OBSERVABLE DYNAMICS AND THE GENERIC COINCIDENCE OF MILNOR, STATISTICAL, AND PHYSICAL ATTRACTORS

MAGDALENA FORYŚ-KRAWIEC, JANA HANTÁKOVÁ, MICHAŁ KOWALEWSKI, AND PIOTR OPROCHA

Abstract. We study the observable long-term behavior of typical continuous dynamical systems on the interval $[0,1]$. For a residual subset of $C([0,1])$, the Milnor, statistical, and physical (in the sense of Ilyashenko) attractors coincide and are equal to the non-wandering set. This unified attractor governs the time-averaged dynamics of almost all initial conditions and depends continuously on the map with respect to the Hausdorff metric. From the physical viewpoint, it represents the ensemble of observable steady states describing the system's long-term statistical behavior. Nevertheless, it is not Lyapunov stable and contains no dense orbits, implying the generic absence of Palis attractors. Thus, generic continuous dynamics admit a well-defined observable attractor even when all classical mechanisms of stability fail, showing how observable statistical behavior persists in the absence of SRB measures or hyperbolic structure.

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1. Introduction

A *(discrete) dynamical system* is a pair (X, f) where X is a compact metric space and f is a continuous map acting on X. To understand the dynamical properties of such a system, one must analyze the behavior of trajectories of points $x \in X$ under the iteration of f. *Limit sets* of trajectories provide a useful tool for this purpose, as they describe the long-term behavior of the system.

The study of trajectories and their limit sets dates back to Henri Poincaré, who, in the late nineteenth century, examined the evolution of sets of initial points rather than individual orbits, revealing the possibility of highly complex (now called *chaotic*) dynamics. We call a limit set of a "large" set of initial points an *attractor*, where "large" may be interpreted in various ways (topologically or measure-theoretically).

Attractors have played a central role in the theory of dynamical systems, though a consensus on their precise definition emerged only later. A major step was John Milnor's seminal paper *On the Concept of Attractor* (1985) [25], which introduced a definition ignoring initial conditions on a set of zero measure—analogous to a meager set in the topological setting. This reflects the idea that physically observable behavior should be insensitive to negligible sets of initial conditions, a principle natural to experiments where initial states cannot be specified exactly.

A stronger notion of attractor, proposed by Arnold et al. [3], requires the convergence of almost all trajectories (up to a null set), while ignoring regions of the space visited with zero frequency. This notion, often termed a *statistical attractor*, captures sets approached with non-vanishing time-average frequency.

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Statistical and Milnor attractors are therefore attempts to formalize the "observable" long-term behavior of trajectories. However, substantial portions of such attractors may still be practically unobservable. For instance, [23] presents an open set of dynamical systems whose topological attractors contain large unobservable subsets. Seeking a definition that better reflects physical observability, Ilyashenko [22] introduced the notion of a *minimal* or *physical attractor*, using the classical Krylov–Bogolyubov approach to construct invariant measures.

A set A is called a *physical attractor* if it is the closure of the union of supports of all invariant measures μ_∞, where μ_∞ arises as a weak* limit of some subsequence of

$$\frac{1}{n}\sum_{i=0}^{n-1}\mu_i, \quad \text{where } \mu_0 \text{ is the Lebesgue measure and } \mu_n = f_*^n\mu_0.$$

The measures μ_∞ are called physical when the set of points generating them has positive Lebesgue measure. However, such physical measures are generically absent in the C^0 topology [1], even though almost every trajectory still has a well-defined statistical behavior. This naturally leads to the question of which set governs the observable dynamics; the notion of a physical attractor addresses precisely this. Physical attractors in this sense were studied by Ilyashenko and Gorodetski [16], who showed that physical and statistical attractors generally differ [24], with the physical attractor always a subset of the statistical one.

We also note that, in the interval setting, Catsigeras and Troubetzkoy [12] introduced *pseudo-physical measures* to describe time averages on sets of positive measure (see also [13] for the general C^0 setting) and studied their typical entropy properties, whereas our focus here is on the topological structure and robustness of attractors.

Within this framework, a natural question arises: which properties of attractors can be regarded as *typical*? Following the topological approach, we call a property *generic* if the corresponding set of maps forms a residual subset of $C(X)$—that is, a dense G_δ set. The question raised in [34, 20] asks whether, generically, these different definitions of attractor coincide.

Problem 1.1 (Problem 1, [20])**.** *Is it true that, for generic dynamical systems, all the various types of attractors coincide?*

In some classes of systems the answer is affirmative. For instance, Okunev [28] proved that for skew-product maps with a circle as fiber over an Anosov diffeomorphism, the Milnor and statistical attractors coincide, are Lyapunov stable, and either have zero Lebesgue measure or coincide with the entire phase space. As we will see, however, Lyapunov stability need not hold in general.

We next recall Palis's celebrated conjecture on the finitude of attractors, formulated for interval dynamics as follows [31].

Conjecture 1.2 ([31], Conjecture on the Finitude of Attractors (for interval maps))**.** *For a generic k-parameter family $(f_t)_{t\in[-1,1]^k} \subset C^r([0,1])$, we have that for Lebesgue almost every $t \in [-1,1]^k$, the function f_t has only finitely many attractors, each either regular (a periodic orbit) or stochastic (carrying an invariant probability measure absolutely continuous with respect to the Lebesgue measure on $[0,1]$).*

Note that Palis's definition of attractor differs from Milnor's: a set A is an attractor if it has a basin of positive Lebesgue measure and is transitive, i.e., possesses a dense orbit. The existence of such attractors is not guaranteed—indeed, the identity map has none.

Palis's conjecture parallels Problem 1.1 but considers typicality across parameter families rather than individual maps, which corresponds to the notion of k-*prevalence*, a measure-theoretic analogue of genericity in function spaces [17]. Note that prevalence may correspond to nowhere dense set, as any dense, countable union of Cantor sets can always be transformed into a set of full Lebesgue measure in $[-1,1]^k$ by Oxtoby-Ulam theorem.

