Generalized Berwald Projective Weyl $(GB\widetilde{W})$ metrics

Nasrin Sadeghzadeh[1*]

[1*]Department of Mathematics, University of Qom, Alghadir Bld., Qom, Iran.

Corresponding author(s). E-mail(s): nsadeghzadeh@qom.ac.ir;

This article has been published in *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, DOI:10.1007/s40010-024-00896-6.

1 Introduction

Finsler spaces possess numerous projective invariants that are widely recognized, including Douglas curvature, Weyl curvature [2], and generalized Douglas-Weyl (GDW) curvature [3]. Additionally, Akbar-Zadeh introduced H-curvature as a C-projective invariant [1], and a new C-projectively invariant quantity, known as $\widetilde{W}$-curvature or C-projective Weyl curvature, has been introduced to characterize Finsler metrics of constant flag curvature ($n > 2$) [7].
The equations $D_j{}^i{}_{kl} = 0$, $W_j{}^i{}_{kl} = 0$ and $D_j{}^i{}_{kl|m}y^m = T_{jkl}y^i$ for some tensors T_{jkl}, are projectively invariant. The symbol $|$ denotes the horizontal covariant derivatives with respect to the Berwald connection.
Put simply, if a Finsler metric F satisfies one of the aforementioned equations, then any Finsler metric that is projectively equivalent to F must satisfy the same equations.

The study of GDW metrics is presented in [8], where it is demonstrated that all Finsler metrics of scalar curvature (Weyl metrics) belong to this class of metrics. Furthermore, B*á*cs*ó*, in [3], shows that GDW metrics are closed under projective relations. This paper introduces a novel quantity in Finsler geometry, called the generalized Berwald projective Weyl $(GB\widetilde{W})$ curvature, which is a C-projective invariant. For a manifold M, let $\mathrm{GB}\widetilde{W}(M)$ denote the class of all Finsler metrics that satisfy

$$B_j{}^i{}_{kl} = \beta_j{}^i{}_{kl} + b_{jkl}y^i,$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$; where $\beta_j{}^i{}_{kl|m}y^m = 0$. This leads to a natural question: "How extensive is $\mathrm{GB}\widetilde{W}(M)$, and what types of interesting metrics belong to this class?"

It is evident that all Berwald metrics are included in this class; however, Berwald metrics are not considered C-projective invariants. It is noteworthy that the class of Berwald metrics is a proper subset of the class of $GB\widetilde{W}$ metrics, which is illustrated in the following example.

Example 1.1 [10] The standard Funk metric on the Euclidean unit ball is defined as follows

$$F = \frac{\sqrt{\mathsf{y}^2 - (\mathsf{x}^2\mathsf{y}^2 - <x,y>^2)}}{1-\mathsf{x}^2} + \frac{<x,y>}{1-\mathsf{x}^2}, \quad y \in T_xB^n \simeq R^n$$

where $<,>$ and $|\,.\,|$ denote the Euclidean inner product and norm on R^n, respectively. F is a Randers metric of constant flag curvature, and consequently $GB\widetilde{W}$ metric. It is easily seen that β is a closed 1-form. By a simple calculation, it follows that $H = 0$ while F is not Berwald metric.

On the other hand, it has been proven that all Finsler metrics of dimension $n > 2$ with constant flag curvature ($\widetilde{W} = 0$) belong to this class of Finsler metrics. As an illustration, consider the following Finsler metric F_ε (Example 1.2), which represents a particular family of Bryant's metrics expressed in a local coordinate system. This metric is locally projectively flat and has a constant flag curvature, thereby satisfying $\widetilde{W} = 0$. Consequently, it is a $GB\widetilde{W}$ metric.

