TORUS THETA-CURVES

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Abstract. A natural approach to construct a prime theta-curve is to an add an arc to a prime knot. We study that approach for knots and arcs on a standard torus in the 3-sphere. We prove that each nontrivial torus knot union an essential arc is a prime theta-curve. By comparing the constituent knots of those theta-curves, we obtain infinitely many prime theta-curves. We then classify all theta-curves that lie on a standard torus. In particular, Kinoshita's theta-curve does not lie on a standard torus.

1. Introduction

Classical knot theory is the placement problem for a circle in Euclidean 3-space or in the 3-sphere. That problem has been generalized in numerous ways. Knotted graph theory replaces the circle with a graph. Knotting of certain graphs has been well-studied. That includes the theta-graph consisting of two vertices connected by three edges—a copy of the letter θ. A **theta-curve** is an embedding of the theta-graph in Euclidean 3-space or in the 3-sphere. Connected sum of knots yields the notion of a prime knot. Similarly, one may sum two theta-curves at a vertex and one may sum a theta-curve and a knot along an edge—precise definitions are given below. Those two operations yield the notion of a prime theta-curve. The work herein was prompted by the question: *how can one add an arc to a prime knot to obtain a prime theta-curve?*

Simple examples show that care must be taken. In Figure 1.1 (left), the added arc is topologically parallel to an arc in the knot K. The resulting theta-curve is a sum of the

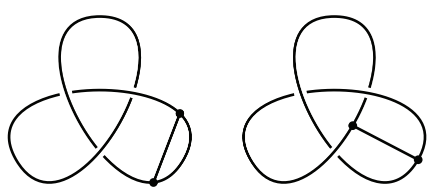

Figure 1.1. Trefoil knot K union an inessential arc (left) and K union an essential arc (right).

unknotted theta-curve and the knot K—see also Figure 2.7 below—and thus is not prime. On the other hand, adding the arc as in Figure 1.1 (right) does produce a prime theta-curve by Theorem 3.1 below. Therefore, the added arc must be judiciously chosen to ensure the resulting theta-curve is prime.

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Torus knots—by virtue of their definition as lying on a standard torus T—are particularly amenable to study and enjoy several nice properties including: they are prime and invertible (see Figure 2.10 and the discussion below it). Thus, it is natural to consider theta-curves that lie on a standard torus T, which we call **torus theta-curves**. A first step towards our motivating question is to consider a torus theta-curve that is the union of a nontrivial torus knot K and an arc on T that is essential in the annulus obtained by cutting T along K as in Figure 1.2. In Theorem 3.1, we prove that each such torus theta-curve is prime.

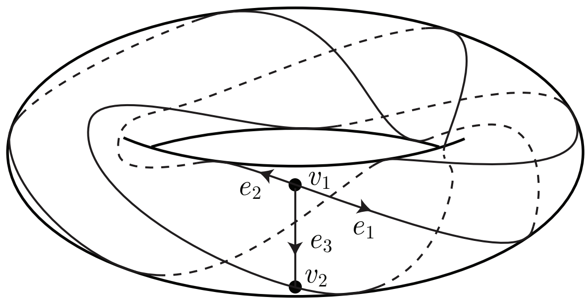

FIGURE 1.2. Theta-curve $\theta(3,5)$ equal to the torus knot $t(3,5)$ union the arc e_3.

We compute the constituent knots of those torus theta-curves and then completely classify torus theta-curves up to both ambient isotopy and homeomorphism allowing reflections. We include examples as well as proofs of some general facts on theta-curves for their further study. Consecutive Fibonacci numbers produce a charming sequence of prime torus theta-curves. Using classical Fibonacci identities, we prove that their constituent knots are torus knots themselves based on consecutive Fibonacci numbers—see Examples 4.3.

As the theta-graph is one of the simplest graphs after the circle, its embeddings in 3-space have been well-studied and arise in various contexts. Kinoshita [Kin58, Kin72] discovered the first knotted theta-curve with all three constituent knots unknotted. Wolcott [Wol87] studied and generalized Kinoshita's theta-curve and studied the sum of two theta-curves at a vertex. Kauffman, Simon, Wolcott, and Zhao [KSWZ93] studied surfaces associated to theta-curves. Theta-curves arose naturally in Scharlemann's work [Sch01] on knots with both tunnel number one and genus one. Moriuchi [Mor09] enumerated theta-curves with up to seven crossings. Matveev and Turaev [MT11] and Turaev [Tur12] introduced and studied knotoids which are intimately related to theta-curves. The first author and Metcalf-Burton [CMB16] proved a folklore conjecture of Thurston on primeness of theta-curves and double branched covers. Baker, Buck, Moore, O'Donnol, and Taylor [BBOMT25] studied primeness of unknotting number one theta-curves and discussed the appearance of theta-curves in DNA replication.

This paper is organized as follows. Section 2 recalls background material and fixes definitions. Section 3 proves our first main result, Theorem 3.1, on primeness of certain torus theta-curves. Section 4 computes the constituent knots of those torus theta-curves, presents examples, and classifies those torus theta-curves. Section 5 proves our second main result, Theorem 5.1, which classifies all torus theta-curves up to homeomorphism of S^3—allowing reflections—and up to ambient isotopy. While our focus is the classification of unoriented and unlabeled theta-curves in S^3, orientations turn out to play a key role. For that reason, care with orientations is taken throughout.

2. Background and definitions

We work in the smooth category DIFF, although it's convenient at times to utilize the piecewise linear PL and topological (locally flat) TOP categories. Those three categories are equivalent in our relevant dimensions ≤ 3. As usual, D^n denotes the closed unit disk in $\mathbb{R}^n$. The boundary of D^n is the $(n-1)$-sphere denoted $\partial D^n = S^{n-1}$. We fix an orientation on S^3. Each embedded 3-disk $B \subseteq S^3$ is oriented as a codimension-zero submanifold of S^3, and its boundary 2-sphere ∂B is given the boundary orientation using the outward normal first convention. Using terminology of Hirsch [Hir76, pp. 30–31], a manifold A embedded in a manifold B is **neat** provided $A \cap \partial B = \partial A$ and that intersection is transverse. If A has empty boundary, then **neat** means that A is embedded in the manifold interior of B.

The closed unit interval is $[0,1] \subseteq \mathbb{R}$. An **arc** is a copy of $[0,1]$. A **simple closed curve**—denoted **scc**—is a copy of S^1. In this paper, **surfaces** are compact and orientable, perhaps with boundary. A scc C in the manifold interior of a surface Σ is **inessential** in Σ provided C bounds an embedded 2-disk[1] or C union a boundary component of Σ bound an embedded annulus in Σ as in Figure 2.1 (left). Otherwise, C is **essential** in Σ. A scc C in a connected

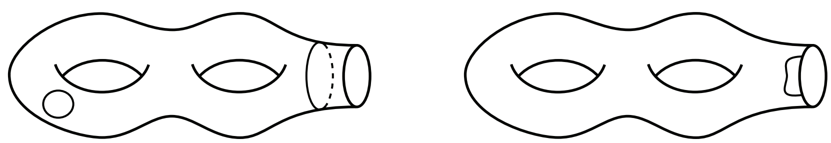

Figure 2.1. Two inessential sccs in a surface Σ with one boundary component (left) and an inessential arc in Σ (right).

surface Σ is **separating** provided $\Sigma - C$ is disconnected and is **nonseparating** provided $\Sigma - C$ is connected as in Figure 2.2. An arc a neatly embedded in a surface Σ is **inessential**

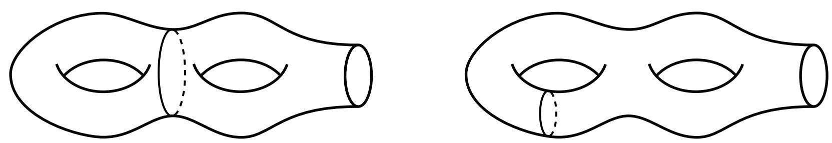

Figure 2.2. Separating essential scc (left) and nonseparating essential scc (right).

in Σ provided a union an arc b in the boundary of Σ bound an embedded 2-disk in Σ as in Figure 2.1 (right). Otherwise, a is **essential** in Σ. A neatly embedded arc a in a connected surface Σ is **separating** provided $\Sigma - a$ is disconnected and is **nonseparating** provided $\Sigma - a$ is connected as in Figure 2.3. Each neatly embedded inessential arc is separating.

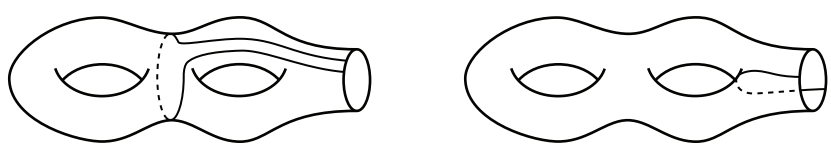

Figure 2.3. Separating essential arc (left) and nonseparating essential arc (right).

The **theta-graph** is the graph in Figure 2.4. Each edge is oriented and points from v_1 to v_2.

[1] Such a 2-disk is unique unless Σ is the 2-sphere.

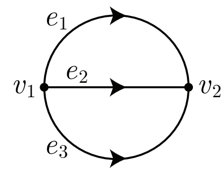

FIGURE 2.4. The theta-graph.

A **theta-curve** is an embedding of the theta-graph in S^3. We are interested in theta-curves up to ambient isotopy and up to homeomorphism of S^3 allowing reflections where we ignore graph labelings and graph orientations. Nevertheless, those labelings and orientations are useful for calculations after which they may be forgotten. Every theta-curve θ contains three **constituent knots** k_1, k_2, and k_3 where each is obtained by ignoring one of the three edges of θ. More precisely, $k_i = e_{i+1} - e_{i+2}$ using the cyclic order in Figure 2.5.

$$k_1 = e_2 - e_3$$
$$k_2 = e_3 - e_1$$
$$k_3 = e_1 - e_2$$

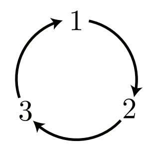

FIGURE 2.5. Constituent knot conventions.

In more detail, consider—possibly oriented—knots K and K' in S^3. Let $K = K'$ indicate that K and K' are equal as sets. Let $K \sim K'$ indicate that an ambient isotopy of S^3 carries K to K'. Let $K \approx K'$ indicate that a homeomorphism of S^3—perhaps reversing orientation of S^3—carries K to K'. If the knots K and K' are oriented, then those three equivalence relations are assumed to respect the orientations of K and K' unless context dictates otherwise. Evidently, $K = K'$ implies $K \sim K'$, and $K \sim K'$ implies $K \approx K'$.