All of the attractor notions discussed so far do not impose any Lyapunov or structural stability requirements: their basins may have intricate topology, and they need not admit an attracting neighborhood. The classical notion of attractor requires isolation: an attractor must admit a neighborhood containing no other invariant pieces. Here, a "classical attractor" refers to an asymptotically stable invariant set, i.e., an invariant set with an attracting neighborhood and Lyapunov stability. However, a number of fundamental examples (due to Newhouse, Shub, and Guckenheimer–Williams) show that such isolation or stability typically fails; the chain recurrent set often contains infinitely many classes accumulating on one another.

This motivates replacing isolated attractors with a more flexible notion. Conley introduced *quasi-attractors* as intersections of nested attracting regions, capturing the chain-transitive core of recurrent dynamics even when no genuine isolated attractor exists. For C^0-generic homeomorphisms of a compact manifold, Hurley [19] proved that the basins of chain–transitive quasi-attractors form a residual subset of the phase space; however, such quasi-attractors may still have zero Lebesgue measure and therefore exhibit no physically observable behavior. Bonatti and Crovisier [11] obtained the analogous result in the C^1 setting and further identified isolated quasi-attractors with homoclinic classes. Bonatti, Li, and Yang [10] showed that, on any compact 3-manifold, there exists a non-empty C^1-open set of diffeomorphisms for which C^r-generic systems (for any $r \geq 1$) admit *no stable classical attractors* at all. Nevertheless, Potrie [32] proved that for a C^2-open class of partially hyperbolic diffeomorphisms on $\mathbb{T}^3$, there exists a unique Milnor attractor supporting a unique SRB measure; in this case, the Milnor attractor coincides with the physical attractor.

In the purely continuous setting, however, the situation is more extreme. Abdenur and Andersson [1] showed that a C^0-generic map on any compact manifold admits no physical (SRB) measures at all: although time averages exist for Lebesgue almost every point, they do not converge to a single invariant measure on any set of positive Lebesgue measure. Thus, even the notion of "observable statistical behavior" breaks down generically in C^0 dynamics. The authors of [1] call such dynamical systems *weird*, provided that additionally there is a basin of attraction of full Lebesgue measure whose preimage is of zero measure. It is also worth mentioning, that for the interval maps result of this type (in different terminology) was obtained first in [26].

In this work, we show that for continuous interval maps one can still identify a canonical observable attractor, although it does not arise from Lyapunov stability or from the existence of a physical measure. More precisely, we prove that for a residual subset of continuous maps, the global Milnor, statistical, and physical attractors all coincide and are equal to the non-wandering set. This attractor is a Cantor set. Solenoidal Cantor attractors were analyzed in the smooth one-dimensional setting by Blokh and Lyubich [9], and solenoidal structures will also play a key role in our arguments.

Main Results. We denote by $C([0,1])$ the space of continuous self-maps of the interval endowed with the uniform topology.

Theorem A. *There exists a residual subset $\mathcal{R} \subset C([0,1])$ such that for every $f \in \mathcal{R}$ the non-wandering set $\Omega(f)$ is a Cantor set of zero Hausdorff dimension, and the Milnor attractor, the statistical attractor, and the physical attractor of f all coincide and equal $\Omega(f)$.*

In most contexts, *robustness* refers to the persistence of key dynamical features under small perturbations, a notion particularly relevant in modeling where systems are subject to noise and uncertainty. Milnor [25] asked whether attractors that persist under perturbation must be Lyapunov stable. In general, ω-limit sets are rarely robust; small perturbations may drastically alter their structure, especially when they are neither minimal nor isolated [37]. In contrast to Milnor's expectation, we show that for continuous interval maps the generic observable attractor is robust but not Lyapunov stable. Moreover, these generic attractors contain no dense orbits and, therefore, cannot be Palis attractors, in particular, for such maps, no Palis attractors exist. This stands in sharp contrast to the smooth setting, where Cao–Mi–Yang [14] obtained positive evidence toward Palis' conjecture by establishing the abundance of SRB measures for C^1 diffeomorphisms. In our one-dimensional, nonsmooth setting, the generic observable attractor exists, but is never a Palis attractor.

Theorem B. *For every f in the residual family $\mathcal{R}$, the global Milnor attractor $\Omega(f)$ is robust with respect to the Hausdorff metric, but it is not Lyapunov stable and contains no dense orbits. In particular, Palis attractors are generically absent.*

Notably, unlike in Hurley's results on generic quasi-attractors for homeomorphisms [19], the generic attractor we obtain is not a quasi-attractor at all.

Organization of the paper. Section 2 recalls basic definitions and properties of limit sets and attractors, and summarizes their fundamental inclusion relations and stability notions. Section 3 establishes the generic structure of the attractor for continuous interval maps and contains the main result (Theorem 3.2), from which Theorems A and B stated above follow directly. Section 4 provides explicit examples showing that the inclusions $A_{\min} \subsetneq A_{\mathrm{stat}} \subsetneq \Lambda$ may be strict.

2. Preliminaries

2.1. **Basic definitions and properties.** Let X be a smooth compact manifold and $f : X \to X$ a continuous function. On a smooth compact manifold, a Riemannian metric induces a natural volume form, which defines a measure that locally coincides with the Lebesgue measure in coordinate charts. Let λ denote this 'Lebesgue' measure on X. The *orbit of* x, denoted by $\mathrm{Orb}_f(x)$, is the set $\{f^n(x) \colon n \geq 0\}$, and the *$\omega$-limit set of* x, denoted by $\omega_f(x)$, is defined as the intersection $\bigcap_{n\geq 0} \overline{\{f^m(x) \colon m \geq n\}}$. If the map f is clear from the context, we denote the orbit of a point x by $\mathrm{Orb}(x)$ and the ω-limit set by $\omega(x)$. The point x is *fixed* if $f(x) = x$. The point x is called *periodic* if $f^p(x) = x$ for some $p \in \mathbb{N}$. The smallest such integer p is called the *prime period* of x. Note that throughout this paper $\mathbb{N}$ denotes the set of positive integers. The point x is *recurrent* if $x \in \omega_f(x)$. Recall that a point x is called *non-wandering* if for every neighborhood U of x there exists an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. By $\mathrm{Fix}(f)$, $\mathrm{Per}(f)$, $\mathrm{Rec}(f)$, $\Omega(f)$ we denote the set of fixed, periodic, recurrent and non-wandering points of f respectively. A point $c \in [0, 1]$ is called a *critical point* if f is not locally injective at c, equivalently, if f is not strictly monotone on any neighborhood of c.

