The zipper condition for 4-tensors in two-dimensional topological order and the higher relative commutants of a subfactor arising from a commuting square

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Dedicated to the memory of Hikosaburo Komatsu

Abstract

They recently study two-dimensional topological order in condensed matter physics in terms of tensor networks involving certain 3- and 4-tensors. Their 3-tensors satisfying the "zipper condition" play an important role there. We identify their 4-tensors with bi-unitary connections in Jones' subfactor theory in operator algebras with precise normalization constants. Then we prove that their tensors satisfying the zipper condition are the same as flat fields of strings in subfactor theory which correspond to elements in the higher relative commutants of the subfactor arising from the bi-unitary connection. This is what we expect, since the zipper condition is a kind of pentagon relations, but we clarify what conditions are exactly needed for this — we do not need the flatness or the finite depth condition for the bi-unitary connection. We actually generalize their 4-tensors so that the four index sets of the 4-tensors can be all different and work on a "half-version" of the zipper condition.

1 Introduction

Fusion categories [5] (with or without braiding) have emerged as new types of symmetries in mathematics and physics. Both quantum field theory and condensed matter physics have seen such symmetries recently and they are often called *non-invertible symmetries* in the physics literature, and also called quantum symmetries in various mathematics literature. Particularly, many researchers in *two-dimensional topological order* in condensed matter physics are interested in such studies using *tensor networks* recently as in [2], [24], [29].

It has been well-known that subfactor theory of Jones [13], [14] in operator algebras gives appropriate tools to study structures of fusion categories, and it is indeed this theory which led to the discovery of the *Jones polynomial*, the first mathematical realization of quantum symmetry. This approach is closely related to operator algebraic studies of quantum field theory [21], [22], [7], since early days of subfactor theory through Doplicher-Haag-Roberts theory in *algebraic quantum field theory*.

In a usual operator algebraic study of fusion categories, we realize an object of a fusion category as a bimodule over (type II_1) factors or an endomorphism of a (type III) factor [6], [7]. Another approach [1] based on bi-unitary connections [25], [31], [15] is less common, but contains the same information as these two methods and has an advantage that everything is *finite dimensional*. This finite dimensionality enabled us to construct the Haagerup subfactor [1], which is still one of the most mysterious quantum symmetries today. Recall that a bi-unitary connection gives a characterization [31], [6] of a non-degenerate commuting square, which was initially studied in [27] in a different context and has complete information to recover an amenable subfactor of type II_1 [28].

It has been pointed out in [16], [18] that the 4-tensors in [2] are mathematically the same as bi-unitary connections up to slight change of normalization, and identification of some natural Hilbert spaces in condensed matter physics and subfactor theory has been given in [17]. A characterization of such 4-tensors as certain generalized quantum $6j$-symbols has been also given in [19]. This shows that *anyons* [20] are studied with such 4-tensor networks [9]. Our correspondence among various mathematical approaches to study fusion categories are summarized in Tables 1 and 2.

Table 1: Correspondence among endomorphisms, bimodules and connections

endomorphism	bimodules	connections
identity	identity bimodule	trivial connection
direct sum	direct sum	direct sum
composition	relative tensor product	composition
conjugate endomorphism	dual bimodule	dual connection
dimension	(Jones index)$^{1/2}$	Perron-Frobenius eigenvalue
intertwiner	intertwiner	flat field of strings

Table 2: Correspondence between connections, commuting squares and 4-tensors

connections	commuting square	4-tensor
trivial connection	commuting square	trivial 4-tensor
direct sum	direct sum	direct sum
composition	composition	concatenation
dual connection	basic construction	complex conjugate tensor
Perron-Frobenius eigenvalue	(Pimsner-Popa index)$^{1/2}$	Perron-Frobenius eigenvalue
flat fields of strings	relative commutant	tensors with the zipper condition

The aim is this paper is threefold and to complete the above Tables as follows.

(1) Give precise normalization constants in various formulas.

(2) Characterize the morphism property in terms of the "if and only if" form in comparison to the open string bimodule framework and the zipper condition [2, (2)].

(3) Give the most general setting of assumptions under which our arguments work.

The first one is only a technical issue, but important for actual computations. Examples of concrete computations in various papers arise from 3-cocycles on finite groups, where all the normalizing constants are 1 and this issue can be ignored, but we need them in a more general setting.

The second is important from a theoretical viewpoint. The bimodule approach involves infinite dimensional operator algebras and Hilbert spaces, so it is not clear whether this approach gives the same morphisms as in the tensor network framework.

For the third aim, we may have four different index sets for our 4-tensors, as long as we have bi-unitarity as in Fig. 26 and 27, and we do not need the finite depth condition or the flatness condition [6] for bi-unitary connections. The lack of the former condition means that the initial data can produce countably many irreducible objects in our tensor category, and the lack of the latter condition means that our bi-unitary connection, a kind of quantum $6j$-symbols, does not have to be in a canonical form.

