n - \partial_n) \sum_{\substack{qwr \preceq y_{n-1} \cdots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} \text{split}_n(\pi(y_n q)w) \cdot \pi(rb)$$

Now we have obtained a specific term in the result:

$$a \cdot \sum_{\substack{qwr \preceq y_{n-1} \cdots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} (\text{split}_n(\pi(y_n q)w) \cdot r) \cdot b.$$

Let us analyze the diagrammatic meaning of this term. In this case, w must again be a type I ambiguity. The expression $\text{split}_n(\pi(y_n q)w) \cdot r$ means that the path $y_{n-1} \cdots y_2 y_1$ is reduced to qwr such that $w \in S_{n-1}$ is a type I ambiguity and $q, r \in A$ are irreducible and then fully reduce $y_n q$ so that w , together with the first arrow of $\pi(y_n q)$, forms an ambiguity in S_n , which is subsequently split off.

In terms of Example 8.8, this means that the dots at the top cannot drop down (as they are created after applying $\pi(y_n q)$), whereas all other dots originally at lower positions can freely move to the bottom (as they are produced during the G_{n-1} step). Hence, this term corresponds to the first diagrams among the resulting terms in (186)–(188).

We are now left with the second term in (200). To evaluate this term, we first apply $\delta_n - \partial_n$ to $\sum_{\substack{qwr \preceq y_{n-1} \cdots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} \text{split}_n(\pi(y_n q)w) \cdot \pi(rb)$, and then apply γ_{n-1} . Recall that $\delta_n - \partial_n$ acts on the ambiguities by

$$\begin{aligned} (\delta_n - \partial_n)(1 \otimes Q_i^n \otimes 1) &= (-1)^n 1 \otimes sQ_i^{n-1} \otimes 1 \\ (\delta_n - \partial_n)(1 \otimes sQ_i^n \otimes 1) &= (-1)^n q_i \otimes sP_{i+1}^{n-1} \otimes 1 + (-1)^n 1 \otimes Q_i^{n-1} \otimes s_{i+\varsigma} \end{aligned} \quad (201)$$

where $i + \varsigma$ marks the position of the source of Q_i^{n-1} , which is

$$\varsigma = \begin{cases} 1, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

If we interchange $P \leftrightarrow Q$, $i \leftrightarrow i + 1$, ς should be replaced by $-\varsigma$.

Note that $\delta_n - \partial_n$ maps a type I ambiguity to a type II ambiguity, and a type II ambiguity to a sum of one type I term and one type II term. The input

$$\sum_{\substack{qwr \preceq y_{n-1} \cdots y_1 \\ q \otimes w \otimes r \in P_{n-1}}} \text{split}_n(\pi(y_n q)w) \cdot \pi(rb)$$

may contain both type I and type II terms. However, as observed earlier, only type I terms survive after applying γ_{n-1} . Since only type II terms produce type I terms under the action of $\delta_n - \partial_n$, we conclude that only the type II components in the input are relevant. In the diagrammatic language of Example 8.8, these correspond precisely to the terms that contain at least one dot in the top position.

In general, a type II term takes the form

$$a_{kj}s_j^\beta \otimes sQ_j^n \otimes a_{j+\varsigma,i}s_i^\alpha,$$

and applying $\gamma_{n-1}(\delta_n - \partial_n)$ to it yields

$$\gamma_{n-1}(\delta_n - \partial_n)(a_{kj}s_j^\beta \otimes sQ_j^n \otimes a_{j+\varsigma,i}s_i^\alpha) \quad (202)$$

$$= (-1)^n \gamma_{n-1}(\pi(a_{kj}s_j^\beta q_j) \otimes sP_{j+1}^{n-1} \otimes a_{j+\varsigma,i}s_i^\alpha + a_{kj}s_j^\beta \otimes Q_j^{n-1} \otimes a_{j+\varsigma,i}s_i^{\alpha+1}) \quad (203)$$

$$= (-1)^{2n}(\text{split}_n(\pi(a_{kj}s_j^\beta q_j) \otimes sP_{j+1}^{n-1}) \cdot a_{j+\varsigma,i}s_i^\alpha + \text{split}_n(a_{kj}s_j^\beta Q_j^{n-1}) \cdot a_{j+\varsigma,i}s_i^{\alpha+1}) \quad (204)$$

$$= \text{split}_n(a_{kj}s_j^\beta Q_j^{n-1}) \cdot a_{j+\varsigma,i}s_i^{\alpha+1} \quad (205)$$

$$= a_{kj}s_j^{\beta-1} \otimes sQ_j^n \otimes a_{j+\varsigma,i}s_i^{\alpha+1} \quad (206)$$

which is, precisely dropping one dot at the top position to the bottom. This gives all the second diagrams in (186)–(188).