2.2. **Measures.** Let (X, μ) be a measure space and $f \colon X \to Y$ a continuous function. The *pushforward* of the measure μ under f is defined as $f_*\mu = \mu \circ f^{-1}$, which is a measure on Y capturing the distribution of f-images of μ-typical points.

In our context, we are particularly interested in the case where $X = Y$ and $f \colon X \to X$ is a continuous transformation of a compact metric space. Given a reference measure, typically the Lebesgue measure λ, we define the sequence of pushforward measures $\mu_n = f_*^n \lambda$, and consider the averages:

$$\mu_n^{\text{avg}} = \frac{1}{n}\sum_{i=0}^{n-1} \mu_i = \frac{1}{n}\sum_{i=0}^{n-1} f_*^i \lambda.$$

By the Krylov–Bogolyubov procedure, any weak-* accumulation point of this sequence is an f-invariant measure. These limiting measures form the basis for the definition of the *physical* (or *minimal*) attractor, which is given by the closure of the union of the supports of all such limiting measures. An equivalent characterization of the physical attractor will be introduced later via the notion of essential sets.

2.3. **Limit sets.** Recall that the ω-limit set of a point $x \in X$ can equivalently be written as

$$\omega(x) = \{z \in X : \limsup_{n\to\infty} \chi_U(f^n(x)) > 0, \forall U \in Nb_z\},$$

where χ_U denotes the indicator function of the set U, and Nb_z is the system of all neighborhoods of a point z.
We define the σ-limit set of a point $x \in X$ as

$$\sigma(x) = \{z \in X : \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k(x)) > 0, \forall U \in Nb_z\}.$$

The σ-limit set is also known as the *minimal center of attraction* [36] or simply the *statistically limit set* [34].
An ω-limit set $\omega(x) \subset [0,1]$ is called a *solenoid* if it can be written in the form

$$\omega(x) = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{k_i} I_{ij}$$

where I_{ij} are closed intervals, for $i \in \mathbb{N}, j = 1, \dots, k_i$, such that the sequence $\{k_i\}_{i=1}^{\infty}$ is unbounded, and

$$\bigcup_{j=1}^{k_i} I_{ij} \subset \bigcup_{j=1}^{k_{i+1}} I_{i+1,j}, \text{ for every } i \in \mathbb{N} \tag{1}$$

$$f(I_{ij}) \subset I_{i,j+1} \mod k_i, \text{ for every } i \in \mathbb{N}, j = 1, \dots, k_i \tag{2}$$

$$\lim_{i\to\infty} \max_j \operatorname{diam} I_{ij} = 0. \tag{3}$$

Note that the condition (3) is sometimes omitted, in which case $\omega(x)$ is only required to be covered by the nested intersection of periodic intervals $\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{k_i} I_{ij}$. This approach is used, for example, in the work of Blokh or Smítal. However, in the case when $\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{k_i} I_{ij}$ has a nonempty interior, any interval contained in it is a wandering interval. We exclude the possibility of a wandering interval in our construction by requiring condition (3).
The following result, stated as Lemma 1 in [26], establishes that for solenoids, the statistical and the ω-limit sets coincide:

Lemma 2.1. [26, Lemma 1] *If $\omega(x)$ is a solenoid, then $\sigma(x) = \omega(x)$.*

2.4. **Attractors.** For any nonempty set $U \subset X$, the *basin of attraction* of U is defined as the set of all points $x \in X$ for which the ω-limit set is contained in U; that is,

$$B(U) = \{x \in X : \omega(x) \subseteq U\}.$$

There are generally two perspectives when defining attractors in dynamical systems. A *local attractor* describes the long-term behavior of a set of initial conditions of positive (Lebesgue) measure, whereas a *global attractor* governs the behavior of almost all initial conditions, with its basin of attraction covering the entire space up to a set of measure zero. One can naturally view a global attractor as a maximal attractor (with respect to inclusion), and local attractors as its subsets. While the following definitions may not always distinguish explicitly between local and global versions, this distinction can typically be made in context.

Definition 2.2 (Milnor local attractor)**.** *The Milnor attractor $A \subset X$ is a closed invariant set with $\lambda(B(A)) > 0$, such that there is no closed proper subset $A' \subset A$ for which $B(A)$ coincides with $B(A')$ up to a set of measure zero.*

Definition 2.3 (Milnor global attractor, likely limit set)**.** *The Milnor global attractor, or the likely limit set, $\Lambda(f) \subset X$ of f is the smallest closed set containing the ω-limit sets of λ-almost all orbits.*

The definition of a Palis attractor was introduced in [30]; however, in this paper, we adopt the formal version as stated in [10].

Definition 2.4 (Palis attractor)**.** *A closed invariant set $A \subset X$ is called a Palis attractor if:*

(1) *there exists an ergodic invariant probability measure μ such that* $\mathrm{supp}(\mu) = A$*;*
(2) *the basin of attraction $B(A)$ has positive Lebesgue measure, where*

$$x \in B(A) \iff \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} = \mu,$$

with δ_z denoting the Dirac measure at z.

Definition 2.5 (Statistical attractor)**.** *The statistical attractor $A_{stat}(f)$ of f is the smallest closed set such that λ-almost all orbits spend an average time 1 in any neighborhood of $A_{stat}(f)$, formally, for every neighbourhood U of $A_{stat}(f)$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k(x)) = 1$$

for almost all $x \in X$.

The following theorem shows how to characterize the statistical attractors using σ-limit sets in the same way as the Milnor global attractors are defined using ω-limit sets. The idea is derived from the work on continuous flows by Karabacak and Ashwin [4].

