LOCALLY UNIFORM ELLIPTICITY OF THE FRACTIONAL HESSIAN OPERATORS

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Abstract. In [1], Caffarelli-Charro introduced a fractional Monge-Ampère operator. Later, Wu [17] generalized it to a fractional analogue of k-Hessian operators and proved the strict ellipticity for $k = 2$. In this paper, we introduce a fractional analogue of general Hessian operators and prove the stability. We also show that the fractional analogue k-Hessian operators defined in [17] are strictly elliptic with respect to convex solutions for all $2 \leq k \leq n$. Furthermore, we provide a new proof for the case $k = 2$ without the convexity condition.
Keywords: Fractional Hessian operators; Strict ellipticity; Integro-differential equations.

1. introduction

In this paper, we consider a fractional setting for the Hessian operators

$$f(\lambda(D^2u)) \tag{1.1}$$

by writing it as a concave envelope of linear operators as in [1] and [17], where f is a symmetric smooth function of n variables, D^2u is the Hessian of a function u defined on $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$ and $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$ denotes the eigenvalues of D^2u. Following [2], we assume f to be defined in an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin, containing

$$\Gamma_n := \{\lambda \in \mathbb{R}^n : \text{ each component } \lambda_i > 0\} \neq \mathbb{R}^n,$$

and to satisfy the standard structure conditions throughout the paper:

$$f_i := \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma,\ 1 \leq i \leq n, \tag{1.2}$$

$$f \text{ is concave in } \Gamma \tag{1.3}$$

and

$$f \in C^2(\Gamma) \cap C^0(\overline{\Gamma}), f > 0 \text{ in } \Gamma \text{ and } f = 0 \text{ on } \partial\Gamma. \tag{1.4}$$

Furthermore, we assume that f satisfies

$$f \text{ is homogeneous of degree one.} \tag{1.5}$$

Let F be the function defined by

$$F(A) = f(\lambda(A))$$

for a symmetric $n \times n$ matrix $A = \{A_{ij}\}$ with $\lambda(A) \in \Gamma$ and

$$F^{ij}(A) = \frac{\partial F}{\partial A_{ij}},\ 1 \leq i, j \leq n.$$

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For simplicity, we still denote Γ to be the set of all symmetric $n \times n$ matrices $A = \{A_{ij}\}$ with $\lambda(A) \in \Gamma$. By (1.2) and (1.3), we find the matrix $\{F^{ij}(A)\}$ is positive definite for any $A \in \Gamma$ and F is also concave in Γ. (The reader is referred to [13] for details.) By (1.5), we find

$$F^{ij}A_{ij} = F(A), \text{ for each } A \in \Gamma.$$

Thus, for any $A = \{A_{ij}\}, B = \{B_{ij}\} \in \Gamma$, we have

$$F^{ij}(B)(A_{ij} - B_{ij}) \geq F(A) - F(B)$$

and

$$F(A) = \inf_{M \in \mathcal{M}} M^{ij}A_{ij}, \tag{1.6}$$

where $\mathcal{M}$ is a subset of all positive definite symmetric $n \times n$ matrices defined by

$$\mathcal{M} := \{M = \{M^{ij}\} : \text{ there exists } T \in \Gamma \text{ such that } M^{ij} = F^{ij}(T)\}.$$

Therefore, for $u \in C^2(\mathbb{R}^n)$, we have

$$F(D^2u(x)) = \inf_{M \in \mathcal{M}} \operatorname{trace}(MD^2u(x)) = \inf_{M \in \mathcal{M}} \Delta(u \circ \sqrt{M})(\sqrt{M}^{-1}x). \tag{1.7}$$

Let us recall the definition of the fractional Laplacian (c.f. [7]):

$$\begin{aligned} -(-\Delta)^s u(x) &= c_{n,s} \, \text{P.V.} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y - x|^{n+2s}} \, dy \\ &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} \, dy \end{aligned}$$

for $0 < s < 1$, where

$$c_{n,s} = \frac{4^s \Gamma(\frac{n}{2} + s)}{\pi^{n/2}\Gamma(-s)}$$

is a normalization constant. We then define the fractional Hessian operator as [1] and [17] by

$$\begin{aligned} F_s[u](x) &= \inf_{M \in \mathcal{M}} \Big\{ - c_{n,s}^{-1}(-\Delta)^s(u \circ \sqrt{M})(\sqrt{M}^{-1}x) \Big\} \\ &= \inf_{M \in \mathcal{M}} \Big\{ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{|\sqrt{M}^{-1}y|^{n+2s}} \det \sqrt{M}^{-1} \, dy \Big\} \\ &= \inf_{M \in \mathcal{M}} \Big\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u,x,y)}{|\sqrt{M}^{-1}y|^{n+2s}} \det \sqrt{M}^{-1} \, dy \Big\}, \end{aligned} \tag{1.8}$$

where

$$\delta(u, x, y) := u(x+y) + u(x-y) - 2u(x).$$

Our first result is the stability of the operator F_s.

Theorem 1.1. *Assume that f satisfies* (1.2)-(1.5). *If u is admissible (See Definition 2.2 in Section 2), asymptotically linear and $\frac{1}{2} < s < 1$, then*

$$\lim_{s \to 1} ((1-s)F_s[u](x)) = \frac{\omega_n}{4} F\left(D^2u(x)\right) \tag{1.9}$$

holds in the viscosity sense.

In this work, we shall study the existence and regularity of solutions to the Dirichlet problem

$$\begin{cases} F_s[u](x) = u(x) - \phi(x) & \text{in } \mathbb{R}^n \\ (u-\phi)(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.10}$$

where $\phi \in C^{2,\alpha}(\mathbb{R}^n)$ is the prescribed boundary data at infinity. As [1], ϕ is assumed to satisfy that ϕ is strictly convex in compact sets and $\phi = \psi + \eta$ near infinity, with $\psi(x)$ a cone and

$$|\eta(x)| \leq a|x|^{-\epsilon},\ |\nabla\eta(x)| \leq a|x|^{-(1+\epsilon)},\ |D^2\eta(x)| \leq a|x|^{-(2+\epsilon)}$$

for some constant $a > 0$ and $0 < \epsilon < n$. In particular, as $|x| \to +\infty$,

$$-(-\Delta)^s\eta(x) = O(|x|^{-(2s+\epsilon)})$$

and

$$a_1|x|^{1-2s} \leq -(-\Delta)^s\psi(x) \leq a_2|x|^{1-2s},$$

where a_1 and a_2 are some positive constants depending on the strict convexity of the section of Γ. ϕ is normalized such that $\phi(0) = 0$ and $\nabla\phi(0) = 0$.

Using same arguments as in [1] (see [17] also), we have the following existence theorem.

Theorem 1.2. *Assume ϕ is semi-concave (The reader is referred to Definition 2.1 for the definition of semi-concavity) and Lipschitz continuous, then there exists a unique viscosity solution to* (1.10)*. Furthermore, the solution is Lipschitz continuous and semi-concave with the same constants as ϕ.*

Typical examples satisfying (1.2)-(1.5) are given by $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n$ defined in the cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k\},$$

where $\sigma_k(\lambda)$ are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k},\ k = 1, \ldots, n.$$

In particular, the operator (1.8) coincides with the one defined in [17] as $f = \sigma_k^{1/k}$.

