Witness Set in Monotone Polygons: Exact and Approximate

Udvas Das [*] Binayak Dutta [†] Satyabrata Jana [‡] Debabrata Pal [§] Sasanka Roy [¶]

Abstract

Given a simple polygon $\mathcal{P}$ on n vertices, two points x and y within $\mathcal{P}$ are *visible* to each other if the line segment between x and y is contained in $\mathcal{P}$. The *visibility region* of a point x includes all points in $\mathcal{P}$ that are visible from x. A point set Q within a polygon $\mathcal{P}$ is said to be a *witness set* for $\mathcal{P}$ if each point in $\mathcal{P}$ is visible from at most one point from Q. In other terms, the visibility region of any two points $s, t \in Q$ are non-intersecting within $\mathcal{P}$. The problem of finding the largest size witness set in a given polygon was introduced by Amit et al. [Int. J. Comput. Geom. Appl. 2010]. Recently, Daescu et al. [Comput. Geom. 2019] gave a linear-time algorithm for this problem on monotone mountains, a proper subclass of monotone polygons. In this study, we contribute to this field by obtaining the largest witness set within both continuous and discrete models.

In the Witness Set (WS) problem, the input is a polygon $\mathcal{P}$, and the goal is to find a maximum-sized witness set in $\mathcal{P}$. In the Discrete Witness Set (DisWS) problem, one is given a finite set of points S alongside $\mathcal{P}$, and the task is to find a witness set $Q \subseteq S$ that maximizes $|Q|$. We investigate DisWS in simple polygons, but consider WS specifically for monotone polygons. Our main contribution is as follows: (1) a polynomial time algorithm for DisWS for general polygons and (2) the discretization of the WS problem for monotone polygons, leading to several algorithmic developments. Our approach only employs basic geometric primitives and circumvents the need for algebraic tools, unlike the methods used in simple polygon guarding by Abrahamsen et al.[J. ACM. 2022], Efrat and Sariel Har-Peled [Inf. Process. Lett. 2006]. This implies that the Witness Problem admits discretization for monotone polygon and conjecture that, unlike polygon guarding as shown by Bonnet and Miltzow [SoCG 2017], WS admits a finite discretization for general polygon. Our discretization, along with its properties, may independently interest those solving WS in general polygon contexts and polygon guarding scenarios like smooth or robust polygon guarding [Das et al., SoCG 2024, Dobbins et al., arXiv:1811.01177]. Specifically, we determine the following for WS when restricted to n-vertex monotone polygons with r reflex vertices, where the size of the maximum witness set is k.

- Generate a point set Q with size $r^{\mathcal{O}(k)} \cdot n$ such that Q contains an optimal solution.
- An exact algorithm running in time $r^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.
- A $(1+\epsilon)$-approximation algorithm with running time $r^{\mathcal{O}(1/\epsilon)} \cdot n^2$.

DisWS is equivalent to finding a maximum independent set (MIS) of visibility region intersection graphs, where each visibility region is a vertex and there exists an edge between a pair of vertices if and only if the corresponding visibility regions intersect. Thus, we obtain a polynomial-time algorithm for computing MIS in visibility region intersection graphs. This characterization motivates the study of which graph classes are subgraphs of visibility region intersection graphs. To corroborate our claim, we demonstrate that the visibility region intersection graph of a polygon can be classified as an outerstring graph. The string model we construct is consistent with the one outlined by Hengeveld and Miltzow [SoCG 2021]. For monotone polygons, these graphs turn into co-comparable graphs. This enables us to resolve the DisWS problem in polynomial time for general polygons, and with enhanced running time for monotone polygons.

[*] Indian Statistical Institute, Kolkata, India, udvas.das@gmail.com
[†] Christ University, Bangalore, India, binayak66@gmail.com
[‡] University of Warwick, UK, satyamtma@gmail.com.
[§] Indian Statistical Institute, Kolkata, India, debabratapal4521@gmail.com
[¶] Indian Statistical Institute, Kolkata, India, sasanka.ro@gmail.com

1 Introduction

Given a simple polygon $\mathcal{P}$, we say that two points $p, q \in \mathcal{P}$ are *visible to each other* if the line segment $\overline{pq}$ is contained in $\mathcal{P}$. A set of points $W \subseteq \mathcal{P}$ is said to be a *witness set* in $\mathcal{P}$ if every point $p \in \mathcal{P}$ is visible from at most one witness $w \in W$. The points of W are called *witnesses*. It may happen that some points in $\mathcal{P}$ are not visible from any witness of W. A witness set of $\mathcal{P}$ is optimal if it is a maximum cardinality witness set of $\mathcal{P}$. In our work, we consider the Witness Set problem: given a simple polygon $\mathcal{P}$, find a maximum-sized witness set in $\mathcal{P}$.

The concept of a Witness Set is intricately related to several well-known problems in computational geometry, as extensively explored in the literature. These include the Art Gallery [AMP10], Hidden Set [BKMP23], and Convex Cover [Abr21]. In Art Gallery problem, we are given a simple polygon $\mathcal{P}$ and our task is to find a minimum size point set (*guard set*) G in $\mathcal{P}$ such that every point $p \in \mathcal{P}$ is visible from at least one guard $g \in G$. Given a simple polygon $\mathcal{P}$, the Convex Cover problem seeks to cover $\mathcal{P}$ with the fewest convex polygons (*convex cover set*) that lie within $\mathcal{P}$ whereas the Hidden Set problem asks to place maximum number of points (*hidden set*) in $\mathcal{P}$ such that no pair of them is visible to each other. Let $\texttt{ws}(\mathcal{P}), \texttt{gs}(\mathcal{P}), \texttt{cc}(\mathcal{P}), \texttt{hs}(\mathcal{P})$ denote the size of a maximum witness set, minimum guard set, minimum convex cover set, and maximum hidden set, respectively. It is easy to observe that since no two witnesses can be visible from a single point of $\mathcal{P}$, the cardinality of a maximum witness set is a lower bound on the minimum guard set in a polygon. Similarly, as two witnesses cannot be visible from each other, the cardinality of a maximum witness set is a lower bound on the maximum hidden set in a polygon. At the same time, since there is at most one hidden point within any convex subset of $\mathcal{P}$, we must have the following basic inequalities: $\texttt{ws}(\mathcal{P}) \leq \texttt{gs}(\mathcal{P})$ as well as $\texttt{ws}(\mathcal{P}) \leq \texttt{hs}(\mathcal{P}) \leq \texttt{cc}(\mathcal{P})$.

Witness Set (WS)

Input: A polygon $\mathcal{P}$.
Task: Find a maximum sized set W of points in $\mathcal{P}$ such that for every pair of points in W their visibility regions inside $\mathcal{P}$ do not intersect.

The Witness Set problem was first introduced by Amit et al. [AMP10] in 2010. Although the authors in [AMP10] inadvertently referred to this problem as NP-hard, this claim appears to be folklore [1]. So the computational complexity of this problem still remains open. There are only very few results in the literature on this problem. Recently, in 2019, Daescu et al. [DFM$^+$19] studied this problem on monotone mountains, a proper subclass of monotone polygons, and gave a linear-time algorithm. An x-monotone polygon is uni-monotone if one of its two chains is a single horizontal segment. Monotone mountains are uni-monotone polygons in which the segment-chain is not necessarily horizontal. These

[1]Joseph S. B. Mitchell strongly believe that this problem should have a polytime solution and our polynomial time solution for DisWS indicates an assertion towards that belief.

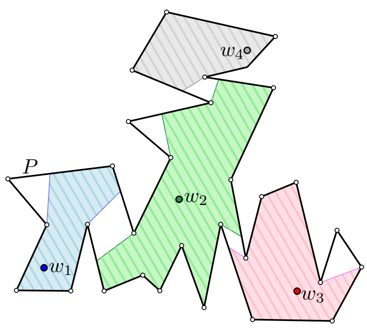


Figure 1: Example of a polygon having four witnesses w_1, w_2, w_3, w_4 with their visibility regions.

algorithmic results motivate us to look at the problems in other classes of polygons. We also explore this problem within discrete models, where we have given a set of points S with $\mathcal{P}$ as input, and the goal is to identify a witness set $W \subseteq S$ that maximizes $|W|$.

Discrete Witness Set (DisWS)

Input: A polygon $\mathcal{P}$, and a finite point set $S \subseteq \mathcal{P}$.
Task: Find a maximum sized $W \subseteq S$ such that for every pair of points in W their visibility regions inside $\mathcal{P}$ do not intersect.

In this work, we mainly focus on two kinds of problems.

Question 1. Can we solve the Discrete Witness Set in polynomial time?

We address Question 1 specifically for both simple polygons and monotone polygons and provide a positive answer. Following this, having solved Question 1, we proceed to explore a broader issue (discretization).

Question 2. Given a polygon $\mathcal{P}$, can we obtain a finite point set $S \subseteq \mathcal{P}$ in polynomial time such that there exists a solution W of the Witness Set in $\mathcal{P}$ satisfying $W \subseteq S$?

We focus primarily on Question 2 concerning monotone polygons and provide a positive answer. We hypothesize that this type of discretization is also applicable to a simple polygon. Consider a polygon $\mathcal{P}$. If (i) we resolve Question 2 in time $g(|S|, n)$, and (ii) we address Question 1 in time $f(|S|, n)$, then it follows that we can solve Witness Set in a total time of $f(|S|, n) + g(|S|, n)$. This forms the main theme of our work in this paper.

Motivations. We use MGS($\mathcal{P}$) to denote the problem of finding a minimum-sized guard set in a given polygon $\mathcal{P}$. There exists an algorithm for MGS($\mathcal{P}$) given by Efrat and Har-Peled [EH06] which is attributed to Micha Sharir. This algorithm uses heavy tools from algebraic geometry. Due to the recent breakthrough result by Abrahamsen et al. [AAM22] that the decision version of MGS($\mathcal{P}$) is $\exists\mathbb{R}$-complete, it is clear that discretization (similar to one that we do in our work) for the gsp might be possible to solve the MGS($\mathcal{P}$). As quoted in [AAM22], As a corollary of our construction, we prove that for any real algebraic number α, there is an instance of the Art Gallery problem where one of the coordinates of the guards equals α in any guard set of minimum cardinality. That rules out many natural geometric approaches to the problem, as it shows that any approach based on constructing a finite set of candidate points for placing guards has to include points with coordinates as roots of polynomials of arbitrary degree. Considering MGS($\mathcal{P}$), Bonnet and Miltzow [BM17] gave an $\mathcal{O}(\log(\texttt{opt}))$ factor approximate algorithm where opt is the size of the optimum solution. Their in-depth analysis of the problem led to the formulation of the necessary assumptions for polynomial time approximations for MGS($\mathcal{P}$). Furthermore, Bonnet and Miltzow [BM17] have pointed out that MGS($\mathcal{P}$)is challenging even in highly restricted settings, such as when a polygon requires only a constant number of guards. Indeed, the problem is known to be difficult for as few as two guards [Bel91]. We hope that our characterizations for solving Witness Set in monotone polygons would be able to bring new insights for MGS($\mathcal{P}$). Recently, Hengeveld and Miltzow [HM21] considered the problem of computing practical algorithms for computing MGS($\mathcal{P}$) , and our characterizations may help devise better practical heuristic algorithms. We hope that if we look for MGS($\mathcal{P}$) through the lens of our algorithm for Witness Set in monotone polygons, then we may be able to devise approximation algorithms that need fewer assumptions than the one by Bonnet and Miltzow [BM17] for guarding simple polygons. MGS($\mathcal{P}$) has been well-studied in literature. For details on early literature, see the classical books [Gho07, O'R87]; for the recent literature one can look [AAM22, BM17, DFM+19, HM21]. To the best of our knowledge, it is not known if one can avoid using the tools in [BM17] for obtaining an optimal solution of

MGS($\mathcal{P}$) using discretization even for the case of monotone polygons, although constant factor approximation for MGS($\mathcal{P}$) is known for monotone polygons due to Krohn and Nilsson [KN13]. To be more precise, is MGS($\mathcal{P}$) $\exists\mathbb{R}$-complete? We anticipate that our geometric characterizations will prove useful in polygon guarding scenarios, such as smooth or robust guarding [DFKM24, DHM18].

Our Results and Methods: Our contributions are as follows.

Section 3. We consider the DisWS problem in simple polygons. We first prove that the visibility intersection graph of a simple polygon is an outerstring graph. Since a maximum independent set in outerstring graphs can be determined in polynomial time, as established by [KMPV17], we derive the subsequent conclusion.

Theorem 1.1. Discrete Witness Set *is solvable in* $\mathcal{O}(|S|^3 \cdot |V(\mathcal{P})|^3)$ *time.*

Section 4. We consider the DisWS problem specifically for a monotone polygon. Initially, we show that the visibility intersection graph of a monotone polygon is co-comparable. Given that a maximum independent set can be solved in polynomial time for co-comparable graphs, we derive the following result.

Theorem 1.2. Discrete Witness Set *in a monotone polygon is solvable in* $\mathcal{O}(|S|^2 + |V(\mathcal{P})||S|)$ *time.*

Section 5. We consider the Witness Set problem in a monotone polygon with a suitable parameter. It is known that the Art Gallery problem is W[1]-hard when parametrized by the number of guards, and Giannopoulos [Gia16] posed the open question of whether the Art Gallery admits a fixed-parameter tractable (FPT) algorithm when parameterized by r. Recently, Agrawal et al. [AKL+24] studied a variation called the Vertex-Vertex Art Gallery and gave an FPT algorithm with running time $r^{\mathcal{O}(r^2)}n^{\mathcal{O}(1)}$. Inspired by this, we explore the Witness Set problem in a monotone polygon with r, the number of reflex vertices, as the parameter.

Although our objective is to design an algorithm with a runtime of $f(r)n^{\mathcal{O}(1)}$ for some computable function f, we have obtained an algorithm with an even better running time that is $r^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$, where k represents the size of the maximum witness set. This outcome shows that we can find the maximum witness set for a monotone polygon in polynomial time ($n^{\mathcal{O}(1)}$) whenever $k\log(r) = \mathcal{O}(\log n)$. As k is at most r, the algorithm achieves a runtime of $f(r)n^{\mathcal{O}(1)}$, for some computable function f.

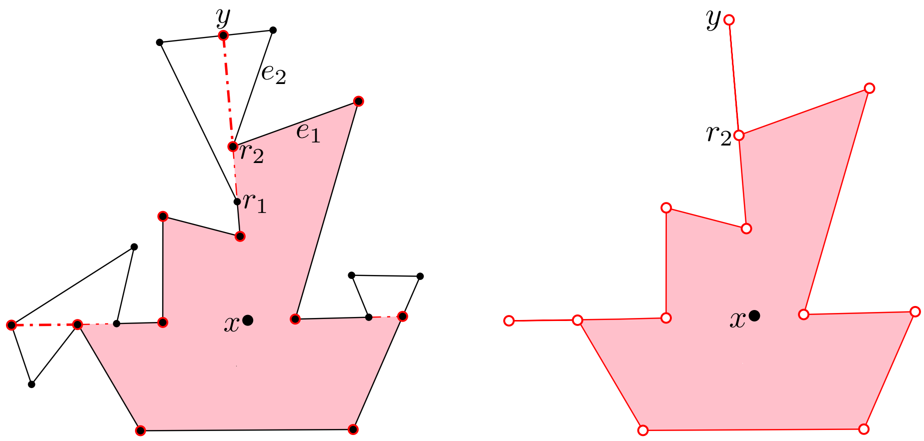
Theorem 1.3. Witness Set *in a monotone polygon* $\mathcal{M}$ *with* n *vertices is solvable in* $r^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ *time, where* r *denotes the number of reflex vertices in* $\mathcal{M}$ *and* k *is the size of a maximum witness set in* $\mathcal{M}$.

The proof of Theorem 1.3 consists of two components. First, we try to find a discrete set of size $r^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$ inside a monotone polygon $\mathcal{M}$, that will suffice to contain a maximum witness set of $\mathcal{M}$. We were able to achieve this with the help of reflex vertices and a repeating *line arrangement* procedure $\mathcal{O}(k)$ many times. The term "line arrangement" refers to connecting line segments between reflex vertices and a newly established set of points on $\mathcal{M}$. Secondly, once we ascertain this discrete set, we apply Theorem 1.2 to derive Theorem 1.3.

Section 6. Using a clever *vertical decomposition* technique, we design $(1+\epsilon)$-factor approximation algorithm (PTAS) for the Witness Set problem in monotone polygon running in time $r^{\mathcal{O}(1/\epsilon)} \cdot n^2$ time, where r and n denotes the number of reflex vertices and vertices, respectively of input monotone polygon.

2 Preliminaries

Polygons. In this paper, we only consider simple polygons. A *simple polygon* $\mathcal{P}$ is a flat shape consisting of n straight, *non-intersecting* line segments that are joined pairwise to form a closed path. The line segments that make up a polygon, called *edges*, meet only at their endpoints, called *vertices*. A simple polygon $\mathcal{P}$ encloses a region, called its *interior*, that has a measurable area. We do not consider the boundary of $\mathcal{P}$ as part of its interior. We use $\partial(\mathcal{P})$, $\mathsf{iN}(\mathcal{P})$, $\mathsf{eX}(\mathcal{P})$ to denote the boundary, interior, and exterior region of $\mathcal{P}$, respectively. A vertex $v \in V(\mathcal{P})$ is a *reflex* vertex if the interior angle of $\mathcal{P}$ at v is larger than 180 degrees, else it is called a *convex* vertex. A polygon $\mathcal{P}$ in the plane is called *monotone* with respect to a straight line L, if every line orthogonal to L intersects the boundary of $\mathcal{P}$ at most twice. Unless otherwise stated, we refer to a polygon that is monotone with respect to the line $y = 0$ (or, the x-axis) as a monotone polygon. For any point p in the plane, $x(p)$ denotes the x-coordinate of p.



(a) x sees both reflex vertices r_1, r_2. (b) $\overline{r_2 y}$ is a *polygonal arm*.

Figure 2: An instance where the visibility region of a point x is not a simple polygon.

We now motivate the definition of *polygonal arms* of a visibility region $\mathsf{Vis}(x)$. We will show an instance where the visibility region of a point x in a simple polygon $\mathcal{P}$ might not be a simple polygon. In the following paragraph, we try to give an example of such a scenario. It is important to note that we are just trying to create an example, i.e., an example of a simple polygon $\mathcal{P}$ and point $x \in \mathcal{P}$ to show the origin of a *polygonal arm* of a visibility region. It is not necessarily true for any arbitrary point x inside a simple polygon $\mathcal{P}$. So, let $x \in \mathcal{P}$ be a point such that it satisfies the following properties: (i) x sees two distinct reflex vertices of $\mathcal{P}$, r_1 and r_2; (ii) the pairs of edges containing r_1 and r_2 lies on either side of the line joining x, r_1 and r_2 (See Figure 2 for an illustration). Furthermore, r_1 and r_2 are the two closest reflex vertices respectively from x on the line $\overline{xr_1r_2}$; and,(iii) Let the pair of edges that contain the reflex vertex r_2 be e_1 and e_2 respectively. Then neither e_1 nor e_2 is part of the extended line $\overline{xr_1r_2}$. Now suppose the line $\overline{xr_1r_2}$ hit the boundary of $\mathcal{P}$ on the point y (See Figure 2a). Then, the segment $\overline{r_2y}$ is not a part of any polygon in $\mathsf{Vis}(x)$. However, this line $\overline{r_2y}$ is indeed visible from x and hence a part of $\mathsf{Vis}(x)$. So, we observe that $\mathsf{Vis}(x)$ doesn't always need to be a simple polygon in $\mathcal{P}$ for any $x \in \mathcal{P}$. Therefore, the above scenario leads us to the following definition. For a simple polygon $\mathcal{P}$ and a point $x \in \mathcal{P}$, if a line segment or an edge is visible from x such that it is not a part of the simple polygon inside $\mathsf{Vis}(x)$, we call that line segment/edge a *polygonal arm* of $\mathsf{Vis}(x)$. In Figure 2b, the line segment/edge $\overline{r_2y}$ is a *polygonal arm* of $\mathsf{Vis}(x)$.

Visibility. Let $\mathcal{P} = (V, E)$ be a simple polygon. We say that a point p *sees* (or is *visible* to) a point q if every point of the line segment $\overline{pq}$ belongs to $\mathcal{P}$ (including the boundary). More generally, a set of points S *sees* a set of points Q if every point in Q is seen by at least one point in S. Note that if a point

p sees a point q, then the point q sees the point p as well. Moreover, a vertex $v \in V$ necessarily sees itself and its two neighbors in $V(\mathcal{P})$. For a point $x \in \mathcal{P}$ we use $\mathsf{Vis}(x)$, to denote the visibility region for the point x, defined by $\mathsf{Vis}(x) = \{q \in \mathcal{P} \mid q \text{ is visible from } x\}$. For a subset $A \subset \mathcal{P}$, we define $\mathsf{Vis}(A) := \bigcup_{x \in A} \mathsf{Vis}(x)$. For a set of points F in $\mathcal{P}$, the *visibility intersection graph* corresponding to F is denoted by $\mathsf{VIG}(F)$ and defined as follows: the points in F correspond to the vertices of the graph, and there is an edge between a pair of vertices $a, b \in F$ in $\mathsf{VIG}(F)$ if and only if $\mathsf{Vis}(a) \cap \mathsf{Vis}(b) \neq \emptyset$.

Throughout the paper, we use $\mathcal{M}$ to denote an x-monotone polygon. The notation $\widetilde{G}$ denotes the complement graph of G, i.e., $V(\widetilde{G}) = V(G)$ and $uv \in E(\widetilde{G})$ if and only $uv \notin E(G)$.

3 Discrete Witness Set in a Simple Polygon

This section aims to demonstrate that the Discrete Witness Set within a simple polygon can be solved in polynomial time (Theorem 1.1). Initially, we prove that the visibility intersection graph in a simple polygon is an outerstring graph (Theorem 3.12). With the use of the known result (Theorem 3.14) by Keil et al. [KMPV17] that a maximum independent set can be found in polynomial time (in $|N|$) for outerstring graphs that is given with an intersection model consisting of polygonal arcs with a total of $|N|$ segments, we achieve our objective, which is the following.

Theorem 1.1. Discrete Witness Set *is solvable in* $\mathcal{O}(|S|^3 \cdot |V(\mathcal{P})|^3)$ *time.*

We start with some basic definitions that we need.

Definition 3.1 (Primary edge and Window)**.** Consider a polygon $\mathcal{P}$ and a point $x \in \mathcal{P}$. We call an edge of $\mathsf{Vis}(x)$ a *primary edge* if that edge coincides entirely with the boundary of $\mathcal{P}$, otherwise we call the edge a *window* of x. Among the two end-points of a window, the one that is closest to x is called a *base* and the other is called an *end*.

For a point $x \in \mathcal{P}$, we define $\min_{pr}(x)$ to be the length of the shortest length primary edge(s) in $\mathsf{Vis}(x)$. See Figure 3b for an illustration of Theorem 3.1, where notice that e_3 is a *polygonal arm* and e_4, e_5 have some parts which are not part of $\partial(\mathcal{P})$. Observe that a polygonal arm is also a window of x. We say a point α is collinear to a line segment l, if any two points of l and α are collinear. i.e., if we extend l then it passes through α. It is easy to observe that every window of point x is collinear with x (also mentioned in [BLM02]). However, a connection exists between a primary edge and a polygonal arm.

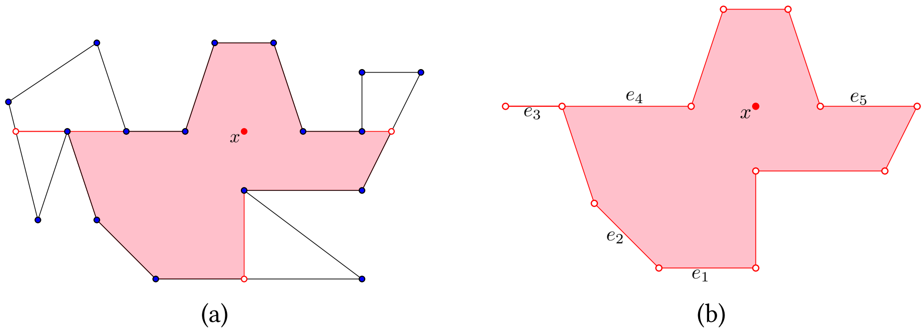


Figure 3: Example of primary edges (e_1, e_2) in $\mathsf{Vis}(x)$ and windows (e_3, e_4, e_5) of x.