Example 1.2 [9] Let ε be an arbitrary number with $|\,\varepsilon\,| < 1$. Let

$$F_\varepsilon := \frac{\sqrt{\Psi\big(\frac{1}{2}(\sqrt{\Phi^2 + (1-\varepsilon^2)\mathsf{y}^4} + \Phi)\big) + (1-\varepsilon^2) <x,y>^2} + \sqrt{1-\varepsilon^2} <x,y>}{\Psi},$$

where,

$$\Phi := \varepsilon\mathsf{y}^2 + (\mathsf{x}^2\mathsf{y}^2 - <x,y>^2), \qquad \Psi := 1 + 2\varepsilon\mathsf{x}^2 + \mathsf{x}^4.$$

$F_\varepsilon = F_\varepsilon(x,y)$ is a Finsler metric on R^n. Note that if $\varepsilon = 1$, then $F_1 = \alpha_{+1}$ is the spherical metric on R^n.

Next, we notice the metrics, which are not of constant curvature (i.e. $\widetilde{W} \neq 0$), are still of $GB\widetilde{W}$ type. Based on Lemma 3.5, it is clear that such a metric could not be of scalar curvature in dimension $n > 2$. In other words, every Finsler metric of scalar curvature and non-constant curvature is not of $GB\widetilde{W}$

type while it is GDW metric. This shows that the class of $GB\widetilde{W}$ metric is the proper subset of the class of GDW. The following example illuminates this point.

Example 1.3 [9] Let

$$F = \frac{\sqrt{\left(\mathsf{x}^2 < a, y > -2 < a, x >< x, y >\right)^2 + \mathsf{y}^2\left(1 - \mathsf{a}^2\mathsf{x}^4\right)}}{1 - \mathsf{a}^2\mathsf{x}^4}$$
$$- \frac{\left(\mathsf{x}^2 < a, y > -2 < a, x >< x, y >\right)}{1 - \mathsf{a}^2\mathsf{x}^4},$$

where, a is a constant vector in R^n, which has very important properties. It is of scalar curvature and isotropic S-curvature; however, the flag curvature and the S-curvature are not constant. We have

$$S = (n+1) < a, x > F,$$

Then by lemma 9.1 in [9], one has $E_{ij} = \frac{n+1}{2} < a, x > F_{ij}$. Then $H \neq 0$; it is not of $GB\widetilde{W}$ type while it is of GDW metric.

It will be demonstrated below that the class of GDW metrics encompasses the class of $GB\widetilde{W}$ metrics. However, it is important to note that there exist numerous GDW metrics that are not of the $GB\widetilde{W}$ type.

Theorem 1.1 Every $GB\widetilde{W}$ metric is GDW metric.

Theorem 1.2 The class of $\mathrm{GB}\widetilde{W}(M)$ on Finsler manifold (M, F) is closed under C-projective changes.

The subsequent section establishes that the class of $GB\widetilde{W}$ metrics encompasses all Finsler metrics with a constant flag curvature ($n > 2$). Additionally, an extension of the well-known Numata's theorem is proven.

Theorem 1.3 All non-Riemannian GDW metrics with vanishing Landsberg curvature are of R-quadratic type ($n > 2$).

2 Preliminaries

A Finsler metric on a manifold M is a nonnegative function F on TM with the following properties

1. F is C^∞ on $TM \setminus \{0\}$;
2. $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0,\ \ y \in TM$;

3. For each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\Big[F^2(y+su+tv)\Big] \mid_{s,t=0}, \qquad u,v \in T_xM. \tag{1}$$

At each point $x \in M$, $F_x := F \mid_{T_xM}$, is an Euclidean norm, if and only if $\mathbf{g}_y$ is independent of $y \in T_xM \setminus \{0\}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big] \mid_{t=0}, \qquad u,v,w \in T_xM. \tag{2}$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM\setminus\{0\}}$ is called the *Cartan torsion.* A curve $c(t)$ is called a *geodesic* if it satisfies

$$\frac{d^2c^i}{dt^2} + 2G^i(\dot{c}(t)) = 0, \tag{3}$$

where, $G^i(y)$ denotes local functions on TM given by

$$G^i(y) := \frac{1}{4}g^{il}(y)\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\}, \quad y \in T_xM. \tag{4}$$