The infinite sum $1 + \sum_{i \geq 1} (\gamma_{n-1}(\delta_n - \partial_n))^i$ iteratively carries out this process, each time dropping one dot from the top to the bottom. The procedure terminates once no dots remain in the top position. At this stage, if there is a turning, we split out a type I ambiguity, applying $\gamma_{n-1}(\delta_n - \partial_n)$ then yields zero; if there is not, the result vanishes already.

Hence, this construction produces exactly all the possible terms described in Theorem 8.4 and Example 8.8. □

9 Components of the A_∞ -natural transformations

Now we are ready to write down the explicit components of the natural transformations. Although one could in principle write out all cocycles, for simplicity we present only one representative from each cohomology class.

9.1 Nat(id, id)

For a degree g natural transformation η , η^d has a degree shift $[g - d]$. Since in the subcategory $\mathcal{C}$, all morphisms have degree 0, thus for a degree g natural transformation $\eta: \text{id} \Rightarrow \text{id}$, only the η^g component is nonzero on morphisms between generators T_i .

9.1.1 Degree 0

Recall from Section 7.1.2 that the cocycles $\phi \in \text{Hom}(P_0, \Delta)$ are parametrized by $\{\epsilon_\ell\}_{\ell \geq 0} \in \mathbb{K}^{\mathbb{Z}_{\geq 0}}$,

$$\epsilon_\ell = \phi_\ell(e_i), \quad \ell \geq 0, \quad \forall 0 \leq i \leq |\mathbf{x}|. \quad (207)$$

Since $G_0 = \text{id}_{A \otimes A}$, the result is:

Theorem 9.1. *The degree 0 natural transformations*

$$\eta_{\{\epsilon_\ell\}}: \text{id} \Rightarrow \text{id}, \quad \{\epsilon_\ell\}_{\ell \geq 0} \in \mathbb{K}^{\mathbb{Z}_{\geq 0}} \cong \text{HH}^0(\Delta)$$

are given by

$$\eta^0(T_i) = \sum_{\ell=0}^{\infty} \epsilon_\ell s_i^\ell \quad (208)$$

$$\eta^1 = \eta^2 = \eta^3 = \dots = 0 \quad (209)$$

These are the natural transformations in the traditional sense.

9.1.2 Degree 1

Theorem 9.2. *The degree 1 natural transformations*

$$\eta_{\{\sigma_\ell\}}: \text{id} \Rightarrow \text{id}, \quad \{\sigma_\ell\}_{\ell \geq 0} \in \mathbb{K}^{\mathbb{Z}_{\geq 0}} \cong \text{HH}^1(\Delta)$$

are given by

$$\eta^1(a_{ji}s^\alpha) = \mathfrak{q}(a_{ji}s^\alpha) \sum_{\ell=0}^{\infty} \frac{1}{2} \sigma_\ell a_{ji}s^{\alpha+\ell} \quad (210)$$

$$\eta^0 = \eta^2 = \eta^3 = \dots = 0 \quad (211)$$

where

$$\mathfrak{q}(a_{ji}s^\alpha) := 2\alpha + |i - j|. \quad (212)$$

Remark 9.3. Here, $\mathfrak{q}(a)$ is the $\mathfrak{q}$ -grading of the morphism a , which is crucial in Aganagic's categorification of Khovanov cohomology. We have discovered it naturally.

Proof. The degree 1 cocycles are

$$\begin{cases} \phi_0(s_i) = 0, \\ \phi_\ell(s_i) =: \sigma_{\ell-1}, & \ell > 0, \forall 0 \leq i \leq |\mathbf{x}|, \\ \phi_\ell(q_i) + \phi_\ell(p_{i+1}) = \sigma_\ell, & \ell \geq 0, \forall 0 \leq i < |\mathbf{x}|. \end{cases} \quad (213)$$

where $\{\sigma_\ell\}_{\ell \geq 0} \in \mathbb{K}^{\mathbb{Z}_{\geq 0}}$ parametrize cohomology. For a representative, we can take $\phi_\ell(q_i) = \phi_\ell(p_{i+1}) = \sigma_{\ell+1}/2$.