Observation 3.2. *For a point* $x \in \mathcal{P}$, $\mathsf{Vis}(x)$ *contains a primary edge, i.e., not all edges of* $\mathsf{Vis}(x)$ *can be windows.*

Proof. Clearly $\mathsf{Vis}(x)$ consists of a polygon (say $\mathcal{P}_x$) and maybe some polygonal arms (see Figure 2). $\mathcal{P}_x$ is a connected region with nonempty interior bounded by line segments in $\mathcal{P}$. So, it must contain at least three edges that, when extended, do not pass through the same point. But, since any window of x is collinear to x, we must have an edge that is not a window, that is, a primary edge. So we can conclude that $\mathcal{P}_x$ must have a primary edge and no edge of $\mathcal{P}_x$ is a polygonal arm of $\mathsf{Vis}(x)$. $\square$

Definition 3.3 (String, String-like)**.** A *string* is a polygonal curve that does not cross itself. A *string-like* structure refers to a combination of strings $\mathcal{S}$ that satisfies a certain condition. Specifically, there exists a common string $\lambda \in \mathcal{S}$ such that none of the strings in the set $\mathcal{S} \setminus \{\lambda\}$ intersect with one another, and each string in $\mathcal{S} \setminus \{\lambda\}$ has exactly one endpoint in λ. In other words, by *string-like*, we mean a structure that is a string or a string with branches.

String-like ($\mathsf{Str}^*(w)$) Creation on $\mathsf{Vis}(w)$. For every input point $w \in \mathcal{P}$, our goal is to create a unique string denoted by $\mathsf{Str}(w)$. To that end, we first create two structures consecutively associated with the point w, which are string-likes and we denote as $\mathsf{Str}^*(w)$. Let us first look at the visibility region $\mathsf{Vis}(w)$ and the topological boundary of the region, denoted by $\partial(\mathsf{Vis}(w))$. We arbitrarily choose a primary edge, say e, of $\mathsf{Vis}(w)$ and then we break $\partial(\mathsf{Vis}(w))$ on e by cutting a very tiny open part of length ϵ where $0 < \epsilon < \min_{pr}(w)$, so that it forms a string-like $\mathsf{Str}^*(w)$ corresponding to w, starting at a point, say v, of $\partial(\mathsf{Vis}(w))$ (see Figure 4). We call v the origin of $\mathsf{Str}^*(w)$, and by ϵ-gap we mean the gap created by cutting that open segment of length ϵ. We make a restriction here by not choosing any vertex of $\mathsf{Vis}(w)$ as the origin of $\mathsf{Str}^*(w)$, we can do this because the length of any primary edge is strictly greater than ϵ. Also, for creating ϵ-gap, we do not choose any polygonal arm of $\mathsf{Vis}(w)$ to cut from, as the existence of a primary edge, which is not a polygonal arm, is ensured by Theorem 3.2.

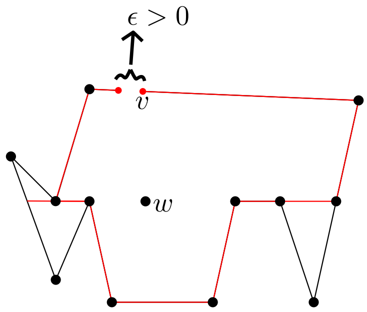


Figure 4: $\mathsf{Str}^*(w)$ (red) is formed from $\partial(\mathsf{Vis}(w))$ by cutting at a primary edge by a tiny length ϵ.

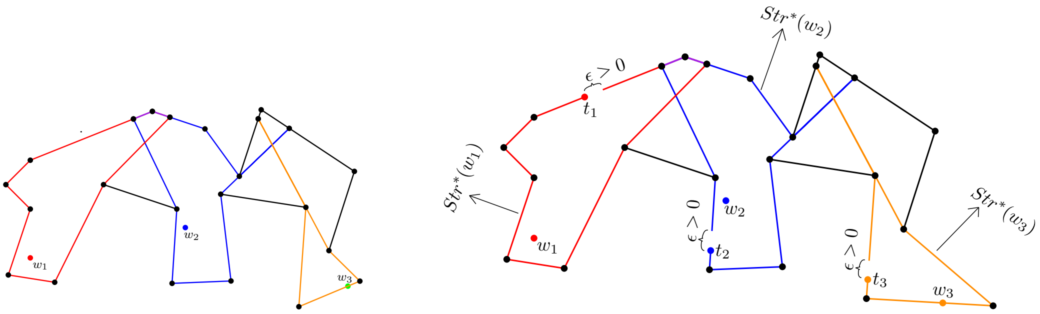
Consider that we are given a simple polygon $\mathcal{P}$ and a set $F = \{w_1, w_2, \cdots, w_m\}$ of m points in $\mathcal{P}$. Our main goal is to construct a unique string corresponding to each point in F such that the following property is satisfied: for each pair of points $x, y \in F$,

$$\mathsf{Vis}(x) \cap \mathsf{Vis}(y) \neq \emptyset \text{ if and only if } \mathsf{Str}(x) \cap \mathsf{Str}(y) \neq \emptyset$$

To this end, we follow the three-step procedure described below.

Step 1. (String-like formation): Our construction requires that each string-like originates at a unique point of $\partial(\mathcal{P})$. In the worst-case scenario, we might have that all the string-likes originate at one particular common primary edge. So we want to choose a common ϵ in such a way that we have enough space to create multiple ϵ-gaps from one various primary edge. Therefore, we choose ϵ so that $0 < \epsilon < \frac{1}{m} \min_{w \in F}(min_{pr}(w))$.

Now for each $w_i \in F$, we choose a primary edge of $\mathsf{Vis}(w_i)$ and construct the string-like, $\mathsf{Str}^*(w_i)$ by creating an ϵ-gap on that edge (See Figure 5). The choice of ϵ ensures that it is possible to create all the string-likes together without overlapping any two ϵ-gaps as $m\epsilon < \min_{w_j \in F}(min_{pr}(w_j)) < min_{pr}(w_i) <$ *length of any primary edge of* $\mathsf{Vis}(w_i)$. We denote the origin of $\mathsf{Str}^*(w_i)$ by t_i. The following observation is immediate.



(a) $\mathsf{Vis}(w_i), i \in [3]$

(b) $\mathsf{Str}^*(w_1), \mathsf{Str}^*(w_2)$ and $\mathsf{Str}^*(w_3)$.

Figure 5: Example of string-like formations for w_1, w_2, w_3 from their visibility regions..

Observation 3.4. *For each $w \in F$ the length of any primary edge of $\mathsf{Vis}(w)$ is more than ϵ.*

Now define S_i := Set of all vertices of $\mathsf{Vis}(w_i)$, $S = \bigcup_{i=1}^{m} S_i$, $T = \{t_1, ..., t_m\}$. Let $d_{t_i} = \min_{s \in S}(d(t_i, s))$ and $d = \min_{t_i \in T} d_{t_i}$, where $d(x, y)$ denotes the length of $\overline{xy}$. Since the choice of ϵ is in our hands, we choose ϵ such that the value of ϵ is even smaller than d. Let $0 < \epsilon < \min\{d, \frac{1}{m} \min_{w \in F}(min_{pr}(w))\}$. The above restriction ensures that for any two points $w_i, w_j \in F$, the distance between the origin t_i of $\mathsf{Str}^*(w_i)$ and any vertex of $\mathsf{Vis}(w_j)$ is at least ϵ. Thus, we have the following observation.

Observation 3.5. *For any pair of points, $w_i, w_j \in F$, $\mathsf{Vis}(w_j)$ has no vertex which is in the ϵ-gap beside the origin of $\mathsf{Str}^*(w_i)$ on $\partial(\mathsf{Vis}(w_i))$.*

Now we prove the following claim.

Claim 3.6. *For $w_i, w_j \in F$, we have $\mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_j) \neq \emptyset$ if and only if $\mathsf{Str}^*(w_i) \cap \mathsf{Str}^*(w_j) \neq \emptyset$.*

Proof. In the forward direction, we show that $\mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_j) \neq \emptyset$ implies $\mathsf{Str}^*(w_i) \cap \mathsf{Str}^*(w_j) \neq \emptyset$. Let $\mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_j) \neq \emptyset$. Consider the case where $\mathsf{Str}^*(w_i) \cap \mathsf{Str}^*(w_j) = \emptyset$. It is easy to observe that $\mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_j) \neq \emptyset$ implies $\partial(\mathsf{Vis}(w_i)) \cap \partial(\mathsf{Vis}(w_j)) \neq \emptyset$. Now in this case, boundaries of $\mathsf{Vis}(w_i)$ and $\mathsf{Vis}(w_j)$ intersect but their corresponding string-likes do not. The source of this conflict must lie in the ϵ-gaps introduced into the visibility regions during the construction of the string-likes. This implies that the boundaries of these regions can only intersect within these specific gaps. This can only happen when there exists a primary edge or a vertex of $\mathsf{Vis}(w_j)$ strictly contained in the ϵ-gap beside the origin of $\mathsf{Str}^*(w_i)$ on $\partial(\mathsf{Vis}(w_i))$ or, there exists a primary edge or a vertex of $\mathsf{Vis}(w_i)$ strictly contained in the ϵ-gap beside the origin of $\mathsf{Str}^*(w_j)$ on $\partial(\mathsf{Vis}(w_j))$. Now, there are only a finite number of such (i, j) pairs. If there is a primary edge contained in the ϵ-gap, then the length of that primary edge is less than ϵ, which contradicts Theorem 3.4. If there is a vertex inside the ϵ-gap, then it contradicts Theorem 3.5. Therefore, $\mathsf{Str}^*(w_i) \cap \mathsf{Str}^*(w_j) \neq \emptyset$.

In the reverse, let us assume for contradiction that $\mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_j) = \emptyset$ however, $\mathsf{Str}^*(w_i) \cap \mathsf{Str}^*(w_j) \neq \emptyset$. Since these *string-likes* are the boundaries of the visibility regions with tiny parts of their edges removed, therefore if $\mathsf{Str}^*(w_i) \cap \mathsf{Str}^*(w_j) \neq \emptyset$ then there exists at least one point in the boundaries of $\mathsf{Vis}(w_j)$ and $\mathsf{Vis}(w_i)$ where they intersect. Therefore, $\mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_j) \neq \emptyset$, a contradiction. This completes the proof. $\square$

Step 2. (Inflation of $\mathcal{P}$): Once the construction of the string-likes is done, we scale up (inflate) the polygon $\mathcal{P}$ by a tiny factor, say δ, without altering the string-likes. We call this modified polygon $\mathsf{Inf}_\delta(\mathsf{P})$. The formal description of inflating the polygon $\mathcal{P}$ is given below.

- For each edge e in $\mathcal{P}$, draw a perpendicular to the outside of $\mathcal{P}$.

- On each of those perpendiculars drawn on an edge e (say) of $\mathcal{P}$, draw another line $\hat{e}$ parallel to that corresponding edge, at a predetermined tiny distance, δ.
- This set of new edges, $\{\hat{e} : e \text{ is an edge in } \mathcal{P}\}$, are the edges of the inflated polygon, $\mathsf{Inf}_\delta(\mathsf{P})$, and the points where any two adjacent such edges meet are the vertices of $\mathsf{Inf}_\delta(\mathsf{P})$.

Now, in $\mathsf{Inf}_\delta(\mathsf{P})$, for each $w_i \in F$, we modify $\mathsf{Str}^*(w_i)$ by joining its origin t_i with the nearest point of t_i on $\partial(\mathsf{Inf}_\delta(\mathsf{P}))$. Note that, by our construction, t_i is not any vertex of $\mathcal{P}$, so t_i must have a unique nearest point in $\partial(\mathsf{Inf}_\delta(\mathsf{P}))$ (the unique nearest point is the intersection point of the perpendicular on t_i, and $\partial(\mathsf{Inf}_\delta(\mathsf{P}))$). We denote the modified string-like by $\mathsf{Str}^{**}(w)$ and the point on $\partial(\mathsf{Inf}_\delta(\mathsf{P}))$ connected to t_i by s_i. Hereon, we call s_i as the origin of the string-like $\mathsf{Str}^{**}(w_i)$. Observe that the procedure of modifying $\mathcal{P}$ presented above preserves the intersection

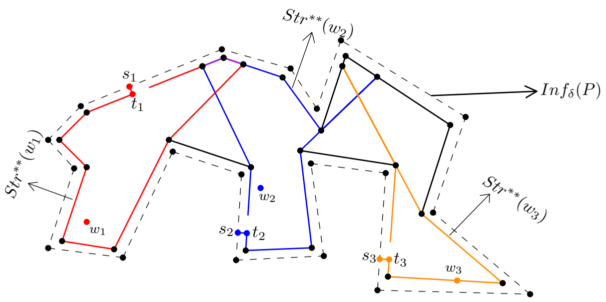
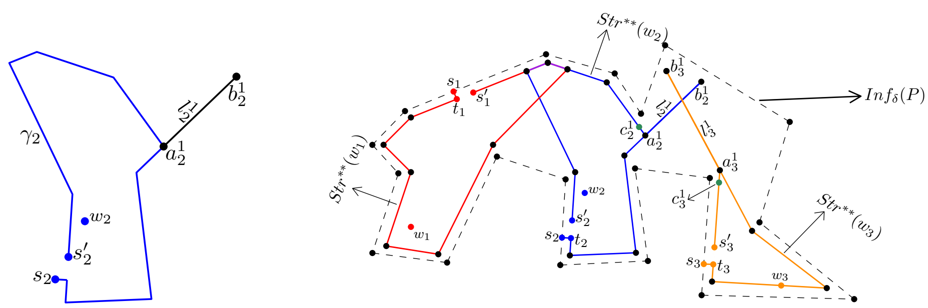


Figure 6: Inflation of $\mathcal{P}$ and $\mathsf{Str}^{**}(w)$ formation.

of string-likes in $\mathcal{P}$, since the string-likes are not perturbed and only $\mathcal{P}$ is modified. The strings that initially intersected in $\mathcal{P}$ still intersected after inflating $\mathcal{P}$. The same goes for the string-likes that do not intersect each other. So we have the following observation, which is immediate.

Observation 3.7. $\mathsf{Str}^*(w_i) \cap \mathsf{Str}^*(w_j) \neq \emptyset$ *if and only if* $\mathsf{Str}^{**}(w_i) \cap \mathsf{Str}^{**}(w_j) \neq \emptyset$.

Step 3. (String formation): In this step, for each $w_i \in F$, we create a string, $\mathsf{Str}(w_i)$ from the string-like, $\mathsf{Str}^{**}(w_i)$. Notice that, after creating the ϵ-gap on $\partial(\mathsf{Vis}(w_i))$, two end-points of $\mathsf{Str}^*(w_i)$ were created. One of them was considered as the origin, t_i and let the other end-point be denoted by s'_i. Then s'_i is also an endpoint of $\mathsf{Str}^{**}(w_i)$. Now, in the string-like $\mathsf{Str}^{**}(w_i)$, there is a unique string (excluding the branches) joining the end-points s_i and s'_i; we call it γ_i (see Figure 7a). Essentially, $\gamma_i \cup \{branches\ of\ \mathsf{Str}^{**}(w_i)\} = \mathsf{Str}^{**}(w_i)$. These branches are nothing but the polygonal arms of $\mathsf{Vis}(w_i)$. Let $l_i^1, l_i^2, \ldots, l_i^k$ be the branches of $\mathsf{Str}^{**}(w_i)$ and l_i^j is joined with γ_i at the point a_i^j. Let the other endpoint of l_i^j be b_i^j. We denote the branch l_i^j by $\overline{a_i^j b_i^j}$ (since a branch is a polygonal arm of a visibility region, it is a line segment).



(a) γ_2 (blue string). (b) $\mathsf{Str}^{**}(w_1), \mathsf{Str}^{**}(w_2)$ and $\mathsf{Str}^{**}(w_3)$.

Figure 7: Construction of $\mathsf{Str}(w)$ from $\mathsf{Str}^{**}(w)$.

Observe that, as there are only a finite number of string-like structures and hence a finite number of branches in total, there exists an open neighborhood U_i^j around a_i^j (the point of intersection of a branch l_i^j and γ_i) such that for any point $c_i^j \in U_i^j \cap \gamma_i$ (See Figure 7b), we can replace $\overline{a_i^j c_i^j}$ by $\overline{c_i^j b_i^j}$ (See Figure 8) to create a new string $\mathsf{Str}(w_i)$ such that it satisfies the following properties: (i) for all l such that $\mathsf{Str}^{**}(w_i) \cap \mathsf{Str}^{**}(w_l) \neq \emptyset$ we have $\mathsf{Str}(w_i) \cap \mathsf{Str}(w_l) \neq \emptyset$; (ii) for all l such that $\mathsf{Str}^{**}(w_i) \cap \mathsf{Str}^{**}(w_l) = \emptyset$ we have $\mathsf{Str}(w_i) \cap \mathsf{Str}(w_l) = \emptyset$. We can ensure property (i) because we only have a finite number of string-like structures that intersect with $\mathsf{Str}^{**}(w_i)$. So, it cannot happen that for any arbitrarily small neighborhood U_i^j, by adding the line $\overline{c_i^j b_i^j}$, it always intersects with a new string-like $\mathsf{Str}^{**}(w_{i'})$. Also, we can ensure property (ii) by choosing a small neighborhood U_i^j such that the segment $\overline{a_i^j c_i^j}$ does not intersect with any other string-like structure. In ensuring both properties, the fundamental idea was that the number of string-like structures is finite. We formally describe the construction of $\mathsf{Str}(w_i)$ from $\mathsf{Str}^{**}(w_i)$ below (see Figure 8).

- For each j, add $\overline{c_i^j b_i^j}$ with $\mathsf{Str}^{**}(w_i)$.
- For each j, delete $\overline{a_i^j c_i^j}$ from $\mathsf{Str}^{**}(w_i)$ except the points a_i^j and c_i^j, i.e., delete the open segment $(a_i^j c_i^j)$.

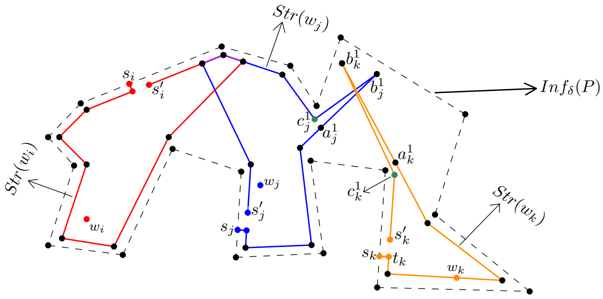


Figure 8: Replacing $\overline{a_i^j c_i^j}$ by $\overline{c_i^j b_i^j}$.

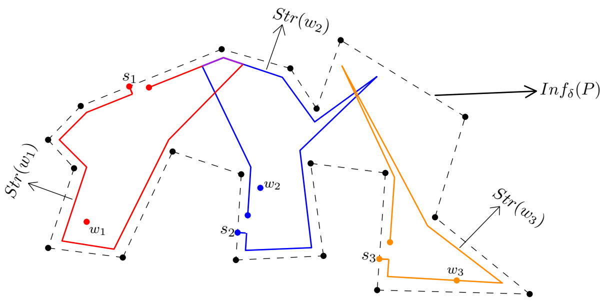


Figure 9: $\mathsf{Str}(w_1)$, $\mathsf{Str}(w_2)$ and $\mathsf{Str}(w_3)$.

Observe that $\mathsf{Str}(w_i)$ is a string as it does not have any branches anymore. s_i is the origin of $\mathsf{Str}(w_i)$. Since the construction of $\mathsf{Str}(w_i)$ for each $w_i \in F$ respects the intersection-invariant property (i.e., two strings intersect if and only if their respective string-likes also intersect, and the procedure described for constructing the strings does not change their respective intersections), we have the following observation.

Observation 3.8. *$\mathsf{Str}^{**}(w_i) \cap \mathsf{Str}^{**}(w_j) \neq \emptyset$ if and only if $\mathsf{Str}(w_i) \cap \mathsf{Str}(w_j) \neq \emptyset$.*

Combining Theorem 3.6, Observations 3.7 and 3.8 we obtain the following.

Lemma 3.9. *Let $F = \{w_1, w_2, \cdots, w_m\}$ be a set of m points in a simple polygon $\mathcal{P}$. For each pair of points $w_i, w_j \in F$, $\mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_j) \neq \emptyset$ if and only if $\mathsf{Str}(w_i) \cap \mathsf{Str}(w_j) \neq \emptyset$.*

Definition 3.10 (String Intersection Graph)**.** For a set $\mathcal{S}$ of strings, the string intersection graph corresponding to $\mathcal{S}$ is denoted by $\mathsf{SIG}(\mathcal{S})$; and is defined as follows: each string in $\mathcal{S}$ corresponds to a vertex of $\mathsf{SIG}(\mathcal{S})$ and there is an edge between a pair of vertices in $\mathsf{SIG}(\mathcal{S})$ if and only if their corresponding strings intersect.

Let $\mathcal{S} = \{\mathsf{Str}(w_1), \dots, \mathsf{Str}(w_m)\}$. Notice that for each point $w \in F$, there exists a corresponding $\mathsf{Vis}(w)$ in $\mathcal{P}$ and $\mathsf{Str}(w)$ in $\mathsf{Inf}_\delta(\mathsf{P})$. Also, $\mathsf{Vis}(w)$ and $\mathsf{Str}(w)$ correspond to a vertex in $\mathsf{VIG}(F)$ and $\mathsf{SIG}(\mathcal{S})$, respectively. With the help of Theorem 3.9, we have the following.

Lemma 3.11. *The graphs $\mathsf{VIG}(F)$ and $\mathsf{SIG}(\mathcal{S})$ are isomorphic.*

Now we show that the string intersection graph corresponding to $\mathcal{S}$ is an outerstring graph. The *outerstring graphs* are the intersection graphs of curves in the plane that lie inside a circle such that each curve intersects the boundary of the circle at one of its endpoints. However, here, instead of a circle, what we have is a simple polygon. Even then, the intersection graph of curves forms an outerstring graph. The reason is that we can translate the underlying polygon to a circle such that the pairwise intersections among the curves are preserved in either case.

Claim 3.12. *$\mathsf{SIG}(\mathcal{S})$ is an outerstring graph.*

Proof. Every vertex of $\mathsf{SIG}(\mathcal{S})$ corresponds to a point $w \in F$ where w corresponds to a string $\mathsf{Str}(w)$. Now the string $\mathsf{Str}(w)$ has one endpoint at $\partial(\mathcal{P})$ and another endpoint at the interior of $\mathcal{P}$. Therefore, the graph $\mathsf{SIG}(Z)$ is an outerstring graph. Hence, the claim. □

Now we show that any independent set on the graph $\mathsf{SIG}(\mathcal{S})$ corresponds to a witness set on $\mathcal{P}$ of the same size.

Lemma 3.13. *Any independent set in $\mathsf{SIG}(\mathcal{S})$ gives a witness set in $\mathcal{P}$ with the same size.*

Proof. Any independent set in the graph $\mathsf{SIG}(\mathcal{S})$ gives us the set of strings in $\mathcal{P}$ that do not intersect each other. Now each string has a corresponding visibility region in $\mathcal{P}$. Since the strings returned by the independent set do not intersect each other, their corresponding visibility regions do not intersect. Hence, the proof follows. □

Proposition 3.14 ([KMPV17])**.** *Given the geometric representation $(\mathcal{P}, S)$ of a weighted outerstring graph G, an independent set with maximum weight for G can be found in $\mathcal{O}(n^3)$ time, where n is the number of segments used to represent the strings of S and the polygon $\mathcal{P}$.*

Proof of Theorem 1.1. Given an outerstring graph G and its polygonal representation in a polygon $\mathcal{P}$, we can compute the maximum independent set in $\mathcal{O}(N^3)$ time [KMPV17], where N is the number of segments used to represent the strings of G and the polygon $\mathcal{P}$. In our problem, each individual string may have $\mathcal{O}(n)$ segments and there are m strings in total, where n is the number of vertices in $\mathcal{P}$ and m is the number of input points. Hence $N = \mathcal{O}(mn)$. Now applying Theorem 3.13, computing the maximum independent set in the graph $\mathsf{SIG}(\mathcal{S})$ returns the maximum witness set in $\mathcal{P}$. Therefore, the maximum witness set problem in a simple polygon $\mathcal{P}$ for a set of m points can be solved in $\mathcal{O}(m^3n^3)$ time. □

4 Discrete Witness Set in a Monotone Polygon

This section aims to improve the running time of the result already obtained in the previous section (Theorem 1.1), for the class of monotone polygons. Initially, we prove that the visibility intersection graph of a monotone polygon is co-comparable (Theorem 4.16). Then, with the use of a known result (Theorem 4.2) that a maximum independent set can be found in polynomial time for co-comparable graphs, we achieve our objective, which is as follows.

Theorem 1.2. Discrete Witness Set *in a monotone polygon is solvable in* $\mathcal{O}(|S|^2 + |V(\mathcal{P})||S|)$ *time.*

Definition 4.1 (Co-comparable Graph)**.** A *comparability* graph is an undirected graph that connects pairs of elements that are comparable to each other in a partial order. A graph G is *co-comparable* if the complement of the graph G is a comparability graph.