If $K \subseteq S^3$ is an oriented knot, then $-K$ denotes the **reverse knot** of K—namely, the same knot with the reverse orientation. An oriented knot K is **invertible** provided it is ambient isotopic to its reverse. Evidently, the classifications—up to ambient isotopy of S^3 and separately up to homeomorphism of S^3 allowing reflections—of invertible knots regarded as oriented or unoriented knots coincide. That is relevant here since torus knots are invertible.

We use those same three equivalence relations—equality, isotopy, and homeomorphism—for theta-curves, except we always ignore graph labelings and graph orientations on theta-curves. In particular, $\theta = \theta'$ implies $\theta \sim \theta'$, and $\theta \sim \theta'$ implies $\theta \approx \theta'$.

Let $U \subseteq S^3$ denote the unknot. A knot $K \subseteq S^3$ is **trivial** or **unknotted** provided $K \sim U$ or equivalently $K \approx U$. Otherwise, K is **nontrivial** or **knotted**. A theta-curve θ is **trivial** or **unknotted** provided it lies in a 2-sphere embedded in S^3, and then we denote it by θ_U. Otherwise, θ is **nontrivial** or **knotted**. All trivial theta-curves are isotopic in S^3.

The constituent knots of theta-curves are important invariants. An isotopy of S^3 carrying one theta-curve to another carries the unoriented constituent knots of the former to the latter. That also holds for any homeomorphism of S^3. More precisely, if $\theta \sim \theta'$, then there is a permutation π of $\{1, 2, 3\}$ such that $k_i \sim k'_{\pi(i)}$ as unoriented knots for $i = 1, 2, 3$. Similarly, if $\theta \approx \theta'$, then there is a permutation π of $\{1, 2, 3\}$ such that $k_i \approx k'_{\pi(i)}$ as unoriented knots for $i = 1, 2, 3$. Paraphrasing, theta-curves with distinct constituent knots are distinct. Constituent knots, however, do not form a complete set of invariants for theta-curves. For example, Kinoshita's theta-curve in Figure 2.6 has trivial constituent knots yet is nontrivial (see Kinoshita [Kin58, Kin72] and Ozawa and Taylor [OT19]). Since the

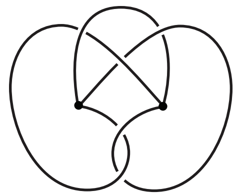

FIGURE 2.6. Kinoshita's theta-curve.

time of Kinoshita's pioneering work in 1958, a rich collection of such theta-curves has been revealed by Wolcott [Wol87] in 1985 and more recently in 2016 by Jang, Kronaeur, Luitel, Medici, Taylor, and Zupan [JKLMTZ16].

Next, we recall three operations one may perform on knots and theta-curves—including the classical connected sum of knots. In each of these operations, we delete the interiors of unknotted ball-prong pairs in separate copies of S^3 and glue together the resulting boundary 2-spheres using an *orientation reversing* homeomorphism. These operations coincide with the intuitive notion of nicely splicing together two knots or graphs in a single copy of S^3 and separated by an embedded 2-sphere. That splicing glues the two halves of S^3 by an orientation reversing homeomorphism of their boundaries.

A **k-prong** is obtained from k disjoint arcs by gluing together one endpoint from each arc. The point in a k-prong at which the endpoints are glued is the **prong point**. A 2-prong is an arc with a distinguished interior point. A **ball-prong pair** (B, P) is a copy B of D^3 with a neatly embedded k-prong P for some integer $k \geq 2$. The prong point of P lies in the interior of B, and the k boundary points of P lie in the boundary of B. A ball-prong pair (B, P) is **unknotted** provided it is homeomorphic to (D^3, R) where $R \subseteq D^2 \subseteq D^3$ is a k-prong with prong point the origin and radial arcs. In particular, an unknotted ball-arc pair is homeomorphic to (D^3, D^1). The following lemma is useful for identifying unknotted ball-prong pairs.

Lemma 2.1. *A ball-prong pair (B, P) is unknotted if and only if there exists a neatly embedded 2-disk Δ in B that contains P.*

Proof. The forward implication is clear. For the reverse implication, one builds the desired homeomorphism h in stages using the following tools: (i) the 2- and 3-dimensional Schoenflies theorems [Moi77, pp. 68 & 117], [Hat00, Thm. 1.1], and [Cer68, Ch. III], and (ii) the facts that every homeomorphism of S^1 and S^2 extend to D^2 and D^3 respectively [Moi77, pp. 44–45], [Mun60, Sma59], and [Thu97, Thm. 3.10.11]. Using those tools, begin by defining a homeomorphism $h : \partial\Delta \to \partial D^2$, extend h to $P \to R$ respecting the cyclic orders of $P \subseteq \Delta$ and $R \subseteq D^2$, extend to $\Delta \to D^2$, extend to $\partial B \to \partial D^3$, and finally extend to $B \to D^3$. $\square$

Let $K \subseteq S^3$ and $J \subseteq S^3$ be knots in separate copies of S^3. A **connected sum** of K and J is a knot in S^3 defined as follows. Let (B_K, a_K) be an unknotted ball-arc pair where

a_K is an arc in K, and similarly define (B_J, a_J). Delete the interiors of B_K and B_J and then glue together the resulting ball-arc pairs along their boundaries using an orientation reversing homeomorphism matching up the two points of K and of J therein. A resulting knot in S^3 is denoted $K\#J$. Without additional data—such as orientations on K and J—the connected sum of K and J could mean two different knots. A knot K is **prime** provided if $K \approx J_1\#J_2$, then $J_1 \approx U$ or $J_2 \approx U$. We adopt the usual convention that the unknot U is not prime. While the connected sum of knots is commutative, associative, and unital, it does not have inverses. Namely, if $K\#J \approx U$, then $K \approx U$ and $J \approx U$. One standard proof of that fact uses additivity of the genus of a knot [BZ03, p. 93].

Let $\theta \subseteq S^3$ be a theta-curve and let $K \subseteq S^3$ be a knot in separate copies of S^3. A **2-connected sum** of θ and K is a theta-curve in S^3 defined as follows. Let (B_θ, a_θ) be

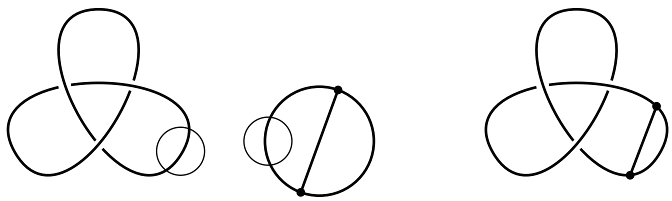

FIGURE 2.7. A knot K and a theta-curve θ with unknotted ball-arc pairs (left) and a 2-connected sum $\theta\#_2K$ of θ and K (right).

an unknotted ball-arc pair where a_θ is an arc in the interior of some edge e_i of θ, and let (B_K, a_K) be an unknotted ball-arc pair where a_K is an arc in K. Delete the interiors of B_θ and B_K and then glue together the resulting ball-arc pairs along their boundaries using an orientation reversing homeomorphism matching up the two points of θ and of K therein as in Figure 2.7. A resulting theta-curve in S^3 is denoted $\theta\#_2K$. Without additional data, $\theta\#_2K$ could mean up to six theta-curves. Observe that 2-connected sum does not have inverses in the following sense. If $\theta\#_2K \approx \theta_U$, then $\theta \approx \theta_U$ and $K \approx U$. That's a consequence of the fact that knots don't have inverses under connected sum.

Let $\theta_1 \subseteq S^3$ and $\theta_2 \subseteq S^3$ be theta curves in separate copies of S^3. A **3-connected sum** of θ_1 and θ_2 is a theta-curve in S^3 defined as follows. Let (B_1, P_1) be an unknotted ball-prong

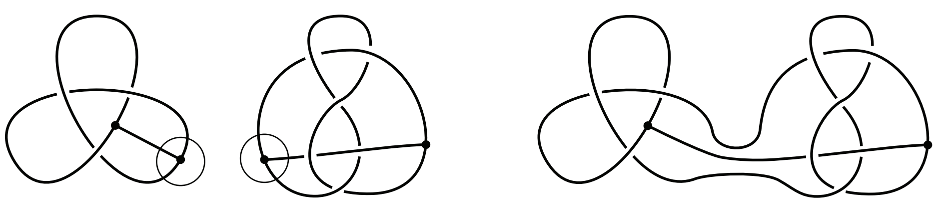

FIGURE 2.8. Two theta-graphs θ_1 and θ_2 with unknotted ball-prong pairs (left) and a 3-connected sum $\theta_1\#_3\theta_2$ of θ_1 and θ_2 (right).

pair where $P_1 \subseteq \theta_1$ is a 3-prong. Similarly define (B_2, P_2). Delete the interiors of B_1 and B_2 and then glue together the resulting ball-prong pairs along their boundaries using an orientation reversing homeomorphism matching up the three points of θ_1 and of θ_2 therein as in Figure 2.8. A resulting theta-curve in S^3 is denoted $\theta_1\#_3\theta_2$. Without additional data, $\theta_1\#_3\theta_2$ could mean up to 24 theta-curves. Wolcott [Wol87, §4] proved that $\#_3$ is well-defined provided one specifies the vertices in θ_1 and θ_2 at which to sum and one specifies the matching of their edges. Observe that 3-connected sum does not have inverses in the following sense.

Lemma 2.2. *If θ_1 and θ_2 are theta-curves and $\theta_1 \#_3 \theta_2 \approx \theta_U$, then $\theta_1 \approx \theta_U$ and $\theta_2 \approx \theta_U$.*

Wolcott [Wol87, Thm. 4.2] proved Lemma 2.2 using **locally unknotted** theta-curves—meaning theta-curves in which all constituent knots are trivial—and work of Lickorish [Lic81] on prime tangles. Motohashi [Mot98, Lem. 2.1] proved a generalization of Lemma 2.2. For the convenience of the reader, we include a straightforward proof using a classical innermost circle argument.

