CONIC LINEAR SERIES AND PENCILS OF PLANE QUARTICS

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Abstract. We study linear systems cut out by cones of fixed degree on a smooth complex curve $C \subset \mathbb{P}^3$. We develop a systematic study of the families of such systems, considering their limits, their infinitesimal behaviour and some associated geometric structures. As an application, we prove the existence of a non-isotrivial pencil of quartics with only one base point, all whose members are irreducible and whose general member is smooth.

Contents

1. Introduction

Let $C \subset \mathbb{P}^3$ be a smooth non-degenerate projective complex curve of degree d. We call a *conic linear system* on C a linear system which is cut out on C by cones of fixed degree whose vertex is disjoint from C (see Equation (4.1)). In this paper we carry on a detailed study of these objects and their limits as the vertex specializes to a point of C.

Our original motivation comes from the following problem: construct, if possible, non-isotrivial pencils of plane curves with only one base point, all whose members are irreducible and with general member smooth. The case of cubics, which we studied extensively in the recent paper [20], appeared many times in the literature from different perspectives: starting with Cayley [7] and ending - to our knowledge - with Kollár [18, Example 46]. The construction in [20] is related to the torsion points of order 9 on a plane cubic (see also [12]).

We wondered if the method of [20] could also be modified to treat the case of higher degree plane curves. In order to work with torsion points as before, it seemed unavoidable to use elliptic curves. We thus tried to work in $\mathbb{P}^{n-1}$ with elliptic curves of degree n and hypersurfaces, coming back to $\mathbb{P}^2$ by projections. This naturally led us to study cones, and divisors cut out on the curve by them. We soon realized that our wishful thinking was difficult to implement in full generality. In this paper we set the theoretical framework to work with curves of any degree in $\mathbb{P}^3$, obtaining some general results on cones and divisors for any smooth curve in $\mathbb{P}^3$. As an application we solve our original problem for $d = 4$.

We now describe in detail the content of the paper. As first result we describe linear limits of cones over a curve C. This is used a lot later in the paper for working with divisors.

Proposition 1.1 (Equation (3.4))**.** *Let $p \in C \subset \mathbb{P}^3$ and let t_p be the projective tangent line to C at p. Consider a line $\ell \neq t_p$ through p. Then, the limit of a cone of degree d over C as its vertex specializes to p along ℓ is the union of the cone over C with vertex in p and the plane spanned by ℓ and t_p.*

We give two proofs of this result: one geometric in Equation (3.4) and one analytical in Section 3.1. While the second approach is less elegant, it introduces techniques that are essential. One of the outcomes of Equation (3.4) is a way to describe a projective model of $\widetilde{\mathbb{P}}^3$, the blow up of $\mathbb{P}^3$ along C. This is described in Section 3.2: denote by $U \subseteq \mathbb{P}^3 \smallsetminus C$ the locus of points from which the linear projection map restricted to C is birational. Let I be the ideal of C in $\mathbb{P}^3$ and let $c\colon U \to \mathbb{P}(I(d))$ be the map assigning to each $p \in U$ the cone over C with vertex at p. In Equation (3.16) we describe an extension of c to an immersion $\widetilde{c}\colon \widetilde{\mathbb{P}}^3 \to \mathbb{P}(I(d))$.

We then turn to the structure of *conic linear systems*: denote by $R_k(q)$ the conic linear system cut on C by the cones of degree k in $\mathbb{P}^3$ with vertex $q \in U$, see Equation (4.1). We are interested in their limits $R^\ell_k(p)$ on points $p \in C$ along lines $\ell \neq t_p$, with the additional hypothesis that the restriction to C of the linear projection from p is birational, see Equation (4.3).

A central part of the paper is devoted to the study of such $R^\ell_k(p)$ for $k = d-1$ and $k = d$. In particular, we show in Equation (4.6) and Equation (4.8) that every element of $\mathbb{P}(R^\ell_k(p))$ has evaluation at least $d-2$ at p. This last result is crucial for the application in the last section.

Another important and natural object in our setting is the *cone map*, defined for every integer $k > 1$ on the open $U \subset \mathbb{P}^3$ with image in a suitable Grassmannian

$$\rho_k\colon U \longrightarrow G(n_k, H^0(C, \mathcal{O}_C(k))),$$

where $n_k = \dim R_k(q)$. This map takes any $q \in U$ to $R_k(q)$. The results described above on the limits allow us to give in Section 5 a partial completion of the cone map from the blow up of $\mathbb{P}^3$ along C:

$$\widetilde{\rho}_k\colon \widetilde{\mathbb{P}}^3 \dashrightarrow G(n_k, H^0(C, \mathcal{O}_C(k))). \tag{1.2}$$

In the case $k = d$, we then encode the geometrical structure of the conic linear series in a morphism

$$\Phi : \mathbb{P}_U(\mathcal{E}) \longrightarrow \mathbb{P}(H^0(C, \mathcal{O}_C(d))),$$

where $\mathcal{E}$ is a vector bundle whose fibres are linear systems of degree d cones with vertex $p \in U$, modulo those cones that contain the curve C. So, the fibres are naturally isomorphic to $R_d(p)$. The morphism Φ assigns to each class of cones the conic divisor cut out on C. By using a combination of classical methods (e.g.

Castelnuovo's bound in Equation (5.12)) and some subtle computations we study the differential of Φ:

Theorem 1.3 (Equation (6.8)). *The differential of Φ is generically of maximal rank. For a general $y \in \mathbb{P}_U(\mathcal{E})$ the following holds:*

1. *if $d \geq 5$ or $d = 4$ and $g = 0$, then $d\Phi_y$ is generically injective;*
2. *if $d = 4$ and $g = 1$, then $\dim\ker d\Phi_y = 1$;*
3. *if $d = 3$ and $g = 0$, then $\dim\ker d\Phi_y = 2$.*

Using the map $\widetilde{\rho_d}$ of Equation (1.2), we obtain a partial compactification of the construction and define an extension Φ'' of Φ, whose domain parametrizes linear limits of systems of cones. This yields the following result:

Corollary 1.4 (Equation (6.9), Equation (6.14)). *The following facts are equivalent:*

(i) *The map Φ'' is surjective,*
(ii) *the map Φ is dominant,*
(iii) *the degree $d \leq 4$.*

By exploiting this result together with the geometric structure of these maps, we show that every effective degree 16 divisor on C an elliptic normal curve of degree 4 is cut out by a cone of degree 4:

Theorem 1.5. *(Equation (7.5)) Let $C \subset \mathbb{P}^3$ a degree 4 genus 1 smooth curve. Let D be any effective divisor of degree 16 on C. There is a 1-dimensional family of cones in $B \subset \mathbb{P}_U(\mathcal{E})$ such that:*

(i) *the fibre projection of B in U is again 1-dimensional;*
(ii) *by calling $\widetilde{K}_t$ the cone corresponding to $t \in B$, we have that $\Phi(\widetilde{K}_t) = D$, for any $t \in B$.*

Eventually, we apply our results to elliptic normal curves and, with some additional work, we obtain our geometric application:

Theorem 1.6 (Equation (7.8)). *There exists a pencil of quartics in $\mathbb{P}^2$ such that:*

1. *the base locus is set-theoretically reduced to one point;*
2. *all the quartics of the pencil are irreducible;*
3. *the general element of the pencil is smooth;*
4. *the pencil is non-isotrivial.*

The idea of the proof is the following. Let $C \subset \mathbb{P}^3$ be a smooth elliptic quartic. Fix the origin of its group structure at a flex point O. Let $\sim$ denote the linear equivalence between divisors on C. Choose a point $q \in C$ such that $16q \sim 16O$ but $8q \not\sim 8O$ (in direct analogy with the construction used for plane cubics in [20]). By Equation (1.5) there exists a 1-dimensional family of cones in the inverse image of $16q$ under the map Φ. Fixing one such cone and projecting from its vertex we obtain a plane quartic which intersects the projection of C precisely on the image of q. These two curves generate the desired pencil. The irreducibility of all its members comes from the assumption we made on q. The smoothness of the general member relies on a delicate argument that uses heavily all the results of the previous sections. In particular we will need the mentioned above result saying that every element of $\mathbb{P}(R^{\ell}_k(p))$ has evaluation in p at least 2 (Equation (4.8)), the existence of the 1-parameter family of cones proved Equation (1.5) and an explicit description of the inverse image $\Phi''^{-1}(16q)$ given in Equation (7.7). The non-isotriviality is ensured by the existence of a reduced fibre with geometric genus 1 (the image of C).

We believe that conic linear systems are interesting by themselves and will have further applications beyond the scope of this work. In the last section we list some open questions and describe some possible developments that our theory can have.

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2. Preliminaries

Let $C \subset \mathbb{P}^3$ be a smooth irreducible non-plane curve of genus g and degree d defined over the field $\mathbb{C}$ of complex numbers. Consider the short exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

where I is the ideal sheaf of C. Let $\mathcal{O}_{\mathbb{P}^3}(1)$ denote the tautological line bundle and let $\mathcal{O}_C(1)$ be its restriction to C. Let $V \subset H^0(C, \mathcal{O}_C(1))$ be the 4-dimensional vector space that defines the embedding $C \subset \mathbb{P}^3 = \mathbb{P}(V^*)$. In particular, $|V| = \mathbb{P}(V) \cong \mathbb{P}^3$ is the linear system on C of divisors given by intersecting C with the planes of $\mathbb{P}^3$. A point $p \in \mathbb{P}(V^*)$ corresponds to a 3-dimensional subspace $W(p) \subset V$, namely $W(p)$ consist of the sections of V vanishing at p (see also Equation (5.8)). Dually, we have a surjection $V^* \twoheadrightarrow W(p)^*$ and a projection map

$$\Pi_p \colon \mathbb{P}(V^*) \dashrightarrow \mathbb{P}(W(p)^*),$$

defined on $\mathbb{P}(V^*) \setminus \{p\}$. Consider the restriction to C of Π_p, which we denote by $\pi_p \colon C \dashrightarrow \mathbb{P}(W(p)^*)$. When $p \notin C$, the map π_p is already a morphism, while when $p \in C$, as p is a smooth point of C, π_p can be extended to a morphism. We will call $C_p := \pi_p(C) \subset \mathbb{P}(W(p)^*)$ the image of this morphism.

Consider a partition of $\mathbb{P}^3$ in three subsets, according to the properties of the map π_p. Let us call $S \subset \mathbb{P}^3$ the set of points in which π_p is not birational, let C' be the set $C \smallsetminus S$, and U be the complement $U = \mathbb{P}^3 \setminus (C \cup S)$, so that we have

$$\mathbb{P}^3 = S \sqcup C' \sqcup U. \tag{2.1}$$

Lemma 2.2. *We have the following:*

(1) $p \in U \iff \pi_p$ is birational and C_p has degree d;
(2) $p \in C' \iff \pi_p$ is birational and C_p has degree $d-1$;
(3) $p \in S \iff \pi_p$ is not birational.

Moreover, S is a finite set of points.

Proof. The first three claims follow from the properties of linear projections. The finiteness of $S \cap C$ comes from the trisecant lemma. For the finiteness of $S \smallsetminus C$ we can use the result of [22]. An alternative argument can be done by means of the theory of focal loci that is presented for instance in [9], see also [8]. $\square$

Remark 2.3. For $p \in U$, the plane curve C_p is singular and reduced of degree d, and when $p \in C'$, the curve C_p is reduced of degree $d-1$.

For any integer $k \geq 1$ let us consider the space $V_k := \mathrm{Sym}^k(V) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$, which parametrizes hypersurfaces of degree k in $\mathbb{P}^3$. For a linear subspace $W \subset V$, we have $W_k := \mathrm{Sym}^k(W) \subset V_k$, parametrizing hypersurfaces of degree k in $\mathbb{P}(W^*)$, or, equivalently, *cones* of degree k in $\mathbb{P}(V^*)$ with vertex $\mathbb{P}Ann(W) \subset \mathbb{P}(V^*)$.

Lemma 2.4. *We have the following characterization of the decomposition* (2.1)*:*

(1) $p \in U \iff \dim W(p)_d \cap I(d) = 1$*;*
(2) $p \in C' \iff \dim W(p)_d \cap I(d) = 3$*;*
(3) $p \in S \iff \dim W(p)_d \cap I(d) \geq 6$.

Proof. Let $p \in \mathbb{P}^3$. Let $C_p \subset \mathbb{P}(W(p)^*) \cong \mathbb{P}^2$ be the image curve and $f_p = 0$ an associated equation. As (2.1) is a partition, it is enough to prove all the implications from left to right in the three points of the statement.