Remark. There is an equivalent definition of a comparability graph, which is as follows. A comparability graph is a graph that has a transitive orientation, i.e., an assignment of directions to the edges of the graph (i.e. an orientation of the graph) such that the adjacency relation of the resulting directed graph is transitive: whenever there exist directed edges (x, y) and (y, z), there must exist an edge (x, z).

Proposition 4.2 ([MS99])**.** *A maximum independent set in a co-comparable graph with n vertices can be found in $\mathcal{O}(n^2)$ time.*

Let F represent a set of f points within a monotone polygon $\mathcal{M}$. Initially, we construct the visibility intersection graph $\mathsf{VIG}(F)$ in the following way. The vertices of $\mathsf{VIG}(F)$ are the points in F and there is an edge between two vertices $a, b \in F$ if and only if $\mathsf{Vis}(a) \cap \mathsf{Vis}(b) \neq \emptyset$. Subsequently, we demonstrate that the graph $\mathsf{VIG}(F)$ is co-comparable. Next, we introduce some notations and definitions. Some of which will be used in the subsequent section.

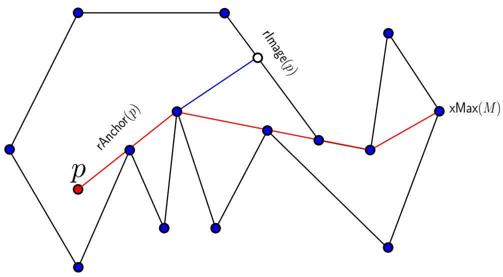


Figure 10: $\Pi_{\mathsf{max}}(p)$ is denoted by the red path. $\mathsf{rBdry}(p)$ is the chord between $\mathsf{rAnchor}(p)$, $\mathsf{rImage}(p)$.

Definition 4.3 ($\mathsf{xMax}()$ and $\mathsf{xMin}()$)**.** For a region A, $\mathsf{xMax}(A)$ (resp., $\mathsf{xMin}(A)$) denotes a point having maximum (resp., minimum) x-coordinate among all the points in A.

Observe that there may be multiple points in the set A which possess the highest (or lowest) x-coordinate. In such situations, we arbitrarily choose $\mathsf{xMax}(A)$. A similar rationale applies to $\mathsf{xMin}(A)$. We represent a monotone polygon by $\mathcal{M}$. For a point $p \in \mathcal{M}$, a new edge is created when the visibility of p is blocked by a reflex vertex c of $\mathcal{M}$. The other endpoint of this newly formed edge, which lies on the boundary of $\mathcal{M}$, is called the *image of p through c*. We will also carry forward all the definitions and notations defined in this section to the following sections.

Definition 4.4 ($\Pi_{\mathsf{max}}()$ and $\Pi_{\mathsf{min}}()$)**.** For a point $p \in \mathcal{M}$, $\Pi_{\mathsf{max}}(p)$ (resp., $\Pi_{\mathsf{min}}(p)$) denotes a shortest path in $\mathcal{M}$ from p to $\mathsf{xMax}(\mathcal{M})$ (resp., $\mathsf{xMin}(\mathcal{M})$).

Definition 4.5 (rAnchor() and ℓAnchor()). For a point $p \in \mathcal{M}$, we define a vertex c of $\mathcal{M}$ as rAnchor(p) if $c \neq$ xMax$(\mathcal{M})$ and c is the leftmost vertex (having smallest x-coordinate) of $\mathcal{M}$ other than p that is in $\Pi_{\mathsf{max}}(p)$. Similarly, we define a vertex c of $\mathcal{M}$ as ℓAnchor(p) if $c \neq$ xMin$(\mathcal{M})$ and c is the rightmost (having largest x-coordinate) vertex of $\mathcal{M}$ other than p that is in $\Pi_{\mathsf{min}}(p)$.

Definition 4.6 (rImage() and ℓImage()). For a point $p \in \mathcal{M}$, if there exist rAnchor(p) and ℓAnchor(p), then we call the points, *image of p through* rAnchor(p) and *image of p through* ℓAnchor(p) as rImage(p) and ℓImage(p), respectively.

This leads to the following observations.

Observation 4.7. For a point $p \in \mathcal{M}$, if there exists no rAnchor(p), then xMax(Vis(p)) = xMax$(\mathcal{M})$. Similarly, if there exists no ℓAnchor(p) then xMin(Vis(p)) = xMin$(\mathcal{M})$.

Observation 4.8. Given a monotone polygon $\mathcal{M}$ and a set of witnesses $w_1, w_2, \ldots, w_k$ with $x(w_1) < x(w_2) < \ldots < x(w_k)$, both ℓAnchor() and rAnchor() are present for each witness w_i where $1 < i < k$. However, this may not hold for the extreme points w_1 and w_k, which are the leftmost and rightmost witnesses, respectively (due to Theorem 4.7).

Definition 4.9 (rBdry() and ℓBdry()). For a point $p \in \mathcal{M}$, we define rBdry(p) as the chord with endpoints at rAnchor(p) and rImage(p). Similarly, we define ℓBdry(p) as the chord with endpoints ℓAnchor(p)) and ℓImage(p).

For an illustration of Definitions 4.4, 4.5, 4.6, and 4.9 see Figure 10. Observe that it might happen that for some point p, there is no rBdry(p) or ℓBdry(p)(due to Theorem 4.7).

Definition 4.10. We define $\ell(p)$ as the vertical chord passing through a point $p \in \mathcal{M}$.

The following observation is immediate.

Observation 4.11. *For any point $p \in \mathcal{M}$, we have $\ell(p) \subset$ Vis(p) .*

In the following Theorem 4.12 and Theorem 4.13, we show that for every pair of points $a, b \in \mathcal{M}$, we can preprocess visibility regions Vis(a) and Vis(b) during the computation of Vis(a) and Vis(b) that helps us to check if Vis$(a) \cap$ Vis$(b) = \emptyset$ in $\mathcal{O}(1)$ time.

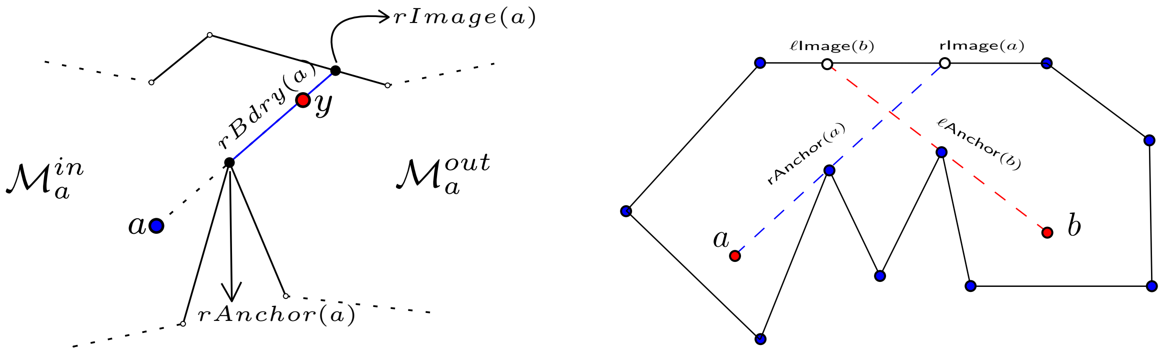
Observation 4.12. *Consider a pair of points $a, b \in \mathcal{M}$ with $x(a) < x(b)$. If $x(b) \leq x($xMax(Vis(a))), then* Vis$(a) \cap$ Vis$(b) \neq \emptyset$

Proof. If $x(b) \leq x($xMax(Vis(a))), then $l(b) \cap$ Vis$(a) \neq \emptyset$ and hence, Vis$(b) \cap$ Vis$(a) \neq \emptyset$ as from Theorem 4.11 we have that $\ell(b) \subset$ Vis(b). ☐

Lemma 4.13. *Consider a pair of points $a, b \in \mathcal{M}$ with $x(a) < x(b)$ and $x(b) > x($xMax(Vis(a))). Then* Vis$(a) \cap$ Vis$(b) \neq \emptyset$ *if and only if* rBdry(a) *intersects with* ℓBdry(b).

Proof. First, let us assume that Vis$(a) \cap$ Vis$(b) \neq \emptyset$. Then, for a pair of points $a, b \in \mathcal{M}$ such that $x(a) < x(b)$ and $x(b) > x($xMax(Vis(a))), the line segment rBdry(a) partitions $\mathcal{M}$ into two subpolygons $\mathcal{M}_a^{in}, \mathcal{M}_a^{out}$ such that $a \in \mathcal{M}_a^{in}$ and $b \in \mathcal{M}_a^{out}$. Also observe that, this rBdry(a) forms the boundary between the two partitions $\mathcal{M}_a^{in}$ and $\mathcal{M}_a^{out}$. We consider rBdry(a) to belong to the partition $\mathcal{M}_a^{in}$. Similarly, we can partition $\mathcal{M}$ into two subpolygons $\mathcal{M}_b^{in}$ and $\mathcal{M}_b^{out}$, by ℓBdry(b) as well, where we consider ℓBdry(b) to belong to the partition $\mathcal{M}_b^{in}$. We see that Vis$(a) \subseteq \mathcal{M}_a^{in}$ and Vis$(b) \subseteq \mathcal{M}_b^{in}$. This is true because if there exists a point $y \in$ Vis$(a) \cap \mathcal{M}_a^{out}$ then, the line of visibility from a to y lies entirely in $\mathcal{M}$ and hence must intersect rBdry(a) (as it partitions $\mathcal{M}$). This can only happen if y lies in rBdry(a), as rBdry(a) is collinear to a (see Figure 11a). But this leads to a contradiction as rBdry$(a) \subseteq \mathcal{M}_a^{in}$. A similar argument shows that Vis$(b) \subseteq \mathcal{M}_b^{in}$. Now, as Vis$(a) \cap$Vis$(b) \neq \emptyset$, it implies that $\mathcal{M}_a^{in} \cap \mathcal{M}_b^{in} \neq \emptyset$. But as $\mathcal{M}_a^{in}$ and $\mathcal{M}_b^{in}$ are two partitions of the polygon $\mathcal{M}$ whose boundaries are rBdry(a) and ℓBdry(b) respectively, they must intersect. (See Figure 11b).

Conversely, let us assume that rBdry(a) intersects with ℓBdry(b) at $c \in \mathcal{M}$. Now, observe that the point c is visible from both a and b. Hence, we conclude that Vis$(a) \cap$ Vis$(b) \neq \emptyset$. ☐



(a) a sees no point in the partition $\mathcal{M}_a^{out}$.   (b) Intersection of $\mathsf{rBdry}(a)$ and $\ell\mathsf{Bdry}(b)$.

Figure 11: Necessary and sufficient condition for two visibility regions to intersect.

Therefore, we have established a necessary and sufficient condition to determine whether $\mathsf{Vis}(a) \cap \mathsf{Vis}(b)$ is empty.

Lemma 4.14. *$\mathsf{VIG}(F)$ can be constructed in $\mathcal{O}(n|F| + |F|^2)$ time.*

Proof. Let $|F| = f$. The visibility polygon for a point $p \in \mathcal{M}$, denoted as $\mathsf{Vis}(p)$, can be determined in $\mathcal{O}(n)$ time, as demonstrated by [Gho07]. Consequently, computing visibility regions for all the points in F requires $\mathcal{O}(fn)$ time, which is also the time complexity for constructing the vertices of $\mathsf{VIG}(F)$. While determining $\mathsf{Vis}(p)$, we generate $\mathsf{rBdry}(p)$ and $\ell\mathsf{Bdry}(p)$. By utilizing Theorem 4.12 and Theorem 4.13, we can check whether $\mathsf{Vis}(a) \cap \mathsf{Vis}(b)$ is empty in $\mathcal{O}(1)$ time. Therefore, computing the edges of $\mathsf{VIG}(F)$ for F takes $\mathcal{O}(f^2)$ time. Thus, the graph $\mathsf{VIG}(F)$ can be fully constructed within $\mathcal{O}(nf + f^2)$ time. □

Observation 4.15. *Consider three distinct points p, q, r in a monotone polygon $\mathcal{M}$ with $x(p) < x(q) < x(r)$. If $\mathsf{Vis}(p) \cap \mathsf{Vis}(q) = \emptyset$ and $\mathsf{Vis}(q) \cap \mathsf{Vis}(r) = \emptyset$ then $\mathsf{Vis}(p) \cap \mathsf{Vis}(r) = \emptyset$.*

Proof. Since $\mathsf{Vis}(p) \cap \mathsf{Vis}(q) = \emptyset$ and $\mathsf{Vis}(q) \cap \mathsf{Vis}(r) = \emptyset$, the line $l(q)$ divides the polygon $\mathcal{M}$ into two separate regions, where one contains p and the other contains r. Thus, for $\mathsf{Vis}(p)$ to have an intersection with $\mathsf{Vis}(r)$, one of them must intersect $l(q)$, as $l(q)$ is responsible for partitioning $\mathcal{M}$ into distinct regions, each containing either p or r. Hence the proof. □

Theorem 4.15 directly implies the following. Consider a monotone polygon $\mathcal{M}$ and a witness set $W = \{w_1, w_2, \dots, w_k\}$ in $\mathcal{M}$. Then, for any witness point w_i where $i \in \{2, \dots, k-1\}$, if we can find another point q such that $x(w_{i-1}) < x(q) < x(w_{i+1})$ with $\mathsf{Vis}(w_{i-1}) \cap \mathsf{Vis}(q) = \emptyset$ and $\mathsf{Vis}(w_{i+1}) \cap \mathsf{Vis}(q) = \emptyset$ then $W \setminus \{w_i\} \cup \{q\}$ is also a witness set in $\mathcal{M}$. Below, we show that the visibility intersection graph of a monotone polygon is co-comparable.

Lemma 4.16. *Let F be a set of points in a monotone polygon. Then $\mathsf{VIG}(F)$ is a co-comparable graph.*

Proof. We show that the complement graph $\widetilde{\mathsf{VIG}(F)}$ of $\mathsf{VIG}(F)$ is comparable. To that end, we prove that the edges of $\widetilde{\mathsf{VIG}(F)}$ has a transitive orientation, i.e., an assignment of directions to the edges of the graph (i.e. an orientation of the graph) such that the adjacency relation of the resulting directed graph is transitive: whenever there exist directed edges (p, q) and (q, r), there must exists an edge (p, r) where p, q, r are the vertices in $\widetilde{\mathsf{VIG}(F)}$.

We assign a transitive orientation to the edges of $\widetilde{\mathsf{VIG}(F)}$ in the following method: Begin by ordering the vertices in $\widetilde{\mathsf{VIG}(F)}$ based on sorting the points in F according to their x-coordinates, with a vertex p preceding a vertex q if $x(p) > x(q)$. When an edge exists between any two vertices p and q where p comes before q, we direct the edge from p to q. Following this directional assignment, if there is a directed edge from p to q and another from q to r in $\widetilde{\mathsf{VIG}(F)}$, it follows that $x(p) > x(q) > x(r)$ and $\mathsf{Vis}(p) \cap \mathsf{Vis}(q) = \emptyset$ as well as $\mathsf{Vis}(q) \cap \mathsf{Vis}(r) = \emptyset$. By applying Theorem 4.15, we deduce $\mathsf{Vis}(p) \cap \mathsf{Vis}(r) = \emptyset$, necessitating a directed edge from p to r in the orientation. Because p, q, r are chosen arbitrarily, we can conclude that $\widetilde{\mathsf{VIG}(F)}$ forms a comparability graph. □

Proof of Theorem 1.2. By Theorem 4.14, the graph $\mathsf{VIG}(S)$ can be formed in $\mathcal{O}(|V(\mathcal{P})||S| + |S|^2)$ time. Given that $\mathsf{VIG}(S)$ is co-comparable (due to Theorem 4.16), Theorem 4.2 implies that a maximum independent set in $\mathsf{VIG}(Q)$ can be determined in $\mathcal{O}(|S|^2)$ time [MS99]. Thus, the overall running time is $\mathcal{O}(|S|^2 + |V(\mathcal{P})||S|)$. □

5 Witness Set in a Monotone Polygon

Overview of this section. In this section, our aim is to find an algorithm for the Witness Set in monotone polygons. To do that, we first build the preliminaries needed to develop our algorithm. We try to find a discrete set inside a monotone polygon $\mathcal{M}$ that will suffice to contain a solution of the Witness Set in $\mathcal{M}$. We propose an algorithm (Algorithm 1)for the Witness Set in $\mathcal{M}$ running in time $r^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$, accordingly it creates a potential witness set of $\mathcal{M}$, i.e., it generates a set of points which contain the set of vertices of $\mathcal{M}$, the midpoints of the line segments joining any two reflex vertices of $\mathcal{M}$, and a new set of points on $\mathcal{M}$, which we will denote by $V(\mathcal{M})$, $\mathcal{R}_{mid}$ and $\mathcal{Z}_{\mathtt{mid}}$, respectively (defined later). Note that for a witness set $X = \{w_1, \ldots, w_\ell\}$ with $x(w_1) < \ldots < x(w_\ell)$, a point $w \notin X$ to check whether $X \cup \{w\}$ becomes a witness set, we only need to check the intersection of $\mathsf{Vis}(w)$ with one pair of witnesses in X, which are the left and right points of w in X, that is, w_i and w_{i+1}, where $x(w_i) < x(w) < x(w_{i+1})$ (due to Theorem 4.15). We then prove the correctness of our algorithm. After obtaining a potential witness set of $\mathcal{M}$, we use the result from the previous sections to obtain the visibility intersection graph $\mathsf{VIG}(\mathcal{M})$. Since $\mathsf{VIG}(\mathcal{M})$ is an outerstring graph (due to Theorem 3.11 and Theorem 3.12), we can obtain maximum independent set via an algorithm presented by the authors in [KMPV17], to obtain an optimal Witness set of $\mathcal{M}$ (due to Theorem 3.13). Finally, we conclude with the main result of this section.

Theorem 1.3. Witness Set *in a monotone polygon $\mathcal{M}$ with n vertices is solvable in $r^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time, where r denotes the number of reflex vertices in $\mathcal{M}$ and k is the size of a maximum witness set in $\mathcal{M}$.*

In the following, we formally describe our procedure for finding a maximum witness set in a monotone polygon $\mathcal{M}$. Let $W = \{w_1, \ldots, w_k\}$ be a solution of the Witness Set in $\mathcal{M}$ where $\mathtt{ws}(\mathcal{M}) = k$ and $x(w_1) < \ldots < x(w_k)$ (due to $x-$monotonicity). We characterize a region of special importance for each witness in W (called $\mathsf{tRegion}(\cdot,\cdot)$, formally defined later, Theorem 5.2), which in turn will help us to obtain a discretization of $\mathcal{M}$ into a finite point set Q such that the size of a maximum witness set $Z \subseteq Q$ is k.

Step 1: Discretization

We start by introducing some essential definitions and notation for our purposes.

To start with, we define a set $\mathsf{Territory}(w, W)$ and a region denoted as $\mathsf{tRegion}(w)$ for a witness $w \in W$, as described below.

Definition 5.1 ($\mathsf{Territory}(\cdot,\cdot)$). Let W be a witness set of a monotone polygon $\mathcal{M}$. For $w \in W$, $\mathsf{Territory}(w, W)$ is a non-empty connected region in $\mathcal{M}$ that satisfies

1. $\mathsf{Vis}(w) \subseteq \mathsf{Territory}(w, W)$
2. $\mathsf{Territory}(w, W) \cap \mathsf{Vis}(w') = \emptyset$ for every $w' \in W \setminus \{w\}$

In Theorem 5.1, condition (1) requires that $\mathsf{Territory}(w, W)$ fully contains $\mathsf{Vis}(w)$, and condition (2) ensures it does not intersect $\mathsf{Vis}(w')$ for any $w' \in W \setminus \{w\}$. Note that $\mathsf{Vis}(w)$ itself satisfies both conditions, so we treat $\mathsf{Territory}(w, W) = \mathsf{Vis}(w)$. For each $w \in W$, we define a connected region $\mathsf{tRegion}(w, W)$, and in Section 5.1, we show that such a region always exists.

Definition 5.2 ($\mathsf{tRegion}(\cdot, \cdot)$)**.** Let W be a witness set of a monotone polygon $\mathcal{M}$. For $w \in W$, $\mathsf{tRegion}(w, W)$ is a connected region in $\mathsf{Territory}(w, W)$ such that $\mathsf{Vis}(\mathsf{tRegion}(w, W)) \subseteq \mathsf{Territory}(w, W)$.

The definition mentioned above leads to the following observation, which is easy to follow.

Observation 5.3. *For a witness set W of a monotone polygon $\mathcal{M}$ and $w \in W$, if* $\mathsf{tRegion}(w, W) \neq \emptyset$ *then for any point $z (\neq w) \in \mathsf{tRegion}(w, W)$, $(W \setminus \{w\}) \cup \{z\}$ is also a witness set in $\mathcal{M}$.*

5.1 Characterizations of tRegion()

In this section, we characterize and define the $\mathsf{tRegion}(w, W)$ for every $w \in W$, for which both $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$ exists, in detail. The case for the leftmost and the rightmost witness will be dealt with later. Recall that for any point $z \in \mathsf{tRegion}(w, W)$, $\mathsf{Vis}(z) \subseteq \mathsf{Territory}(w, W)$ whereas $\mathsf{tRegion}(w, W)$ is a connected region that is fully contained in $\mathsf{Territory}(w, W)$. Below we define a notion of $\mathsf{rChord}(p)$ and $\ell\mathsf{Chord}(p)$ for any point p in $\mathcal{M}$.

Definition 5.4 ($\mathsf{rChord}()$ and $\ell\mathsf{Chord}()$)**.** For a point p in a monotone polygon $\mathcal{M}$, the maximal chord in $\mathcal{M}$ passing through the vertices p and $\mathsf{rImage}(p)$ is defined as $\mathsf{rChord}(p)$. Similarly, the maximal chord in $\mathcal{M}$ passing through the vertices p and $\ell\mathsf{Image}(p)$ is defined as $\ell\mathsf{Chord}(p)$.

Definition 5.5 ($\mathsf{pChain}_u()$ and $\mathsf{pChain}_\ell()$)**.** For a pair of points $a, b \in \mathsf{uC}(\mathcal{M})$ (resp, $\in \ell\mathsf{C}(\mathcal{M})$), we use $\mathsf{pChain}_u(a, b)$ (resp, $\mathsf{pChain}_\ell(a, b)$) to denote the path from a to b along $\mathsf{uC}(\mathcal{M})$ (resp, $\ell\mathsf{C}(\mathcal{M})$).

Definition 5.6 (hill)**.** For a pair of distinct points x, y in a polygon $\mathcal{P}$, which cannot see each other, i.e., $y \notin \mathsf{Vis}(x)$, and vice-versa, the line joining x and y, must intersect the exterior $\mathsf{eX}(\mathcal{P})$ of $\mathcal{P}$. Specifically, $\overline{xy} \cap \mathsf{eX}(\mathcal{P}) = \{\ell_1, \cdots, \ell_j\}$, where ℓ_i's are open line segments[2] with their endpoints on $\partial(\mathcal{P})$. For any line segment ℓ_i, we name its two endpoints as h_i^1 and h_i^2. For any $i = 1, \cdots, j$, the polygonal chain (part of $\partial(\mathcal{P})$) from h_i^1 to h_i^2 is called a *hill* $\mathcal{H}_\rangle$ corresponding to a pair of mutually invisible points x and y (Refer to Figure 12 for an illustration).

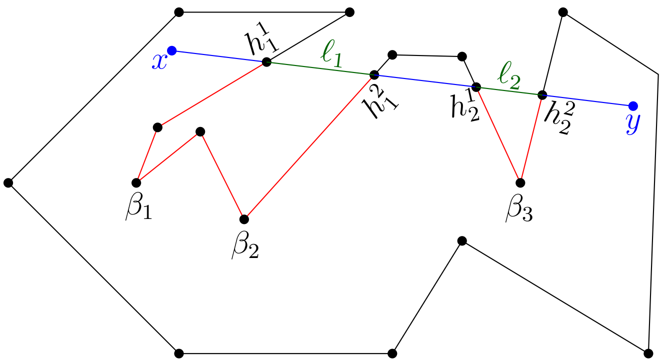


Figure 12: The points $x, y \in \mathcal{P}$ cannot see each other. $\ell_1 \cup \ell_2 = \overline{xy} \cap \partial(\mathcal{P})$. The polygonal chains from h_1^1 to h_1^2 and h_2^1 to h_2^2 are two *hills* $\mathcal{H}_\infty, \mathcal{H}_\in$ corresponding to x, y. β_1, β_2 are two reflex vertices present on $\mathcal{H}_\infty$ and β_3 is another reflex vertex present on the hill $\mathcal{H}_\in$.

The immediate observation concerns when two points in $\mathcal{P}$ cannot see each other.

[2]By an open line segment (a, b) we mean the line segment $\overline{ab}$ except the two endpoints a and b.