To obtain higher regularity, we usually need to use the nonlocal Evans-Krylov theory developed in [3] and [4]. For this purpose, we have to prove the strict ellipticity of (1.8). For fractional operator (1.8), the strictly elliptic operator is defined as follows.

$$\begin{aligned} F_s^\theta[u](x) &= \inf_{M\in\mathcal{M}} \left\{ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|\sqrt{M}^{-1}y|^{n+2s}} \det\sqrt{M}^{-1}\,dy \,\Big|\, \lambda_{\min}(M) \geq \theta \right\} \\ &= \inf_{M\in\mathcal{M}} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u,x,y)}{|\sqrt{M}^{-1}y|^{n+2s}} \det\sqrt{M}^{-1}\,dy \,\Big|\, \lambda_{\min}(M) \geq \theta \right\}, \end{aligned}$$

where θ is a positive constant. The main results of the current work are contained in the following theorem.

Theorem 1.3. *Suppose $f = \sigma_k^{1/k}$, for $2 \leq k \leq n$. Assume that $\frac{1}{2} < s < 1$ and u is convex, Lipschitz continuous, semi-concave, and*

$$(1-s)F_s[u](x) \geq \eta_0 > 0, \quad \forall x \in \Omega \tag{1.11}$$

in the viscosity sense for some constant $\eta_0 > 0$ and $\Omega \subset \mathbb{R}^n$. Then there exists a positive constant θ depending only on n, k, η_0, s and the Lipschitz and semi-concavity constants of u such that

$$F_s[u](x) = F_s^{\theta}[u](x), \quad \forall x \in \Omega. \tag{1.12}$$

Furthermore, Theorem 1.3 is stable as $s \to 1$, i.e., the constant θ will not go to 0 as $s \to 1$.

By Theorem 1.3, Proposition 6.3 in [1] and nonlocal Evans-Krylov theorem proved in [3] and [4], we obtain the $C^{2s+\alpha}$ estimates for convex solutions to (1.10) with $f = \sigma_k^{1/k}$ and then if the solutions are convex and then are classical.

The assumption that u is convex is expected to be removed. It is only used to show (4.8). In [17], Wu proved Theorem 1.3 for $k = 2$ without the condition that u is convex as follows.

Theorem 1.4. *Suppose $f = \sigma_2^{1/2}$. Assume that $\frac{1}{2} < s < 1$ and u is Lipschitz continuous, semi-concave, and satisfies (1.11). Then (1.12) holds.*

In the current work, we will provide an alternative proof of Theorem 1.4 in Section 4. It is of interest to remove the convexity condition for $3 \leq k \leq n-1$.

We remark that when $f = \sigma_n^{1/n}$, (1.1) is the Monge-Ampère operator and (1.8) becomes the fractional Monge-Ampère operator defined in [1]. Another class of interesting examples of f are

$$\tilde{f}(\lambda) = f\Big(\sum_{j\neq 1} \lambda_j, \ldots, \sum_{j\neq n} \lambda_j\Big),$$

where f satisfies (1.2)-(1.5). In particular, as $f = \sigma_n^{1/n}$, $\tilde{f}$ is called the $(n-1)$ Monge-Ampère operator.

Classical Hessian equations, especially the Monge-Ampère equation, appear in many geometric problems as well as the optimal transportation problem. The Dirichlet problem for Hessian equations was first studied by Ivochkina [11] and Caffarelli-Nirenberg-Spruck [2], followed by work in [8, 15, 16], etc. Guan [9] proved the existence of smooth solutions to the Dirichlet problem for Hessian equations only under conditions (1.2)-(1.4) and that there exists a subsolution.

Fractional Monge-Ampère operator was introduced by Caffarelli-Charro [1]. Later, Caffarelli-Silvestre [5] defined another nonlocal fractional Monge-Ampère operator

$$MA_s u(x) = c_{n,s} \inf_{K \in \mathcal{K}_n^s} \left\{ \int_{\mathbb{R}^n} (u(y) - u(x) - y \cdot \nabla u(x)) K(y)\, dy \right\}$$

by observing that

$$(\det(D^2 u(x)))^{1/n} = \inf\{\Delta[\tilde{u} \circ \varphi](0) : \text{ for all } \varphi \text{ measure preserving s.t. } \varphi(0) = 0\}.$$

Here

$$\tilde{u}(y) = u(x+y) - u(x) - y \cdot \nabla u(x)$$

and

$$\mathcal{K}_n^s = \left\{K : \mathbb{R}^n \to \mathbb{R}_+ : \left|\{x \in \mathbb{R}^n : K(x) > r^{-n-2s}\}\right| = |B_r| \text{ for all } r > 0\right\}.$$

Their definition was further generalized by Caffarelli and Soria-Carro [6] recently. A different nonlocal version of the Monge-Ampère operator was also considered in [10]. Besides, fractional k-Hessian operators (the cases $f = \sigma_k^{1/k}$) were introduced and studied by Wu [17]. In view of (1.7), the Hessian operators (1.1) can be regarded as a class of degenerate Bellman operators. It is interesting to consider general fractional Bellman operators.

The rest of this paper is organized as follows. In Section 2, we provide the notations of this paper and some preliminaries. Section 3 is devoted to proving Theorem 1.1. In Section 4, we consider the strict ellipticity of (1.8) and prove Theorem 1.3 and 1.4.

2. Preliminaries

In this section we introduce the notations of this paper and recall some basic results and definitions.

For square matrices, $M > 0$ means positive definite and $M \geq 0$ positive semi-definite. $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the smallest and largest eigenvalues of M, respectively.

We shall denote the r-th-dimensional ball of radius 1 and center 0 by $B_1^r(0) = \{x \in \mathbb{R}^r : |x| \leq 1\}$ and the corresponding $(r-1)$-th-dimensional sphere by $\partial B_1^r(0) = \{x \in \mathbb{R}^r : |x| = 1\}$. Whenever r is clear from context, we shall simply write $B_1(0)$ and $\partial B_1(0)$. $\mathcal{H}^r$ stands for the r-dimensional Haussdorff measure. We shall denote $\omega_r = |B_1^r(0)|$.

Definition 2.1. Let $A \subset \mathbb{R}^n$ be an open set. We say that a function $u : A \to \mathbb{R}$ is semiconcave if it is continuous in A and there exists $SC \geq 0$ such that $\delta(u, x, y) \leq SC|y|^2$ for all $x, y \in \mathbb{R}^n$ such that $[x - y, x + y] \subset A$. The constant SC is called a semiconcavity constant for u in A.

Alternatively, a function u is semiconcave in A with constant SC if $u(x) - \frac{SC}{2}|x|^2$ is concave in A. Geometrically, this means that the graph of u can be touched from above at every point by a paraboloid of the type $a + \langle b, x\rangle + \frac{SC}{2}|x|^2$.

Next, we give the definition of non-smooth admissible functions (in viscosity sense).

Definition 2.2. An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is called admissible in $\Omega \subset \mathbb{R}^n$ if $\lambda(D^2\psi(x)) \in \Gamma$ for all $x \in \Omega$ and $\psi \in C^2(N)$ satisfying $u - \psi$ has a local maximum at x, where N is some open neighborhood of x in Ω.

It is easy to find $u \in C^2(\Omega)$ is admissible in Ω if and only if $\lambda(D^2u(x)) \in \Gamma$ for all $x \in \Omega$.

Remark 2.3. An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is called k-convex if u satisfies the conditions in Definition 2.2 with the cone Γ replaced by Γ_k.