(1) If $p \in U$, then f_p has degree d, and it represents the unique cone with vertex p over C. So $W(p)_d \cap I(d) = \langle f_p \rangle$.

(2) If $p \in C'$, then the cones with vertex p over C all have equation $f_p h = 0$ where $h \in W(p)$. Hence its dimension is 3.

(3) If $p \in S$, then C_p has degree $k < d-1$. So the space $W(p)_d \cap I(d)$ consists in the polynomials of the form $f_p g$, where $g \in W(p)_{d-k}$. Now we have, as wanted,

$$\dim W(p)_{d-k} = \binom{d-k+2}{2} \geq \binom{4}{2} = 6.$$

$\square$

Definition 2.5. For any $p \in \mathbb{P}^3$, we denote by $\widetilde{C}_p \subset \mathbb{P}^3$ the cone over C with vertex p. Let $0 \neq f_p$ be a polynomial describing the hypersurface $\widetilde{C}_p$.

The polynomial f_p is defined up to a scalar and has degree d if $p \in \mathbb{P}^3 \smallsetminus C$, degree $d-1$ for $p \in C$. When $p \in S$, the cone $\widetilde{C}_p$ is non-reduced; see Equation (2.3).

Remark 2.6. Recalling that $\Pi_p \colon \mathbb{P}(V^*) \dashrightarrow \mathbb{P}(W(p)^*)$ is the prohection map, we have that set-theoretically

$$\widetilde{C}_p = \overline{{\Pi_p}^{-1}(C_p)}.$$

By choosing a plane $\Sigma \subset \mathbb{P}^3$ not containing p in $\mathbb{P}^3$ and identifying it with $\mathbb{P}(W(p)^*)$, one can also think as $\widetilde{C}_p$ as the cone in $\mathbb{P}^3$ with base the plane curve $C_p \subset \Sigma$ and vertex p. Moreover, $f_p = 0$ can be thought as both: (1) the equation of the cone with vertex p over C in $\mathbb{P}^3$; (2) the equation of the projection curve $C_p \subset \mathbb{P}(W(p)^*) \cong \mathbb{P}^2$.

Definition 2.7. If $p \in U$, the polynomial f_p is a generator of $W(p)_d \cap I(d)$, see Equation (2.4). The correspondence $p \mapsto W(p)_d \cap I(d)$ thus defines a map

$$\begin{aligned} c : U &\longrightarrow \mathbb{P}(I(d)) \\ p &\mapsto [f_p]. \end{aligned} \tag{2.8}$$

In Equation (3.16) we will extend c to the blow up of $\mathbb{P}^3$ along C and compute the associated linear system.

3. Linear limits of cones

There are two families of dimension 3 of cones of degree d containing C: the first one consists of cones $\widetilde{C}_p$ with $p \in U$, the second one is made of cones of the form $\widetilde{C}_p \cup H$ with vertex in $p \in C$, and H a plane belonging to $W(p)$. As a consequence the elements of the second family can not all be limits of elements of the first one. In this section we study the limits of the cones of the first family as the vertex specializes linearly to a point of C' and see that they become certain elements of the second family.

Fix a point $p \in C$. The limit cone for a point moving to p depends on the direction. Let us consider the following situation: suppose we have a point $p_t \in \mathbb{P}^3$ going to p along a line $\ell \subset \mathbb{P}^3$. Consider Δ a small disk in $\mathbb{C}$ centred in p, and an affine coordinate $t \in \Delta$. The line ℓ near p_t is parametrized by the affine coordinate t as

$$\ell = \{p + tv, t \in \mathbb{C}\}, \tag{3.1}$$

where v is the direction of ℓ. Suppose that for any $t \in \Delta^*$ we have $p_t \in U$. Then the polynomial $f_t := f_{p_t}$ of Equation (2.5) belongs to V_d for $t \neq 0$.

Lemma 3.2. *Given $p \in C$ on a line $\ell \subset \mathbb{P}^3$, there exists a unique (flat-)limit of $[f_t]$ in $\mathbb{P}(V_d)$ for $t \to 0$, that is a degree d cone with vertex p and containing C.*

Proof. First observe that the limit of the family $[f_t] \in \mathbb{P}(V_d)$ exists by the valuative criterion of properness, and that the locus

$$\mathcal{C} := \{[f] \in \mathbb{P}(V_d) \mid Z(f) \text{ is a cone}\} = \cup_{p\in\mathbb{P}^3} W_d(p)$$

is closed in $\mathbb{P}^3$, so the limit also is a degree d cone. Finally, it is clear that the vertex of this limit cone contains p, and that the limit cone contains C. □

Definition 3.3. We call the construction above *linear limit of* $\widetilde{C}_{p_t}$. We denote the corresponding hypersurface as $\widetilde{C}_0^\ell$.

As in Equation (2.6), we fix a general plane $\mathbb{P}^2 \subset \mathbb{P}^3$ as the target space. We will use the same notation $\Pi_{p_t} \colon \mathbb{P}^3 \smallsetminus \{p_t\} \to \mathbb{P}^2$. The cone $\widetilde{C}_{p_t}$ is completely determined by the data $(p_t, \Pi_{p_t}(C))$; indeed, it is precisely the cone with vertex p_t over $\Pi_{p_t}(C)$, and this remains true for $t = 0$.

Proposition 3.4. *Let $p \in C$ and let t_p be the projective tangent line to C at p. Let $\ell \neq t_p$ be a line passing through p. Then*

$$\widetilde{C}_0^\ell = \widetilde{C}_p \cup H,$$

where $H = \langle \ell, t_p\rangle$ is the unique plane in $\mathbb{P}^3$ containing t_p and ℓ.

Proof. Since $\widetilde{C}_p$ is contained in the limit $\widetilde{C}_0^\ell$ and $\deg \widetilde{C}_p = d - 1$, we must have $\widetilde{C}_0^\ell = \widetilde{C}_p \cup K$ for some plane K through p. Since the line ℓ lies in $\widetilde{C}_{p_t}$ for every $t \neq 0$, it also lies in $\widetilde{C}_0^\ell$, so K contains ℓ. It remains to prove that $K = \langle \ell, t_p\rangle$, i.e., that K also contains t_p. The assumption $\ell \neq t_p$ implies that $\langle \ell, t_p\rangle$ is a plane. As a consequence, $\widetilde{C}_0^\ell$ is determined by its intersection with the target $\mathbb{P}^2$, which consists of two components: the curve $\Pi_p(C)$ of degree $d-1$ and the line $K' := K \cap \mathbb{P}^2$. Let $q := \Pi_{p_t}(p)$ for $t \neq 0$. Notice that $q \notin \Pi_p(C)$, hence $q \in K$. The projection $\overline{r} := \pi_{P_t}(t_p)$ is a line in $\mathbb{P}^2$ independent of $t \neq 0$; it contains q and it is tangent to $\Pi_{p_t}(C)$ at q by construction. Thus $\overline{r} \subset \widetilde{C}_0^\ell \cap \mathbb{P}^{n-1}$, forcing $K' = \overline{r}$ and hence $K = \langle \ell, t_p \rangle$. □

Remark 3.5. The same argument works in any dimension n provided C is an irreducible codimension 2 subvariety of $\mathbb{P}^n$.

We also need to cover the case in which we take the limit along the tangent line.

Lemma 3.6. *Let $p \in C$ and t_p the projective tangent line at p. Then*

$$\widetilde{C}_0^{t_p} = \widetilde{C}_p \cup H,$$

where H is the the osculating plane to C at p.

Proof. The claim is a consequence of the definition of osculating plane, see [2]. □

3.1. **Analytical proof of Equation (3.4).** We now give another proof of Equation (3.4). This new proof involves an explicit computation, with a suitable choice of coordinates. Similar, but slightly more involved, computations are used in the arguments of Section 4 and 6. Fix homogeneous coordinates $(x : y : z : w)$ of $\mathbb{P}^3$. We can assume the coordinates of p to be $(0 : 0 : 0 : 1)$, so that $W(p) = \langle x, y, z \rangle$, equivalently, f_p does not depend on the variable w. We will also fix the tangent line to the curve C at p to be $t_p := \{x = y = 0\}$. Notice that with this choice, we have that $\pi_p(p) = (0 : 0 : 1) \in \mathbb{P}^2$. Let us consider the line $\ell := \{p_t = (-at : -bt : -ct : 1), t \in \mathbb{C}\}$ in $\mathbb{P}^3$ as in Equation (3.1). When moving the point p_t along ℓ, we get a deformation of $W(p)$ given by

$$W(p_t) = \langle x + atw, y + btw, z + ctw \rangle.$$

The tangent vector to this deformation is

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}. \tag{3.7}$$

If we assume $\ell \neq t_p$, by taking into account all the previous assumptions, we may set $a = 1, b = c = 0$. Then, $p_t = (-t : 0 : 0 : 1)$, $W(p_t) = \langle x + tw, y, z \rangle$. The line ℓ has equation $\{y = z = 0\}$.

Setting 3.8. Here is an overview of all the choices we have discussed.

1. $p = (0 : 0 : 0 : 1)$, a fixed point on C;
2. $t_p = \{x = y = 0\}$, the tangent line to C at p;
3. $\ell = \{y = z = 0\}$ the implicit equations of the line along which we deform;
4. $p_t = (-t : 0 : 0 : 1)$, the corresponding linear deformation of p;
5. $W(p_t) = \langle x + tw, y, z \rangle$ the corresponding deformation of $W(p)$.

Remark 3.9. Given any homogeneous polynomial $l \in V_k$ we will denote by $\nu_p(l)$ the evaluation of l at p on C. This is the intersection multiplicity of C with the hypersurface $Z(l)$ in the point p. With the choices of Equation (3.8), we have $\nu_p(w) = 0$, $\nu_p(z) = 1$, $\nu_p(x) \geq 2$ and $\nu_p(y) \geq 2$.

Proof of Equation (3.4). Without loss of generality, we assume Setting 3.8. Firstly we prove the result under the following assumptions, valid for a general $p \in C$ and ℓ:

(1) $p \in C'$;
(2) the tangent in p does not meet the rest of the curve elsewhere; (see [16] and [4])
(3) ℓ is not bisecant to C.

From (1) and Equation (2.2) the projection curve $C_p = \pi_p(C)$ has degree $d-1$. From (2), $\pi_p(p)$ is a smooth point of C_p.

Let $f_t = 0$ be an equation of degree d of the cone $\widetilde{C}_{p_t}$, for $t \neq 0$. The fact that p_t is moving along $\{y = z = 0\}$ can be encoded in the polynomial f_t by highlighting the contribution of t in the first variable: $f_t(x + tw, y, z)$.

By Equation (3.2) the limit cone $\widetilde{C}_0^\ell$ has degree d. Since it contains the degree $d-1$ cone $\widetilde{C}_p$, it also contains a plane $H := \{h = 0\}$ passing through p. Since $h(p) = 0$, and $p = (0 : 0 : 0 : 1)$, we have $h = Ax + By + Cz$. Let $f = 0$ be an equation of $\widetilde{C}_p$. It follows that $f_0 = h(x, y, z)f(x, y, z)$.

By assumption (3) we have that the point $(1 : 0 : 0)$ does not belong to the tangent line of C_p at $\pi_p(p) = (0 : 0 : 1)$. This is equivalent to $\frac{\partial f}{\partial x}|_{(0:0:1)} \neq 0$. This means that, up to a non-zero constant, we can write

$$f = xz^{d-2} + x^2L(x, y, z) + yM(x, y, z), \tag{3.10}$$

where $\deg L = d-3$, $\deg M = d-2$.

The following formula follows from taking Taylor expansions in t, and it is proved in Equation (3.14).

$$f_t(x + tw, y, z) - h(x, y, z)f(x, y, z) = t\left(w\frac{\partial(hf)}{\partial x} + g(x, y, z)\right) \mod t^2. \tag{3.11}$$

It follows that $w\frac{\partial(hf)}{\partial x} = w(\frac{\partial h}{\partial x}f + h\frac{\partial f}{\partial x}) = wAf + wh\frac{\partial f}{\partial x}$, and so

$$w\frac{\partial(hf)}{\partial x} + g = wAf + wh\frac{\partial f}{\partial x} \in I(d).$$

In particular we have

$$wh\frac{\partial f}{\partial x} + g \in I(d). \tag{3.12}$$

By evaluating Equation (3.12) at p, we obtain

$$\nu_p(w) + \nu_p(h) + \nu_p\left(\frac{\partial f}{\partial x}\right) = \nu_p(g) \geq d.$$

By Equation (3.10), we have $\nu_p(\frac{\partial f}{\partial x}) = d-2$, and so we obtain $\nu_p(h) \geq 2$. This means that $h(x, y, z) = Ax + By$, equivalently H contains t_p. Moreover, $\ell \subset \widetilde{C}_{p_t}$ for all $t \neq 0$, so $\ell \subset \widetilde{C}_p^\ell = \widetilde{C}_p \cup H$.