Precomposing with G_1 , one gets

$$\eta_{\{\sigma_\ell\}}^1(a_{ji}s^\alpha) = \phi_{\{\sigma_\ell\}} \circ G_1(a_{ji}s^\alpha) \quad (214)$$

$$= \phi_{\{\sigma_\ell\}} \left(\sum_{\beta=0}^{\alpha-1} a_{ji}s^{\alpha-\beta-1} \otimes s_i \otimes s_i^\beta + \sum_{k=i+\text{sgn}(i-j)}^j a_{jk} \otimes \frac{q_k}{p_k} \otimes a_{k+\text{sgn}(i-j),i} s_i^\alpha \right) \quad (215)$$

$$= \sum_{\beta=0}^{\alpha-1} a_{ji}s^{\alpha-\beta-1} \cdot \sum_{\ell=0}^{\infty} \sigma_\ell s_i^{\ell+1} \cdot s_i^\beta + \sum_{k=i\pm 1}^j a_{jk} \cdot \sum_{\ell=0}^{\infty} \frac{1}{2} \sigma_\ell \frac{q_k}{p_k} s_{k+\text{sgn}(i-j)}^\ell \cdot a_{k+\text{sgn}(i-j),i} s_i^\alpha \quad (216)$$

$$= (2\alpha + |i - j|) \sum_{\ell=0}^{\infty} \frac{1}{2} \sigma_\ell a_{ji}s_i^{\alpha+\ell} \quad (217)$$

where in (215), we choose q_k if $i > j$, or p_k if $i < j$. $\square$

9.1.3 Degree 2

Theorem 9.4. *The degree 2 natural transformations*

$$\eta_{\{\vartheta_i\}}: \text{id} \Rightarrow \text{id}, \quad \{\vartheta_1, \dots, \vartheta_{|\mathbf{x}|-1}\} \in \mathbb{K}^{|\mathbf{x}|-1} \cong \text{HH}^2(\Delta),$$

are given by

$$\eta^2(a_{kj}s^\beta, a_{ji}s^\alpha) = \mathfrak{C}(a_{kj}s^\beta, a_{ji}s^\alpha) a_{ki}s^{\beta+\alpha+\delta(i,j,k)-1} \quad (218)$$

$$\eta^0 = \eta^1 = \eta^3 = \dots = 0 \quad (219)$$

where

$$\mathfrak{C}(a_{kj}s^\beta, a_{ji}s^\alpha) := \begin{cases} -\beta\vartheta_j - \sum_{t=1}^{j-k} 2(\beta+t)\vartheta_{j-t} - (\beta+j-k)\vartheta_{k-1}, & i < k \leq j, \\ \beta\vartheta_j + \sum_{t=1}^{k-j} 2(\beta+t)\vartheta_{j+t} + (\beta+k-j)\vartheta_{k+1}, & j \leq k < i, \\ -\beta\vartheta_j - \sum_{t=1}^{j-i} 2(\beta+t)\vartheta_{j-t} + (\beta+j-i)\vartheta_i, & k \leq i < j, \\ \beta\vartheta_j + \sum_{t=1}^{i-j} 2(\beta+t)\vartheta_{j+t} - (\beta+i-j)\vartheta_i, & j < i \leq k, \\ 0, & \text{else.} \end{cases} \quad (220)$$

Here we set $\vartheta_0 = \vartheta_{|\mathbf{x}|} = 0$.

Remark 9.5. *The coefficient $\mathfrak{C}(a_2, a_1)$ is best summarized diagrammatically. For the strand diagram representing the path a_2a_1 , slide the black strand across one red strand at a time. If there is a subpath of the form $p_i s_i^\alpha$ in the middle, it contributes $\alpha(\vartheta_i + \vartheta_{i-1})$ to $\mathfrak{C}$. Moreover, if there is an additional q_{i-1} above $p_i s_i^\alpha$, add another ϑ_{i-1} to the total. Summing all contributions, and if a_2a_1 bends to the right (i.e. $j > i$), there is an overall minus sign.*

For an illustration, consider $\mathfrak{C}(a_{24}s^2, a_{41}s)$. Following Fig. 18, we get

$$\mathfrak{C}(a_{24}s^2, a_{41}s) = -4\vartheta_1 - 8\vartheta_2 - 6\vartheta_3 - 2\vartheta_4. \quad (221)$$

If there are only 4 punctures, then set $\vartheta_4 = 0$.