Proposition 2.4. *Assume that* $1/2 < s < 1$. *If* u *is Lipschitz continuous and semi-concave, then* $F_s[u](x)$ *is well-defined.*

Proof. For all $\rho > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\delta(u,x,y)}{|\sqrt{M}^{-1}y|^{n+2s}} \det \sqrt{M}^{-1}\, dy \\
=& \int_{\mathbb{R}^n} \frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\, dy \\
=& \int_{B_\rho(0)} \frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\, dy + \int_{\mathbb{R}^n\setminus B_\rho(0)} \frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\, dy \\
\leq& \int_{B_\rho(0)} \frac{C|\sqrt{M}y|^2}{|y|^{n+2s}}\, dy + \int_{\mathbb{R}^n\setminus B_\rho(0)} \frac{2L|\sqrt{M}y|}{|y|^{n+2s}}\, dy \\
\leq& \int_{B_\rho(0)} \frac{C\lambda_{\max}(M)}{|y|^{n+2s-2}}\, dy + \int_{\mathbb{R}^n\setminus B_\rho(0)} \frac{2L\lambda_{\max}(M)^{1/2}}{|y|^{n+2s-1}}\, dy < +\infty.
\end{aligned}$$

Therefore, $F_s[u](x)$ is well-defined. □

We recall from [3] the definition of viscosity solutions.

Definition 2.5. A function $u : \mathbb{R}^n \to \mathbb{R}$, upper (resp. lower) semicontinuous in $\overline{\Omega}$, is said to be a subsolution (supersolution) to $\mathcal{F}_s u = f$, and we write $\mathcal{F}_s u \geq f$ (resp. $\mathcal{F}_s u \leq f$), if every time all the following happen,

- x is a point in Ω,
- N is an open neighborhood of x in Ω,
- ψ is some C^2 function in $\overline{N}$,
- $\psi(x) = u(x)$,
- $\psi(y) > u(y)$ (resp. $\psi(y) < u(y)$) for every $y \in N \setminus \{x\}$,

and if we let

$$v := \begin{cases} \psi & \text{in } N \\ u & \text{in } \mathbb{R}^n \setminus N, \end{cases}$$

then we have $\mathcal{F}_s[v](x) \geq f(x)$ (resp. $\mathcal{F}_s[v](x) \leq f(x)$). A solution is a function u that is both a subsolution and a supersolution.

The following lemma states that $\mathcal{F}_s u$ can be evaluated classically at those points x where u can be touched by a paraboloid. The details of the proof can be found in [3].

Lemma 2.6. *Let $1/2 < s < 1$ and $u : \mathbb{R}^n \to \mathbb{R}$ with asymptotically linear growth. If we have $\mathcal{F}_s u \geq f$ in $\mathbb{R}^n$ (resp. $\mathcal{F}_s u \leq f$) in the viscosity sense and ψ is a C^2 function that touches u from above (below) at a point x, then $\mathcal{F}_s[u](x)$ is defined in the classical sense and $\mathcal{F}_s[u](x) \geq f(x)$ (resp. $\mathcal{F}_s[u](x) \leq f(x)$).*

3. Stability of fractional Hessian operators

In this section, we use similar ideas of [1] to prove Theorem 1.1.

Proof of Theorem 1.1. We first consider the case $u \in C^2$. By (1.7), we have

$$F\left(D^2u(x)\right) = \inf_{M\in\mathcal{M}} \left\{\operatorname{trace}\left(MD^2u(x)\right)\right\}$$

provided $u \in C^2$ and $\lambda(D^2u(x)) \in \Gamma$. Thus, it suffices to prove:

$$\lim_{s\to 1}(1-s)F_s[u](x) = \frac{w_n}{4}\inf_{M\in\mathcal{M}}\left\{\operatorname{trace}\left(MD^2u(x)\right)\right\}.$$

For suitable $0 \le \rho \le R$ and fixed x, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{\delta(u,x,y)}{\left|\sqrt{M}^{-1}y\right|^{n+2s}}\det\sqrt{M}^{-1}\,dy\\
=&\int_{\mathbb{R}^n}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy\\
=&\int_{B_\rho(0)}\frac{\left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle}{|y|^{n+2s}}\,dy \qquad (3.1)\\
&+\int_{B_\rho(0)}\frac{\delta(u,x,\sqrt{M}y)-\left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle}{|y|^{n+2s}}\,dy\\
&+\int_{B_R(0)\setminus B_\rho(0)}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy+\int_{\mathbb{R}^n\setminus B_R(0)}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy.
\end{aligned}$$

First, we estimate the right-hand side of the equation:

$$\begin{aligned}
&\int_{B_\rho(0)}\frac{\left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle}{|y|^{n+2s}}\,dy\\
=&\int_0^\rho\int_{\partial B_1(0)}\frac{\left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle}{r^{2s-1}}\,dS_y dr\\
=&\frac{\rho^{2-2s}}{2-2s}\int_{\partial B_1(0)}\left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle\,dS_y \qquad (3.2)\\
=&\frac{\rho^{2-2s}}{2-2s}\int_{B_1(0)}\operatorname{div}_y\left[\sqrt{M}^TD^2u(x)\sqrt{M}y\right]\,dy\\
=&\frac{\rho^{2-2s}}{2-2s}\omega_n\operatorname{trace}\left(MD^2u(x)\right),
\end{aligned}$$

where we have used the divergence theorem for the penultimate equality.

Since

$$\delta(u,x,\sqrt{M}y) = \left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle + o\left(\left|My\cdot y^T\right|\right) \quad \text{as} \quad |y|\to 0,$$

we have, for any fixed ε small,

$$\left|\delta(u,x,\sqrt{M}y) - \left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle\right| \le \varepsilon\lambda_{\max}(M)|y|^2$$

provided $|y|<\rho$ and $0<\rho\ll\frac{1}{\lambda_{\max}^{1/2}(M)}$ is sufficiently small. It follows that

$$\begin{aligned}
&\int_{B_\rho(0)}\frac{\left|\delta(u,x,\sqrt{M}y)-\left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle\right|}{|y|^{n+2s}}\,dy\\
\le&\int_{B_\rho(0)}\frac{\varepsilon\lambda_{\max}(M)}{|y|^{n+2s-2}}\,dy\\
=&\frac{\varepsilon w_n}{2-2s}\rho^{2-2s}\lambda_{\max}(M).
\end{aligned}\tag{3.3}$$

Moreover,

$$\int_{B_R(0)\setminus B_\rho(0)}\frac{|\delta(u,x,\sqrt{M}y)|}{|y|^{n+2s}}\,dy\le\frac{2\omega_n}{s}\|u\|_{L^\infty(B_{\lambda_{\max}^{\frac12}(M)R}(x))}\cdot\left(\rho^{-2s}-R^{-2s}\right).\tag{3.4}$$

By the asymptotic linearity of u, there exists $R>0$ such that for $|y|>R$,

$$|\delta(u,x,\sqrt{M}y)|\le 2L|\sqrt{M}y|\le 2L\lambda_{\max}^{\frac12}(M)\cdot|y|.$$

Thus,

$$\int_{\mathbb{R}^n\setminus B_R(0)}\frac{|\delta(u,x,\sqrt{M}y)|}{|y|^{n+2s}}\,dy\le\frac{2Lw_nR^{1-2s}\lambda_{\max}^{\frac12}(M)}{2s-1}.\tag{3.5}$$