Theorem 2.6. *The statistical attractor is the smallest closed set that contains the σ-limit sets of almost all trajectories.*

For the proof, we use the auxiliary lemma derived from [4].

Lemma 2.7. *Let X be a compact metric space and γ be any map from X to the set of closed subsets of X. Let $\{U_i\}$ be a countable base for X. Consider a subset $A \subset X$; then the following statements are equivalent:*

(1) $A^C = \bigcup\{U_i : U_i \cap \gamma(x) = \emptyset$ *for* λ*-almost every* $x \in X\}$,
(2) A *is the smallest closed subset of* X *that contains* $\gamma(x)$ *for* λ*-almost every point in* X.

Proof of Theorem 2.6. By definition of $\sigma(x)$, $z \notin \sigma(x)$ iff there is a neighborhood U of z such that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k(x)) = 0$. It follows that, for every open subset $U \subset X$, we have

$$U \cap \sigma(x) = \emptyset \text{ iff } \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k(x)) = 0.$$

Clearly, if $z \in U$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k(x)) = 0$ then $z \notin \sigma(x)$ and $U \cap \sigma(x) = \emptyset$. On the other hand, if $U \cap \sigma(x) = \emptyset$ then for every $z \in U$ there is a neighborhood V_z of z such that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{V_z}(f^k(x)) = 0$. We have $U \subset \bigcup_{z\in U} V_z$. Let $V_1, \ldots, V_m$ be a finite subcover from $\{V_z\}$ such that $U \subset \bigcup_{i=1}^m V_i$. Then

$$\begin{aligned} 0 \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k(x)) &\le \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{m}\sum_{k=0}^{n-1} \chi_{V_i}(f^k(x)) \\ &= \sum_{i=1}^{m} \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_{V_i}(f^k(x)) = 0, \end{aligned}$$

where the second inequality follows from the fact that, for every $k \ge 0$,

$$\chi_U(f^k(x)) \le \sum_{i=1}^{m} \chi_{V_i}(f^k(x)).$$

Definition 2.5 is equivalent to

$$A^C_{\text{stat}}(f) = \bigcup\{U \subset X : U \text{ is open and } \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_U(f^k(x)) = 0 \text{ for almost all } x \in X\}.$$

Since we have shown that, in fact,

$$A^C_{\text{stat}}(f) = \bigcup\{U \subset X : U \text{ is open and } \sigma(x) \cap U = \emptyset \text{ for almost all } x \in X\},$$

the statement of theorem follows from Lemma 2.7. □

We now turn to the concept of *physical attractors*, which are characterized in terms of invariant measures arising from the global evolution of the Lebesgue measure. An open set $U \subset X$ is called an *essential set* if there exists a sequence $\{k_n\}_{n\ge 0}$ with $k_n \to +\infty$ and some constants $\varepsilon, \delta > 0$ such that for every $n \in \mathbb{N}$,

$$\lambda\left(\left\{x \in X : \frac{1}{k_n}\sum_{i=0}^{k_n-1} \chi_U(f^i(x)) > \varepsilon\right\}\right) > \delta.$$

In other words, an open set U is *not essential* if the time averages $\frac{1}{k}\sum_{i=0}^{k-1} \chi_U \circ f^i$ converge to zero in measure with respect to λ.

We now turn to the concept of *physical attractors*, which encode the observable behavior of Lebesgue-typical orbits. As discussed in 2.2, the Krylov–Bogolyubov procedure applied to the pushforward sequence $(f^n_*\lambda)$ produces f-invariant probability measures as weak* accumulation points of the time averages

$$\mu_n^{\text{avg}} = \frac{1}{n}\sum_{i=0}^{n-1} f^i_*\lambda.$$

The physical attractor is defined in [16, 22] as the smallest closed set containing the supports of all such accumulation points. Rather than working directly with these measures, we use the equivalent formulation via essential sets, following [6], which is more convenient for our purposes.

Definition 2.8 (Physical attractor)**.** *The physical attractor $A_{phys}(f)$ is the set*

$$A_{phys}(f) = X \setminus \left\{ \bigcup W : W \text{ is open and not essential} \right\}.$$

For completeness, we remark that $A_{\mathrm{stat}}(f)$ and $\Lambda(f)$ may also be defined in a similar way; see also [21]. For the statistical attractor,

$$A_{\mathrm{stat}}(f) = X \setminus \left\{ \bigcup W : W \text{ is open and not statistically essential} \right\},$$

where an open set U is not statistically essential if $\frac{1}{k}\sum_{i=0}^{k-1} \chi_U \circ f^i \to 0$ for λ-almost all $x \in X$. For the Milnor attractor,

$$\Lambda(f) = X \setminus \left\{ \bigcup W : W \text{ is open and not Milnor essential} \right\},$$

where an open set U is not Milnor essential if $\chi_U \circ f^k \to 0$ for λ-almost all $x \in X$.

It follows that $A_{\mathrm{phys}}(f)$ is the maximal closed set in X such that any neighborhood of an arbitrary point in $A_{\mathrm{phys}}(f)$ is an essential open set.

When f is clear from the context, we denote the above attractors by Λ, A_{stat} and A_{phys} respectively. We recall the following fact from [21]

Lemma 2.9. *For any continuous map $f : X \to X$, we have*

$$A_{phys}(f) \subseteq A_{stat}(f) \subseteq \Lambda(f) \subseteq \Omega(f).$$

Definition 2.10 (Stability)**.** *A non-empty closed set A with $f(A) = A$ is said to be (Lyapunov) stable if for any open set $U \supset A$ there is an open set $V \supset A$ such that* $\mathrm{Orb}_f(x) \subset U$ *for every $x \in V$.*

A stable set A is asymptotically stable if additionally there is an open set $W \supset A$ contained in its basin of attraction $W \subset B(A)$.

It is well known that A is asymptotically stable iff there exists an arbitrarily small open neighborhood $U \supset A$ such that $f(\overline{A}) \subset A$ and $\bigcap_n f^n(U) = \bigcap_n f^n(\overline{U}) = A$. The reader is referred to chapter V in [7] for some basic properties of asymptotically stable sets (cf. [25]).