G^is called the associated spray to (M,F).
F is called a *Berwald metric* if $G^i(y)$ are quadratic in $y \in T_xM$ for all $x \in M$. Define

$$B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$$
$$B_y(u,v,w) = B_j{}^i{}_{kl}u^jv^kw^l\frac{\partial}{\partial x^i},$$

where, $B^i_{j\,kl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$, and

$$E_y : T_xM \otimes T_xM \to R$$

$$E_y(u,v) = E_{jk}u^jv^k,$$

where, $E_{jk} = \frac{1}{2}B_j{}^m{}_{km}$, $u = u^i\frac{\partial}{\partial x^i}$, $v = v^i\frac{\partial}{\partial x^i}$ and $w = w^i\frac{\partial}{\partial x^i}$. B and E are called the Berwald curvature and the mean Berwald curvature, respectively. F is called a Berwald metric and weakly Berwald (WB) metric if $B = 0$ and $E = 0$, respectively [10]. By means of E-curvature, we can define $\bar{E}$-curvature as follows

$$\bar{E}_y : T_xM \otimes T_xM \otimes T_xM \longrightarrow R$$
$$\bar{E}_y(u,v,w) := \bar{E}_{jkl}(y)u^iv^jw^k = E_{jk|l}u^jv^kw^l.$$

It is worth noting that $\bar{E}_{ijk}$ is not completely symmetric with respect to all three indices. To define the H-curvature, we take the covariant derivative of E along geodesics. Specifically, $H_{ij} = E_{ij|m}y^m$,

$$H_y : T_xM \otimes T_xM \longrightarrow R$$

$$H_y(u,v) := H_{ij}u^iv^j$$

Let

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{1}{n+1}\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}(\frac{\partial G^m}{\partial y^m}y^i).$$

It is easy to verify that $D := D_j{}^i{}_{kl}dx^j \otimes \frac{\partial}{\partial x^i} \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle TM_0. We call D the Douglas tensor, which is a non-Riemannian projective invariant. For two Finsler metrics, F and $\bar{F}$, with the Geodesic coefficients G^i and $\bar{G}^i$, respectively. The diffeomorphism $\phi : F \to \bar{F}$ is called projective mapping if there exists a positive homogeneous scalar function $P(x,y)$ of degree one which is satisfying

$$\bar{G}^i = G^i + Py^i,$$

where, $P = P(x,y)$ is positively y-homogeneous of degree one which called projective factor [4] and [10]. A projective transformation with projective factor P would be C-projective if $Q_{ij} = 0$; where,

$$Q_{ij} = \frac{\partial Q_j}{\partial y^i} - \frac{\partial Q_i}{\partial y^j}, \tag{5}$$

$$Q_i = \frac{\partial P}{\partial x^i} - G^m_i\frac{\partial P}{\partial y^m} - P\frac{\partial P}{\partial y^i}.$$

One could easily show that

$$D_j{}^i{}_{kl} = B_j{}^i{}_{kl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jkl}y^i\}, \tag{6}$$

where $E_{jkl} = \frac{\partial E_{jk}}{\partial y^l}$.

In [13], Weyl introduces a projective invariant for Riemannian metrics. Then Douglas extends Weyl's projective invariant to Finsler metrics [5]. Finsler metrics with vanishing projective Weyl curvature are called Weyl metrics or W metrics. In [11], Szabó proves that Weyl metrics are exactly Finsler metrics of scalar flag curvature. In [7], a new C-projective invariant quantity is defined, namely $\widetilde{W}$ curvature, as follows

$$\widetilde{W}^i_k = K^i_k - \frac{1}{1-n^2}\{y^i\widetilde{K}_{0k} - \delta^i_k\widetilde{K}_{00}\},$$

where, $\widetilde{K}_{jk} = nK_{jk} + K_{kj} + y^r\frac{\partial K_{kr}}{\partial y^j}$.