Proof of Lemma 2.2. By hypothesis, θ_U lies in an embedded sphere $S^2 \subseteq S^3$. As $\theta_1 \#_3 \theta_2 \approx \theta_U$, there is a splitting sphere $\Sigma \approx S^2$ that is transverse to S^2 and θ_U, and meets each edge e_i at exactly one point $p_i \in \operatorname{Int} e_i$. The spheres Σ and S^2 meet in a finite disjoint union of sccs. Evidently, one of those sccs, C, must meet every edge of θ. We can eliminate any other scc in $\Sigma \cap S^2$ without disturbing θ or C as follows. Let C' be one of those other sccs that is innermost in $\Sigma - C$. So, $C' = \partial \Delta$ where Δ is a 2-disk in $\Sigma - C$ and $\Delta \cap S^2 = C'$. The scc C' separates S^2 into two 2-disks D_1 and D_2. By connectedness, θ_U lies in the interior of D_1 or D_2. Without loss of generality, $\theta_U \subseteq \operatorname{Int} D_2$. So, the 2-sphere $\Delta \cup D_1$ is disjoint from θ_U and separates S^3 into two 3-balls. By connectedness, one of those 3-balls, B, is disjoint from θ_U. Use B to push Δ slightly past D_1 to a parallel copy of D_1. That eliminates at least C' from $\Sigma \cap S^2$. Repeat that operation finitely many times so that $\Sigma \cap S^2 = C$. Now, C separates S^2 into two 2-disks containing 3-prongs of θ_U. By Lemma 2.1, $\theta_1 \approx \theta_U$ and $\theta_2 \approx \theta_U$. □

In Section 5, we classify torus theta-curves. In Remarks 5.2, we use Lemma 2.2 and the following lemma to observe that a torus theta-curve is never a 3-connected sum of nontrivial theta-curves.

Lemma 2.3. *If $\theta_U \#_2 K \approx \theta_1 \#_3 \theta_2$ where K is a prime knot and θ_1 and θ_2 are theta-curves, then $\theta_1 \approx \theta_U$ or $\theta_2 \approx \theta_U$.*

Turaev [Tur12, Lemma 5.1] proved Lemma 2.3 in his seminal work on knotoids. We include a proof for the convenience of the reader. Our proof uses the following two lemmas.

Lemma 2.4. *Let (B, a) be a ball-arc pair and let $\Delta \subseteq B$ be a neatly embedded 2-disk that meets a in general position at exactly one point $p \in \operatorname{Int} a$. Then, Δ divides (B, a) into two ball-arc pairs (B_1, a_1) and (B_2, a_2). Furthermore, (B, a) is unknotted if and only if (B_1, a_1) and (B_2, a_2) are both unknotted.*

Proof of Lemma 2.4. The 3-dimensional Schoenflies theorem implies that Δ divides (B, a) into two ball-arc pairs (B_1, a_1) and (B_2, a_2) where $B_1 \cap B_2 = \Delta$, $B_1 \cup B_2 = B$, $a_1 \cap a_2 = p$, and $a_1 \cup a_2 = a$.

For the forward implication, we are given that (B, a) is unknotted. Without loss of generality, we may assume $(B, a) = \left(D^3, D^1\right)$. In particular, $p = \Delta \cap D^1 \in \operatorname{Int} D^1$. We have standard disks $D^1 \subseteq D^2 \subseteq D^3$, and Δ is in general position with respect to D^1 and D^2. We will isotop Δ in D^3 fixing D^1 pointwise and sending $\partial \Delta$ into ∂D^3 at all times. By an isotopy with support near ∂D^3, we arrange for $\partial \Delta$ to be a great circle on ∂D^3. So, $\partial \Delta$ meets D^2 at exactly two points. Hence, $\Delta \cap D^2$ is the disjoint union of a single arc b and finitely many sccs. The arrangement of the endpoints of b and D^1 on ∂D^2 imply that b meets D^1. As Δ meets D^1 at exactly p, we see that p lies on b. An innermost circle argument allows us to further isotop Δ so that $\Delta \cap D^2 = b$. Now, b divides D^2 into two 2-disks, one containing a_1 and the other containing a_2. By Lemma 2.1, (B_1, a_1) and (B_2, a_2) are unknotted, as desired.

For the reverse implication, we are given that (B_1, a_1) and (B_2, a_2) are unknotted. As (B_1, a_1) is unknotted, there is a neatly embedded 2-disk $D_1 \subseteq B_1$ containing a_1. Adjust D_1 in B_1 by an isotopy fixing a_1, sending ∂D_1 into ∂B_1 at all times, and with support near ∂B_1 so that D_1 meets Δ in exactly an arc b neatly embedded in Δ. Similarly, adjust D_2 in B_2. Now, $D_1 \cup D_2$ is a neatly embedded 2-disk in B containing a. By Lemma 2.1, (B, a) is unknotted, as desired. □

Lemma 2.5. *Let $K \subseteq S^3$ be a prime knot. Let (B, b) be an unknotted ball-arc pair such that $B \subseteq S^3$, $b \subseteq K$, and ∂B meets K in general position. Let (B', b') be the ball-arc pair* complementary *to (B, b) in S^3, meaning $B' = S^3 - \operatorname{Int} B$ and $b' = K - \operatorname{Int} b$. Let $\Delta \subseteq B'$ be a neatly embedded 2-disk that meets b' in general position at exactly one point $p \in \operatorname{Int} b'$. Then, Δ divides (B', b') into two ball-arc pairs, exactly one of which is unknotted.*

Proof of Lemma 2.5. By Lemma 2.4, Δ divides (B', b') into two ball-arc pairs (B_1, b_1) and (B_2, b_2). If (B_1, b_1) and (B_2, b_2) are both knotted, then K is not prime, a contradiction. If (B_1, b_1) and (B_2, b_2) are both unknotted, then the reverse implication in Lemma 2.4 implies that (B', b') is unknotted. That implies K is unknotted, a contradiction. □

Proof of Lemma 2.3. Begin with θ_U as in Figure 2.9 (left). Let $\theta' = \theta_U \#_2 K$ where, without loss of generality, the 2-connected sum is performed along e_3 as in Figure 2.9 (middle). That

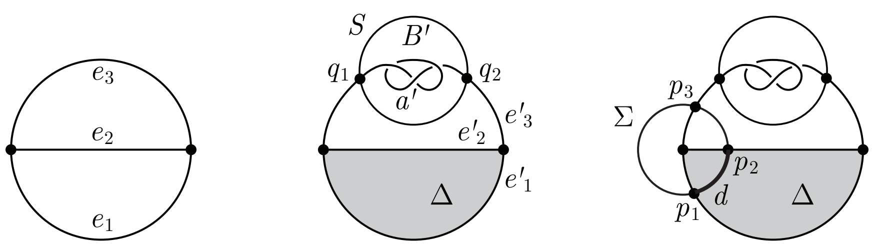

FIGURE 2.9. Unknotted theta-graph θ_U (left), 2-connected sum $\theta' = \theta_U \#_2 K$ where K is a prime knot (middle), and goal in Lemma 2.3.

means an unknotted ball-arc pair (B, a) along e_3 with boundary 2-sphere S is replaced with a knotted ball-arc pair (B', a') obtained from (S^3, K) by removing the interior of an unknotted ball-arc pair. The edges of θ' are $e'_1 = e_1$, $e'_2 = e_2$, and e'_3 equal to e_3 with K tied in it. Also depicted in Figure 2.9 (middle) is a 2-disk Δ with boundary $e'_1 \cup e'_2$ where Δ is disjoint from $\operatorname{Int} e'_3$ and S.

By hypothesis, $\theta' \approx \theta_1 \#_3 \theta_2$. So, there is a splitting 2-sphere $\Sigma \subseteq S^3$ that meets each arc e'_i in general position at exactly one point $p_i \in \operatorname{Int} e'_i$ for $i = 1, 2, 3$. Notice that if Σ and S are disjoint, then the desired result holds. To see that, replace (B', a') with (B, a). Lemma 2.2 implies that Σ splits (S^3, θ_U) into two unknotted ball-prong pairs. One of those pairs is disjoint from B and is the desired unknotted ball-prong pair.

So, it remains to show that we can isotop Σ fixing θ' setwise so that Σ and S are disjoint. Note that Σ meets Δ in exactly one arc d and finitely many sccs $D_1, \ldots, D_m$, and Σ meets S in finitely many sccs $C_1, \ldots, C_m$. As Δ and S are disjoint, each C_i is disjoint from d and from each D_j. Each C_i and D_j has two possibilities on Σ:

(1) It bounds a 2-disk in Σ not containing d or p_3.

(2) It splits Σ into two 2-disks, one containing d and one containing p_3.

And, each C_i has two possibilities on S:

(1) It bounds a 2-disk in S not containing q_1 or q_2.

(2) It splits S into two 2-disks, one containing q_1 and one containing q_2.

The following *observation* will be used repeatedly: a knot in S^3 cannot meet a 2-sphere in general position exactly once. Consider an innermost scc of type (1) on Σ. If it's a D_j, then an isotopy of Σ removes it. If it's a C_i, then it must also be of type (1) on S—by the observation applied to $e_1' \cup e_3'$—and an isotopy of Σ removes it. Repeat that operation until all sccs of type (1) on Σ are removed. By the observation applied to $e_1' \cup e_3'$, no D_j remains and no C_i of type (1) on S remains.

Now, let C_i be innermost on Σ with respect to p_3—meaning C_i bounds a 2-disk Δ_i on Σ such that $p_3 \in \operatorname{Int} \Delta_i$ and Δ_i is disjoint from any other remaining C_j. If Δ_i is contained in B', then Lemma 2.5 implies that Δ_i splits (B', a') into two ball-arc pairs, exactly one of which is unknotted. Use that unknotted ball-arc pair to remove C_i. Otherwise, $\Delta_i \subseteq S^3 - \operatorname{Int} B'$. Let $B'' = S^3 - \operatorname{Int} B'$ and $b'' = B'' \cap (e_1' \cup e_3')$. So, (B'', a'') is an unknotted ball-arc pair and Lemma 2.4 implies that Δ_i splits (B'', a'') into two unknotted ball-arc pairs. One of those two ball-arc pairs is disjoint from Δ. Use that unknotted ball-arc pair to remove C_i. $\square$

A theta-curve $\theta \subseteq S^3$ is **prime** provided: (i) θ is nontrivial, (ii) θ is not homeomorphic to a 2-connected sum of a possibly trivial theta-curve and a nontrivial knot, and (iii) θ is not homeomorphic to a 3-connected sum of nontrivial theta-curves. In other words, (ii) says that if $\theta \approx \theta' \#_2 K$ for a theta-curve θ' and a knot K, then K is trivial. And, (iii) says that if $\theta \approx \theta_1 \#_3 \theta_2$ for theta-curves θ_1 and θ_2, then θ_1 or θ_2 is trivial.