Definition 2.11 (Robustness)**.** *We say that an attractor $A(f)$ is* robust *if the Hausdorff distance between $A(f)$ and $A(g)$ tends to zero whenever $g \to f$ in the uniform topology on $C(X, X)$.*

Another way to understand the robustness of a set $A(f)$ is as follows. Consider the set-valued function

$$\Gamma : C(X, X) \longrightarrow K(X), \qquad \Gamma(g) = A(g),$$

where $C(X, X)$ is the space of continuous self-maps of X equipped with the uniform metric, and $K(X)$ is the space of nonempty compact subsets of X equipped with the Hausdorff metric. Then $A(f)$ is robust if and only if f is a continuity point of Γ.

Example 2.12. *Consider the family of maps*

$$f_n(x) = x^{\frac{n-1}{n}}, \quad x \in [0, 1].$$

As $n \to \infty$, we have $f_n \to f = Id$ uniformly on $[0, 1]$. Therefore, $\Lambda(f) = [0, 1]$, since f is the identity map. For each n, observe that $A(f_n) = \{1\}$. Thus, $\Lambda(f)$ is not robust, and the map $\Gamma(f) = \Lambda(f)$ is not continuous at f.

An *odometer system* is a topological dynamical system (X, τ) satisfying the following properties:

(1) X is a *Cantor set*, i.e., a compact, totally disconnected, perfect metrizable space.
(2) $\tau : X \to X$ is a homeomorphism.
(3) The system is *minimal*: for every $x \in X$, the orbit $\{\tau^n(x) : n \in \mathbb{Z}\}$ is dense in X.
(4) The system is *equicontinuous*: for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d(x, y) < \delta \quad \Rightarrow \quad d(\tau^n(x), \tau^n(y)) < \varepsilon \quad \text{for all } n \in \mathbb{Z},$$
where d is a compatible metric on X.

The following tools will be used to show that our constructed typical dynamical system has a robust Milnor attractor.

Lemma 2.13. *Let $f\colon [0, 1] \to [0, 1]$ be a continuous map. Suppose there exists a compact invariant set $S \subset [0, 1]$ such that the restricted system $(S, f|_S)$ is topologically conjugate to an odometer. Then $S \subset \Lambda(f)$.*

Proof. Let Q be the maximal ω-limit set containing S. Such a set Q always exists for interval maps, and if S is infinite, then Q is uniquely determined and infinite (see [8]). In fact, by *Blokh's spectral decomposition theorem* (Theorem 5.4 in [8]) for ω-limit sets of interval maps, Q must be one of the following types:

(1) A finite periodic orbit;
(2) A basic set (Q is a cycle of intervals);
(3) A solenoidal set (Q is covered by an intersection of a nested sequence of cycles of periodic intervals with increasing periods).

Since $(S, f|_S)$ is conjugate to an odometer, it is minimal and equicontinuous. This rules out the first two cases - Q cannot be a finite periodic orbit, as S is infinite. If S were contained in a basic set, the semiconjugacy would project S onto an uncountable equicontinuous subset $\hat{S}$ of a transitive interval map g (see [33]). Replacing g with g^2, we can assume that g is mixing. (Namely, in the worst case, when g was only transitive, $\hat{S}$ splits into two minimal sets and the domain of g^2 into two invariant intervals, so we restrict to one of them). There exists an $m > 0$ such that $\hat{S}$ splits into at least four invariant Cantor sets for g^m. If we take the convex hulls $I_0 := \mathrm{conv}(\hat{S}_0)$ and $I_1 := \mathrm{conv}(\hat{S}_1)$ of two distinct invariant Cantor sets $\hat{S}_0, \hat{S}_1 \subset \hat{S}$, then, since g^m is mixing, we can find $k > m$ (a multiple of m) such that I_0 and I_1 form a strong horseshoe for g^k. Then there exists an invariant set $C \subset I_0 \cup I_1$ such that $g^k|_C$ is almost conjugate (via a map π) to the full shift on two symbols, and $\hat{S}_0, \hat{S}_1 \subset C$, as they are invariant under g^k and remain in $I_0 \cup I_1$. Note that $\pi(\hat{S}_0)$ is infinite and $(\pi(\hat{S}_0), \sigma)$ is equicontinuous, which is impossible. Hence, Q must be a solenoidal set.

By the structure of solenoidal sets, there exists a sequence of periodic intervals $\{I_{i,j}\}_{j=1}^{k_i}$ of period k_i, with $k_i \to \infty$, such that
$$Q \subset \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{k_i} I_{i,j}.$$
It is possible that $\lim_{i\to\infty} \max_j \mathrm{diam}(I_{i,j}) > 0$ as $i \to \infty$ but we may assume, without loss of generality, that the intervals $I_{i,1}$ form a nested sequence with $\mathrm{diam}(I_{i,1}) \to 0$ as $i \to \infty$.

Let $\Lambda(f)$ denote the Milnor attractor of f. Since each $I_{i,1}$ has positive measure and is eventually mapped into itself, it follows that
$$\lambda(B(\Lambda(f)) \cap I_{i,1}) = \lambda(I_{i,1}),$$

where $B(\Lambda(f))$ is the basin of $\Lambda(f)$. Hence $\Lambda(f) \cap I_{i,1} \neq \emptyset$ for all i. As the intervals $I_{i,1}$ form a nested sequence with diameters decreasing to zero and $\Lambda(f)$ is a closed set, their intersection contains a point of $\Lambda(f)$, and it follows that $S \cap \Lambda(f) \neq \emptyset$. By the minimality of S and by the invariance of $\Lambda(f)$, we conclude that $S \subset \Lambda(f)$. □

Lemma 2.14. *Let $f \in C([0,1])$, and suppose there exists a solenoid $S = \omega(x)$ defined by*

$$S = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{k_i} I_{i,j},$$

where:

