NON-KÄHLER CALABI–YAU MANIFOLDS AND HOLOMORPHIC GEOMETRIC STRUCTURES

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Abstract. We study holomorphic geometric structures on non-Kähler compact complex manifolds with trivial canonical line bundle. For Vaisman Calabi-Yau manifolds we prove that all holomorphic geometric structures of affine type on them are locally homogeneous. Moreover, if the geometric structure is rigid, then the Vaisman manifold must be a Kodaira manifold. The proof uses a Beauville-Bogomolov type decomposition from [Is] together with a weak form of Bochner principle for Vaisman Calabi-Yau manifolds that we prove here. Other results show that a compact complex manifold with self-dual holomorphic tangent bundle bearing a rigid holomorphic geometric structure of affine type have infinite fundamental group. We prove the same result for compact complex manifolds with trivial canonical line bundle having semistable holomorphic tangent bundle, with respect to some Gauduchon metric. We exhibit (non-Kähler) compact complex simply connected manifolds with trivial canonical line bundle that admit non-closed holomorphic one-forms.

Contents

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1. Introduction

The geometry of compact Kähler manifolds with vanishing real first Chern class was intensively investigated in the last decades. A main differential geometric tool in the study of those so-called Kähler Calabi-Yau manifolds was introduced by the celebrated theorem of Yau [Ya] which endows these manifolds with Ricci flat Kähler metric. The Riemannian geometry of the Ricci flat Kähler metric leads to the important Beauville-Bogomolov decomposition theorem for those manifolds which roughly speaking splits the manifold, up to a finite unramified cover, into a product of a flat compact complex torus and a simply connected Kähler manifold with trivial canonical line bundle [Bea, Bog]. Consequently, the canonical line bundle of a Kähler Calabi-Yau manifold has finite order (a finite power of it is holomorphically isomorphic to the trivial holomorphic line bundle). Another powerful tool which plays a role in the Beauville-Bogomolov decomposition theorem is the Bochner principle which asserts that any holomorphic tensor on a compact Kähler Calabi-Yau manifold is parallel with respect to a Ricci flat Kähler metric on it [Bea].

Inspired by the results on geometric structures proved in [Gr, DG], and using the geometry of compact Kähler Calabi-Yau manifolds, the second-named author proved previously the following theorem:

Theorem 1.1 ([Du])**.** *Let M be a compact Kähler manifold M with trivial first Chern class bearing a holomorphic geometric structure ϕ of affine type. Then*

(1) *ϕ is locally homogeneous;*
(2) *if ϕ is rigid, then M is covered by a compact complex torus.*

The proof of the first statement makes use of the Bochner principle, while the second statement uses the Beauville–Bogomolov decomposition theorem [Bea, Bog]. Statement 2 of Theorem 1.1 generalizes a result established in [IKO] for holomorphic affine connections.

Concerning non-Kähler manifolds, the results of Tosatti in [To] triggered attention on compact complex manifolds having vanishing first Chern class in the Bott-Chern cohomology (see Section 3) with the hope that they might inherit some geometric features of Kähler Calabi-Yau manifolds. Following Tosatti, [To], we call those manifolds non-Kähler Calabi-Yau manifolds. In particular, compact complex manifolds with trivial canonical line bundle are non-Kähler Calabi-Yau in this sense. It was proved in [To] that non-Kähler Calabi-Yau manifolds in the Fujiki class $\mathcal{C}$ (i.e., bimeromorphic with Kähler manifolds) have torsion canonical line bundle (meaning some power of the canonical line bundle is holomorphically trivial). Recently, a decomposition theorem and a Bochner principle were proved in [BCDG] for non-Kähler Calabi-Yau manifolds in the Fujiki class $\mathcal{C}$ which are either Moishezon (i.e., birational with projective manifolds) or of dimension not more than four.

It should be mentioned that the *generalized Calabi-Gray manifolds* constructed by Fei in [Fe] provide examples of simply connected non-Kähler Calabi-Yau manifolds with trivial canonical line bundle that admit non-trivial holomorphic one-forms with the property that the form vanishes at some point of the manifold. Consequently, the Bochner principle does not apply to those generalized Calabi-Gray manifolds constructed in [Fe] whose holomorphic tangent bundle is not polystable with respect to any Gauduchon metric on it (see Section 3).

Examples of complex structures on connected sums of $n \geq 2$ copies of $S^3 \times S^3$ giving non-Kähler Calabi-Yau threefolds with trivial canonical line bundle were constructed by Lu and Tian and by Friedman in [Fr, LT1, LT2]. Bozhkov proved that all those examples of simply connected non-Kähler Calabi-Yau threefolds admit a stable holomorphic tangent bundle with respect to any Gauduchon metric [Boz]. Other examples of simply connected non-Kähler Calabi-Yau threefolds with trivial canonical line bundle and of algebraic dimension zero were constructed in [FP]. Notice also that examples of compact complex manifolds with vanishing first Chern class and holomorphic canonical bundle which is not of finite order are given in [Ro].

Recently a decomposition theorem of Beauville-Bogomolov type was proved by Istrati, [Is], for non-Kähler Calabi-Yau manifolds which are Vaisman (see [Va1, Va2] for Vaisman manifolds). More precisely, it is proved in [Is] that Vaisman Calabi-Yau manifolds have the structure of a principal elliptic orbibundle over an orbifold basis which is the product of an abelian variety with a simply connected orbifold with trivial canonical line bundle. This implies, in particular, that the canonical line bundle is of finite order. Her proof uses the orbifold version of the Beauville-Bogomolov decomposition for projective orbifolds proved by Campana in [Ca].

In this article we prove a weak form of the Bochner principle for Vaisman Calabi-Yau manifolds (Theorem 5.1) which says the following: *No nontrivial holomorphic tensor on a compact Vaisman Calabi-Yau manifold vanishes at some point.* Using this Theorem 5.1 and the decomposition theorem of Istrati in [Is], we deduce the following analogous of Theorem 1.1 for compact Vaisman Calabi-Yau manifolds.

Theorem 1.2. *Let M be a compact Vaisman Calabi-Yau manifold bearing a holomorphic geometric structure ϕ of affine type. Then the following two statements hold:*

(1) *ϕ is locally homogeneous and invariant with respect to the canonical elliptic fibration.*
(2) *If ϕ is rigid, then M is an orbifold elliptic bundle over an orbifold abelian variety (so M is a Kodaira manifold).*

Notice that all Kodaira (primary and secondary) surfaces are known to admit Vaisman metrics [Bel]: they are Vaisman Calabi-Yau complex surfaces. On the other hand, all Kodaira primary surfaces admit complex affine structures and other non-flat locally homogeneous holomorphic affine connections [IKO, Vi]. Kodaira manifolds are higher dimensional generalizations of Kodaira surfaces: they have the structure of a nilmanifold with a left invariant complex structure [GMPN]. They are examples of Vaisman Calabi-Yau manifolds which are principal elliptic orbibundles over an abelian variety [Is, Examples 7.2].

A by-product of the proof of Theorem 1.2 is the generalization to the case of a principal compact complex torus orbi-bundle with trivial canonical bundle over a orbifold Kähler Ricci flat manifold of a result in [BD2] (see Theorem 5.2): all holomorphic geometric structures of affine type on them are locally homogeneous and invariant by the principal fibration.

In contrast, minimal complex surfaces, which are principal elliptic bundles over compact Riemann surfaces of genus at least two, and having an odd first Betti number are known to admit non-locally homogeneous holomorphic affine connections [Du1]. They also admit nontrivial holomorphic one-forms with non-empty vanishing locus that are pulled back from the base. It was proved in [Bel] that all those minimal surfaces with Kodaira dimension one and odd first Betti number actually admit Vaisman metrics. Therefore, Theorem 1.2 and the weak Bochner principle (Theorem 5.1) do not hold for all Vaisman manifolds. Other examples of Vaisman manifolds which admit non-locally homogeneous rigid holomorphic geometric structures of affine type and nontrivial holomorphic tensors with non-empty vanishing locus are the diagonal Hopf manifolds (see the construction of those holomorphic geometric structures in Remark 5.3 and recall that diagonal Hopf manifolds are also known to admit Vaisman metrics [Bel]).

