OBSERVABLE SETS FOR THE FREE SCHRÖDINGER EQUATION ON COMBINATORIAL GRAPHS

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ABSTRACT. We study observability for the free Schrödinger equation $\partial_t u = i\Delta u$ on combinatorial graphs $G = (\mathcal{V}, \mathcal{E})$. A subset $E \subset \mathcal{V}$ is observable at time $T > 0$ if there exists $C(T, E) > 0$ such that for all $u_0 \in l^2(\mathcal{V})$,

$$\|u_0\|_{l^2(\mathcal{V})}^2 \le C(T, E) \int_0^T \|e^{it\Delta} u_0\|_{l^2(E)}^2 \, \mathrm{d}t.$$

On the one-dimensional lattice $\mathbb{Z}$, we obtain a sharp threshold for thick sets: if $E \subset \mathbb{Z}$ is γ-thick with $\gamma \ge 1/2$, then E is observable at some time; conversely, for every $\gamma < 1/2$ there exists a γ-thick set that is not observable at any time. This critical threshold marks the exact point where the discrete lattice departs from the real line: on the lattice it must be attained, whereas on $\mathbb{R}$ any γ-thick set with $\gamma > 0$ already suffices.

On $\mathbb{Z}^d$ we show that the complements of finite sets are observable at any time $T > 0$. This parallels the Euclidean setting $\mathbb{R}^d$: any set that contains the exterior of a finite ball is observable at any time.

For finite graphs we give an equivalent characterization of observability in terms of the zero sets of Laplacian eigenfunctions. As an application, we construct unobservable sets of large density on discrete tori, in contrast with the continuous torus $\mathbb{T}^d$, where every nonempty open set is observable.

1. INTRODUCTION

This work can be situated within the framework of observability for the free Schrödinger equation

$$\begin{cases} \partial_t u(x,t) = i\Delta u(x,t), & (x,t) \in X \times (0,\infty), \\ u(x,0) = u_0(x), & x \in X, \end{cases} \tag{1}$$

on various geometric settings X. In this paper, we focus on the discrete case where $X = G = (\mathcal{V}, \mathcal{E})$ is a combinatorial graph. In the discrete setting, the continuum Laplacian is replaced by the graph Laplacian Δ_{disc}, defined for $f : \mathcal{V} \to \mathbb{C}$ by

$$\Delta_{\mathrm{disc}} f(x) := \sum_{y \sim x} \big(f(y) - f(x)\big),$$

where $y \sim x$ means that y and x are adjacent in G. We are motivated by the well-developed observability theory for the free Schrödinger equation in continuous settings such as $\mathbb{R}^d$ and $\mathbb{T}^d$.

We briefly note that the discrete Schrödinger equation arises in tight-binding models of solids and optical lattices, photonic waveguide arrays, and continuous-time quantum walks; see [4, 5, 12, 19, 39, 40]. For the Schrödinger equation on combinatorial graphs, related advances include discrete well-posedness, continuum limit and more [11, 20, 21, 24, 25, 42].

Building on the continuous-case terminology (see for example [26]), we introduce the following observability notions on graphs.

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Definition 1.1. *A subset $E \subset \mathcal{V}$ is called an **observable set at some time** for* (1) *if there exists $T > 0$ and a constant $C_{\rm obs} = C_{\rm obs}(T, E) > 0$ such that the observability inequality*

$$\sum_{x\in\mathcal{V}} |u_0(x)|^2 \;\leq\; C_{\rm obs} \int_0^T \sum_{x\in E} |u(x,t)|^2 \, \mathrm{d}t \tag{2}$$

holds for any solution u to Equation (1) *on G. A subset $E \subset \mathcal{V}$ is called an **observable set at any time** for* (1) *if for any $T > 0$, there exists $C_{\rm obs} = C_{\rm obs}(T, E) > 0$ such that* (2) *holds. Similarly, E is called an **observable set at time** $T > 0$ if there exists $C_{\rm obs} = C_{\rm obs}(T, E) > 0$ such that* (2) *holds.*

Our study analyzes and compares observability properties of the free Schrödinger equation in discrete (combinatorial graphs) and continuous settings. General observability criteria and quantitative inequalities for the free discrete Schrödinger flow on combinatorial graphs remain largely open. In combinatorial graphs, the lack of a smooth differential structure and geodesic flow makes standard PDE tools such as Carleman estimates [29, 31], propagation of singularities [35], and microlocal analysis [7, 9] largely inapplicable; likewise, many uncertainty principles developed in $\mathbb{R}^d$ (e.g., Nazarov [28], Logvinenko–Sereda [34]) do not transfer in a straightforward or equally strong form. Consequently, establishing observability needs to rely on discrete spectral features and arithmetic sampling effects, rendering the analysis qualitatively different and technically more delicate. Our contribution establishes observability inequalities for the free discrete model on common combinatorial graphs and quantitatively compares them with the continuum theory.

Before stating our first main result, we recall the notion of thick sets (see, e.g., [8, 34]). In $\mathbb{R}^d$, thickness is defined via uniform density on cubes, here we use the analogous notion on $\mathbb{Z}^d$.

Definition 1.2. *A subset $E \subset \mathbb{Z}^d$ is called γ**-thick** if there exists $\gamma \in (0, 1]$ and $L \in \mathbb{Z}_{\geq 0}$ such that for every $x \in \mathbb{Z}^d$*

$$\frac{|E \cap Q_L(x)|}{|Q_L(x)|} \;\geq\; \gamma,$$

where $Q_L(x) := \{y \in \mathbb{Z}^d : \|y - x\|_\infty \leq L\}$ is a discrete cube with cardinality $|Q_L(x)| = (2L + 1)^d$. (Here and below, for $E \subset \mathcal{V}$, $|E|$ denotes the cardinality.)

We remark that the concept of thick sets is closely related to the sets with **Beurling density** (see Definition 2.1).

The first main result characterizes the sets that are observable at some time via a thick condition; analogous conclusions in the continuous setting appear in [47] and so on. The key ingredient in the proof of those results heavily relies on a generalization of the Logvinenko-Sereda uncertainty principle due to Kovrijkine [34], which states that if $E \subset \mathbb{R}$ is a thick set, then for all f in $L^2(\mathbb{R})$ with Fourier support contained in a union of finitely many intervals, one has $\|f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(E)}$. Our proof heavily relies on Theorem 2.2 , which can be seen as a discrete Logvinenko-Sereda uncertainty principle and we prove it by Beurling sampling theorem (see e.g. [22, 33]).

Theorem 1.1. *The critical threshold for γ-thickness to imply observability at some time for the Schrödinger equation* (1) *on $\mathbb{Z}$ is $\gamma = \frac{1}{2}$. Specifically:*

- (i) *If $\gamma \geq \frac{1}{2}$, then any γ-thick subset of $\mathbb{Z}$ is observable at some time.*
- (ii) *If $\gamma < \frac{1}{2}$, there exists a γ-thick subset of $\mathbb{Z}$ that is not observable for any time $T > 0$.*

For Theorem 1.1, we have the following remarks.

($\mathbf{R}_1$) In the continuous setting $X = \mathbb{R}$, a measurable set E is observable at some time if and only if it is γ-thick for some $\gamma > 0$, as established by the Logvinenko–Sereda uncertainty principle; see [26, Theorem 1.1]. Furthermore, [43, Theorem 1.2] extends this result to any observability time $T > 0$ for general one-dimensional Schrödinger operators with bounded and continuous potentials. The proof is purely quantitative and yields an upper bound for the control cost in terms of T and the thickness parameter. The discrete threshold $1/2$ is somewhat surprising and reflects sampling and arithmetic phenomena specific to the lattice $\mathbb{Z}$, rather than merely the thickness. The following example shows arithmetic effects for observability: $2\mathbb{Z}$ is not observable at any time (see Proposition 2.3), whereas the set of integers congruent to 0 or 1 mod 4 is observable at any time (see Example 2.1), even though both sets have the same thickness in $\mathbb{Z}$.

($\mathbf{R}_2$) It remains an open challenge to generalize Theorem 1.1(i) to higher dimensions: to the best of our knowledge, the extra sampling (or uncertainty) required has not yet been characterized, and its development lies beyond the scope of the present work. Similar technical barriers have also appeared in many existing studies on $\mathbb{R}$ (see e.g. [26, 43]). On the other hand, Theorem 1.1(ii) extends to $d \geq 1$: one can easily construct γ-thick subsets of $\mathbb{Z}^d$ that are not observable for any $T > 0$ whenever $\gamma < \frac{1}{2}$, by taking the Cartesian product of a γ-thick subset of $\mathbb{Z}$ with $\mathbb{Z}^{d-1}$. Our Corollary 2.1 shows that for any thickness $0 < \gamma < 1$, there exists a set $E \subset \mathbb{Z}$ is observable at any time. An interesting follow-up question is whether every subset $E \subset \mathbb{Z}$ with thickness $\gamma \geq \frac{1}{2}$ is observable at any time, and to characterize observability at some (any) time with thickness $\gamma < \frac{1}{2}$.

There is a substantial literature describing sufficient geometric conditions for observability of the free Schrödinger flow on $\mathbb{R}^d$. For instance, [46, Remark (a6)] shows that if E contains the exterior of a ball $B_r(0)$ for some $r \geq 0$, then E is observable at any time for the free Schrödinger equation. In the discrete setting, an analogous exterior observability phenomenon also holds: complements of finite sets are observable at any time. This is our second result.

Theorem 1.2. *Let $d \geq 1$, $E^c := \mathbb{Z}^d \setminus E \subset \mathbb{Z}^d$ be a finite set. Then E is an observable set at any time $T > 0$.*

Informally, since the discrete Laplacian on $\mathbb{Z}^d$ admits no compactly supported eigenfunctions, mass cannot remain trapped in a finite hole F; its time average necessarily leaks into E, yielding the desired observability. In particular, the observability constant depends only on d, T, and the geometric size of F (e.g., $|F|$), and necessarily satisfies $C_{\text{obs}}(T, E) \geq 1/T$. We have the following remark to contrast the techniques employed in $\mathbb{R}^d$ with their discrete counterparts in $\mathbb{Z}^d$.

($\mathbf{R}_3$) On $\mathbb{R}^d$, the corresponding exterior observability results [46, Theorem 1.1 and Remark (a6)] rely crucially on quantitative Nazarov-type uncertainty principles in higher dimensions (see [28]). The Nazarov-type uncertainty principles state as follows. Given subsets S, $\Sigma \subset \mathbb{R}^d$ with $|S|, |\Sigma| < \infty$, there is a positive constant $C(d, S, \Sigma) := Ce^{C\min\{|S||\Sigma|, |S|^{1/d}w(\Sigma), |\Sigma|^{1/d}w(S)\}}$, (here $w(\cdot)$ denotes mean width) with $C = C(d)$ such that for each $f \in L^2(\mathbb{R}^d; \mathbb{C})$,

$$\int_{\mathbb{R}^d_x} |f(x)|^2 \, \mathrm{d}x \leq C(d, S, \Sigma) \left(\int_{\mathbb{R}^d_x \setminus S} |f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^d_\xi \setminus \Sigma} |\mathcal{F} f(\xi)|^2 \, \mathrm{d}\xi \right).$$

In our lattice setting, we avoid such microlocal machinery: the proof is based on a TT^* operator-norm argument together with specific spectral features of the discrete Laplacian. Concretely, the projection onto the finite hole F has finite rank, so its time-averaged contribution cannot dominate the identity, which yields coercivity and the observability inequality.

Our third result addresses the finite (compact) setting and should be compared with the theory on compact Riemannian manifolds. On a finite graph, the Schrödinger flow evolves in a finite-dimensional Hilbert space, and observability reduces to a purely spectral injectivity question. This yields a sharp characterization, stated as following.

Theorem 1.3. *Let $G = (\mathcal{V}, \mathcal{E})$ be a finite graph with discrete Laplacian Δ_{disc}, and let $E \subseteq \mathcal{V}$. The following are equivalent:*

(i) *E is observable at any time $T > 0$.*
(ii) *For every nonzero eigenvector ψ of Δ_{disc}, its restriction $\psi|_E$ is not identically zero.*

For Theorem 1.3, we have the following remark.

($\mathbf{R}_4$) It is classical that on a compact Riemannian manifold, any open set satisfying the geometric control condition is observable at any time for the wave and Schrödinger equations (see e.g. [35,41]); the same notion extends to metric graphs, e.g. [1]. Furthermore, the geometric control condition is known to be necessary on manifolds with periodic geodesic flows (Zoll manifolds), see [36]. In contrast, for the finite combinatorial graphs there are no continuous geodesic flows nor microlocal ray-propagation mechanisms; accordingly, GCC plays no role in the discrete combinatorial setting.

The final result concerns the sets on discrete tori of large density which are not observable at any time, providing a comparison with the standard tori.