By Assumption (3), we get $\ell \subset H$, and then $H = \langle \ell, t_p \rangle$ is the plane spanned by ℓ and the tangent t_p, since $\ell = \{p_t = (-t : 0 : 0 : 1), t \in \mathbb{C}\}$, we have that $A = 0$

and $H = \{y = 0\}$. So, (3.12) becomes

$$wy\frac{\partial f}{\partial x} + g \in I(d), \tag{3.13}$$

This proves the result under the assumptions (1), (2), (3). By continuity we can drop (2) and (3), and get the result for any $p \in C'$.

In order to drop (1), consider a point $p \in S \cap C$. Choose a small neighbourhood $\mathcal{V} \subset C$ of p such that any $q \in \mathcal{V} \setminus \{p\}$ is in C'. Consider any direction v different from the one of t_p; then we can assume that v is not the direction of t_q for any $q \in \mathcal{V}$. Let ℓ be the line passing through p with direction v. For any $q \in \mathcal{V} \setminus \{p\}$ we have that the linear limit of $\widetilde{C}_{q_t}$ along a line with direction v is the cone $\widetilde{C}_q \cup \langle t_q, \ell\rangle$ by the proof above. Then by specialization the same holds for p. Note that in this case we have $\deg(\pi_p) = e$ and $er = d-1$. The equations of the limit cones have the form $f_p^r h$. As a divisor of $\mathbb{P}^3$ we get $H + r\widetilde{C}_p$. This concludes the proof. □

Lemma 3.14. *With the assumptions and notations of Equation (3.4) we have*

$$f_t(x+tw, y, z) - h(x,y,z)f(x,y,z) = t\left(w\frac{\partial(hf)}{\partial x} + g(x,y,z)\right) \mod t^2.$$

Proof. We highlight the dependence of $f_t(x+tw,y,z)$ on t by writing $f_t = \varphi\circ\alpha$, where $\varphi\colon \mathbb{C}^4 \to \mathbb{C}$ and $\alpha\colon \mathbb{C}^5 \to \mathbb{C}^4$ is given by $\alpha(t,x,y,z,w) = (t, x+tw, y, z)$. Now we use the Taylor expansion of $\varphi\circ\alpha$ with respect to t near $t=0$, calling $(x_1,\dots,x_4)$ the coordinates in $\mathbb{C}^4$:

$$\begin{aligned}
\varphi\circ\alpha(t,x,y,z,w) - \varphi\circ\alpha(0,x,y,z,w) &= \\
&= t\frac{\partial\varphi}{\partial x_1}\alpha(0,x,y,z,w) + tw\frac{\partial\varphi}{\partial x_2}\alpha(0,x,y,z,w) \mod t^2 = \\
&= t\frac{\partial\varphi}{\partial x_1}(0,x,y,z) + tw\frac{\partial\varphi}{\partial x_2}(0,x,y,z) \mod t^2.
\end{aligned}$$

Call $\frac{\partial\varphi}{\partial x_1}(0,x,y,z) = g(x,y,z)$, and observe that

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_2}\alpha(0,x,y,z,w) &= \frac{\partial f_t}{\partial x}\alpha(0,x,y,z,w)\frac{\partial\alpha}{\partial x_2}(0,x,y,z,w) = \\
&= \frac{\partial f_0}{\partial x}(x,y,z) = \frac{\partial}{\partial x}((h\cdot f)(x,y,z)),
\end{aligned}$$

since $\frac{\partial\alpha}{\partial x_2}(0,x,y,z,w) = 1$, formula (3.11) is verified. □

Remark 3.15. Consider the case where $p \in S$, $p \notin C$. Assume that $\deg\pi_p = e$ and $d = er$. Let $\widetilde{C}_p$ be the cone of degree r joining p and C. If we take any limit $p_t \to p$, the equation of the limit cone becomes f_p^e, where $f_p \in I(r)$ is the equation of the reduced cone, and the divisor we get in $\mathbb{P}^3$ is then $\widetilde{C}_p$ with multiplicity e.

3.2. **A projective model of the blow up of $\mathbb{P}^3$ along C.** Let $\nu\colon \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ along C. We now extend the map (2.8) over $\widetilde{\mathbb{P}}^3$.

Proposition 3.16. *The map $c\colon U \to \mathbb{P}(I(d))$ extends to an injective morphism*

$$\widetilde{c}\colon \widetilde{\mathbb{P}}^3 \longrightarrow \mathbb{P}(I(d)).$$

The closure of $c(U) \subset \mathbb{P}(I(d))$ *is* $\widetilde{c}(\widetilde{\mathbb{P}}^3)$. *The divisor associated to* $\widetilde{c}$ *in* $\mathrm{Pic}(\widetilde{\mathbb{P}}^3)$ *is* $d\widetilde{\Sigma} - E$, *where* $\widetilde{H}$ *is the pullback of a hyperplane divisor on* $\mathbb{P}^3$ *and* E *is the exceptional divisor of* ν.

Proof. The exceptional divisor of ν is precisely $E = \mathbb{P}(N_{C|\mathbb{P}^3})$. The extension of c can be derived from Lemma 3.4 as follows. Recall that any $p \in E$ corresponds to a point $p' \in C$ and to a line ℓ different from the tangent $t_{p'}$ to C at p'. Define $\widetilde{c}(p) := [f_{p'}h] \in \mathbb{P}(I(d))$, where $\{h = 0\}$ is the equation of the plane generated by $t_{p'}$ and ℓ. On the finite set $S \smallsetminus C$ the limit is unique as observed in Remark 3.15, so the extension is well defined on the whole $\widetilde{\mathbb{P}}^3$. It is easy to prove that it is injective. Let us just observe that in the case we have two distinct points $p, q \in E$ such that $p' = q'$, then the hyperplanes are different, and so $\tilde{c}(p) \neq \tilde{c}(q)$.

We now want to compute the class of $\widetilde{c}^*\mathcal{O}_{\mathbb{P}(I(d))}(1)$. Recall that

$$\mathrm{Pic}(\widetilde{\mathbb{P}}^3) \cong \mathbb{Z}[\widetilde{L}] \oplus \mathbb{Z}[E],$$

where $\widetilde{L}$ is the pullback of a hyperplane divisor L of $\mathbb{P}^3$ via ν.

Following for instance the results of [13, Chap.4, sec. 6], it can be easily proved that:

$$\widetilde{L}^3 = L^3 = 1, \quad \widetilde{L}^2 \cdot E = 0, \quad \widetilde{L} \cdot E^2 = -d,$$

$$E^3 = -\deg N_{C|\mathbb{P}^3} = -2g + 2 + (K_{\mathbb{P}^3} \cdot C) = -2g + 2 - 4d.$$

Observe that an effective divisor in $|\widetilde{c}^*\mathcal{O}_{\mathbb{P}(I(d))}(1)|$ can be seen as $\widetilde{c}^*\Sigma$, where:

$$\Sigma := \{\text{classes of hypersurfaces of degree } d \text{ containing } r \text{ and } C\} \subset \mathbb{P}(I(d)),$$

where $r \in \mathbb{P}^3$ is a general point.

Let $\alpha, \beta \in \mathbb{Z}$ such that $|\widetilde{c}^*\mathcal{O}_{\mathbb{P}(I(d))}(1)| = |\alpha\widetilde{L} - \beta E|$. Call M a general $\widetilde{c}^*\Sigma \in |\widetilde{c}^*\mathcal{O}_{\mathbb{P}(I(d))}(1)|$. We prove the following two facts:

(1) $M \cdot \widetilde{L} \cdot E = d$.
Indeed, with the notation introduced above, the elements in $M \cap E$ are the pullbacks via $\widetilde{c}$ of the degenerate cones $\widetilde{C}_p \cup H$ where $p \in C$, $p \in H$ and $r \in \widetilde{C}_p \cup H$.

We now intersect with $\widetilde{L} = \nu^*L$. The intersection $L \cap C$ is a set of d points in general position by the general position theorem [13, Chap.2 Sec.3], which we call $p_1, \dots p_d$. For a general choice of L and r we have that r does not belong to any of the cones $\widetilde{C}_{p_i}$ with vertex p_i for any $i = 1, \dots, d$. Therefore, the elements in $M \cdot \widetilde{L} \cdot E$ are the pullback of the set of degenerate cones $\widetilde{C}_{p_i} \cup H_i$ such that $r \in H_i$. Recall that $H = \langle t_{p_i}, \ell \rangle$ for some line $\ell \subset \mathbb{P}^3$. But for any i there exists exactly one line ℓ_i such that $r \in \langle t_{p_i}, \ell_i \rangle$. So, for any $i \in \{1, \dots, d\}$ we have precisely one contribution to $M \cdot \widetilde{L} \cdot E$, and the proof is concluded.

(2) $M^2 \cdot E = 2\left((d-1)^2 - g\right)$.
Indeed, M^2 can be seen as the pullback via $\widetilde{c}$ of the hypersurfaces in $I(d)$ containing two general points r, s in $\mathbb{P}^3$. Since r, s are general, then they are never contained in the same cone $\widetilde{C}_p$ for $p \in \mathbb{P}^3 \smallsetminus C$. On the other hand, there are $2g - 2 + 2d$ degenerate cones $\widetilde{C}_p \cup H$ (whose pullback is in E) such that the plane H contains

both r and s. Indeed, consider the projection from the line $\langle r, s\rangle$:

$$\begin{tikzcd} \mathbb{P}^2 \setminus \langle r, s\rangle \arrow[r] & \mathbb{P}^1 \\ C \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Its restriction to C is a degree d covering with only simple ramifications by the generality of r and s. The cones we want to count correspond to these points of ramifications, so we obtain $2g-2+2d$ such points by the Riemann-Hurwitz formula [13, Preliminaries].

Now we are left to compute the number of degenerate cones $\widetilde{C}_p \cup H$ such that H contains r and $\widetilde{C}_p$ contains s.

Observe that $s \in \widetilde{C}_p$ if and only if the line $\langle p, s\rangle$ intersects the curve C. In other words, the image curve obtained by projecting form s, $\pi_s\colon C \to C_s \subset \mathbb{P}^2$ has a node. Now, the number of nodes of C_s coincides with the difference between its arithmetic and its geometric genus:

$$p_a(C_s)) - g = \frac{(d-1)(d-2)}{2} - g.$$

So, for every node q of C_s we have two points in $M^2 \cap E$: let $\{p, p'\} = \pi_s^{-1}(q)$, we have then precisely the two cones

$$\widetilde{C}_p \cup \langle t_p, \ell_{s,p}\rangle, \quad \widetilde{C}_{p'} \cup \langle t_{p'}, \ell_{s,p'}\rangle.$$

Now, switching the roles of r and s we have another equal contribution. Summing up, we have

$$M^2 \cdot E = (2g-2+2d) + 2\big((d-1)(d-2) - 2g\big) = 2((d-1)^2 - g),$$

as wanted.

Now, let us use the information above. We have from (1) and the relations listed above that

$$-d = M \cdot \widetilde{L} \cdot E = (\alpha\widetilde{H} - \beta E) \cdot \widetilde{L} \cdot E = -d\beta,$$

so $\beta = 1$. From (2) and what we have proven so far we have:

$$2((d-1)^2 - g) = M^2 \cdot E = (\alpha\widetilde{L} - E)^2 \cdot E = 2\alpha d + 2 - 2g - 4d,$$

so we obtain $\alpha = d$ as wanted. □

Remark 3.17. From the theorem above we have in particular that $d\widetilde{L} - E$ is ample (cf. [19, Cor.1.2.15]). Note that $-K_{\widetilde{\mathbb{P}^3}} = 4\widetilde{L} - E$. In case $d = 3, 4$ it is already known that $-K_{\widetilde{\mathbb{P}^3}}$ is ample [3, Prop. 3.1].