Combining (3.1)-(3.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{\delta(u,x,y)}{\left|\sqrt{M}^{-1}y\right|^{n+2s}}\det\sqrt{M}^{-1}\,dy\\
\le&\frac{\rho^{2-2s}}{2-2s}w_n\operatorname{trace}\left(MD^2u(x)\right)+\frac{\varepsilon w_n}{2-2s}\rho^{2-2s}\lambda_{\max}(M)\\
&+\frac{2\omega_n}{s}\|u\|_{L^\infty(B_{\lambda_{\max}^{\frac12}(M)R}(x))}\cdot\left(\rho^{-2s}-R^{-2s}\right)+\frac{2Lw_nR^{1-2s}\lambda_{\max}^{\frac12}(M)}{2s-1}.
\end{aligned}$$

By the definition of $F_s[u](x)$,

$$\begin{aligned}
F_s[u](x)\le&\frac{\rho^{2-2s}w_n}{4(1-s)}\operatorname{trace}\left(MD^2u(x)\right)+\frac{\varepsilon w_n}{4(1-s)}\rho^{2-2s}\lambda_{\max}(M)\\
&+\frac{\omega_n}{s}\|u\|_{L^\infty(B_{\lambda_{\max}^{\frac12}(M)R}(x))}\cdot\left(\rho^{-2s}-R^{-2s}\right)+\frac{L}{2s-1}w_nR^{1-2s}\lambda_{\max}^{\frac12}(M).
\end{aligned}$$

Multiplying both sides by $(1-s)$,

$$\begin{aligned}
&(1-s)F_s[u](x)\\
\le&\frac{w_n}{4}\rho^{2-2s}\operatorname{trace}\left(MD^2u(x)\right)+\frac{\varepsilon w_n}{4}\rho^{2-2s}\lambda_{\max}(M)\\
&+\frac{(1-s)\omega_n}{s}\|u\|_{L^\infty\left(B_{\lambda_{\max}^{\frac12}(M)R}(x)\right)}\cdot\left(\rho^{-2s}-R^{-2s}\right)+\frac{(1-s)w_nL}{2s-1}R^{1-2s}\lambda_{\max}^{\frac12}(M).
\end{aligned}$$

Taking the limit as $s\to1$,

$$\lim_{s\to1}\left((1-s)F_s[u](x)\right)\le\frac{w_n}{4}\operatorname{trace}\left(MD^2u(x)\right)+\frac{\varepsilon w_n}{4}\lambda_{\max}(M).$$

By the arbitrariness of ε,

$$\lim_{s\to 1}\left((1-s)F_s[u](x)\right) \le \frac{w_n}{4}\operatorname{trace}\left(MD^2u(x)\right).$$

Since this holds for all $M \in \mathcal{M}$,

$$\lim_{s\to 1}\left((1-s)F_s[u](x)\right) \le \frac{w_n}{4}\inf_{M\in\mathcal{M}}\operatorname{trace}\left(MD^2u(x)\right).$$

This proves one side of the inequality.

Next, we prove the reverse inequality:

$$\lim_{s\to 1}\left((1-s)F_s[u](x)\right) \ge \frac{w_n}{4}\inf_{M\in\mathcal{M}}\left\{\operatorname{trace}\left(MD^2u(x)\right)\right\}.$$

From the earlier computation (3.1),

$$\begin{aligned}
&\int_{B_\rho(0)} \frac{\left\langle D^2u(x)\cdot\sqrt{M}y, \sqrt{M}y\right\rangle}{|y|^{n+2s}}\,dy \\
&= \int_{\mathbb{R}^n}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy - \int_{B_\rho(0)}\frac{\delta(u,x,\sqrt{M}y) - \left\langle D^2u(x)\cdot\sqrt{M}y,\sqrt{M}y\right\rangle}{|y|^{n+2s}}\,dy \\
&\quad - \int_{B_R(0)\setminus B_\rho(0)}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy - \int_{\mathbb{R}^n\setminus B_R(0)}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy.
\end{aligned}$$

By (3.2)-(3.5) again, for any $M \in \mathcal{M}$, we have

$$\begin{aligned}
\frac{w_n}{2-2s}\rho^{2-2s}\operatorname{trace}\left(MD^2u(x)\right) \le{}& \int_{\mathbb{R}^n}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy + \varepsilon\frac{w_n}{2(1-s)}\rho^{2-2s}\lambda_{\max}(M) \\
&+ \frac{2\omega_n}{s}\|u\|_{L^\infty(B_{\lambda_{\max}^{\frac{1}{2}}(M)R}(x))}\cdot\left(\rho^{-2s} - R^{-2s}\right) \\
&+ \frac{2Lw_nR^{1-2s}\lambda_{\max}^{\frac{1}{2}}(M)}{2s-1}.
\end{aligned}$$

By the definition of $F_s[u](x)$, for any $\varepsilon' > 0$, there exists $M_{\varepsilon'} \in \mathcal{M}$ such that

$$\frac{1}{2}\int_{\mathbb{R}^n}\frac{\delta\left(u,x,\sqrt{M_{\varepsilon'}}y\right)}{|y|^{n+2s}}\,dy \le F_s[u](x) + \varepsilon'.$$

Substituting this into the inequality,

$$\begin{aligned}
\frac{w_n}{4(1-s)}\rho^{2-2s}\operatorname{trace}\left(M_{\varepsilon'}D^2u(x)\right) \le{}& F_s[u](x) + \varepsilon' + \varepsilon\frac{w_n}{4(1-s)}\rho^{2-2s}\lambda_{\max}\left(M_{\varepsilon'}\right) \\
&+ \frac{2\omega_n}{s}\|u\|_{L^\infty(B_{\lambda_{\max}^{\frac{1}{2}}(M)R}(x))}\cdot\left(\rho^{-2s} - R^{-2s}\right) \\
&+ \frac{2Lw_nR^{1-2s}\lambda_{\max}^{\frac{1}{2}}(M_{\varepsilon'})}{2s-1}.
\end{aligned}$$

Then,

$$\begin{aligned}\frac{w_n}{4(1-s)}\rho^{2-2s}\inf_{M\in\mathcal{M}}\left\{\operatorname{trace}\left(MD^2u(x)\right)\right\} \le{}& F_s[u](x)+\varepsilon'+\varepsilon\frac{w_n}{4(1-s)}\rho^{2-2s}\lambda_{\max}\left(M_{\varepsilon'}\right)\\ &+\frac{2\omega_n}{s}\|u\|_{L^\infty(B_{\lambda_{\max}^{\frac{1}{2}}(M)R}(x))}\cdot\left(\rho^{-2s}-R^{-2s}\right)\\ &+\frac{2Lw_nR^{1-2s}\lambda_{\max}^{\frac{1}{2}}(M_{\varepsilon'})}{2s-1}.\end{aligned}$$

Multiplying both sides by $(1-s)$ and taking the limit as $s\to 1$,

$$\frac{w_n}{4}\inf_{M\in\mathcal{M}}\left\{\operatorname{trace}\left(MD^2u(x)\right)\right\}\le\lim_{s\to1}\left((1-s)F_s[u](x)\right)+\varepsilon\frac{w_n}{4}\lambda_{\max}\left(M_{\varepsilon'}\right)+\varepsilon'.$$

By the arbitrariness of ε and ε',

$$\frac{w_n}{4}\inf_{M\in\mathcal{M}}\left\{\operatorname{trace}\left(MD^2u(x)\right)\right\}\le\lim_{s\to1}\left((1-s)F_s[u](x)\right).$$

Thus, the equality (1.9) holds for any admissible function $u\in C^2$.