$\widetilde{W}^i{}_k$ is called projective Weyl curvature or $\widetilde{W}$ curvature which is another candidate for characterizing the Finsler metrics of constant flag curvature.

There is another projective invariant equation in Finsler geometry for some tensors T_{jkl}

$$D_j{}^i{}_{kl|m}y^m = T_{jkl}y^i,$$

where, ${D_j}^i{}_{kl|m}$ denotes the horizontal covariant derivatives of ${D_j}^i{}_{kl}$ with respect to the Berwald connection of F. These metrics are called GDW metrics.
Now a new C-projective invariant equation is introduced as follows

$$
{B_j}^i{}_{kl} = {\beta_j}^i{}_{kl} + b_{jkl}y^i, \tag{7}
$$

for some tensors b_{jkl} and ${\beta_j}^i{}_{kl}$ where ${\beta_j}^i{}_{kl|m}y^m = 0$, or equivalently,

$$
h^i_r {B_j}^r{}_{kl|m}y^m = 0.
$$

Finsler metrics satisfying (7) are called $GB\widetilde{W}$ metrics.
There are numerous Finsler metrics that belong to the GB$\widetilde{W}$ type. Specifically, all Berwald metrics and $\widetilde{W}$ metrics, as well as Finsler metrics with a constant flag curvature ($n > 2$), are included in this class of metrics.

3 Generalized Berwald Projective Weyl ($GB\widetilde{W}$) metrics

This section focuses on $GB\widetilde{W}$ metrics. Firstly, it is demonstrated that all Berwald and $\widetilde{W}$ metrics belong to this noteworthy class of C-projective invariant metrics.

Lemma 3.1 The class of $GB\widetilde{W}$ metrics contains $\widetilde{W}$ metrics ($n > 2$).

Proof Let F be a $\widetilde{W}$ metric ($n > 2$). Based on [7], it is of constant curvature. Then there is a constant λ such that

$$
R^i_k = \lambda(F^2\delta^i_k - y^i y_k).
$$

Then

$$
{R_j}^i{}_{kl.m} = 2\lambda(C_{jlm}\delta^i_k - C_{jkm}\delta^i_l).
$$

Here $_{.m}$ means the differential with respect to y^m. In [12], the following Ricci identity has been stated

$$
{B_j}^i{}_{ml|k} - {B_j}^i{}_{km|l} = {R_j}^i{}_{kl.m}.
$$

Contracting the above equations by y^k one easily can find

$$
{B_j}^i{}_{ml|0} = 2\lambda C_{jlm}y^i.
$$

Thus, $h^i_m {B_j}^m{}_{kl|0} = 0$ which means that F is of $GB\widetilde{W}$ type. □

3.1 Proof of Theorem 1.1

Let F be a Finsler metric of $GB\widetilde{W}$ type; then the Berwald curvature satisfies the following equation

$$
{B_j}^i{}_{kl} = {\beta_j}^i{}_{kl} + b_{jkl}y^i,
$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$ where $\beta_j{}^i{}_{kl|0} = 0$. Then $B_j{}^i{}_{kl|0} = b_{jkl|0}y^i$ and contracting it by y^l yields $b_{jkl|0}y^l = 0$. Thus, by the definition of E-curvature, one gets

$$2E_{jk} = \beta_j{}^m{}_{km} + b_{jkm}y^m.$$

Thus, one may conclude that

$$H_{jk} = E_{jk|0} = 0. \tag{8}$$

Considering (6), one easily gets

$$D_j{}^i{}_{kl} = (\beta_j{}^i{}_{kl} - \frac{2}{n+1}e_j{}^i{}_{kl}) + (b_{jkl} + E_{jkl})y^i = t_j{}^i{}_{kl} + d_{jkl}y^i, \tag{9}$$

where,

$$e_j{}^i{}_{kl} = E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j, \quad and \quad E_{jkl} = \frac{\partial E_{jk}}{\partial y^l} = \frac{1}{2}\frac{\partial^3 S}{\partial y^j \partial y^k \partial y^l}.$$

According to (9) and (8) and considering $\beta_j{}^i{}_{kl|m}y^m = 0$, we will have $t_j{}^i{}_{kl|m}y^m = 0$, meaning that F is GDW metric.