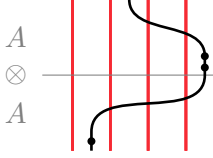
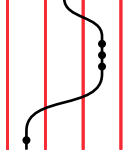
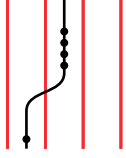
	ϑ_0	ϑ_1	ϑ_2	ϑ_3	ϑ_4
				2	2
			3	3	
			1		
		4	4		
	0	4	8	6	2

Figure 18: Evaluation of coefficient $\mathfrak{C}$.

Below is the proof of Theorem 9.4.

Proof. From (148),

$$- J_\ell(s_0) = V_{0,Q}^{2,\ell} \quad (222)$$

$$- J_\ell(s_i) = V_{i-1,P}^{2,\ell} + V_{i,Q}^{2,\ell}, \quad 0 < i < |\mathbf{x}| \quad (223)$$

$$- J_\ell(s_{|\mathbf{x}|}) = V_{|\mathbf{x}|-1,P}^{2,\ell} \quad (224)$$

where the vectors J_ℓ span the image and the vectors $V^{2,\ell}$ span the kernel.

Thus, we can take the representatives of the Hochschild cohomology as

$$\mathfrak{V}_i := V_{i-1,P}^{2,\ell} - V_{i,Q}^{2,\ell} = [0, 0, \dots, 1, 0, 1, -1, 1, -1, 0, -1, \dots, 0, 0]^\top, \quad 1 \leq i \leq |\mathbf{x}| - 1. \quad (225)$$

A general representative of the cohomology class is given by a linear combination of $\mathfrak{V}_i$,

$$\mathfrak{V} = \sum_{i=1}^{|\mathbf{x}|-1} \vartheta_i \mathfrak{V}_i \quad (226)$$

More concretely, we have

$$\begin{bmatrix} \vdots \\ \phi_0(s_{i-1}q_{i-1}) \\ \phi_0(q_{i-1}p_i) \\ \phi_0(p_iq_{i-1}) \\ \phi_0(s_ip_i) \\ \phi_0(s_iq_i) \\ \phi_0(q_ip_{i+1}) \\ \phi_0(p_{i+1}q_i) \\ \phi_0(s_{i+1}p_{i+1}) \\ \vdots \end{bmatrix} = \begin{bmatrix} \cdots & \mathfrak{V}_{i-1} & \mathfrak{V}_i & \mathfrak{V}_{i+1} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vartheta_{i-1} \\ \vartheta_i \\ \vartheta_{i+1} \\ \vdots \end{bmatrix} \quad (227)$$

$$= \begin{bmatrix} \ddots & & & & & & & & \\ & 1 & 1 & & & & & & \\ & -1 & 0 & & & & & & \\ & 0 & 1 & & & & & & \\ & -1 & -1 & & & & & & \\ & & 1 & 1 & & & & & \\ & & -1 & 0 & & & & & \\ & & 0 & 1 & & & & & \\ & & -1 & -1 & & & & & \\ & & & & \ddots & & & & \end{bmatrix} \begin{bmatrix} \vdots \\ \vartheta_{i-1} \\ \vartheta_i \\ \vartheta_{i+1} \\ \vdots \end{bmatrix} \quad (228)$$

which are

$$\phi_0(q_ip_{i+1}) = -\vartheta_i \quad (229)$$

$$\phi_0(p_{i+1}q_i) = \vartheta_{i+1} \quad (230)$$

$$\phi_0(s_iq_i) = \vartheta_i + \vartheta_{i+1} \quad (231)$$

$$\phi_0(s_{i+1}p_{i+1}) = -\vartheta_i - \vartheta_{i+1} \quad (232)$$

Note that all the cohomologically nontrivial coefficients are $\phi_0: S_2 \rightarrow \mathbb{k}$. Each element in S_2 , after reduction, corresponds to a strand with one dot. Thus, η removes one dot, yielding the $a_{ki}s^{\beta+\alpha+\delta(i,j,k)-1}$ in (218). Combining this with the map G_2 , one obtains the diagrammatic description of the coefficients given precisely in (220) and Remark 9.5. $\square$