Observation 5.7. *For a pair of distinct points x, y in a polygon $\mathcal{P}$ such that $y \notin \mathsf{Vis}(x)$, there exists at least one hill corresponding to x, y (note that there might exist multiple hills corresponding to x, y). Any such hill (which is a chain in $\partial(\mathcal{P})$) contains at least one reflex vertex (See Figure 12).*

Proof. The two endpoints h_i^1, h_i^2 of any hill $\mathcal{H}_\rangle$ are two points on $\partial(\mathcal{P})$ which cannot see each other. If the polygonal chain between them consists of only convex vertices, it means that h_i^1 and h_i^2 can see each other, which is a contradiction. □

We now propose a claim on $\mathsf{tRegion}(\cdot, \cdot)$ which will be crucial in our paper hereon.

Claim 5.8. *For a witness $w \in W$, there exists a non-empty $\mathsf{tRegion}(w, W) \setminus \{w\}$ such that it is either an open line segment or $\mathsf{tRegion}(w, W)$ has a non-empty intersection with $\partial(\mathcal{M})$.*

Proof. Observe that, for any witness w, the intersection of its visibility region $\mathsf{Vis}(w)$ with another visibility region of a witness w' depends on its $\mathsf{rBdry}(w)$ and $\ell\mathsf{Bdry}(w)$ (Due to Theorem 4.13). Now, we consider the following two cases.

Case 1: The witness w, $\mathsf{rAnchor}(w)$ and $\ell\mathsf{Anchor}(w)$ are collinear. In this case, we have that the $\mathsf{rChord}(w)$ and $\ell\mathsf{Chord}(w)$ (as in Theorem 5.4) are essentially the same line (See Figure 17 for an example). So, if we move the witness w to any other point w' along the line segment $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)}$ (except the two points $\mathsf{rAnchor}(w)$ and $\ell\mathsf{Anchor}(w)$), then $\mathsf{rAnchor}(w')$ and $\ell\mathsf{Anchor}(w')$ remain unchanged and therefore their respective $\mathsf{rBdry}(w')$ and $\ell\mathsf{Bdry}(w')$ remains the same as in the case of the witness w. Thus, since the intersection of $\mathsf{Vis}(w')$ with any other visibility region of a witness depends on the position of its $\mathsf{rBdry}(w')$ and $\ell\mathsf{Bdry}(w')$ (due to Theorem 4.13), $\mathsf{Vis}(w')$ does not intersect with any other visibility region of another witness. So, by Theorem 5.1, $\mathsf{Vis}(w') \subseteq \mathsf{Territory}(w, W)$, which means that $w' \in \mathsf{tRegion}(w, W)$. This shows that in this case the open line segment $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)} = \overline{\mathsf{rAnchor}(w')\ell\mathsf{Anchor}(w')} \subseteq \mathsf{tRegion}(w, W)$.

Case 2: The witness w, $\mathsf{rAnchor}(w)$ and $\ell\mathsf{Anchor}(w)$ are not collinear. In this case, we will show that either $w \in \partial(\mathcal{M})$ (in which case we need not show anything in particular) or $\mathsf{tRegion}(w, W)$ is non-empty and that $\mathsf{tRegion}(w, W) \cap \partial(\mathcal{M}) \neq \emptyset$, or an open line segment $(r_1, r_2) \subseteq \mathsf{tRegion}(w, W)$, where r_1 and r_2 are two reflex vertices of $\mathcal{M}$. As w, $\mathsf{rAnchor}(w)$ and $\ell\mathsf{Anchor}(w)$ are not collinear, the respective $\ell\mathsf{Chord}(w)$ and $\mathsf{rChord}(w)$ are two distinct lines intersecting at w. Let the two endpoints of $\ell\mathsf{Chord}(w)$ and $\mathsf{rChord}(w)$ other than $\ell\mathsf{Image}(w)$ and $\mathsf{rImage}(w)$ be a and b, respectively. Now, observe that if we move w to any other point w^* along the $\ell\mathsf{Chord}(w)$ between $\overline{\ell\mathsf{Anchor}(w)a}$ (similarly along $\mathsf{rChord}(w)$ between $\overline{\mathsf{rAnchor}(w)b}$, resp.), then the visibility region changes but the $\ell\mathsf{Chord}(w^*)$ ($\mathsf{rChord}(w^*)$, resp.) does not. Let us look at one of the above specific cases to understand the substitution of w by w^* in a better way. If we move w to a point w^* along $\ell\mathsf{Chord}(w)$ towards a, then the visibility region may increase, or decrease on the right side of $\mathsf{Vis}(w)$, but it remains unchanged on the left side as the $\ell\mathsf{Chord}(w)$ remains unchanged. Similarly, this works if we move w towards $\ell\mathsf{Anchor}(w)$. However, the movement of w in these two opposite directions results in two different outcomes. If the visibility region increases while moving w towards a, then the visibility region decreases if we move w towards $\ell\mathsf{Anchor}(w)$, and vice-versa. This is because the change in visibility region while moving w along $\overline{\ell\mathsf{Anchor}(w)a}$ is actually affected by the rotation of $\mathsf{rChord}(w)$ pivoting about the reflex vertex $\mathsf{rAnchor}(w)$ (See Figure 13). Let us name the movement of w along the $\ell\mathsf{Chord}(w)$ in the two opposite directions by $\ell Cdir_1$ and $\ell Cdir_2$. In a similar manner, let us name the two opposite directions of movement (one movement towards $\mathsf{rAnchor}(w)$ and another movement is towards b) of w along the $\mathsf{rChord}(w)$ in between $\overline{\mathsf{rAnchor}(w)b}$ as $rCdir_1$ and $rCdir_2$. Essentially, we are denoting the *direction of movements* of w towards any four of the points $\ell\mathsf{Anchor}(w), a$ (along $\ell\mathsf{Chord}(w)$) and $\mathsf{rAnchor}(w), b$ (along $\mathsf{rChord}(w)$)

by $\ell Cdir_1$, $\ell Cdir_2$, $rCdir_1$, and $rCdir_2$, respectively (See Figure 14). Note that if we choose w^* by moving w towards any of these four directions, exactly for two of the directions the visibility region $\mathsf{Vis}(w^*)$ increases, and for exactly two other directions, $\mathsf{Vis}(w^*)$ decreases. To be precise, moving w towards the directions $\ell Cdir_1$ and $\ell Cdir_2$ results in opposite outcomes for the change in the visibility area, $\mathsf{Vis}(w)$. The same goes for the two directions $rCdir_1$ and $rCdir_2$. So, if we can choose w^* in such a direction that the visibility region decreases, then we have that $\mathsf{Vis}(w^*) \subseteq \mathsf{Vis}(w) \subseteq \mathsf{Territory}(w, W)$. So, by Theorem 5.2, $w^* \in \mathsf{tRegion}(w, W)$, which means that $\mathsf{tRegion}(w, W) \neq \emptyset$.

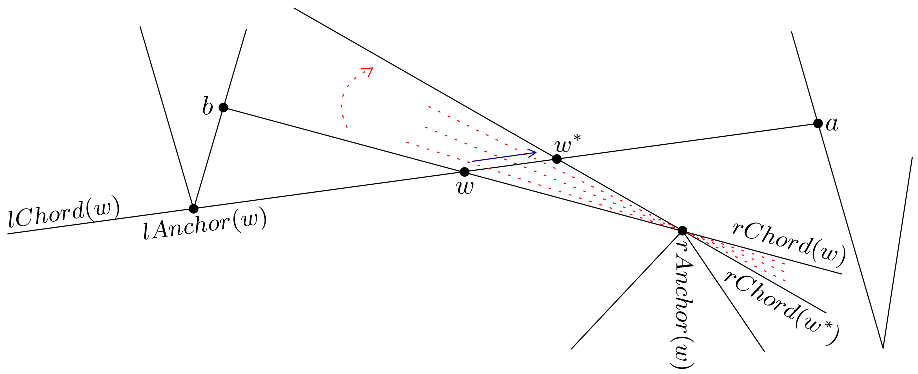


Figure 13: w is the witness. The change in visibility area when w is moved along $\ell\mathsf{Chord}(w)$ to another point w^* is depicted.

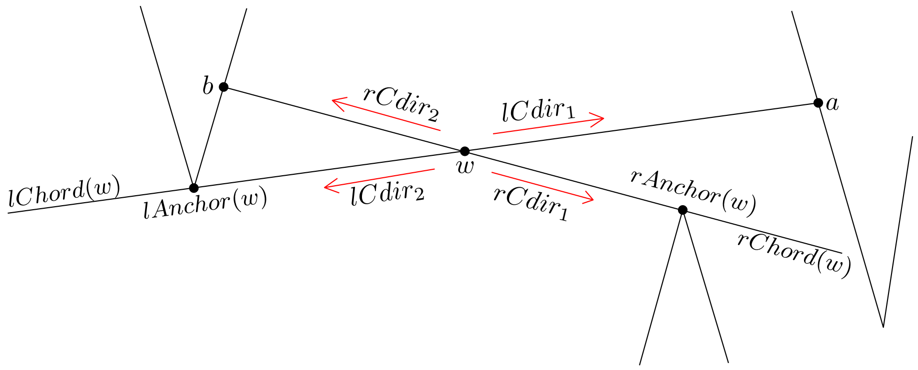


Figure 14: w is the witness. a, b are the endpoints of $\ell\mathsf{Chord}(w)$ and $\mathsf{rChord}(w)$ respectively, other than the endpoints $\ell\mathsf{Image}(w)$ and $\mathsf{rImage}(w)$. The four directions of moving w along either $\ell\mathsf{Chord}(w)$ or $\mathsf{rChord}(w)$ are shown.

We now move forward to our next part of the proof, which is to show that either an open segment $(r_1, r_2) \subseteq \mathsf{tRegion}(w, W)$ where r_1 and r_2 are two reflex vertices of $\mathcal{M}$, or $\mathsf{tRegion}(w, W) \cap \partial(\mathcal{M}) \neq \emptyset$. If our choice of $w^* \in \mathsf{tRegion}(w, W)$ lies on either of $\overline{wa}$ or $\overline{wb}$, then we can choose w^* to be the endpoints a or b itself, as choosing a reduces the visibility region on the right side of $\mathsf{Vis}(w)$ without changing the $\ell\mathsf{Chord}(w)$ and choosing b reduces the visibility region on the left side of $\mathsf{Vis}(w)$ without changing the $\mathsf{rChord}(w)$. So, in that case, as both $a, b \in \partial(\mathcal{M})$, we have that $\mathsf{tRegion}(w, W) \cap \partial(\mathcal{M}) \neq \emptyset$. On the other hand, if our choice of $w^* \in \mathsf{tRegion}(w, W)$ was restricted to the lines $\overline{w\ell\mathsf{Anchor}(w)}$ and $\overline{w\mathsf{rAnchor}(w)}$, then we cannot directly choose w^* to be $\ell\mathsf{Anchor}(w)$ or $\mathsf{rAnchor}(w)$, as then, $\ell\mathsf{Anchor}(w^*) = \ell\mathsf{Anchor}(\ell\mathsf{Anchor}(w))$ would not be same as $\ell\mathsf{Anchor}(w)$ and also $\mathsf{rAnchor}(w^*) = \mathsf{rAnchor}(\mathsf{rAnchor}(w))$ would not be the same as $\mathsf{rAnchor}(w)$. This could result in an increase in the visibility region of w^*, $\mathsf{Vis}(w^*)$, which will not help in our existence and construction of the $\mathsf{tRegion}(w, W)$. To resolve this, we consider two sub-cases.

- **The open segment** $(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$ **lies totally inside** $\mathcal{M}$ (See Figure 15(a)). In this case, any point on the open segment $(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$ can be considered as w^*, where w^* is in $\mathsf{tRegion}(w, W)$. In other words, the open segment $(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w)) \subseteq \mathsf{tRegion}(w, W)$. A short proof of this is as follows:
  Let w^* be any point on $(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$. Then, we draw two lines ℓ_1, ℓ_2 from w^*, one of which is parallel to $\overline{w\ell\mathsf{Anchor}(w)}$ and another one is parallel to $\overline{w\mathsf{rAnchor}(w)}$, respectively. Now let $\ell_1 \cap \overline{w\mathsf{rAnchor}(w)} = p_1$ and $\ell_2 \cap \overline{w\ell\mathsf{Anchor}(w)} = p_2$, respectively. Then, by moving our initial witness w to p_1, we get that $\mathsf{Vis}(p_1) \subseteq \mathsf{Vis}(w)$. Again, by moving p_1 to w^*, we are essentially moving p_1 in the direction parallel to $\ell\mathsf{Anchor}(w)$. So, $\mathsf{Vis}(w^*) \subseteq \mathsf{Vis}(p_1)$. Therefore, $\mathsf{Vis}(w^*) \subseteq \mathsf{Vis}(w)$, which implies that $w^* \in \mathsf{tRegion}(w, W)$, as required.
- **The open segment** $(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$ **does not lie totally inside** $\mathcal{M}$**, i.e.,** $\ell\mathsf{Anchor}(w)$ **and** $\mathsf{rAnchor}(w)$ **is not visible to each other** (See Figure 15(b)). In this case, due to Theorem 5.7, there must exist a hill containing a reflex vertex r that intersects the line $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)}$. Observe that the orientation of this reflex vertex forces it to lie inside the triangle $\triangle\ell\mathsf{Anchor}(w)w\mathsf{rAnchor}(w)$. Then, this reflex vertex r, which also lies on $\partial(\mathcal{M})$, will lie in $\mathsf{tRegion}(w, W)$. This is again by the same argument as provided in the case above. $\mathsf{Vis}(r) \subseteq \mathsf{Vis}(p_1) \subseteq \mathsf{Vis}(w)$. So, $r \in \partial(\mathcal{M})$ lies in $\mathsf{tRegion}(w, W)$, which means that $\mathsf{tRegion}(w, W) \cap \partial(\mathcal{M}) \neq \emptyset$.

□

Thus far, we can see that for a witness $w \in W$, the $\mathsf{tRegion}(w, W)$ can be classified into two categories based on the relative positions of $\ell\mathsf{Anchor}(w)$, $\mathsf{rAnchor}(w)$, and w: (i) open line segments and (ii) connected regions with non-empty interior. Below, we provide a more detailed examples of construction of these classifications.

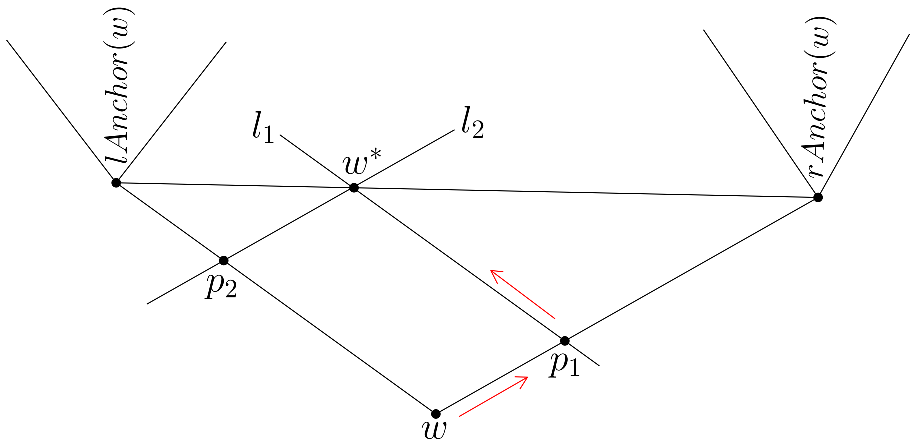
Case 1. Both the vertices $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$ belongs to same chain. There are four sub-cases.

1.1 Here we consider that both belongs to the lower chain $\ell\mathsf{C}(\mathcal{M})$, and the witness w lies inside the region bounded by $\mathsf{pChain}_\ell(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$, $\overline{\mathsf{rAnchor}(w)\ell\mathsf{Anchor}(w)}$. Let a and b be the endpoints of $\mathsf{rChord}(w)$ and $\ell\mathsf{Chord}(w)$, respectively where $a \neq \mathsf{rImage}(w)$ and $b \neq \ell\mathsf{Image}(w)$. Here $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries $\overline{wb}$, $\mathsf{pChain}_\ell(b, a)$ and $\overline{aw}$. See Figure 16a for an illustration.

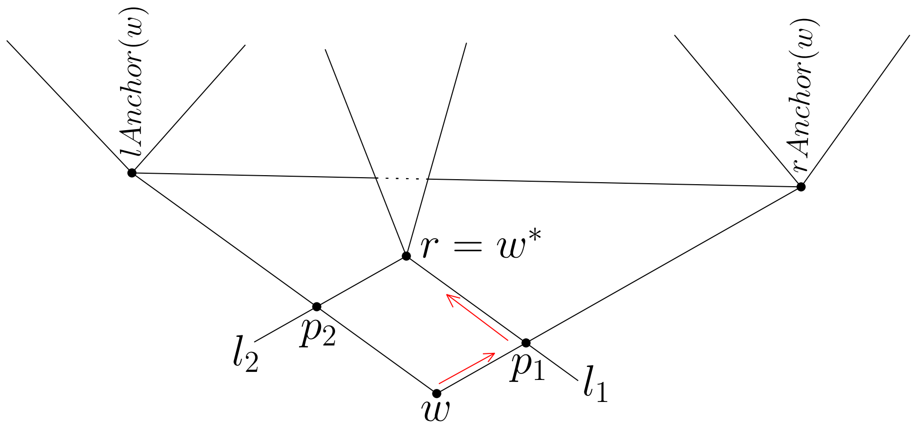
1.2 Here we consider that both belongs to the lower chain $\ell\mathsf{C}(\mathcal{M})$, but the witness w lies outside the region bounded by $\mathsf{pChain}_\ell(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$, $\overline{\mathsf{rAnchor}(w)\ell\mathsf{Anchor}(w)}$. Then, the $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries $\mathsf{pChain}_\ell(\mathsf{rAnchor}(w), \ell\mathsf{Anchor}(w)$, $\overline{\ell\mathsf{Anchor}(w)w}$, $\overline{w\mathsf{rAnchor}(w)}$. See Figure 16b for an illustration.

1.3 Now we consider that both belongs to the upper chain $\mathsf{uC}(\mathcal{M})$ and the witness w lies inside the region bounded by $\mathsf{pChain}_u(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$, $\overline{\mathsf{rAnchor}(w)\ell\mathsf{Anchor}(w)}$. Let a and b be the endpoints of $\mathsf{rChord}(w)$ and $\ell\mathsf{Chord}(w)$, respectively where $a \neq \mathsf{rImage}(w)$ and $b \neq \ell\mathsf{Image}(w)$. Here $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries $\overline{wb}$, $\mathsf{pChain}_u(b, a)$ and $\overline{aw}$. See Figure 16c for an illustration.

1.4 Now we consider that both belongs to the upper chain $\mathsf{uC}(\mathcal{M})$ and the witness w lies outside the region bounded by $\mathsf{pChain}_u(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$, $\overline{\mathsf{rAnchor}(w)\ell\mathsf{Anchor}(w)}$. Then, the $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries



(a) $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)}$ lies totally inside the polygon $\mathcal{M}$.



(b) $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)}$ does not lie totally inside the polygon $\mathcal{M}$.

Figure 15: Cases where $\ell\mathsf{Anchor}(w)$, $\mathsf{rAnchor}(w)$ and $\boldsymbol{w}$ are not collinear and by moving (w) towards $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$ reduces the visibility region.

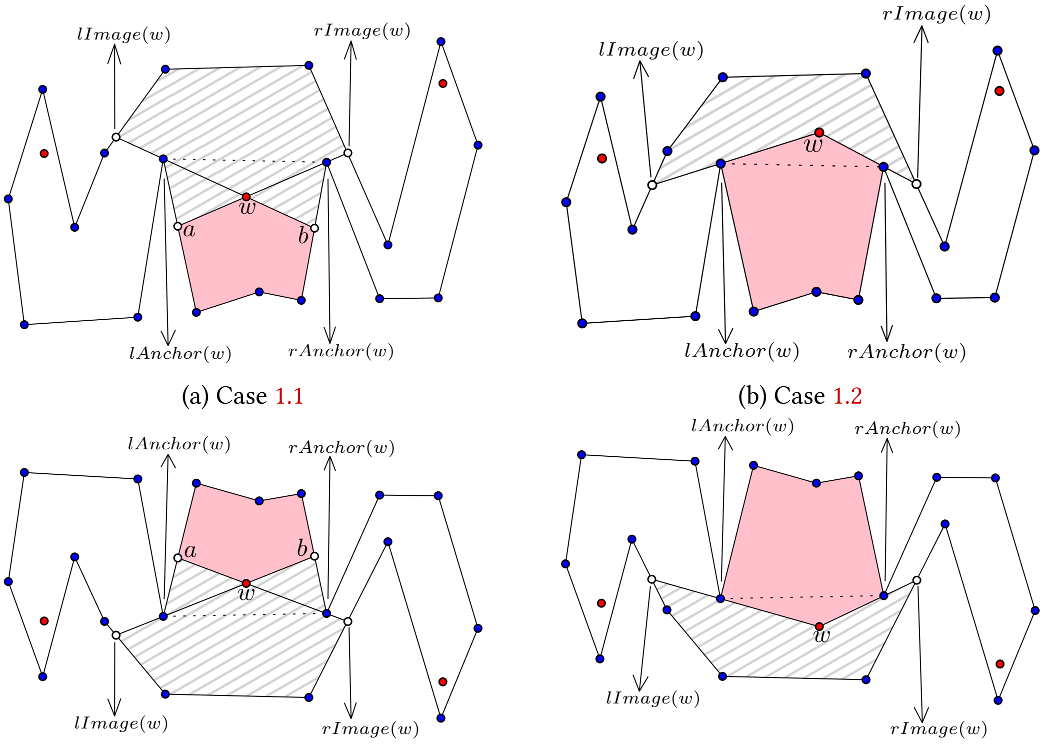
$\mathsf{pChain}_u(\mathsf{rAnchor}(w), \ell\mathsf{Anchor}(w)$, $\overline{\ell\mathsf{Anchor}(w)w}$, $\overline{w\mathsf{rAnchor}(w)}$. See Figure 16d for an illustration.

Case 2. $\boldsymbol{\ell}$Anchor(w) and rAnchor(w) belong to different chains. There are two subcases:

2.1. $\boldsymbol{\ell}$Anchor(w), rAnchor(w) and $\boldsymbol{w}$ are collinear. First we consider the situation that $\ell\mathsf{Anchor}(w) \in \mathsf{uC}(\mathcal{M})$ and $\mathsf{rAnchor}(w) \in \ell\mathsf{C}(\mathcal{M})$. Here $\mathsf{tRegion}(w, W)$ is defined by the open line segment $(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$ (See Figure 17a for an illustration). The argument for $\ell\mathsf{Anchor}(w) \in \ell\mathsf{C}(\mathcal{M})$ and $\mathsf{rAnchor}(w) \in \mathsf{uC}(\mathcal{M})$ in this case is exactly same (See Figure 17b).

2.2. $\boldsymbol{\ell}$Anchor(w), rAnchor(w) and $\boldsymbol{w}$ are not collinear. We will consider four distinct subcases that depend on the location of the witness and the placements of $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$. Although these cases are symmetric with respect to reflection and rotation, we will present each one individually for the sake of clarity and completeness.

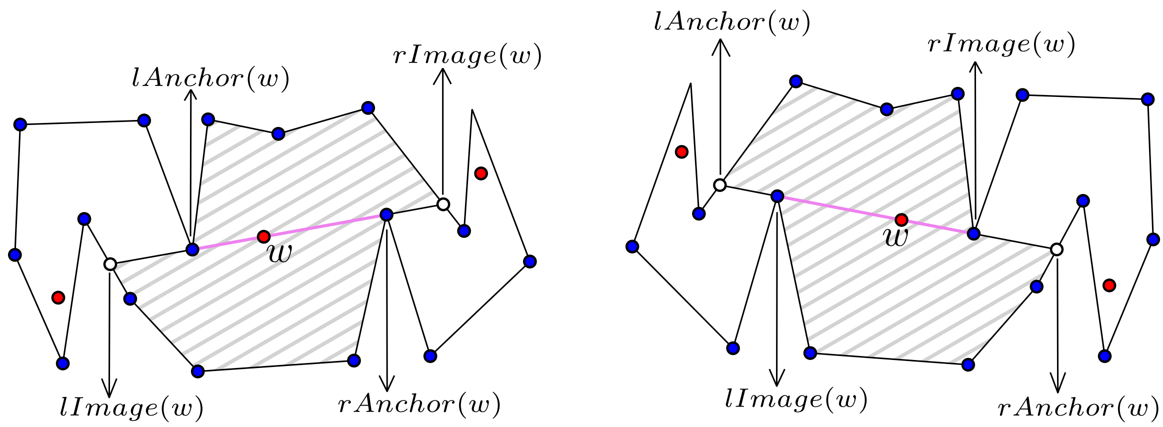
2.2.a Here, we consider $\ell\mathsf{Anchor}(w)$ belongs to the upper chain $\mathsf{uC}(\mathcal{M})$ and $\mathsf{rAnchor}(w)$ belongs to the lower chain $\ell\mathsf{C}(\mathcal{M})$; and the witness w lies inside the region defined by the boundaries $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)}$, $\mathsf{pChain}_\ell(\mathsf{rAnchor}(w), \mathsf{xMin}(\mathcal{M}))$ and $\mathsf{pChain}_u(\mathsf{xMin}(\mathcal{M}), \ell\mathsf{Anchor}(w))$. Let c and d denote the endpoints of $\mathsf{rChord}(w)$ and $\ell\mathsf{Chord}(w)$, respectively where $x(c) < x(w)$ and $x(d) > x(w)$.