Corollary 3.2 Let F be a $GB\widetilde{W}$ metric, then $H = 0$.

Corollary 3.3 Every GDW metric is $GB\widetilde{W}$ metric, if and only if $H = 0$.

Corollary 3.4 Every R-quadratic Finsler metric is $GB\widetilde{W}$ metric.

The class of $\widetilde{W}$ metrics is a proper subset of the class of $GB\widetilde{W}$ metrics. Likewise, the class of W metrics ($W = 0$) is a proper subset of the class of GDW metrics.

Proof of Theorem 1.2
Let F be a Finsler metric of $GB\widetilde{W}$ type; then the Berwald curvature is as follows

$$B_j{}^i{}_{kl} = \beta_j{}^i{}_{kl} + b_{jkl}y^i,$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$ where $\beta_j{}^i{}_{kl|0} = 0$. Under a C-projective transformation, $\widetilde{G}^i = G^i + Py^i$, one has

$$\widetilde{B}_j{}^i{}_{kl} = (\beta_j{}^i{}_{kl} + \rho_j{}^i{}_{kl}) + (b_{jkl} + P_{jkl})y^i = \widetilde{\beta}_j{}^i{}_{kl} + \widetilde{b}_{jkl}y^i, \tag{10}$$

where,

$$\rho_j{}^i{}_{kl} = P_{jk}\delta^i_l + P_{jl}\delta^i_k + P_{kl}\delta^i_j, \quad P_{jk} = \frac{\partial^2 P}{\partial y^j \partial y^k} \quad and \quad P_{jkl} = \frac{\partial^3 P}{\partial y^j \partial y^k \partial y^l}.$$

Based on the previous theorem and our understanding that the GDW property is a projective invariant, we can conclude that both F and $\widetilde{F}$ belong to the GDW metric class. Therefore, taking into account equation (6), we obtain

$$D_j{}^i{}_{kl} = (\beta_j{}^i{}_{kl} - \frac{2}{n+1}e_j{}^i{}_{kl}) + (b_{jkl} - \frac{2}{n+1}E_{jkl})y^i =$$
$$\widetilde{D}_j{}^i{}_{kl} = (\widetilde{\beta}_j{}^i{}_{kl} - \frac{2}{n+1}\widetilde{e}_j{}^i{}_{kl}) + (\widetilde{b}_{jkl} - \frac{2}{n+1}\widetilde{E}_{jkl})y^i,$$

where,

$$\widetilde{e}_j{}^i{}_{kl} = \widetilde{E}_{jk}\delta^i_l + \widetilde{E}_{jl}\delta^i_k + \widetilde{E}_{kl}\delta^i_j, \quad and \quad \widetilde{E}_{jkl} = \frac{\partial \widetilde{E}_{jk}}{\partial y^l}.$$

Since F belongs to both the classes of GDW metric and $GB\widetilde{W}$ metric, we can conclude that

$$\beta_j{}^i{}_{kl|0} = \frac{2}{n+1} e_j{}^i{}_{kl|0} = 0,$$

Moreover, $\widetilde{F}$ is GDW-metric; then

$$\widetilde{\beta}_j{}^i{}_{kl\|0} - \frac{2}{n+1}\widetilde{e}_j{}^i{}_{kl\|0} = 0, \tag{11}$$

where $\|$ denotes the horizontal derivative with respect to $\widetilde{G}^i$. Also, since it is shown $\widetilde{E}_{jk\|0} = 0$; then

$$\widetilde{e}_j{}^i{}_{kl\|0} = \widetilde{E}_{jk0}\delta^i_l + \widetilde{E}_{jl\|0}\delta^i_k + \widetilde{E}_{kl\|0}\delta^i_j = 0.$$