4. Conic linear systems

Now we want to consider divisors on C coming from cones in $\mathbb{P}^3$. For any integer $k \geq 1$, call $S_k := H^0(C, \mathcal{O}_C(k))$, so that $\mathbb{P}(S_k)$ parametrises divisors of degree dk on C. Recall that $V_k = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$; denote by $\phi_k\colon V_k \to S_k$ the natural restriction map. The projectivization of its image consists of the divisors of degree dk on C

coming from the intersection of hypersurfaces of degree k in $\mathbb{P}^3$ with C. We have the sequence

$$0 \longrightarrow I(k) \longrightarrow V_k \xrightarrow{\phi_k} S_k,$$

where $I(k) = \ker \phi_k$ is the space of the homogeneous polynomials of degree k vanishing on C. We know from the Castelnuovo type result in [14] that ϕ_k is surjective for $k \geq d-2$ and from Castelnuovo's results [5], [23, First Theorem of Castelnuovo] that $H^1(C, \mathcal{O}_C(k)) = 0$ for $k \geq \lfloor \frac{d-1}{2} \rfloor$. By applying the Riemann-Roch theorem, we get $\dim S_k = dk - g + 1$, and in particular $\dim S_d = d^2 - (g-1)$. It follows that $I(d) = \ker \phi_d$ has dimension $\binom{d+3}{3} - d^2 + g - 1$.

Definition 4.1. Given a linear subspace $W \subset V$, the *conic linear system* over C of degree dk associated to W is

$$R_k(W) := \phi_k(W_k) \subset S_k.$$

For $p \in \mathbb{P}^3$ we will use the notation $R_k(p) = R_k(W(p)) = \phi_k(W(p)_k)$.

We also call *conic linear series* the projectivization of a conic linear system and *conic divisor* an element of a conic linear series.

Remark 4.2. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I(k) \cap W_k \arrow[r] \arrow[d] & W_k \arrow[r] \arrow[d] & R_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I(k) \arrow[r] & V_k \arrow[r] & S_k &
\end{tikzcd}$$

For $k = d-1$ and $W = W(p)$ we have:

(1) $p \in U$ if and only if $\dim R_{d-1}(p) = \dim W(p)_{d-1} = \binom{d+1}{2}$;
(2) $p \in C'$ if and only if $\dim R_{d-1}(p) = \dim W(p)_{d-1} - 1 = \binom{d+1}{2} - 1 = \frac{(d-1)(d+2)}{2}$;
(3) $p \in S$ if and only if $\dim R_{d-1}(p) \leq \dim W(p)_{d-1} - 2 = \binom{d+1}{2} - 2 = \frac{d^2+d-4}{2}$.

By Equation (2.4) for $k = d$ we have instead:

(1) $p \in U$ if and only if $\dim R_d(p) = \dim W(p)_d - 1 = \binom{d+2}{2} - 1 = \frac{d(d+3)}{2}$;
(2) $p \in C'$ if and only if $\dim R_d(p) = \dim W(p)_d - 3 = \binom{d+2}{2} - 3 = \frac{(d-1)(d+4)}{2}$;
(3) $p \in S$ if and only if $\dim R_d(p) \leq \dim W(p)_d - 6 = \binom{d+2}{2} - 6 = \frac{d^2+3d-10}{2}$.

Limits of conic linear systems. We start by a definition analogous to Equation (3.3).

Definition 4.3. Choose a point $p \in C$ and a line $\ell \neq t_p$ passing through p. We call $R_k^\ell(p)$ the limit of $R_k(p_t)$, in the appropriate Grassmannian, where p_t goes to p along ℓ.

By the same reasoning as in Equation (3.2), we have that the limits $R_k^\ell(p)$ are well-defined. In this section we aim to describing them.

Lemma 4.4. *There is an inclusion $R_k(p) \subseteq R_k^\ell(p)$.*

Proof. Choose coordinates as in Setting 3.8. Fix a cone of degree k given by a polynomial $r(x, y, z) \in W(p)_k$. If we consider $r(x + tw, y, z) \in W(p_t)_k$, for $t \in \mathbb{C}$, we have a family of cones with vertex p_t which has this $r(x, y, z)$ as limit. Therefore we have that $\phi_k(W(p)_k) \subseteq \lim \phi_k(W(p_t)_k)$. □

Remark 4.5. In particular, if $k < d-1$ and $p \in C'$ we have $R_k(p) = R_k^\ell(p)$, because they have the same dimension.

We now deal with the case $k = d - 1$ and $p \in C'$.

Proposition 4.6. *Let $\ell \subset \mathbb{P}^3$ be a line and passing through p, such that $\ell \neq t_p$.*

1. *Assume that p is in C' and ℓ is not bisecant to C. By choosing the coordinates as in Setting 3.8 we have:*
$$R^\ell_{d-1}(p) = \left\langle R_{d-1}(p), w\frac{\partial f}{\partial x} \right\rangle,$$
where $f := f_p$ is the equation of the degree $d-1$ cone $\widetilde{C}_p$ (see Definition 2.5).
2. *For any $p \in C$, if $u \in R^\ell_{d-1}(p)$, then $\nu_p(u) \geq d - 2$.*

Proof. Let us prove (1). We have
$$f(x + tw, y, z) - f(x, y, z) = tw\frac{\partial f}{\partial x}(x, y, z) \mod t^2$$
and then, restricting to C,
$$\lim_{t\to 0} \frac{f(x + tw, y, z)}{t} = w\frac{\partial f}{\partial x}(x, y, z) \in R^\ell_{d-1}(p).$$
By the generality assumption on ℓ, we can suppose that $\frac{\partial f}{\partial x}|_{(0:0:1)} \neq 0$. Then we get:
$$\nu_p\left(w\frac{\partial f}{\partial x}\right) = \nu_p\left(\frac{\partial f}{\partial x}\right) = d - 2, \tag{4.7}$$
which implies that $w\frac{\partial f}{\partial x}(x, y, z) \in R^\ell_{d-1}(p) \smallsetminus R_{d-1}(p)$. Since $\dim R^\ell_{d-1}(p) = \dim R_{d-1}(p)+ 1$, we have
$$R^\ell_{d-1}(p) = \left\langle R_{d-1}(p), w\frac{\partial f}{\partial x} \right\rangle,$$
and the proof of (1) is concluded.

Let us prove (2). By Equation (4.2) and Equation (4.4), for any $p \in C'$ and for any $\ell \neq t_p$, there exists a well defined limit $R^\ell_{d-1}(p) = \langle R_{d-1}, l\rangle$, for some $l \in S_{d-1}$. By continuity, any polynomial $r \in R^\ell_{d-1}(p)$ must vanish of order at least $d - 2$ at p: indeed, the polynomials in $R_{d-1}(p)$ vanish of order at least $d - 1$, and by (4.7) we can assume that $\nu_p(l) \geq d - 2$. So, the divisor associated with r has the form $(d - 2)p + D$, with D effective and
$$\deg D = d(d - 1) - (d - 2) = (d - 1)^2 + 1.$$
By continuity, the same holds for any $p \in C$. □

Next, we turn to the case $k = d$.

Proposition 4.8. *Let $\ell \subset \mathbb{P}^3$ be a line and passing through p, such that $\ell \neq t_p$.*

(1) Assume that p is in C' and that ℓ is not bisecant to C. By choosing the coordinates as in Setting 3.8 we have:

$$R_d^\ell(p) = \left\langle R_d(p), wz\frac{\partial f}{\partial x}, \xi \right\rangle,$$

where $\nu_p(\xi) \geq d-2$.

(2) For any $p \in C$, if $u \in R_d^\ell(p)$, then $\nu_p(u) \geq d-2$.

Proof. Let us prove (1). As observed in Equation (4.2), for any $p \in C'$ and any line ℓ as above,

$$\dim R_d^\ell(p) = \dim \phi_d(W(p)_d) = \binom{d+2}{2} - 3 = \dim R_{d-1}(p) - 2.$$

So we have to find two more independent limit sections that generate $R_d^\ell(p)$ together with $R_{d-1}(p)$. Recall that for the polynomials $f_t \in W(p_t)_d \cap I(d)$ it holds by Equation (3.4)

$$\lim_{t\to 0} f_t = yf,$$

where f is the equation of the cone over C of degree $d-1$ with vertex in p. Consider $F := zf$, which is a form of degree d vanishing on C. We have

$$\begin{aligned} F(x+tw, y, z) - F(x,y,z) &= tw\left(\frac{\partial F}{\partial x}(x,y,z)\right) \mod t^2 = \\ &= twz\frac{\partial f}{\partial x} \qquad\qquad \mod t^2. \end{aligned}$$

By the generality assumption on ℓ we have $\frac{\partial f}{\partial x}(0,0,1) \neq 0$. So we obtain

$$\nu_p\left(zw\frac{\partial f}{\partial x}\right) = \nu_p(z) + \nu_p\left(\frac{\partial f}{\partial x}\right) = d-1.$$

Thus we have that $zw\frac{\partial f}{\partial x} \notin R_d(p)$, because the elements in $R_d(p)$ have vanishing of order d in p.

Take now $F := xf$. By taking derivatives, on C we have

$$\frac{\partial F}{\partial x} = f + x\frac{\partial f}{\partial x} = x\frac{\partial f}{\partial x}.$$

As before, $xw\frac{\partial f}{\partial x} \in R_d^\ell(p)$; we have two possibilities:

(i) $xw\frac{\partial f}{\partial x} \notin \langle R_d(p), zw\frac{\partial f}{\partial x}\rangle$. Then we immediately have the desired conclusion, since in this case

$$R_d^\ell(p) = \left\langle R_d(p), zw\frac{\partial f}{\partial x}, xw\frac{\partial f}{\partial x} \right\rangle,$$

so we can take $\xi := xw\frac{\partial f}{\partial x}$. From Remark 3.9 and Proposition 4.6 we have

$$\nu_p(\xi) = \nu_p(x) + \nu_p\left(w\frac{\partial f}{\partial x}\right) \geq d.$$

(ii) There is an $a \in \mathbb{C}$ such that $(x+az)w\frac{\partial f}{\partial x} \in R_d(p)$. Note that in this case there exists a $q(x,y,z)$ of degree d such that

$$(x+az)w\frac{\partial f}{\partial x}(x,y,z) - q(x,y,z) \in I(d).$$

Observe that

$$\nu_p\left((x+az)w\frac{\partial f}{\partial x}(x,y,z)\right) \geq d.$$

If $a \neq 0$

$$\nu_p\left((x+az)w\frac{\partial f}{\partial x}(x,y,z)\right) = \nu_p\left(zw\frac{\partial f}{\partial x}(x,y,z)\right) = d-1,$$

a contradiction. The only possibility left is $a = 0$ and, as observed above,

$$xw\frac{\partial f}{\partial x}(x,y,z) - q(x,y,z) \in I(d).$$

In this case we compute the Taylor expansion near $t = 0$ of the auxiliary function

$$G(x,y,z,t) = (x+tw)f(x+tw,y,z) - tq(x+tw,y,z),$$

and restrict it to the curve C. Let us start by the first derivative of G:

$$\begin{aligned}\frac{\partial G}{\partial t}(x,y,z,t) = wf(x+tw,y,z) + (x+tw)w\frac{\partial f}{\partial x}(x+tw,y,z)+ \\ - q(x+tw,y,z) - tw\frac{\partial q}{\partial x}(x+tw,y,z).\end{aligned} \tag{4.9}$$

When evaluated in $t = 0$, (4.9) becomes

$$wf(x,y,z) + xw\frac{\partial f}{\partial x}(x,y,z) - q(x,y,z),$$

and completely vanishes on C by our hypotheses. Let us compute the second derivative in $t = 0$:

$$\xi := \left.\frac{\partial^2 G}{\partial t^2}\right|_{t=0} = 2w^2\frac{\partial f}{\partial x}(x,y,z) + xw^2\frac{\partial^2 f}{\partial x^2}(x,y,z) - 2w\frac{\partial q}{\partial x}(x,y,z).$$

This becomes the only nontrivial part of the Taylor expansion of G on the curve C near $t = 0$ mod t^3. As a consequence, we have $\xi(x,y,z,w) \in R_d^\ell(p)$. We now take the evaluation at p to show that ξ is independent of the other parts:

$$\nu_p\left(2w^2\frac{\partial f}{\partial x}\right) = d-2, \quad \nu_p\left(xw^2\frac{\partial^2 f}{\partial x^2}\right) \geq d-1, \quad \nu_p\left(w\frac{\partial q}{\partial x}\right) \geq d-1.$$

It follows that $\nu_p(\xi) = d-2$, hence

$$R_d^\ell(p) = \left\langle R_d(p), wz\frac{\partial f}{\partial x}, \xi\right\rangle,$$

as wanted.

Let us now prove part (2). As in the proof of Equation (4.6), we can specialize by continuity to any $p \in C$ and any $\ell \neq t_p$; in particular, any limit divisor in $R^{\ell}_d(p)$ has vanishing order at least $d-2$ at the vertex p. □

5. The cone map and its differential

Definition of the cone map. For a vector space A and an integer $s \leq \dim A$ we let $G(s, A)$ be the Grassmannian of the s-vector spaces of A.