2 Bi-unitary connections

We prepare notations and conventions on bi-unitary connections as in [1], [6, Chapter 11], [15], [25], [31]. We give complete definitions since our setting is slightly more general than the one in [16], [17].

We have four finite bipartite oriented graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$. We assume that $\mathcal{G}_0$ and $\mathcal{G}_2$ are connected. Each graph can have multiple edges between one pair of vertices and also cycles, but cannot have a loop, an edge from one vertex to the same one, since it is bipartite. We write $E(\mathcal{G})$ for the edge set of a graph $\mathcal{G}$.

We assume that the sets of the source vertices of $E(\mathcal{G}_0)$, $E(\mathcal{G}_1)$, $E(\mathcal{G}_2)$ and $E(\mathcal{G}_3)$ are V_0, V_0, V_1 and V_3, respectively. We further assume that the sets of the range vertices of $E(\mathcal{G}_0)$, $E(\mathcal{G}_1)$, $E(\mathcal{G}_2)$ and $E(\mathcal{G}_3)$ are V_3, V_1, V_2 and V_2, respectively. We draw a diagram as in Fig. 1 to depict this situation. We assume that the numbers of the edges of all the four graphs are larger than one.

$$\begin{tikzcd} V_0 \arrow[r, "\mathcal{G}_0"] \arrow[d, "\mathcal{G}_1"'] & V_3 \arrow[d, "\mathcal{G}_3"] \\ V_1 \arrow[r, "\mathcal{G}_2"'] & V_2 \end{tikzcd}$$

Figure 1: Four graphs

Let $\Delta_{\mathcal{G}_0,xy}$ be the number of edges of $\mathcal{G}_0$ between $x \in V_0$ and $y \in V_3$. Let $\Delta_{\mathcal{G}_1,xy}$ be the number of edges of $\mathcal{G}_1$ between $x \in V_1$ and $y \in V_2$. Let $\Delta_{\mathcal{G}_2,xy}$ be the number of edges of $\mathcal{G}_2$ between $x \in V_0$ and $y \in V_1$. Let $\Delta_{\mathcal{G}_3,xy}$ be the number of edges of $\mathcal{G}_3$ between $x \in V_3$ and $y \in V_2$. We assume that we have the following identities for some positive numbers β_0, β_1. We assume that we have a positive number $\mu(x)$ for each vertex x and that the following identities hold. That is, for each of V_0, V_1, V_2, V_3, the vector given by $\mu(x)$ gives a Perron-Frobenius eigenvector for the adjacency matrix of one of the four graphs, and the numbers β_0, β_1 are the Perron-Frobenius eigenvalues of these matrices. Since all the four graphs have more than one edge, we have $\beta_0, \beta_1 > 1$. We fix one such $\mu(x)$ for all x.

$$\begin{aligned}
\sum_{x \in V_0} \Delta_{\mathcal{G}_0,xy}\mu(x) &= \beta_0(\mu_y), \quad y \in V_3, \\
\sum_{y \in V_3} \Delta_{\mathcal{G}_0,xy}\mu(y) &= \beta_0(\mu_x), \quad x \in V_0, \\
\sum_{x \in V_1} \Delta_{\mathcal{G}_2,xy}\mu(x) &= \beta_0(\mu_y), \quad y \in V_2, \\
\sum_{y \in V_2} \Delta_{\mathcal{G}_2,xy}\mu(y) &= \beta_0(\mu_x), \quad x \in V_1, \\
\sum_{x \in V_0} \Delta_{\mathcal{G}_1,xy}\mu(x) &= \beta_1(\mu_y), \quad y \in V_1, \\
\sum_{y \in V_1} \Delta_{\mathcal{G}_1,xy}\mu(y) &= \beta_1(\mu_x), \quad x \in V_0, \\
\sum_{x \in V_3} \Delta_{\mathcal{G}_3,xy}\mu(x) &= \beta_1(\mu_y), \quad y \in V_2, \\
\sum_{y \in V_2} \Delta_{\mathcal{G}_3,xy}\mu(y) &= \beta_1(\mu_x), \quad x \in V_3,
\end{aligned}$$

Here is one example Fig. 2 of four graphs where all the four graphs are isomorphic and $\beta_0 = \beta_1 = 3$.