Thus, based on equation (11), we can deduce that $\widetilde{\beta}_j{}^i{}_{kl0} = 0$, indicating that $\widetilde{F}$ is a $GB\widetilde{W}$ metric. Next, we aim to demonstrate that $\widetilde{E}_{jk\|0} = 0$. Using equation (10), we obtain

$$\widetilde{E}_{jk} = E_{jk} + \frac{n+1}{2}P_{jk},$$

where, $P_{jk} = \frac{\partial^2 P}{\partial y^j \partial y^k}$. Then we will have

$$\begin{aligned}
\widetilde{E}_{jk\|0} &= y^m\partial_m\widetilde{E}_{jk} - 2(G^m + Py^m)\widetilde{E}_{jk.m} - \widetilde{E}_{jm}(G^m_k + P_k y^m + P\delta^m_k) \\
-\widetilde{E}_{mk}(G^m_j + P_j y^m + P\delta^m_j) &= y^m\partial_m\widetilde{E}_{jk} - 2G^m\widetilde{E}_{jk.m} - \widetilde{E}_{jm}G^m_k - \widetilde{E}_{mk}G^m_j \\
&= \widetilde{E}_{jk|0} = E_{jk|0} + \frac{n+1}{2}P_{jk|0}.
\end{aligned} \tag{12}$$

Based on (5)

$$\begin{aligned}
y^m Q_{jm.k} &= y^m\Big[(\partial_j P_m - \partial_m P_j) - (G^r_j P_{rm} - G^r_m P_{rj})\Big]_{.k} \\
&= -y^m\partial_m P_{jk} + 2G^m P_{jk.m} + G^m_j P_{mk} + G^m_k P_{mj} = -P_{jk|0}.
\end{aligned} \tag{13}$$

Based on the definition of C-projective transformation, which stated in (5), and considering (12), (13) and (8), one can conclude that $\widetilde{E}_{jk\|0} = E_{jk|0} = 0$.

Lemma 3.5 There does not exist a non-constant scalar flag curvature Finsler metric that is a $GB\widetilde{W}$ metric ($n > 2$).

Proof Assuming the existence of a $GB\widetilde{W}$ metric F with scalar flag curvature, Corollary 3.2 implies that the H-curvature would vanish. This would imply that F has constant flag curvature, as demonstrated in [**?**], which states that for a Finsler metric with scalar flag curvature in dimensions $n > 2$, the flag curvature is constant if and only if $H = 0$. However, this contradicts our initial assumption that F has non-constant flag curvature, thus proving the non-existence of a $GB\widetilde{W}$ metric with scalar flag curvature that is non-constant.

Proof of Theorem 1.3

Suppose F is a GDW metric with a Landsberg curvature of zero ($n > 2$). In that case, it also has a vanishing H-curvature, as shown in [12]. Utilizing Corollary 3.3, we can easily conclude that F belongs to the $GB\widetilde{W}$ metric class.

Now assume that F is of scalar curvature. Then by Lemma 3.5, it is of constant flag curvature λ. By Lemma 3.1, we have

$$B_j{}^i{}_{kl|0} = 2\lambda C_{jkl} y^i.$$

Contracting the above equation by y_i yields $\lambda C_{jkl} = 0$. F is not Riemannian, then $\lambda = 0$ and $R^i{}_k = 0$. Then it is of R-quadratic type.
Now let F be of non-scalar curvature. For this $GB\widetilde{W}$ metric F, we have

$$B_j{}^i{}_{kl} = \beta_j{}^i{}_{kl} + b_{jkl} y^i,$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$ where $\beta_j{}^i{}_{kl|0} = 0$. Since F is of Landsberg type, then

$$0 = y_i B_j{}^i{}_{kl} = -2L_{jkl} = y_i \beta_j{}^i{}_{kl} + b_{jkl} F^2,$$

Accordingly, one could get the following

$$B_j{}^i{}_{kl} = \beta_j{}^m{}_{kl} h^i_m, \tag{14}$$

However, since F is of $GB\widetilde{W}$ type, then $\beta_j{}^i{}_{kl|0} = 0$; then