Theorem 1.4. *Let $(\mathbb{Z}/N\mathbb{Z})^d$ be d-dimensional discrete tori with vertex number N^d. For every $d \geq 1$ there exist $c > 0$ and an infinite sequence $N_m \to \infty$ such that there are sets $E_{N_m} \subset (\mathbb{Z}/N_m\mathbb{Z})^d$ with $|E_{N_m}| \geq c\, N_m^d$ for all m, which are not observable at any time.*

The remark below compares the differences for observability between the continuous tori and the discrete tori.

($\mathbf{R}_5$) On continuous tori $\mathbb{T}^d$, every non-empty open set is observable at any time (see, e.g. [2,7,9, 23,27,32,44]). Furthermore, every positive-measure set on $\mathbb{T}^2$ is observable at any time [10], and a key ingredient in the proof is the construction of semiclassical defect measures to enforce mass concentration along rational directions. In contrast, Theorem 1.4 shows that due to arithmetic structure effects, some discrete tori contain high-density subsets that remain not observable at any time as the number of vertices tends to infinity.

We mention that there are also studies about observability results for Schrödinger equations on negatively curved manifolds, see for instance [3, 16, 17, 30]. We also highlight the quantitative Hautus-type resolvent estimates established in [16,17]. A central idea in those works is the fractal uncertainty principle, introduced in [18] and recently extended to higher dimensions in [13].

The rest of the paper is organized as follows. In Section 2, we discuss observable sets for lattices. The proofs of Theorem 1.1(i), Theorem 1.1(ii), and Theorem 1.2 are presented in Subsections 2.3, 2.4, and 2.6, respectively. In particular, we discuss the observability for periodic Bohr sets in Subsection 2.5. In Section 3, we turn to observable sets for finite graphs. The proofs of Theorem 1.3 and Theorem 1.4 can be found in Subsections 3.1 and 3.2, respectively.

2. Observable sets on lattices

2.1. **Fourier transform on the lattice.** For any $f \in l^2\left(\mathbb{Z}^d\right)$, its Fourier transform $\widehat{f} : \mathbb{T}^d \to \mathbb{C}$ is defined by

$$\widehat{f}\,(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} f\,(n)\, e^{in\cdot x} \qquad \text{for } x \in \mathbb{T}^d,$$

and for any $f \in L^2\left(\mathbb{T}^d\right)$, its inverse Fourier transform $f^\vee$ is

$$f^\vee(n) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} f(x)\, e^{-in\cdot x} \mathrm{d}x \qquad \text{for } n \in \mathbb{Z}^d.$$

The Plancherel identity shows that for any $f \in l^2\left(\mathbb{Z}^d\right)$,

$$\|f\|_{l^2\left(\mathbb{Z}^d\right)} = \|\widehat{f}\|_{L^2\left(\mathbb{T}^d\right)}. \tag{3}$$

The convolution $f * g$ of $f, g \in l^2\left(\mathbb{Z}^d\right)$ is defined by

$$f * g\,(n) := \sum_{m \in \mathbb{Z}^d} f(m)\, g(n-m),$$

and hence for $x \in \mathbb{T}^d$, we have

$$(f * g)^\wedge (x) = (2\pi)^{\frac{d}{2}}\, \widehat{f}(x)\, \widehat{g}(x) \qquad \text{for } x \in \mathbb{T}^d.$$

Here and below, we identify $\mathbb{T}^d$ with $[-\pi, \pi)^d$. By basic Fourier analysis, the solution $u(n,t) : \mathbb{Z}^d \times [0, \infty) \to \mathbb{C}$ of Equation (1) with an initial data $u_0 \in l^2\left(\mathbb{Z}^d\right)$ is as follows:

$$u(n,t) = e^{it\Delta_{\mathrm{disc}}} u_0(n) := \left(e^{2it\sum_{j=1}^d (\cos x_j - 1)} \widehat{u_0}(x)\right)^\vee (n) = u_0 * K_t(n),$$

where the kernel function $K_t : \mathbb{Z}^d \to \mathbb{C}$ is defined by

$$K_t(n) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{2it\sum_{j=1}^d (\cos x_j - 1)} e^{-in\cdot x} \mathrm{d}x.$$

We denote the Fourier multiplier of Laplacian Δ_{disc} by

$$\varphi(x) := 2\sum_{j=1}^d (\cos x_j - 1) \tag{4}$$

for simplicity. Let $u \in C^1([0, +\infty); l^2(\mathbb{Z}^d))$ be a solution of Equation (1), we have the l^2 conservation law:

$$\|e^{it\Delta_{\mathrm{disc}}} u_0\|_{l^2\left(\mathbb{Z}^d\right)} = \|u_0\|_{l^2\left(\mathbb{Z}^d\right)}. \tag{5}$$

2.2. **Beurling density for discrete sets.** In this subsection, we recall the notion of Beurling density and prove a discrete Logvinenko-Sereda uncertainty principle.

Definition 2.1. *[Beurling density] For a discrete set $\Lambda \subset \mathbb{Z}$, its lower Beurling density $d^-(E)$ and upper Beurling density $d^+(E)$ are given by*

$$d^-(\Lambda) := \liminf_{R\to\infty} \inf_{x\in\mathbb{R}} \frac{|\Lambda \cap [x-R, x+R]|}{2R} \text{ and } d^+(\Lambda) := \limsup_{R\to\infty} \sup_{x\in\mathbb{R}} \frac{|\Lambda \cap [x-R, x+R]|}{2R}.$$

We say a discrete set $\Lambda \subset \mathbb{Z}$ has Beurling density γ if $d^-(\Lambda) = d^+(\Lambda) = \gamma$.

We remark that using integer or real-valued centers is equivalent in the definition of density.

We use the notation of Paley-Wiener space $\mathcal{PW}_a := \{f \in L^2(\mathbb{R}) : \mathrm{spt}\mathcal{F}f \subset [-\pi a, \pi a]\}$, it consists of restrictions of entire functions of exponential type to the real line. Here and below, $\mathcal{F}f$ is the Fourier transform of f on $L^2(\mathbb{R})$ defined by

$$\mathcal{F}f(y) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixy} \mathrm{d}x \qquad \text{for } y \in \mathbb{R}.$$

Recall a discrete subset $\Lambda \subset \mathbb{R}$ is a *sampling set* for $\mathcal{PW}_a$ if for any $f \in \mathcal{PW}_a$,

$$A \sum_{x\in\Lambda} |f(x)|^2 \leq \int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x \leq B \sum_{x\in\Lambda} |f(x)|^2 ,$$

where the implicit constants A, B are independent of function f.

The following theorem of Beurling gives an almost complete characterization of sampling sets for the Paley-Wiener space, which can be found in [6, 22]

Theorem 2.1 (Beurling)**.** *The notations are given as above, then we have*

(a). *If $\Lambda \subset \mathbb{R}$ is discrete and $d^-(\Lambda) > 1$, then Λ is a sampling set for $\mathcal{PW}_1$.*

(b). *Conversely, if Λ is a sampling set for $\mathcal{PW}_1$, then $d^-(\Lambda) \geq 1$.*

Remark 2.1 (On the separation hypothesis)**.** *The classical Beurling sampling theorem on $\mathbb{R}$ typically assumes that the sampling set is separated, i.e., there exists $q > 0$ such that any two distinct points are at distance at least q. In our discrete setting, every $E \subset \mathbb{Z}$ is automatically 1-separated in the standard metric. After the rescaling $\sigma = \frac{a}{2\pi}$ (which corresponds to the bandwidth normalization used in the proof of Theorem 2.2), the set σE remains $q = \sigma$ separated. Therefore, the separation hypothesis required by the theorem is automatically satisfied in our application; see [6, 22].*

The following basic Propositions will be used in the proof of Theorem 2.2 and Theorem 1.1.

Proposition 2.1. *Let $E \subset \mathbb{Z}$ be a γ-thick set. Then its lower density satisfies $d^-(E) \geq \gamma$. Moreover, whenever $\gamma \geq \frac{1}{2}$, we have the sharper bound $d^-(E) > \frac{1}{2}$.*

Proof. Let $m := 2L + 1$ (so m is odd). By integrality of counts, the thickness condition implies

$$|E \cap [x - L, x + L]| \;\geq\; \lceil \gamma m \rceil \qquad \text{for all } x \in \mathbb{Z}.$$

For each $k \in \mathbb{Z}$, define the m-length integer-centered block

$$B_k := [km - L,\, km + L] \subset \mathbb{R}.$$

Then $|E \cap B_k| \geq \lceil \gamma m \rceil$ for every $k \in \mathbb{Z}$.

Fix $R > L$ and $x \in \mathbb{R}$. A block B_k is fully contained in $[x - R, x + R]$ if and only if $|km - x| \leq R - L$. Hence the set of such indices $K(x, R) := \{k \in \mathbb{Z} : \; B_k \subset [x - R, x + R]\}$ is precisely the set of integers in an interval of length $\frac{2(R-L)}{m}$, whence $|K(x, R)| \geq \left\lfloor \frac{2(R-L)}{m} \right\rfloor$. Summing the per-block lower bounds gives

$$|E \cap [x - R, x + R]| \;\geq\; \sum_{k\in K(x,R)} |E \cap B_k| \;\geq\; |K(x, R)| \lceil \gamma m \rceil \;\geq\; \left\lfloor \frac{2(R - L)}{m} \right\rfloor \lceil \gamma m \rceil.$$

Therefore,

$$\inf_{x\in\mathbb{R}} \frac{|E \cap [x - R, x + R]|}{2R} \;\geq\; \frac{\lceil \gamma m \rceil}{2R} \left\lfloor \frac{2(R - L)}{m} \right\rfloor.$$

Letting $R \to \infty$ and using $\frac{1}{2R} \left\lfloor \frac{2(R-L)}{m} \right\rfloor = \frac{1}{m} + o(1)$, we obtain

$$d^-(E) \;=\; \liminf_{R\to\infty} \inf_{x\in\mathbb{R}} \frac{|E \cap [x - R, x + R]|}{2R} \;\geq\; \frac{\lceil \gamma m \rceil}{m} \;\geq\; \gamma.$$

If $\gamma > \frac{1}{2}$ then $d^-(E) \geq \gamma > \frac{1}{2}$. If $\gamma = \frac{1}{2}$, the oddness of m yields

$$\frac{\lceil \gamma m \rceil}{m} \;=\; \frac{\lceil m/2 \rceil}{m} \;=\; \frac{L + 1}{2L + 1} \;>\; \tfrac{1}{2},$$

so $d^-(E) > \frac{1}{2}$ also in this borderline case. This proves the proposition. $\square$

Proposition 2.2. *Let $E \subset \mathbb{Z}$ have Beurling density γ. Then for any $\gamma' \in (0, \gamma)$, the set E is γ'-thick on $\mathbb{Z}$.*

Proof. Since $d^-(E) = \gamma$, by the definition of the lim inf there exists $S > 0$ such that

$$\inf_{x\in\mathbb{R}} \frac{|E \cap [x-R, x+R]|}{2R} \;\geq\; \gamma - \varepsilon \qquad \text{for all } R \geq S,$$

for any fixed $\varepsilon \in (0, \gamma - \gamma')$. In particular, taking $R = L$ with any integer $L \geq S$, we have

$$\inf_{x\in\mathbb{R}} \frac{|E \cap [x-L, x+L]|}{2L} \;\geq\; \gamma - \varepsilon.$$

Since $\mathbb{Z} \subset \mathbb{R}$, it follows that for all $m \in \mathbb{Z}$,

$$\frac{|E \cap [m-L, m+L]|}{2L} \;\geq\; \gamma - \varepsilon.$$

Passing to the discrete normalization $(2L+1)$ in the denominator, we obtain for all $m \in \mathbb{Z}$,

$$\frac{|E \cap [m-L, m+L]|}{2L+1} \;=\; \frac{2L}{2L+1} \cdot \frac{|E \cap [m-L, m+L]|}{2L} \;\geq\; \frac{2L}{2L+1}\,(\gamma - \varepsilon).$$

Finally, choose $L \geq S$ large enough so that

$$\frac{2L}{2L+1}\,(\gamma - \varepsilon) \;\geq\; \gamma',$$

which is possible because $\frac{2L}{2L+1} \to 1$ as $L \to \infty$ and $\gamma - \varepsilon > \gamma'$. With this choice of L, we conclude that

$$\frac{|E \cap [m-L, m+L]|}{2L+1} \;\geq\; \gamma' \qquad \text{for all } m \in \mathbb{Z},$$

i.e., E is γ'-thick. This completes the proof. ☐

The following theorem, which can be regarded as a discrete version of the Logvinenko-Sereda uncertainty principle, will be used in the proof of Theorem 1.1 (i).