Also in this article we extend the result proved in [BCDG, Theorem D] on rigid holomorphic geometric structures on simply connected manifolds with trivial canonical line bundle and prove the following:

Theorem 1.3. *Let M be a compact complex manifold M with trivial canonical line bundle that admits a rigid holomorphic geometric structure of affine type. Then the following two statements hold:*

(1) *Assume that TM is semistable with respect to some Gauduchon metric on M. Then the fundamental group of M is infinite.*
(2) *Assume that $\dim H^0(M,\, T^*M) \;=\; \dim H^0(M,\, TM)$. Then the fundamental group of M is infinite.*

An analogous result was proved in [BCDG, Theorem D] in several other cases: for threefolds, for manifolds in the Fujiki class $\mathcal{C}$ and for manifolds having algebraic dimension at most one. All those results point in the direction of the conjecture formulated in [BCDG] which asserts that simply connected compact complex manifolds with trivial canonical line bundle do not admit holomorphic rigid geometric structures of affine type.

Theorem 1.2 and Theorem 1.3 also hold for some important geometric structures which are not of affine type (see Definition 2.1), namely for holomorphic projective connections and for holomorphic conformal structures. This is because on manifolds with trivial canonical line bundle, these two geometric structures lift to global representatives which are of affine type, namely, a holomorphic affine connection and a holomorphic Riemannian metric respectively. The particular case of holomorphic Riemannian metrics was settled in [BD3]. Statement (2) of Theorem 1.3 generalizes this result to any self-dual holomorphic tangent bundle (i.e., TM is isomorphic to its dual T^*M).

The statement (1) in Theorem 1.3 is proved in Section 3.1 and the statement (2) is proved in Section 4.1. Theorem 1.2 together with the weak Bochner principle for Vaisman Calabi-Yau manifolds (Theorem 5.1) are proved in Section 5.1. Theorem 6.1 in Section 6 constructs simply connected elliptic principal bundles over a $K3$-surface that admit non-closed holomorphic one-forms.

2. Holomorphic rigid geometric structures

This section introduce the notion of a holomorphic geometric structure. We follow the presentation of rigid geometric structures given in [Gr] (see also [Ben, DG]).

Let M be a connected complex manifold of complex dimension n. Consider, for any positive integer $k \geq 1$, the associated principal bundle of k-th order frames $R^k(M) \longrightarrow M$: it is the holomorphic principal bundle formed by the k-jets of local holomorphic coordinates on the manifold M. The structure group of $R^k(M)$ is the group $D^k(\mathbb{C}^n)$ of k-jets of local biholomorphisms of $\mathbb{C}^n$ fixing the origin. We denote it simply by D^k and we recall that D^k is actually a complex affine algebraic group.

Definition 2.1. A *holomorphic geometric structure* of order k on the complex manifold M is a holomorphic D^k-equivariant map $\phi : R^k(M) \longrightarrow Z$, with Z being a complex algebraic manifold endowed with an algebraic D^k-action. Moreover, the geometric structure ϕ is said to be of *affine type* if Z is a complex affine variety.

All holomorphic tensors on M (meaning holomorphic sections of a bundle $(T^*M)^{\otimes p} \otimes (TM)^{\otimes q}$, for some nonnegative integers p, q) are holomorphic geometric structures of affine type of order one. Indeed, any holomorphic tensor on M can be seen as a holomorphic $\mathrm{GL}(n, \mathbb{C})$-equivariant map from the first order frame bundle $R^1(M)$ to a linear complex algebraic representation of $\mathrm{GL}(n, \mathbb{C})$, namely $((\mathbb{C}^n)^*) \otimes p \otimes \mathbb{C}^{n \otimes q}$, where $\mathbb{C}^n$ is the standard representation of $\mathrm{GL}(n, \mathbb{C})$. Holomorphic affine connections are holomorphic geometric structures of affine type of order two [Gr, DG, Ben]. Holomorphic foliations, holomorphic projective connections and holomorphic conformal structures are holomorphic geometric structures of non-affine type.

There is the following natural notion of symmetry of a holomorphic geometric structure.

A (local) biholomorphism of M preserves a holomorphic geometric structure ϕ if its canonical lift to $R^k(M)$ fixes each fiber of the map ϕ. Such a (local) biholomorphism is called a *(local) automorphism* of ϕ. The group of biholomorphisms of M preserving the geometric structure ϕ is called the automorphism group of (M, ϕ) and will be denoted by $\mathrm{Aut}(M, \phi)$.

A (local) holomorphic vector field on M is called a *Killing vector field* with respect to ϕ if its local flow acts on M by local biholomorphisms preserving ϕ (i.e., acts by local automorphisms of ϕ).

Definition 2.2. A holomorphic geometric structure ϕ is called *rigid* of order l in Gromov's sense if any local biholomorphism preserving ϕ is uniquely determined by its l-jet at any given point.

Holomorphic affine connections are rigid of order one in Gromov's sense (see [Gr, DG, Ben]). The rigidity comes from the fact that local biholomorphisms fixing a point and preserving a connection linearize in exponential coordinates, so they are indeed completely determined by their differential at the fixed point.

Holomorphic Riemannian metrics, holomorphic projective connections and holomorphic conformal structures in dimension ≥ 3 are all rigid holomorphic geometric structures, while holomorphic symplectic structures and holomorphic foliations are non-rigid geometric structures [DG].

The sheaf of local Killing vector fields of a holomorphic rigid geometric structure ϕ is locally constant. Its fiber is a finite dimensional Lie algebra which is called *the Killing algebra* of ϕ [DG, Gr].

An extendibility result, proved first by Nomizu for analytic Riemannian metrics, [No], and generalized in [Am, Gr, DG] for analytic rigid geometric structures, asserts that every point m in M has an open neighborhood V_m in M such that for any connected open subset $W \subset V_m$, any local Killing vector field defined over W uniquely extends to a local Killing vector field defined on V_m. Therefore, by monodromy principle, all local Killing vector fields extend along any path in the manifold, and the extension depends only on the homotopy class of the path. In particular, for simply connected manifolds, local Killing vector fields extend to the entire manifold as global (uni-valued) Killing vector fields.

Assume now that M is compact and simply connected, The global holomorphic vector fields are complete and they define an action of a complex Lie group. We then get that the local holomorphic Killing vector fields for holomorphic rigid geometric structures globalize and define an action of a connected complex Lie subgroup lying in $\text{Aut}(M,\, \phi)$ and acting by automorphisms on M while preserving the rigid geometric structure. The corresponding Lie algebra is the vector space of globally defined holomorphic Killing vector fields on M.

In this context the following result was proved in [BCDG, Theorem 5.1].

Theorem 2.3 ([BCDG])**.** *Let M be a compact simply connected complex manifold with trivial canonical line bundle K_M. Assume that M bears a holomorphic rigid geometric structure ϕ of affine type. Then the following three statements hold:*

(i) *There exists a holomorphic submersion $\pi\, :\, M \longrightarrow B$ to a simply connected Moishezon manifold B with globally generated canonical line bundle K_B such that the fibers of π are complex tori.*

(ii) *The above fibration π is not isotrivial. Equivalently, K_B is not trivial.*

(iii) *There exists a maximal connected abelian subgroup A of the automorphism group* Aut(M, ϕ) *whose orbits coincide with the fibers of π. Moreover, A is noncompact and its (real) maximal compact subgroup K acts freely and transitively on the fibers of π (hence M is a C^∞ principal K–bundle over B).*

We will use this result to prove Theorem 1.3. Statement (1) in Theorem 1.3 is proved in Section 3 and statement (2) is proved in in Section 4.