- *Each $I_{i,j} \subset [0,1]$ is a closed, nondegenerate interval,*
- *$f(I_{i,j}) \subset \operatorname{int} I_{i,j+1 \bmod k_i}$ for all i, j,*
- *$\max_j \operatorname{diam}(I_{i,j}) \to 0$ as $i \to \infty$.*

Let $\{f_n\}_{n \in \mathbb{N}} \subset C([0,1])$ be a sequence converging uniformly to f, and let $\Lambda(f_n)$ denote the Milnor attractor of $f_n$for each $n \in \mathbb{N}$. Then:

$$S \subset \liminf_{n \to \infty} \Lambda(f_n).$$

Proof. Fix any $z \in S$ and $\varepsilon > 0$. Since $\max_j \operatorname{diam}(I_{i,j}) \to 0$, there exists a level $i \in \mathbb{N}$ and an index $j \in \{1, \dots, k_i\}$ such that

$$z \in I_{i,j}, \quad \text{and} \quad \operatorname{diam}(I_{i,j}) < \varepsilon.$$

By the assumption,

$$f(I_{i,j}) \subset \operatorname{int}(I_{i,j+1}).$$

Because $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|f_n - f\|_\infty < \delta,$$

for some $\delta > 0$ small enough that $f_n(I_{i,j}) \subset I_{i,j+1}$. Thus, for all $n \geq N$, the original family $\{I_{i,j}\}_{j=1}^{k_i}$ satisfies the same cyclic structure under f_n:

$$f_n(I_{i,j}) \subset I_{i,j+1 \bmod k_i}.$$

Define

$$I^{(n)} := \bigcup_{j=1}^{k_i} I_{i,j}.$$

Then $f_n(I^{(n)}) \subset I^{(n)}$, so $I^{(n)}$ is forward-invariant under f_n. Since each $I_{i,j}$ is nondegenerate, we have $\lambda(I^{(n)}) > 0$, and thus $\Lambda(f_n) \cap I^{(n)} \neq \emptyset$. Moreover, since f_n maps each $I_{i,j}$ into $I_{i,j+1 \bmod k_i}$, any forward orbit in $I^{(n)}$ visits all the $I_{i,j}$. Hence, the ω-limit set of such an orbit intersects every $I_{i,j}$, and in particular $\Lambda(f_n) \cap I_{i,j} \neq \emptyset$. Let $B_\varepsilon(z)$ denote the open ε–ball around z. Since $I_{i,j} \subset B_\varepsilon(z)$, we obtain

$$\Lambda(f_n) \cap B_\varepsilon(z) \neq \emptyset.$$

As this holds for all $\varepsilon > 0$, we conclude $z \in \liminf_{n \to \infty} \Lambda(f_n)$. and since $z \in S$ was arbitrary, we have $S \subset \liminf_{n \to \infty} \Lambda(f_n)$. □

3. Properties of typical attractor on interval

The results of this section are inspired by the construction in [26, Theorem 1], however, with more local approach to perturbations. This allows for careful control of periodic points in the perturbed map. As a consequence, our modification yields a residual set of maps $g \in C([0,1])$ for which the Milnor attractor, the statistical attractor, and the physical attractor coincide, and this common attractor is precisely the closure of the set of periodic points(which, as we show, coincides with the nonwandering set). Hence, the typical attractor is maximal possible for each typical map.

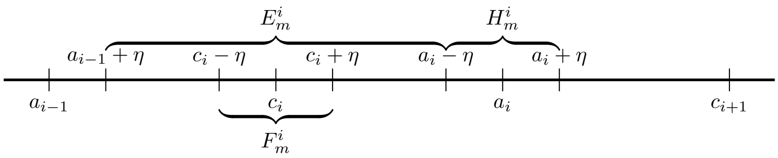
In the following proofs, we write $d(x, y)$ for the Euclidean distance on $[0, 1]$, and $\|f-g\|_\infty = \sup_{x\in[0,1]} d(f(x), g(x))$ for the C^0 metric on $C([0, 1])$. For compact sets $A, B \subset [0, 1]$ we denote by $d_H(A, B)$ the Hausdorff distance.

We begin by introducing the combinatorial structure of intervals that allows precise control of some properties of perturbations. These intervals, following the notation of [26], provide a partition of the unit interval into small building blocks with controlled diameters.

Notation. For $m \in \mathbb{N}$, define

$$
\begin{aligned}
M &= 2^m, \quad \eta = \eta_M = 2^{-2m(m+1)}\\
a_i &= \frac{i}{M}, \quad i = 1, \dots, M\\
c_i &= \frac{1}{2}(a_{i-1} + a_i), \quad i = 1, \dots, M\\
E^i_m &= [a_{i-1} + \eta, a_i - \eta], \quad F^i_m = [c_i - \eta, c_i + \eta], \quad i = 1, \dots, M\\
H^i_m &= (a_i - \eta, a_i + \eta), \quad i = 1, \dots, M\\
H^0_m &= [0, \eta), \quad H^M_m = (1 - \eta, 1]\\
E_m &= \bigcup_{i=1}^{M} E^i_m, \quad F_m = \bigcup_{i=1}^{M} F^i_m, \quad H_m = \bigcup_{i=0}^{M} H^i_m\\
L_m &= \bigcup_{i\geq 0} E^{2i+1}_m, \quad G_m = \bigcup_{i\geq 0} F^{2i+1}_m\\
R_m &= \bigcup_{i\geq 1} E^{2i}_m, \quad D_m = \bigcup_{i\geq 1} F^{2i}_m\\
C_m &= \{c_1, c_2, \dots, c_M\}
\end{aligned}
$$

The structure of those blocks is presented on the figure below:



The sets defined above have the following properties:

(1) the intervals E^i_m and H^i_m form together a partition of $[0, 1]$,
(2) $F^i_m \subset E^i_m$, $D^i_m \subset R^i_m$, $G^i_m \subset L^i_m$,
(3) $\operatorname{diam} H^i_m < 2^{-m}$ and $\sum_{i=1}^{M} (\operatorname{diam} H^i_m)^{\frac{1}{m}} < 2^{-m}$,
(4) $\operatorname{diam} F^i_m < 2^{-m}$ and $\sum_{i=1}^{M} (\operatorname{diam} F^i_m)^{\frac{1}{m}} < 2^{-m}$,
(5) if $n \geq 2m(m + 1)$ then for every i, the intersections $F^i_m \cap G_n$, $F^i_m \cap D_n$ are nonempty and consist of precisely those components of G_n or D_n which have nonempty intersections with F^i_m.
(6) $L_m \cap R_m = D_m \cap G_m = \emptyset$.