Theorem 2.2 (Logvinenko-Sereda)**.** *Suppose $f \in l^2(\mathbb{Z})$ with $\operatorname{spt}\widehat{f} \subset I \subsetneq \mathbb{T}$ and $|I| = a$, and let $E \subset \mathbb{Z}$ be a γ-thick set. If $2\pi\gamma > a$, then*

$$\|f\|_{l^2(\mathbb{Z})} \lesssim_{a,E} \|f\|_{l^2(E)}.$$

The implicit constant depends only on a, E, is independent of the location of I.

Proof of Theorem 2.2 via the Beurling sampling theorem. Let f be the function which satisfies the assumptions, note that without loss of generality, we can assume that $\operatorname{spt}\widehat{f} \subset \left[-\frac{a}{2}, \frac{a}{2}\right]$. Define the following function

$$\widetilde{f}\,(y) = \mathcal{F}(\mathbb{1}_{[-\pi,\pi]}\widehat{f})\,(y) = \sum_{m\in\mathbb{Z}} f\,(m)\,\mathrm{sinc}\,(\pi\,(y-m)) \qquad \text{where } \mathrm{sinc}\,x := \frac{\sin x}{x}, \tag{6}$$

then $\widetilde{f}|_{\mathbb{Z}} = f|_{\mathbb{Z}}$. One checks that $\operatorname{spt}\mathcal{F}\widetilde{f} \subset \left[-\frac{a}{2}, \frac{a}{2}\right]$ and $\widetilde{f} \in L^2(\mathbb{R})$, so we derive $\widetilde{f} \in \mathcal{PW}_{\frac{a}{2\pi}}$. Consider the scaling $(\widetilde{f})_{\frac{a}{2\pi}}$, $\mathcal{F}(\widetilde{f})_{\frac{a}{2\pi}} = (\mathcal{F}\widetilde{f})^{\frac{a}{2\pi}}$, its Fourier support satisfies that $\operatorname{spt}\mathcal{F}(\widetilde{f})_{\frac{a}{2\pi}} \subset [-\pi, \pi]$, so we obtain that $(\widetilde{f})_{\frac{a}{2\pi}} \in \mathcal{PW}_1$. Here, $(\cdot)_\sigma$ denotes scaling $g_\sigma(x) := g(\sigma x)$, hence $\mathcal{F}g_\sigma(\xi) = \sigma^{-1}\mathcal{F}g(\xi/\sigma)$. By the fact that E is a γ-thick set, we have $d^-(E) \geq \gamma$, which implies that

$$d^-\left(\frac{a}{2\pi}E\right) = \frac{2\pi}{a}d^-(E) \geq \frac{2\pi\gamma}{a} > 1.$$

By the Beurling sampling theorem 2.1, the subset $\frac{a}{2\pi}E$ is a sampling set of $\mathcal{PW}_1$. In particular, as for $(\widetilde{f})_{\frac{a}{2\pi}}$, we deduce

$$A\sum_{x\in\frac{a}{2\pi}E}\left|\widetilde{f}\left(\frac{2\pi x}{a}\right)\right|^2\leq\int_{\mathbb{R}}\left|\widetilde{f}\left(\frac{2\pi x}{a}\right)\right|^2\mathrm{d}x\leq B\sum_{x\in\frac{a}{2\pi}E}\left|\widetilde{f}\left(\frac{2\pi x}{a}\right)\right|^2,\tag{7}$$

where the constants A,B are depend only on a and E. Calculating directly, we get

$$\sum_{x\in\frac{a}{2\pi}E}\left|\widetilde{f}\left(\frac{2\pi x}{a}\right)\right|^2=\sum_{x\in E}\left|\widetilde{f}\left(x\right)\right|^2=\sum_{x\in E}\left|f\left(x\right)\right|^2.\tag{8}$$

It follows from the Plancherel formula that

$$\int_{\mathbb{R}}\left|\widetilde{f}\left(\frac{2\pi x}{a}\right)\right|^2\mathrm{d}x=\frac{a}{2\pi}\|\widetilde{f}\|^2_{L^2(\mathbb{R})}=\frac{a}{2\pi}\|(\widetilde{f})^\wedge\|^2_{L^2(\mathbb{R})}=\frac{a}{2\pi}\|\widehat{f}\|^2_{L^2(\mathbb{T})}=\frac{a}{2\pi}\sum_{x\in\mathbb{Z}}\left|f\left(x\right)\right|^2.\tag{9}$$

Combining (7), (8) and (9), we derive

$$\frac{2\pi A}{a}\sum_{x\in E}\left|f\left(x\right)\right|^2\leq\sum_{x\in\mathbb{Z}}\left|f\left(x\right)\right|^2\leq\frac{2\pi B}{a}\sum_{x\in E}\left|f\left(x\right)\right|^2.$$

The constant $\frac{2\pi B}{a}$ depends only on a and E, and is independent of the location of I. This completes the proof. $\square$

2.3. **Proof of Theorem 1.1(i).** In the functional framework from Miller [38], he considered the observability of the following system:

$$\dot{x}-i\mathscr{A}x\left(t\right)=0,\ x\left(0\right)=x_0\in X,\ y\left(t\right)=\mathscr{C}x\left(t\right),\tag{10}$$

where X is a Hilbert space and $\mathscr{A}:D\left(\mathscr{A}\right)\to X$ is a self-adjoint operator. Miller proved that

Theorem 2.3. *[38, Theorem 5.1] The system* (10) *is exactly observable, i.e., the observable inequality holds at some $T>0$:*

$$\forall x_0\in X,\ \|x_0\|^2_X\leq\kappa_T\int_{(0,T)}\|y\|^2_X\mathrm{d}t,\tag{11}$$

if and only if, there exist $m,M>0$ such that the following observable resolvent estimate holds:

$$\forall\ x\in D\left(\mathscr{A}\right),\forall\lambda\in\mathbb{R},\ \ \|x\|^2_X\leq M\|\left(\mathscr{A}-\lambda\right)x\|^2_X+m\|\mathscr{C}x\|^2_X.\tag{12}$$

Moreover, for all $\epsilon>0$, there is a $C_\epsilon>0$ such that (12) *implies* (11) *for all $T>\sqrt{M\left(\pi^2+\epsilon\right)}$ with $\kappa_T=C_\epsilon mT/(T^2-M\left(\pi^2+\epsilon\right))$.*

In our cases, we set $X=l^2\left(\mathbb{Z}\right)$, $\mathscr{A}=\Delta_{\mathrm{disc}}$ and $\mathscr{C}=\mathbb{1}_E\cdot\mathrm{Id}_{l^2(\mathbb{Z})}$. Here and below, the indicator function $\mathbb{1}_E(x)=1$ if $x\in E$, otherwise it equals to 0.

The following lemma analyzes the Fourier multiplier properties of Δ_{disc}, and will be used in the proof of Theorem 1.1 (i).

Lemma 2.1 (Analysis of the Fourier multiplier of Δ_{disc})**.** *Suppose $\varphi=2\cos x-2$ is the Fourier multiplier of Δ_{disc}, defined in* (4)*. For any $\epsilon>0$, there exists a collection of intervals $\{I_x\}_{x\in\mathbb{T}}$ such that the following properties hold:*

1. *$x\in I_x$ for each $x\in\mathbb{T}$.*
2. *There are only finitely many possible lengths for these intervals, and $\sup_{x\in\mathbb{T}}|I_x|\leq\epsilon$.*

3. *There exists $M_\epsilon > 0$ such that*

$$\inf_{x\in\mathbb{T}} \left\{ \min_{y\in\mathbb{T}\setminus(I_x\cup I_{-x})} |\varphi(y) - \varphi(x)| \right\} \geq M_\epsilon. \tag{13}$$

Here we identify I_π with $I_{-\pi}$.

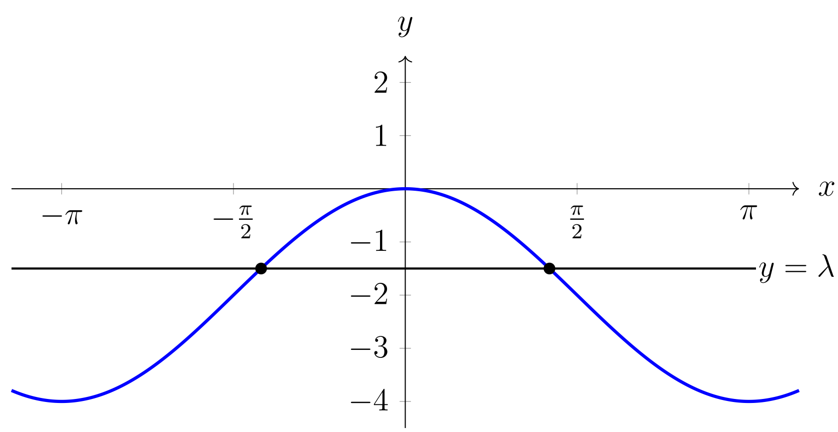
Proof. For any $\epsilon > 0$, if $x \in \{0, -\pi\}$, we choose I_x to be the interval centered at x such that

$$|I_x| = \min\left\{\frac{1}{100}, \epsilon\right\} \implies \min_{y\in\mathbb{T}\setminus(I_x\cup I_{-x})} \{|\varphi(y) - \varphi(x)|\} \gtrsim_\epsilon 1.$$

If $x \in B_{\frac{|I_0|}{4}}(0)$, we also define $I_x := B_{\frac{|I_0|}{2}}(0) (= I_0)$. Similarly, denote $I_x := B_{\frac{|I_0|}{2}}(-\pi) (= I_{-\pi})$ when $x \in B_{\frac{|I_0|}{4}}(-\pi)$. For any $x \in \mathbb{T} \setminus \left(\frac{1}{2}I_0 \cup \frac{1}{2}I_{-\pi}\right)$, noting that the function $\varphi'(x) = -2\sin x$ is nonzero on $\mathbb{T} \setminus \left(\frac{1}{4}I_0 \cup \frac{1}{4}I_{-\pi}\right)$, we set $\delta_x = \frac{1}{8}|I_0|$, $I_x := B_{\delta_x}(x)$, and then

$$\min_{y\in\mathbb{T}\setminus(I_x\cup I_{-x})} |\varphi(y) - \varphi(x)| \geq \frac{1}{8}|I_0| \cdot \min_{z\in\mathbb{T}\setminus\left(\frac{1}{4}I_0\cup\frac{1}{4}I_{-\pi}\right)} |\varphi'(z)| \gtrsim_\epsilon 1.$$

This collection of intervals satisfies the assumptions. ∎



$\varphi(x) = 2\cos x - 2$

Proof of Theorem 1.1 (i). Let $E \subset \mathbb{Z}$ be a γ-thick set with $\gamma \geq \frac{1}{2}$. Then by Proposition 2.1, there exists a sufficiently small $\epsilon > 0$ such that

$$d^-(E) > \frac{\pi + 2\epsilon}{2\pi}.$$

We choose a collection of intervals $\{I_x\}_{x\in\mathbb{T}}$ satisfies the assumptions in Lemma 2.1 with respect to ϵ above. By Theorem 2.3, for showing $E \subset \mathbb{Z}$ is observable at some time T, we just need to prove there are $M, m > 0$ such that for any $\lambda \in \mathbb{R}$, $f \in l^2(\mathbb{Z})$, the following inequality holds

$$\|f\|^2_{l^2(\mathbb{Z})} \leq M\|(\Delta_{\text{disc}} - \lambda) f\|^2_{l^2(\mathbb{Z})} + m\|\mathscr{C}f\|^2_{l^2(\mathbb{Z})}, \text{ where } \mathscr{C} = \mathbb{1}_E \cdot \mathrm{Id}_{l^2(\mathbb{Z})}. \tag{14}$$

When $\lambda \leq -4$, we define the projection $P_\lambda f := \left(\mathbb{1}_{I_{-\pi}}\widehat{f}\right)^\vee$. Then by Plancherel identity (3), we deduce

$$\|f\|^2_{l^2(\mathbb{Z})} = \|\widehat{f}\|^2_{L^2(\mathbb{T})} = \|\mathbb{1}_{I_{-\pi}}\widehat{f}\|^2_{L^2(\mathbb{T})} + \|\left(1 - \mathbb{1}_{I_{-\pi}}\right)\widehat{f}\|^2_{L^2(\mathbb{T})}. \tag{15}$$