3. Semistable holomorphic tangent bundle

Let us first recall the notion of stability in the framework of compact complex (non necessarily Kähler) manifolds.

Consider a compact complex connected manifold M of complex dimension n. A Hermitian metric on the holomorphic tangent bundle TM is called a Gauduchon metric if the corresponding real (1, 1)–form ω on M satisfies the equation

$$\partial\overline{\partial}\omega^{n-1} = 0.$$

Gauduchon proved in [Ga] that any compact complex manifold admits a Gauduchon metric [Ga]. In the sequel we fix such a Gauduchon metric on M, and denote by ω_0 the (1, 1)–form on M associated to the Gauduchon metric.

Let $\Omega^{1,1}_{cl}(M, \mathbb{R})$ denote the space of all globally defined d–closed real (1, 1)–forms on M. The Bott–Chern cohomology of M is defined as

$$H^{1,1}_{BC}(M, \mathbb{R}) := \frac{\Omega^{1,1}_{cl}(M, \mathbb{R})}{\{\sqrt{-1}\partial\overline{\partial}\alpha \mid \alpha \in C^\infty(M, \mathbb{R})\}}.$$

The integration mapping

$$\Omega^{1,1}_{cl}(M, \mathbb{R}) \longrightarrow \mathbb{R}, \quad \alpha \longmapsto \int_M \alpha \wedge \omega_0^{n-1}$$

descends to a linear form on the Bott-Chern cohomology $H^{1,1}_{BC}(M, \mathbb{R})$.

For any holomorphic line bundle L on M, choose a Hermitian metric h_L on L, and consider the associated element $c(L) \in H^{1,1}_{BC}(M, \mathbb{R})$ given by the curvature of the Chern connection on L associated to h_L. This cohomology class $c(L)$ does not depend on the choice of the Hermitian metric h_L. Clearly, $c(L)$ vanishes for the trivial holomorphic bundle.

For any torsionfree coherent analytic sheaf F on the manifold M, define

$$\text{degree}(F) := \int_M c(\det F) \wedge \omega_0^{n-1} \in \mathbb{R},$$

where $\det F$ is the determinant line bundle for F [Ko, p. 166, Proposition 6.10]. The real number degree(F)/rank(F) is called the *slope* of F and it is denoted by $\mu(F)$.

A torsionfree coherent analytic sheaf V on M is called *stable* if

$$\mu(F) < \mu(V)$$

for all coherent analytic subsheaf $F \subset V$ with $0 < \text{rank}(F) < \text{rank}(V)$. A torsionfree coherent analytic sheaf V on M is called *semistable* if

$$\mu(F) \leq \mu(V)$$

for all coherent analytic subsheaf $F \subset V$ with $0 < \text{rank}(F) < \text{rank}(V)$.

If V is a direct sum of stable sheaves of the same slope, then it is called *polystable*.

The reader is referred to [LT] for more details and results about the stability.

Let us now give the proof of statement (i) in Theorem 1.3.

3.1. **Proof of statement (i) in Theorem 1.3.** Let M be a simply connected compact complex manifold with trivial canonical line bundle and Ψ a rigid holomorphic geometric structure of affine type on M.

Let

$$\pi : M \longrightarrow B \tag{3.1}$$

be the holomorphic submersion given by Theorem 2.3, and denote by A the connected abelian Lie subgroup of $\text{Aut}(M, \Psi)$ in Theorem 2.3(iii) whose orbits coincide with the fibers of π.

Fix a Gauduchon metric ω on M. Assume that the holomorphic tangent bundle TM of M is semistable with respect to ω.

Denote by $\mathfrak{a}$ the Lie algebra of A. The trivial holomorphic vector bundle on M

$$M \times \mathfrak{a} \longrightarrow M$$

with fiber $\mathfrak{a}$ will be denoted by $\mathcal{A}$. The differential of the action of A on M produces an $\mathcal{O}_M$–linear holomorphic homomorphism

$$\Phi : \mathcal{A} \longrightarrow TM. \tag{3.2}$$

Notice that $\Phi(\mathcal{A})$ coincides with T_π, with $T_\pi \subset TM$ is the vertical tangent bundle for the projection π in (3.1).

Consider the kernel

$$\mathcal{K} := \text{kernel}(\Phi) \subset \mathcal{A} \tag{3.3}$$

of the homomorphism Φ in (3.2). We $\text{degree}(TM) = 0$ because the holomorphic line bundle $\det(TM) = \bigwedge^{\text{top}} TM$ is trivial (recall the assumption that the canonical line bundle of M is trivial). Since the holomorphic tangent bundle TM is semistable of degree zero, and $\Phi(\mathcal{A}) \subset TM$, it follows that

$$\text{degree}(\Phi(\mathcal{A})) \leq 0. \tag{3.4}$$

On the on other hand, $\Phi(\mathcal{A})$ is a quotient of the holomorphically trivial vector bundle $\mathcal{A}$, which implies that

$$\text{degree}(\Phi(\mathcal{A})) \geq 0, \tag{3.5}$$

because the trivial vector bundle is semistable. Combining (3.4) and (3.5) we conclude that $\text{degree}(\Phi(\mathcal{A})) = 0$. Since

$$\text{degree}(\mathcal{A}) = 0 = \text{degree}(\Phi(\mathcal{A})),$$

it now follows that

$$\text{degree}(\mathcal{K}) = 0 \tag{3.6}$$

(see (3.3)).

Note that $\Phi(\mathcal{A})$ is torsionfree because it is a subsheaf of the torsionfree sheaf TM. This and the fact that $\mathcal{A}$ is locally free together imply that $\mathcal{K}$ is reflexive [Ko, Ch. V, p. 153, Proposition 4.13]. Since the trivial vector bundle $\mathcal{A}$ is polystable of degree zero, and $\mathcal{K}$ is a reflexive subsheaf of it of degree zero (see (3.6)), it follows that $\mathcal{K}$ is a polystable subbundle of $\mathcal{A}$ of degree zero. All stable subbundles of $\mathcal{A}$ of degree zero are of the form $M \times \ell$, where $\ell \subset \mathfrak{a}$ is a line. Consequently, there is a linear subspace $V_0 \subset \mathfrak{a}$ such that $\dim V_0 = \text{rank}(\mathcal{K})$ and

$$\mathcal{K} = M \times V_0 \subset M \times \mathfrak{a} = \mathcal{A}.$$

Take any $v \in V_0 \subset \mathfrak{a}$. Note that the vector field $\Phi(v) \in H^0(M, TM)$ (see (3.2)) vanishes identically because $\Phi(v) \in H^0(M, \mathcal{K})$. This implies that $v = 0$. Therefore, we conclude that Φ defines an isomorphism

$$\mathcal{A} \simeq T_\pi \subset TM, \tag{3.7}$$

where T_π as before is the vertical tangent bundle for the projection π in (3.1). From (3.7) it follows that T_π is a trivial bundle, and consequently,

$$\det T_\pi = \mathcal{O}_M.$$

Hence we have

$$\det(\pi^* TB) = \pi^* \det(TB) = \det(TM) \otimes (\det T_\pi)^* = \mathcal{O}_M. \tag{3.8}$$

Since the fibers of π are connected, from (3.8) it follows that $\det(TB) = \mathcal{O}_B$.

This is in contradiction with the statement (ii) in Theorem 2.3 and completes the proof in the case where M is simply connected.

If the fundamental group of M is finite, then there exists a finite unramified covering of M which is simply connected and endowed with the pull-back of Φ which is a rigid holomorphic geometric structure of affine type. This leads to a contradiction as seen above.