Remark 9.6. One can directly verify the A_∞ -naturality condition (60) on the results of Theorems 9.1–9.4. As an example, consider a general degree 2 natural transformation η acting on $a_{13}s \otimes a_{30} \otimes a_{02}$. In this case, equation (60) reduces to One can directly verify the condition for A_∞ -natural transformations (60) on the results Theorems 9.1–9.4. For example, consider a general

degree 2 natural transformation η acting on $a_{13}s \otimes a_{30} \otimes a_{02}$. Here (60) reduces to

$$(-\mu^2(\eta^2 \otimes \text{id}) + \mu^2(\text{id} \otimes \eta^2) - \eta^2(\mu^2 \otimes \text{id}) + \eta^2(\text{id} \otimes \mu^2))(a_{13}s \otimes a_{30} \otimes a_{02}) \quad (233)$$

$$= -\eta^2(a_{13}s, a_{30}) \cdot a_{02} + a_{13}s \cdot \eta^2(a_{30}, a_{02}) - \eta^2(a_{10}s^3, a_{02}) + \eta^2(a_{13}s, a_{32}s^2) \quad (234)$$

$$= (3\vartheta_0 + 6\vartheta_1 + 4\vartheta_2 + \vartheta_3)a_{10}s^2 \cdot a_{02} + a_{13}s \cdot (2\vartheta_1 + 2\vartheta_2)a_{32}s \quad (235)$$

$$- (3\vartheta_0 + 8\vartheta_1 + 4\vartheta_2)a_{12}s^3 - (2\vartheta_2 + \vartheta_3)a_{12}s^3$$

$$= (3\vartheta_0 + 6\vartheta_1 + 4\vartheta_2 + \vartheta_3 + 2\vartheta_1 + 2\vartheta_2 - 3\vartheta_0 - 8\vartheta_1 - 4\vartheta_2 - 2\vartheta_2 - \vartheta_3)a_{12}s^3 \quad (236)$$

$$= 0 \quad (237)$$

9.2 Nat(id, β_{i-})

Away from the punctures, the natural transformations to β_{i-} looks much similar to the ones to id. By degree reasons, for $|\eta| = g$, only η^g and η^{g+1} components can be nonzero acting on chains of morphisms between generators. We will not give all the proofs for conciseness.

In all expressions of the natural transformation $\text{id} \Rightarrow \beta_{i-}$, for a similar reason as in Remark 3.5, the choice of the $[\frac{1}{2}, \frac{1}{2}]$ -morphism into $\beta_{i-}T_i$ in η^g is merely conventional. One may choose a different partition; accordingly, the component η^{g+1} will change in a compatible way.

9.2.1 Degree 0

Theorem 9.7. *The degree 0 natural transformations*

$$\eta_{\{\epsilon_\ell\}}: \text{id} \Rightarrow \beta_{i-}, \quad \{\epsilon_\ell\}_{\ell \geq 1} \in \mathbb{k}^{\mathbb{Z}_{\geq 1}} \cong \text{HH}^0(\mathfrak{B}_{i-})$$

are given by

$$\eta^0(T_j) = \begin{cases} \sum_{\ell=1}^{\infty} \epsilon_\ell s_j^\ell, & j \neq i, \\ \sum_{\ell=1}^{\infty} \epsilon_\ell \left[\frac{1}{2}q_{i-1}s^{\ell-1}, \quad \frac{1}{2}p_{i+1}s^{\ell-1} \right], & j = i, \end{cases} \quad (238)$$

$$\eta^1(a_{kj}s^\alpha) = \begin{cases} 0, & k \neq i, \\ \text{sgn}(i-j) \sum_{\ell=1}^{\infty} \frac{1}{2}\epsilon_\ell a_{ij}s^{\alpha+\ell-1}, & k = i, \end{cases} \quad (239)$$

$$\eta^2 = \eta^3 = \eta^4 = \dots = 0 \quad (240)$$

For example, we can check the naturality condition for $\eta: \text{id} \Rightarrow \beta_{2-}$ with only $\epsilon_2 = 2$ on morphism $a := a_{24}s \in \text{Hom}(T_4, T_2)$,