Taking use of Theorem 2.2, we have

$$\|f\|_{l^2(\mathbb{Z})}^2 \leq C_{1,\epsilon}\|P_\lambda f\|_{l^2(E)}^2 + \|\left(1-P_\lambda\right)f\|_{l^2(\mathbb{Z})}^2, \tag{16}$$

where the implicit constant $C_{1,\epsilon}$ comes from the Theorem 2.2, which only depends on ϵ, $|I_{-\pi}|$ and E. It is directly to calculate that

$$\begin{aligned} C_{1,\epsilon}\|P_\lambda f\|_{l^2(E)}^2 + \|\left(1-P_\lambda\right)f\|_{l^2(\mathbb{Z})}^2 =& C_{1,\epsilon}\|f-\left(1-P_\lambda\right)f\|_{l^2(E)}^2 + \|\left(1-P_\lambda\right)f\|_{l^2(\mathbb{Z})}^2 \\ \leq& 4C_{1,\epsilon}\|f\|_{l^2(E)}^2 + \left(1+4C_{1,\epsilon}\right)\|\left(1-P_\lambda\right)f\|_{l^2(\mathbb{Z})}^2 \quad \text{(triangle inequality)} \\ =& 4C_{1,\epsilon}\|f\|_{l^2(E)}^2 + \left(1+4C_{1,\epsilon}\right)\|\left(1-\mathbb{1}_{I_{-\pi}}\right)\widehat{f}\|_{L^2(\mathbb{T})}^2. \end{aligned} \tag{17}$$

By using (13), we obtain

$$|\varphi-\lambda| \geq |\varphi+4| \geq M_\epsilon \text{ on } \mathbb{T}\setminus I_{-\pi}.$$

Hence

$$\|\left(1-\mathbb{1}_{I_{-\pi}}\right)\widehat{f}\|_{L^2(\mathbb{T})}^2 \leq \frac{1}{M_\epsilon^2}\|\left(\varphi-\lambda\right)\widehat{f}\|_{L^2(\mathbb{T})}^2 = \frac{1}{M_\epsilon^2}\|\left(\Delta_{\text{disc}}-\lambda\right)f\|_{l^2(\mathbb{Z})}^2. \tag{18}$$

Combining (16), (17) and (18), we derive

$$\|f\|_{l^2(\mathbb{Z})}^2 \leq 4C_{1,\epsilon}\|f\|_{l^2(E)}^2 + \frac{1+4C_{1,\epsilon}}{M_\epsilon^2}\|\left(\Delta_{\text{disc}}-\lambda\right)f\|_{l^2(\mathbb{Z})}^2 \text{ when } \lambda \leq -4. \tag{19}$$

The proof of the case when $\lambda \geq 0$ is similar to above argument where the only distinction is that we replace the interval $I_{-\pi}$ to I_0. Thus, we deduce

$$\|f\|_{l^2(\mathbb{Z})}^2 \leq 4C_{2,\epsilon}\|f\|_{l^2(E)}^2 + \frac{1+4C_{2,\epsilon}}{M_\epsilon^2}\|\left(\Delta_{\text{disc}}-\lambda\right)f\|_{l^2(\mathbb{Z})}^2 \text{ when } \lambda \geq 0. \tag{20}$$

Here the constant $C_{2,\epsilon}$ depends on ϵ, $|I_0|$ and E. For the remain case when $-4<\lambda<0$, the level set $\{\varphi=\lambda\}$ consists with two points $x_1, x_2 \in \mathbb{T}$ where $x_1=-x_2$. For these intervals I_{x_1}, I_{x_2}, we can cover them by a larger interval $\widetilde{I_{x_1,x_2}} \subset \mathbb{T}$ such that $\left|\widetilde{I_{x_1,x_2}}\right| = \pi+2\epsilon$, and

$$\min_{y\in\mathbb{T}\setminus\widetilde{I_{x_1,x_2}}} |\varphi(y)-\lambda| \geq M_\epsilon. \tag{21}$$

We define the projection $P_\lambda f := \left(\mathbb{1}_{\widetilde{I_{x_1,x_2}}}\widehat{f}\right)^\vee$. Using the same argument in inequality (19), we get

$$\|f\|_{l^2(\mathbb{Z})}^2 \leq 4C_{3,\epsilon}\|f\|_{l^2(E)}^2 + \frac{1+4C_{3,\epsilon}}{M_\epsilon^2}\|\left(\Delta_{\text{disc}}-\lambda\right)f\|_{l^2(\mathbb{Z})}^2, \tag{22}$$

where the new implicit constant $C_{3,\epsilon}$ only depends on ϵ and E. Concluding (19), (20) and (22), we deduce that

$$\|f\|_{l^2(\mathbb{Z})}^2 \leq 4\max\left\{C_{1,\epsilon}, C_{2,\epsilon}, C_{3,\epsilon}\right\}\cdot\|f\|_{l^2(E)}^2 + \frac{1+4\max\left\{C_{1,\epsilon}, C_{2,\epsilon}, C_{3,\epsilon}\right\}}{M_\epsilon^2}\|\left(\Delta_{\text{disc}}-\lambda\right)f\|_{l^2(\mathbb{Z})}^2,$$

for any $\lambda\in\mathbb{R}$, $f\in l^2(\mathbb{Z})$. This completes the observable resolvent estimate (14), thus we finish the proof. $\square$

Remark 2.2. *Although in the proof of Theorem 1.1 (i) we apply the pigeonhole principle to construct such $\widetilde{I_{x_1,x_2}} \subset \mathbb{T}$, this is sharp in the following sense. When we choose $\lambda=-2$, then $\{\varphi=\lambda\} = \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$, and if we hope to cover $I_{-\frac{\pi}{2}}, I_{\frac{\pi}{2}}$ with a large interval $\widetilde{I_{-\frac{\pi}{2},\frac{\pi}{2}}}$ on $\mathbb{T}$, its length would be greater than π.*

2.4. **Proof of Theorem 1.1 (ii).** Below we construct a subset $E \subset \mathbb{Z}$ with the Beurling density $\frac{q}{p}$ where $1 \leq q \leq p$ for some $p, q \in \mathbb{Z}_{\geq 1}$ firstly. Consider the following periodic Bohr set

$$E_{R,p} := \bigcup_{s \in R} (p\mathbb{Z} + s) \text{ where } R \subset \{0, 1, \cdots, p-1\} \text{ with } |R| = q. \tag{23}$$

It is directly to check that

$$d^{+}(E_{R,p}) = d^{-}(E_{R,p}) = \frac{q}{p}.$$

We now define operator

$$T_p^{(s)} : l^2(\mathbb{Z}) \to L^2(\mathbb{T}),\ T_p^{(s)} f(x) := \frac{1}{p} \sum_{r=0}^{p-1} e^{-is\left(x - \frac{2\pi r}{p}\right)} \widehat{f}\left(x - \frac{2\pi r}{p}\right), \tag{24}$$

where $s \in \{0, 1, \ldots, p-1\}$. Here, $T_p^{(s)}$ is the congruence-class extraction operator

$$\left(T_p^{(s)} f\right)^{\vee}(n) = \frac{f(n+s)}{p} \sum_{r=0}^{p-1} e^{-i\frac{2n\pi}{p} \cdot r} = f(n+s)\, \mathbb{1}_{p\mathbb{Z}}(n). \tag{25}$$

By (25) and the Plancherel identity (3), it isolates the l^2-energy of the subsequence indexed by $p\mathbb{Z} + s$: for any $f \in l^2(\mathbb{Z})$,

$$\|T_p^{(s)} f\|_{L^2(\mathbb{T})} = \|f\|_{l^2(p\mathbb{Z}+s)}. \tag{26}$$

The next proposition yields that there exists a γ-thick set with $\gamma \in \mathbb{Q} \cap (0, 1/2)$ that is not observable at any time $T > 0$.

Proposition 2.3. *Let $1 \leq q < p$ such that $\frac{q}{p} \leq \frac{1}{2}$. Then there is a periodic Bohr set defined as* (23) *with density $\frac{q}{p}$ is not observable at any time.*

Proof. If p is even, we choose the following subset $E_{R,p} \subset \mathbb{Z}$ where

$$R := \{2k : 0 \leq k \leq q-1\} \implies |R| = q.$$

If $E_{R,p}$ is observable at some time $T > 0$, that is, there exists some constant $C_T > 0$ such that for any initial data $u_0 \in l^2(\mathbb{Z})$,

$$\sum_{n \in \mathbb{Z}} |u_0(n)|^2 \leq C_T \int_0^T \sum_{n \in E_{R,p}} \left| e^{it\Delta_{\mathrm{disc}}} u_0(n) \right|^2 \mathrm{d}t. \tag{27}$$

Consider a initial data v_0 whose Fourier transform is given by

$$\widehat{v_0}(x) = \chi\Big(x - \frac{\pi}{2}\Big) - \chi\Big(x + \frac{\pi}{2}\Big) \qquad \text{on } \mathbb{T},$$

where χ is the 2π-periodization of a smooth, even bump function χ_0 supported in the ball $B_\delta(0)$ for a sufficiently small $\delta > 0$. Noting that $2\|\chi\|^2_{L^2(\mathbb{T})} = \|\widehat{v_0}\|^2_{L^2(\mathbb{T})} = \|v_0\|^2_{l^2(\mathbb{Z})}$, since $\mathrm{spt}\chi\left(\cdot - \frac{\pi}{2}\right) \cap \mathrm{spt}\chi\left(\cdot + \frac{\pi}{2}\right) = \varnothing$. Using (24), (26) and definitions of v_0, we have

$$\sum_{n \in p\mathbb{Z}+s} \left| e^{it\Delta_{\mathrm{disc}}} v_0(n) \right|^2 \mathrm{d}t = \|T_p^{(s)}\left(e^{it\Delta_{\mathrm{disc}}} v_0\right)\|^2_{L^2(\mathbb{T})}$$

$$= \frac{1}{p^2} \int_{\mathbb{T}} \left| \sum_{r=0}^{p-1} e^{-is\left(x - \frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x - \frac{2\pi r}{p}\right) - 1\right)} \left(\chi\left(x - \frac{2\pi r}{p} - \frac{\pi}{2}\right) - \chi\left(x - \frac{2\pi r}{p} + \frac{\pi}{2}\right)\right) \right|^2 \mathrm{d}x.$$

Noting that when $s\in R$ is even, $e^{is\pi}=1$, we obtain

$$
\begin{aligned}
&\sum_{r=0}^{p-1} e^{-is\left(x-\frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x-\frac{2\pi r}{p}\right)-1\right)} \left(\chi\left(x-\frac{2\pi r}{p}-\frac{\pi}{2}\right)-\chi\left(x-\frac{2\pi r}{p}+\frac{\pi}{2}\right)\right)\\
=&\sum_{r=0}^{\frac{p}{2}-1} e^{-is\left(x-\frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x-\frac{2\pi r}{p}\right)-1\right)} \chi\left(x-\frac{2\pi r}{p}-\frac{\pi}{2}\right)-\sum_{r=\frac{p}{2}}^{p-1} e^{-is\left(x-\frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x-\frac{2\pi r}{p}\right)-1\right)} \chi\left(x-\frac{2\pi r}{p}+\frac{\pi}{2}\right)\\
&+\sum_{r=\frac{p}{2}}^{p-1} e^{-is\left(x-\frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x-\frac{2\pi r}{p}\right)-1\right)} \chi\left(x-\frac{2\pi r}{p}-\frac{\pi}{2}\right)-\sum_{r=0}^{\frac{p}{2}-1} e^{-is\left(x-\frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x-\frac{2\pi r}{p}\right)-1\right)} \chi\left(x-\frac{2\pi r}{p}+\frac{\pi}{2}\right)\\
=&\sum_{r=0}^{p-1} e^{-is\left(x-\frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x-\frac{2\pi r}{p}\right)-1\right)} \left(1-e^{-4it\cos\left(x-\frac{2\pi r}{p}\right)}\right) \chi\left(x-\frac{2\pi r}{p}-\frac{\pi}{2}\right).
\end{aligned}
$$

Hence, it holds that

$$
\begin{aligned}
&\sum_{n\in p\mathbb{Z}+s} \left|e^{it\Delta_{\mathrm{disc}}} v_0(n)\right|^2 \mathrm{d}t\\
=&\frac{1}{p^2}\int_{\mathbb{T}} \left|\sum_{r=0}^{p-1} e^{-is\left(x-\frac{2\pi r}{p}\right)} e^{2it\left(\cos\left(x-\frac{2\pi r}{p}\right)-1\right)} \left(1-e^{-4it\cos\left(x-\frac{2\pi r}{p}\right)}\right) \chi\left(x-\frac{2\pi r}{p}-\frac{\pi}{2}\right)\right|^2 \mathrm{d}x\\
=&\frac{1}{p^2}\int_{\mathbb{T}} \sum_{r=0}^{p-1} \left|\left(1-e^{-4it\cos\left(x-\frac{2\pi r}{p}\right)}\right) \chi\left(x-\frac{2\pi r}{p}-\frac{\pi}{2}\right)\right|^2 \mathrm{d}x \qquad \left(\text{by setting } \delta\le\frac{\pi}{100p}\right)\\
\le&\frac{16t^2\delta^2}{p}\int_{\mathbb{T}} |\chi(x)|^2 \mathrm{d}x \qquad \left(\left|1-e^{-4it\cos\left(x+\frac{\pi}{2}\right)}\right|\le 4t|\sin x|\le 4t\delta \text{ when } x\in \operatorname{spt}\chi\cap\mathbb{T}\right)\\
=&\frac{16t^2\delta^2}{p}\|\chi\|_{L^2(\mathbb{T})}^2=\frac{8t^2\delta^2}{p}\|v_0\|_{l^2(\mathbb{Z})}^2.
\end{aligned}
$$

Therefore, we derive

$$
\sum_{n\in\mathbb{Z}} |v_0(n)|^2 \le C_T\sum_{s\in R}\int_0^T \sum_{n\in p\mathbb{Z}+s} \left|e^{it\Delta_{\mathrm{disc}}} v_0(n)\right|^2 \mathrm{d}t \le \frac{8qC_T\delta^2T^3}{3p}\sum_{n\in\mathbb{Z}} |v_0(n)|^2.
$$

Defining

$$
\delta:=\min\left\{\left(\frac{p}{8qC_TT^3}\right)^{\frac{1}{2}}, \frac{\pi}{100p}\right\},
$$

we have

$$
\sum_{n\in\mathbb{Z}} |v_0(n)|^2 \le \frac{1}{3}\sum_{n\in\mathbb{Z}} |v_0(n)|^2.
$$

This is a contradiction, which implies (27) does not hold.