Next, we consider the case that u is non-smooth. Let ψ be any function satisfying that

$$\psi(x)=u(x);\quad \psi(y)>u(y)\ (\text{or } \psi(y)<u(y))\ \forall y\in\mathcal{N}\setminus\{x\},$$

where $\mathcal{N}$ is a neighborhood of x and $\psi\in C^2(\overline{\mathcal{N}})$. Define

$$v(x):=\begin{cases}\psi(x), & x\in\mathcal{N}\\ u(x), & x\in\mathbb{R}^n\setminus\mathcal{N}\end{cases}$$

Taking $B_\rho(0)\subset\mathcal{N}$, similar to the classical proof, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{\delta(v,x,y)}{\left|\sqrt{M}^{-1}y\right|^{n+2s}}\det\sqrt{M}^{-1}\,dy\\ =&\int_{B_\rho(0)}\frac{\left\langle D^2\psi(x)\sqrt{M}y,\sqrt{M}y\right\rangle}{|y|^{n+2s}}\,dy\\ &+\int_{B_\rho(0)}\frac{\delta(\psi,x,\sqrt{M}y)-\left\langle D^2\psi(x)\sqrt{M}y,\sqrt{M}y\right\rangle}{|y|^{n+2s}}\,dy\\ &+\int_{B_R(0)\setminus B_\rho(0)}\frac{\delta(v,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy+\int_{\mathbb{R}^n\setminus B_R(0)}\frac{\delta(u,x,\sqrt{M}y)}{|y|^{n+2s}}\,dy.\end{aligned}$$

By analogous arguments to the classical case, we can prove

$$\lim_{s\to1}\left((1-s)F_s[v](x)\right)\ge\frac{w_n}{4}\inf_{M\in\mathcal{M}}\operatorname{trace}\left(MD^2\psi(x)\right)$$

and

$$\lim_{s\to1}\left((1-s)F_s[v](x)\right)\le\frac{w_n}{4}\inf_{M\in\mathcal{M}}\operatorname{trace}\left(MD^2\psi(x)\right).$$

Thus, (1.9) holds in viscosity sense. $\square$

4. Strict ellipticity

In this section, we prove the strict ellipticity of (1.8) with

$$\mathcal{M} = \mathcal{M}_k := \{M = \{M^{ij}\} : \text{ there exists } T \in \Gamma_k \text{ such that } M^{ij} = (\sigma_k^{1/k})^{ij}(T)\}.$$

Indeed, we shall prove Theorem 1.3.

Without loss of generality, we set $x = 0$.

Lemma 4.1. *Let u be a function satisfying the conditions in Theorem 1.3 but not necessary being convex. Then there exist positive constants μ_0 depending only on η_0, n, k, s, SC and L and μ_1 depending only on n, k, s, SC and L such that*

$$0 < \mu_0 \le (1-s)\int_{\mathbb{R}^{n-k+1}} \frac{u(y_1e_1 + \cdots + y_{n-k+1}e_{n-k+1}) - u(0)}{|\bar{y}|^{n-k+1+2s}} d\bar{y} \le \mu_1, \tag{4.1}$$

for any choice $e_1, \ldots, e_{n-k+1}$ of $(n-k+1)$ orthonormal vectors in $\mathbb{R}^n$. Here $\bar{y}$ denotes $\bar{y} = (y_1, \ldots, y_{n-k+1}) \in \mathbb{R}^{n-k+1}$.

Proof. For any small $\epsilon > 0$, let

$$B = \mathrm{diag}\{\underbrace{0, \ldots, 0}_{n-k}, \epsilon^{k-1}, \underbrace{\frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon}}_{k-1}\} \in \Gamma_k.$$

We find $S_k(B) = 1$ and

$$M := \{S_k^{ij}\} = \frac{1}{k}\mathrm{diag}\{\underbrace{g(\epsilon), \ldots, g(\epsilon)}_{n-k}, \frac{1}{\epsilon^{k-1}}, \underbrace{\epsilon, \ldots, \epsilon}_{k-1}\} \in \mathcal{M}_k,$$

where $g(\epsilon) = \frac{1}{\epsilon^{k-1}} + (k-1)\epsilon$. It follows that

$$\sqrt{M}^{-1} = \sqrt{k}\mathrm{diag}\{\underbrace{g^{-1/2}(\epsilon), \ldots, g^{-1/2}(\epsilon)}_{n-k}, \epsilon^{\frac{k-1}{2}}, \underbrace{\epsilon^{-1/2}, \ldots, \epsilon^{-1/2}}_{k-1}\}.$$

Denote $y'' = (y_1, \ldots, y_{n-k+1})$, $y' = (y_{n-k+2}, \ldots, y_n)$ and $y = (y'', y')$. We have, by (1.11),

$$\begin{aligned}
\frac{\eta_0}{1-s} \le\, & g^{-\frac{n-k}{2}}(\epsilon)k^{\frac{n}{2}} \int_{\mathbb{R}^n} \frac{u(y'', y') - u(0)}{\left(g^{-1}(\epsilon)(y_1^2 + \ldots + y_{n-k}^2) + \epsilon^{k-1}y_{n-k+1}^2 + \epsilon^{-1}|y'|^2\right)^{\frac{n+2s}{2}}} dy \\
=\, & g^{-\frac{n-k}{2}}(\epsilon)k^{\frac{n}{2}} \int_{\mathbb{R}^n} \frac{u(y'', y') - u(y'', 0)}{\left(g^{-1}(\epsilon)(y_1^2 + \ldots + y_{n-k}^2) + \epsilon^{k-1}y_{n-k+1}^2 + \epsilon^{-1}|y'|^2\right)^{\frac{n+2s}{2}}} dy \\
& + g^{-\frac{n-k}{2}}(\epsilon)k^{\frac{n}{2}} \int_{\mathbb{R}^n} \frac{u(y'', 0) - u(0)}{\left(g^{-1}(\epsilon)(y_1^2 + \ldots + y_{n-k}^2) + \epsilon^{k-1}y_{n-k+1}^2 + \epsilon^{-1}|y'|^2\right)^{\frac{n+2s}{2}}} dy \\
:=\, & I_1 + I_2.
\end{aligned} \tag{4.2}$$

Now we estimate I_1. By the Lipschitz continuity and semi-concavity of u, we have

$$I_1 \le \frac{g^{-\frac{n-k}{2}}(\epsilon)k^{\frac{n}{2}}}{2} \int_{\mathbb{R}^n} \frac{\min\{2L|y'|, SC|y'|^2\}}{\left(g^{-1}(\epsilon)(y_1^2 + \ldots + y_{n-k}^2) + \epsilon^{k-1}y_{n-k+1}^2 + \epsilon^{-1}|y'|^2\right)^{\frac{n+2s}{2}}} dy.$$