4. Self-dual holomorphic tangent bundle

Now assume that

$$\dim H^0(M, T^*M) = \dim H^0(M, TM). \tag{4.1}$$

Notice that an interesting particular case where the condition (4.1) is satisfied is when the holomorphic tangent bundle TM is self-dual.

Let us now give the proof of point (ii) in Theorem 1.3.

4.1. Proof of statement (ii) in Theorem 1.3.

As before, assume that M is simply connected with trivial canonical bundle and Ψ a rigid holomorphic geometric structure of affine type on it. Consider the holomorphic submersion in (3.1). We use the same notation as in the previous Section.

We have seen that M bears global Killing holomorphic vector fields: they are the fundamental vector fields for the holomorphic action of the abelian complex Lie subgroup $A \subset \mathrm{Aut}(M, \phi)$ acting transitively on the fibers of π. Therefore,

$$\dim H^0(M, T^*M) = \dim H^0(M, TM) \geq \dim A \geq \dim M - \dim B > 0.$$

Take any nonzero holomorphic 1-form

$$0 \neq \theta \in H^0(M, \Omega^1_M) = H^0(M, T^*M). \tag{4.2}$$

We will show that there is some $b \in B$ such that the section of $\Omega^1_{\pi^{-1}(b)}$ obtained by restricting θ (see (4.2)) to the compact complex torus $\pi^{-1}(b)$ is nonzero. To prove this by contradiction, assume that the section of $\Omega^1_{\pi^{-1}(x)}$ obtained by restricting θ to the compact complex torus $\pi^{-1}(x)$ is zero for all $x \in B$. Then there is a section $\theta' \in H^0(B, \Omega^1_B)$ such that

$$\theta = \pi^*\theta'.$$

Since B is Moishezon, holomorphic forms on B are closed. The manifold B being also simply connected, this implies that $H^0(B,\, \Omega^1_B) \,=\, 0$. Consequently, θ' and (hence also) θ vanish identically. This contradicts (4.2), and hence there is some $b \,\in\, B$ such that the section of $\Omega^1_{\pi^{-1}(b)}$ obtained by restricting θ to $\pi^{-1}(b)$ is nonzero.

A stronger version of the above statement will be proved. We will now show that the section of $\Omega^1_{\pi^{-1}(z)}$ obtained by restricting θ is nonzero for *every* $b \,\in\, B$.

To prove this by contradiction, assume that the section of $\Omega^1_{\pi^{-1}(z)}$ obtained by restricting θ is identically zero for some $b \,\in\, B$. Take any $v \,\in\, \mathfrak{a}$, and consider the holomorphic function $\theta(v)$ on M. Since the section of $\Omega^1_{\pi^{-1}(z)}$ obtained by restricting θ is identically zero, and the vector field v is vertical for the projection π, we know that the function $\theta(v)$ vanishes on $\pi^{-1}(z)$. This implies that the (constant) function $\theta(v)$ vanishes identically on M.

On the other hand, the image of the homomorphism Φ (see (3.2)) is the entire T_π (see (3.7)). Therefore, that above observation that the function $\theta(v)$ vanishes identically on N for all $v \,\in\, \mathfrak{a}$ implies that the restriction of θ to the entire T_π vanishes identically. But this contradicts the earlier observation that there is some $b \,\in\, B$ such that the section of $\Omega^1_{\pi^{-1}(b)}$ obtained by restricting θ is nonzero. Hence we conclude that the section of $\Omega^1_{\pi^{-1}(z)}$ obtained by restricting θ is nonzero for every $z \,\in\, B$.

Since the section of $\Omega^1_{\pi^{-1}(z)}$ obtained by restricting θ is nonzero for every $z \,\in\, B$, we have

$$\dim H^0(M,\, \Omega^1_M) \,\leq\, \dim H^0(\pi^{-1}(z),\, \Omega^1_{\pi^{-1}(z)}) \,=\, \dim M - \dim B. \tag{4.3}$$

On the other hand,

$$\dim H^0(M,\, \Omega^1_M) \,=\, \dim H^0(M,\, TM) \,\geq\, \dim M - \dim B \tag{4.4}$$

(see (4.1)). Combining (4.3), (4.4) and (4.1) we have the following:

$$\dim H^0(M,\, \Omega^1_M) \,=\, \dim H^0(M,\, TM) \,=\, \dim M - \dim B.$$

Hence the homomorphism Φ in (3.2) is an isomorphism between $\mathcal{A}$ and T_π. In particular, the holomorphic vector bundle T_π is trivial.

We now have (as in (3.8))

$$\det(\pi^*TB) \,=\, \pi^*\det(TB) \,=\, \det(TM)\otimes(\det T_\pi)^* \,=\, \mathcal{O}_M.$$

Since the fibers of π are connected, this implies that $\det(TB) \,=\, \mathcal{O}_B$. This is in contradiction with statement (ii) in Theorem 2.3, and hence the proof is complete.

Remark 4.1. Notice that the above proof is simplified if TM is moreover assumed to be self-dual. Indeed, we have seen that the homomorphism Φ in (3.2) is not an isomorphism between $\mathcal{A}$ and T_π. This implies that there exists a holomorphic vector field on M defined by the holomorphic action of A which vanishes at some point in M. Since T^*M is self-dual, this constructs a nonzero holomorphic one-form θ on M which vanishes at some point in M.

Now take any $v \,\in\, \mathfrak{a}$, and consider the holomorphic function $\theta(v)$ on M. Since θ vanishes at some point, the function $\theta(v)$ is identically zero. This implies that θ vanishes on the fibers of π, and consequently θ coincides with the pull-back — through π — of a nonzero holomorphic one-form θ' defined on B. Since B is simply connected and Moishezon, we have $H^0(B, \Omega^1_B) = 0$. This contradicts that $\theta' \neq 0$.

5. Vaisman Calabi-Yau manifolds

Vaisman manifolds were introduced in [Va1]. This is a class of Hermitian non-Kähler manifolds which do not satisfy the $\partial\overline{\partial}$-lemma, but still possess a cohomological behaviour close to that of Kähler manifolds.

Recall that a Hermitian manifold M is called Vaisman if its fundamental $(1, 1)$–form Ω is not close, but there exists a closed one-form θ, called the *Lee form*, satisfying the conditions that $d\Omega = \Omega \wedge \theta$ and θ is parallel with respect to the Levi-Civita connection. The vector field dual to the Lee form — known as the *Lee vector field* — is real holomorphic and Killing [Va2]: it generates a complex one-parameter subgroup in the holomorphic automorphism group of the complex manifold M which does not depend on the chosen Vaisman metric (see [Ts]). This one-parameter complex group generated by the Lee vector field — called *the Lee group* of the manifold — acts locally freely on M: its action defines a holomorphic foliation — known as the *canonical foliation* — which is actually transversally Kähler [Va2]. If the Lee group is compact, then the Vaisman manifold is called *quasi-regular* and in this case the quotient of M by the action of the Lee group is in fact a complex orbifold. If, moreover, the action of the compact Lee group is free, then M is called a *regular* Vaisman manifold, and the quotient of M by the Lee group is smooth: therefore, in the regular case, M has the structure of a holomorphic principal elliptic bundle over a complex smooth base. For more informations about Vaisman manifolds and more generally about locally conformally Kähler manifolds we refer the reader to the book [OV].

Following Tosatti, [To], compact complex manifolds with vanishing first Chern class in Bott-Chern cohomology are called Calabi-Yau.