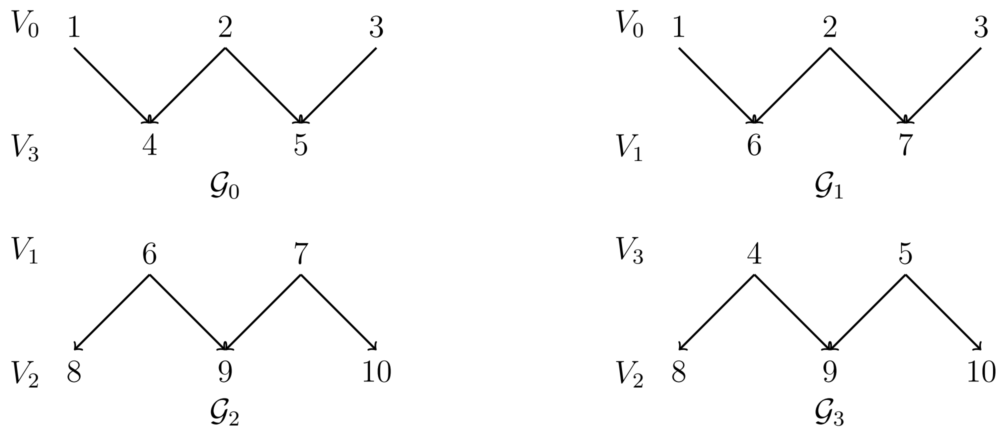

Figure 2: Example 1:How four graphs are connected

Another example of four graphs is given in Fig. 3, where all the four graphs are different. We have $\beta_0 = 2\cos(\pi/12)$ and $\beta_1 = (3+\sqrt{3})^{1/2}$.

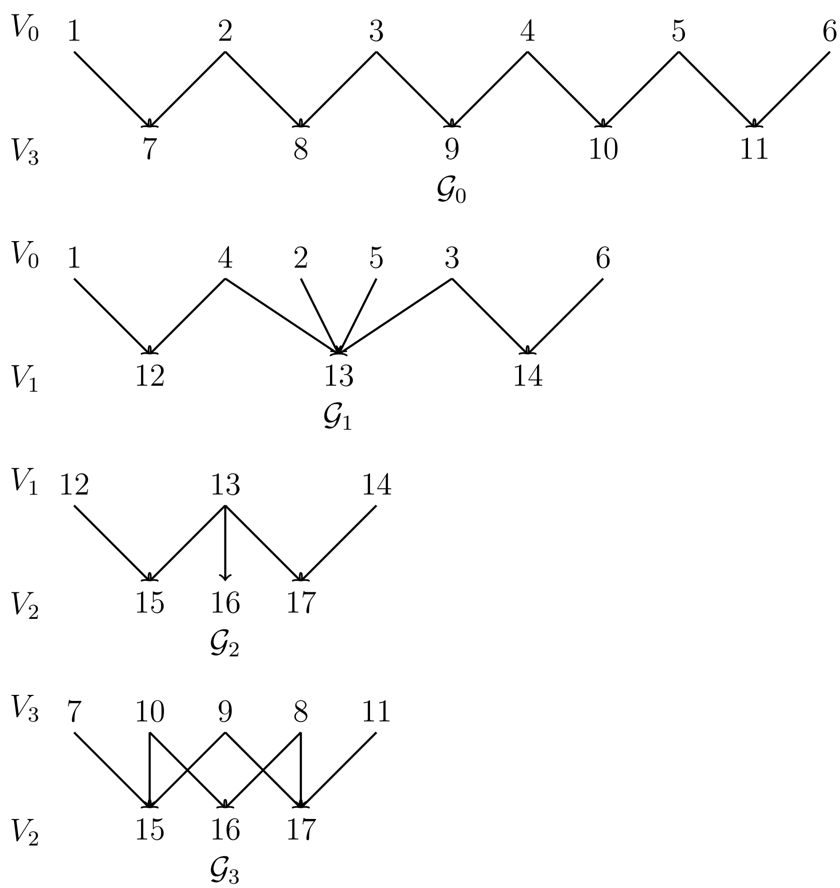

Figure 3: Example 2:How four graphs are connected

For an edge ξ of one of the graphs $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, we write $s(\xi)$ and $r(\xi)$ for the source and the range. Let $\xi_0, \xi_1, \xi_2, \xi_3$ be edges of $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$, respectively. If we have $s(\xi_0) = x_0 \in V_0$, $r(\xi_0) = x_3 \in V_3$, $s(\xi_1) = x_0 \in V_0$, $r(\xi_1) = x_1 \in V_1$, $s(\xi_2) = x_1 \in V_1$, $r(\xi_2) = x_2 \in V_2$, $s(\xi_3) = x_3 \in V_3$, and $r(\xi_3) = x_2 \in V_2$, then we call a combination of ξ_i a *cell*, as in Fig. 4.

$$\begin{tikzcd} x_0 \arrow[r, "\xi_0"] \arrow[d, "\xi_1"'] & x_3 \arrow[d, "\xi_3"] \\ x_1 \arrow[r, "\xi_2"'] & x_2 \end{tikzcd}$$