Using the change of variables

$$\begin{cases} z' = y' \\ z_j = \frac{y_j}{|y'|}\sqrt{\epsilon} g^{-\frac{1}{2}}(\epsilon), j = 1, \ldots, n-k \\ z_{n-k+1} = \frac{y_{n-k+1}}{|y'|}\epsilon^{\frac{k}{2}} \end{cases}$$

we get

$$dz = \frac{dy}{|z'|^{n-k+1}}\epsilon^{\frac{n}{2}} g^{-\frac{n-k}{2}}(\epsilon)$$

and

$$\begin{aligned} I_1 &\leq \frac{\epsilon^{-\frac{n}{2}} k^{\frac{n}{2}}}{2} \int_{\mathbb{R}^n} \frac{\min\{2L|z'|, SC|z'|^2\}|z'|^{n-k+1}}{\epsilon^{-\frac{n+2s}{2}}|z'|^{n+2s}(1+|z''|^2)^{\frac{n+2s}{2}}} dz \\ &= \frac{\epsilon^s k^{\frac{n}{2}}}{2} \int_{\mathbb{R}^{k-1}} \frac{\min\{2L|z'|, SC|z'|^2\}}{|z'|^{k+2s-1}} dz' \int_{\mathbb{R}^{n-k+1}} \frac{1}{(1+|z''|^2)^{\frac{n+2s}{2}}} dz'' \\ &\leq \frac{C_1 \epsilon^s}{1-s}, \end{aligned} \tag{4.3}$$

where C_1 is the positive constant given by

$$C_1 := \frac{(1-s)k^{\frac{n}{2}}}{2} \int_{\mathbb{R}^{k-1}} \frac{\min\{2L|z'|, SC|z'|^2\}}{|z'|^{k+2s-1}} dz' \int_{\mathbb{R}^{n-k+1}} \frac{1}{(1+|z''|^2)^{\frac{n+2s}{2}}} dz'' < \infty.$$

Now we consider I_2. Let

$$\begin{cases} z'' = y'' \\ z_j = \sqrt{\frac{g(\epsilon)}{\epsilon}} \frac{y_j}{|y''|}, j = n-k+2, \ldots, n. \end{cases}$$

We have

(4.4)

$$\begin{aligned} I_2 = &\int_{\mathbb{R}^n} \frac{\epsilon^{\frac{k-1}{2}} g^{-\frac{n-1}{2}}(\epsilon) k^{\frac{n}{2}} |z''|^{k-1}(u(z'',0)-u(0))}{\left(g^{-1}(\epsilon)|z''|^2(1+|z'|^2) + (\epsilon^{k-1} - g^{-1}(\epsilon))z_{n-k+1}^2\right)^{\frac{n+2s}{2}}} dz \\ \leq &k^{\frac{n}{2}} \int_{\{u(z'',0)\geq u(0)\}} \epsilon^{-(k-1)s} h^{\frac{n+2s}{2}}(\epsilon) \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}(1+|z'|^2)^{\frac{n+2s}{2}}} dz \\ &+ k^{\frac{n}{2}} \int_{\{u(z'',0)<u(0)\}} \epsilon^{-(k-1)s} h^{-\frac{n-1}{2}}(\epsilon) \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}(1+|z'|^2)^{\frac{n+2s}{2}}} dz \\ = &k^{\frac{n}{2}} \epsilon^{-(k-1)s} h^{-\frac{n-1}{2}}(\epsilon) \int_{\mathbb{R}^n} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}(1+|z'|^2)^{\frac{n+2s}{2}}} dz \\ &+ k^{\frac{n}{2}} \epsilon^{-(k-1)s} \left(h^{\frac{n+2s}{2}}(\epsilon) - h^{-\frac{n-1}{2}}(\epsilon)\right) \int_{\{u(z'',0)\geq u(0)\}} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}(1+|z'|^2)^{\frac{n+2s}{2}}} dz \\ = &C_2 \int_{\mathbb{R}^{n-k+1}} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}} dz'' \\ &+ k^{\frac{n}{2}} \epsilon^{-(k-1)s} \left(h^{\frac{n+2s}{2}}(\epsilon) - h^{-\frac{n-1}{2}}(\epsilon)\right) \int_{\{u(z'',0)\geq u(0)\}} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}(1+|z'|^2)^{\frac{n+2s}{2}}} dz, \end{aligned}$$

where the constants $h(\epsilon) := 1 + (k-1)\epsilon^k$ and

$$C_2 = C_2(n,k,s,\epsilon) := k^{\frac{n}{2}} \epsilon^{-(k-1)s} h^{-\frac{n-1}{2}}(\epsilon) \int_{\mathbb{R}^{k-1}} \frac{1}{(1+|z'|^2)^{\frac{n+2s}{2}}} dz'.$$

Note that

$$h^{\frac{n+2s}{2}}(\epsilon) - h^{-\frac{n-1}{2}}(\epsilon) \le C_3\epsilon^k,$$

for some positive constant C_3 depending only on n and k. By the Lipschitz continuity and semi-concavity of u again, we find

$$k^{\frac{n}{2}} \int_{\{u(z'',0)\ge u(0)\}} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}(1+|z'|^2)^{\frac{n+2s}{2}}} dz \le \frac{C_4}{1-s},$$

for some positive constant C_4 depending only on n, k, s, L and C. Thus, (4.4) becomes

$$I_2 \le C_2 \int_{\mathbb{R}^{n-k+1}} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}} dz'' + \frac{C_3C_4\epsilon^s}{1-s}. \tag{4.5}$$

Combing (4.2), (4.3) and (4.4), we obtain

$$C_2 \int_{\mathbb{R}^{n-k+1}} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}} dz'' \ge \frac{\eta_0}{1-s} - \frac{(C_1+C_3C_4)\epsilon^s}{1-s}.$$

Fixing ϵ sufficiently small, we get a positive constant μ_0 depending only on η_0, n, k, s, C and L such that

$$\int_{\mathbb{R}^{n-k+1}} \frac{u(z'',0)-u(0)}{|z''|^{n+2s-k+1}} dz'' \ge \frac{\mu_0}{1-s}.$$

By an orthonormal transformation, it is easy to show that if $e_1, \dots, e_{n-k+1}$ are orthonormal vectors in $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^{n-k+1}} \frac{u(y_1e_1+\cdots+y_{n-k+1}e_{n-k+1})-u(0)}{|\bar{y}|^{n-k+1+2s}} d\bar{y} \ge \frac{\mu_0}{1-s}.$$

The other part of (4.1) can be proved easily by the Lipschitz continuity and semi-concavity of u. □

Remark 4.2. Special cases of Lemma 4.1 with $k=n$ and $k=2$ were proved in [1] and [17] respectively.

With the purpose to prove Theorem 1.3, We will show when $\lambda_{\min}(M) \to 0$,

$$(1-s)\int_{\mathbb{R}^n} \frac{u(y)-u(0)}{\left|\sqrt{M}^{-1}y\right|^{n+2s}} \det \sqrt{M}^{-1}\, dy \to +\infty.$$

On the other hand, we will show that there exists $\tilde{C} > 0$ such that

$$(1-s)\int_{\mathbb{R}^n} \frac{u(y)-u(0)}{\left|\sqrt{M}^{-1}y\right|^{n+2s}} \det \sqrt{M}^{-1}\, dy \le \tilde{C}.$$

This contradiction will complete the proof.