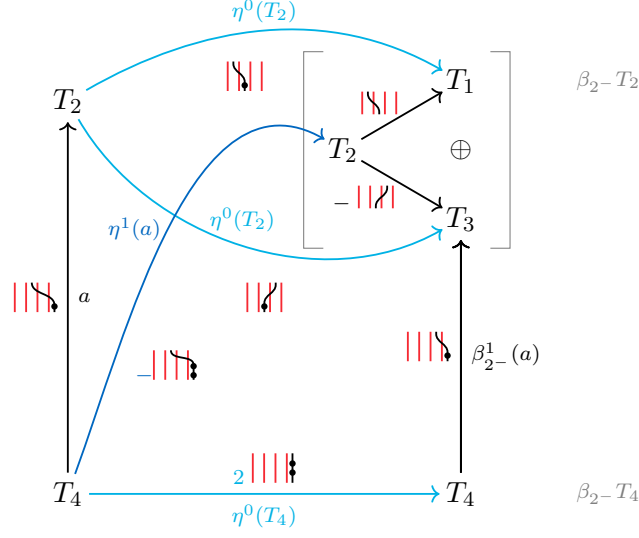


Figure 19: Naturality condition for $\eta \in \text{Nat}^0(\text{id}, \beta_{i-})$

We can see that

$$\mu_\delta^2(\beta^1(a), \eta^0(T_4)) - \mu_\delta^2(\eta^0(T_2), a) + \mu_\delta^1(\eta^1(a)) = 0. \quad (241)$$

For the length 2 input, one have for example,

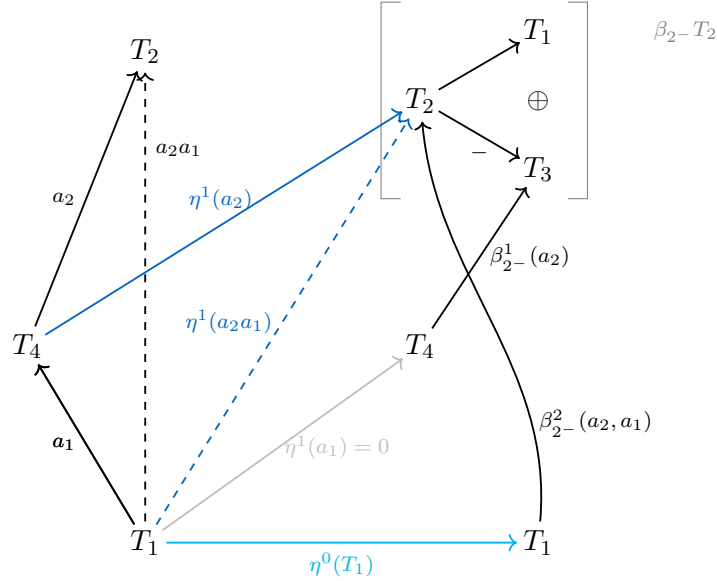


Figure 20: Naturality condition for $\eta \in \text{Nat}^0(\text{id}, \beta_{i-})$

One can check directly that

$$-\mu_\delta^2(\eta^1(a_2), a_1) - \mu_\delta^2(\beta_{i-}^1(a_2), \eta^1(a_1)) + \mu_\delta^2(\beta_{i-}^2(a_2, a_1), \eta^0(X_0)) + \eta^1(\mu^2(a_2, a_1)) = 0 \quad (242)$$

Remark 9.8. *Even when considering degree 0 natural transformations, we cannot restrict ourselves to the classical notion in ordinary categories, because nontrivial higher components (e.g. η^1) arise in the $\text{id} \Rightarrow \beta_{i-}$ case.*

9.2.2 Degree 1

Theorem 9.9. *The degree 1 natural transformations*

$$\eta_{\{\sigma_\ell\}} : \text{id} \Rightarrow \beta_{i-}, \quad \{\sigma_\ell\}_{\ell \geq 0} \in \mathbb{K}^{\mathbb{Z}_{\geq 0}} \cong \text{HH}^1(\mathfrak{B}_{i-})$$