Vaisman Calabi-Yau manifolds have been studied by Istrati in [Is]. By Theorem [Is, Theorem C], the connected component of the automorphism group of any Vaisman Calabi-Yau is compact one dimensional and it coincides with the Lee group. Therefore, any Vaisman Calabi-Yau manifold is quasi-regular. Moreover, a Beauville-Bogomolov type decomposition Theorem for Vaisman Calabi-Yau manifolds is proved in [Is, Theorem D]. More precisely, [Is, Theorem D] shows that any Vaisman Calabi-Yau manifold M admits a finite unramified cover which is a principal elliptic orbi-bundle over an orbifold Y which is the product of an abelian variety and a simply connected projective orbifold with trivial canonical class. This implies, in particular, that the canonical bundle of a Vaisman Calabi-Yau manifold is of finite order.

Now we make use of the above results of Istrati [Is] to prove Theorem 1.2.

5.1. **Proof of Theorem 1.2.** Assume that the Vaisman Calabi-Yau manifold M bears a holomorphic geometric structure of affine type Ψ. We pull-back the holomorphic geometric structure on the finite unramified cover given by the decomposition theorem [Is, Theorem D]. Therefore, we assume — without any loss of generality — that M has a holomorphically trivial canonical line bundle and M has the structure of a principal elliptic orbi-bundle

$$\pi : M \longrightarrow Y \tag{5.1}$$

over the projective orbifold Y with trivial canonical class.

Denote by T the elliptic curve which is the structure group of the above elliptic orbi-bundle, and also denote by X the fundamental holomorphic vector field on M defined by the holomorphic T-action on M. The holomorphic vector field X does not vanish (since the T-action is locally free) and it spans the kernel of $d\pi$ (see (5.1)), defining the vertical subbundle $V \subset TM$ for the projection π.

To prove by contradiction, assume that the holomorphic geometric structure of affine type Ψ is not locally homogeneous on M. Then [Du, Lemma 3.2] proves that there exist two integers $a,\, b \geq 0$ (at least one of them being positive) and a nontrivial holomorphic section of the holomorphic vector bundle $(TM)^{\otimes a} \otimes (T^*M)^{\otimes b}$ which vanishes at some point in M.

Observe that all holomorphic sections of $(TM)^{\otimes a} \otimes (T^*M)^{\otimes b}$ are automatically T-invariant. Indeed, to any holomorphic section ψ of this vector bundle and any $t \in T$ one associates the section $t^*\psi$, given by the linear T-action on the vector space $\mathrm{H}^0(M,\, (TM)^{\otimes a} \otimes (T^*M)^{\otimes b})$ canonically induced by the T-action on M. Therefore, we get a complex Lie group homomorphism $T \longrightarrow \mathrm{GL}(\mathrm{H}^0(M,\, (TM)^{\otimes a} \otimes (T^*M)^{\otimes b})$. But any such homomorphism from a compact complex torus to a complex affine group must be the trivial homomorphism. This implies that $t^*\psi = \psi$, for all $t \in T$ and all $\psi \in \mathrm{H}^0(M,\, (TM)^{\otimes a} \otimes (T^*M)^{\otimes b})$. Consequently, every holomorphic tensor on M must be T-invariant.

The canonical line bundle K_M being trivial, there is a canonical contraction isomorphism between TM and $\bigwedge^{n-1}(T^*M)$. Through this isomorphism, the nontrivial holomorphic section of the holomorphic vector bundle $(TM)^{\otimes a} \otimes (T^*M)^{\otimes b}$ — constructed above — provides a nontrivial holomorphic section ϕ of $(T^*M)^{\otimes m}$, with $m = (n-1)a + b \geq 1$, such that ϕ vanishes at some point $m_0 \in M$. Recall that this section ϕ was proved to be T-invariant.

We will get a contradiction using an induction on the above integer $m \geq 1$.

First consider the case where $m = 1$. In this case ϕ is a nontrivial holomorphic one-form on M vanishing at $m_0 \in M$. Note that $\phi(X)$, where X as before is the fundamental vector field on M defined by the holomorphic T-action on M, is a holomorphic function on M vanishing at m_0, and therefore it vanishes identically on M. This proves that the vertical space V (for the projection π in (5.1)) is in the kernel of ϕ. The fibers of π being compact and connected we can project ϕ on Y using $d\pi$. This constructs a nontrivial holomorphic one-form ϕ' on Y which vanishes at $\pi(m_0) \in Y$ and satisfies the condition that $\pi^*\phi' = \phi$. Notice that at the points in M having a nontrivial (finite) stabilizer $\mathrm{Stab}(m)$ in T, the holomorphic tensor ϕ being T-invariant is, in particular, $\mathrm{Stab}(m)$–invariant. Consequently, ϕ descends as a holomorphic tensor ϕ' on Y which is well-defined on the orbifold Y.

Since Y is a projective Calabi-Yau orbifold, it admits a Ricci flat orbifold Kähler metric; see [Ca, Théorème 4.1] for an extension of Yau's Theorem, [Ya], to the orbifold case. Holomorphic tensors on Y are parallel with respect to any Ricci flat orbifold Kähler metric on it (see [GGK, Theorem 8.2] and [CGGN, Theorem A]). Hence the tensor ϕ' cannot vanish at the point $\pi(m_0)$ without being trivial. For a detailed explanation of this, notice that in the case where $\mathrm{Stab}(m_0)$ in T is a nontrivial (finite) subgroup in T, a neighborhood of $\pi(m_0)$ in Y is isomorphic to a quotient of a ball $U \subset \mathbb{C}^n$ centered in the origin by the action of the finite group $\mathrm{Stab}(m_0)$ fixing the origin. The holomorphic tensor ϕ pulls back as a $\mathrm{Stab}(m_0)$–invariant holomorphic tensor on U which is parallel with respect to the pull-back, on U, of the Kähler Ricci flat metric (which is $\mathrm{Stab}(m_0)$–invariant as well). Therefore, ϕ cannot vanish at m_0 without being identically zero.

So we get a contradiction when $m = 1$.

Now consider the case where $m > 1$. The induction hypothesis is that a nontrivial holomorphic section of $(T^*M)^{\otimes k}$ does not vanish if $0 < k < m$.

It will be shown that for all $0 < r \leq m$, any contraction of ϕ with r-copies of the vector field X vanishes identically on M.

First consider the case of $r = m$. When contracted with m-copies of X, the tensor ϕ produces the holomorphic function $\phi(X, X, \cdots, X)$ on M which vanishes at m_0. Therefore the function $\phi(X, X, \cdots, X)$ vanishes identically on M.

Now assume that $0 < r < m$. When contracted with any r-copies of X, the tensor ϕ produces a holomorphic section of $(T^*M)^{\otimes m-r}$ vanishing at the point m_0. Since $0 < m-r < m$, the induction hypothesis holds for $m - r$. Consequently, this section of $(T^*M)^{\otimes m-r}$ vanishes identically on M.

Since all the above contractions vanish identically, it follows that our tensor ϕ is a holomorphic section of $((TM/V)^*)^{\otimes m}$. Moreover, we proved the T-invariance of ϕ.

The fibers of π being compact and connected, we can project the previous tensor on Y using the differential $d\pi$. This produces a nontrivial holomorphic section of $((TY)^*)^{\otimes m}$ which vanishes at the point $\pi(m_0) \in Y$. But, as before, this holomorphic tensor should be parallel with respect to any Kähler Ricci flat metric on the projective orbifold Y (see [GGK, Theorem 8.2] and [CGGN, Theorem A]): a contradiction. This completes the proof of the local homogeneity property of the holomorphic geometric structure Ψ of affine type.

Let us prove that the geometric structure Ψ is T-invariant. The proof is similar to the one already given for tensors.

Assume that Ψ is defined as a holomorphic D^k-equivariant map $\phi : R^k(M) \longrightarrow Z$, with Z being a complex affine variety endowed with an algebraic D^k-action (see Definition 2.1).