Proof of Theorem 1.3. We first consider the case that M is diagonal. Let $M \in \mathcal{M}_k$ be given by

$$M = \operatorname{diag}\{f_1, \dots, f_n\} \text{ with } f_1 \ge \cdots \ge f_n = \epsilon.$$

Let

$$\sqrt{M}^{-1} = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}.$$

Thus,

$$\lambda_1 \le \cdots \le \lambda_n = \epsilon^{-1/2}.$$

By Proposition 2.1 (4) of [16], we see

$$\prod_{j=1}^{n} f_j \geq \frac{1}{n^n}(C_n^k)^{\frac{n}{k}} := c_0 > 0$$

and thus,

$$0 < c_0 \leq f_1 \cdots f_n \leq f_1^{n-1} f_n = \epsilon f_1^{n-1}.$$

It follows that

$$f_1 \geq c_1 \epsilon^{-\frac{1}{n-1}},$$

where $c_1 := c_0^{1/(n-1)}$. Without loss of generality, we assume $\eta = (\eta_1, \ldots, \eta_n) \in \Gamma_k$ with $\eta_1 \leq \cdots \leq \eta_n$ such that $\sigma_k(\eta) = 1$ and $f_i = \frac{1}{k}\sigma_{k-1;i}(\eta)$ for $i = 1, \ldots, n$. Theorem 1 in [12] shows that

$$f_i = \frac{1}{k}\sigma_{k-1;i}(\eta) \geq \theta f_1 \geq \theta c_1 \epsilon^{-\frac{1}{n-1}}, \ i = 1, \ldots, n-k+1$$

for some positive constant θ depending only on n and k. Therefore,

$$\frac{\lambda_1}{\lambda_{n-k+1}} = \sqrt{\frac{f_{n-k+1}}{f_1}} \geq \sqrt{\theta_0}, \quad \frac{1}{\lambda_{n-k+1}} \geq \sqrt{\theta c_1}\epsilon^{-\frac{1}{2(n-1)}}. \tag{4.6}$$

Denote $y'' = (y_1, \ldots, y_{n-k+1})$, $y' = (y_{n-k+2}, \ldots, y_n)$ and $y = (y'', y')$, we find

$$\begin{aligned}
I &:= \prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{(\lambda_1^2 y_1^2 + \cdots + \lambda_n^2 y_n^2)^{\frac{n+2s}{2}}} dy \\
&= \prod_{j=1}^{n} \lambda_j \left[\int_{\mathbb{R}^n} \frac{u(y'', y') - u(y'', 0)}{(\lambda_1^2 y_1^2 + \cdots + \lambda_n^2 y_n^2)^{\frac{n+2s}{2}}} dy + \int_{\mathbb{R}^n} \frac{u(y'', 0) - u(0)}{(\lambda_1^2 y_1^2 + \cdots + \lambda_n^2 y_n^2)^{\frac{n+2s}{2}}} dy\right] \\
&:= I_1' + I_2'.
\end{aligned}$$

We first consider I_1'. Let

$$\begin{cases} z' = y', \\ z_j = \frac{\lambda_j y_j}{(\lambda_{n-k+2}^2 y_{n-k+2}^2 + \cdots + \lambda_n^2 y_n^2)^{\frac{1}{2}}}, \quad j = 1, \cdots, n-k+1. \end{cases}$$

We have

$$dz = \frac{\lambda_1 \cdots \lambda_{n-k+1}}{\left(\lambda_{n-k+2}^2 y_{n-k+2}^2 + \cdots + \lambda_n^2 y_n^2\right)^{\frac{n-k+1}{2}}} dy$$

and

$$\begin{aligned}
I_1' &= \prod_{j=n-k+2}^{n} \lambda_j \int_{\mathbb{R}^n} \frac{u(z'', z') - u(z'', 0)}{\left(\lambda_{n-k+2}^2 z_{n-k+2}^2 + \cdots + \lambda_n^2 z_n^2\right)^{\frac{2s+k-1}{2}} \cdot (1 + |z''|^2)^{\frac{n+2s}{2}}} dz \\
&= \int_{\mathbb{R}^{n-k+1}} \frac{1}{(1 + |z''|^2)^{\frac{n+2s}{2}}} \cdot \int_{\mathbb{R}^{k-1}} J(z'', z') dz' dz'',
\end{aligned} \tag{4.7}$$

where

$$J(z'', z') := \prod_{j=n-k+2}^{n} \lambda_j \frac{u(z'', z') - u(z'', 0)}{\left(\lambda_{n-k+2}^2 z_{n-k+2}^2 + \cdots + \lambda_n^2 z_n^2\right)^{\frac{2s+k-1}{2}}}.$$

Since u is convex in $\mathbb{R}^n$, there exists a vector $a = a(z'') \in \mathbb{R}^n$ such that $u(z'', z') - u(z'', 0) \geq a \cdot z'$ for all $z' \in \mathbb{R}^{k-1}$. Therefore, we have

$$I_1^{'} \geq 0. \tag{4.8}$$

Next, we consider $I_2^{'}$. Let

$$\begin{cases} z'' = y'', \\ z_j = \frac{\lambda_j y_j}{\left(\lambda_1^2 y_1^2 + \cdots + \lambda_{n-k+1}^2 y_{n-k+1}^2\right)^{\frac{1}{2}}}, j = n-k+2, \ldots, n. \end{cases}$$

We have

$$\begin{aligned}
I_2^{'} = \prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^n} & \frac{(u(z'', 0) - u(0)) \cdot \left(\lambda_1^2 z_1^2 + \cdots + \lambda_{n-k+1}^2 z_{n-k+1}^2\right)^{\frac{k-1}{2}} \cdot \frac{1}{\lambda_{n-k+2} \cdots \lambda_n}}{\left[\lambda_1^2 z_1^2 + \cdots + \lambda_{n-k+1}^2 z_{n-k+1}^2 + \left(\lambda_1^2 z_1^2 + \cdots + \lambda_{n-k+1}^2 z_{n-k+1}^2\right) |z'|^2\right]^{\frac{n+2s}{2}}} dz \\
= \prod_{j=1}^{n-k+1} \lambda_j \int_{\mathbb{R}^n} & \frac{u(z'', 0) - u(0)}{\left(\lambda_1^2 z_1^2 + \cdots + \lambda_{n-k+1}^2 z_{n-k+1}^2\right)^{\frac{n+2s-k+1}{2}} \cdot (1 + |z'|^2)^{\frac{n+2s}{2}}} dz \\
\geq \prod_{j=1}^{n-k+1} \lambda_j \int_{\mathbb{R}^{k-1}} & \frac{dz'}{(1 + |z'|^2)^{\frac{n+2s}{2}}} \int_{\mathbb{R}^{n-k+1}} \frac{u(z'', 0) - u(0)}{\lambda_{n-k+1}^{n+2s-k+1} |z''|^{n+2s-k+1}} dz'' \\
\geq \left(\frac{\lambda_1}{\lambda_{n-k+1}}\right)^{n-k+1} & \cdot \frac{1}{\lambda_{n-k+1}^{2s}} \cdot \frac{\mu_0}{1-s} \int_{\mathbb{R}^{k-1}} \frac{dz'}{(1 + |z'|^2)^{\frac{n+2s}{2}}},
\end{aligned}$$

where the last inequality follows from Lemma 4.1. By (4.6), we have

$$I_2^{'} \geq \frac{\mu_3}{1-s} \epsilon^{-\frac{s}{n-1}}. \tag{4.9}$$

It follows that

$$(1 - \epsilon) I = (1 - \epsilon)(I_1^{'} + I_2^{'}) \geq \mu_3 \epsilon^{-\frac{s}{n-1}} \to \infty \text{ as } \epsilon \to 0.$$

Now we consider general $M \in \mathcal{M}_k$ with $\lambda_{\min}(M) = \epsilon$. Since $\sqrt{M}$ is symmetric, there exists an orthogonal matrix P, such that

$$P^T \sqrt{M}^{-1} P = J := \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}.$$