are given by

$$\eta^1(a_{kj}s^\alpha) = \begin{cases} \mathfrak{q}(a_{kj}s^\alpha) \sum_{\ell=0}^{\infty} \frac{1}{2} \sigma_\ell a_{kj} s^{\alpha+\ell}, & k \neq i, \\ \mathfrak{q}(a_{ij}s^\alpha) \sum_{\ell=0}^{\infty} \frac{1}{2} \sigma_\ell a_{i+\text{sgn}(j-i),j} s^{\alpha+\ell}, & k = i \neq j \\ \mathfrak{q}(s^\alpha) \sum_{\ell=0}^{\infty} \frac{1}{2} \sigma_\ell \left[\frac{1}{2} q_{i-1} s^{\alpha+\ell-1}, \quad \frac{1}{2} p_{i+1} s^{\alpha+\ell-1} \right], & k = j = i. \end{cases} \quad (243)$$

$$\eta^2(a_{lk}s^\beta, a_{kj}s^\alpha) = \begin{cases} 0, & l \neq i, \\ \mathfrak{Q}(a_{ik}s^\beta, a_{kj}s^\alpha) \sum_{\ell=0}^{\infty} \frac{1}{2} \sigma_\ell a_{ij} s^{\beta+\alpha+\delta(j,k,i)+\ell-1}, & l = i. \end{cases} \quad (244)$$

$$\eta^0 = \eta^3 = \eta^4 = \dots = 0 \quad (245)$$

where

$$\mathfrak{Q}(a_2, a_1) = \mathfrak{Q}(a_{ik}s^\beta, a_{kj}s^\alpha) \quad (246)$$

$$= \begin{cases} \text{sgn}(i-k) \mathfrak{q}(a_2 a_1), & k < i < j \text{ or } j < i < k, \\ \text{sgn}(j-i) \frac{1}{2} \mathfrak{q}(a_2), & k = i, \\ \text{sgn}(i-k) \frac{1}{2} \mathfrak{q}(a_2 a_1), & j = i, \\ 0, & \text{else.} \end{cases} \quad (247)$$

one can check if $\beta + \alpha + \delta(j, k, i) = 0$, then $\mathfrak{Q} = 0$, thus the sum in (247) will not contain the undefined term s^{-1} .

One can verify the naturality conditions on inputs of length 2 and 3 by writing down the corresponding commutative diagrams. For example,

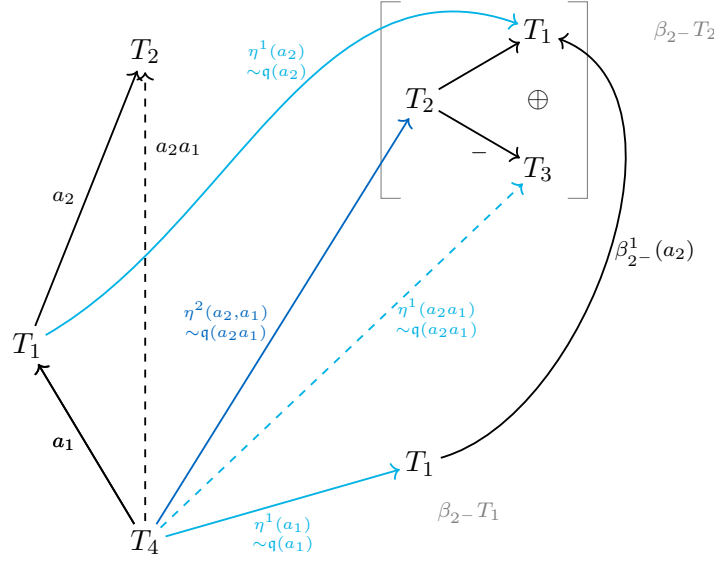


Figure 21: Naturality condition for $\eta \in \text{Nat}^1(\text{id}, \beta_{i-})$

In Fig. 21, we only write the coefficient before each diagram. We can check the naturality condition (60), which is

$$\mu_\delta^2(\eta^1(a_2), a_1) + \mu_\delta^2(\beta_{i-}^1(a_2), \eta^1(a_1)) + \mu_\delta^1(\eta^2(a_2, a_1)) - \eta^1(\mu^2(a_2, a_1)) = 0, \quad (248)$$

$$\implies \eta^1(a_2) \cdot a_1 + \beta_{i-}^1(a_2) \cdot \eta^1(a_1) = \delta \eta^2(a_2, a_1) + \eta^2(a_2, a_1) \delta + \eta^1(a_2 \cdot a_1). \quad (249)$$

The length 3 case is straightforward to verify, so we omit it here.