Remark 2.6. Some authors use an alternative definition of prime theta-curve, omitting requirement (ii) above. For example, see Motohashi [Mot98, p. 204], Matveev and Turaev [MT11], and Turaev [Tur12, p. 205].

Let $T \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a standard, unknotted torus with oriented longitude l and meridian m as in Figure 2.10. In particular, T divides S^3 into two solid tori $T_1 \approx S^1 \times D^2$ and $T_2 \approx S^1 \times D^2$ such that $T_1 \cup T_2 = S^3$ and $T_1 \cap T_2 = T$. Given relatively prime integers p and q, let $t(p,q) \subseteq T$ denote the oriented **torus knot** that wraps p times around T in the longitudinal direction and q times around T in the meridinal direction. The scc $e_1 - e_2$ in Figure 1.2 is the torus knot $t(3,5)$. Figure 2.11 depicts T as a square with the usual identifications (left) and $t(3,5) = e_1 - e_2$ in T (right).

Recall three involutions of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that are fundamental for the classification of torus knots and for our purposes. Each restricts to an involution of the torus T. The first two are described on $\mathbb{R}^3$ and fix the point at infinity.

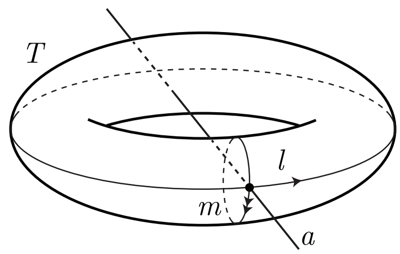

FIGURE 2.10. Standard torus $T \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ with longitude l, meridian m, and the axis a of the rotation ρ by π radians.

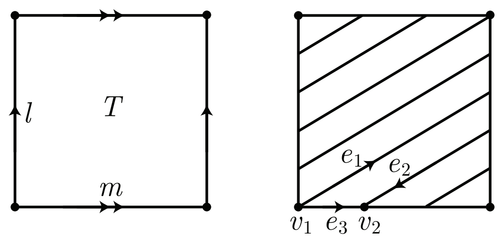

FIGURE 2.11. Standard torus T with longitude l and meridian m (left) and torus knot $t(3,5) = e_1 - e_2$ and torus theta-curve $\theta(3,5)$ in T (right).

- $\rho : S^3 \to S^3$ is the rotation of $\mathbb{R}^3$ by π radians about the axis a in Figure 2.10.
- $R : S^3 \to S^3$ is the reflection of $\mathbb{R}^3$ across the horizontal plane containing the longitude l in Figure 2.10.
- $\sigma : S^3 \to S^3$ is the rotation of S^3 that swaps the core circles of T_1 and T_2, swaps T_1 and T_2, and swaps l and m.

The rotations ρ and σ of S^3 are isotopic to the identity, but the reflection R is not. Note that ρ and R commute, ρ and σ commute, but R and σ do not commute.

Recall some facts about torus knots and their classification. Consider the oriented torus knot $t(p,q)$ where p and q are relatively prime integers. We have:

$$\begin{aligned} -t(p,q) &= t(-p,-q) && \\ t(p,q) &\sim -t(p,q) && \text{by the rotation } \rho \text{ of } S^3 \\ t(p,q) &\sim t(q,p) && \text{by the rotation } \sigma \text{ of } S^3 \\ t(p,q) &\approx t(p,-q) && \text{by the reflection } R \text{ of } S^3 \end{aligned} \tag{2.1}$$

The rotation ρ of S^3 shows that torus knots are invertible, meaning they are ambient isotopic to their reverse. Evidently, the classifications—up to ambient isotopy of S^3 and separately up to homeomorphism of S^3 allowing reflections—of invertible knots regarded as oriented or unoriented knots coincide. Torus knots are classified—see Burde-Zieschang [BZ03, Ch. 3E] or Murasugi [Mur08, Ch. 7].

Theorem 2.7 (Classification of torus knots)**.** *Let $t(p,q)$ be a torus knot where p and q are relatively prime integers. If p or q equals 0 or ± 1, then $t(p,q)$ is the unknot. Otherwise, $|p|, |q| \geq 2$ and $t(p,q)$ is a prime knot. Consider another such torus knot $t(p', q')$. Then,*

$t(p',q') \sim t(p,q)$—as oriented knots—if and only if $\{p',q'\}$ equals $\{p,q\}$ or $\{-p,-q\}$. And, $t(p',q') \approx t(p,q)$—as oriented knots—if and only if $\{p',q'\}$ equals $\{p,q\}$, $\{-p,-q\}$, $\{p,-q\}$, or $\{-p,q\}$. The classifications of unoriented torus knots are exactly the same since torus knots are invertible.

Given a torus knot $t(p,q)$ where p and q are relatively prime integers and $|p|,|q| \geq 2$, we define the **torus theta-curve** $\theta(p,q)$ as follows. In short, $\theta(p,q)$ is $t(p,q)$ union a little arc e_3 in T as in Figures 1.2, 2.11, and 2.12. More precisely, consider first the case where $p > 0$ and $q > 0$. We choose a basepoint, denoted v_1, in $t(p,q)$ where l and m meet. Follow m from v_1 (using the orientation of m) until $t(p,q)$ is first encountered; call that point v_2. The oriented arc just described in m from v_1 to v_2 is denoted e_3. The oriented arc in $t(p,q)$ out of v_1 and ending at v_2 is denoted e_1. The remaining arc in $t(p,q)$ out of v_1 and ending at v_2 is denoted e_2. That completes our definition of $\theta(p,q)$ in case $p > 0$ and $q > 0$. Figure 2.12

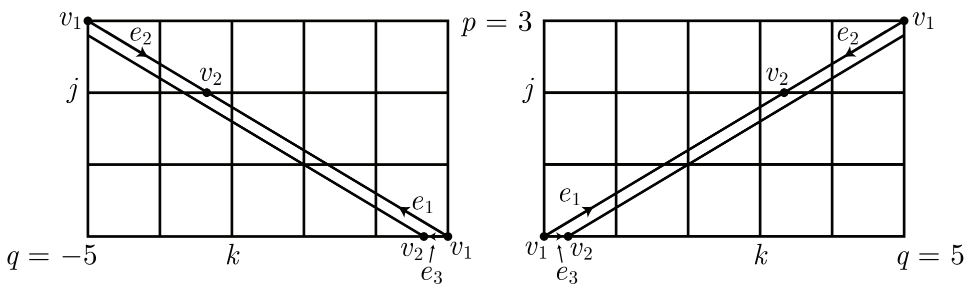

FIGURE 2.12. Portions of the universal cover $\mathbb{R}^2$ of T containing lifts of the arcs of torus theta-curves $\theta(3,-5)$ (left) and $\theta(3,5)$ (right).

(right) depicts lifts of the arcs of $\theta(p,q)$ to the universal cover $\mathbb{R}^2$ of T. For notational convenience, lifts of v_1, v_2, e_1, e_2, and e_3 are denoted by the same symbols. Now, we define torus theta-curves for arguments of other sign combinations using the involutions ρ and R which commute. Using the same relatively prime integers $p \geq 2$ and $q \geq 2$, we define:

$$\begin{aligned} \theta(p,-q) &= R\left(\theta\left(p,q\right)\right) \\ \theta(-p,-q) &= \rho\left(\theta\left(p,q\right)\right) \\ \theta(-p,q) &= R\left(\theta\left(-p,-q\right)\right) = \rho\left(\theta\left(p,-q\right)\right) \end{aligned} \tag{2.2}$$

Figure 2.13 depicts local pictures of the definitions (2.2) in T near v_1.

Thus, we have the following relations between theta-curves where a and b are relatively prime integers and $|a|,|b| \geq 2$.

$$\begin{aligned} &\theta(a,b) \sim \theta(-a,-b) && \text{by the rotation } \rho \text{ of } S^3 \\ &\theta(a,b) \sim \theta(b,a) && \text{by the rotation } \sigma \text{ of } S^3 \\ &\theta(a,b) \approx \theta(a,-b) && \text{by the reflection } R \text{ of } S^3 \\ &\theta(a,b) \approx \theta(-a,b) && \text{by the composition } \rho \circ R = R \circ \rho \end{aligned} \tag{2.3}$$

In general, sccs in T are classified up to isotopy of T by their intersection numbers with l and m. Let $H_1(T)$ denote the dimension one integer homology group of T. Equip $H_1(T) \cong \mathbb{Z}^2$ with the ordered basis $[l,m]$ so that l corresponds to $(1,0) \in \mathbb{Z}^2$ and m corresponds to

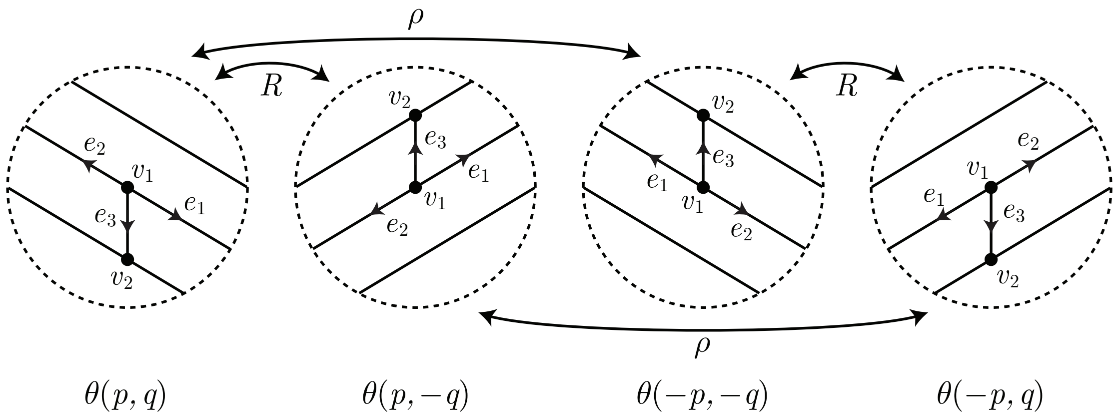

FIGURE 2.13. Close-up view of $\theta(p,q)$ in T near v_1 in case $p>0$ and $q>0$ (left) along with the images $\theta(p,-q)$, $\theta(-p,-q)$, and $\theta(-p,q)$ of $\theta(p,q)$ under the involutions ρ and R.