Let $\tilde{u}(y) = u(Py)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{u(y) - u(0)}{\left|\sqrt{M}^{-1} y\right|^{n+2s}} \det \sqrt{M}^{-1} dy &= \int_{\mathbb{R}^n} \frac{u(Py) - u(0)}{\left(\sum_{j=1}^n \lambda_j^2 y_j^2\right)^{\frac{n+2s}{2}}} \prod_{j=1}^n \lambda_j \, dy \\
&= \int_{\mathbb{R}^n} \frac{\tilde{u}(y) - \tilde{u}(0)}{\left(\sum_{j=1}^n \lambda_j^2 y_j^2\right)^{\frac{n+2s}{2}}} \prod_{j=1}^n \lambda_j \, dy \\
&\geq \frac{\mu_3}{1-s} \epsilon^{-\frac{s}{n-1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{u(y)-u(0)}{\left|\sqrt{M}^{-1}y\right|^{n+2s}} \det \sqrt{M}^{-1}\, dy\\
\leq{}& \frac{1}{2}\int_{\mathbb{R}^n} \frac{\min\{2L|y|, SC|y|^2\}}{\left|\sqrt{M_0}^{-1}y\right|^{n+2s}} \det \sqrt{M_0}^{-1}\, dy\\
\leq{}& \frac{\tilde{C}}{1-s}
\end{aligned}$$

which is a contradiction if ϵ is chosen small enough. Consequently, there exists a positive constant θ such that

$$F_s[u](x) = F_s^{\theta}[u](x)$$

and Theorem 1.3 is proved. □

Proof of Theorem 1.4. In the case $k=2$, (4.7) becomes

$$I_1^{'} = \lambda_n \int_{\mathbb{R}^n} \frac{u(z'', z_n) - u(z'', 0)}{\lambda_n^{1+2s}|z_n|^{1+2s}\cdot(1+|z''|^2)^{\frac{n+2s}{2}}} dz.$$

Thus, by the Lipschitz continuity and semi-concavity of u, we obtain

$$\begin{aligned}
|I_1^{'}| &\leq \frac{\lambda_n^{-2s}}{2}\int_{\mathbb{R}} \frac{\min\{2L|z_n|, SC|z_n|^2\}}{|z_n|^{1+2s}} dz_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|z''|^2)^{\frac{n+2s}{2}}} dz''\\
&= \frac{\tilde{C}_1 \epsilon^s}{1-s},
\end{aligned} \tag{4.10}$$

where $\tilde{C}_1$ is the positive constant given by

$$\tilde{C}_1 := \frac{(1-s)}{2}\int_{\mathbb{R}} \frac{\min\{2L|z_n|, SC|z_n|^2\}}{|z_n|^{1+2s}} dz_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|z''|^2)^{\frac{n+2s}{2}}} dz'' < \infty.$$

Combining (4.9) and (4.10), we get

$$I \geq \frac{\mu_3}{2(1-s)}\epsilon^{-\frac{s}{n-1}}$$

provided ϵ is sufficiently small. Therefore, Theorem 1.4 follows as Theorem 1.3. □

We finish this paper by some remarks. With the purpose to remove the convexity condition in Theorem 1.3, we reconsider $I_1^{'}$ with $k \geq 3$. The first recall the following Proposition proved in [1].

Proposition 4.3 (Proposition B.1 of [1])**.** *Let $\lambda_j > 0$ for $j = 1, \ldots, r$. Then,*

$$\frac{\omega_r}{r} \leq \frac{\prod_{j=1}^r \lambda_j}{\sum_{j=1}^r \lambda_j^{-2s}} \int_{\partial B_1^r(0)} \frac{1}{(\sum_{j=1}^r \lambda_j^2 x_j^2)^{\frac{r+2s}{2}}} d\mathcal{H}^{r-1}(x) \leq \frac{\pi^{r/2}}{s\Gamma(2-s)\Gamma(\frac{r}{2}+s)}. \tag{4.11}$$

Since u is Lipschitz continuous and semi-concave, we have

$$\begin{aligned}\int_{\mathbb{R}^{k-1}} |J(z'', z')| dz' &\leq \prod_{j=n-k+2}^{n} \lambda_j \int_{\mathbb{R}^{k-1}} \frac{\min\{2L|z'|, SC|z'|^2\}}{\left(\lambda_{n-k+2}^2 z_{n-k+2}^2 + \cdots + \lambda_n^2 z_n^2\right)^{\frac{2s+k-1}{2}}} dz' \\ &= \int_0^\infty \frac{\min\{2Lr, SCr^2\}}{r^{2s+1}} dr \int_{\partial B_1^{k-1}(0)} \frac{\prod_{j=n-k+2}^n \lambda_j}{(\sum_{j=n-k+2}^n \lambda_j^2 x_j^2)^{\frac{k-1+2s}{2}}} d\mathcal{H}^{k-2}(x) \\ &= \tilde{C}_2 \int_{\partial B_1^{k-1}(0)} \frac{\prod_{j=n-k+2}^n \lambda_j}{(\sum_{j=n-k+2}^n \lambda_j^2 x_j^2)^{\frac{k-1+2s}{2}}} d\mathcal{H}^{k-2}(x),\end{aligned}$$

where $\tilde{C}_2$ is the positive constant defined by

$$\tilde{C}_2 := \int_0^\infty \frac{\min\{2Lr, SCr^2\}}{r^{2s+1}} dr.$$

By (4.11), we find

$$\int_{\partial B_1^{k-1}(0)} \frac{\prod_{j=n-k+2}^n \lambda_j}{(\sum_{j=n-k+2}^n \lambda_j^2 x_j^2)^{\frac{k-1+2s}{2}}} d\mathcal{H}^{k-2}(x) = O\Big(\sum_{j=n-k+2}^{n} \lambda_j^{-2s}\Big),$$

however, $\sum_{j=n-k+2}^n \lambda_j^{-2s}$ can be very large. For instance, let

$$\tilde{B} = \mathrm{diag}\{\underbrace{0, \ldots, 0}_{n-k}, \underbrace{(k\epsilon)^{\frac{1}{k-1}}, \ldots, (k\epsilon)^{\frac{1}{k-1}}}_{k-1}, \frac{1}{k\epsilon}\} \in \Gamma_k.$$

We find $S_k(\tilde{B}) = 1$ and

$$\begin{aligned}\tilde{M} &:= \{S_k^{ij}\} \\ &= \mathrm{diag}\{\underbrace{\epsilon + (k-1)\tilde{g}(\epsilon), \ldots, \epsilon + (k-1)\tilde{g}(\epsilon)}_{n-k}, \underbrace{\tilde{g}(\epsilon), \ldots, \tilde{g}(\epsilon)}_{k-1}, \epsilon\} \in \mathcal{M}_k,\end{aligned}$$

where

$$\tilde{g}(\epsilon) := \frac{(k\epsilon)^{-\frac{1}{k-1}}}{k} \sim \epsilon^{-\frac{1}{k-1}}$$

Thus,

$$\sqrt{\tilde{M}}^{-1} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$$

with $\lambda_1 \leq \cdots \leq \lambda_n$ and

$$\lambda_{n-k+1} = \cdots = \lambda_{n-1} = \tilde{g}^{-1/2}(\epsilon), \lambda_n = \epsilon^{-1/2}.$$

It follows that

$$\sum_{j=n-k+2}^{n} \lambda_j^{-2s} \sim \epsilon^{-\frac{s}{k-1}} \to \infty \text{ as } \epsilon \to 0.$$

It means that we cannot use similar arguments as (4.10) to deal with the case $k \geq 3$.

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