Figure 4: A cell

For each cell, we assign a complex number. We call this map a *connection* and write W for this. We also write as in Fig. 5 for the number assigned by W to this cell. If one of the conditions $s(\xi_0) = s(\xi_1)$, $r(\xi_0) = s(\xi_3)$, $r(\xi_1) = s(\xi_2)$ and $r(\xi_2) = r(\xi_3)$ fails, we understand that the diagram in Fig. 5 denotes the number 0.

$$\begin{tikzcd} \cdot \arrow[r, "\xi_0"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "W"] & \cdot \arrow[d, "\xi_3"] \\ \cdot \arrow[r, "\xi_2"'] & \cdot \end{tikzcd}$$

Figure 5: A connection value

We first require *unitarity* of W as in Fig. 6, where the bar on the right cell denotes the complex conjugate.

$$\sum_{\xi_1,\xi_2} \begin{tikzcd} \cdot \arrow[r, "\xi_0"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "W"] & \cdot \arrow[d, "\xi_3"] \\ \cdot \arrow[r, "\xi_2"'] & \cdot \end{tikzcd} \overline{\begin{tikzcd} \cdot \arrow[r, "\xi'_4"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "W"] & \cdot \arrow[d, "\xi'_3"] \\ \cdot \arrow[r, "\xi_2"'] & \cdot \end{tikzcd}} = \delta_{\xi_3,\xi'_3}\delta_{\xi_4,\xi'_4}$$

Figure 6: Unitarity

We define a new connection W' as in Fig. 7, where $\tilde{\xi}_0$ denotes the edge ξ_0 with its orientation reversed. We also require that this W' satisfies unitarity. When unitarity holds for W and W', we say W satisfies *bi-unitarity* and call W a *bi-unitary connection.* Since we consider only connections with bi-unitarity, we simply write a connection for a bi-unitary connection. Ocneanu and Haagerup found that a bi-unitary connection characterizes a non-degenerate commuting squares of finite dimensional C^*-algebras with a trace as in [6, Section 11.2],

$$
\begin{tikzcd} \cdot \arrow[r, "\tilde{\xi}_0"] \arrow[d, "\xi_3"'] \arrow[dr, phantom, "W'"] & \cdot \arrow[d, "\xi_1"] \\ \cdot \arrow[r, "\tilde{\xi}_2"'] & \cdot \end{tikzcd}
= \sqrt{\frac{\mu(s(\xi_0))\mu(r(\xi_2))}{\mu(r(\xi_0))\mu(s(\xi_2))}}
\overline{\begin{tikzcd} \cdot \arrow[r, "\xi_0"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "W"] & \cdot \arrow[d, "\xi_3"] \\ \cdot \arrow[r, "\xi_2"'] & \cdot \end{tikzcd}}
$$

Figure 7: Renormalization (1)

$$
\begin{tikzcd} \cdot \arrow[r, "\xi_2"] \arrow[d, "\tilde{\xi}_1"'] \arrow[dr, phantom, "\bar{W}"] & \cdot \arrow[d, "\tilde{\xi}_3"] \\ \cdot \arrow[r, "\xi_0"'] & \cdot \end{tikzcd}
= \sqrt{\frac{\mu(s(\xi_0))\mu(r(\xi_2))}{\mu(r(\xi_0))\mu(s(\xi_2))}}
\overline{\begin{tikzcd} \cdot \arrow[r, "\xi_0"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "W"] & \cdot \arrow[d, "\xi_3"] \\ \cdot \arrow[r, "\xi_2"'] & \cdot \end{tikzcd}}
$$

Figure 8: Renormalization (2)

$$
\begin{tikzcd} \cdot \arrow[r, "\tilde{\xi}_2"] \arrow[d, "\tilde{\xi}_3"'] \arrow[dr, phantom, "\bar{W}'"] & \cdot \arrow[d, "\tilde{\xi}_1"] \\ \cdot \arrow[r, "\tilde{\xi}_0"'] & \cdot \end{tikzcd}
=
\begin{tikzcd} \cdot \arrow[r, "\xi_0"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "W"] & \cdot \arrow[d, "\xi_3"] \\ \cdot \arrow[r, "\xi_2"'] & \cdot \end{tikzcd}
$$

Figure 9: Renormalization (3)

We also define new connections $\bar{W}$ and $\bar{W}'$ as in Fig. 8 and 9. They both satisfy bi-unitarity automatically. We also define a value of another diagram as in Fig. 10. Note that we have Fig. 11 due to Fig. 7 and 10.

$$
\begin{tikzcd} \cdot \arrow[d, "\xi_3"'] \arrow[dr, phantom, "W"] & \cdot \arrow[l, "\xi_0"'] \arrow[d, "\xi_1"] \\ \cdot & \cdot \arrow[l, "\xi_2"] \end{tikzcd}
=
\overline{\begin{tikzcd} \cdot \arrow[r, "\xi_0"] \arrow[d, "\xi_1"'] \arrow[dr, phantom, "W"] & \cdot \arrow[d, "\xi_3"] \\ \cdot \arrow[r, "\xi_2"'] & \cdot \end{tikzcd}}
$$