If p is odd, Consider $2p$ (which is even), and take

$$
R\subset\{2k: 0\le k\le 2q-1\} \qquad \text{with } |R|=2q,
$$

Applying the same argument we complete the proof. □

The next theorem states the classical Weyl equidistribution law together with its dynamical formulation for irrational rotations of the circle. If $\alpha\notin\mathbb{Q}$, then the sequence $\{\{n\alpha\}\}_{n\in\mathbb{Z}}$ is equidistributed mod 1 [48]. Moreover, the rotation $R_\alpha : x\mapsto x+\alpha$ on $\mathbb{T}$ is uniquely ergodic with respect to the normalized Lebesgue measure, so for every $f \in \mathscr{C}(\mathbb{T})$ the Birkhoff averages $M^{-1}\sum_{k=0}^{M-1} f(R_\alpha^k x)$ converge uniformly in the initial point x to $\int_{\mathbb{T}} f(x)\,\mathrm{d}x/(2\pi)$ [45, Thm. 6.19]. Identifying $\mathbb{T}$ with $[-\pi,\pi)$ via $x \mapsto e^{ix}$ and taking $x = 2\pi\{s\alpha\}$ yields the uniformity in s stated below.

Theorem 2.4. *[Weyl's equidistribution law] Let $\alpha \in \mathbb{R}$ with $\alpha \notin \mathbb{Q}$, then for any $\gamma > 0$, the subset $\{\{n\alpha\} : n \in \mathbb{Z}\}$ is equidistributed. Moreover, for any $f \in C^0(\mathbb{T})$, we have*

$$\frac{1}{M}\sum_{n=s}^{s+M-1} f\left(2\pi\left\{n\alpha\right\}\right) \to \int_{\mathbb{T}} f\left(x\right)\frac{\mathrm{d}x}{2\pi} \qquad \text{uniformly in } s \in \mathbb{Z} \text{ as } M \to \infty.$$

For any fixed positive, irrational number $\alpha \in \mathbb{R}$ and irrational number $\gamma \in (0, 1/2)$, we consider the following subset $E_\gamma \subset 2\mathbb{Z}$,

$$E_\gamma := \left\{n \in 2\mathbb{Z} : \{n\alpha\} \in \left(\frac{1}{2}-\gamma, \frac{1}{2}+\gamma\right)\right\}. \tag{28}$$

Proposition 2.4. *E_γ defined as* (28) *has Beurling density γ and is not observable at any time.*

Proof. We prove E_γ has Beurling density γ firstly. Note that for any $x \in \mathbb{R}$ and $R \geq 100$, we have

$$|E_\gamma \cap [x-R, x+R]| = \sum_{y\in[x-R,x+R]\cap 2\mathbb{Z}} \mathbb{1}_{\left(\frac{1}{2}-\gamma,\frac{1}{2}+\gamma\right)}\left(\{y\alpha\}\right) = \sum_{y\in[x-R,x+R]\cap 2\mathbb{Z}} \mathbb{1}_{(\pi-2\pi\gamma,\pi+2\pi\gamma)}\left(2\pi\left\{y\alpha\right\}\right).$$

Denote that $s := \lceil\frac{x-R}{2}\rceil$ and $s+M-1 := \lfloor\frac{x+R}{2}\rfloor$, then we have

$$M = \lfloor\frac{x+R}{2}\rfloor - \lceil\frac{x-R}{2}\rceil + 1 = R + O\left(1\right).$$

So we derive

$$\sum_{y\in[x-R,x+R]\cap 2\mathbb{Z}} \mathbb{1}_{(\pi-2\pi\gamma,\pi+2\pi\gamma)}\left(2\pi\left\{y\alpha\right\}\right) = \sum_{n=s}^{s+M-1} \mathbb{1}_{(\pi-2\pi\gamma,\pi+2\pi\gamma)}\left(2\pi\left\{2n\alpha\right\}\right).$$

For any $\epsilon > 0$, choose nonnegative, continuous functions f_ϵ^+, f_ϵ^- on $\mathbb{T}$ such that

$$f_\epsilon^- \leq \mathbb{1}_{(\pi-2\pi\gamma,\pi+2\pi\gamma)} \leq f_\epsilon^+ \text{ on } \mathbb{T} \text{ and } \int_{\mathbb{T}}\left|f_\epsilon^+ - f_\epsilon^-\right| < \epsilon.$$

Applying the Weyl's Theorem, for any sufficiently large R, we have

$$\left|\frac{1}{M}\sum_{n=s}^{s+M-1} f_\epsilon^\pm\left(2\pi\left\{2n\alpha\right\}\right) - \int_{\mathbb{T}} f_\epsilon^\pm\left(x\right)\frac{\mathrm{d}x}{2\pi}\right| < \epsilon.$$

Hence

$$\begin{aligned}
&\left|\frac{|E_\gamma \cap [x-R,x+R]|}{2R} - \gamma\right| \\
=&\frac{1}{2}\left|\frac{1}{R}\sum_{n=s}^{s+M-1}\mathbb{1}_{(\pi-2\pi\gamma,\pi+2\pi\gamma)}\left(2\pi\left\{2n\alpha\right\}\right) - \int_{\mathbb{T}}\mathbb{1}_{(\pi-2\pi\gamma,\pi+2\pi\gamma)}\frac{\mathrm{d}x}{2\pi}\right| \\
\leq&\frac{1}{2}\left(\frac{1}{R}\sum_{n=s}^{s+M-1}\left(f_\epsilon^+ - f_\epsilon^-\right)\left(2\pi\left\{2n\alpha\right\}\right) + \int_{\mathbb{T}}\left|f_\epsilon^+ - f_\epsilon^-\right|\frac{\mathrm{d}x}{2\pi}\right) \leq C\epsilon,
\end{aligned}$$

for some constant $C>0$. Thus we deduce

$$\lim_{R\to\infty}\sup_{x\in\mathbb{R}}\frac{|E_\gamma\cap[x-R,x+R]|}{2R}=\lim_{R\to\infty}\inf_{x\in\mathbb{R}}\frac{|E_\gamma\cap[x-R,x+R]|}{2R}=\gamma,$$

that is, the Beurling density $d\left(E_\gamma\right)=\gamma$.

By setting $p=2$ and $R=\{0\}\subset\{0,1\}$ in Proposition 2.3, the Bohr set $E_{p,R}=2\mathbb{Z}$ is unobservable at any time $T>0$. It follows from a basic inequality

$$\int_0^T\sum_{n\in E_\gamma}\left|e^{it\Delta_{\mathrm{disc}}}u_0\left(n\right)\right|^2\mathrm{d}t\le\int_0^T\sum_{n\in 2\mathbb{Z}}\left|e^{it\Delta_{\mathrm{disc}}}u_0\left(n\right)\right|^2\mathrm{d}t$$

that $E_\gamma\subset 2\mathbb{Z}$ is unobservable at any time. This completes the proof. □

Proof of Theorem 1.1 (ii). Combining the constructions for rational density (Proposition 2.3) and irrational density (Proposition 2.4) with Proposition 2.1, we obtain the assertion of Theorem 1.1 (ii). □

2.5. **Observability for Bohr sets.** In this subsection, we provide a complete characterization for the observability of any periodic Bohr set (see (23)), that is,

Theorem 2.5 (Complete arithmetic criterion). *Let $p\in\mathbb{Z}_{\ge 1}$ and $R\subset\{0,\dots,p-1\}$ be not empty set. Then the following are equivalent:*

(a) *$E_{p,R}$ is observable at any time.*
(b) *The subgroup of $\mathbb{Z}/p\mathbb{Z}$ generated by $D(R):=\left\{\,s_1-s_0 \pmod{p}:\ s_0,s_1\in R\right\}\subset\mathbb{Z}/p\mathbb{Z}$ equals the whole $\mathbb{Z}/p\mathbb{Z}$.*

Denote the map $\mathcal{B}f\left(n\right):=\left(f(pn),f(pn+1),\dots,f(pn+p-1)\right)^\top\in\mathbb{C}^p$ for any $n\in\mathbb{Z}$, then the Fourier transform $\widehat{\mathcal{B}f}:\mathbb{T}\to\mathbb{C}^p$ can be given by

$$\widehat{\mathcal{B}f}(x):=\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\mathcal{B}f(n)e^{inx}\text{ for }x\in\mathbb{T}.$$

By the Plancherel formula (3), we have

$$\|f\|_{l^2(\mathbb{Z})}=\|\mathcal{B}f\|_{l^2(\mathbb{Z})}=\|\widehat{\mathcal{B}f}\|_{L^2(\mathbb{T})}.$$

For each $p\ge 2$, for the solution $u(n,t)=e^{it\Delta_{\mathrm{disc}}}u_0(n)$, it satisfies that

$$\partial_t\widehat{\mathcal{B}u}(x,t)=i\Phi(x)\widehat{\mathcal{B}u}(x,t),\ \widehat{\mathcal{B}u}(x,0)=\widehat{\mathcal{B}u_0}(x)\text{ for }x\in\mathbb{T}.$$

Here the matrix-valued function $\Phi:\mathbb{T}\to\mathbb{C}^{p\times p}$ is defined by

$$\Phi(x):=2I_p-\begin{pmatrix}0&1&0&\cdots&0&e^{ix}\\1&0&1&\cdots&0&0\\0&1&0&\ddots&\vdots&\vdots\\\vdots&\ddots&\ddots&\ddots&1&0\\0&\cdots&\cdots&1&0&1\\e^{-ix}&0&0&\cdots&1&0\end{pmatrix}.$$

Note for $p=2$,

$$\Phi(x)=\begin{pmatrix}2&-1-e^{ix}\\-1-e^{-ix}&2\end{pmatrix}.$$

Through direct computation, we derive the following lemma, characterizing the eigenvalues and eigenvectors of matrix Φ.

Lemma 2.2. *Let Φ be defined as above. For $x \in \mathbb{T}$ and $s = 0, 1, \dots, p-1$, set*

$$\eta_s(x) := \frac{-x + 2\pi s}{p}, \qquad U_s(x) := \left(e^{ir\,\eta_s(x)}\right)_{r=0}^{p-1} \in \mathbb{C}^p, \qquad \mu_s(x) := 2 - 2\cos\left(\eta_s(x)\right).$$

Then $\{U_s(x)\}_{s=0}^{p-1}$ is an orthogonal basis of $\mathbb{C}^p$ and

$$\Phi(x)\, U_s(x) = \mu_s(x)\, U_s(x), \qquad s = 0, \dots, p-1.$$

Moreover, all eigenvalues are simple when $x \notin \{-\pi, 0\}$. For $x \in \{-\pi, 0\}$, each $\kappa \in (0, \pi)$ yields a two-dimensional eigenspace $\operatorname{span}\{u_\kappa, u_{-\kappa}\}$*, while $\kappa \in \{-\pi, 0\}$ remains one-dimensional.*

For any periodic Bohr set $E_{R,p}$, we define the restriction operator related to $E_{R,p}$

$$\mathcal{R} : \mathbb{C}^p \to \mathbb{C}^p,\ \mathcal{R}(x_1, x_2, \dots, x_p) := (\mathbb{1}_R(0) \cdot x_1, \mathbb{1}_R(1) \cdot x_2, \dots, \mathbb{1}_R(p-1) \cdot x_p).$$

Using this operator, one has

$$\sum_{n \in E_{R,p}} |f(n)|^2 = \sum_{n \in \mathbb{Z}} |\mathcal{R}\mathcal{B}f(n)|^2 = \int_{\mathbb{T}} \left|\mathcal{R}\widehat{\mathcal{B}f}(x)\right|^2 \mathrm{d}x,$$

and then

$$\begin{aligned}
\int_0^T \sum_{n \in E_{R,p}} \left|e^{it\Delta_{\mathrm{disc}}} u_0(n)\right|^2 \mathrm{d}t &= \int_0^T \int_{\mathbb{T}} \left|\mathcal{R}\left(e^{it\Phi(\cdot)}\widehat{\mathcal{B}u_0}\right)\right|^2 \mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{T}} \int_0^T \left(\widehat{\mathcal{B}u_0}(x)\right)^H \left((\mathcal{R}e^{it\Phi(x)})^H (\mathcal{R}e^{it\Phi(x)})\right) \widehat{\mathcal{B}u_0}(x)\, \mathrm{d}t\mathrm{d}x \\
&= \int_{\mathbb{T}} \left(\widehat{\mathcal{B}u_0}(x)\right)^H W_T(x) \widehat{\mathcal{B}u_0}(x) \mathrm{d}x,
\end{aligned}$$

Here and below $(\cdot)^H$ denotes the conjugate transpose and we define $W_T(x) \in \mathbb{C}^{p \times p}$ by

$$W_T(x) := \int_0^T e^{-it\Phi(x)} \mathcal{R}^H \mathcal{R} e^{it\Phi(x)}\, \mathrm{d}t. \tag{29}$$

The following proposition is a standard result in linear algebra; a similar discussion will arise in the proof of Theorem1.3.