Note that for a fixed $m \in \mathbb{N}$ and every $i = 1, \dots, M$ we have:

$$\operatorname{diam} E^i_m = \frac{1}{M} - 2\eta = \frac{2^{m(m+1)} - 2}{2^{2m(m+1)}} \quad \text{and} \quad \operatorname{diam} H^i_m = 2\eta = \frac{2}{2^{2m(m+1)}}.$$

From the definition it is clear that $\operatorname{diam} E^i_m + \operatorname{diam} H^i_m = \frac{1}{M}$.

We now state the main result obtained by modifying Mizera's construction.

Theorem 3.1. *There is a residual set $G \in C([0,1])$ such that for every $g \in G$ there is a set $Z(g) \subset [0,1]$ such that*

(a) the set $Z(g)$ has full Lebesgue measure,
(b) the set $g^{-1}(Z(g))$ has zero Lebesgue measure
(c) $\omega(x)$ is a solenoid, for every $x \in Z(g)$,
(d) the basin of attraction $B(\omega(x))$ of every $x \in Z(g)$, has zero Lebesgue measure.

Moreover, the set $A(g) = \overline{\bigcup_{x\in Z(g)}\{\omega(x)\}}$ has the following properties:

(e) $A(g)$ has zero Hausdorff dimension,
(f) $A(g) = \overline{\mathrm{Per}(g)} = \Omega(g)$ and consequently the basin of attraction $B(A(g)) = [0,1]$,
(g) every open set U with $U \cap A(g) \neq \emptyset$ is essential.

Proof. We start the proof with the construction of the set G and consequently the set $Z(g)$ for $g \in G$.

For every $K \geq 1$ let $\mathcal{F}_K \subset C([0,1])$ be a dense set of piecewise linear maps with slope at least 3 such that for any $f \in \mathcal{F}_K$ critical points are not k-periodic, $f^k(0) \neq 0$, $f^k(1) \neq 1$ for $k \leq K$. For technical convenience we regard each $\mathcal{F}_K$ as a sequence, and write $\mathcal{F}_K = \{f_{j,K} : j \geq 1\}$ for every K.

From now on, we proceed with the construction of an open dense set $G_K \subset C([0,1])$, fixing any $K \geq 1$, an arbitrary map $f_{j,K} \in \mathcal{F}_K$, and some fixed $\gamma_{j,K} < \frac{1}{4j}$. For simplicity of notation, we write $f = f_{j,K}$ and $\gamma = \gamma_{j,K}$. All further constants will depend on f. While we do not write this dependence directly (e.g. we write m_K instead of m_K^f, etc.), the reader should keep in mind that such dependence exists.

Consider the set $\mathrm{Fix}(f^K)$ of K-periodic points and by $\mathrm{Crit}(f^K)$ denote the set of critical points for map f together with 0 and 1. Note that between any two consecutive (with respect to the natural order on the interval) points form $\mathrm{Fix}(f^K)$ there is exactly one point from $\mathrm{Crit}(f^K)$. Moreover, there is at least one and at most two points from $\mathrm{Crit}(f^K)$ in each of the nondegenerate intervals $[0, \min \mathrm{Fix}(f^K)]$ and $[\max \mathrm{Fix}(f^K), 1]$.

Let θ_f be the maximal slope of f. If necessary, we further decrease γ so that for every $h \in C([0,1])$ the implication

$$\|h - f\|_\infty < \gamma \Rightarrow d_H(\mathrm{Fix}(h^K), \mathrm{Fix}(f^K)) < \frac{1}{4K} \tag{4}$$

holds. This adjustment is possible because f is piecewise linear. Next, define

$$\beta_K = \frac{1}{2} d_H\left(\mathrm{Fix}(f^K), \mathrm{Crit}(f^K)\right) > 0.$$

Choose $\alpha_K < \min\{\beta_K, \frac{\gamma}{2}, \frac{1}{K}\}$ and $\delta_K < \alpha_K$ with the following properties:

(i) $d(x,y) < \delta_K$ implies $d(f^i(x), f^i(y)) < \frac{\alpha_K}{4}$ for every $x, y \in [0,1]$, $i = 1, \ldots, K$,
(ii) for every $i = 0, \ldots, K$ and every point $y \in \mathrm{Fix}(f^K)$ we have
$$f^i\left((y - \delta_K, y + \delta_K)\right) \subset \left(f^{i \pmod K}(y) - \alpha_K, f^{i \pmod K}(y) + \alpha_K\right),$$
(iii) for every δ_K-pseudo-orbit $\{x_0, \ldots, x_K\}$ of length $K+1$ we have
$$d(f^i(x_0), x_i) < \frac{\alpha_K}{4},\ i = 1, \ldots, K.$$

We select $m_K > K$ large enough so that the following inequalities hold:

$$4(\operatorname{diam} E^i_{m_K} + \operatorname{diam} H^i_{m_K}) = \frac{4}{M_K} < \frac{\delta_K}{4}, \tag{5}$$

$$M_K = 2^{m_K} > \frac{4}{K\alpha_K}, \tag{6}$$

$$\frac{2\theta_f + 3}{M_K} < \frac{\alpha_K}{4}. \tag{7}$$

The condition (5) ensures that $\operatorname{diam}(E^i_{m_K}) < \delta_K$ for all i, so property (i) applies on each $E^i_{m_K}$, and, for every $y \in \operatorname{Fix}(f^K)$, the δ_K-neighborhood $(y - \delta_K, y + \delta_K)$ contains at least four consecutive sets from $\{E^i_{m_K}\}_{i=1}^{M_K}$.