$(0,1) \in \mathbb{Z}^2$. In $H_1(T)$, we have $t(p,q) = e_1 - e_2$. Recall the definitions of the constituent knots k_1, k_2, and k_3 of $\theta(p,q)$ in Figure 2.5. In $H_1(T)$, we have:

$$\begin{aligned} k_1 + k_2 &= -k_3 \\ k_2 + k_3 &= -k_1 \\ k_3 + k_1 &= -k_2 \end{aligned}$$

Each constituent knot k_i of $\theta(p,q)$ is a torus knot since $\theta(p,q)$ lies in the torus T. Cutting T along $t(p,q)$ yields an **annulus** A. The arc e_3 is neatly embedded and essential in A (e_3 has one boundary point in each boundary circle of A). The algebraic intersection number—see Stillwell [Sti93, p. 209]—of two torus knots $t = t(p,q)$ and $t' = t(p',q')$ is:

$$I(t,t') = \det \begin{bmatrix} p & p' \\ q & q' \end{bmatrix}$$

Note that $I(l,m) = 1$ and $I(m,l) = -1$.

We close this section with a lemma that will be useful for proving the theta-curves $\theta(p,q)$ are prime. Recall that T divides S^3 into two solid tori T_1 and T_2.

Lemma 2.8. *Let $\theta = \theta(p,q)$ be a torus theta-curve where p and q are relatively prime integers and $|p|, |q| \geq 2$. There does not exist a 2-disk Δ neatly embedded in T_1 or T_2 such that $C = \partial\Delta$ is in general position with respect to θ and meets θ in exactly one point.*

Proof. Suppose, by way of contradiction, that there exists such a 2-disk Δ. If C meets e_1 or e_2, then C meets $t(p,q)$ once. That implies C is essential in T. As $C = \partial\Delta$ and Δ is neatly embedded in T_1 or T_2, C is isotopic in T to l or m. That is a contradiction since $t(p,q)$ has algebraic intersection number $-q$ with l and p with m. So, assume C meets e_3 once. As e_3 is essential in the annulus A, C is topologically parallel to the boundary circles of A. Thus, Δ union an annulus in A is a 2-disk embedded in S^3 with boundary $t(p,q)$. That is a contradiction since $t(p,q)$ is knotted in S^3. $\square$

3. Prime torus theta-curves

In this section, we prove the following theorem.

Theorem 3.1. *If p and q are relatively prime integers and $|p|, |q| \geq 2$, then the torus theta-curve $\theta = \theta(p,q)$ is prime.*

Proof. The constituent knot $k_3 = t(p,q)$ is a non-trivial torus knot, so θ is knotted.

Suppose, by way of contradiction, that $\theta = \theta' \#_2 K$ where $\theta' \subseteq S^3$ is a possibly unknotted theta-curve and $K \subseteq S^3$ is a nontrivial knot. Then, there is a 2-sphere $\Sigma \subseteq S^3$ that is transverse to T and θ such that Σ meets exactly one arc e_i of θ in exactly two points $p_1, p_2 \in \operatorname{Int} e_i$. Further Σ separates S^3 into two 3-balls B_1 and B_2. Without loss of generality, B_1 meets θ in a subarc α of e_i, (B_1, α) is a knotted ball-arc pair, and v_1 and v_2 lie in $\operatorname{Int} B_2$. We have $\Sigma \cap T$ is finitely many pairwise disjoint sccs. Each such scc either meets θ or is disjoint from θ. Each of the latter type is inessential in $T - \theta \approx \operatorname{Int} D^2$. We will eliminate all of the latter type by ambient isotopy of Σ leaving θ unmoved.

Let C be a scc of the latter type that is innermost in $T - \theta$. So, $C = \partial\Delta$ where Δ is a 2-disk in $T - \theta$ and $\Delta \cap \Sigma = C$. As Σ separates S^3 into B_1 and B_2, Δ lies in one of those 3-balls denoted B_j. The scc $C \subseteq \Sigma$ separates Σ into two 2-disks, D_1 and D_2. By connectedness, p_1 and p_2 both lie in D_1 or D_2. Without loss of generality, let p_1 and p_2 both lie in D_1. So $\Delta \cup D_2$ is a 2-sphere in S^3 that bounds a 3-disk E that is disjoint from θ. Use E to push D_2 slightly past Δ to a parallel copy of Δ. That eliminates at least C from $\Sigma \cap T$ as desired. Repeat that operation finitely many times to eliminate all sccs in $\Sigma \cap T$ that do not meet θ.

Now, each scc in $\Sigma \cap T$ meets θ. As $\Sigma \cap \theta = \{p_1, p_2\}$ we see that $\Sigma \cap T$ consists of one or two sccs. Suppose, by way of contradiction, that $\Sigma \cap T$ consists of two sccs, C_1 and C_2 where $p_1 \in C_1$ and $p_2 \in C_2$. Then in Σ we obtain two different 2-disks that contradict Lemma 2.8. Thus, $\Sigma \cap T$ is one scc, C, that meets θ in exactly $p_1, p_2 \in e_i$. There are three cases to consider.

Case 1. C meets e_1 twice as in Figure 3.1 (left). In that figure, endpoints $1, 2, 3$, and 4

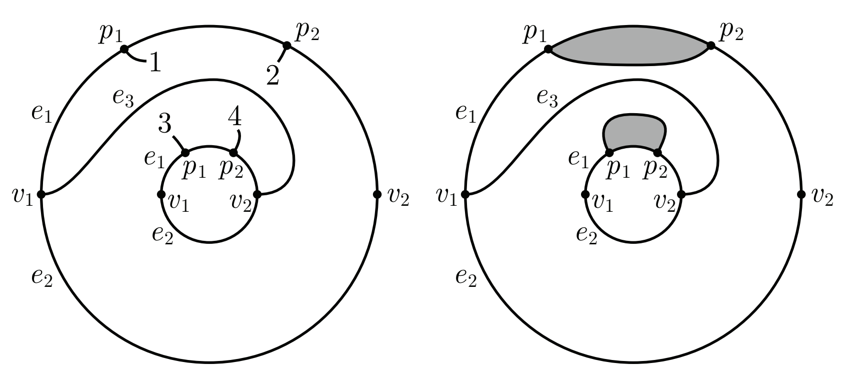

FIGURE 3.1. The annulus A in case C meets e_1 twice (left) and the disks bounded by arcs of C and e_3 (right).

of four arcs are indicated. Those arcs are subarcs of C that must connect in A to form C without meeting θ at any points other than p_1 and p_2. If 1 connects to 2, then 3 must connect to 4. Then the two arcs of C in the annulus bound two 2-disks along arcs of e_1 as in Figure 3.1 (right). Those two 2-disks form a neatly embedded 2-disk, Δ, contained in B_1 such that Δ contains α. That contradicts Lemma 2.1, since (B_1, α) is a knotted ball-arc pair. If 1 connects to 3, then C does not contain p_1, a contradiction. If 1 connects to 4, then 2 and 3 are prevented from connecting, a contradiction.

Case 2. C meets e_2 twice. This case is almost identical to Case 1.

Case 3. C meets e_3 as in Figure 3.2 (left). If 1 connects to 2, then 3 must connect to

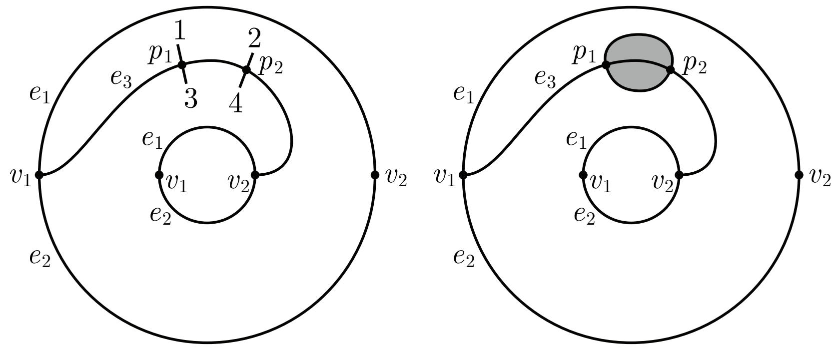

FIGURE 3.2. The annulus A in case C meets e_3 twice (left) and the disk bounded by C (right).

4. Then, C bounds a 2-disk, Δ, in the annulus A as in Figure 3.2 (right) that is neatly embedded in B_1 and contains α. That contradicts Lemma 2.1, since (B_1, α) is a knotted ball-arc pair. If 1 connects to 3, then C does not contain p_2, a contradiction. If 1 connects to 4, then 2 and 3 are prevented from connecting, a contradiction.

Each of the three cases yielded a contradiction. Hence, $\theta \not\approx \theta' \#_2 K$.

Suppose, by way of contradiction, that $\theta \approx \theta_1 \#_3 \theta_2$ where θ_1 and θ_2 are knotted theta-curves. Then, there is a 2-sphere $\Sigma \subseteq S^3$ that is transverse to T and θ such that Σ meets each e_i at exactly one point $p_i \in \operatorname{Int} e_i$. Further, Σ separates S^3 into two 3-balls B_1 and B_2 where $v_1 \in \operatorname{Int} B_1$ and $v_2 \in \operatorname{Int} B_2$, and $(B_1, B_1 \cap \theta)$ and $(B_2, B_2 \cap \theta)$ are knotted ball-prong pairs. We have $\Sigma \cap T$ is finitely many pairwise disjoint sccs. Each such scc either meets θ or is disjoint from θ. Each of the latter type is inessential in $T - \theta \approx \operatorname{Int} D^2$. We will eliminate all of the latter type by ambient isotopy of Σ leaving θ unmoved.

Let C be a scc of the latter type that is innermost in $T - \theta$. So, $C = \partial\Delta$ where Δ is a 2-disk in $T - \theta$ and $\Delta \cap \Sigma = C$. As Σ separates S^3 into B_1 and B_2, Δ lies in one of those 3-balls denoted B_j. The scc $C \subseteq \Sigma$ separates Σ into two 2-disks, D_1 and D_2. By connectedness, p_1, p_2, and p_3 all lie in D_1 or D_2. Without loss of generality, assume that they all lie in D_1. So $\Delta \cup D_2$ is a 2-sphere in S^3 that bounds a 3-disk E that is disjoint from θ. Use E to push D_2 slightly past Δ to a parallel copy of Δ. That eliminates at least C from $\Sigma \cap T$ as desired. Repeat that operation finitely many times to eliminate all sccs in $\Sigma \cap T$ that do not meet θ.