$$B_j{}^i{}_{kl|0} = 0, \tag{15}$$

Taking the vertical derivative of the above equation with respect to y^m, we obtain the following

$$B_j{}^i{}_{kl|m} + B_j{}^i{}_{kl|p.m} y^p = 0. \tag{16}$$

But one has

$$B_j{}^i{}_{kl|p.m} y^p = B_j{}^i{}_{kl.m;\,p} y^p - 2G^r B_j{}^i{}_{kl.m.r} - N^r_m B_j{}^i{}_{kl.r} - N^r_j B_r{}^i{}_{kl.m} - N^r_k B_j{}^i{}_{rl.m}$$
$$-N^r_l B_j{}^i{}_{kr.m} + N^i_r B_j{}^r{}_{kl.m},$$

where $B_j{}^i{}_{kl.m;\,p} = \frac{\partial}{\partial x^p} B_j{}^i{}_{kl.m}$. Then one can conclude that

$$B_j{}^i{}_{kl|m.p} y^p = B_j{}^i{}_{km|l.p} y^p. \tag{17}$$

On the other hand, (16) and (17) yield

$$B_j{}^i{}_{kl|m} = B_j{}^i{}_{km|l}, \tag{18}$$

which putting in the following Ricci identity for Berwald connection [12] yields

$$R_j{}^m{}_{lk.p} = B_j{}^m{}_{pk|l} - B_j{}^m{}_{pl|k} = 0.$$

It means that F is of R-quadratic type. □

References

[1] Akbar-Zadeh, H.: Champ de vecteurs projectifs sur le fibre unitaire. Journal de Mathématiques Pures et Appliquées **65**, 47-79 (1986)(In French).

[2] Antonelli, PL., Ingarden, RS., Matsumoto, M.,: The Theorey of Sprays and Finsler Spaces with Applications in Physics and Biology. Kluwer Academic Publishers, (1993).

[3] Bácsó, S., Pape, I.: A note on generalized Douglas space. Periodica Mathematica Hungarica **48**, 181-184 (2004).

[4] Cheng, X., Shen, Z.: On Douglas metrics. Publicationes Mathematicae Debrecen **66** 3-4, 503-512 (2005).

[5] Douglas, J.: The general geometry of paths. Annals of Mathematics **29**, 143-168 (1927-28).

[6] Najafi, B., Shen, Z., Tayebi, A.: Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties. Geometriae Dedicata **131**, 87-97 (2008).

[7] Najafi, B., Tayebi, A.: A new quantity in Finsler geometry. Comptes Rendus Mathematique 349 **1**, 81-83 (2011).

[8] Sakaguchi, T.: On Finsler spaces of scalar curvature. Tensor N. S **38**, 211-219 (1982).

[9] Shen, Z.: Landsberg curvature, S-curvature, and Riemann curvature. in A Sampler of Finsler Geometry, Mathematical Sciences Research Institute Publications, 50. Cambridge University Press, Cambridge, 303-355 (2004).

[10] Shen, Z.: Differential Geometry of Spray and Finsler Spaces. Kluwer Aca-demic Publishers (2001).

[11] Szabó, Z.: Ein Finslerscher Raum ist gerade dann von skalarer Krümmung, wennseine Weyl sche ProjectivKrümmung verschwindet, Acta Sci. Math. **39**, 163-168 (1977).

[12] Tayebi A., Sadeghi, H., Peyghan, E.: On Generalized Douglas-Weyl Spaces. Bulletin of the Malaysian Mathematical Sciences Society, Heidelberg Vol **36(3)**, 587-594 (2013).

[13] Weyl, H.: Zur Infinitesimal geometrie; Einordnung der projektiven und der konformen Auffassung. Göttinger Nachrichten 99-112 (1921).

Authors' addresses: *Nasrin Sadeghzadeh*, University of Qom, Qom, Iran.
e-mail: `nsadeghzadeh@qom.ac.ir`.