Proposition 2.5. *Let Φ and $\mathcal{R}$ defined as above. For $T > 0$, the following are equivalent:*

(i) *For every eigenvector u of Φ, $\mathcal{R}u \neq 0$.*
(ii) *W_T defined in 29 is positive definite.*
(iii) *There exists $C(T) > 0$ such that $\int_0^T \left|\mathcal{R}e^{-it\Phi}z\right|^2 \mathrm{d}t \geq C(T)\,|z|^2$ for all $z \in \mathbb{C}^p$.*

Proof. Diagonalize $\Phi = \sum_\lambda \lambda P_\lambda$ with orthogonal spectral projectors P_λ. Then

$$W_T = \sum_{\lambda,\mu} \left(\int_0^T e^{-it(\lambda-\mu)}\, dt\right) P_\lambda \mathcal{R}^H \mathcal{R} P_\mu.$$

For $\lambda \neq \mu$, the oscillatory integral is bounded, while for $\lambda = \mu$ it equals T. Moreover, restricted to each eigenspace $E_\lambda = P_\lambda \mathbb{C}^p$ we have $P_\mu z = 0$ for $\mu \neq \lambda$ and all $z \in E_\lambda$, hence the cross terms vanish on E_λ and

$$W_T|_{E_\lambda} = T\,(\mathcal{R}|_{E_\lambda})^H (\mathcal{R}|_{E_\lambda}).$$

Thus $W_T > 0$ if and only if $\mathcal{R}$ is injective on every E_λ, i.e. (i). The equivalence with (iii) follows from the Rayleigh quotient characterization of the smallest eigenvalue of W_T. □

Now we begin to prove Theorem 2.5.

Proof of Theorem 2.5. By Proposition 2.5, (a) holds if and only if for every $x \in [-\pi, \pi)$, the pair $(\Phi(x), \mathcal{R})$ satisfies that every eigenvector u of $\Phi(x)$ obeys $\mathcal{R}u \neq 0$.

We consider nondegenerate fibers $x \notin \{0, -\pi\}$ firstly. By Lemma 2.2, each eigenspace is one-dimensional and is spanned by a vector $U_s(x)$ whose entries are all nonzero. $\mathcal{R}U_s(x) \neq 0$ since $R \neq \emptyset$, so every eigenvector u of $\Phi(x)$ obeys $\mathcal{R}u \neq 0$.

Now we discuss degenerate fibers $x \in \{0, -\pi\}$. For each such x and $\kappa \in (0, \pi)$, the eigenspace is $E_\kappa = \operatorname{span}\{U_{+\kappa}, U_{-\kappa}\}$ where $U_{\pm\kappa} = (e^{\pm ir\kappa})_{r=0}^{p-1}$. An element $z = a_+ U_{+\kappa} + a_- U_{-\kappa}$ lies in $\ker \mathcal{R}$ if and only if

$$a_+ \, e^{ir\kappa} + a_- \, e^{-ir\kappa} = 0, \qquad \forall\, r \in R.$$

Thus $\mathcal{R}|_{E_\kappa}$ is injective if and only if there exist $r_0, r_1 \in R$ such that

$$\det \begin{pmatrix} e^{ir_0\kappa} & e^{-ir_0\kappa} \\ e^{ir_1\kappa} & e^{-ir_1\kappa} \end{pmatrix} = -2i \, \sin\big((r_1 - r_0)\kappa\big) \neq 0,$$

equivalently, there exists $d \in D(R)$ with $\sin(d\kappa) \neq 0$.

For $x = 0$, the two-dimensional eigenspaces correspond to

$$\kappa = \frac{m\pi}{p}, \qquad m \in \{2, 4, \dots\} \cap [1, p-1],$$

and for $x = -\pi$, they correspond to

$$\kappa = \frac{m\pi}{p}, \qquad m \in \{1, 3, \dots\} \cap [1, p-1].$$

For such κ,

$$\sin(d\kappa) = 0 \iff d\,\kappa \in \pi\mathbb{Z} \iff p \mid m\,d.$$

Therefore, the injectivity of $\mathcal{R}$ on every degenerate eigenspace on both fibers is equivalent to the requirement

$$(*) \qquad \forall\, m \in \{1, \dots, p-1\},\ \exists\, d \in D(R) \text{ such that } p \nmid m\,d.$$

Let $G = \langle D(R) \rangle \leq \mathbb{Z}/p\mathbb{Z}$ be the subgroup generated by $D(R)$. For each $m \in \{1, \dots, p-1\}$, set $\operatorname{Ker}(m) := \{x \in \mathbb{Z}/p\mathbb{Z} : \ m\,x = 0\}$. Then $(*)$ is equivalent to

$$G \not\subseteq \operatorname{Ker}(m) \quad \text{for every } m = 1, \dots, p-1.$$

Since $\mathbb{Z}/p\mathbb{Z}$ is cyclic, there exists $g \in \mathbb{Z}_{\geq 1}$ with $g \mid p$ such that $G = \{t\,g \pmod p) : \ t \in \mathbb{Z}\}$, where

$$g = \gcd\big(p, \{r_1 - r_0 : \ r_0, r_1 \in R\}\big).$$

Moreover, $\operatorname{Ker}(m) = \{t\,(p/\gcd(p,m)) \pmod p : \ t \in \mathbb{Z}\}$. In a cyclic group, $H_a \subset H_b$ if and only if $b \mid a$. Hence

$$G \subset \operatorname{Ker}(m) \quad \iff \quad \frac{p}{\gcd(p,m)} \ \mid \ g.$$

If $g = 1$, then for every m one has $\frac{p}{\gcd(p,m)} \geq 2$ and thus does not divide g, so $G \not\subseteq \operatorname{Ker}(m)$ and $(*)$ holds. Conversely, if $g > 1$, choose $m = p/g \in \{1, \dots, p-1\}$. Then $\frac{p}{\gcd(p,m)} = g \mid g$, so $G \subset \operatorname{Ker}(m)$, violating $(*)$. Thus (a)$\Leftrightarrow$(b) follows by checking, on each fiber, that $\mathcal{R}$ is injective on the eigenspaces of $\Phi(x)$.

Finally, when (b) holds, $W_T(x)$ is positive definite for every $x \in \mathbb{T}$ and $T > 0$ by Proposition 2.5. The map $x \mapsto W_T(x)$ is continuous on the compact set $\mathbb{T}$, hence its smallest eigenvalue

$$\lambda_{\min}(x) := \min_{|z|=1} \langle W_T(x)z, z \rangle$$

is continuous and strictly positive on $\mathbb{T}$. Therefore $m_T := \min_{x\in\mathbb{T}} \lambda_{\min}(x) > 0$, and

$$\int_0^T \sum_{n\in E_{R,p}} |u(n,t)|^2 \, dt = \int_{\mathbb{T}} \left\langle W_T(x)\widehat{\mathcal{B}u_0}(x), \widehat{\mathcal{B}u_0}(x) \right\rangle \, dx \geq m_T \, \|\widehat{\mathcal{B}u_0}\|^2_{L^2(\mathbb{T})} = m_T \, \|u_0\|^2_{\ell^2(\mathbb{Z})}.$$

Taking $C(T) = 1/m_T$ gives (a). □

By Theorem 2.5, an intriguing example is the following:

Example 2.1. *The set of integers congruent to* 0 *or* 1 *modulo* 4 *is observable at any time.*

This example has the same thickness as $2\mathbb{Z}$, yet its observability differs. The proof of the following corollary parallels that of Theorem 1.1(ii).

Corollary 2.1. *For any* $q \in (0,1)$ *there exists a set* $E \subset \mathbb{Z}$ *with thickness* q *that is observable at any time.*

Proof. By Proposition 2.2, it suffices to construct a set $E \subset \mathbb{Z}$ that is observable at any time and has arbitrary Beurling density $q \in (0,1)$.

Choose an integer $p \geq \lceil 2/q \rceil$, and set $m := \lfloor qp \rfloor$. Then $2 \leq m \leq p-1$. Let $R := \{0, 1, \ldots, m-1\} \subset \{0, \ldots, p-1\}$. Since $1 \in D(R)$, the subgroup generated by $D(R)$ is the whole $\mathbb{Z}/p\mathbb{Z}$. By Theorem 2.5, $E_{p,R}$ is observable at any time with Beurling density $r = \frac{m}{p} \leq q$.

Define the deficit $\delta := q - r \in [0, 1-r)$ and the proportion

$$\theta := \frac{\delta}{1-r} \in [0,1).$$

Let $C := \{0, \ldots, p-1\} \setminus R$ be the set of residues not used by $E_{p,R}$. Fix an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and an interval $J \subset \mathbb{T}$ of length $|J| = \theta$. For each $c \in C$ define

$$F_c := \left\{ n \in \mathbb{Z} : \ n \equiv c \ (\mathrm{mod}\ p) \ \text{ and } \ \{(n-c)\alpha/p\} \in J \right\},$$

and let

$$F := \bigcup_{c\in C} F_c.$$

By construction $F \cap E_{p,R} = \varnothing$. Similar to the proof of Proposition 2.4, by using Weyl's equidistribution law (Theorem 2.4), we derive F_c has Beurling density θ/p, for each $c \in \{0, \ldots, p-1\}$. Set $E := E_{p,R} \cup F$, then E has Beurling density q. The fact E is observable at any time follows from a basic inequality

$$\int_0^T \sum_{n\in E} \left| e^{it\Delta_{\mathrm{disc}}} u_0\,(n) \right|^2 \mathrm{d}t \geq \int_0^T \sum_{n\in E_{p,R}} \left| e^{it\Delta_{\mathrm{disc}}} u_0\,(n) \right|^2 \mathrm{d}t.$$

This completes the proof. □

2.6. **Observability for complements of finite sets.** We begin the proof of Theorem 1.2 with a basic spectral observation.

Fact 2.1. *The discrete Laplacian* Δ_{disc} *does not admit any nonzero eigenfunction in* $l^2(\mathbb{Z}^d)$.

Indeed, if not, let $f \in l^2\left(\mathbb{Z}^d\right) \setminus \{0\}$ satisfies $\Delta_{\mathrm{disc}} f = \lambda f$. Applying the Fourier transform, the eigenfunction f satisfies

$$\varphi\,(x)\,\widehat{f}\,(x) = \lambda \widehat{f}\,(x)\,,$$

where the Fourier multiplier φ we have defined in (4). This implies that

$$\mathrm{spt}\widehat{f} \subset \left\{ x \in \mathbb{T}^d : \varphi\,(x) = \lambda \right\}.$$

Then $\widehat{f}=0$ *a.e.* on $\mathbb{T}^d$, and by the Plancherel identity (3) we derive $\|\widehat{f}\|_{L^2(\mathbb{T}^d)}=\|f\|_{l^2(\mathbb{Z}^d)}=0$, which contradicts to the assumption.