9.2.3 Degree 2

Theorem 9.10. *The degree 2 natural transformations*

$$\eta_{\{\vartheta_i\}}: \text{id} \Rightarrow \beta_{i-}, \quad \{\vartheta_1, \dots, \vartheta_{i-1}, \vartheta_{i+1}, \dots, \vartheta_{|\mathbf{x}|-1}\} \in \mathbb{k}^{|\mathbf{x}|-2} \cong \text{HH}^2(\mathfrak{B}_{i-}),$$

are given by

$$\eta^2(a_{lk}s^\beta, a_{kj}s^\alpha) = \begin{cases} \mathfrak{C}(a_{lk}s^\beta, a_{kj}s^\alpha) a_{lj}s^{\beta+\alpha+\delta(j,k,l)-1}, & l \neq i, \\ \mathfrak{C}(a_{ik}s^\beta, a_{kj}s^\alpha) a_{i+\text{sgn}(j-i),j}s^{\beta+\alpha+\delta(j,k,i)-1}, & l = i \neq j \\ \mathfrak{C}(a_{ik}s^\beta, a_{ki}s^\alpha) \left[\frac{1}{2}q_{i-1}s^{\beta+\alpha+|i-k|-2}, \quad \frac{1}{2}p_{i+1}s^{\beta+\alpha+|i-k|-2} \right], & l = j = i. \end{cases} \quad (250)$$

$$\eta^0 = \eta^1 = \eta^4 = \dots = 0 \quad (251)$$

where we set $\vartheta_0 = \vartheta_i = \vartheta_{|\mathbf{x}|} = 0$ in the expression for $\mathfrak{C}(a_2, a_1)$ (220). A nontrivial $\eta^3(a_3, a_2, a_1)$ arises whenever $a_3 \in \text{Hom}(T_l, T_i)$, and it is determined by

$$\delta \eta^3(a_3, a_2, a_1) = \eta^2(a_3, a_2) \cdot a_1 - \beta_{i-}^1(a_3) \cdot \eta^2(a_2, a_1) + \eta^2(a_3 \cdot a_2, a_1) - \eta^2(a_3, a_2 \cdot a_1). \quad (252)$$

This relation determines η^3 and makes it explicitly computable on arbitrary inputs; a closed formula is omitted for brevity.

References

- [1] Mina Aganagic. Knot categorification from mirror symmetry, part II: Lagrangians. *arXiv:2105.06039*.
- [2] Mina Aganagic, Ivan Danilenko, Yixuan Li, Vivek Shende, and Peng Zhou. Quiver Hecke algebras from Floer homology in Coulomb branches. *arXiv:2406.04258*.
- [3] Michael J. Bardzell. The alternating syzygy behavior of monomial algebras. *Journal of Algebra*, 188(1):69–89, 1997.
- [4] Severin Barmeier and Zhengfang Wang. Deformations of path algebras of quivers with relations. *arXiv:2002.10001*.
- [5] Sergio Chouhy and Andrea Solotar. Projective resolutions of associative algebras and ambiguities. *Journal of Algebra*, 432:22–61, 2015.
- [6] Elise LePage and Vivek Shende. Aganagic’s invariant is Khovanov homology. *arXiv:2505.00327*.
- [7] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups I. *Representation Theory*, 13(14):309–347, 2009.
- [8] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups II. *Transactions of the American Mathematical Society*, 363(5):2685–2700, 2011.
- [9] Mikhail Khovanov and Richard Thomas. Braid cobordisms, triangulated categories, and flag varieties. *arXiv:0001043*.
- [10] Paul Seidel. Fukaya Categories and Picard–Lefschetz Theory. *European Mathematical Society*, 2008.
- [11] Paul Seidel and Richard Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Mathematical Journal*, 108(1):37–108, 2001.
- [12] Raphaël Rouquier. 2-Kac-Moody algebras. *arXiv:0812.5023*.
- [13] Yujin Tong. Hamiltonian flow between standard module Lagrangians. *arXiv:2511.06431*.
- [14] Ben Webster. Weighted Khovanov–Lauda–Rouquier algebras. *arXiv:1209.2463*.
- [15] Ben Webster. Knot invariants and higher representation theory. *arXiv:1309.3796*.
- [16] Ben Webster. Tensor product algebras, Grassmannians and Khovanov homology. *Physics and Mathematics of Link Homology* 680, 23–58 (2016).