Figure 10: Conjugate convention

$$
\begin{array}{ccc} & \tilde{\xi}_0 & \\ \xi_1 & \boxed{W} & \xi_3 \\ & \tilde{\xi}_2 & \end{array}
= \sqrt{\frac{\mu(s(\xi_0))\mu(r_(\xi_2))}{\mu(r(\xi_0))\mu(s(\xi_2))}}
\begin{array}{ccc} & \xi_0 & \\ \xi_1 & \boxed{W} & \xi_3 \\ & \xi_2 & \end{array}
$$

Figure 11: Renormalization convention

Suppose we have two connections W_1 and W_2 as depicted in Fig. 13.

We now define unitary equivalence of two connections W_1 and W_2 on the same graphs depicted as in Fig. 1. Suppose we have two unitary matrices U, V whose index sets are the edge sets of $\mathcal{G}_1$, $\mathcal{G}_3$, respectively. Furthermore, we assume $U_{\xi_1,\xi_1'} = 0$ if $s(\xi_1) \neq s(\xi_1')$ or $r(\xi_1) \neq r(\xi_1')$ and a similar property for V. Then we say W_1 and W_2 are equivalent if the identity as in Fig. 12 holds.

$$
\begin{array}{ccc} & \xi_0 & \\ \xi_1 & \boxed{W_1} & \xi_3 \\ & \xi_2 & \end{array}
= \sum_{\xi_1',\xi_3'} U_{\xi_1,\xi_1'}
\begin{array}{ccc} & \xi_0 & \\ \xi_1' & \boxed{W_2} & \xi_3' \\ & \xi_2 & \end{array}
V_{\xi_3',\xi_3}
$$

Figure 12: Unitary equivalence of W_1 and W_2

$$
\begin{array}{ccc} V_0 & \mathcal{G}_0 & V_3 \\ \mathcal{G}_1 & \boxed{W_1} & \mathcal{G}_3 \\ V_1 & \mathcal{G}_2 & V_2 \end{array}
\qquad
\begin{array}{ccc} V_1 & \mathcal{G}_4 & V_2 \\ \mathcal{G}_5 & \boxed{W_2} & \mathcal{G}_7 \\ V_4 & \mathcal{G}_0 & V_5 \end{array}
$$

Figure 13: Two connections

$$
\begin{array}{ccc} & \xi_0 & \\ \xi_1 & & \xi_3 \\ \xi_5 & & \xi_7 \\ & \xi_4 & \end{array}
= \sum_{\xi_2}
\begin{array}{ccc} & \xi_0 & \\ \xi_1 & \boxed{W_1} & \xi_3 \\ & \xi_2 & \end{array}
\begin{array}{ccc} & \xi_2 & \\ \xi_5 & \boxed{W_2} & \xi_7 \\ & \xi_4 & \end{array}
$$

Figure 14: The product connection of W_1 and W_2

We now assume that we have two connections W_1 and W_2 as in Fig. 15.

$$
\begin{array}{ccc}
V_0 & \mathcal{G}_0 & V_3 \\
\mathcal{G}_1 & \boxed{W_1} & \mathcal{G}_3 \\
V_1 & \mathcal{G}_2 & V_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
V_0 & \mathcal{G}_0 & V_3 \\
\mathcal{G}'_1 & \boxed{W_2} & \mathcal{G}'_3 \\
V_1 & \mathcal{G}_2 & V_2
\end{array}
$$

Figure 15: Two connections

We define the sum graph $\mathcal{G}''_1$ of $\mathcal{G}_1$ and $\mathcal{G}'_1$ as follows. This is a bipartite graph with the two disjoint vertex sets V_0 and V_1 and the edge set being the disjoint union of $E(\mathcal{G}_1)$ and $E(\mathcal{G}'_1)$. We similarly define the sum graph $\mathcal{G}''_3$ of $\mathcal{G}_3$ and $\mathcal{G}'_3$ We next define the direct sum connection $W_{\oplus}W_2$ as in Fig.17 on the four graphs in Fig. 16.

$$
\begin{array}{ccc}
V_0 & \mathcal{G}_0 & V_3 \\
\mathcal{G}''_1 & \boxed{} & \mathcal{G}''_3 \\
V_1 & \mathcal{G}_2 & V_2
\end{array}
$$

Figure 16: The four graphs for $W_1 \oplus W_2$

$$
\begin{array}{ccc}
 & \xi_0 & \\
\xi''_1 & \boxed{} & \xi''_3 \\
 & \xi_2 &
\end{array}
=
\begin{cases}
\begin{array}{ccc}
 & \xi_0 & \\
\xi''_1 & \boxed{W_1} & \xi'''_3 \\
 & \xi_2 &
\end{array} & \text{if } \xi''_1 \in E(\mathcal{G}_1), \xi''_3 \in E(\mathcal{G}_3), \\[2em]
\begin{array}{ccc}
 & \xi_0 & \\
\xi''_1 & \boxed{W_2} & \xi''_3 \\
 & \xi_2 &
\end{array} & \text{if } \xi''_1 \in E(\mathcal{G}'_1), \xi''_3 \in E(\mathcal{G}'_3), \\[2em]
0 & \text{otherwise.}
\end{cases}
$$

Figure 17: The direct sum connection

If a connection W is written as $W_1 \oplus W_2$, we call it a *direct sum decomposition*. For a connection W, if none of the connection unitarily equivalent to W have a direct sum decomposition, we say that W is *irreducible*.