A classical result of algebraic group representations furnishes a D^k–equivariant algebraic embedding of Z into a complex vector space W endowed with a D^k–algebraic linear action (see, for instance, [PV]). This identifies Ψ with a holomorphic section of the holomorphic vector bundle W_M over M with fiber type W associated to the principal bundle $R^k(M)$ via the D^k–algebraic linear action on W. As in the above proof for holomorphic tensors (corresponding to the case $k = 1$), observe that any complex Lie group homomorphism $T \longrightarrow \mathrm{GL}(H^0(M, W_M))$ is trivial, and therefore $t^*(\Psi) = \Psi$, for any $t \in T$ (here the T–action on $H^0(M, W_M)$ is inherited by the canonical lift of the T–action on the k–frame bundle $R^k(M)$). Consequently, T acts via automorphisms for the holomorphic geometric structure Ψ. In other words, Ψ is T–invariant. This completes the proof of statement (1) in Theorem 1.2.

In order to prove the statement (2) in Theorem 1.2, consider the sheaf of local Killing vector fields for Ψ. Since Ψ is locally homogeneous this sheaf is transitive on M. We proved that the fundamental vector field X is a Killing vector field and therefore it is a global section of the sheaf of local Killing vector fields.

Since X is a global holomorphic vector field defined on M, the holomorphic geometric structure $(\Psi,\, X)$ obtained by the juxtaposition of Ψ and X is also a holomorphic geometric structure on M [Gr, DG, Ben]. Moreover, if Ψ is rigid, then $(\Psi,\, X)$ is also rigid.

The holomorphic geometric structure $(\Psi,\, X)$ is locally homogeneous by statement (i) in Theorem 1.2 applied to (Ψ, X). This means that the sheaf of local Killing vector fields for $(\Psi,\, X)$ is transitive on M and X is a global section of it. Observe that this sheaf is the centralizer of X in the sheaf of local Killing vector fields for Ψ (in other words, the sheaf of local Killing vector fields for Ψ that commute with X). Denote this sheaf of vector fields by Kill_X and consider its direct image ν through the fibration $\pi\,:\, M \,\longrightarrow\, Y$. Since the local sections of Kill_X commute with X, the sheaf Kill_X is actually T-invariant. Therefore, the above direct image ν is a well-defined sheaf of local holomorphic vector fields on the orbifold Y.

Let us now prove — under the assumption that Ψ is rigid — that the local sections of ν uniquely extend to global sections of ν over all simply connected open subsets of Y. For this purpose, we prove the following extension property for the sheaf ν: Every point y in Y has an open neighborhood V_y in Y such that, for any connected open subset $W\,\subset\, V_y$, any local section of ν over W uniquely extends to a local section of ν over V_y.

First consider a regular point $y\,\in\, Y$. Chose a simply connected open subset $V_y\,\subset\, Y$ such that $y\,\in\, V_y$ and $\pi^{-1}(V_y)\,=\, V_y\times T$. The fundamental group of $\pi^{-1}(V_y)$ is generated by the fundamental group of the fiber T. More precisely, if the elliptic curve T is uniformized as a quotient of $\mathbb{C}$ by a normal lattice Γ in $\mathbb{C}$, then the universal cover of $\pi^{-1}(V_y)$ is biholomorphic to $V_y\times\mathbb{C}$ and $\pi^{-1}(V_y)\,=\,V_y\times T$ is the quotient of $V_y\times\mathbb{C}$ by the fundamental group identified with Γ acting trivially on the first factor V_y and by translations on the second factor $\mathbb{C}$. By the extendibility result of local Killing vector fields for the holomorphic rigid geometric structure $(\Psi,\, X)$ (see [Am, Gr, DG, No]), the pull-back of the sheaf Kill_X on the universal cover $V_y\times\mathbb{C}$ of $\pi^{-1}(V_y)$ is generated by its global sections, i.e., any local section of the sheaf actually uniquely extends to a global holomorphic vector field on $V_y\times\mathbb{C}$ whose flow preserves the pull-back of $(\Psi,\, X)$. Moreover, these global sections commute with the action of $\mathbb{C}$ by translations on the fibers (generated by the pull-back of X). In particular, the global sections of the pull-back of Kill_X on $V_y\times\mathbb{C}$ are Γ–invariant; consequently, they descend as global sections of Kill_X over $\pi^{-1}(V_y)\,=\,V_y\times T$. In other words, the local sections of the sheaf Kill_X restricted to $\pi^{-1}(V_y)$ uniquely extend as holomorphic Killing vector fields on entire $\pi^{-1}(V_y)$. Consequently, the restriction of the above defined sheaf ν to V_y (recall that ν is the direct image of Kill_X under the projection π) shares the property that all its local sections uniquely extend to global sections (defined on entire V_y).

Next consider a point $y \in Y$ such that the points in the fiber $\pi^{-1}(y)$ have a nontrivial (finite) stabilizer Stab(y) $\subset T$ (since T is abelian, points of $\pi^{-1}(y)$ have a common stabilizer). In this case y admits a simply connected open neighborhood V_y in Y such that $\pi^{-1}(V_y) = (V_y \times T)/\text{Stab(y)}$, with the cyclic group Stab(y) acting by rational rotations on V_y (identified with a ball) and it acts by translations on the fiber T. As before, the sheaf obtained by the pull-back of Kill_X (restricted to $\pi^{-1}(V_y)$) to $V_y \times T$ is generated by its global sections. Since those global sections are T–invariant (they commute with the pull-back of X), they are in particular Stab(y)–invariant. Therefore, the sheaf Kill_X restricted to $\pi^{-1}(V_y) = (V_y \times T)/\text{Stab(y)}$ is generated by its global sections. We conclude as before that its direct image ν is generated by its global sections on V_y.

This implies that the sheaf ν on Y has the extendibility property: Its sections extend by analytic continuation along any path in Y. Moreover, by the monodromy principle, this extension only depends on the homotopy type (with fixed endpoints) of the chosen path. In particular, any local section of ν defined on an open subset V_y in Y extends as a global holomorphic vector field on any connected simply connected open subset containing V_y.

This completes the proof of Theorem 1.2.

Notice that in the proof of the proof of the statement (1) of Theorem 1.2 the following weak version of Bochner principle is proved:

Theorem 5.1. *Let M be a compact Vaisman Calabi-Yau manifold. Then any nontrivial holomorphic tensor on M has empty vanishing locus.*

Another by-product of the proof of the statement (1) of Theorem 1.2 is the following.

Theorem 5.2. *Let M be a compact complex manifold, with trivial canonical line bundle, which is a principal torus orbi-bundle over a Kähler orbifold with trivial canonical class. Then any holomorphic geometric structure of affine type on M is locally homogeneous and invariant by the principal fibration.*

Indeed, the proof of Theorem 1.2(1) directly generalizes from the case of a holomorphic principal T–orbi-bundle, with T an elliptic curve, to the case where T is a compact complex torus. The above statement extends to orbi-bundles a result obtained earlier in [BD2, Theorem 1.2].

Remark 5.3. Diagonal Hopf manifolds M_D are defined as quotients of $\mathbb{C}^n \setminus \{0\}$ by a diagonal linear contraction matrix D. Hopf manifolds possess a natural complex affine structure (given by a torsionfree flat affine connection ∇_0) which is constructed as the descend of the standard affine structure of the universal cover $\mathbb{C}^n \setminus \{0\}$. Moreover the commuting holomorphic vector fields $z_i \dfrac{\partial}{\partial z_i}$, $i \in \{1, \cdots, n\}$, descend from $\mathbb{C}^n \setminus \{0\}$ to the quotient M_D as commuting holomorphic vector fields X_i on M_D which are linearly independent outside the divisor S given by the projection of the coordinate axis. The holomorphic geometric structure Ψ given by the juxtaposition (∇_0, X_i) of the complex affine structure and the family of commuting holomorphic vector fields X_i, $i \in \{1, \cdots, n\}$, is a rigid holomorphic geometric structure of affine type on M [DG, Gr] which is locally homogeneous on $M \setminus S$.