Definition 5.1. Let $k \leq d$ and take $p \in U$. The correspondence

$$p \mapsto R_k(p) \subset S_k$$

defines a morphism:

$$\rho_k \colon U \longrightarrow G(n_k, S_k), \tag{5.2}$$

where $n_k := \dim R_k(p)$. We call this map *the k-th cone map.*

By Equation (4.2) we have that $n_k = \binom{k+2}{2}$ for $k < d$, and that $n_d = \binom{d+2}{2} - 1$.

Remark 5.3. The cone map associates to any $p \in U$ the conic linear system which is the image $R_k(p)$ of $W(p)_k$ in S_k. Observe that in case $k = d$, we have that $R_d(p)$ is isomorphic to $W(p)_d/\langle f_p \rangle$, where f_p is as usual an equation (unique up to $\mathbb{C}^*$) of the cone over C with vertex in p.

In Section 4 we studied the limits of the conic linear series, i.e. the limits of the images of the map $\rho_k \colon U \to G(n_k, S_k)$, defined above, where $k \leq d$. Via these results it is natural to partially extend ρ_k to the blow up of $\mathbb{P}^3$ along C:

$$\widetilde{\rho_k} \colon \widetilde{\mathbb{P}^3} \smallsetminus \nu^{-1}(S) \longrightarrow G(n_k, S_k), \tag{5.4}$$

as follows. First observe that $\widetilde{\mathbb{P}^3} \smallsetminus \nu^{-1}(S) = \nu^{-1}(U) \cup \mathbb{P}(N_{C'|\mathbb{P}^3})$. If we consider a point in the projectivization of the normal bundle $\mathbb{P}(N_{C'|\mathbb{P}^3})$, it corresponds to a point $p \in C'$ and a line $\ell \neq t_p$. We then define the image of this point via $\widetilde{\rho_k}$ as the limit $R^{\ell}_k(p)$.

By Equation (4.5), the morphism ρ_k for $k < d-1$ extends trivially to $\widetilde{\mathbb{P}^3} \smallsetminus \nu^{-1}(S \cap C)$.

Proposition 5.5. *The morphism ρ_d is generically injective for $d \geq 4$. Moreover, for any $d \geq 3$ and any $p \in U$ and $q \in \widetilde{\mathbb{P}^3} \smallsetminus \nu^{-1}(U)$, we have that $\widetilde{\rho_d}(p) \neq \widetilde{\rho_d}(q)$.*

Proof. Assume that p and q are distinct points of U and $\rho_d(p) = \rho_d(q)$. As a consequence we have that $R_d(p) = R_d(q)$, i.e. the linear systems cut out by $W(p)_d$ and $W(q)_d$ on C are the same. In particular, the points that are identified by the morphisms $\pi_p \colon C \to C_p \subset \mathbb{P}(W(p)^*)$ and $\pi_q \colon C \to C_q \subset \mathbb{P}(W(q)^*)$ are the same. Now, two points of C are identified by π_p if and only if there exists a line r passing through p intersecting C in two distinct points. If $\rho_d(p) = \rho_d(q)$ necessarily π_q identifies these points, so q belongs to r. For $d \geq 4$ and p general, there exist at least two bisecants r, s passing through p, because the image curve is nodal and has at least two nodes. So we have that $q \in r \cap s = \{p\}$.

Let us now come to the second statement. Fix $p \in U$ and $q \in \widetilde{\mathbb{P}^3} \smallsetminus \nu^{-1}(U) \subseteq \mathbb{P}(N_{C|\mathbb{P}^3})$, corresponding to a point $q' \in C$ and a plane $H = \langle t_{q'}, \ell \rangle$, for some $\ell \neq t_{q'}$.

As proved in Equation (4.8) any element of $R^{\ell}_d(q')$ has a base point in q'. On the other hand, there exists an element in $W(p)_d$ not containing q', so $R_d(p)$ has an element without q' as a base point, and the proof is concluded. $\square$

Remark 5.6. With the same arguments as in Equation (5.5) it is easy to prove that $\widetilde{\rho}_d$ separates points in $\mathbb{P}(N_{C'|\mathbb{P}^3})$ corresponding to different points of C'. It would be interesting, but not necessary for this work, to prove injectivity on the whole fibres of the normal bundle.

In the first part of the proof of Equation (5.5) we proved the general injectivity of ρ_d for $d \geq 4$ by using that fact that there exist at least two bisecants passing from a general point of a curve in $\mathbb{P}^3$ of degree at least 4. This last statement is false for the rational normal cubic in $\mathbb{P}^3$. We have the following result:

Lemma 5.7. *The morphism ρ_3 is generically $3:1$.*

Proof. Fix a general point $p \in U$. From p there exists a unique bisecant to C, call it r. We have seen in the proof of Equation (5.5) that any point $q \in U$ such that $\rho_3(p) = \rho_3(q)$ has to lie on r. Now we see that there are three points in the preimage $\rho_3^{-1}(\rho_3(p))$. Let us fix the coordinates so that $p = (1 : 0 : 0 : -1)$, and consider $p' := (a : 0 : 0 : -1)$, $p'' = (a^2 : 0 : 0 : -1)$, where a is a primitive cubic root of 1. We have that $\rho_3(p) = \rho_3(p') = \rho_3(p'')$. Indeed, observe first that

$$W(p) = \langle y, z, x + w\rangle,\ W(p') = \langle y, z, x + aw\rangle,\ W(p'') = \langle y, z, x + a^2 w\rangle.$$

Let us consider the affine open set $w = 1$; if we substitute $x = t^3, y = t^2, z = t, w = 1$ in the generators of $W(p)_3$, $W(p')_3$ and $W(p'')_3$ respectively, we obtain in any case

$$R_3(p) = R_3(p') = R_3(p'') = \langle t, t^2, t^3, \ldots, t^8, t^9 + 1\rangle.$$

If we consider $q = (b : 0 : 0 : -1)$, with $b^3 \neq 1$, it is easy to see that over C we obtain

$$R_3(q) = \langle t, t^2, t^3, \ldots, t^8, t^9 + b^3\rangle \neq R_3(p).$$

So the proof is concluded. $\square$

Differential of the cone map. We now turn to the study of the the differential of $\rho := \rho_d$. We first give a standard definition.

Definition 5.8. Let D be any effective divisor on C. We set

$$V(-D) := \{s \in V : (s) \geq D\},$$

i.e. the subspace of the section vanishing at D. Accordingly, for any $W \subseteq V$, we set $W(-D) := V(-D) \cap W$. Note that $V(-p) = W(p)$ for any $p \in \mathbb{P}^3$.

We will now use an explicit setting which is analogous to Equation (3.8), but for $p \in U$.

Setting 5.9.

(1) $p = (0 : 0 : 0 : 1)$, a fixed point in U;
(2) $\ell = \{y = z = 0\}$ the implicit equations of the line along which we deform;
(3) $p_t = (-t : 0 : 0 : 1)$, the corresponding linear deformation of p;
(4) $W(p_t) = \langle x + tw, y, z\rangle$ the corresponding deformation of $W(p) = \langle x, y, z\rangle$.

Theorem 5.10. *The differential of ρ_d is injective over U.*

Proof. Let $p \in U$. We make the coordinate choice as in Equation (5.9), and consider the tangent vector X as in Equation (3.7). We have

$$d_p\rho(X) \in \mathrm{Hom}(R_d(p), S_d/R_d(p)) = T_{\rho(p)}G(n_d, S_d).$$

We assume $d_p\rho(X) = 0$ and show that $X = 0$. By contradiction suppose that $X \neq 0$. Let us consider $A(x, y, z) \in W(p)_d$, so that $A(x + tw, y, z) \in W(p_t)_d$. By taking the Taylor expansion as in Subsection 3.1 we get

$$A(x + tw, y, z) = A(x, y, z) + twX(A) \mod t^2. \tag{5.11}$$

Clearly for any $G(x, y, z)$ of degree $d - 1$ we can find A such that $X(A) = G$. In particular from Equation (5.11) we have that $wG \in R_d(p)$ and therefore

$$\phi_d(wW(p)_{d-1}) \subseteq R_d(p).$$

This means that $wW(p)_{d-1} \subseteq (W(p)_d + I(d))$, and we now show that this inclusion cannot happen. Let us fix two points $q_1, q_2 \in C \cap \{z = 0\}$ such that

$$q_1 \in \{x = 0\},\ q_2 \in \{y = 0\},\ \text{but } q_1 \notin \{y = 0\},\ q_2 \notin \{x = 0\}.$$

We can choose the coordinate w such that

$$q_1 \in \{w = 0\},\ q_2 \notin \{w = 0\},$$

and fix a third point $q_3 \in C \cap \{z = 0\}$ such that

$$q_3 \in \{w = 0\},\ q_3 \notin \{x = 0\} \cup \{y = 0\}.$$

In other words, the points are in the following configuration in the plane $z = 0$:

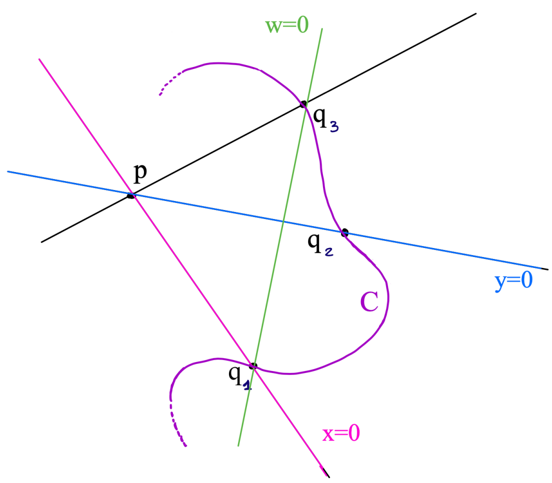

FIGURE 1. Configuration of points in $z = 0$

By what observed above, we have that

$$wx^{d-1} = s(x, y, z) + g(x, y, z, w),$$

where $s \in \mathrm{Sym}^d\langle x, y, z\rangle$ and $g \in I(d)$. Now, observe that wx^{d-1} vanishes on q_1, and hence $s(q_1) = 0$. This implies that s does not contain the monomial y^d, i.e. it is of the form:

$$s(x, y, z) = \sum_{i=0}^{i<d} p_{d-i}(x, z)y^i,$$

where $p_k(x, z)$ is a homogeneous polynomial of degree k in x and z. On the other hand, wx^{d-1} vanishes of order $d-1$ on p, while s vanishes of order d on p, so we get the desired contradiction. □

The key point of the above argument is that for $p \in U$

$$wW(p)_{d-1} \not\subseteq (W(p)_d + I(d)).$$

An equivalent way of phrasing this result is by stating that $\dim(\Gamma(p)) > 0$, where we set

$$\Gamma(p) := \frac{wW(p)_{d-1}}{(W(p)_d + I(d)) \cap wW(p)_{d-1}}.$$

By assuming p to be general and d to be large enough, we can prove a stronger result, which will be needed in Section 6.

Proposition 5.12. *With the notations above, let $p \in U$ be a general point. Then we have:*

(1) if $d > 3$ then $\dim(\Gamma(p)) > 1$,

(2) if $d > 4$ or $d = 4$ and $g = 0$ then $\dim(\Gamma(p)) > 2$.

In case (2) we find a base of $W(p) = \langle x_1, x_2, x_3\rangle \subset \langle x_1, x_2, x_3, x_4\rangle = V$ such that

$$x_4x_1^{d-1}, x_4x_2^{d-1}, x_4x_3^{d-1}$$

are independent $\mod (I(d) + W(p)_d)$.

Proof. Assume $d > 3$; then for $p \in U$ general there are two distinct bisecant lines r_i, for $i = 1, 2$, that intersect C in $p_i + q_i$. Then we can write $W := W(p) = \langle x, y, z\rangle$ such that $z \in W(-(p_1 + p_2 + q_1 + q_2))$, $x \in W(-p_1 - q_1)$ and $y \in W(-p_2 - q_2)$. Finally, we take $w \in V(-(p_1 + q_1))$.

We will show that wx^{d-1} and wy^{d-1} are independent in

$$\frac{wW_{d-1}}{(I(d) + W_d) \cap wW_{d-1}},$$

and this will imply $dim(\Gamma(p)) > 1$. Assume by contradiction that

$$w(ax^{d-1} + by^{d-1}) = g(x, y, z) \mod I(d),$$

where $g \in W_{d-1}$. Since the left member vanishes on p_1 and q_1, the polynomial g cannot have the terms in x^d and y^d, and we can write $g(x, y, z) = zs(x, y, z) + xyt(x, y)$ and

$$w(ax^{d-1} + by^{d-1}) = zs(x, y, z) + xyt(x, y) = g(x, y, z) \mod I(d).$$

Therefore g must vanish on q_1 and q_2. This implies that $ax^{d-1} + by^{d-1}$ vanishes on q_1 and q_2 forcing $a = b = 0$.