At the end of this Section, we present how a flat field of strings in the sense of [6, page 563] acts on an open string bimodule in [1, Claim 1 on page 19], because this action is not explicitly written in [1].

We start with a connection W as in the above. Then we choose initial vertices $*_0 \in V_0$ and $*_1 \in V_1$, and construct an open string bimodule X^W as in [1, page 14]. Take a general element in X^W before the completion as in Fig. 18. Without loss of generality, we may assume that the horizontal length of this string is $2k$, even, and thus $x \in V_0$.

$$\begin{tikzcd} *_0 \arrow[r] & {} \arrow[r] & \cdots \arrow[r] & x \arrow[d] \\ *_1 \arrow[r] & {} \arrow[r] & \cdots \arrow[r] & y \end{tikzcd}$$

Figure 18: An element of X^W

Let f be a flat field of strings on V_0. Label the edges from x to y on V_1 as $\xi_1, \xi_2, \ldots, \xi_n$. Then the part of f starting with x and ending with y is written as in Fig. 19.

$$\sum_{i,j} c_{ij} \left(\downarrow \xi_i \; , \quad \downarrow \xi_j \right)$$

Figure 19: A part of a flat field f

We now define an action of f on the element in Fig. 18. We may assume that the edge from x to y in Fig. 18 is ξ_k. Then the result of this action is defined to be as in Fig. 20.

$$\sum_i c_{ik} \begin{tikzcd} *_0 \arrow[r] & {} \arrow[r] & \cdots \arrow[r] & x \arrow[d, "\xi_i"] \\ *_1 \arrow[r] & {} \arrow[r] & \cdots \arrow[r] & y \end{tikzcd}$$

Figure 20: The result of an action of f

We prove that this action is well-defined. Consider a vector represented by Fig. 18 and write s for this. We also label the edge from x to y in Fig. 18 as ξ_i. We rewrite the vector s using the basis corresponding to the diagram in Fig. 21. This element is represented as in Fig. 22. We now consider the action of f on this element. Due to the well-definedness of the action of string algebras on the open string bimodule, this action is given by the action of f written in terms of the basis corresponding to the diagram in Fig. 23, but this is simply the action of a parallel transport f' of f on the vector s written in terms of basis as in Fig. 21. (See [6, Definition 11.18] for the notion of parallel transport.) Then this is exactly equal to the action of f on s considered with respect to the basis corresponding to the diagram in Fig. 23. The same argument shows the well-definedness of the step from $2k$ to $2k+1$, so we have proved the following Proposition.

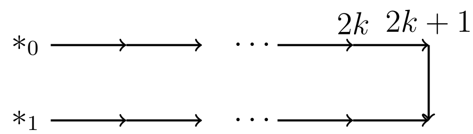

Figure 21: A basis for a finite dimensional subspace of X^W

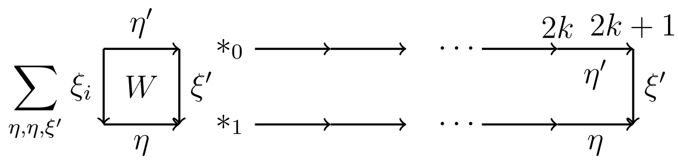

Figure 22: The element s written in terms of a new basis

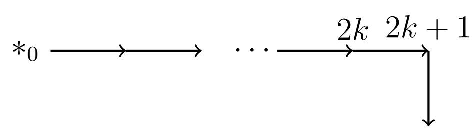

Figure 23: A basis for a finite dimensional string subalgebra

Proposition 2.1 *The above action of flat fields of strings gives a self-intertwiner of X^W commuting with the left and right actions of hyperfinite II_1 factors arising from the string algebras on $\mathcal{G}_0$ and $\mathcal{G}_2$, and all self-intertwiners of X^W arise in this way.*