Now, each scc in $\Sigma \cap T$ meets θ. As $\Sigma \cap \theta = \{p_1, p_2, p_3\}$, we see that $\Sigma \cap T$ consists of one, two, or three sccs, and those topological possibilities are depicted in Figure 3.3. The first three possibilities each contain a 2-disk that contradicts Lemma 2.8. Thus, $\Sigma \cap T$ is

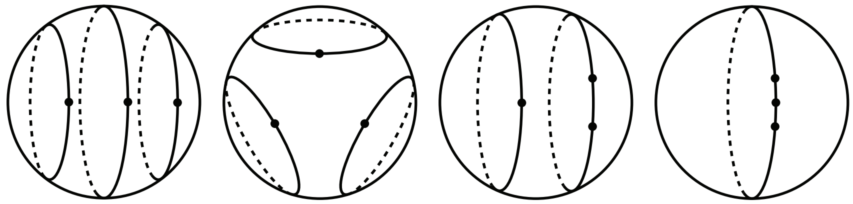

FIGURE 3.3. The sphere Σ in cases where $\Sigma \cap T$ consists of three, two, or one scc.

one scc, C, that meets each e_i at exactly one point p_i for $i = 1, 2, 3$ as in Figure 3.4 (left). In that figure, endpoints $1, 2, \ldots, 6$ of five arcs are indicated. Those arcs are subarcs of C

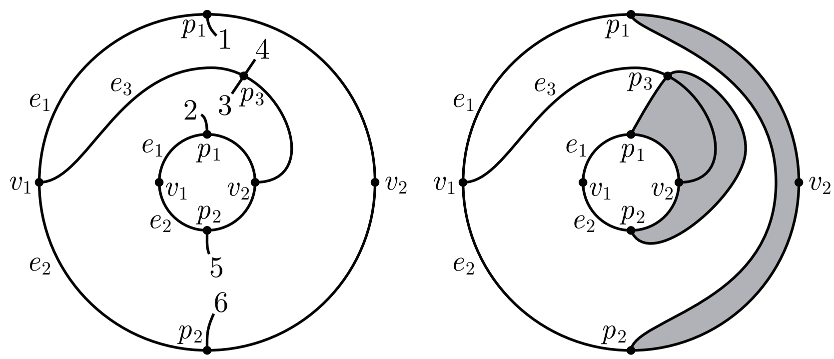

FIGURE 3.4. The annulus A where C meets each arc e_i once (left) and 2-disks bounded by arcs of C, e_1, and e_2 (right).

that must connect in A to form C without meeting θ at any points other than p_1, p_2, and p_3. Notice that 1 cannot connect to 2 since otherwise p_2 and p_3 do not lie in C, 1 cannot connect to 3 since otherwise 6 is closed off, and 1 cannot connect to 5 since otherwise 4 is closed off. There are two remaining cases to consider.

Case 1. 1 connects to 6. Then, 4 cannot connect to 2 since otherwise 5 is closed off, and 4 cannot connect to 3 since p_1, p_2, p_3 all lie in C. So, 4 connects to 5, and 2 connects to 3. Now, the two arcs of C in the annulus bound two 2-disk along arcs of $e_1 \cup e_2$ as in Figure 3.4 (right). Those two 2-disks form a neatly embedded 2-disk, Δ contained in B_1 and containing the prong $B_1 \cap \theta$. That contradicts Lemma 2.1 since $(B_1, B_1 \cap \theta)$ is a knotted ball-prong pair.

Case 2. 1 connects to 4. Then, 3 cannot connect to 2 since p_1, p_2, p_3 must all lie in C, and 3 cannot connect to 5 since otherwise 2 is closed off. So, 3 connects to 6, and 2 connects to 5. As in the previous case, that yields two 2-disks in the annulus whose union Δ is a neatly embedded 2-disk in B_1 containing the prong $B_1 \cap \theta$. That contradicts Lemma 2.1 since $(B_1, B_1 \cap \theta)$ is a knotted ball-prong pair.

Both cases yielded a contradiction. Hence, $\theta \not\approx \theta_1 \#_3 \theta_2$. Therefore, θ is a prime theta-curve. $\square$

4. Constituent knots

We determine the constituent knots of each torus theta-curve $\theta(p,q)$ and compute examples. As $\theta(p,q)$ lies in the torus T, each constituent knot k_i of $\theta(p,q)$ is a torus knot—perhaps the unknot. For each positive integer n, let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ denote the ring of integers modulo n. Let $[k]_n \in \mathbb{Z}_n$ denote the element in $\mathbb{Z}_n$ represented by the integer k. Recall that $[k]_n \in \mathbb{Z}_n$ has a multiplicative inverse if and only if k and n are relatively prime. In that case, the multiplicative inverse of $[k]_n$ is denoted $[k]_n^{-1}$ and $[-k]_n^{-1} = -[k]_n^{-1}$.

Proposition 4.1. *Let p and q be relatively prime integers where $p \geq 2$ and $q \geq 2$. Define the integers j and k by:*

$$\begin{aligned} [j]_p &= [q]_p^{-1} \quad \in \mathbb{Z}_p \quad \text{where } 1 \leq j \leq p-1 \\ [k]_q &= [-p]_q^{-1} \in \mathbb{Z}_q \quad \text{where } 1 \leq k \leq q-1 \end{aligned}$$

Then, the constituent knots of $\theta(p,q)$ are:

$$\begin{aligned} k_3 &= e_1 - e_2 = \quad t(p,q) \\ k_2 &= e_3 - e_1 = -t(j,k) \\ k_1 &= e_2 - e_3 = -t(p-j, q-k) \end{aligned}$$

In particular, the set of unoriented constituent knots of $\theta(p,q)$ is:

$$\{\ t(p,q),\ t(j,k),\ t(p-j,q-k)\ \}$$

where $1 \leq p-j \leq p-1$ and $1 \leq q-k \leq q-1$.

Furthermore, the set of unoriented constituent knots of $\theta(p,-q)$ is:

$$\{\ t(p,-q),\ t(j,-k),\ t(p-j,k-q)\ \}$$

the set of unoriented constituent knots of $\theta(-p,-q)$ is:

$$\{\ t(-p,-q),\ t(-j,-k),\ t(j-p,k-q)\ \}$$

and the set of unoriented constituent knots of $\theta(-p,q)$ is:

$$\{\ t(-p,q),\ t(-j,k),\ t(j-p,q-k)\ \}$$

where $1-p \leq -j \leq -1$, $1-q \leq -k \leq -1$, $1-p \leq j-p \leq -1$, and $1-q \leq k-q \leq -1$.

Proof. Lift $t(p,q)$ to the universal cover $\mathbb{R}^2$ of T as in Figure 2.12 (right). The lift of $t(p,q)$ is the graph of $f(x) = \frac{p}{q}x$. Thus, we seek the integer $1 \leq k \leq q-1$ such that $\frac{p}{q}\left(k + \frac{1}{p}\right) = j$ for some integer $1 \leq j \leq p-1$. Equivalently, we seek the integer $1 \leq k \leq q-1$ such that:

$$1 = jq - kp$$

for some integer $1 \leq j \leq p-1$. The unique solutions to the latter are those claimed in the proposition. That determines the constituent knots of $\theta(p,q)$.

The remaining claims now follow by applying the reflection R and the rotation ρ to $\theta(p,q)$ and its constituent knots (recall definition (2.2) and relations (2.1)). $\square$

Corollary 4.2. *Consider torus theta-curves $\theta(p,q)$ where p and q are relatively prime integers and $|p|, |q| \geq 2$. Among those theta-curves, $\{|p|, |q|\}$ is a homeomorphism invariant of $\theta(p,q)$, and whether the signs of p and q are the same or are opposite is an isotopy invariant of $\theta(p,q)$.*

Proof. Let $\theta(p,q)$ be such a theta-curve. The idea is that $\theta(p,q)$ contains a unique "largest" constituent knot from which we can recover $\{|p|,|q|\}$. In more detail, let $C(p,q)$ be the set of pairs (a,b) of integers such that $t(a,b) \approx k_i$ for some *nontrivial* constituent knot k_i of $\theta(p,q)$. We exclude trivial constituent knots since $t(\pm 1, n)$ is trivial for each integer n and we seek a "largest" element of $C(p,q)$. By Proposition 4.1 and the classification of torus knots, there are exactly two pairs (a,b) in $C(p,q)$ for which $a+b$ is maximal, namely $(|p|,|q|)$ and $(|q|,|p|)$. If $\theta(p',q')$ is another such a theta-curve and $\theta(p',q') \approx \theta(p,q)$, then $C(p',q') = C(p,q)$ and $\{|p'|,|q'|\} = \{|p|,|q|\}$, as desired.

Similarly, let $D(p,q)$ be the set of pairs (a,b) of integers such that $t(a,b) \sim k_i$ for some *nontrivial* constituent knot k_i of $\theta(p,q)$. By Proposition 4.1 and the classification of torus knots, either each pair (a,b) in $D(p,q)$ has a and b with the same sign or each pair (a,b) in $D(p,q)$ has a and b with opposite signs. If $\theta(p',q')$ is another such a theta-curve and $\theta(p',q') \sim \theta(p,q)$, then $D(p',q') = D(p,q)$ and the sign property is identical for both sets, as desired. $\square$

Examples 4.3.

(1) *Adjacent integers: $p = n$ and $q = n+1$.* Consider the torus theta-curve $\theta(n, n+1)$ where $n \geq 2$. Recall that adjacent integers are relatively prime. As in Proposition 4.1, we have $j = 1$, $k = 1$, and the set of unoriented constituent knots of $\theta(n, n+1)$ is:

$$\{\ t(1,1),\ t(n-1,n),\ t(n,n+1)\ \}$$

By Corollary 4.2, the torus theta-curves $\theta(n, n+1)$ for $n \geq 2$ are pairwise inequivalent up to homeomorphism of S^3.

(2) *Consecutive Fibonacci numbers: $p = F_n$ and $q = F_{n+1}$.* Recall the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-2} + F_{n-1}$ for each $n \geq 2$. The Euclidean algorithm implies that consecutive Fibonacci numbers are relatively prime. Consider the torus theta-curve $\theta(F_n, F_{n+1})$ where $n \geq 3$ (so $F_n \geq 2$). Using Proposition 4.1 and the computer algebra system MAGMA, we obtained the data in Table 1.