Now we consider the case $d > 4$ or $d = 4$ and $g = 0$. For the general point $p \in U$ there are 3 bisecant lines r_i, for $i = 1, 2, 3$ passing through p not lying on the same plane. Indeed, the image curve $C_p \subset \mathbb{P}(W(p)^*)$ has at least $(d-1)(d-2)/2 - g$ nodes. Under our assumptions these are more than 3 nodes. Now by Castelnuovo's bound [13, Chap. 2, Sec 3]

$$\frac{(d-1)(d-2)}{2} - g \geq \frac{(d-1)(d-2)}{2} - \frac{(d-1)(d-3)}{4} = \frac{(d-1)(3d-5)}{4}.$$

These nodes can not be all contained in the same line in $\mathbb{P}(W(p)^*)$. Indeed, if this was the case, the plane in $\mathbb{P}^3$ corresponding to this line would intersect the curve in at least $(d-1)(3d-5)/2 > d$ points, a contradiction. So, let $p_i + q_i = r_i \cap C$ and take x_i defined by $r_j \cup r_k$ where $\{i,j,k\} = \{1,2,3\}$. Take x_4 such that $x_4(p_i) = 0$ for $i = 1,2,3$. Then $W(p) = \langle x_1, x_2, x_3 \rangle$. If we suppose as before that

$$\sum_{i=1}^{3} a_i x_4 x_i^{d-1} = g(x_1, x_2, x_3) \text{ on } C,$$

then the coefficients of the terms x_i^d in g are necessarily 0. Therefore we have

$$g(x_1, x_2, x_3) = x_1 x_2 s_3(x_1, x_2, x_3) + x_2 x_3 s_1(x_1, x_2, x_3) + x_1 x_3 s_2(x_1, x_2, x_3).$$

Then g vanishes on q_i for all $i = 1,2,3$; this implies that necessarily $a_i = 0$ for all $i = 1,2,3$. $\square$

6. Structural results on conic linear series

We now develop a machinery to describe families of conic divisors on a non-degenerate smooth curve $C \subset \mathbb{P}^3$ of degree d. In particular, we will prove that the general degree d^2 divisor on C is conic if and only if $d \leq 4$.

We begin by defining the space parametrizing cones in $\mathbb{P}^3$ of degree d which have vertex in U:

$$Y_d := \bigcup_{p \in U} \mathbb{P}(W(p)_d) \subset \mathbb{P}V_d.$$

Recall that $V = V_1 = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and for any $k \geq 1$, $V_k = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$, and that we denoted by $W(p)$ the sections of V vanishing at p, see Section 2.

Consider the tautological sequence of $\mathcal{O}_U(1)$ over U

$$0 \longrightarrow \mathcal{S} \longrightarrow V \otimes \mathcal{O}_U \longrightarrow \mathcal{O}_U(1) \longrightarrow 0. \tag{6.1}$$

The stalk of $\mathcal{S}$ over p is $W(p)$. By taking the the d-th symmetric product and the projectivization over U of (6.1), we see that Y_d is the image of the induced natural map τ_d:

$$\begin{tikzcd} \mathbb{P}_U(\mathrm{Sym}^d(\mathcal{S})) \arrow[rr, "\tau_d"] \arrow[dr, two heads] & & \mathbb{P}(V_d) \\ & Y_d \arrow[ur, hook] & \end{tikzcd}$$

Clearly τ_d is birational onto its image Y_d, and in particular we get:

$$\dim Y_d = \dim \mathbb{P}_U(\mathrm{Sym}^d(\mathcal{S})) = \binom{d+2}{2} + 2. \tag{6.2}$$

Moreover, for any $p \in U$ we have that $W(p)_d \cap I(d)$ is a line in $W(p)_d$ and so it defines a line sub-bundle of $\mathrm{Sym}^d(\mathcal{S})$ over U:

$$0 \to \mathcal{L} \to \mathrm{Sym}^d(\mathcal{S}).$$

Let $\mathcal{E} := \mathrm{Sym}^d(\mathcal{S})/\mathcal{L}$ be the quotient bundle. Observe that

$$\dim \mathbb{P}_U(\mathcal{E}) = \binom{d+2}{2} + 1. \tag{6.3}$$

Remark 6.4. Note that $\mathcal{E}$ could also be defined as the pullback via ρ_d (Equation (5.1)) of the tautological bundle $\mathcal{P}$ on the Grassmannian: $\mathcal{E} = \rho_d^* \mathcal{P}$. Indeed, the fibre of $\rho_d^* \mathcal{P}$ over $p \in U$ is

$$\frac{W(p)_d}{W(p)_d \cap I(d)} \cong R_d(p)$$

by Equation (5.3). We will use this point of view in Equation (6.11) to extend $\mathcal{E}$ to (a blow up of) $\widetilde{\mathbb{P}}^3$.

We have an injection $j \colon U \to Y_d$ sending $p \mapsto [f_p]$, where f_p is as usual an equation of the cone over C with vertex in p. We set

$$\mathcal{V} := Y_d \smallsetminus j(U) \subset \mathbb{P}(V_d). \tag{6.5}$$

In other words, $\mathcal{V}$ is the locus of cones in $\mathbb{P}(V_d)$ which have vertices in U and are not cones over C. As $j(U)$ is closed in Y_d we have that $\mathcal{V}$ is an open subset of Y_d and that $\dim \mathcal{V} = \dim Y_d$.

If $[f] \in \mathcal{V}$, the restriction to C gives a well-defined morphism

$$\Psi \colon \mathcal{V} \longrightarrow \mathbb{P}(S_d), \tag{6.6}$$

where we recall that $S_d := H^0(C, \mathcal{O}_C(d))$. The image of Ψ consists of the conic divisors of degree d^2 over C with vertex in U.

We remark that, if $[g] \in \mathbb{P}(W(p)_d) \smallsetminus \{[f_p]\}$, then Ψ is constant on the line generated by $[g]$ and $[f_p]$, since f_p vanishes on C. This implies that the rational map $\Psi \circ \tau_d \colon \mathbb{P}_U(\mathrm{Sym}^d(\mathcal{S})) \dashrightarrow \mathbb{P}(S_d)$ factorizes to give a morphism $\Phi : \mathbb{P}_U(\mathcal{E}) \longrightarrow \mathbb{P}(S_d)$. We summarise all this in the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}_U(\mathrm{Sym}^d(\mathcal{S})) \arrow[r, two heads] \arrow[rr, bend left, "\tau_d"] \arrow[dd, two heads] \arrow[drr, dashed, bend right] & Y_d \arrow[r, hook] & \mathbb{P}(V_d) \\
& \mathcal{V} \arrow[u, hook] \arrow[r, "\Psi"] & \mathbb{P}(S_d) \\
\mathbb{P}_U(\mathcal{E}) \arrow[urr, bend right, "\Phi"'] & &
\end{tikzcd} \tag{6.7}$$

We want to understand when Φ and Ψ are dominant. To this aim, we study the differential of Φ.

Theorem 6.8. *The differential of Φ is generically of maximal rank. More precisely, for the general point $y \in \mathbb{P}_U(\mathcal{E})$, if we let*

$$m := \dim(\ker(d\Phi_y)),$$

we have that:

(1) if $d \geq 5$ or $d = 4$ and $g = 0$, then $m = 0$ ($d\Phi_y$ is generically injective);
(2) if $d = 4$ and $g = 1$ (i.e. C is the elliptic normal curve), then $m = 1$;
(3) if $d = 3$ and $g = 0$ (i.e. C is the rational normal curve), then $m = 2$.

Proof. The differential of the fibre bundle map $\pi : \mathbb{P}_U(\mathcal{E}) \to U$ gives the sequence

$$0 \longrightarrow T' \longrightarrow T_{\mathbb{P}_U(\mathcal{E})} \longrightarrow T_U \longrightarrow 0,$$

where T' is the tangent along the fibre of π and $T_U = T_{\mathbb{P}^3}|_U$. Fix a point $[g] \in \mathbb{P}_U(\mathcal{E})$, call $p := \pi([g]) \in U$ and assume $[g] \neq 0$, i.e. $g \notin \langle f_p \rangle$. We have

$$T_{U,p} \cong \mathrm{Hom}(W(p), V/W(p)).$$

Since $[g] \neq 0$, we have

$$T'_{[g]} \cong W(p)_d/\langle g, f_p \rangle.$$

We remark that any fibre of $\mathbb{P}_U(\mathcal{E})$ embeds in $\mathbb{P}(S_d)$ via Φ, so

$$d\Phi|_{T'_{[g]}} : T'_{[g]} \longrightarrow T_{\mathbb{P}(S_d),[g]} \cong S_d/\langle g \rangle$$

is injective. Recall that by Equation (4.1), $R_d := \phi_d(W(p)_d)$, we obtain

$$d\Phi(T'_{[g]}) = R_d/\langle g \rangle \subset S_d/\langle g \rangle.$$

We consider case (1): $d > 4$ or $d = 4$ and $g = 0$. We have to find a point $[g] \in \mathbb{P}_U(\mathcal{E})$ such that $d\Phi_{[g]}$ is injective: as we have already seen that Φ is injective on the fibres of π, we have to find three "horizontal" tangent vectors X_i in $[g]$ such that $d\Phi_{[g]}(X_i) \in S_d/\langle g \rangle$ are independent mod $d\Phi_{[g]}(T'_{[g]}) = R_d/\langle g \rangle$. This is equivalent to the $d\Phi_{[g]}(X_i)$'s being independent in S_d/R_d.

Arguing as in Proposition 5.12 (2), for the general point $p \in U$ there are 3 non-coplanar bisecants to C passing through p, which we call r_1, r_2, r_3. We can choose coordinates x_1, x_2, x_3 such that:

- $\langle r_j, r_k \rangle = \{x_i = 0\}$, where $\{i, j, k\} = \{1, 2, 3\}$.
- $W(p) = \langle x_1, x_2, x_3 \rangle$.

Let us name the intersection points as follows: $r_i \cap C := \{p_i, q_i\}$. Then we take x_4 such that $V = \langle x_1, x_2, x_3, x_4 \rangle$ and $x_4(p_i) = 0,\ 1 \leq i \leq 3$. Note that with this choices the p_i's are the three first coordinates points of $\mathbb{P}^3$.

We fix a $[g] \in \mathbb{P}_U(\mathcal{E})$ as follows:

$$g(x_1, x_2, x_3) = \frac{1}{d}(x_1^d + x_2^d + x_3^d).$$

Observe that $[g]$ is not zero, because $g(p_i) \neq 0$ for $i = 1, 2, 3$.

We define for any fixed triple of numbers $(a_1, a_2, a_3) \neq (0, 0, 0)$ the curve

$$\gamma(t) := [g(x_1 + a_1 t x_4, x_2 + a_2 t x_4, x_3 + a_3 t x_4)].$$

Note that $\pi(\gamma(t))$ is the point corresponding to the annihilator of the subspace $\langle x_1 + atx_4, x_2 + btx_4, x_3 + ctx_4\rangle$. Consider the tangent vector $X := \gamma'(0)$. We have

$$d\Phi_{[g]}(X) = \left[a_1x_4x_1^{d-1} + a_2x_4x_2^{d-1} + a_3x_4x_3^{d-1}\right] \mod \langle g\rangle.$$

By Equation (5.12) we have that

$$x_4x_1^{d-1}, x_4x_2^{d-1}, x_4x_3^{d-1}$$

are independent modulo $(I(d)+W_d)$. As a consequence, their classes are independent in $S_d \mod R_d$. By a suitable choice of the a_i's we can find three independent tangent vectors X_i such that $d\Phi_{[g]}(X_i) = x_4x_i^{d-1}$. Hence $\ker(d\Phi_{[g]})) = 0$.

The other cases can be proved similarly by using Equation (5.12) for the case $d = 4$ and $g = 1$, and Equation (5.10) for $d = 3$. This concludes the proof. □

Corollary 6.9. *The maps Φ and Ψ are dominant $\iff$ $d \leq 4$.*

Proof. It is clear from Diagram (6.7), Ψ is dominant if and only if Φ is.