3 4-tensors

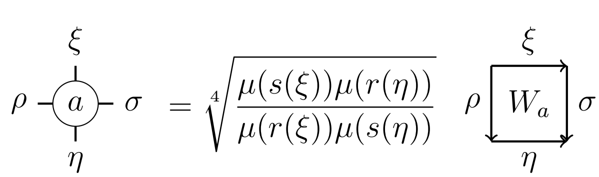

Figure 24: The 4-tensor a and the connection W_a

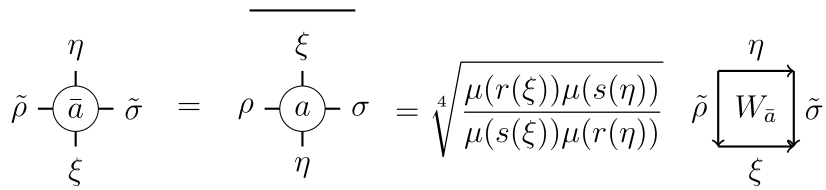

Figure 25: The 4-tensors $a, \bar a$ and the connection $W_{\bar a}$

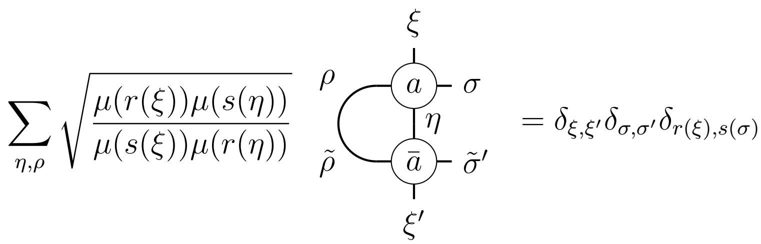

Figure 26: Bi-unitarity (1)

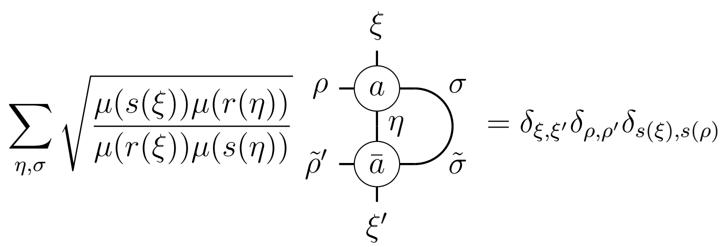

Figure 27: Bi-unitarity (2)

We can simply represent bi-unitarity in the two identity as in a diagram in Fig.28, where we drop all labels and the Kronecker δ's.

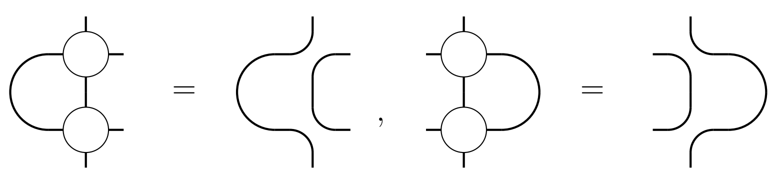

Figure 28: Graphical representation of bi-unitarity in a simplified form

4 Flatness and the zipper condition

Theorem 4.1 *The following are equivalent for a* 2*-tensor* F *and the corresponding field* f *of strings defined as above.*

(1) [The half zipper condition] *There exists another* 2 *tensor* $\tilde{F}$ *so that the* 2*-tensors* $F, \tilde{F}$ *satisfy the intertwining property as in Fig. 34.*

(2) [The zipper condition] *The* 2*-tensors* F *satisfies the invariance property as in Fig. 35.*

(3) [Half flatness] *There exists another field* $\tilde{f}$ *of strings so that we have the half flatness as in Fig. 29.*

(4) [Flatness] *The field* f *of strings satisfies the flatness as in Fig. 30.*

Proof. We first show equivalence of (3) and (4) Recall that this has been essentially proved in [6, pages 563–564], but we give more details in the current context. Using the initial connection W_a, we apply the string algebra construction in [6, Section 11.3], but allow all vertices in V_0 to be starting vertices. We then have a double sequence $\{A_{jk}\}_{j,k=1,2,\dots}$ of finite dimensional C^*-algebras and A_{00} is an abelian algebra $\mathbb{C}^{|}V_0|$, where $|V_0|$ denotes the cardinality of V_0.

We assume (3). The fields of strings f and $\tilde{f}$ give the corresponding same elements in the algebras A_{10} and A_{11}. We use the symbol f for this. Half flatness implies f commutes with A_{01}. The first horizontal Jones projection e_1 commutes with A_{10}, so it commutes with f, in particular. This means f commutes with A_{02} which is generated by A_{01} and e_1. This shows f produces another field of strings $\bar{f}$ on $\mathcal{G}_1$. The argument for $z = z'$ in [6, Fig. 11.16] shows that the field of strings $\bar{f}$ is equal to the field of strings f. This implies (4).

Conversely, we assume (4). In the same way to the above argument, we construct string algebras $\{A_{jk}\}_{j,k=1,2,\dots}$. The flatness of f shows that f gives an element in A_{10} which commutes with A_{02}. In particular, it commutes with A_{01}, and produces a field $\tilde{f}$ of strings satisfying the half flatness condition.