Proof of Theorem 1.2. Let E be a subset of $\mathbb{Z}^d$ satisfying the assumption in Theorem 1.2. By the conservation law (5) of the free Schrödinger equation, we have

$$\int_0^T\sum_{n\in E}\left|e^{it\Delta_{\text{disc}}}u_0(n)\right|^2\mathrm{d}t=T\sum_{n\in\mathbb{Z}^d}|u_0(n)|^2-\int_0^T\sum_{n\in E^c}\left|e^{it\Delta_{\text{disc}}}u_0(n)\right|^2\mathrm{d}t,$$

for any time $T>0$. We define the operator

$$U:l^2\left(\mathbb{Z}^d\right)\to L^2\left((0,T),\mathbb{C}^{|E^c|}\right),\quad Uf(t):=\left\{e^{it\Delta_{\text{disc}}}f(n)\right\}_{n\in E^c}.$$

U is well-defined via the trivial estimate

$$\|Uf(t)\|_{\mathbb{C}^{|E^c|}}\leq\|f\|_{l^2(\mathbb{Z}^d)}. \tag{30}$$

It is straightforward to calculate that

$$\int_0^T\sum_{n\in E^c}\left|e^{it\Delta_{\text{disc}}}u_0(n)\right|^2\mathrm{d}t=\|Uu_0\|^2_{L^2\left((0,T),\mathbb{C}^{|E^c|}\right)}=\|\mathbb{1}_{E^c}e^{it\Delta_{\text{disc}}}u_0\|^2_{L^2\left((0,T),l^2\left(\mathbb{Z}^d\right)\right)}$$

$$=\langle\int_0^T e^{-it\Delta_{\text{disc}}}\left(\mathbb{1}_{E^c}e^{it\Delta_{\text{disc}}}\right)u_0\mathrm{d}t,u_0\rangle_{l^2(\mathbb{Z}^d)}\leq\|\int_0^T e^{-it\Delta_{\text{disc}}}\left(\mathbb{1}_{E^c}e^{it\Delta_{\text{disc}}}\right)\mathrm{d}t\|_{l^2(\mathbb{Z}^d)\to l^2(\mathbb{Z}^d)}\cdot\|u_0\|^2_{l^2(\mathbb{Z}^d)}.$$

Indeed, we apply the TT^* method in the last step. Below, we show that

$$\|S_{E^c}(T)\|_{l^2(\mathbb{Z}^d)\to l^2(\mathbb{Z}^d)}<T\text{ where }S_{E^c}(T):=\int_0^T e^{-it\Delta_{\text{disc}}}\left(\mathbb{1}_{E^c}e^{it\Delta_{\text{disc}}}\right)\mathrm{d}t.$$

First, $\|S_{E^c}(T)\|_{l^2(\mathbb{Z}^d)\to l^2(\mathbb{Z}^d)}\leq T$ holds since the duality argument

$$\begin{aligned}\|S_{E^c}(T)\|_{l^2(\mathbb{Z}^d)\to l^2(\mathbb{Z}^d)}&=\sup_{\|u_0\|_{l^2(\mathbb{Z}^d)},\|v_0\|_{l^2(\mathbb{Z}^d)}\leq1}\left|\langle\int_0^T e^{-it\Delta_{\text{disc}}}\left(\mathbb{1}_{E^c}e^{it\Delta_{\text{disc}}}\right)u_0\mathrm{d}t,v_0\rangle_{l^2(\mathbb{Z}^d)}\right|\\&\leq\sup_{\|u_0\|_{l^2(\mathbb{Z}^d)},\|v_0\|_{l^2(\mathbb{Z}^d)}\leq1}\int_0^T\|e^{it\Delta_{\text{disc}}}u_0\|_{l^2(E^c)}\|e^{it\Delta_{\text{disc}}}v_0\|_{l^2(E^c)}\mathrm{d}t\leq T.\end{aligned}$$

Second, $\|S_{E^c}(T)\|_{l^2(\mathbb{Z}^d)\to l^2(\mathbb{Z}^d)}<T$ is proved by contradiction, exploiting specific properties of $S_{E^c}(T)$. We define the kernel function $K_{E^c}(T):\mathbb{T}^d\times\mathbb{T}^d\to\mathbb{C}$,

$$K_{E^c}(T)(x,y):=\sum_{n\in E^c}\frac{e^{in\cdot(x-y)}}{(2\pi)^d}\int_0^T e^{it(\varphi(x)-\varphi(y))}\mathrm{d}t.$$

Using a simple observation

$$\int_0^T e^{it(\varphi(x)-\varphi(y))}\mathrm{d}t=\begin{cases}\frac{e^{iT(\varphi(x)-\varphi(y))}-1}{i(\varphi(x)-\varphi(y))}, & \text{when }\varphi(x)\neq\varphi(x),\\ T, & \text{when }\varphi(x)=\varphi(y),\end{cases}$$

we have

$$|K_{E^c}(T)(x,y)|\leq\frac{|E^c|}{(2\pi)^d}\cdot T\implies\|K_{E^c}(T)\|_{L^2(\mathbb{T}^d\times\mathbb{T}^d)}\leq\frac{|E^c|}{(2\pi)^{\frac{d}{2}}}\cdot T,$$

that is, K_{E^c} belongs to $L^2\left(\mathbb{T}^d\times\mathbb{T}^d\right)$. Hence $S_{E^c}(T)$ is a Hilbert-Schmidt operator, which is a compact operator directly. Moreover, being positive and self-adjoint, $S_{E^c}(T)$ attains its norm as an eigenvalue. If $\|S_{E^c}(T)\|_{l^2(\mathbb{Z}^d)\to l^2(\mathbb{Z}^d)} = T$, then there exists a non-zero normalized eigenfunction $u \in l^2(\mathbb{Z}^d)$ such that

$$S_{E^c}(T)\, u = Tu.$$

By the definition of $S_{E^c}(T)$,

$$T = \langle S_{E^c}(T)\, u, u\rangle_{l^2(\mathbb{Z}^d)} = \int_0^T \|Uf(t)\|^2_{\mathbb{C}^{|E^c|}} \mathrm{d}t,$$

combining inequality (30), we have

$$\|Uu(t)\|_{\mathbb{C}^{|E^c|}} = 1 \qquad \text{a.e. on } (0,T).$$

The continuity of $\|Uu(t)\|_{\mathbb{C}^{|E^c|}}$ gives that $\|Uu(t)\|_{\mathbb{C}^{|E^c|}} \equiv 1$ on $(0,T)$. Hence, $e^{it\Delta_{\mathrm{disc}}}u \equiv 0$ on E for $t \in (0,T)$, that is, $e^{it\Delta_{\mathrm{disc}}}u \in \mathrm{Ran}U$ for $t \in (0,T)$. Since $G(t) := (\mathbb{1}_E\, e^{it\Delta_{\mathrm{disc}}})u$ is entire-analytic and vanishes on $(0,T)$, uniqueness of analytic property forces $e^{it\Delta_{\mathrm{disc}}}u \equiv 0$ on E for $t \in \mathbb{R}$. Let $W := \mathrm{span}\{e^{it\Delta_{\mathrm{disc}}}u :\ t \in \mathbb{R}\}$. Then W is a finite-dimensional subspace of $l^2(\mathbb{Z}^d)$ (since $\mathrm{Ran}(\mathbf{1}_{E^c})$ is finite-dimensional), invariant under the unitary group $\{e^{is\Delta_{\mathrm{disc}}}\}_{s\in\mathbb{R}}$. By spectral theorem in finite dimension, the generator $\Delta_{\mathrm{disc}}|_W$ has pure point spectrum and u is a finite linear combination of l^2-eigenvectors of Δ_{disc}. This contradicts the fact that the discrete Laplacian on $\mathbb{Z}^d$ has no nonzero l^2-eigenfunctions (Fact 2.1). Hence $\|S_{E^c}(T)\| < T$.

Therefore, we derive

$$\int_0^T \sum_{n\in E} \left|e^{it\Delta_{\mathrm{disc}}}u_0(n)\right|^2 dt \ \geq\ \left(T - \|S_{E^c}(T)\|\right) \|u_0\|^2_{l^2(\mathbb{Z}^d)},$$

which is the desired observability inequality at any time $T > 0$, with observability constant $C_{\mathrm{obs}}(T,E) = \left(T - \|S_{E^c}(T)\|\right)^{-1}$. This completes the proof. $\square$

3. Observable sets for finite graph

3.1. An equivalent criterion for observability on finite graphs. In this subsection, we study the free Schrödinger equation (1) on a finite graph $G = (\mathcal{V}, \mathcal{E})$. In this setting, the discrete Laplacian Δ_{disc} is a finite-dimensional self-adjoint operator, and its spectral decomposition is standard.

Proof of Theorem 1.3. (i) $\Rightarrow$ (ii) Let $\{\psi_k\}_{k=1}^n$ be an orthonormal eigenbasis of Δ_{disc} in $l^2(\mathcal{V})$ with real eigenvalues $\{\lambda_k\}_{k=1}^n$, where $n = |\mathcal{V}|$. Expand the initial datum as

$$u_0 = \sum_{k=1}^n c_k\psi_k, \qquad \boldsymbol{c} = (c_1, \ldots, c_n)^\top, \qquad \|u_0\|^2_{l^2(\mathcal{V})} = \boldsymbol{c}^H\boldsymbol{c}.$$

The solution of Schrödinger equation (1) is

$$u(t) = \sum_{k=1}^n c_k e^{it\lambda_k}\psi_k.$$

We introduce the Gramian quadratic form as follows

$$\int_0^T \|u(t)\|^2_{l^2(E)}\,\mathrm{d}t \ =\ \boldsymbol{c}^H G\boldsymbol{c}, \qquad G_{jk} \ =\ \left\langle \psi_k|_E, \psi_j|_E\right\rangle_{l^2(E)} \int_0^T e^{it(\lambda_k-\lambda_j)}\,\mathrm{d}t.$$

We claim that G is positive-definite. Suppose $\boldsymbol{c}^H G\boldsymbol{c} = 0$. Then

$$\int_0^T \|u(t)\|^2_{l^2(E)}\,\mathrm{d}t = 0.$$

Since $t \mapsto \|u(t)\|^2_{l^2(E)}$ is nonnegative and continuous, it follows that $\|u(t)\|_{l^2(E)} = 0$ for all $t \in [0, T]$, i.e.

$$\sum_{k=1}^{n} c_k e^{it\lambda_k} \psi_k(x) \;=\; 0 \quad \text{for all } x \in E,\ t \in [0, T].$$

Let $\{\Lambda_m\}$ be the distinct eigenvalues and group terms accordingly:

$$\sum_m e^{it\Lambda_m}\, w_m(x) \;=\; 0, \qquad w_m \;=\; \sum_{k:\lambda_k = \Lambda_m} c_k \psi_k, \quad x \in E.$$

The exponentials $\{e^{it\Lambda_m}\}_m$ with distinct frequencies are linearly independent on any interval of positive length. Hence $w_m(x) = 0$ for all $x \in E$ and all m, i.e. $w_m|_E \equiv 0$. Each w_m lies in the eigenspace of Λ_m and thus is a Laplace eigenfunction; by assumption (ii), no nonzero eigenfunction can vanish on E. Therefore $w_m = 0$ for all m, and consequently $\boldsymbol{c} = \boldsymbol{0}$.

This shows G is positive definite. Let $\mu_{\min} > 0$ be its smallest eigenvalue. Then

$$\int_0^T \|u(t)\|^2_{l^2(E)}\, \mathrm{d}t \;=\; \boldsymbol{c}^H G \boldsymbol{c} \;\geq\; \mu_{\min}\, \boldsymbol{c}^H \boldsymbol{c} \;=\; \mu_{\min}\, \|u_0\|^2_{l^2(\mathcal{V})}.$$

Setting $C_E(T) = 1/\mu_{\min}$ yields (2). Since the linear independence of exponentials holds on any interval of positive length, the same argument gives (i) for every $T > 0$.

(ii) $\Rightarrow$ *(i)* We argue by contradiction. Suppose there exists a nonzero eigenvector ψ_0 with eigenvalue λ_0 such that $\psi_0|_E \equiv 0$. Taking $u_0 = \psi_0$ yields $u(t) = e^{it\lambda_0}\psi_0$, hence

$$\|u(t)\|^2_{l^2(E)} \;=\; \sum_{x \in E} \left|e^{it\lambda_0}\psi_0(x)\right|^2 \;=\; \sum_{x \in E} |\psi_0(x)|^2 \;=\; 0 \quad \text{for all } t \geq 0.$$

Thus the right-hand side of (2) is zero, whereas the left-hand side equals $\|\psi_0\|^2_{l^2(\mathcal{V})} > 0$. No positive constant $C_E(T)$ can satisfy (2), so E is not observable. This completes the proof. □

Remark 3.1 (Equivalence for observability at some time and any time)**.** *Combining the two implications shows that (i) holding for some $T > 0$ implies (ii), which in turn implies (i) for every $T > 0$. Thus observability at some time $T > 0$ is equivalent to observability for any time $T > 0$.*

3.2. **A positive density construction for unobservable sets on discrete tori.** We specialize to the discrete d-dimensional tori $(\mathbb{Z}/N\mathbb{Z})^d$ with nearest-neighbor graph structure: $x, y \in (\mathbb{Z}/N\mathbb{Z})^d$ are adjacent iff $y = x \pm e_j$ for some standard basis vector e_j, with arithmetic taken modulo N.