θ	$\theta(2,3)$	$\theta(3,5)$	$\theta(5,8)$	$\theta(8,13)$	$\theta(13,21)$
k_3	$t(2,3)$	$t(3,5)$	$t(5,8)$	$t(8,13)$	$t(13,21)$
k_2	$-t(1,1)$	$-t(2,3)$	$-t(2,3)$	$-t(5,8)$	$-t(5,8)$
k_1	$-t(1,2)$	$-t(1,2)$	$-t(3,5)$	$-t(3,5)$	$-t(8,13)$

TABLE 1. Constituent knots of torus theta-curves arising from consecutive Fibonacci numbers.

It appears that the constituent knots of $\theta(F_n, F_{n+1})$ are $k_3 = t(F_n, F_{n+1})$, $-t(F_{n-1}, F_n)$, and $-t(F_{n-2}, F_{n-1})$ where the roles of k_2 and k_1 alternate depending on the parity of n. Indeed, that is the case as we now prove.

Recall two classical Fibonacci identities:

$$F_{n-1}F_{n+1} - F_nF_n = (-1)^n \quad \text{for } n \geq 1$$
$$F_{n-1}F_n - F_{n-2}F_{n+1} = (-1)^n \quad \text{for } n \geq 2$$

The former is Cassini's identity from 1680 and the latter is a special case of Tagiuri's identity from 1901. Both identities are readily proven by induction as follows. For Cassini's identity, consider the determinant of the matrix $\begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$. For the inductive step, add the second column to the first and then swap columns. For Tagiuri's identity, consider the determinant of the matrix $\begin{bmatrix} F_{n-1} & F_{n-2} \\ F_{n+1} & F_n \end{bmatrix}$. For the inductive step, add the first column to the second and then swap columns.

If $n \geq 4$ is even, then Cassini's identity is $F_{n-1}F_{n+1} - F_nF_n = 1$. So, $j = F_{n-1} = F_{n+1}^{-1} \in \mathbb{Z}_{F_n}$ and $k = F_n = -F_n^{-1} \in \mathbb{Z}_{F_{n+1}}$. By Proposition 4.1, the constituent knots of $\theta(F_n, F_{n+1})$ are $k_3 = t(F_n, F_{n+1})$, $k_2 = -t(F_{n-1}, F_n)$, and $k_1 = -t(F_{n-2}, F_{n-1})$.

If $n \geq 3$ is odd, then Tagiuri's identity is $F_{n-2}F_{n+1} - F_{n-1}F_n = 1$. So, $j = F_{n-2} = F_{n+1}^{-1} \in \mathbb{Z}_{F_n}$ and $k = F_{n-1} = -F_n^{-1} \in \mathbb{Z}_{F_{n+1}}$. By Proposition 4.1, the constituent knots of $\theta(F_n, F_{n+1})$ are $k_3 = t(F_n, F_{n+1})$, $k_2 = -t(F_{n-2}, F_{n-1})$, and $k_1 = -t(F_{n-1}, F_n)$.

Hence, for each $n \geq 3$, the set of unoriented constituent knots of $\theta(F_n, F_{n+1})$ is:

$$\{\ t(F_{n-2}, F_{n-1}),\ t(F_{n-1}, F_n),\ t(F_n, F_{n+1})\ \}$$

By Corollary 4.2, the torus theta-curves $\theta(F_n, F_{n+1})$ for $n \geq 3$ are pairwise inequivalent up to homeomorphism of S^3.

We now classify torus theta-curves of the form $\theta(p, q)$.

Theorem 4.4. *Consider a torus theta-curve $\theta(p,q)$ where p and q are relatively prime integers and $|p|, |q| \geq 2$. Let $\theta'(p', q')$ be another such torus theta-curve. Then, $\theta(p', q') \approx \theta(p, q)$ if and only if $\{p', q'\}$ equals $\{p, q\}$, $\{-p, -q\}$, $\{p, -q\}$, or $\{-p, q\}$. And, $\theta(p', q') \sim \theta(p, q)$ if and only if $\{p', q'\}$ equals $\{p, q\}$ or $\{-p, -q\}$.*

Proof. If $\theta(p', q') \approx \theta(p, q)$, then Corollary 4.2 implies that $\{|p'|, |q'|\} = \{|p|, |q|\}$, as desired. The converse follows from the relations (2.3).

If $\theta(p', q') \sim \theta(p, q)$, then Corollary 4.2 implies that $\{|p'|, |q'|\} = \{|p|, |q|\}$ and either p' and q' have the same signs and p and q have the same signs, or p' and q' have opposite signs and p and q have opposite signs, as desired. The converse follows from the relations (2.3). □

5. Classification of torus theta-curves

In this section, we prove the following classification of torus theta-curves. Recall that a **torus theta-curve** is any theta-curve that lies on the standard torus $T \subseteq S^3$. A knot or theta-curve is **knotted** provided it is knotted in S^3. Each constituent knot of a torus theta-curve is a torus knot—perhaps unknotted.

Theorem 5.1 (Classification of torus theta-curves)**.** *If θ is a torus theta-curve, then exactly one of the following holds:*

(1) All three constituent knots of θ are unknotted, in which case θ itself is unknotted.

(2) At least one constituent knot of θ is knotted and another is inessential in T, in which case $\theta = \theta_U \#_2\, t(a, b)$ is a 2-connected sum of the trivial theta-curve θ_U and a nontrivial torus knot $t(a, b)$.

(3) At least one constituent knot of θ is knotted and each is essential in T, in which case θ is isotopic in T to $\theta(p,q)$ for some relatively prime integers p and q where $|p|, |q| \geq 2$.

Remarks 5.2.

(1) Prime torus theta-curves—up to isotopy on T—are exactly the $\theta(p,q)$ for relatively prime integers p and q where $|p|, |q| \geq 2$. That follows immediately from Theorems 3.1 and 5.1.

(2) A torus theta-curve is never a 3-connected sum of nontrivial theta-curves. By Theorem 5.1, there are three cases to consider. The first case follows by Lemma 2.2, the second follows by Lemma 2.3 since nontrivial torus knots are prime, and the third follows by Theorem 3.1.

(3) Given a torus theta-curve θ, it is straightforward to determine which of the three mutually exclusive possibilities in Theorem 5.1 holds for θ. For each constituent knot k_i of θ, compute the signed intersection numbers $I(k_i, l)$ and $I(k_i, m)$ where l is a longitude and m is a meridian of T as in Figure 2.10. Those two intersection numbers classify k_i up to isotopy on T (see the end of Section 2). Note that k_i is inessential in T if and only if both of those intersection numbers are zero. Now, apply the classification of torus knots to the constituent knots of θ.

(4) Theorem 5.1 combines with results above to completely classify torus theta-curves up to homeomorphism of S^3—allowing reflections—and up to isotopy of S^3. First, all unknotted theta-curves are isotopic in S^3. Second, theta-curves of the form $\theta_U \#_2 K$ are classified by the knot type of K, namely:

$$\begin{aligned} \theta_U \#_2 K \approx \theta_U \#_2 K' \quad &\Leftrightarrow \quad K \approx K' \\ \theta_U \#_2 K \sim \theta_U \#_2 K' \quad &\Leftrightarrow \quad K \sim K' \end{aligned}$$

Proofs of those two facts are straightforward. (An observation in Section 2 is helpful: $\theta \#_2 K \approx \theta_U$ if and only if $K \approx U$ and $\theta \approx \theta_U$.) Now, apply the classification of torus knots. Third, Theorem 4.4 classified torus theta-curves of the form $\theta(p,q)$ where p and q are relatively prime and $|p|, |q| \geq 2$. In summary, torus theta-curves are completely classified up to homeomorphism and up to isotopy in S^3 by two properties of their constituent knots: their knot types in S^3 and whether they are essential in T.

The remainder of this section is devoted to proving Theorem 5.1. We will state three lemmas, prove Theorem 5.1 using those lemmas, and then prove the lemmas. Let θ denote an arbitrary theta-curve, typically in S^3 but in a surface where indicated. Let Σ_g denote the closed, orientable surface of genus $g \geq 0$. We state and prove the first lemma for theta-curves in Σ_g. While we only use the lemma with $g = 1$, our proof works for arbitrary genus and the general result may be useful for further study of theta-curves.

Lemma 5.3. *Let $\theta \subseteq \Sigma_g$ where $g \geq 0$. If θ contains at least two constituent knots that are inessential in Σ_g, then there is a constituent knot of θ that bounds a 2-disk $\Delta \subseteq \Sigma_g$ containing the third arc of θ. In particular, all three constituent knots of θ are inessential in Σ_g.*

Lemma 5.4. *If all three constituent knots of a torus theta-curve θ are unknotted, then θ is unknotted.*

Corollary 5.5. *If θ is knotted but all three constituent knots of θ are unknotted, then θ cannot lie on T. In particular, Kinoshita's theta-curve in Figure 2.6 is not a torus theta-curve.*

Remark 5.6. A torus theta-curve may have two unknotted constituent knots and still be knotted. Proposition 4.1 implies that for each positive integer n, $\theta(2, 2n+1)$ is an example. In fact, those are all such examples of the form $\theta(p, q)$ where p and q are relatively prime integers and $2 \leq p < q$. To see that, note that in Proposition 4.1, $j = 1$ and $p - 1 = 1$ yield the examples $\theta(2, 2n + 1)$. And, $j = 1$ and $q - k = 1$ is not possible since then the key equation $1 = jq - kp$ becomes $1 = q - (q - 1)p$. The latter implies $q(p - 1) = p - 1$, a contradiction.

Next, we define a collection of torus theta-curves, denoted $\theta(p, q, r)$, where p and q are relatively prime integers, $p, q \geq 2$, and r is an integer. They generalize torus theta-curves $\theta(p, q)$ since $\theta(p, q, 0) = \theta(p, q)$. Define $\theta(p, q, r)$ in the same way that $\theta(p, q)$ was defined in

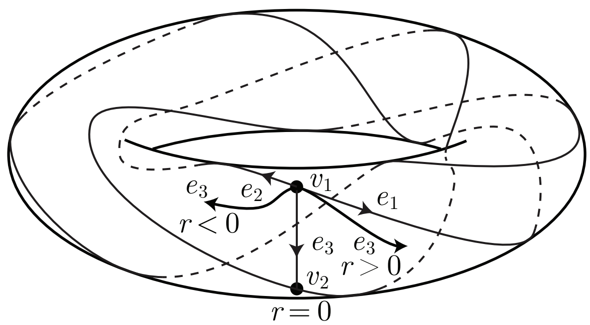

FIGURE 5.1. Torus theta-curve $\theta(p, q, r)$ where p and q are relatively prime integers, $p, q \geq 2$, and r is an integer. Also indicated are various possibilities for the essential arc e_3 determined by the integer r.