Indeed, the complex abelian Lie group A generated by the holomorphic vector fields X_i acts transitively on $M\setminus S$ preserving Ψ: A coincides with the connected component of the identity element in the automorphism group $\mathrm{Aut}(M,\, \Psi)$. Moreover, Ψ is not locally homogeneous on entire M since the holomorphic section $X_1 \wedge \ldots \wedge X_n$ of the canonical line bundle $\bigwedge^n T^*M_D$ (which is invariant by all local automorphisms of Ψ) vanishes on S. Therefore, M_D admits non-locally homogeneous rigid holomorphic geometric structures of affine type and admit nontrivial holomorphic tensors with non-empty vanishing locus. On the other hand diagonal Hopf manifolds are known to admit Vaisman metrics [Bel]. Therefore, Theorem 1.2 and Theorem 5.1 do not hold for all Vaisman manifolds.

6. Principal torus bundles over Kähler Calabi-Yau manifolds

Recall that compact complex simply connected threefolds with trivial canonical bundle do not admit *rigid* holomorphic geometric structures [BCDG, Theorem D]. The following result exhibits (non-closed) holomorphic one-forms on some compact simply connected threefolds with trivial canonical bundle which are holomorphic elliptic bundles over $K3$ surfaces. It is inspired by a construction of Brunella in [Br, page 648].

Theorem 6.1. *There exists a simply connected principal elliptic bundle M over a projective $K3$ surface B admitting a (non closed) nonsingular holomorphic one-form ω.*

Proof. Let us first construct a projective $K3$-surface B bearing a holomorphic two-form Ω such that the group of periods of Ω is a lattice Λ in $\mathbb{C}$.

One way to obtain such a pair $(B,\, \Omega)$ is to consider a Kummer surface. Indeed, let Γ be the standard lattice in $\mathbb{C}^2$ generated by $(1,\, 0)$, $(\sqrt{-1},\, 0)$, $(0,\, 1)$, $(0,\, \sqrt{-1})$. Consider the quotient of the abelian variety $Y = \mathbb{C}^2/\Gamma$ by the holomorphic involution $i\, :\, \mathbb{C}^2/\Gamma \longrightarrow \mathbb{C}^2/\Gamma$ defined by $(x,\, y) \longmapsto (-x,\, -y)$.

The above involution i admits 16 fixed points which are the order 2 points in Y. Let B be the blow-up of Y/i at these 16 points. The smooth complex projective surface B is a K3 surface (in particular, it is simply connected). A holomorphic 2-form Ω on B is given by the pull-back of the translation volume form $dz_1 \wedge dz_2$ on Y, which is also invariant under the involution i.

The homology group $H_2(B,\, \mathbb{Z})$ is generated by the 16 copies of $\mathbb{C}P^1$ (the exceptional divisors for the above mentioned blow-up) and by the images in $H_2(B,\, \mathbb{Z})$ of the 6 standard generators of $H_2(Y,\, \mathbb{Z})$. The integral of Ω on each of the exceptional divisors is zero because the form Ω is pulled back from Y/i and it hence vanishes on the exceptional divisors. On the generators of $H_2(Y,\, \mathbb{Z})$ the integral of Ω is one of the numbers 0, $1/4$, $-1/4$, $\sqrt{-1}/4$, $-\sqrt{-1}/4$. Consequently, the group of periods of Ω form a lattice Λ in $\mathbb{C}$.

Now consider the above pair $(B,\, \Omega)$, with B being the projective K3 surface and Ω the (nonsingular) holomorphic two-form whose group of periods is a lattice Λ in $\mathbb{C}$. The corresponding elliptic curve $\mathbb{C}/\Lambda$ will be denoted by T.

Let us choose an open cover $\{U_i\}_{i\in I}$ of B, by open subsets in the analytic topology, such that all U_i and all connected components of $U_i \cap U_j$ are contractible. Then on each open subset U_i there exists a holomorphic one-form ω_i such that $\Omega_i = d\omega_i$. On nontrivial intersections $U_i \cap U_j$, we get that $\omega_i - \omega_j = dF_{ij}$, with F_{ij} a holomorphic function defined on $U_i \cap U_j$.

On intersections $U_i \cap U_j \cap U_k$, the sum $F_{ij} + F_{jk} + F_{ki}$ is a locally constant 2-cocycle that represents the cohomology class in $H^2(B, \mathbb{C})$ of the closed two-form Ω. Therefore, we can choose the local one-forms ω_i and the associated holomorphic functions F_{ij} such that — for every triple (i, j, k) — the value of the 2-cocycle $F_{ij} + F_{jk} + F_{ki}$ lies in the lattice $\Lambda \subset \mathbb{C}$ formed by the periods of Ω.

Consider the local holomorphic function $\{F_{ij}\}$ as a 1-cocycle taking values in the translation group of T and form the associated holomorphic principal elliptic bundle $M \longrightarrow B$ with typical fiber T associated to this one-cocycle. The local forms on $U_i \times T$ which are $\pi_1^*(\omega_i) + \pi_2^* dz$, where π_1 (respectively, π_2) is the projection on the first (respectively, second) factor and dz is a translation invariant holomorphic one-form on the elliptic curve T, glue compatibly to produce a globally defined holomorphic one-form ω on M. By construction, the differential $d\omega$ projects on B to Ω and does not vanish. Hence, the holomorphic one-form ω is not closed; more precisely, its differential $d\omega$ is the pull-back on M of the form Ω. Moreover, $\omega \wedge d\omega$ does not vanish at any point in M and provides a holomorphic section trivializing the canonical line bundle K_M.

Observe that the total space M of the above holomorphic principal bundle with fiber type T is not Kähler because the holomorphic one-form ω on M is not closed.

Moreover, the above manifold M is simply connected, which is proved in [Ho, Theorem 12.3 and Remark 12.4]. More precisely, employing the notation of [Ho] observe that, by construction, the topological bundle M is characterized by a characteristic class $c = a_1 \otimes \lambda_1 + a_2 \otimes \lambda_2 \in H^2(M, \mathbb{Z}) \otimes \Lambda$, given by $\lambda_1 = 1/4, \lambda_2 = \sqrt{-1}/4$ and the evaluation of a_1 equals 1 on the pull-back of the class of the elliptic curve E_1 in Y defined by the quotient of $\mathbb{R}(1,0) \oplus \mathbb{R}(1,0)$, equals -1 on the pull-back of the class of the elliptic curve E_2 in Y defined by the quotient of $\mathbb{R}(0, 1) \oplus \mathbb{R}(0, 1)$ and a_1 vanishes on the complement in $H^2(M, \mathbb{Z})$ of the direct summand generated by the pull-back of the classes of E_1 and E_2. The element a_2 is defined in a similar way using the pull-back of the classes of the elliptic curves generated by E_3 and E_4 which are the images in Y of $\mathbb{R}(1, 0) \oplus \mathbb{R}(0, 1)$ and $\mathbb{R}(0, 1) \oplus \mathbb{R}(1, 0)$, respectively. Notice that in our situation the obstruction Δ in [Ho, Proposition 6.6 and Remark 12.4] vanishes, by construction, and since a_1, a_2 form the basis of a direct summand in $H^2(M, \mathbb{Z})$ the manifold M is simply connected by [Ho, Theorem 12.3]. □

Remark 6.2. The manifold M in Theorem 6.1 is not Kähler because it bears a non-closed holomorphic one-form ω. Its algebraic dimension therefore is 2 (the projection of M on B coincides with the algebraic reduction of M). Notice that it comes from the proof that $\omega \wedge d\omega$ provides a holomorphic trivialization of the canonical line bundle of M. In particular, ω is not integrable (its kernel is not integrable and therefore it does not define a foliation on M). Recall that [Br, Proposition 1] proves that on compact complex manifolds M of algebraic dimension $\dim M - 1$, all integrable holomorphic one-forms are closed. Also recall that [BD2, Theorem 1.2] proves that on holomorphic principal torus bundles over compact

Kähler Calabi-Yau manifolds all holomorphic geometric structures of affine type are actually locally homogeneous. This implies that those, among these manifolds, that admit a rigid holomorphic geometric structure of affine type must have infinite fundamental group [BD2, Corollary 1.3]. The manifolds in Theorem 6.1 being simply connected do not admit any rigid holomorphic geometric structure of affine type. Moreover, the holomorphic one-form ω is locally homogeneous.