We next prove that (3) implies (1). We first assume half flatness for f and $\tilde{f}$. We define a 2-tensor F from f as in Fig. 31. We similarly define a 2-tensor $\tilde{F}$ from $\tilde{f}$.

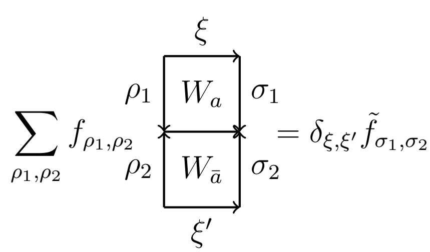

Figure 29: Half flatness of f

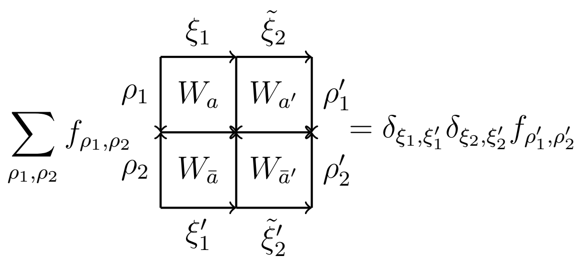

Figure 30: Flatness of f

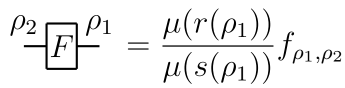

Figure 31: The 2-tensor F arising from the field $\sum_{\rho_1,\rho_2} f_{\rho_1,\rho_2}(\rho_1,\rho_2)$ of strings

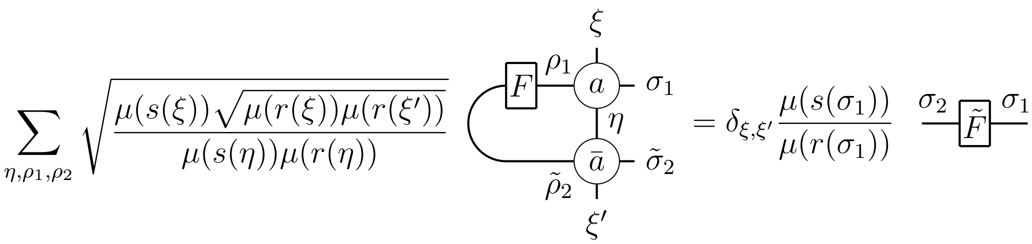

Figure 32: Half flatness for F

We rewrite the coefficient within the summation on the left-hand side as

$$\frac{\mu(s(\xi))}{\mu(s(\eta))}\sqrt{\frac{\sqrt{\mu(r(\xi))\mu(r(\xi'))}\mu(s(\eta))}{\mu(s(\xi))\mu(r(\eta))}},$$

and multiply the number in Fig. 33 to the numbers on the both hands sides of Fig. 32 and sum them over ξ', σ_2.

Then by bi-unitarity (2), only the terms for $\eta = \eta'$ and $\rho_2 = \rho_3$ remain, and we have the identity as in Fig. 34 by dividing the both hand sides by

$$\frac{\mu(r(\sigma_1))}{\mu(s(\sigma_1))}\sqrt[4]{\frac{\mu(r(\xi))\mu(s(\eta'))}{\mu(s(\xi))\mu(r(\eta'))}}.$$

Note that only the term $\xi = \xi'$ remains on the right hand side due to $\delta_{\xi,\xi'}$. This proves (1).

Hh]

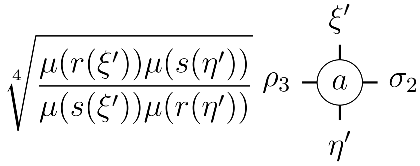

Figure 33: The multiplier

Since the above graphical manipulation amounts to a multiplication of a unitary matrix, we also have the converse direction. That is, we know that (1) implies (3).

A similar argument to the proof of equivalence of (1) and (3) shows equivalence of (2) and (4).

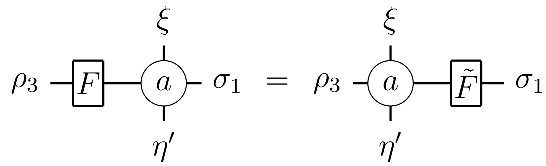

Figure 34: Intertwining property for $F, \tilde{F}$

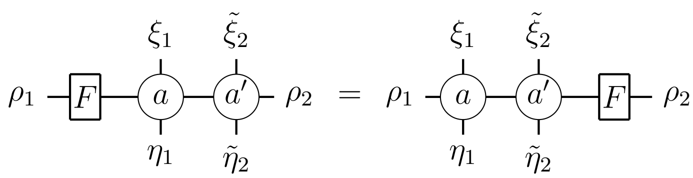

Figure 35: Intertwining property for F

□

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