(a) Case 1.1 (b) Case 1.2

(c) Case 1.3 (d) Case 1.4

Figure 16: Red points are set of witnesses. Here $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$ belongs to same chain. Pink region is defined for $\mathsf{tRegion}(w)$ which is a part of $\mathsf{Territory}(w)$.



(a) $\ell\mathsf{Anchor}(w) \in \mathsf{uC}(\mathcal{M}), \mathsf{rAnchor}(w) \in \ell\mathsf{C}(\mathcal{M})$ (b) $\ell\mathsf{Anchor}(w) \in \ell\mathsf{C}(\mathcal{M}), \mathsf{rAnchor}(w) \in \mathsf{uC}(\mathcal{M})$

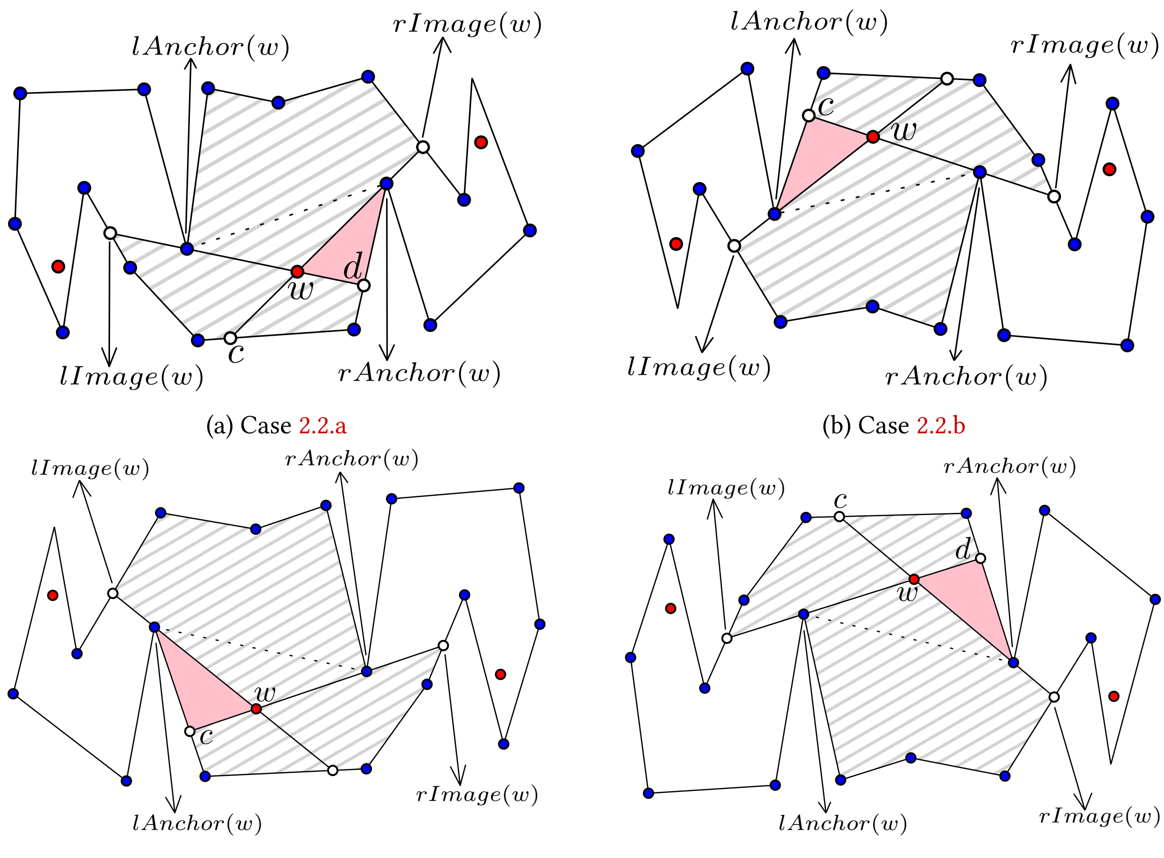
Figure 17: Case 2.1. $\ell\mathsf{Anchor}(w)$, $\mathsf{rAnchor}(w)$ and $\boldsymbol{w}$ are collinear.

Then, the $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries $\overline{wd}$, $\mathsf{pChain}_\ell(d, \mathsf{rAnchor}(w))$ and $\overline{\mathsf{rAnchor}(w)w}$, except the point $\mathsf{rAnchor}(w)$. (See Figure 18a for an illustration).

2.2.b Here, we consider $\ell\mathsf{Anchor}(w)$ belongs to the upper chain $\mathsf{uC}(\mathcal{M})$ and $\mathsf{rAnchor}(w)$ belongs to the lower chain $\ell\mathsf{C}(\mathcal{M})$; and the witness w lies inside the region defined by the boundaries $\overline{\mathsf{rAnchor}(w)\ell\mathsf{Anchor}(w)}$, $\mathsf{pChain}_u(\ell\mathsf{Anchor}(w), \mathsf{xMax}(\mathcal{M}))$ and $\mathsf{pChain}_\ell(\mathsf{xMax}(\mathcal{M}), \mathsf{rAnchor}(w))$. Let c denote the endpoint of $\mathsf{rChord}(w)$ where $x(c) < x(w)$. Then, the $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries $\overline{wc}$, $\mathsf{pChain}_\ell(c, \ell\mathsf{Anchor}(w))$ and $\overline{\ell\mathsf{Anchor}(w)w}$, except the point $\ell\mathsf{Anchor}(w)$. (See Figure 18b for an illustration).

2.2.c Here, we consider $\ell\mathsf{Anchor}(w)$ belongs to the lower chain $\ell\mathsf{C}(\mathcal{M})$ and $\mathsf{rAnchor}(w)$ belongs to the upper chain $\mathsf{uC}(\mathcal{M})$; and the witness w lies inside the region defined by the boundaries $\overline{\mathsf{rAnchor}(w)\ell\mathsf{Anchor}(w)}$, $\mathsf{pChain}_\ell(\ell\mathsf{Anchor}(w), \mathsf{xMax}(\mathcal{M}))$ and $\mathsf{pChain}_u(\mathsf{xMax}(\mathcal{M}), \mathsf{rAnchor}(w))$. Let c denote the endpoint of $\mathsf{rChord}(w)$ where $x(c) < x(w)$. Then, the $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries $\overline{wc}$, $\mathsf{pChain}_\ell(c, \ell\mathsf{Anchor}(w))$ and $\overline{\ell\mathsf{Anchor}(w)w}$, except the point $\ell\mathsf{Anchor}(w)$. (See Figure 18c for an illustration).

2.2.d Here, we consider $\ell\mathsf{Anchor}(w)$ belongs to the lower chain $\ell\mathsf{C}(\mathcal{M})$ and $\mathsf{rAnchor}(w)$ belongs to the upper chain $\mathsf{uC}(\mathcal{M})$; and the witness w lies inside the region defined by the boundaries $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)}$, $\mathsf{pChain}_u(\mathsf{rAnchor}(w), \mathsf{xMax}(\mathcal{M}))$ and $\mathsf{pChain}_\ell(\mathsf{xMax}(\mathcal{M}), \ell\mathsf{Anchor}(w))$. Let c and d denote the endpoints of $\mathsf{rChord}(w)$ and $\ell\mathsf{Chord}(w)$, respectively where $x(c) < x(w)$ and $x(d) > x(w)$. Then, the $\mathsf{tRegion}(w, W)$ is the connected region defined by the boundaries $\overline{wd}$, $\mathsf{pChain}_\ell(d, \mathsf{rAnchor}(w))$ and $\overline{\mathsf{rAnchor}(w)w}$, except the point $\mathsf{rAnchor}(w)$.(See Figure 18d for an illustration).



(a) Case 2.2.a

(b) Case 2.2.b

(c) Case 2.2.c

(d) Case 2.2.d

Figure 18: Case 2.2. $\ell\mathsf{Anchor}(w)$, $\mathsf{rAnchor}(w)$ and $\boldsymbol{w}$ are collinear. The red points are a set of witnesses. Here $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$ belong to the same chain. Pink region is defined for $\mathsf{tRegion}(w, W)$.

Observe that for any witness $w \in W$, the region $\mathsf{tRegion}(w, W)$ is either of following three types;(1) It contains some vertex from $V(\mathcal{M})$ (see Figures 16, 18a, and 18d), (2) It contains no vertex from $V(\mathcal{M})$ (see (Figures 18b and 18c), and (3) It contains all points of the segment $\overline{\beta\beta'}$ except the points $\{\beta, \beta'\}$ where β and β' is a pairwise reflex vertices that is visible to each other (See Figure 17).

Based on the above discussion, we partition the set W into $W_{\texttt{int}}$, $W^{\texttt{good}}_{\texttt{bdry}}$, and $W^{\texttt{bad}}_{\texttt{bdry}}$.

Definition 5.9 ($W_{\texttt{int}}$, $W^{\texttt{good}}_{\texttt{bdry}}$ and $W^{\texttt{bad}}_{\texttt{bdry}}$)**.** We partition the witness set W into three subsets, labeled $W_{\texttt{int}}$, $W^{\texttt{good}}_{\texttt{bdry}}$ and $W^{\texttt{bad}}_{\texttt{bdry}}$, based on the nature of their respective $\mathsf{tRegion}(\cdot,\cdot)$. Each witness w in W is sorted into either of these subsets.

1. $W_{\texttt{int}}$**:** Set of the witnesses $w \in W$ for which w, $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$ are collinear and $\ell\mathsf{Anchor}(w)$ and $\mathsf{rAnchor}(w)$ belongs to the different chains $\mathsf{uC}(\mathcal{M})$ and $\ell\mathsf{C}(\mathcal{M})$. So, if $w \in W_{\texttt{int}}$, then $\mathsf{tRegion}(w, W)$ is obtained by the open line segment $(\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w))$.

2. $W^{\texttt{good}}_{\texttt{bdry}}$**:** Set of the witnesses w in W for which $\mathsf{tRegion}(w, W)$ intersects $V(\mathcal{M})$.

3. $W^{\texttt{bad}}_{\texttt{bdry}}$**:** Set of the witnesses w in W for which $\mathsf{tRegion}(w, W)$ has a non-empty intersection with $\partial(\mathcal{M})$ but contains no vertex from $V(\mathcal{M})$.

Now, we introduce the concept of *potential witness set*, which contains a solution to the Witness Set.

Definition 5.10 (potential witness set)**.** For a monotone polygon $\mathcal{M}$, a finite point set Q in $\mathcal{M}$ is said to be a *potential witness set* of $\mathcal{M}$, if there exists a subset $Q' \subseteq Q$ such that Q' is a solution of the Witness Set in $\mathcal{M}$.

Definition 5.11 ((W, S)-potential witness set)**.** Consider a witness set W and a subset $S \subseteq W$. A set of points Z is said to be a (W, S)-potential witness set if there exists a subset $Z' \subseteq Z$ satisfying (i) $(W \setminus S) \cup Z'$ is a witness set of $\mathcal{M}$, and (ii) $|(W \setminus S) \cup Z'| = |W|$.

Note that if W is a solution to the Witness Set within $\mathcal{M}$ and $S = W$, then (W, S)-potential witness set corresponds precisely to the potential witness set of $\mathcal{M}$. Thus, Theorem 5.11 can be viewed as a generalization of Theorem 5.10. Now we define the notion of a potential witness set for W in $\mathcal{M}$.

Definition 5.12 (W-potential witness set)**.** Consider a witness set W in $\mathcal{M}$. A set of points Z is said to be a W-potential witness set in $\mathcal{M}$ if for each witness w in W, there exists a $q \in Z$ such that $(W \setminus w) \cup \{q\}$ is a witness set in $\mathcal{M}$.

In the next section, we create both $W_{\texttt{int}}$-potential witness set and $W^{\texttt{bad}}_{\texttt{bdry}}$-potential witness set.

5.2 Generating a Potential Witness Set of $\mathcal{M}$

Let $\mathcal{R}$ denote the set of all reflex vertices within $\mathcal{M}$, and let r be its cardinality. The main objective of a potential witness set Q for $\mathcal{M}$ is described as follows: for any solution W of the Witness Set problem, each witness w in W should have an alternative q in Q such that substituting w with q in W results in a valid solution, specifically, $(W \setminus \{w\}) \cup \{q\}$ remains a solution for the Witness Set in $\mathcal{M}$, due to Theorem 5.3.

Few Conventions. Consider the witness set $W = \{w_1, \dots, w_k\}$ with a specific order such that $x(w_1) < \dots < x(w_k)$. To demonstrate that Q is a W-potential witness set of $\mathcal{M}$, we proceed as follows: for each w_i $i \in [k]$, we find a substitute $w_i^* \in Q$ such that $(W \setminus \{w_i\}) \cup \{w_i^*\}$ is a witness set within $\mathcal{M}$. If $w_i \in Q$ then there is nothing to show; in that case $w_i = w_i^*$. This implies every witness $w \in W$ can be appropriately replaced by a $w^* \in Q$. Within our proof, we adhere to the following convention: For any two points a_1, a_2 within $\mathcal{M}$, we assert that a_1 lies to the left of a_2 or equivalently a_2 lies to the right of a_1 if $x(a_1) < x(a_2)$. We will now present two important definitions – *visibility region of a line segment* and *clone* that we use often in our proof.

Definition 5.13 (Visibility of a line segment from another line segment)**.** Consider a pair of line segments, L_1 and L_2, both contained entirely within $\mathcal{M}$. The notation $\mathsf{Vis}(L_1)|_{L_2}$ stands for the visibility region of L_1 when restricted to L_2, i.e., $\mathsf{Vis}(L_1) \cap L_2$, with respect to the subspace topology on L_2. Here, $\mathsf{Vis}(L_1)$ denotes the visibility region of $L_1 \subseteq \mathcal{M}$ as defined in Section 2. Essentially this captures the portion of L_2 that can be seen from L_1 in $\mathcal{M}$.

The following observation is immediate from the above definition.

Observation 5.14. *For any two line segments L_1, L_2, the region $\mathsf{Vis}(L_1)|_{L_2}$ is closed and indeed a connected segment.*

Definition 5.15 (Clone of a vertex/point)**.** Let v be a vertex or a point of $\partial(\mathcal{M})$. We denote v^+ (resp., v^-) to be an *infinitesimally close point* near v on $\partial(\mathcal{M})$ with $x(v^+) > x(v)$ (resp., $x(v^-) < x(v)$). We call v^+ as the *right clone* of v (resp., v^- as the *left clone* of v).

We refer to the clone of a vertex as *clone point*. We now present a useful claim.

Claim 5.16. *Consider a witness $w_i \in W$ with $w_i \notin W_{\texttt{int}}$ $(1 < i < k)$. Then, as $\mathsf{tRegion}(w_i) \cap \partial(\mathcal{M}) \neq \emptyset$, we can assume, w.l.o.g. that w_i lies on $\partial(\mathcal{M})$. Let e_i be the edge of $\mathcal{M}$ on which the witness w_i lies. Then $\mathsf{Vis}(\mathsf{rBdry}(w_{i-1}))|_{e_i} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(w_{i+1}))|_{e_i} \neq e_i$.*

Proof. Suppose for the sake of contradiction that $\mathsf{Vis}(\mathsf{rBdry}(w_{i-1}))|_{e_i} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(w_{i+1}))|_{e_i} = e_i$. This indicates that $w_i \in \mathsf{Vis}(\mathsf{rBdry}(w_{i-1}))|_{e_i} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(w_{i+1}))|_{e_i}$, suggesting that w_i is visible from $\mathsf{rBdry}(w_{i-1})$ or $\ell\mathsf{Bdry}(w_{i+1})$. Consequently, this implies that there exists a point $x \in \mathsf{rBdry}(w_{i-1})$ or a point $y \in \ell\mathsf{Bdry}(w_{i+1})$ such that x (or y) is visible from w_i. Also, every point on $\mathsf{rBdry}(w_{i-1})$ is visible from w_{i-1} and every point on $\ell\mathsf{Bdry}(w_{i+1})$ is visible from w_{i+1}. So, this means that $(x \in) \mathsf{Vis}(w_{i-1}) \cap \mathsf{Vis}(w_i) \neq \emptyset$ or $(y \in) \mathsf{Vis}(w_i) \cap \mathsf{Vis}(w_{i+1}) \neq \emptyset$, which is a contradiction in either case. ☐

Definition 5.17 ($\texttt{Reg}(e), e_i, \texttt{Reg}_i, \texttt{Reg}_i^c$)**.** For an edge e in $\mathcal{M}$, $\texttt{Reg}(e)$ denotes the region determined by all the points lying on the edge e. For each witness w_i, if it is indeed an internal point of some edge in $\mathcal{M}$, we use e_i to denote such an edge that contains the witness w_i. For each integer $i \in \{2, \dots, k-1\}$, we define two regions $\texttt{Reg}_i$ and $\texttt{Reg}_i^c$ as follows.

- $\texttt{Reg}_i := \mathsf{Vis}(\mathsf{rBdry}(w_{i-1}))|_{e_i} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(w_{i+1}))|_{e_i}$.
- $\texttt{Reg}_i^c := \texttt{Reg}(e_i) \setminus \texttt{Reg}_i$.

As $\mathsf{tRegion}(w, W)$ of each witness $w \in W_{\texttt{bdry}}^{\texttt{good}}$ intersects $V(\mathcal{M})$, the following holds.

Claim 5.18. *$V(\mathcal{M})$ is a $(W, W_{\texttt{bdry}}^{\texttt{good}})$-potential witness set.*

Now we show that when $|W| \geq 2$, the size of the set $W_{\texttt{bdry}}^{\texttt{good}}$ is at least two. To be precise, both the vertices w_1 and w_k belong to $W_{\texttt{bdry}}^{\texttt{good}}$. We prove this fact formally in Theorem 5.19.

Claim 5.19. $\{w_1, w_k\} \subseteq W_{\texttt{bdry}}^{\texttt{good}}$

Proof. First, we demonstrate that w_1 can be substituted with w_1^*, where w_1^* belongs to $V(\mathcal{M})$. Symmetrically, this argument applies to w_k, the last witness from left to right. If w_1 is already in $V(\mathcal{M})$, we have nothing to prove and $w_1 = w_1^*$. Otherwise, if $w_1 \notin V(\mathcal{M})$, we determine w_1^* through a two-step process.

Step 1. Finding a replacement of w_1 on $\partial(\mathcal{M})$. If $w_1 \in \partial(\mathcal{M})$, then we are done. Otherwise, consider the chord in $\mathcal{M}$ that passes through $w_1, \mathsf{rAnchor}(w_1)$ and has one endpoint at $\mathsf{rImage}(w_1)$. Let $\mathcal{R}(w_1)$ represent the other endpoint. Clearly, $\mathcal{R}(w_1) \in \partial(\mathcal{M})$. It is apparent that $\mathsf{Vis}(\mathcal{R}(w_1)) \subseteq \mathsf{Territory}(w_1, W)$, as w_1 is the leftmost witness and $\mathsf{rBdry}(w_1) = \mathsf{rBdry}(\mathcal{R}(w_1))$. Thus, w_1 has a substitute on $\partial(\mathcal{M})$ concerning W, and $\mathcal{R}(w_1)$ is one such substitute.

Step 2. Finding a replacement of $\mathcal{R}(w_1)$ on $V(\mathcal{M})$. Consider $\mathcal{R}(w_1)$ positioned on the edge $e_1 = \overline{u_1 v_1}$ of $\mathcal{M}$. We examine $\mathsf{Vis}(\ell\mathsf{Bdry}(w_2))|_{e_1}$ and assert that it cannot be equal to $\mathtt{Reg}(e_1)$. If it did, it would imply $w_1 \in \mathsf{Vis}(\ell\mathsf{Bdry}(w_2))|_{e_1}$, thereby obtaining (at least) one point x on $\ell\mathsf{Bdry}(w_2)$ which sees w_1, and vice-versa. Now, every point on $\ell\mathsf{Bdry}(w_2)$ is visible from w_2. Thus, this would mean $(x \in)\mathsf{Vis}(w_1) \cap \mathsf{Vis}(w_2) \neq \emptyset$, which would lead to a contradiction. Based on Theorem 5.14, $\mathsf{Vis}(\ell\mathsf{Bdry}(w_2))|_{e_1}$ is a connected line segment, and since $\mathsf{Vis}(\ell\mathsf{Bdry}(w_2))|_{e_1} \neq e_1$, either one of u_1 or v_1 must not be visible from $\ell\mathsf{Bdry}(w_2)$. Assume v_1 is not visible from $\ell\mathsf{Bdry}(w_2)$ (refer to Figure 12). Consequently, a replacement for $\mathcal{R}(w_1)$ in $V(\mathcal{M})$ is obtained, which serves as a substitute for w_1 in $\mathcal{M}$ with respect to W.

Hence the proof. ◻

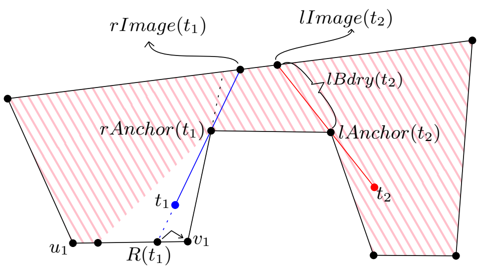


Figure 19: Illustration of the proof of Theorem 5.19. Replacing t_1 by $v_1 \in V(\mathcal{M})$. The pink region denotes visibility region of $\ell\mathsf{Bdry}(t_2)$.

5.2.1 Construction of a $(W, W_{\mathtt{int}})$-Potential Witness Set

The $\mathsf{tRegion}(w, W)$ of each witness $w \in W_{\mathtt{int}}$ consists of an open line segment. It is important to note that there can be at most r^2 of these segments, with each $w \in W_{\mathtt{int}}$ lying within one of these r^2 regions. Recall that the $\mathsf{tRegion}(w, W)$ is associated with a pair of visible reflex vertices where $\ell\mathsf{Anchor}(w), \mathsf{rAnchor}(w)$, and w are collinear, and $\ell\mathsf{Anchor}(w)$, $\mathsf{rAnchor}(w)$ belong to the different chains, $\mathsf{uC}(\mathcal{M})$ and $\ell\mathsf{C}(\mathcal{M})$. We know that $\mathsf{Vis}(\mathsf{tRegion}(w, W)) \subseteq \mathsf{Territory}(w, W)$, implying that for any point $z \in \mathsf{tRegion}(w)$, $\mathsf{Vis}(z) \subseteq \mathsf{Territory}(w, W)$. We will now construct a point set, denoted as $\mathcal{R}_{mid}$, in the following manner:

$$\mathcal{R}_{mid} = \{p \ : \ p \text{ is the midpoint of } \overline{ss'} \text{ for all } s, s' \in \mathcal{R} \text{ with } s' \in \mathsf{Vis}(s)\}$$

Let $\mathcal{R}_{mid}$ denote the set of all midpoints of segments connecting visible reflex vertex pairs. The $\mathcal{R}_{mid}$ can be computed in $n^{\mathcal{O}(1)}$ time. The following observation is immediate.

Observation 5.20. $|\mathcal{R}_{mid}| \leq r^2$.

In the following Theorem 5.21, we demonstrate that the collection $\mathcal{R}_{mid}$ is indeed a $(W, W_{\texttt{int}})$-potential witness set. Specifically, there exists a subset $Q' \subseteq \mathcal{R}_{mid}$ such that: (i) $(W \setminus W_{\texttt{int}}) \cup Q'$ forms a witness set in $\mathcal{M}$, and (ii) $|(W \setminus W_{\texttt{int}}) \cup Q'|$ does not exceed $|W|$.

Claim 5.21. *$\mathcal{R}_{mid}$ is a $(W, W_{\texttt{int}})$-potential witness set.*

Proof. We show that for any $w \in W_{\texttt{int}}$, we can find a substitute $w^* \in \mathcal{R}_{mid}$ such that $(W \setminus \{w\} \cup w^*$ is a valid witness set of $\mathcal{M}$. So, now let $w \in W_{\texttt{int}}$ be an arbitrary witness. We have that $\mathsf{tRegion}(w, W)$ is an open line segment $(\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)})$. In particular, the mid-point p of $\overline{\ell\mathsf{Anchor}(w)\mathsf{rAnchor}(w)}$ is in $\mathsf{tRegion}(w, W)$. Therefore, we can successfully substitute w by the point $p(= w^*$ in this case) to get a witness set of $\mathcal{M}$ without changing the cardinality of W. This proves our claim. $\square$

Using Theorem 5.20 and Theorem 5.21, we arrive at the following lemma.