The combinatorial Laplacian Δ_{disc} acts on $f : (\mathbb{Z}/N\mathbb{Z})^d \to \mathbb{C}$ by

$$(\Delta_{\text{disc}} f)(x) \;=\; \sum_{j=1}^{d} \big(f(x + e_j) + f(x - e_j)\big) - 2d\, f(x).$$

For $k := (k_1, \ldots, k_d) \in (\mathbb{Z}/N\mathbb{Z})^d$, define the character

$$\varphi_k(x) \;=\; \exp\!\Big(\frac{2\pi i}{N}\, \langle k, x\rangle\Big), \qquad x \in (\mathbb{Z}/N\mathbb{Z})^d,$$

where $\langle k, x\rangle = k_1 x_1 + \cdots + k_d x_d$ (interpreted modulo N). A direct computation gives that

$$\Delta_{\text{disc}} \varphi_k = \left(2 \sum_{j=1}^{d} \cos\!\Big(\frac{2\pi k_j}{N}\Big) - 2d\right) \varphi_k := \mu(k)\varphi_k.$$

The imaginary parts $\psi_k^{\sin}(x) = \sin(\frac{2\pi}{N}\, \langle k, x\rangle)$ are real-valued eigenfunctions with eigenvalue $\mu(k)$. By the standard structure theory for finite abelian groups, the characters $\{\varphi_k\}_{k \in (\mathbb{Z}/N\mathbb{Z})^d}$ form an orthogonal basis of the Hilbert space $L^2\big((\mathbb{Z}/N\mathbb{Z})^d\big)$.

Fix $k \in (\mathbb{Z}/N\mathbb{Z})^d$. Consider the group homomorphism

$$F_k : (\mathbb{Z}/N\mathbb{Z})^d \longrightarrow \mathbb{Z}/N\mathbb{Z}, \qquad F_k(x) = \langle k, x\rangle \pmod N,$$

and write

$$d_0 = \gcd(k_1, \dots, k_d, N) \in \mathbb{Z}_{\geq 1}.$$

On the additive group $\mathbb{Z}/N\mathbb{Z}$, we introduce

$$S_{\sin} \;:=\; \left\{ r \in \mathbb{Z}/N\mathbb{Z} : \; \sin\!\left(\frac{2\pi r}{N}\right) = 0 \right\} \;=\; \begin{cases} \{ 0 \}, & 2 \nmid N, \\ \{ 0, \; N/2 \}, & 2 \mid N, \end{cases}$$

Here $N/2$ denotes the unique residue in $\mathbb{Z}/N\mathbb{Z}$ satisfying $2 \cdot (N/2) \equiv 0$.

We have the following Proposition for counting the size of zero set $Z(\psi_k^{\sin})$.

Proposition 3.1 (Zero set size via group homomorphisms)**.** *With the notation above, we have*

$$Z\big(\psi_k^{\sin}\big) \;=\; F_k^{-1}\big(S_{\sin} \cap \operatorname{Im} F_k\big), \qquad \big|Z(\psi_k^{\sin})\big| \;=\; \big|S_{\sin} \cap \operatorname{Im} F_k\big| \cdot d_0\, N^{d-1},$$

Proof. Consider the subgroup $H \subset \mathbb{Z}/N\mathbb{Z}$ generated by the residues $[k_1], \dots, [k_d]$. The preimage of H under the natural projection $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ is the ideal $\langle k_1, \dots, k_d, N\rangle = d_0\mathbb{Z}$, where $d_0 = \gcd(k_1, \dots, k_d, N)$. Hence $H = \langle [d_0]\rangle$ and therefore $|\operatorname{Im} F_k| = |H| = N/d_0$. Since F_k is a homomorphism between finite groups, all fibers over residues in $\operatorname{Im} F_k$ have the same cardinality; explicitly,

$$\big|F_k^{-1}(r)\big| \;=\; |\ker F_k| \;=\; \frac{|(\mathbb{Z}/N\mathbb{Z})^d|}{|\operatorname{Im} F_k|} \;=\; d_0\, N^{d-1} \qquad \text{for every } r \in \operatorname{Im} F_k.$$

Next, by the elementary characterizations $\sin\theta = 0$ iff $\theta \in \pi\mathbb{Z}$, a residue $r \in \mathbb{Z}/N\mathbb{Z}$ satisfies $\sin(2\pi r/N) = 0$ exactly when $r \equiv 0$ (and also $r \equiv N/2$ if $2 \mid N$). These are precisely the set $S_{\sin}$ appearing in the statement.

Finally, since $\psi_k^{\sin}(x) = \sin\!\big(\frac{2\pi}{N} F_k(x)\big)$, an element $x \in (\mathbb{Z}/N\mathbb{Z})^d$ is a zero of $\psi_k^{\sin}$ if and only if $F_k(x)$ lies in $S_{\sin}$. Thus

$$Z\big(\psi_k^{\sin}\big) \;=\; F_k^{-1}\big(S_{\sin} \cap \operatorname{Im} F_k\big).$$

Because the preimage of a subset is the disjoint union of the fibers over its elements, and each fiber has size $d_0\, N^{d-1}$, the cardinalities are

$$\big|Z(\psi_k^{\sin})\big| \;=\; \big|S_{\sin} \cap \operatorname{Im} F_k\big| \cdot d_0\, N^{d-1},$$

as claimed. □

Now we begin to construct high-density unobservable sets via product eigenfunctions. For $r = (r_1, \dots, r_d) \in (\mathbb{Z}/N\mathbb{Z})^d$, we consider

$$\Psi_r(x) = \prod_{j=1}^{d} \sin\!\Big(\frac{2\pi r_j x_j}{N}\Big), \qquad x = (x_1, \dots, x_d) \in (\mathbb{Z}/N\mathbb{Z})^d. \tag{31}$$

Expanding the product shows that

$$\Psi_r(x) \;=\; \sum_{s \in \{\pm 1\}^d} c_s \, \exp\!\Big(\frac{2\pi i}{N}\, \langle s \circ r, x\rangle\Big),$$

where $(s \circ r)_j = s_j r_j$ and $c_s \neq 0$. Since $\mu(s \circ r) = 2\sum_{j=1}^{d} \cos(\frac{2\pi r_j}{N}) - 2d$ is independent of s, Ψ_r is a Laplace eigenfunction with eigenvalue $\mu(r)$.

Set $Z_j(r) := \{x \in (\mathbb{Z}/N\mathbb{Z})^d : \ \sin(\frac{2\pi r_j x_j}{N}) = 0\}$ and

$$E_{\mathrm{prod}}(r) := \bigcup_{j=1}^{d} Z_j(r).$$

Then Ψ_r vanishes on $E_{\mathrm{prod}}(r)$, so $E_{\mathrm{prod}}(r)$ is unobservable by Theorem 1.3.

Theorem 3.1 (Product construction and its size)**.** *Let $N \geq 3$, $d \geq 1$, and let $p_{\min}$ denote the smallest odd prime divisor of N (if N is odd prime, then $p_{\min} = N$). Define r_j coordinate-wise by*

$$r_j \ = \ \begin{cases} N/4, & \text{if } 4 \mid N, \\ N/p_{\min}, & \text{if } 4 \nmid N, \end{cases} \qquad j = 1, \ldots, d,$$

and set $r = (r_1, \ldots, r_d)$ and $E_{\mathrm{prod}} := E_{\mathrm{prod}}(r)$.

Then E_{prod} is unobservable and its cardinality is

$$|E_{\mathrm{prod}}| \ = \ \begin{cases} N^d - (N/2)^d \ = \ N^d\big(1 - 2^{-d}\big), & \text{if } 4 \mid N, \\ N^d - \big(N - N/p_{\min}\big)^d \ = \ N^d\Big[1 - \big(1 - \frac{1}{p_{\min}}\big)^d\Big], & \text{if } 4 \nmid N. \end{cases}$$

Proof. We already showed Ψ_r defined in (31) is an eigenfunction and $\Psi_r|_{E_{\mathrm{prod}}} \equiv 0$. It remains to count.

If $4 \mid N$ and $r_j = N/4$, then $\sin(\frac{2\pi r_j x_j}{N}) = \sin(\frac{\pi x_j}{2})$ vanishes iff x_j is even, i.e. $|Z_j(r)| = N/2$. Independence across coordinates shows the complement of E_{prod} consists of points with all x_j odd, of size $(N/2)^d$, yielding $|E_{\mathrm{prod}}| = N^d - (N/2)^d$.

If $4 \nmid N$, take $r_j = N/p_{\min}$. Then, as discussed in Proposition 3.1, since $\operatorname{Im} F_{r_j}$ has order $p_{\min}$ and $p_{\min}$ is odd, the only sine zero in that image is 0, so $Z_j(r) = \{x : \ x_j \equiv 0 \pmod{p_{\min}}\}$, of size $N/p_{\min}$. Again independence across coordinates gives $|((\mathbb{Z}/N\mathbb{Z})^d) \setminus E_{\mathrm{prod}}| = \big(N - N/p_{\min}\big)^d$, hence the stated formula. ∎

Proof of Theorem 1.4. Let $N_m \equiv 0 \pmod 4$. The nonzero function $\psi(x) = \prod_{j=1}^d \sin(\pi x_j/2)$ is a Laplacian eigenfunction. Lettling $E_{N_m} := E_{\mathrm{prod}}$, Theorem 3.1 implies that E_{N_m} is not observable for any $T > 0$ and $|E_{N_m}| = N_m^d(1 - 2^{-d})$. This completes the proof. ∎

We remark that product construction over primes yields vanishing density. More precisely, assume N is odd prime and take $r_j = N/p_{\min} = 1$ for all j. Then $|E_{\mathrm{prod}}| = N^d - (N-1)^d \sim d\, N^{d-1}$ $(N \to \infty)$, that is, $|E_{\mathrm{prod}}|/N^d \to 0$ as $N \to \infty$.

For $r = (r_1, \ldots, r_d) \in (\mathbb{Z}/N\mathbb{Z})^d$, we introduce

$$\mathcal{V}_r \ = \ \operatorname{span}\big\{\, \varphi_{s \circ r} : \ s \in \{\pm 1\}^d \,\big\}.$$

Then $\dim \mathcal{V}_r \leq 2^d$ (in fact $= 2^d$ if all $r_j \not\equiv 0$ and $r_j \not\equiv N/2$ when $2 \mid N$). To conclude this paper, we show that, when $4 \mid N$ and $r_j = N/4$, the eigenfunction Ψ_r minimizes the number of nonzero entries among all nonzero vectors in $\mathcal{V}_r$; equivalently, it maximizes the zero set $Z(\psi)$ within $\mathcal{V}_r$.

Proposition 3.2 (An uncertainty bound and optimality in $\mathcal{V}_r$)**.** *Fix r as in Theorem 3.1. Then for any $0 \neq \psi \in \mathcal{V}_r$, we have*

$$|\operatorname{supp} \psi| \ \geq \ \frac{N^d}{2^d}.$$

If moreover $4 \mid N$ and $r_j = N/4$ for all j, then Ψ_r satisfies $|\operatorname{supp} \Psi_r| = (N/2)^d = N^d/2^d$, hence it attains the uncertainty lower bound in $\mathcal{V}_r$. Consequently, among all nonzero $\psi \in \mathcal{V}_r$, the zero set $Z(\psi)$ is maximized by Ψ_r.

Proof. Let $G = (\mathbb{Z}/N\mathbb{Z})^d$ and let $\widehat{G}$ be its dual (identified with $(\mathbb{Z}/N\mathbb{Z})^d$ via characters). For any nonzero $f : G \to \mathbb{C}$ the Donoho–Stark uncertainty principle (see for example [15, 37]) gives $|\operatorname{supp} f| \cdot |\operatorname{supp} \widehat{f}| \geq |G| = N^d$.

Every $\psi \in \mathcal{V}_r$ has Fourier support contained in the set $\{\, s \circ r : \; s \in \{\pm 1\}^d \,\}$, hence $|\operatorname{supp} \widehat{\psi}| \leq 2^d$. Applying the uncertainty inequality yields $|\operatorname{supp} \psi| \geq N^d / 2^d$.

When $4 \mid N$ and $r_j = N/4$, we computed in Theorem 3.1 that $\Psi_r(x) \neq 0$ iff all coordinates x_j are odd, so $|\operatorname{supp} \Psi_r| = (N/2)^d$. Thus Ψ_r attains the lower bound in $\mathcal{V}_r$, which implies maximality of $|Z(\psi)|$ within $\mathcal{V}_r$. ☐

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