Let us now consider the case when Φ has generically injective differential: using Equation (6.8) this is equivalent to $d \geq 5$ or $d = 4$ and $g = 0$. In this case, Φ is dominant if and only if $\dim(\mathbb{P}_U(\mathcal{E})) \geq \dim(\mathbb{P}(S_d))$. On one hand,

$$\dim(\mathbb{P}_U(\mathcal{E})) = \dim Y_d - 1 = \dim \mathcal{V} - 1 = \binom{d+2}{2} + 1,$$

where the last equality is from Equation (6.2). On the other hand, $\dim(\mathbb{P}(S_d)) = h^0(C, \mathcal{O}_C(d)) - 1 = d^2 - g$. It follows that

$$\dim(\mathbb{P}_U(\mathcal{E})) \geq \dim(\mathbb{P}(S_d)) \iff g \geq d^2 - \binom{d+2}{2} - 1 = \frac{d^2 - 3d - 4}{2}.$$

From Castelnuovo's bound we have: $g \leq \frac{(d-2)^2}{4}$. Putting these inequalities together we get that necessarily $d \leq 4$; so we are in the case $d = 4$, $g = 0$. In this case $\dim(\mathbb{P}_U(\mathcal{E})) = 16 = \dim(\mathbb{P}(S_d))$, so Φ is dominant in this case, as wanted.

The differential of Φ is surjective when $d \leq 4$, $g \geq 1$ if we verify that

$$\dim \mathcal{V} = \dim \mathbb{P}(S_d) + \dim \ker \Phi$$

in the three cases of Equation (6.8), showing that Φ is generically submersive. This is easily proved. For instance, if $d = 4$ and $g = 1$ we have $\dim \mathbb{P}_U(\mathcal{E}) = 16$, $\dim \mathbb{P}(S_d) = 15$, and $\dim \ker d\Phi = 1$. □

Remark 6.10. If $d \geq 5$ the conic divisors on C form a subvariety of dimension $\dim(\mathbb{P}_U(\mathcal{E})) = \binom{d+2}{2} + 1$ inside $\mathbb{P}(S_d)$.

We partially extend Φ to $\widetilde{\mathbb{P}^3}$, by using the results of the previous sections. Over U we defined the vector bundle $\mathcal{E} \to U$. By taking limits of conic linear series we can construct a vector bundle $\mathcal{E}'$ over $U' := \widetilde{\mathbb{P}^3} \smallsetminus \nu^{-1}(S)$, where S is, as usual, the set of non-birational projection points.

This vector bundle can also be constructed using the extension of the cone map constructed in Section 5:

$$\widetilde{\rho}_d \colon U' \longrightarrow G(n_d, S_d) :$$

by taking the pull-back of the tautological bundle over $G(n_d, S_d)$ as in Equation (6.4), we obtain $\mathcal{E}'$. Take the projectivization $\mathbb{P}_{U'}(\mathcal{E}')$ and extend Φ to a map

$$\Phi' \colon \mathbb{P}_{U'}(\mathcal{E}') \longrightarrow \mathbb{P}(S_d).$$

Notice that the set $\nu^{-1}(S \setminus C)$ is finite. Let $\tau \colon \widehat{\mathbb{P}}^3 \to G(n_d, S_d)$ be a suitable series of blow ups that extends $\widetilde{\rho}_d$ to $U'' := \widehat{\mathbb{P}}^3 \setminus (\tau \circ \nu)^{-1}(S \cap C)$:

$$\widehat{\rho}_d \colon U'' \longrightarrow G(n_d, S_d).$$

Definition 6.11. Define $\mathcal{E}''$ to be the extension of $\mathcal{E}'$ over U'' obtained via $\widehat{\rho}_d$, and Φ'' to be the corresponding extension of Φ':

$$\Phi'' \colon \mathbb{P}_{U''}(\mathcal{E}'') \longrightarrow \mathbb{P}(S_d). \tag{6.12}$$

Remark 6.13. Consider a point q of $C' \setminus S$: it is in the projectivization of the normal bundle $\mathbb{P}(N_{C'|\mathbb{P}^3})$ and so it corresponds to a point $q' \in C'$ and a line $\ell \neq t_{q'}$. The elements of the fibre $\mathbb{P}(\mathcal{E}') \otimes \mathbb{C}(q)$ are elements of the linear series $\mathbb{P}(R_d^{\ell}(q'))$. They can be cones over q', or some of the mysterious elements described in Equation (4.6) and Eq. (4.8), arising from the limit construction. The elements in the fibres over the points coming from the blow ups of $S \setminus C$ could be even more elusive.

Eventually we can prove the following result.

Corollary 6.14. *The following are equivalent:*

(i) $d \leq 4$;
(ii) the map Φ'' *is proper and surjective.*

Proof. We already proved in Equation (6.9) that (i) is equivalent to Φ being dominant. Now observe that in general if $d-1$ is prime, then we have that $S' = S$ (i.e. $S \cap C$ is empty). Indeed, assume by contradiction that p is a point in C such that the projection $\pi_p \colon C \to \mathbb{P}^2$ is non-birational. Let Γ the normalization of the image of π_p. We have the diagram

$$\begin{tikzcd} C \arrow[rr, "\pi_p"] \arrow[dr, "\alpha"'] & & \mathbb{P}^2 \\ & \Gamma \arrow[ur, "\beta"'] & \end{tikzcd}$$

where α is a covering of degree a and β is generically injective, and $\beta(\Gamma)$ has degree b in $\mathbb{P}^2$. We have that $ab = d-1$. By the contradiction assumption $a > 1$, and we have $d-1$ prime, so necessarily $a = d-1$ and $b = 1$, so Γ is a line in $\mathbb{P}^2$. This implies that $C \subset \mathbb{P}^3$ is degenerate, a contradiction. Therefore, the map Φ'' is proper any time $d-1$ is prime. In particular for $d = 3, 4$ the map Φ'' is proper and dominant, hence surjective. We have proven that $(i) \Rightarrow (ii)$. The converse implication is clear because if Φ'' is surjective then necessarily Φ is dominant. $\square$

Remark 6.15. Let p be any point on $E \setminus \nu^{-1}(S) \subset \widetilde{\mathbb{P}}^3$. By Equation (4.8), any divisor in the image via Φ' of the fibre of $\mathcal{E}'$ has vanishing on p of order at least $d-2$. This will be very important in the next section.

Corollary 6.16. *Let* $p \in C$. *If* $D \in \Phi'\big(\mathbb{P}(\mathcal{E}'_{|\mathbb{P}(N_p)})\big)$, *i.e.* D *belongs to* $\mathbb{P}(R_d^{\ell}(p))$, *for some line* $\ell \neq t_p$. *Then* $D = (d-2)p + R$ *where* R *is effective.*

Proof. By applying part (2) of Equation (4.8) it follows that $D-(d-2)p$ is still effective. This concludes the proof. □

Remark 6.17. For any point $p \in C'$ we let $\mathbb{P}(N_p) \cong \mathbb{P}^1$ be the projectivization of the normal bundle N_p. We have that the restriction

$$\mathcal{E}'_{|\mathbb{P}(N_p)} \to \mathbb{P}(N_p) \tag{6.18}$$

is a vector bundle. Making the point p vary in C' in Equation (6.18) we get a natural map coming from the bundle projection

$$\mu : \mathbb{P}_{U'}(\mathcal{E}') \smallsetminus \mathbb{P}_U(\mathcal{E}) \longrightarrow C. \tag{6.19}$$

7. An application: some special pencils of plane quartics

Take an elliptic curve C with origin O, define the embedding $j : C \to |4O| = \mathbb{P}^3$, so that $j(C)$ is the elliptic normal curve and O is an inflection point of $j(C)$. We identify C and $j(C)$. Recall that it is a complete intersection of two quadrics

$$C = \{Q_1 = Q_2 = 0\}.$$

Remark 7.1. In the proof of Equation (6.14) we have seen that the non-birational projection points S are not contained in C for any smooth plane curve of degree smaller than or equal to 4. In this case we can be more precise: S consists of the vertices of the four singular quadrics in the pencil

$$\lambda Q_1 + \mu Q_2, \qquad (\lambda : \mu) \in \mathbb{P}^1.$$

After a suitable coordinate choice, we have

$$Q_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2;\ Q_2 = a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2, \quad a_i \neq a_j, \iff i \neq j.$$

The singular quadrics are given by $(\lambda, \mu) \equiv (-a_i, 1)$ and the vertices are the coordinate points.

We have $\dim H^0(C, \mathcal{O}_C(4)) = 16$ and so $\mathbb{P}(S_4) \cong \mathbb{P}^{15}$. We recall that $\dim \mathcal{V} = 17$ (see Equation (6.2)), and that the maps $\Psi \colon \mathcal{V} \to \mathbb{P}(S_4)$ and $\Phi \colon \mathbb{P}_U(\mathcal{E}) \to \mathbb{P}(S_4)$ are both dominant.

Remark 7.2. The extension of Φ made in Equation (6.11) is in this case a surjective morphism

$$\Phi'' \colon \mathbb{P}_{\widehat{\mathbb{P}^3}}(\mathcal{E}'') \longrightarrow \mathbb{P}(S_4),$$

because U'' coincides with $\widehat{\mathbb{P}^3}$ by Equation (7.1). Thus we know that a general divisor in $\mathbb{P}(S_4)$ is a conic divisor and that for any effective divisor $D \in \mathbb{P}(S_4)$ the inverse image $\Phi''^{-1}(D)$ is non empty. From how we constructed Φ'' we are not a priori sure that we have a cone in $\Phi''^{-1}(D)$: see Equation (6.13).

Now we prove that *any* divisor D in $\mathbb{P}(S_4)$ is a conic divisor, that is $\Phi''^{-1}(D) \cap \mathbb{P}_U(\mathcal{E})$ is not empty. Before proving this result, let us establish this simple lemma:

Lemma 7.3. *Let X, Y, Z be smooth varieties, and $\varphi \colon X \to Y$ and $\mu \colon X \to Z$ be proper surjective morphisms such that for a general $y \in Y$ we have*

$$\dim(\mu(\varphi^{-1}(y))) = \dim \varphi^{-1}(y) > 0. \tag{7.4}$$

Then $\dim(\mu(\varphi^{-1}(y))) \geq 1$ holds for any $y \in Y$.

Proof. Let us assume by contradiction that there exists a point $\overline{y} \in Y$ such that $\dim(\mu(\varphi^{-1}(\overline{y}))) = 0$. So $\mu(\varphi^{-1}(\overline{y}))$ consists of a finite number of points. Consider an affine open subset $\mathcal{U} \subset Z$ which contains such points. The set $\phi(\mu^{-1}(\mathcal{U}))$ is open in Y and contains $\overline{y}$. For any $y \in \phi(\mu^{-1}(\mathcal{U}))$, then $\mu(\phi^{-1}(y))$ is a proper subvariety in the affine $\mathcal{U}$, so it has dimension zero, a contradiction. □

Theorem 7.5. *Fix an effective divisor $D \in \mathbb{P}(S_4)$ on C. There is a 1-dimensional variety $B \subset \mathbb{P}_U(\mathcal{E})$ such that:*

(i) the fibre projection of B over U is again 1-dimensional;

(ii) by calling $\widetilde{K}_t$ the cone in $\mathbb{P}^3$ corresponding to $t \in B$, we have that $\Phi''(\widetilde{K}_t) = \Phi(\widetilde{K}_t) = D$ for all $t \in B$.

Proof. Set $G := (\Phi'')^{-1}(D) \subset \mathbb{P}_{\widehat{\mathbb{P}^3}}(\mathcal{E}'')$. The set G is non-empty because Φ'' is surjective. Since $\dim \mathbb{P}_{\widehat{\mathbb{P}^3}}(\mathcal{E}'') = \dim \mathbb{P}(S_4) + 1$ by Equation (6.3), we have that $\dim G \geq 1$. We apply Equation (7.3) with $X := \mathbb{P}_{\widehat{\mathbb{P}^3}}(\mathcal{E}'')$, $Y := \mathbb{P}(S_4)$ and $Z := \mathbb{P}^3$. The map φ is Φ'' and the map μ is the restriction of the fibre projection of X composed with the blow down to $\mathbb{P}^3$. It is clear that for $p \in U$ the hypothesis in Equation (7.4) holds, so the dimension of $\mu(G)$ is greater or equal than 1. We want to prove that $\mu(G) \cap U \neq \emptyset$. Assume by contradiction that $\mu(G) \subset S \cup C$. Since $\dim \mu(G) \geq 1$, necessarily C must be contained in $\mu(G)$. However, by Equation (6.16) $\mu(G)$ must be contained in the support of D, a contradiction. So, there is at least a 1-dimensional complete subvariety $\overline{B}$ of G such that $\mu(\overline{B}) \cap U \neq \emptyset$. The corresponding $B = \overline{B} \cap \mathbb{P}_U(\mathcal{E})$ is the family we want. Properties (i) and (ii) are clear from the construction. □

Remark 7.6. Notice that Equation (7.5) can also be proven for the rational normal curve of degree 3 in $\mathbb{P}^3$.