Section 2 except now e_3 winds $r \in \mathbb{Z}$ times around the annulus A obtained by cutting T along $t(p, q)$ as in Figure 5.1. Note that e_3 has a boundary point in each of the two boundary circles of A. So, e_3 is essential in A. Proposition 4.1 and the signed intersection numbers $I(k_i, l)$ and $I(k_i, m)$ determine the constituent knots of $\theta(p, q, r)$. As in Proposition 4.1, define the integers j and k by:

$$\begin{aligned} [j]_p &= [q]_p^{-1} \in \mathbb{Z}_p \quad \text{where } 1 \leq j \leq p-1 \\ [k]_q &= [-p]_q^{-1} \in \mathbb{Z}_q \quad \text{where } 1 \leq k \leq q-1 \end{aligned}$$

The constituent knots of $\theta(p, q, r)$ are:

$$\begin{aligned} k_3 &= e_1 - e_2 = t(p, q) \\ k_2 &= e_3 - e_1 = t(rp - j, rq - k) \\ k_1 &= e_2 - e_3 = t(j - (r+1)p, k - (r+1)q) \end{aligned} \tag{5.1}$$

Lemma 5.7. *If θ is a torus theta-curve, θ contains a knotted constituent knot, and each constituent knot of θ is essential in T, then θ is isotopic on T to $\theta(p, q)$ for some relatively prime integers p and q where $|p|, |q| \geq 2$.*

We now use the three lemmas to prove Theorem 5.1.

Proof of Theorem 5.1. If all three constituent knots of θ are unknotted, then θ is unknotted by Lemma 5.4. Otherwise, θ contains at least one knotted constituent knot. Without loss of generality, assume $k_3 = e_1 - e_2$ is knotted. Then, $k_3 = t(a, b)$ for some relatively prime integers a and b where $|a|, |b| \geq 2$. Let A denote the annulus obtained by cutting T along k_3. As k_3 is essential in T, θ has a constituent knot inessential in T if and only if e_3 is inessential in A. So, if θ has a constituent knot inessential in T, then $\theta = \theta_U \#_2 t(a, b)$, as desired. Otherwise, each constituent knot of θ is essential in T and the result follows by Lemma 5.7. $\square$

It remains to prove the three lemmas.

Proof of Lemma 5.3. If $g = 0$, then the result follows by the 2-dimensional Schoenflies theorem [Moi77, p. 71]. So, assume $g > 0$. Let $\pi : \widetilde{\Sigma}_g \to \Sigma_g$ be the universal covering map where $\widetilde{\Sigma}_g \approx \mathbb{R}^2$. Without loss of generality, assume $k_1 = e_2 - e_3$ and $k_2 = e_3 - e_1$ are inessential in Σ_g. Let Δ_1 and Δ_2 be the 2-disks in Σ bounded by k_1 and k_2 respectively. If Δ_1 contains e_1 or Δ_2 contains e_2, then we are done. So, assume e_1 does not lie in Δ_1 and e_2 does not lie in Δ_2. The 2-disks Δ_1 and Δ_2 are given by embeddings $f_1 : D^2 \to \Sigma_g$ and $f_2 : D^2 \to \Sigma_g$ respectively. Lift f_1 to $\widetilde{f_1} : D^2 \to \widetilde{\Sigma}_g$ so $f_1 = \pi \circ \widetilde{f_1}$. Note that $\widetilde{f_1}$ is an embedding, and the restriction of $\widetilde{f_1}$ to ∂D^2 is a lift of k_1 to a scc in $\widetilde{\Sigma}_g$. Let $\widetilde{v}_1$ and $\widetilde{v}_2$ denote the lifts under f_1 of v_1 and v_2 respectively. Lift f_2 to $\widetilde{f_2} : D^2 \to \widetilde{\Sigma}_g$ so $f_2 = \pi \circ \widetilde{f_1}$ and v_1 lifts to $\widetilde{v}_1$ (see Moise [Moi77, p. 176]). Note that $\widetilde{f_2}$ is an embedding, and the restriction of $\widetilde{f_2}$ to ∂D^2 is a lift of k_2 to a scc in $\widetilde{\Sigma}_g$. Restricting f_1 and f_2 to arcs in the boundary of D^2 corresponding to e_3, uniqueness of lifts implies that f_2 lifts v_2 to $\widetilde{v}_2$. Thus, we may restrict f_1 and f_2 to ∂D^2 and paste together those restrictions to obtain a lift $\widetilde{\theta}$ of θ embedded in $\widetilde{\Sigma}_g$. Note that π restricted to $\widetilde{\theta}$ is injective. As $\widetilde{\Sigma}_g \approx \mathbb{R}^2$, Moise [Moi77, pp. 20 & 68] implies that exactly one constituent knot of $\widetilde{\theta}$ bounds a 2-disk $\widetilde{\Delta} \subseteq \widetilde{\Sigma}_g$ containing the third arc of $\widetilde{\theta}$. As π restricted to $\partial\widetilde{\Delta}$ is injective, Lemma 1.6 of Epstein [Eps66] implies that π restricted to $\widetilde{\Delta}$ is injective. Define $\Delta = \pi\left(\widetilde{\Delta}\right)$ which is a 2-disk embedded in Σ, bounded by a constituent knot of θ, and containing the third arc of θ. Recall that e_1 does not lie in Δ_1, e_2 does not lie in Δ_2, and each inessential scc in Σ_g (where $g > 0$) bounds a unique 2-disk in Σ_g. Hence, $\partial\Delta = k_3 = e_1 - e_2$ and Δ contains e_3, as desired. $\square$

Proof of Lemma 5.4. If at least two constituent knots of θ are inessential in T, then θ lies in a 2-disk in T by Lemma 5.3 and hence is unknotted. So, assume θ contains a constituent knot that is essential in T. Recall the standard longitude l and meridian m of T in Figure 2.11. Evidently, if a constituent knot of θ is isotopic in T to m or to l, then θ is unknotted. By the classification of torus knots and the relations (2.1), it remains, without loss of generality, to consider the case $k_3 = e_1 - e_2 = t(1, q)$ where q is a positive integer. Let A denote the annulus obtained by cutting T along k_3. If e_3 is inessential in A, then θ is unknotted. So, assume e_3 is essential in A. That implies e_3 has a boundary point in each of the two boundary circles of A. Hence, up to isotopy on T fixing k_3 pointwise, e_3 is determined by an integer r that records the number of times e_3 winds around A as in Figure 5.2. Note that for any integer r, e_3 always terminates at v_2 *on the same side* of k_3 since e_3 is essential in A. The signed intersection numbers $I(k_i, l)$ and $I(k_i, m)$ determine the constituent knots

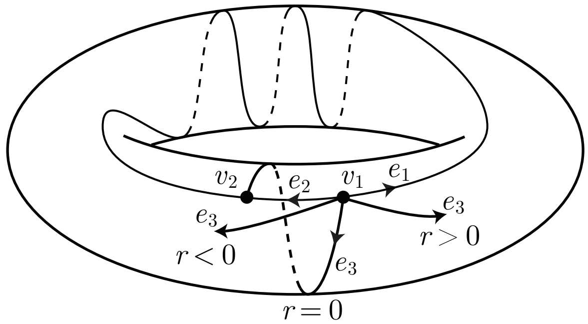

FIGURE 5.2. Constituent knot $k_3 = e_1 - e_2 = t(1, q)$ where q is a positive integer ($q = 3$ is depicted) and various possibilities for the essential arc e_3 determined by the integer r.

of θ:

$$\begin{aligned} k_3 &= e_1 - e_2 = t(1, q) \\ k_2 &= e_3 - e_1 = t(r - 1, q(r - 1) + 1) \\ k_1 &= e_2 - e_3 = t(-r, -qr - 1) \end{aligned} \tag{5.2}$$

In each case $q \in \mathbb{Z}_+$ and $r \in \mathbb{Z}$, either a constituent knot of θ is isotopic on T to m or l implying θ is unknotted, or a constituent knot of θ is knotted which contradicts the hypotheses. In more detail, if $r = 0$, then $k_1 = m$. If $r = 1$, then $k_2 = m$. If $r \geq 2$, then k_1 is knotted. If $r = -1$ and $q = 1$, then $k_1 = l$. If $r = -1$ and $q \geq 2$, then k_2 is knotted. If $r \leq -2$, the k_2 is knotted. $\square$

Proof of Lemma 5.7. By hypothesis, $\theta \subseteq T$ has a constituent knot $t(p', q')$ where p' and q' are relatively prime integers and $|p'|, |q'| \geq 2$. The desired conclusion is invariant under reflection of S^3. So, we can and do assume $p', q' \geq 2$. Without loss of generality, assume $k_3 = t(p', q')$. Let A denote the annulus obtained by cutting T along k_3. By hypothesis, each constituent knot of θ is essential in T. Thus, e_3 is essential in A and e_3 has a boundary point in each of the two boundary circles of A. Up to isotopy on T fixing k_3 pointwise, e_3 is determined by an integer r' that records the number of times e_3 winds around A as in Figure 5.1. So, $\theta = \theta(p', q', r')$ where p' and q' are relatively prime integers, $p', q' \geq 2$, and r' is an integer. If $r' = 0$, then we're done. So, assume $r' \neq 0$.

The constituent knots of $\theta(p', q', r')$ were determined in (5.1). Among those three torus knots $t(a, b)$, there is one that is "largest" in the sense that $|a| + |b|$ is maximal. Direct inspection of (5.1) shows that k_1 is largest when $r' > 0$ and k_2 is largest when $r' < 0$. Furthermore, direct inspection of (5.1) shows that in both of those cases the largest constituent knot is knotted. Isotop $\theta(p', q', r')$ on T to $\theta(p, q, r)$ so that the largest constituent knot of $\theta(p', q', r')$ is carried to k_3 of $\theta(p, q, r)$. As the largest constituent knot of $\theta(p', q', r')$ is knotted, we have that p and q are relatively prime integers, $p, q \geq 2$, and r is an integer. Now, let k_i refer to a constituent knot of $\theta(p, q, r)$. As k_3 is the largest constituent knot of $\theta(p, q, r)$, direct inspection of (5.1) implies that $r = 0$, as desired. $\square$

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