Lemma 5.22. *Consider a monotone polygon $\mathcal{M}$ with n vertices. For any hypothetical witness set W in $\mathcal{M}$ and subset $W_{\texttt{int}} \subseteq W$ as specified in Theorem 5.9, there exists a $(W, W_{\texttt{int}})$-potential witness set of size at most r^2. Moreover, this set can be found in $n^{\mathcal{O}(1)}$ time.*

Observe that if there is a solution W for the Witness Set in $\mathcal{M}$ such that $W^{\text{bad}}_{\texttt{bdry}} = \emptyset$, then we can solve the Witness Set for $\mathcal{M}$ in $n^{\mathcal{O}(1)}$ time. This is true due to the fact that $Z := V(\mathcal{M}) \cup \mathcal{R}_{mid}$ is a W-potential witness set of $\mathcal{M}$ (due to Theorem 5.18 and Theorem 5.21) with $|Z| \leq n + r^2$ (due to Theorem 5.20).

As every witness in $W^{\text{good}}_{\texttt{bdry}} \cup W_{\texttt{int}}$ has a potential replacement in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$, we can assume that for each witness $w_i \in W$ where $i \in [k]$, we have exactly one of the following.

1. $w_i \in V(\mathcal{M}) \cup \mathcal{R}_{mid}$, or

2. $w_i \in W^{\text{bad}}_{\texttt{bdry}}$, i.e., it has no substitute in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$, moreover $w_i \in \texttt{Reg}(e_i) \subseteq \partial(\mathcal{M})$ (as $\mathsf{tRegion}(w_i, W) \cap \partial(\mathcal{M}) \neq \emptyset$).

Transforming into Sub-instances. Initially, we have a given monotone polygon $\mathcal{M}$. Then we consider an hypothetical solution of the Witness Set $W = \{w_1, \ldots, w_k\}$. We have already shown that if $W^{\text{bad}}_{\texttt{bdry}} = \emptyset$ we can solve the problem in $n^{\mathcal{O}(1)}$ time. Now we consider $W^{\text{bad}}_{\texttt{bdry}} \neq \emptyset$, and $|W^{\text{good}}_{\texttt{bdry}} \cup W_{\texttt{int}}| = (\ell < k)$ and $W^{\text{good}}_{\texttt{bdry}} \cup W_{\texttt{int}} := \{w_{i_1}, w_{i_2}, \ldots, w_{i_\ell}\}$ where $i_1 < i_2 < \ldots < i_\ell$. It is clear from our above discussion that $W_{\texttt{int}} \cup W^{\text{good}}_{\texttt{bdry}} \neq \emptyset$ as $w_1, w_k \in V(\mathcal{M})$ (by Theorem 5.19), so $i_1 = 1$ and $i_\ell = k$. So, a chain of witnesses $w_{i_j+1}, w_{i_j+2}, \ldots, w_{i_{j+1}-1}$ between two consecutive $w_{i_j}, w_{i_{j+1}} \in W^{\text{good}}_{\texttt{bdry}} \cup W_{\texttt{int}}$ lies in $W^{\text{bad}}_{\texttt{bdry}}$ for any $1 \leq j \leq l - 1$.

In the following, we give an algorithm that produces a set of points called $\mathcal{Z}_{\texttt{mid}}$ such that $\mathcal{Z}_{\texttt{mid}}$ serves as a $(W, W^{\text{bad}}_{\texttt{bdry}})$-Potential Witness Set. We show that (later in Section 5.2.3) for any such chain of witnesses $w_{i_j+1}, w_{i_j+2}, \ldots, w_{i_{j+1}-1} \in W^{\text{bad}}_{\texttt{bdry}}$, we can find a suitable replacement for them in $\mathcal{Z}_{\texttt{mid}}$. Basically we are trying to find a witness set $W' = \{w'_i : i \in [k]\}$ such that $w'_i \in \mathcal{Z}_{\texttt{mid}}$ when $i' \notin \{i_1, i_2, \ldots, i_\ell\}$ and $w'_i \in V(\mathcal{M}) \cup \mathcal{R}_{mid}$ when $i' \in \{i_1, i_2, \ldots, i_\ell\}$. Thus, so far, we have that our witness set in $\mathcal{M}$ satisfies the following properties:

$W = \{w_1, \ldots, w_k\}$ is a witness set in $\mathcal{M}$ such that $\{w_{i_1}, w_{i_2}, \ldots, w_{i_l}\} \subseteq V(\mathcal{M}) \cup \mathcal{R}_{mid}$ and each $w_i, i \notin \{i_1, \ldots, i_l\}$ the point w_i is in $\texttt{Reg}(e_i)$ (as in Theorem 5.17) where w_i could not be replaced by any vertex in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$. So $W^{\text{bad}}_{\texttt{bdry}} = W \setminus \{w_{i_1}, \ldots, w_{i_l}\}$. From now on, we choose any particular chain of witnesses $w_{i_j+1}, w_{i_j+2}, \ldots, w_{i_{j+1}-1} \in W^{\text{bad}}_{\texttt{bdry}}$, and show that we can find a suitable replacement for each of them in $\mathcal{Z}_{\texttt{mid}}$.

5.2.2 Construction of a $(W, W^{\text{bad}}_{\text{bdry}})$-Potential Witness Set

We have that each witness $w_i \in W^{\text{bad}}_{\text{bdry}}$ lies on $\partial(\mathcal{M})$. Below in Algorithm 1 we describe in more detail how to find the set $\mathcal{Z}_{\texttt{mid}}$ which is a $(W, W^{\text{bad}}_{\text{bdry}})$-potential witness set.

Overview of Algorithm 1. Here we give a short description of Algorithm 1. Let A_i denote the arrangement[3] of i^{th} iteration and Q_i denotes a set of points added on $\partial(\mathcal{M})$. At the beginning, $A_0 = \partial(\mathcal{M})$ and $Q_0 = V(\mathcal{M})$. Finally we return the point set Q_{2k} as $\mathcal{Z}_{\texttt{mid}}$. For each $i \in [2k]$, we compute the arrangement A_i by creating chords that connect every possible pair of visible points v and β, where $v \in Q_{i-1}$ and $\beta \in \mathcal{R}$ (the set of reflex vertices in $\mathcal{M}$). Define the set $Q^{\texttt{chord}}_{i-1}$ in this manner: for each chord L drawn in A_i, we include the intersection point $L \cap \partial(\mathcal{M})$ into $Q^{\texttt{chord}}_{i-1}$. Let $Q^{\texttt{midpt}}_{i-1}$ represent the set of all midpoints (the exact central point of the line segment) between any two points u and v in Q_{i-1}, where both u and v are on the same edge of $\mathcal{M}$ and are contiguous (i.e., the line segment $\overline{uv}$ does not contain any other point of Q_{i-1}). The set Q_i is defined in the following way (see Algorithm 1).

$$Q_i = Q_{i-1} \cup Q^{\texttt{midpt}}_{i-1} \cup Q^{\texttt{chord}}_{i-1}$$

Algorithm 1: $\texttt{WitGen}(\mathcal{M}, k)$: Generation of $\mathcal{Z}_{\texttt{mid}}$

Input : A monotone polygon $\mathcal{M}$ and a non-negative integer k
Output: A point set $\mathcal{Z}_{\texttt{mid}}$

```
Initialization: A_0 = ∂(M), Q_0 = V(M) ;
for i ← 1 to 2k do
    Q^midpt_{i-1} := set of all mid-points between a pair adjacent points of Q_{i-1} that lie on the
      same edge of M.
    Q^chord_{i-1} = ∅
    For each possible pair of vertices v and β where v ∈ Q_{i-1} and β ∈ R ;      // R
      denotes the set of all reflex vertices in V(M).
    if v is visible to β then
        Add a maximal chord L in M joining v and β in A_i ;     // Maximal means
          no other chord in M contains L.
        Add the point L ∩ ∂(M) into Q^chord_{i-1}.
    else do nothing;
    Q_i = Q_{i-1} ∪ Q^midpt_{i-1} ∪ Q^chord_{i-1}
return Z_mid = Q_2k
```

Observation 5.23. *$|\mathcal{Z}_{\texttt{mid}}| \leq |V(\mathcal{M})| \cdot (2+r)^{2k}$, where $r = |\mathcal{R}|$.*

Proof. As $Q_i = Q_{i-1} \cup Q^{\texttt{midpt}}_{i-1} \cup Q^{\texttt{chord}}_{i-1}$. Now $|Q^{\texttt{midpt}}_{i-1}| \leq |Q_{i-1}|$ and $|Q^{\texttt{chord}}_{i-1}| \leq |Q_{i-1}| \cdot r$. This implies $|Q_i| \leq 2|Q_{i-1}| + |Q_{i-1}| \cdot r = |Q_{i-1}| \cdot (2+r)$. As $\mathcal{Z}_{\texttt{mid}} = Q_{2k}$ and $|Q_0| = V(\mathcal{M})$, hence $\mathcal{Z}_{\texttt{mid}} = |V(\mathcal{M})| \cdot (2+r)^{2k}$. □

We will now demonstrate that the collection $\mathcal{Z}_{\texttt{mid}}$ is indeed a $(W, W^{\text{bad}}_{\text{bdry}})$-potential witness set. Specifically, there exists a subset $Q \subseteq \mathcal{Z}_{\texttt{mid}}$ such that: (i) the union $(W \setminus W^{\text{bad}}_{\text{bdry}}) \cup Q$ forms a witness set in $\mathcal{M}$, and (ii) $|W^{\text{bad}}_{\text{bdry}}| = |Q|$ (as per Theorem 5.11). The lemma is stated below, while the proof will be presented in the subsequent section (Section 5.2.3). We want to mention that we prove the following lemma as follows: for each witness $w \in W^{\text{bad}}_{\text{bdry}}$ there is a replacement (which will also be referred to as "substitute") q of w in Q, i.e., $(W \setminus \{w\}) \cup \{q\}$ is a witness set.

[3] Here, by arrangement, we mean adding some line segments in our instance in a specific way.

Lemma 5.24. *$\mathcal{Z}_{\texttt{mid}}$ is a $(W, W^{\texttt{bad}}_{\texttt{bdry}})$-potential witness set.*

Using Theorem 5.23 and Theorem 5.24, we arrive at the following lemma.

Lemma 5.25. *Consider a monotone polygon $\mathcal{M}$ with n vertices. In $r^{\mathcal{O}(k)}n^{\mathcal{O}(1)}$ time we can construct a point set $\mathcal{Z}_{\texttt{mid}}$ of size at most $n \cdot (2+r)^{2k-1}$ such that for any witness set W in $\mathcal{M}$ and its subset $W^{\texttt{bad}}_{\texttt{bdry}} \subseteq W$ as specified in Theorem 5.9, $\mathcal{Z}_{\texttt{mid}}$ can serve as a $(W, W^{\texttt{bad}}_{\texttt{bdry}})$-potential witness set. Here r denotes the number of reflex vertices in $\mathcal{M}$.*

In the next subsection, we show that $\mathcal{Z}_{\texttt{mid}}$ is $(W, W^{\texttt{bad}}_{\texttt{bdry}})$-potential witness set. That is, for each witness w in W, there exists a $q \in \mathcal{Z}_{\texttt{mid}}$ such that $(W \setminus w) \cup \{q\}$ is a witness set in $\mathcal{M}$. Hereafter, every point in $\mathcal{Z}_{\texttt{mid}} \setminus V(\mathcal{M})$ will be referred to as a *vertex* of $\mathcal{Z}_{\texttt{mid}}$. Unless stated otherwise, we will refer to each vertex in $V(\mathcal{M})$ simply as a *vertex*.

5.2.3 Correctness

Overview of the proof of Theorem 5.24. We first consider an Optimal Witness Set $W = \{w_1, w_2, \ldots, w_k\}$ of size k in $\mathcal{M}$ (Here, $x(w_1) < x(w_2) < \ldots < x(w_k)$). Then, we try to show that for every witness $w \in W$, we can obtain a substitute $w^* \in Q = \mathcal{R}_{mid} \cup \mathcal{Z}_{\texttt{mid}}$ ($\mathcal{Z}_{\texttt{mid}}$ contains $V(\mathcal{M})$). Observe that, for all witnesses in $W_{\texttt{int}} \cup W^{\texttt{good}}_{\texttt{bdry}}$, we have already found a suitable replacement in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$ in $n^{\mathcal{O}(1)}$ time. Let these witnesses be denoted by the set $\{w_{i_1}, w_{i_2}, \ldots, w_{i_l}\}$. Then, we pick a chain of witnesses $w_{i_j+1}, w_{i_j+2}, \ldots, w_{i_{j+1}-1} \in W^{\texttt{bad}}_{\texttt{bdry}}$ for some $1 \le j \le l-1$. For notational simplicity, from now onward, we will denote these witnesses by $t_2, t_3, \ldots t_{k'-1}$, respectively. Note that $t_1 = w_{i_j}$ and $t'_k = w_{i_{j+1}}$ lies in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$. In the proof provided below, we will show that we can find a replacement for $t_i \in W^{\texttt{bad}}_{\texttt{bdry}}$ in $\mathcal{Z}_{\texttt{mid}}$ for all $1 < i < k'$. Since the choice of this *chain* of witnesses in $W^{\texttt{bad}}_{\texttt{bdry}}$ was done arbitrarily, our proof will hold for any such *chain* of witnesses in $W^{\texttt{bad}}_{\texttt{bdry}}$.
In the beginning of our proof, we introduce a crucial concept of *line visibility*, which means the part of a line which is visible from another line of $\mathcal{M}$ (Theorem 5.13). We then proceed in the following way: For finding the substitutes of the witnesses $t_i \in W^{\texttt{bad}}_{\texttt{bdry}}$ in $\mathcal{Z}_{\texttt{mid}}$, we might not be able to find them directly in all cases. So, in those cases, we will create an intermediary set of points, known as *clone points* (Theorem 5.15), which will temporarily suffice as substitutes for the witnesses $t_2, \ldots, t_{k'-1}$. We first show that a substitute of t_2 is present in a clone point of a vertex in $\mathcal{Z}_{\texttt{mid}}$, assuming the fact that $t_1 \in V(\mathcal{M}) \cup \mathcal{R}_{mid}$. Next, we proceed to the next set of witnesses from left to right,t_3, t_4, and so on. After finding the substitutes of the witnesses in these intermediate set of points, i.e., clone points, we then proceed our arguments similarly again, by moving from right to left this time, again assuming the fact that $t'_k \in V(\mathcal{M}) \cup \mathcal{R}_{mid}$. That is, we try to find substitutes for $t_{k'-1}, t_{k'-2}$, and so on. We then show that, for those particular cases where we had chosen clone points as substitutes, we can find a substitute for the witness by choosing any point from an open line segment, the endpoints of which lie on the set $\mathcal{Z}_{\texttt{mid}}$. So, we then substitute those witnesses by the mid-points of the open segments, which were also considered in our Algorithm 1, and get added to our final output $\mathcal{Z}_{\texttt{mid}} = Q_{2k}$. Therefore, this would prove that we can successfully find a substitute for any witness $t_i \in W^{\texttt{bad}}_{\texttt{bdry}}$ in $\mathcal{Z}_{\texttt{mid}}$. We then conclude that our algorithm is indeed correct. The details of the proof are given below. Here onward, unless mentioned otherwise, we will assume that as for any $t_i \in W^{\texttt{bad}}_{\texttt{bdry}}$, $\mathsf{tRegion}(t_i, W) \cap \partial(\mathcal{M}) \neq \emptyset$, t_i lies on the edge $e_i = \overline{u_i v_i}$ of $\mathcal{M}$.

We now present the detailed proof of Theorem 5.24.

Determining Substitutes for t_2. Here, we show that t_2 can be substituted with a clone of a vertex from Q_2. By our definition, $e_2 = \overline{u_2 v_2}$ is the edge of $\mathcal{M}$ on which the witness t_2 lies. Consider the pair of segments $\mathsf{rBdry}(t_1)$ and $\ell\mathsf{Bdry}(t_3)$, and their corresponding visibility regions $\mathsf{Vis}(\mathsf{rBdry}(t_1))$ and $\mathsf{Vis}(\ell\mathsf{Bdry}(t_3))$. We begin by noting a few observations as outlined below.

Observation 5.26. $|\mathsf{Vis}(\mathsf{rBdry}(t_1)) \cap \{u_2, v_2\}| \leq 1$ *and* $|\mathsf{Vis}(\ell\mathsf{Bdry}(t_3)) \cap \{u_2, v_2\}| \leq 1$.

Proof. Our objective is to show that at most one of u_2 and v_2 can be visible from $\mathsf{rBdry}(t_1)$ (or equivalently, from $\ell\mathsf{Bdry}(t_3)$). For the sake of contradiction, let's assume that $|\mathsf{Vis}(\mathsf{rBdry}(t_1)) \cap \{u_2, v_2\}| = 2$. We have established that for any two lines, L_1 and L_2, the visibility $\mathsf{Vis}(L_1)|_{L_2}$ remains connected, according to Theorem 5.14. Consequently, this implies that every point on e_2 is visible from $\mathsf{rBdry}(t_1)$, or in notation, $\mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2} = \mathtt{Reg}(e_2)$, contradicting Theorem 5.16. The proof to show $|\mathsf{Vis}(\ell\mathsf{Bdry}(t_3)) \cap \{u_2, v_2\}| \leq 1$ follows a similar argument. □

Observation 5.27. *If both the points u_2 and v_2 lies in the region* $\mathtt{Reg}_2$ *(as per Theorem 5.17) then the region* $\mathtt{Reg}_2^c$ *is a non-empty, open and connected subregion of* $\mathtt{Reg}(e_2)$.

Proof. Recall in Theorem 5.17 that $\mathtt{Reg}_2 = \mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(t_3))|_{e_2}$. According to Theorem 5.14, both $\mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2}$ and $\mathsf{Vis}(\ell\mathsf{Bdry}(t_3))|_{e_2}$ are closed and connected segments. The region $\mathtt{Reg}_2^c$ is formed by subtracting two closed connected segments, one containing u_2 and the other containing v_2 (both of them simultaneously cannot be contained in a single closed connected segment due to Theorem 5.26), where u_2 and v_2 are the endpoints of e_2, resulting in an open, connected segment within $\mathtt{Reg}(e_2)$. Furthermore, Theorem 5.16 indicates that $\mathtt{Reg}_2^c$ is not empty. □

Observation 5.28. *For every point $x \in$* $\mathtt{Reg}_2^c$ *we have* $\mathsf{Vis}(x) \cap \mathsf{Vis}(t_i) = \emptyset$ *where $i \in \{1, 3\}$.*

Proof. If $\mathsf{Vis}(x) \cap \mathsf{Vis}(t_1) \neq \emptyset$, it follows from Theorem 4.13 that $\mathsf{rBdry}(t_1)$ and $\ell\mathsf{Bdry}(x)$ intersect. This indicates that x is visible from $\mathsf{rBdry}(t_1)$ along the line $\ell\mathsf{Chord}(x)$, meaning $x \in \mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2}$. Consequently, $x \notin \mathtt{Reg}_2^c$, leading to a contradiction. An analogous argument applies to t_3 to prove that $\mathsf{Vis}(x) \cap \mathsf{Vis}(t_3) = \emptyset$. □

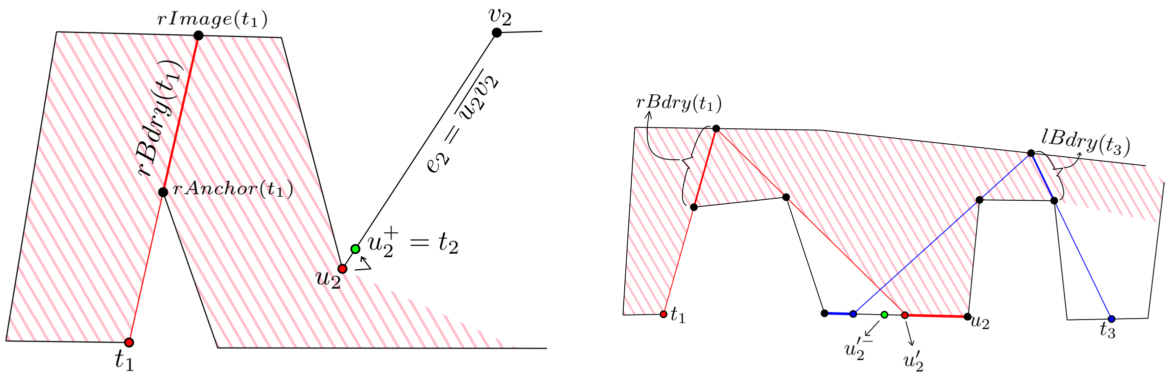
According to Theorem 5.28, t_2 can be substituted with any point from $\mathtt{Reg}_2^c$. Our primary objective is to demonstrate the existence of a point $q \in Q_2$ that allows t_2 to be replaced by a clone of q. It should be noted that our ultimate objective is to show that t_2 has a replacement within $\mathcal{Z}_{\mathtt{mid}}$. Note that we are dealing with the situation when both u_2 and v_2 are contained in the region defined by union of $\mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2}$ and $\mathsf{Vis}(\ell\mathsf{Bdry}(t_3))|_{e_2}$. Here $\mathsf{rBdry}(t_1)$ sees one vertex of $\{u_2, v_2\}$ whereas $\ell\mathsf{Bdry}(t_3)$ sees the other. So, w.l.o.g., assume that $u_2 \in \mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2}$ and $v_2 \in \mathsf{Vis}(\ell\mathsf{Bdry}(t_3))|_{e_2}$. Recall that by Theorem 5.27, the region $\mathtt{Reg}_2^c$ is a non-empty connected open segment. Now we deal with the following two subcases.

Case I. $\mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2} = \{u_2\}$**.** Here the only point of the edge e_2 that is visible from $\mathsf{rBdry}(t_1)$ is u_2. Consequently, there is an open connected segment in $\mathtt{Reg}(e_2)$ beside[4] u_2 within which t_2 can be substituted with any point. Thus, t_2 can be swapped with either u_2^+ or u_2^- (i.e., clone point), depending on which is positioned on the edge e_2.(See Figure 20a)

Case II. $\{u_2\} \subset \mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2}$**.** Let $\mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2} = \overline{u_2 u_2'}$, where $u_2' \neq u_2$ and is located in e_2. In this situation, there is an open connected segment within $\mathtt{Reg}(e_2)$ beside u_2' that allows t_2 to be substituted by any point within this segment. Therefore, we have the option to substitute t_2 with either $u_2'^+$ or $u_2'^-$ (a clone point), one of which does not fall within $\mathsf{Vis}(\mathsf{rBdry}(t_1))|_{e_2}$. Now, w.l.o.g., assume $x(u_2') < x(u_2)$. For this, we can substitute t_2 with $u_2'^-$ (see Figure 20b). Otherwise, if $x(u_2) < x(u_2')$, then we can substitute t_2 with $u_2'^+$.

In the following (Theorem 5.30), we demonstrate that the point u_2' introduced in Case II is indeed a vertex of $\mathcal{Z}_{\mathtt{mid}}$ (more specifically, a point of Q_2). This will establish that t_2 can be replaced by a clone point of Q_2. Before that, we first make an important observation here.

[4]by this, we mean that the open, connected segment $\mathtt{Reg}_2^c$ has one of its endpoints as u_2.



(a) Case I : u_2^+ is a replacement of t_2.

(b) Case II: $u_2'^-$ is a replacement of t_2.

Figure 20: Pink regions are the visibility regions of $\mathsf{rBdry}(t_1)$.