In Equation (7.5) we proved the existence of cones in $\mathbb{P}_U(\mathcal{E})$, i.e. cones with vertex in the open U cutting on C any divisor $D \in \mathbb{P}(S_4)$. For what follows we need to study elements in the complement $\mathbb{P}_{U'}(\mathcal{E}') \smallsetminus \mathbb{P}_U(\mathcal{E})$, which correspond to a specific divisor D, under the map Φ''.

Lemma 7.7. *Fix $q \in C$ such that $16q \sim 16O$ and set $D := 16q$. Let $\pi_q \colon C \to \mathbb{P}^2$ be the linear projection and $G := (\Phi'')^{-1}(D) \subset \mathbb{P}_{\widehat{\mathbb{P}^3}}(\mathcal{E}'')$. Consider the set*

$$\Gamma := G \cap (\mathbb{P}_{U'}(\mathcal{E}') \smallsetminus \mathbb{P}_U(\mathcal{E}))$$

whose fibres parametrize the elements in $\mathbb{P}(R^\ell_p)$, with $p \in C$, $\ell \neq t_p$ containing p, which are $16q$ as divisors on C. Then Γ is a rational curve and every element of Γ is the class of a cone over a fixed quartic $\mathcal{Q} \subset \mathbb{P}(W(q)^*) \cong \mathbb{P}^2$ intersecting C in $16q$.

Proof. As in the proof of Equation (7.5), G is non-empty and $\dim G \geq 1$. Recall that $\mu \colon (\mathbb{P}_{U'}(\mathcal{E}') \smallsetminus \mathbb{P}_U(\mathcal{E})) \to C$ is the restriction of the fibre projection. Since D has support only on q, from Equation (6.16) we get $\Gamma \subseteq \mu^{-1}(q)$. Indeed, if K is an element of Γ, the divisor $\Phi'(K)$ belongs to $\mathbb{P}(R^\ell_4(p))$ for some p in C and some line ℓ with direction in $\mathbb{P}(N_p)$, so by Equation (6.16) is of the form $2p + R$, with R effective. On the other hand by assumption $\Phi'(K) = 16q$, so necessarily $p = q$.

The linear system $|4O - q|$ gives the linear projection $\pi_q \colon C \to \mathbb{P}(W(q)^*) \cong \mathbb{P}^2$, with image a smooth cubic $C_q = \pi_q(C)$. The map $\pi_q \colon C \to C_q$ is an isomorphism.

Let $\overline{O}$ be the image of O and $\overline{q}$ be the image of q. We have by assumption that $16\overline{q} \sim 16\overline{O}$ on C_q, and hence

$$4(4\overline{O} - \overline{q}) \sim 12\overline{q}.$$

This implies that we can find a quartic curve $\mathcal{Q} \subset \mathbb{P}(W(q)^*)$ intersecting C_q in $12\overline{q}$.

Now observe that the cone $K_{\mathcal{Q}} \subset \mathbb{P}^3$ corresponding to $\mathcal{Q}$ cuts the curve C in $16q$. Indeed, the elements in $\mathbb{P}(W(q)_4)$ have a vanishing of order 4 in q. So the class of $K_{\mathcal{Q}}$ is an element of $G \smallsetminus \mathbb{P}_U(\mathcal{E})$, and being a cone, its class also is an element of the linear system $\mathbb{P}(R_4(q))$. Now recall from Eq. (4.8) that $R_4(q)$ is contained in all the $R_4^\ell(q)$'s for all lines ℓ with directions parametrised by the exceptional divisor $\mathbb{P}(N_q)$. Hence the class of $K_{\mathcal{Q}}$ defines a (constant) section of

$$\mathbb{P}_{\widetilde{\mathbb{P}^3}}(\mathcal{E}'')_{|\mathbb{P}(N_q)} \to \mathbb{P}(N_q),$$

and therefore a rational curve $\Gamma \subseteq G \cap Z$. Now recall that the restriction of Φ'' to the fibre $\mathbb{P}(\mathcal{E}'' \otimes \mathbb{C}(q)) \cong \mathbb{P}(R_4^\ell(q'))$ to $\mathbb{P}(S_4)$ is injective, and so we have that $\Gamma = G \cap Z$. This concludes the proof. $\square$

We are now able to prove the main result of this section, which provides a degree 4 analogue of [20, Theorem 1.3].

Theorem 7.8. *There exists a pencil of quartics in $\mathbb{P}^2$ such that*

(1) the base locus is set-theoretically one point;
(2) all the quartics of the pencil are irreducible;
(3) the general element of the pencil is smooth;
(4) the pencil is non-isotrivial.

Proof. As above, we take an elliptic curve C with origin O, and define the embedding

$$j \colon C \longrightarrow |4O| = \mathbb{P}^3,$$

so that $j(C)$ is the elliptic normal curve and O is an inflection point of $j(C)$. For simplicity write $C = j(C)$. Fix a point $q \in C$ such that $16q \sim 16O$ but $8q \not\sim 8O$; so, q is a torsion point of order exactly 16. By applying Equation (7.5) to $D = 16q$, we find a 1-dimensional family of quartic cones, which we call $\widetilde{K}_t$, with $t \in B \subset \mathbb{P}_U(\mathcal{E})$, such that $\widetilde{K}_t \cdot C = 16q$, for any $t \in B$. Let $\widetilde{K}_t = \{k_t = 0\}$, so that $\Phi([k_t]) = 16q$. Fix $\widetilde{K} = \{k = 0\}$ one of this cones and let $p \in U$ be its vertex. Let $\Pi_p \colon \mathbb{P}^3 \smallsetminus \{p\} \to \mathbb{P}^2$ be the projection. We have that $\Pi_p(\widetilde{K} \setminus \{p\}) = K_p$ is a quartic and so is the projection $\Pi_p(C) = C_p$.

The pencil we want is defined by $\lambda k + \mu f_p = 0$ in $\mathbb{P}^2$. It satisfies (1) by construction, because set-theoretically $C_p \cap K_p = \{\Pi_p(q)\}$. Let $T_{\lambda,\mu} \sim \lambda K_p + \mu C_p$ be an element of this pencil.

We now prove (2). If this were not the case, then some $T_{\lambda,\mu}$ would be either the union of a line and a cubic or of two conics. In the first case we would have $4O \sim 4q$ and in the second case $8O \sim 8q$, both contradicting our assumption on q.

We now prove (3). Assume by contradiction that all the pencils constructed above have only singular members. Let us fix one of these pencils. It is generated by the projection from $p \in U$ of C and by the projection of a cone $K_{t'}$, with $t' \in B$, with vertex p. It is convenient to fix in this case a plane $\Pi \cong \mathbb{P}^2$ in $\mathbb{P}^3$ not containing p nor q, and consider the projections over this fixed frame.

By Bertini's theorem the only common singularity of the elements of the pencil must be at the point $\pi_p(q) = \overline{q}$. Then C_p is singular in $\overline{q}$ only if the line ℓ joining p and q is the tangent line to C at q. By construction, for all $t \in B$, all the cones $\widetilde{K}_t$ cut C in $16q$ and by the contradiction assumption they are all singular in $\overline{q}$. This means that ℓ is also contained in all the other cones $\widetilde{K}_t$, for any $t \in B$. In other words, the fibre projection of B over U is contained in $U \cap \ell$. Since B corresponds to a 1-dimensional family of quartic cones cutting C on $16q$ and all the pencils associated to any of these cones have all members singular, we conclude that the vertices of these cones must all lie on ℓ.

We now specialise the projection point p to q along ℓ. Call $t = 0$ the point of the closure of B corresponding to q. Consider the plane pencil $\mathcal{V}$ in Π generated by the curves obtained by projecting $\widetilde{K}_0^\ell$ and the limit cone $\widetilde{C}_0^\ell$ from q.

Call as usual $\pi_q \colon C \to \Pi$ the extension of the projection from q, and $C_q \subset \Pi$ the image curve, which is a plane smooth cubic isomorphic to C. Observe that C_q passes through $\overline{q}$ and has a simple tangent in this point. Indeed the osculating plane in $\mathbb{P}^3$ to C at q intersects C in the divisor $3q + q'$, with $q' \neq q$, because of the hypothesis $4q \not\sim 4O$.

By Equation (3.6), the projection of $\widetilde{C}_0^\ell$ is the reducible quartic $C_q \cup L$, where L is the tangent line to C_q at $\overline{q}$ (the projection of the osculating plane). The intersection divisor with C_q is thus $2\overline{q} + \pi_q(q')$.

Since we have taken a limit of K_t along B going to a point over C, we have that K_0^ℓ necessarily lies in the rational curve Γ that we found in Equation (7.7). The pencil $\mathcal{V}$ is thus generated by the quartic $\mathcal{Q}$ of Equation (7.7) and by $C_q \cup L$. The general element of $\mathcal{V}$ is necessarily reduced because $C_q \cup L$ is. Moreover, Equation (7.7) shows that the base locus of $\mathcal{V}$ is set-theoretically $\overline{q}$.

We know that $D \cdot (C_q + L) = 16\overline{q}$, and as a consequence $D \cdot C_q = 12\overline{q}$ and $D \cdot L = 4\overline{q}$. Therefore $D \cdot 3L = 12\overline{q}$ and so there is a pencil of cubics osculating D at $\overline{q}$ generated by C_q and $3L$. This pencil covers $\mathbb{P}^2$, so there exists a member of the pencil passing through $\overline{q}$ and another point q' of D. This is a contradiction. Note that this argument is essentially the well-known uniqueness of the osculating cubic to a quartic.

We now prove (4). To show that the pencil constructed is non-isotrivial, we remark that this pencil contains a reduced member whose geometric genus is 1 (the image of C itself). The stable reduction of this member necessarily is a singular curve, while by point (3) above the general member is smooth. □

Remark 7.9. Observe that for any degree d smooth curve in $\mathbb{P}^3$ there does *not* exist an irreducible curve $C' \subset \mathbb{P}^3$ of degree d' such that the intersection of C with C' consists of dd' points counted with multiplicity, see [11, 21, 15]. Instead, under the assumptions of Theorem 7.8, we have proved that there exists a cone over a smooth plane quartic that intersects C in p with multiplicity 16.

Remark 7.10. The properties of the members of the pencils studied in this section are related to the higher Weierstrass points, as studied for instance in [17]. There it is discussed an explicit construction of a smooth plane quartic curve $\mathcal{Q}$ that intersects a plane cubic at a point q such that $4(4O-q) \sim 12q$ but $8O$ is not linearly equivalent to $8q$.

8. Conclusions and open problems

The technique of using cones to study specific divisors on curves seems very promising. We conclude our work with a list of open problems highlighting potential directions for future research.

Problem 8.1. Study the map Φ for degree greater than 5 and for curves embedded in $\mathbb{P}^n$, with $n \geq 4$. As observed in Equation (6.10), the closure of the image of Φ is a subvariety of $\mathbb{P}(S_d)$ of codimension $d^2 - g - \left(\binom{d+2}{2} + 1\right) = \frac{(d-1)(d-4)}{2} - g$. It would be interesting to find geometric conditions under which a divisor belongs to this subvariety.

A precise description of the limits of the conic linear series in higher degree could be interesting:

Problem 8.2. Find results analogue to Equation (4.6) and Equation (4.8) for cones of degree $d+1$ and higher.

As for the results in Section 7, it would be interesting to generalize the setting to other curves which are not elliptic, for instance:

Problem 8.3. Construct a similar framework as in Section 7, starting with C a (possibly non smooth) rational curve.

Problem 8.4. The pencils constructed in Section 7 are degree 4 integrable foliations on the projective plane minus one point. We wonder if some of our results can be related to the study of foliations as done for instance in [1] by C. Alcántara and A. Zamora.

Finally, we remark that in the recent paper [10], W. Chen uses the pencil of cubics with one base points studied in [20] to give an example related to the hyperbolicity of the Hirzebruch surface $\mathbb{F}_1$ minus a curve B such that $(\mathbb{F}_1, B)$ is of log general type. This is related to the Lang conjecture for pairs of log general type $(\mathbb{P}^2, B)$ where B is a curve with at least three components, as treated for example in [6].

Problem 8.5. Investigate whether the generalization of [20] to quartics given in Equation (7.8) can be applied to problems related to the hyperbolicity of the complement of a curve in a rational surface.

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