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References

[Am] A. M. Amores, Vector fields of a finite type G-structure, *Jour. Diff. Geom.* **14** (1980), 1–6.

[At1] M. F. Atiyah, On the Krull-Schmidt theorem with application to sheaves, *Bull. Soc. Math. Fr.* **84** (1956), 307–317.

[At2] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[Bea] A. Beauville, Variétés kähleriennes dont la première classe de Chern est nulle, *Jour. Diff. Geom.* **18** (1983), 755–782.

[Bel] F. Belgun, On the metric structure of non-Kähler complex surfaces, *Math. Ann.* **317** (2000), 1–40.

[Ben] Y. Benoist, Orbites des structures rigides (d'après M. Gromov), *Progr. Math.* **145** (1997), 1–17.

[BCDG] I. Biswas, J. Cao, S. Dumitrescu and H. Guenancia, Geometry of K-trivial Moishezon manifolds: decomposition theorem and holomorphic geometric structures, *Math. Annalen*, **391** (2025), 3181–3220.

[BD1] I. Biswas and S. Dumitrescu, Holomorphic Affine Connections on Non-Kähler manifolds, *Internat. J. Math.* **27** (2016), no. 11.

[BD2] I. Biswas and S. Dumitrescu, Fujiki class $\mathcal{C}$ and holomorphic geometric structures, *Internat. J. Math.* **31** (2020), no. 5.

[BD3] I. Biswas and S. Dumitrescu, Holomorphic Riemannian metric and fundamental group, *Bull. Soc. Math. Fr.* **147** (2019), 455–468.

[BG] I. Biswas and T. L. Gómez, Connections and Higgs fields on a principal bundle, *Ann. Global Anal. Geom.* **33** (2008), 19–46.

[Bog] F. A. Bogomolov, Kähler manifolds with trivial canonical class, *Izv. Akad. Nauk. SSSR* **38** (1974), 11–21; English translation in *Math. USSR Izv.* **8** (1974), 9–20.

[Boz] Y. Bozhkov, The geometry of certain three-folds, *Rend. Istitu. Math. Univ. Trieste* **26**, (1994), 79–93.

[Br] M. Brunella, On holomorphic forms on compact complex threefolds, *Comment. Math. Helv.* **74** (1999), 642–656.

[Ca] F. Campana, Orbifoldes à première de classe de Chern nulle, *The Fano conference*, University Torino, Torino, 339–351.

[CGGN] B. Claudon, P. Graf, H. Guenancia and P. Naumann, Kähler spaces with zero first Chern class: Bochner principle, Albanese map and fundamental groups, *J. Reine Angew. Math.*, **786**, (2022), 245–275.

[DG] G. D'Ambra and M. Gromov, *Lectures on transformations groups: geometry and dynamics*, Surveys in Differential Geometry, Cambridge MA, (1991).

[Du] S. Dumitrescu, Structures géométriques holomorphes sur les variétés complexes compactes, *Ann. Scient. Ec. Norm. Sup* **34**, (2001), 557–571.

[Du1] S. Dumitrescu, Connexions affines et projectives sur les surfaces complexes compactes, *Math. Zeit.* **264** (2010), 301–316.
[Fe] T. Fei, Remarks on generalized Calabi-Gray manifolds, arxiv:2307.13265
[FP] J. Fine and D. Panov, Hyperbolic geometry and non-Kähler manifolds, *Geometry and Topology*, **14** (2010), 1723–1763.
[Ga] P. Gauduchon, Le théorème de l'excentricité nulle, *Com. Ren. Acad. Sci. Paris* **285** (1977), 387–390.
[GMPN] G. Grantcharov, C. McLaughin, H. Pedersen and Y. Poon, Deformations of Kodaira manifolds, *Glasgow Math. J.* **46** (2004), 259–281.
[Gr] M. Gromov, Rigid Transfomations Groups, Editors D. Bernard and Y. Choquet Bruhat, *Géométrie Différentielle*, **33**, Hermann, (1988), 65–139.
[Ho] T. Höfer, Remarks on torus principal bundles, *J. Math. Kyoto Univ.*, **33** (1993), 227–259.
[HL] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.
[IKO] M. Inoue, S. Kobayashi and T. Ochiai, Holomorphic affine connections on compact complex surfaces, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **27** (1980), 247–264.
[Is] N. Istrati, Vaisman manifolds with vanishing first Chern class, *Jour. Eur. Math. Soc.* (2025), DOI 10.4171/JEMS/1633
[Fr] R. Friedman, On threefolds with trivial canonical bundle, *Complex geometry and Lie theory*, Proc. Sympos Pure Math., **53**, American Mathematical Society, Providence, (1991), 103–134.
[GGK] D. Greb, H. Guenancia and S. Kebekus, Klt varieties with trivial canonical class: holonomy, differential forms and fundamental groups, *Geom. Topol.* **23** (2019), 2051–2124.
[Ko] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, Princeton, NJ, Iwanami Shoten, Tokyo, 1987.
[LY] J. Li and S.-T. Yau, Hermitian-Yang-Mills connections on non-Kahler manifolds, *Mathematical aspects of string Theory*, Adv. Ser. Math. Phys., ed.: S.-T. Yau, World Scientific, (1987), 560–573.
[LT] M. Lübke and A. Teleman, *The Kobayashi-Hitchin correspondence*, World Scientific, 1995.
[LT1] P. Lu and G. Tian, The complex structure on a connected sum of $S^3 \times S^3$ with trivial canonical bundle, *Math. Ann.* **298** (1994), 761–764.
[LT2] P. Lu and G. Tian, Complex structure on connected sums of $S^3 \times S^3$, *Manifolds and geometry*, 284–293, Symposia Math., XXXVI,, Cambridge, Cambridge Univ. Press, 1996.
[No] K. Nomizu, On local and global existence of Killing vector fields, *Ann. of Math.* **72** (1960), 105–120.
[OV] L. Ornea and M. Verbitsky, *Principles of locally conformally Kähler geometry*, Birkhäuser, (2024).
[PV] V. Popov and E. Vinberg, Invariant Theory, *Algebraic Geometry IV*, **55**, Editors I. Shafarevich, A. Parshin, E.M.S. Springer-Verlag, (1991), 123–280.
[Ro] S. Rollenske, The Kuranishi space of complex parallelisable nilmanifolds, *J. Eur. Math. Soc.* **13** (2011), 513–531.
[To] V. Tosatti, Non-Kähler Calabi–Yau manifolds, *Contemp. Math.* **644** (2015), 261–277.
[Ts] K. Tsukada, Holomorphic maps of compact generalized Hopf manifolds, *Diff. Geom. Appl.* **69** (1997), 61–71.
[Va1] I. Vaisman, On locally and globally locally Kähler manifolds, *Trans. Amer. Math. Soc.* **262** (1980), 533–542.
[Va2] I. Vaisman, Generalized Hopf manifolds, *Geom. Dedicata* **13** (1982), 231–255.
[Vi] A. Vitter, Affine structures on compact complex manifolds, *Invent. Math.*, **17**, (1972), 231–244.
[Wa] H.-C. Wang, Complex Parallelisable manifolds, *Proc. Amer. Math. Soc.* **5** (1954), 771–776.
[Ya] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

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