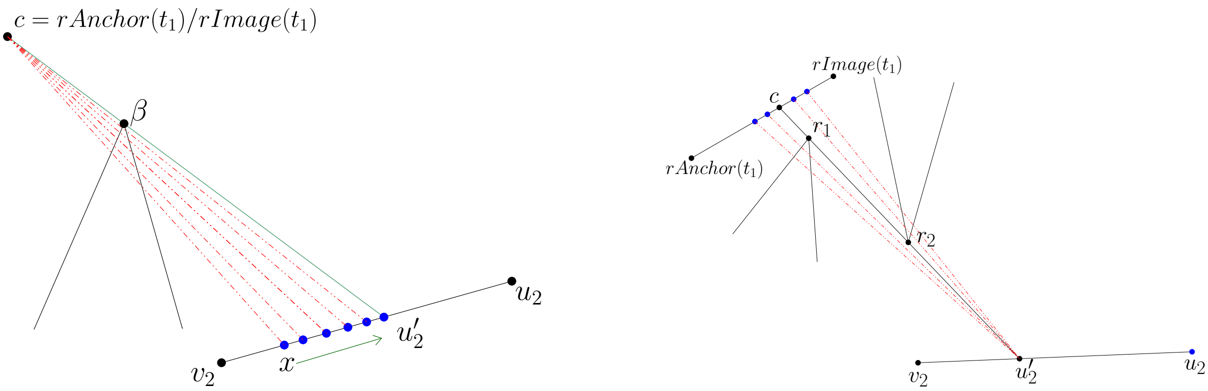
Observation 5.29. *Both* $\mathsf{rAnchor}(t_1)$ *and* $\mathsf{rImage}(t_1)$ *is a vertex of* $\mathcal{Z}_{\mathtt{mid}}$ *(similarly,* $\ell\mathsf{Anchor}(t_k)$ *and* $\ell\mathsf{Image}(t_k)$ *for* t_k*, and consequently for any* $w_i \in V(\mathcal{M}) \cup \mathcal{R}_{mid}$*, their respective* $\mathsf{rAnchor}(w_i)$, $\ell\mathsf{Anchor}(w_i)$ *are vertices of* $\mathcal{Z}_{\mathtt{mid}}$*).*

Proof. Clearly, as $\mathsf{rAnchor}(t_1) \in \mathcal{R} \subseteq V(\mathcal{M}) = Q_0$, $\mathsf{rAnchor}(t_1) \in \mathcal{Z}_{\mathtt{mid}}$. Now, we need to show that $\mathsf{rImage}(t_1)$ is also a vertex of $\mathcal{Z}_{\mathtt{mid}}$. We know that $t_1 \in V(\mathcal{M}) \cup \mathcal{R}_{mid}$. Therefore, in either case, $\mathsf{rChord}(t_1)$ passes through two vertices of $\mathcal{M}$, which are visible to each other. That is, if $t_1 \in V(\mathcal{M})$, then $\mathsf{rChord}(t_1)$ passes through the two mutually visible vertices t_1 itself, and $\mathsf{rAnchor}(t_1)$. Otherwise, if $t_1 \in \mathcal{R}_{mid}$, it implies that t_1, $\mathsf{rAnchor}(t_1)$ and $\ell\mathsf{Anchor}(t_1)$ are collinear; so, $\mathsf{rChord}(t_1)$ passes through two mutually visible vertices $\ell\mathsf{Anchor}(t_1)$ and $\mathsf{rAnchor}(t_1)$. As $\mathsf{rImage}(t_1)$ is an endpoint of $\mathsf{rChord}(t_1)$ on $\partial(\mathcal{M})$, distinct from t_1, $\ell\mathsf{Anchor}(t_1)$ and $\mathsf{rAnchor}(t_1)$, $\mathsf{rImage}(t_1) \in Q_1 \subseteq \mathcal{Z}_{\mathtt{mid}}$. □

Claim 5.30. u_2' *is a vertex of* $\mathcal{Z}_{\mathtt{mid}}$*. More specifically* $u_2' \in Q_2$*.*

Proof. Since u_2' is visible from $\mathsf{rBdry}(t_1)$, u_2' also sees some point(s) of $\mathsf{rBdry}(t_1)$. So, we examine the sub-region of $\mathsf{rBdry}(t_1)$ that is visible from u_2', namely $\mathsf{Vis}(u_2')|_{\mathsf{rBdry}(t_1)}$. Now, w.l.o.g., let us assume that $x(u_2') < x(u_2)$. The other case is symmetric.

Case A: u_2' **sees exactly one point on** $\mathsf{rBdry}(t_1)$**.** Let c be the unique point on $\mathsf{rBdry}(t_1)$ that is visible from u_2', i.e., $\mathsf{Vis}(u_2')|_{\mathsf{rBdry}(t_1)} = \{c\}$. We now differentiate our cases based on whether the point c is a point of the set $\{\mathsf{rAnchor}(t_1), \mathsf{rImage}(t_1)\}$.



(a) **Case** A1: $c = \mathsf{rAnchor}(t_1)$ or $\mathsf{rImage}(t_1)$.

(b) **Case** A2: $c \neq \mathsf{rAnchor}(t_1)$ or $\mathsf{rImage}(t_1)$.

Figure 21: u_2' is a vertex of $\mathcal{Z}_{\mathtt{mid}}$.

Case A1: c is either of $\mathsf{rAnchor}(t_1)$ and $\mathsf{rImage}(t_1)$. u_2 is a vertex on the boundary $\partial(\mathcal{M})$. Given that c is a point either from $\mathsf{rAnchor}(t_1)$ or $\mathsf{rImage}(t_1)$, to demonstrate that u_2' is a vertex of $\mathcal{Z}_{\mathtt{mid}}$, it suffices to show that the segment $\overline{cu_2'}$ intersects a reflex vertex in $\mathcal{M}$ and is wholly contained within $\mathcal{M}$ (as $\mathsf{rAnchor}(t_1)$ and $\mathsf{rImage}(t_1)$ lies in $\mathcal{Z}_{\mathtt{mid}}$, due to Theorem 5.29). Observe that c is visible from u_2' and vice versa, yet no point left of u_2' is visible from c due to $x(u_2') < x(u_2)$, and only $\overline{u_2'u_2}$ of e_2 is visible from $\mathsf{rBdry}(t_1)$. Thus, if c is joined to any point $x \in e_2$ left of u_2', the segment $\overline{cx}$ cannot be entirely inside $\mathcal{M}$. Consequently, as mentioned in Theorem 5.7, there exists an obstacle, i.e., a hill containing a reflex vertex that obstructs visibility from x to c (here, the hills might be different for different x, but that won't matter in our arguments below). However, since u_2' is visible from c, transforming $\overline{cx}$ towards $\overline{cu_2'}$ as x moves along $\overline{v_2u_2}$ means the segment becomes part of $\mathcal{M}$ when x reaches u_2'. This implies that $\overline{cu_2'}$ must intersect reflex vertex r since $\overline{cx}$ is wholly contained in $\mathcal{M}$ only when the segment $\overline{cx}$ intersect exactly at the point of intersection of two edges of a hill, which is precisely a reflex vertex β(refer to Figure 21a). Now if $c = \mathsf{rAnchor}(t_1)$, then c itself is a reflex vertex of $\mathcal{M}$. If $c = \mathsf{rImage}(t_1)$, then $c \in Q_1 \subseteq \mathcal{Z}_{\mathtt{mid}}$ as $t_1 \in \mathcal{R}_{mid} \cup V(\mathcal{M})$, (note that this t_1 was obtained by replacing w_{i_j} by a point in $\mathcal{R}_{mid} \cup V(\mathcal{M})$). Therefore, as $\overline{cu_2'}$ passes through a reflex vertex r, it signifies that $u_2' \in Q_2 \subseteq \mathcal{Z}_{\mathtt{mid}}$.

Case A2. c is neither of $\mathsf{rAnchor}(t_1)$ and $\mathsf{rImage}(t_1)$. In this scenario where the point $c \neq \mathsf{rAnchor}(t_1)$ or $\mathsf{rImage}(t_1)$, we aim to establish that $u_2' \in \mathcal{Z}_{\mathtt{mid}}$. This will be shown by demonstrating that the line segment $\overline{cu_2'}$ intersects a pair of distinct reflex vertices of $\mathcal{M}$ and remains completely contained in $\mathcal{M}$. Since c is the sole point on $\mathsf{rBdry}(t_1)$ that is visible from u_2', there are no visible points to either the left or right of c on $\mathsf{rBdry}(t_1)$ from u_2'. As argued in **Case** A1, the non-visibility of any point to the left of c on $\mathsf{rBdry}(t_1)$ from u_2' implies that $\overline{cu_2'}$ passes through a reflex vertex, say r_1 of $\mathcal{M}$. Likewise, since no point to the right of c on $\mathsf{rBdry}(t_1)$ is visible from u_2', a similar rationale suggests it also must intersect another reflex vertex, say r_2, distinct from r_1, in $\mathcal{M}$. The fact that the two reflex vertices r_1 and r_2 are distinct is guaranteed by the divergent orientation of the hills containing them, as illustrated in Figure 21b. Therefore, u_2' is derived by connecting two reflex vertices, r_1 and r_2, confirming that $u_2' \in \mathcal{Z}_{\mathtt{mid}}$.

Case B: u_2' sees more than one point on $\mathsf{rBdry}(t_1)$. Let $\mathsf{Vis}(u_2')|_{\mathsf{rBdry}(t_1)} = \overline{rs} \subseteq \mathsf{rBdry}(t_1)$, i.e., u_2' sees a part of (segment) or whole of $\mathsf{rBdry}(t_1)$.

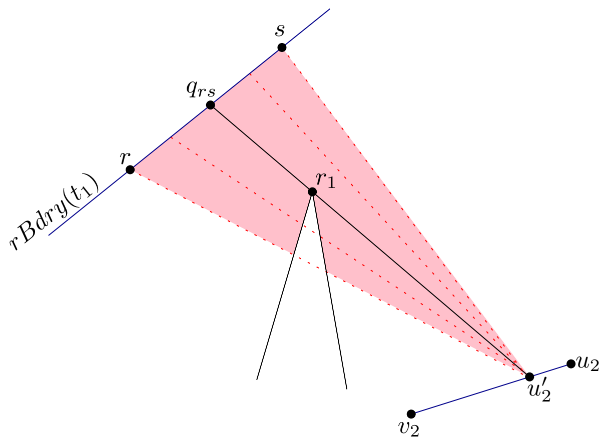


Figure 22: **Case** B: u_2' sees a part of (segment) or whole of $\mathsf{rBdry}(t_1)$.

In this case, we essentially show that this scenario does not arise. Assume that either $x(r) = x(s)$ and $y(r) < y(s)$ or $x(r) < x(s)$. But $x(r) = x(s)$ is not possible because that would mean that the entire $\mathsf{rChord}(t_1)$ is a vertical line (with same x-coordinates); and as $\mathsf{rChord}(t_1) \subsetneq \mathcal{M}$ intersects $\partial(\mathcal{M})$ on at least three points, namely $\mathsf{rImage}(t_1), \mathcal{R}(t_1)$ (the two distinct endpoints of $\mathsf{rChord}(t_1)$) and $\mathsf{rAnchor}(t_1)$, it contradicts the fact that the polygon $\mathcal{M}$ is x-monotone. So, we only consider the case where $x(r) < x(s)$.

Now, we have that the entire $\overline{rs}$ is visible from u_2'. Therefore, if we join any point q_{rs} on $\overline{rs}$ from u_2', the line $\overline{u_2' q_{rs}}$ lies in $\mathcal{M}$. So, essentially the solid triangle $\triangle u_2' rs$ lies entirely inside $\mathcal{M}$. Let us now consider any point q_{rs} on the open segment (r, s). In particular, for the sake of clarity, let q_{rs} be the mid-point of the line $\overline{rs}$. Note that this point q_{rs} is visible from u_2' and vice-versa. Since no point to the left of u_2' is visible from q_{rs}, then, by the argument of **Case** A1 we will get a reflex vertex r_1 through which $\overline{q_{rs} u_2'}$ must pass (r_1 must lie on the open segment (q_{rs}, u_2')). But then, observe that a part of the hill which contains the reflex vertex r_1 must lie inside the $\triangle u_2' rs$ (See Figure 22). But this contradicts the fact that $\triangle u_2' rs$ lies entirely inside $\mathcal{M}$. Thus, we conclude that this **Case** B does not occur.

This completes the proof that $u_2' \in \mathcal{Z}_{\mathtt{mid}}$. Now, as per our description of Algorithm 1, the point u_2' has been identified during the arrangement A_2, hence $u_2' \in Q_2$. □

Thus, we got a replacement of t_2 in a clone point of a vertex from $Q_2 \subseteq \mathcal{Z}_{\mathtt{mid}}$.

Notice that when attempting to find an alternative for t_2, the key requirement is having $\mathsf{rAnchor}(t_1)$ and $\mathsf{rImage}(t_1)$ as a vertex of Q_1. Here, this fact is ensured by Theorem 5.29. Similarly, in the search for a substitute for t_3, it is crucial to have $\mathsf{rAnchor}(t_2)$ and $\mathsf{rImage}(t_2)$ as vertices of Q_i for some $i \in [2k]$, although acquiring these is not as straightforward as obtaining $\mathsf{rImage}(t_1)$ since the substitute of t_2 may be a clone point which is not a vertex of $V(\mathcal{M}) \cup \mathcal{Z}_{\mathtt{mid}}$.

Let t_2^* (a clone point) be a valid substitute for t_2 within W. In the following section, we will determine the substitute for t_3 by considering all possible positions of t_2^*.

Determining Substitutes for t_3. Here, we show that t_3 can be substituted with a clone point of a vertex from Q_4. Recall that $t_3 \in W^{\mathtt{bad}}_{\mathtt{bdry}}$ and $t_2^* = u_2'^{+}/u_2'^{-}$, where u_2' is a vertex of Q_2. Also t_3 lies on the edge $e_3 = \overline{u_3 v_3}$ of $\mathcal{M}$. We first observe the following.

Observation 5.31. $\mathsf{Vis}(\mathsf{rBdry}(u_2'))|_{e_3} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(t_4))|_{e_3} \neq \mathtt{Reg}(e_3)$

Proof. If $\mathsf{Vis}(\mathsf{rBdry}(u_2'))|_{e_3} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(t_4))|_{e_3} = \mathtt{Reg}(e_3)$, then, it means that t_3 is visible from either $\mathsf{Vis}(\mathsf{rBdry}(u_2'))|_{e_3}$ or $\mathsf{Vis}(\ell\mathsf{Bdry}(t_4))|_{e_3}$. But t_3 cannot be visible from any point of $\ell\mathsf{Bdry}(t_4)$. So, t_3 must be visible from some point of $\mathsf{rBdry}(u_2')$. Using similar kind of argument in Theorem 5.28 we get that $\mathsf{rBdry}(u_2') \cap \ell\mathsf{Bdry}(t_3) \neq \emptyset$. So, it means u_2' is visible from $\ell\mathsf{Bdry}(t_3)$. Also, u_2' is visible from $\mathsf{rBdry}(t_1)$. But then, we recall the case where we needed t_2 to be replaced by a clone point. In that case, $\mathsf{rBdry}(t_1)$ sees a vertex of e_2 and $\ell\mathsf{Bdry}(t_3)$ sees the other vertex of e_2. But as both intersect in at least one point, i.e., u_2', and, as $\mathsf{Vis}(\mathsf{rBdry}(t_1)|_{e_2}, \mathsf{Vis}(\ell\mathsf{Bdry}(t_3)|_{e_2}$ are connected, this violates Theorem 5.16. Thus, we get a contradiction. [Note that as $\mathsf{Vis}(\mathsf{rBdry}(u_2')|_{e_3}, \mathsf{Vis}(\ell\mathsf{Bdry}(t_4)|_{e_3}$ are closed, the non visible region of $\mathtt{Reg}(e_3)$ from $\mathsf{rBdry}(u_2')$ and $\ell\mathsf{Bdry}(t_4)$ is open.] □

Using a similar argument, we can claim the following as well.

Observation 5.32. $\mathsf{Vis}(\mathsf{rBdry}(t_2^*))|_{e_3} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(t_4))|_{e_3} \neq \mathtt{Reg}(e_3)$

Let $\mathtt{Reg}_{2,4} := \mathsf{Vis}(\mathsf{rBdry}(t_2^*))|_{e_3} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(t_4))|_{e_3}$ and $\mathtt{Reg}^c_{2,4} := \mathtt{Reg}(e_3) \setminus (\mathsf{Vis}(\mathsf{rBdry}(t_2^*))|_{e_3} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(t_4))|_{e_3})$. If at most one of $\{u_3, v_3\}$ is contained in $\mathtt{Reg}_{2,4}$, then the region $\mathsf{tRegion}(t_3, W)$ contains either u_3 or v_3. Hence, we have shown that $V(\mathcal{M})$ contains a replacement of t_3. But as we have already assumed that $t_3 \in W^{\mathtt{bad}}_{\mathtt{bdry}}$, this case doesn't arise. So, we consider that $\{u_3, v_3\} \subseteq \mathtt{Reg}_{2,4}$.

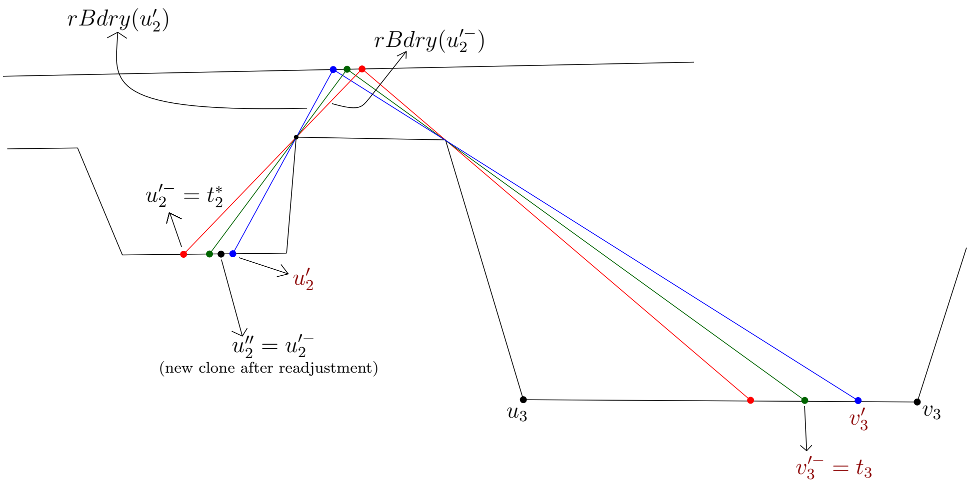


Figure 23: Readjusting the position of t_2^* after replacing t_3 by a clone point. Here, our initial choice of substitute for t_2 was $u_2'^-$, which changes to u_2'' after replacing t_3 by $v_3'^-$.

We know that $t_2^* = u_2'^+/u_2'^-$. For the time being, let us assume that t_2^* is u_2', a vertex of $Q_2 \subseteq \mathcal{Z}_{\mathtt{mid}}$. Then t_3 can be replaced in a manner analogous to our replacement of t_2, where we already knew the substituted position of $t_1 = w_{i_j}$. So, temporarily, we do that itself by assuming $t_2^* = u_2'$. That is, we look at the closed, connected region on $\mathtt{Reg}(e_3)$, $\mathsf{Vis}(\mathsf{rBdry}(u_2')|_{e_3} = \overline{v_3 v_3'}$ (say). Then, we replace t_3 by a clone of v_3', either $v_3'^+$ or $v_3'^-$, whichever lies on the open connected region $\mathtt{Reg}^c_{2,4}$ (Note that $\mathtt{Reg}^c_{2,4}$ is an open connected region due to Theorem 5.31). So, w.l.o.g., we assume that the substitute of t_3 is $v_3'^+$ (the other case will be analogous, as our arguments provided below will evidently show). Now, this arrangement of u_2', $v_3'^+$ and t_4 will satisfy $\mathsf{Vis}(u_2') \cap \mathsf{Vis}(v_3'^+) = \emptyset$ and $\mathsf{Vis}(v_3'^+) \cap \mathsf{Vis}(t_4) = \emptyset$. In particular, as $\mathsf{Vis}(u_2') \cap \mathsf{Vis}(v_3'^+) = \emptyset$, there exists an open region on $\mathtt{Reg}(e_2)$ where any replacement of u_2' would successfully preserve the non-intersection property with $\mathsf{Vis}(v_3'^+)$. So, effectively, we can choose a clone point of u_2', such that it still preserves the non-intersection property with $\mathsf{Vis}(v_3'^+)$ (due to Theorem 5.32). Since we can choose this clone point from an open segment around u_2' on $\mathtt{Reg}(e_2)$, we particularly choose one such that it lies to the left of u_2' (recall that our original substitute of t_2 was also positioned on the left of u_2'). So, we can follow the same procedure of replacement as we had done for t_2, without affecting the non-intersection property with both $\mathsf{Vis}(t_1)$ (as any point to the left of u_2' that lies on $\mathtt{Reg}(e_2)$ is not visible from $\mathsf{rBdry}(t_1)$), as well as $\mathsf{Vis}(v_3'^+)$. It may happen that after choosing $v_3'^+$ as our replacement for t_3, we have to update the previous choice of replacement of t_2, by choosing a point closer to u_2', but even then it can still be considered as a clone point of u_2'. Thus, ultimately, this allows us to effectively replace t_3 with a clone point of a vertex of Q_4, illustrated in Figure 23.

Determining Substitutes for t_4 and onwards So, proceeding similarly, we replace the next set of witnesses t_4 onward. Recall that our goal is to demonstrate that each witness in $W^{\mathtt{bad}}_{\mathtt{bdry}}$ has an appropriate replacement in $\mathcal{Z}_{\mathtt{mid}}$. So, to do that, we had chosen a chain of such consecutive witnesses in $W^{\mathtt{bad}}_{\mathtt{bdry}}$, $t_2, t_3, \ldots, t_{k'-1}$ (t_1 and t'_k lies in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$). We then tried to find a suitable substitute for each t_i $(1 < i < k')$ in $W^{\mathtt{bad}}_{\mathtt{bdry}}$ in $\mathcal{Z}_{\mathtt{mid}}$. However, as described thus far, the replacement is not consistent with vertices from $V(\mathcal{M}) \cup \mathcal{Z}_{\mathtt{mid}}$ and includes clone points. Some witnesses may find their replacements in clone points, yet our algorithm never produces clone points. Our next aim is to show that we can obtain a viable alternative for these clone points of the vertices of z_{mid}, in $\mathcal{Z}_{\mathtt{mid}}$ itself.

Left $\to$ Right: Find a replacement for t_i and updating all previous Clones for $t_j, j < i$. We have seen that when we try to find a replacement of t_3 as a clone point of some vertices of Q_4, after finding the suitable clone, we might have to readjust the position of the clone point which was initially considered to be a replacement for t_2. Similarly, we proceed to the next set of witnesses from left to right. Assume that we are at $A_{2i}{}^{\text{th}}$ arrangement where we aim to find a replacement of t_i. Once we find the clone point, which is a replacement t_i, we might need to update all the clone points corresponding to the replacement of each t_j where $1 < j < i$. But we know that we do not have to find a replacement for t'_k as t'_k has already been substituted by a point in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$. So in $A_{2i}{}^{\text{th}}$ arrangement we actually find all the replacement of t_i for each $i \in \{2, 3, \ldots, k' - 1\}$ in some clone point, say u_i^* (either u_i^+ or u_i^-) where u_i is a vertex obtained $A_{2i}{}^{\text{th}}$ arrangement.

Recall that our primary goal is to demonstrate that each vertex in W has an appropriate replacement in $\mathcal{Z}_{\texttt{mid}}$. In the following, we illustrate that witnesses with potential replacements in clone points also have substitutes in $\mathcal{Z}_{\texttt{mid}}$. More precisely, every clone point has a viable alternative in $\mathcal{Z}_{\texttt{mid}}$. Further details are provided below.

Right $\rightarrow$ Left: Generating replacement of Clones in $\mathcal{Z}_{\texttt{mid}}$. Note that t'_k is a point in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$. Now, focus on the witness $t_{k'-1}$. According to the previously described method, $t_{k'-1}$ was initially replaced by a clone point of a vertex of $\mathcal{Z}_{\texttt{mid}}$, say $p_{k'-1}(\in Q_{2k'})$, while moving from left to right. Note that the vertex of $\mathcal{Z}_{\texttt{mid}}$, $p_{k'-1}$, was determined by analyzing the visibility regions from $\mathsf{rBdry}(t_{k'-2})$ on the edge $e_{k'-1} = \overline{u_{k'-1}v_{k'-1}}$ of $\mathcal{M}$, where $t_{k'-1}$ lies. However, since $t_{k'}$ belongs to $V(\mathcal{M}) \cup \mathcal{R}_{mid}$, we can opt for a clone of another vertex of $\mathcal{Z}_{\texttt{mid}}$, say $q_{k'-1}$ from Q_1 to do a replacement of $t_{k'-1}$ where $q_{k'-1}$ lies on the edge $e_{k'-1}$ and obtained in the arrangement A_1. Specifically, $q_{k'-1}$ is determined by examining the part of $e_{k'-1}$ that is visible from $\ell\mathsf{Bdry}(t_{k'})$. The clone of $q_{k'-1}$ would adequately replace $t_{k'-1}$. This approach is symmetric to the replacement method of t_2 by using the fact that t_1 lies in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$. Now we can claim the following.

Claim 5.33. *These two choices of replacement of $t_{k'-1}$, i.e., clone of $p_{k'-1}$ (obtained from $\mathsf{rBdry}(t_{k'-2})|_{e_{k'-1}}$) and clone of $q_{k'-1}$ (obtained from $\ell\mathsf{Bdry}(t_{k'})|_{e_{k'-1}}$) are distinct.*

Proof. As we were not able to replace $t_{k'-1}$ with a vertex of $\mathcal{M}$, it implies that $\mathsf{rBdry}(t_{k'-2})$ sees exactly one vertex of $e_{k'-1}$ while $\ell\mathsf{Bdry}(t_{k'})$ sees the other. If both $p_{k'-1}, q_{k'-1} \in V(\mathcal{M})$, then they are the two endpoints of the edge $e_{k'-1}$ (since $p_{k'-1} \in V(\mathcal{M})$ is visible from $\mathsf{rBdry}(t_{k'-2})$ while $q_{k'-1} \in V(\mathcal{M})$ is visible from $\ell\mathsf{Bdry}(t_{k'})$), and hence, are distinct. Otherwise, if $p_{k'-1} = q_{k'-1}$, then, $\overline{u_{k'-1}p_{k'-1}}$ is visible from $\mathsf{rBdry}(t_{k'-2})$ and $\overline{v_{k'-1}q_{k'-1}}$ is visible from $\ell\mathsf{Bdry}(t_{k'})$, or, $\overline{v_{k'-1}p_{k'-1}}$ is visible from $\mathsf{rBdry}(t_{k'-2})$ and $\overline{u_{k'-1}q_{k'-1}}$ is visible from $\ell\mathsf{Bdry}(t_{k'})$. But then, $\texttt{Reg}_{k'-1} = \mathsf{Vis}(\mathsf{rBdry}(t_{k'-2})|_{e_{k'-1}} \cup \mathsf{Vis}(\ell\mathsf{Bdry}(t_{k'})|_{e_{k'-1}} = \texttt{Reg}(e_{k'-1})$, as $p_{k'-1} = q_{k'-1}$ and the respective regions $\mathsf{Vis}(\mathsf{rBdry}(t_{k'-2})|_{e_{k'-1}}$ and $\mathsf{Vis}(\ell\mathsf{Bdry}(t_{k'})|_{e_{k'-1}}$ are connected, which contradicts Theorem 5.16. So, as $p_{k'-1} \neq q_{k'-1}$ we can claim that their respective clones are also distinct. □

Thus, we can replace $t_{k'-1}$ by a clone of $p_{k'-1}$, or a clone of $q_{k'-1}$ with $p_{k'-1} \neq q_{k'-1}$. Since we cannot replace the witness with a vertex of the edge $e_{k'-1}$, the region in which we can place a witness, i.e., $\texttt{Reg}_{k'-1}$ is open and connected (due to Theorem 5.27). This means that we can replace $t_{k'-1}$ by any point that lie in between $\overline{p_{k'-1}q_{k'-1}}$. Specifically, we can replace $t_{k'-1}$ by the midpoint of $\overline{p_{k'-1}q_{k'-1}}$. (See Figure 24) Now, as the mid-points of two adjacent vertices of $\mathcal{Z}_{\texttt{mid}}$ on the same edge of $\mathcal{M}$ in the $A_i{}^{\text{th}}$ arrangement get added to our set $\mathcal{Z}_{\texttt{mid}}$, we can consider $t_{k'-1}$ to be replaced by a vertex from $\mathcal{Z}_{\texttt{mid}}$ which is the midpoint (say, $\alpha_{k'-1}$) between two adjacent vertices of Q_i generated in the preceding step of Algorithm 1) itself. Then, by following the same argument as we replaced $t_{k'-1}$, we move towards the set of witnesses to the left of $t_{k'-1}$ again. In more detail, once we try to find a replacement for $t_{k'-1}$ in $\mathcal{Z}_{\texttt{mid}}$, specifically in $Q^{\texttt{midpt}}_{k'+1}$, we use the position of the point t'_k. We call t'_k is the *important* point for $t_{k'-1}$. In the next steps, when trying to find a replacement of $t_{k'-2}$, we need such an important point. For that, we apply the same argument as before, considering the vertex $\alpha_{k'-1}$ as an important point for $t_{k'-2}$. Continuing this way we can get a set of points $\alpha_{k'-2}, \alpha_{k'-3}$, …, and so on, which are distinct from the choices we had obtained by replacing the witnesses while moving from left to right. Since choices q_i and p_i are distinct (due to Theorem 5.33), existence of such an $\alpha_i \in \mathcal{Z}_{\texttt{mid}}$ is always ensured. Proceeding this way, we can get a choice of a substitute for any witness of W by a vertex from $\mathcal{Z}_{\texttt{mid}}$. From the above discussion, we obtain the following conclusions for each $i \in \{2, 3, \ldots, k' - 1\}$:

- α_i is a mid-point of two vertices, say p_i and q_i of $\mathcal{Z}_{\texttt{mid}}$ where $p_i \in Q_i$, $q_i \in Q_{2k-i}$.
- $\alpha_i \in \mathcal{Z}_{\texttt{mid}}$. More specifically, $\alpha_i \in Q_{2k-i}^{\texttt{midpt}}$.
- the point α_i is a replacement of the witness t_i in $\mathcal{Z}_{\texttt{mid}}$.
- the point α_{i+1} is an important point for t_i ($\alpha'_k = t'_k$).

Proof of Theorem 5.24. We have already shown that for a witness $w \in W_{\texttt{int}} \cup W_{\texttt{bdry}}^{\texttt{good}}$, w has a valid substitute in $V(\mathcal{M}) \cup \mathcal{R}_{mid}$. And, for any chain of consecutive witnesses $t_i \in W_{\texttt{bdry}}^{\texttt{bad}} (1 < i < k')$, we have a valid substitute in $\mathcal{Z}_{\texttt{mid}}$. Therefore, every $w \in W$ has a valid substitute in $V(\mathcal{M}) \cup \mathcal{R}_{mid} \cup \mathcal{Z}_{\texttt{mid}}$. This shows the correctness of our Algorithm 1. Hence, the proof follows. ☐

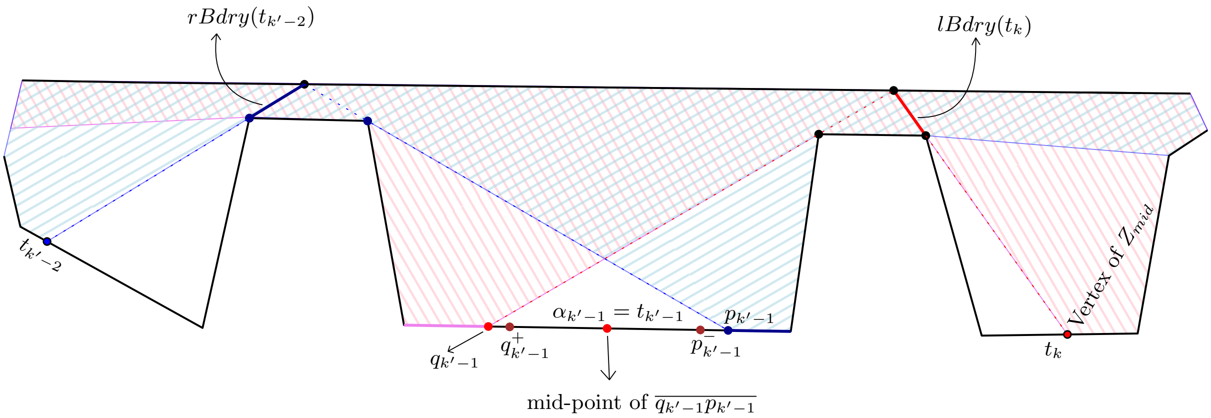


Figure 24: Replacement of $t_{k'-1}$ with the mid-point of two adjacent vertices of $\mathcal{Z}_{\texttt{mid}}$.

Running Time. The running time of the Algorithm 1 is influenced by both the number of arrangements and the cardinality of points generated in each arrangement. According to Theorem 5.23, the number of points in $\mathcal{Z}_{\texttt{mid}}$ is bounded by $r^{\mathcal{O}(k)}n$, where $|V(\mathcal{M})| = n$ and r represent the count of reflex vertices in $\mathcal{M}$. Given that the number of arrangements is limited to $2k$, and each arrangement is quadratically dependent on the number of existing points, the running time is bounded by $r^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$.

Proof of Theorem 1.3. Consider a hypothetical solution W for the Witness Set of $\mathcal{M}$, divided into parts $W = W_{\texttt{int}} \cup W_{\texttt{bdry}}^{\texttt{good}} \cup W_{\texttt{bdry}}^{\texttt{bad}}$, as defined in Theorem 5.9. According to Theorem 5.18, we understand that $V(\mathcal{M})$ functions as a $(W, W_{\texttt{bdry}}^{\texttt{good}})$-potential witness set. Furthermore, as detailed in Theorem 5.22, there is a $(W, W_{\texttt{int}})$-potential witness set with a size of at most r^2, which can be determined in $n^{\mathcal{O}(1)}$ time. Additionally, by Theorem 5.25, it is possible to generate a point set $\mathcal{Z}_{\texttt{mid}}$ with a maximum size of $n \cdot (2+r)^{2k-1}$, ensuring that $\mathcal{Z}_{\texttt{mid}}$ is a $(W, W_{\texttt{bdry}}^{\texttt{bad}})$-potential witness set, and this set can be constructed within $r^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time. Hence, we have shown that if the largest witness set has size k, then the set $V(\mathcal{M}) \cup \mathcal{R}_{mid} \cup \mathcal{Z}_{\texttt{mid}}$ (as derived by Algorithm 1) contains a witness set of the same size, k.

Now, the algorithm for the Witness Set in $\mathcal{M}$ works as follows: Define O_k as the output (the set of vertices) from $\texttt{WitGen}(\mathcal{M}, k)$. We aim to identify the minimum k (by guessing the value k) such that: (i) $V(\mathcal{M}) \cup \mathcal{R}_{mid} \cup O_k$ encompasses a witness set of size k, while (ii) $V(\mathcal{M}) \cup \mathcal{R}_{mid} \cup O_{k+1}$ does not contain any witness set of size $k + 1$. The correctness of this approach is assured because if the maximum possible size of a witness set exceeds k, then $V(\mathcal{M}) \cup \mathcal{R}_{mid} \cup O_{k+1}$ must include a witness set of at least size $k + 1$. Since the DISCRETE WITNESS SET problem in a monotone polygon $\mathcal{M}$ can be solved in $\mathcal{O}(|S|^2 + |V(\mathcal{M})||S|)$ time (as demonstrated by Theorem 1.2), we thus infer that the WITNESS SET problem in a monotone polygon is solvable in $r^{\mathcal{O}(k)} \cdot n^{\mathcal{O}(1)}$ time. □

6 Approximation Algorithms for WITNESS SET in Monotone Polygons

Here we first obtain a polynomial-time factor-2 approximation and a polynomial-time approximation scheme (PTAS) for the WITNESS SET in Monotone Polygons. Both algorithms employ a discretization method similar to that of the optimal algorithm.. The only difference in this discretization is as follows: we first draw the vertical chords $\ell(v)$ through each vertex $v \in V(\mathcal{M})$ (vertical decomposition). Let $H = \cup_v \{h_v\}$ be the set of other end points h_v[5] of $\ell(v)$. After this, the generation of the i-th iteration of the arrangement remains the same as the generation of arrangements in the optimal solution. We can sort all the vertical chords according to their x-coordinate. Let the ordering be $\ell(v_1) < \ldots < \ell(v_n)$. The following observation is immediate.

Observation 6.1. *If $\ell(v_j)$ and $\ell(v_{j+1})$ are the two consecutive chords and p is a point in $\mathcal{M}$ that lies in the vertical strip $(\{z \in \mathcal{M} : x(v_j) \leq x(z) \leq x(v_{j+1})\})$ then $\ell(v_j),\ \ell(v_{j+1}) \subseteq \mathsf{Vis}(p)$.*

Definition 6.2 (well-defined)**.** Let W be a witness set in $\mathcal{M}$. For a witness $w \in W$, we say w is *well-defined* in a set Q with respect to W if w has a replacement in Q, that is, there exists a point $q \in Q$ such that $(W \setminus \{w\}) \cup \{q\}$ is a witness set in $\mathcal{M}$.

Now, let Q_1 be the set of all points generated from the monotone polygon $\mathcal{M}$ as follows:

- For each $v \in V(\mathcal{M}) \cup H$ and $x \in \mathcal{R}$, if v is visible to x, add a maximal chord L in $\mathcal{M}$ joining v and x, where "maximal" means no other chord in $\mathcal{M}$ contains L.
- Add the point $(L \cap \partial(\mathcal{M})) \setminus \{v\}$ into Q_1.

6.1 Factor-2 Approximation

Lemma 6.3. *Any witness set W of size k contains a subset $W_{odd} \subseteqq W$ such that $|W_{odd}| \geq \frac{k}{2}$ and each witness in W_{odd} are well-defined in Q_1 with respect to W_{odd}.*

Proof. Let $W = \{w_1, \cdots, w_k\}$ be a witness set of size k, with $x(w_i) < x(w_{i+1})$, where $1 \leq i < k$. Let $W_{odd} = \{w_i \mid w_i \in W, 1 \leq i \leq k \text{ and } i \text{ is odd}\}$. Obviously W_{odd} is a witness set in $\mathcal{M}$ of size at least $\frac{k}{2}$. Now we show each witness in W_{odd} is well-defined in Q_1 with respect to W_{odd}, i.e., Q_1 is a W_{odd}-potential witness set in Q_1. Surely, the witness w_1 and w_k is well-defined in $V(\mathcal{M})$ and so has replacemnt in Q_1. Now we consider the witnesses that are in between w_1 and w_k, i.e, witnesses from $W_{odd} \setminus \{w_1, w_k\}$.

Let w_{2t+1} be an arbitrary vertex in W_{odd} for some positive integer t where $1 < 2t + 1 < k$. Now there must exist a pair of consecutive vertical chords, say $\ell(v_j)$ and $\ell(v_{j+1})$ such that w_{2t} lies in the vertical strip formed by $\ell(v_j)$ and $\ell(v_{j+1})$ with $\ell(v_j), \ell(v_{j+1}) \subset \mathsf{Vis}(w_{2t})$ (follows from Theorem 6.1). Note that w_{2t+1} may not be well-defined in Q_1 with respect to the witness set W. But if we consider the witness $W \setminus \{w_{2t}, w_{2t+2}\}$, then we can show that w_{2t+1} is well-defined in Q_1 with respect to the witness set $W \setminus \{w_{2t}, w_{2t+2}\}$. Below, we prove that.

Notice that $\mathsf{Vis}(w_{2t-1}) \cap \ell(v_{j+1}) = \emptyset$ in fact $\mathsf{Vis}(w_{2t-1}) \cap \ell(v_j) = \emptyset$. Also $\mathsf{Vis}(w_{2t+1}) \cap \ell(v_{j+1}) = \emptyset$. Now consider $\ell\mathsf{Chord}(w_{2t+1})$. It must pass through some reflex vertex, say x^*. Now consider the line L (not segment) passing through w_{2t+1} and x^*. If we rotate the line L anti-clockwise through x^* (i.e., the

[5]The chord through $\mathsf{xMin}(\mathcal{M})$ or $\mathsf{xMax}(\mathcal{M})$ could be a point which is the vertex itself.

point x^* remains in the line when rotating), it must hit some vertices in $V(\mathcal{M}) \cup H$. Now consider the point p where the rotating line hits some vertices in $V(\mathcal{M}) \cup H$ at the very first time. Either $p \in V(\mathcal{M})$ or p is a vertex of H, more specifically a vertex from the line segment $\ell(v_{j+1})$ or $\ell(v_{j+2})$ (this is true because while rotating it may happen that the $\ell\mathsf{Chord}(w_{2t+1})$ hits a vertex of $V(\mathcal{M}) \cup H$ on the left side or right side). Now we can find a vertex w^* in the rotated line such that w^* is a replacement of w_{2t+1} in Q_1 with respect to the witness set $W \setminus \{w_{2t}, w_{2t+2}\}$. This is true due the fact of combining (i) As $\mathsf{Vis}(w_{2t-1}) \cap \ell(v_j) = \emptyset$ and w^* can see at most one point of $\ell(v_{j+1}) \cup \ell(v_{j+2})$ that is also from H, $\ell(v_j) \cap \mathsf{Vis}(w^*) = \emptyset$ that indeed imply $\mathsf{Vis}(w_{2t-1}) \cap \mathsf{Vis}(w^*) = \emptyset$, similarly $\ell(v_{j+3}) \cap \mathsf{Vis}(w^*) = \emptyset$ that indeed imply $\mathsf{Vis}(w_{2t+3}) \cap \mathsf{Vis}(w^*) = \emptyset$, (ii) as we rotate the line L anti-clockwise through x^*, $\mathsf{Vis}(w_{2t-1}) \cap \mathsf{Vis}(w^*) = \emptyset$ and $\mathsf{Vis}(w_{2t+3}) \cap \mathsf{Vis}(w^*) = \emptyset$. □

We have shown that Q_1 contains a witness set of size at least $k/2$ where k is the size of the maximum witness set. So even for the optimum value, the argument works. Hence, Q_1 contains a factor-2 solution (witness set). It remains to find a maximum witness set that is contained in Q_1. Now since $|Q_1| = \mathcal{O}(nr)$ and the Discrete Witness Set in monotone polygons is solvable in $\mathcal{O}(|Q_1|^2 + |V(\mathcal{P})||Q_1|)$ time (by Theorem 1.2). We have the following theorem.

Theorem 6.4. *There is a 2-approximation algorithm for the* Witness Set *problem in monotone polygons running in time $\mathcal{O}(r^2 n^2)$, where r and n denote the number of reflex vertices and vertices, respectively, of the input monotone polygon.*

6.2 Polynomial-time Approximation Scheme

The polynomial-time approximation scheme algorithm follows a similar idea to the factor-2 approximation algorithm. Intuitively, when we try to design 2-factor algorithm we show that for any witness set $W = \{w_i : i \in [k]\}$ if we consider the witness set $W_{odd} = \{w_i \mid w_i \in W, 1 \le i \le k\}$ the set Q_1 forms a W_{odd}-potential witness set. Here we show that when we try $(1+\epsilon)$-factor approximation algorithm we show that for any witness set $W = \{w_i : i \in [k]\}$ if we consider the witness set $W_{apx} = \{w_j \mid w_j \in W, j \bmod (i+1) \neq 0\}$ then Q_{2i} forms a W_{apx}-potential witness set, where $i = \frac{1}{\epsilon}$.

Lemma 6.5. *Any witness set W of size k contains a subset $W_{apx} \subseteq W$, such that $|W_{apx}| \geq \frac{ik}{i+1}$ and each witness in W_{apx} is well-defined in Q_{2i} with respect to W_{apx}.*

Proof. Let $W_{apx} = \{w_j \mid w_j \in W, j \bmod (i+1) \neq 0\}$. W_{apx} is a witness set of size at least $\frac{ik}{i+1}$. Now we show each witness in W_{apx} is well-defined in Q_{2i} with respect to W_{apx}, i.e., has a replacement in Q_{2i}. Surely, the witness w_1 and w_k is well-defined in Q_{2i} with respect to W_{apx}. Now fix a j and consider the witnesses $w_{j\times(i+1)+1}, w_{j\times(i+1)+2}, \cdots w_{j\times(i+1)+i} \in W_{apx}$ whose indexes are of the form $\{(j \times (i+1)+1), (j \times (i+1)+2), \cdots, (j \times (i+1)+i)\}$, (except w_1 and w_k). If we can show that the vertex $w_{j\times(i+1)+1} \in W_{apx}$ is well-defined on Q_2 then it trivially follows that $\{(j \times (i+1)+1), \cdots, (j \times (i+1)+i)\}$ is well-defined in $Q^*_{2\times 1}, Q^*_{2\times 3}, \cdots, Q_{2\times i-2}$ is well-defined on Q_{2i}. Towards proving that we apply a similar kind of argument as in Theorem 6.3. In Theorem 6.3, we show that if we consider w_{2t-1} and w_{2t+1} not as a witness, then we can find a replacement of w_{2t} in Q_1. Here, we also do the exact same thing.

Consider the witness $w_{j\times(i+1)}$. Now there must exists a pair of consecutive vertical chords, say $\ell(v_l)$ and $\ell(v_{l+1})$ such that $w_{j\times(i+1)}$ lies in the vertical strip formed by $\ell(v_l)$ and $\ell(v_{l+1})$ with $\ell(v_l), \ell(v_{l+1}) \subset \mathsf{Vis}(w_{j\times(i+1)})$ (follows from Theorem 6.1). Note that $w_{j\times(i+1)+1}$ may not be well-defined in Q_1 with respect to the witness set W. But if we consider the witness $W \setminus \{w_{j\times(i+1)}\}$, then we can show that $w_{j\times(i+1)+1}$ is well-defined in Q_1 with respect to the witness set $W \setminus \{w_{j\times(i+1)}\}$. Below, we prove that.

Notice that $\mathsf{Vis}(w_{j\times(i+1)-1}) \cap \ell(v_{l+1}) = \emptyset$ in fact $\mathsf{Vis}(w_{j\times(i+1)-1}) \cap \ell(v_l) = \emptyset$. Also $\mathsf{Vis}(w_{j\times(i+1)+1}) \cap \ell(v_{l+1}) = \emptyset$. Now consider $\ell\mathsf{Chord}(w_{j\times(i+1)+1})$. It must pass through some reflex vertex, say x^*. Now consider the line L (not segment) passing through $w_{w_{j\times(i+1)+1}}$ and x^*. If we rotate the line L anti-clockwise through x^* (i.e., the point x^* must be in the line when rotating), it must hit some vertices in $V(\mathcal{M}) \cup H$. Now consider the point p where the rotating line hits some vertices in $V(\mathcal{M}) \cup H$ for the first time. Either $p \in V(\mathcal{M})$ or p is a vertex of H; more specifically, a vertex from the line segment $\ell(v_{l+1})$. Now we can find a vertex w^* in the rotated line such that w^* is a replacement of $w_{j\times(i+1)+1}$ in Q_1 with respect to the witness set $W \setminus \{w_{j\times(i+1)}\}$. This is true due the fact of combining (i) As $\mathsf{Vis}(w_{j\times(i+1)-1}) \cap \ell(v_l) = \emptyset$ and w^* can see at most one point of $\ell(v_{l+1})$ that is also from H, $\ell(v_l) \cap \mathsf{Vis}(w^*) = \emptyset$ that indeed imply $\mathsf{Vis}(w_{j\times(i+1)-1}) \cap \mathsf{Vis}(w^*) = \emptyset$, (ii) as we rotate the line L anti-clock wise through x^*, it may happen that L hits a vertex of $V(\mathcal{M}) \cup H$ for the first time on the right side. Even then, by a symmetric argument we can conclude that $\mathsf{Vis}(w_{j\times(i+1)+2}) \cap \mathsf{Vis}(w^*) = \emptyset$. Given that $w_{j\times(i+1)+1}$ is well-defined in Q_1, we can apply the reasoning used to prove the correctness of Algorithm 1 to show that witnesses $w_{j\times(i+1)+2}, \cdots, w_{j\times(i+1)+i} \in W_{apx}$ whose indexes are of the form $\{(j \times (i+1)+1), \cdots, (j \times (i+1)+i)\}$ is well-defined in $Q_3, Q_5, \cdots, Q_{2i}$. This completes the proof. $\square$

In Theorem 6.5, we are able to show that Q_{2i} contains a witness set of size at least $\frac{ik}{i+1}$ where the value k can be arbitrary. So even for the optimum value, the argument works. Hence Q_{2i} contains a factor $\frac{i+1}{i}$ solution (witness set). It remains to find a maximum witness set that is contained in Q_{2i}. Now since $|Q_{2i}| = \mathcal{O}(nr^{\mathcal{O}(i)})$ and the Discrete Witness Set in monotone polygons is solvable in $\mathcal{O}(|Q|^2 + |V(\mathcal{P})||Q|)$ time (by Theorem 1.2). Thus, we have the following result.

Theorem 6.6. *For every fixed $\epsilon > 1$, there is a $(1+\epsilon)$-approximation algorithm for the* Witness Set *problem in monotone polygons running in time $r^{\mathcal{O}(1/\epsilon)} \cdot n^2$, where r and n denote the number of reflex vertices and vertices, respectively, of input polygon.*

7 Conclusion

In the literature, although Amit et al.[AMP10] claim the problem to be NP-hard, there was no NP-hardness result for the Witness Set problem in any kind of polygons. We roughly argue that it is difficult to build polynomial-size gadgets to show NP-hardness reduction for this problem. The reason is the following: let Π' be a gadget that we construct for the Witness Set problem in a simple polygon $\mathcal{P}$ from an NP-complete problem Π. Now Π' should consist of a polynomial size point set Q where witnesses of $\mathcal{P}$ in the desired solution are constrained to lie on. As shown in Theorem 1.2, the Discrete Witness Set problem can be solved in polynomial time for any polygon, so we will be unable to show the reduction between Π and Π'. That is why we feel that NP-hardness of this problem does not hold for simple polygons, though we do not rule out the hardness result for simple polygons. The preceding discussion implies that any NP-hardness proof for this problem would likely require methods beyond standard polynomial-size discretization techniques.

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