We now construct a perturbation $\tilde{f}$ of f following the approach of [26, Theorem 1], with adjustments near periodic points in order to produce disjoint periodic orbits absorbed by attracting set F_{m_K}. Define the map $\tilde{f} : [0,1] \to [0,1]$ by the following rules:

(M1) $\tilde{f}(c_i)$ is the closest to $f(c_i)$ element of $\begin{cases} D_{m_K} \cap C_{m_K} & , \text{ if } c_i \in D_{m_K} \\ G_{m_K} \cap C_{m_K} & , \text{ if } c_i \in G_{m_K} \end{cases}$,

(M2) $\tilde{f}(x) = \tilde{f}(c_i)$ for every $x \in E^i_{m_K}$,

(M3) $\tilde{f}$ is linear on every $H^i_{m_K}$ and $\tilde{f}$ is continuous,

(M4) $\tilde{f}(0), \tilde{f}(1)$ are respectively the closest to $f(0)$ and $f(1)$ elements of C_{m_K}.

Note that the map $\tilde{f}$ obeying the rules (M1)-(M4) is uniquely defined, up to possible choice in (M1) between points with equal distance to $f(c_i)$.

We will now show that $\|f - \tilde{f}\|_\infty \leq \frac{\alpha_K}{4}$. Observe that since $d(c_i, c_{i-1}) = \frac{1}{M_K}$ and points of C_{m_K} with fixed parity are spaced by $\frac{2}{M_K}$, we obtain

$$d(f(c_i), \tilde{f}(c_i)) \leq \frac{1}{M_K}.$$

Now, for every $x \in E^i_{m_K}$ we have $d(x, c_i) < \frac{1}{2M_K}$ so $d(f(x), f(c_i)) \leq \frac{\theta_f}{2M_K}$. Hence, by (7), for every $x \in E^i_{m_K}$ we obtain

$$d(f(x), \tilde{f}(x)) \leq d(f(x), f(c_i)) + d(f(c_i), \tilde{f}(c_i)) = \frac{\theta_f + 2}{2M_K} \leq \frac{\alpha_K}{4}.$$

Similarly, for every $x \in H^i_{m_K}$ we have $x \in [a_i - \eta, a_i + \eta] \subset (c_i, c_{i+1})$ and $d(c_i, c_{i+1}) = \frac{1}{M_K}$. As $\tilde{f}$ is constant on each $E^i_{m_K}$ and $E^{i+1}_{m_K}$ and linear on $H^i_{m_K}$, we obtain

$$\operatorname{diam}(\tilde{f}(H^i_{m_K})) = d(\tilde{f}(a_i - \eta), \tilde{f}(a_i + \eta)) = d(\tilde{f}(c_i), \tilde{f}(c_{i+1})).$$

Using (M1), we have

$$d(\tilde{f}(c_{i+1}), \tilde{f}(c_i)) \leq d(f(c_{i+1}), f(c_i)) + d(f(c_{i+1}), \tilde{f}(c_{i+1})) + d(f(c_i), \tilde{f}(c_i)) \leq \frac{\theta_f + 2}{M_K}.$$

Thus,

$$d(\tilde{f}(x), \tilde{f}(c_i)) \leq \operatorname{diam}(\tilde{f}(H^i_{m_K})) \leq \frac{\theta_f + 2}{M_K},$$

while $d(x, c_i) < \frac{1}{M_K}$ implies $d(f(x), f(c_i)) \leq \frac{\theta_f}{M_K}$. Therefore,

$$d(f(x), \tilde{f}(x)) \leq \frac{\theta_f}{M_K} + \frac{1}{M_K} + \frac{\theta_f + 2}{M_K} = \frac{2\theta_f + 3}{M_K}.$$

As a consequence, by (7),

$$d(f(x), \tilde{f}(x)) \leq \frac{2\theta_f + 3}{M_K} \leq \frac{\alpha_K}{4}.$$

Therefore, choosing m_K large enough, we obtain

$$\|f-\tilde{f}\|_\infty \leq \frac{\alpha_K}{4}. \tag{8}$$

Now we introduce a further perturbation on $\tilde{f}$ to obtain the map $\hat{f}$ with the property that $K-$periodic orbits of f are traced by $K-$periodic points of $\hat{f}$ located in C_{m_k}.

For any $y \in \mathrm{Fix}(f^K)$ take $i_0 \in \{1,\dots,M_K\}$ such that $\mathrm{dist}(E^{i_0}_{m_K}, y)$ is the smallest among all sets from $\{E^{i_0}_{m_K}\}_{i=1}^{M_K}$ contained in $(y-\delta_K, y+\delta_K)$. Let p be the prime period of y and let $i_1,\dots,i_{p-1} \in \{1,\dots,M_K\}$ be pairwise different and such that

$$f(E^{i_j}_{m_K}) \subset E^{i_{j+1}}_{m_k}, j=1,\dots,p-2.$$

Such indexes i_j exist by the definition of α_K and the structure of the blocks from $\{E^i_{m_K}\}_{i=1}^{M_K}$. It follows that points $\{c_{i_0},\dots,c_{i_{p-1}}\}$ form a finite portion of the $\tilde{f}-$orbit of point c_{i_0}. Since $\|f-\tilde{f}\|_\infty \leq \alpha_K/4 \leq \delta_K$, the sequence $x_0 = c_{i_0},\ x_{j+1} := \tilde{f}(x_j)$ is a δ_K–pseudo-orbit of f. Recall that for map f every δ_K-pseudo-orbit of length $K+1$ is $\frac{\alpha_K}{4}-$traced. In particular, as $d(c_{i_0}, y) < \frac{1}{M_K} < \delta_K$ we have, by (iii) and (i),

$$d(f^K(c_{i_0}), \tilde{f}^K(c_{i_0})) < \frac{\alpha_K}{4},$$
$$d(f^K(c_{i_0}), y) < \frac{\alpha_K}{4},$$

which implies $d(\tilde{f}^K(c_{i_0}), y) < \frac{\alpha_K}{2}$ and, altogether, we obtain

$$d(\tilde{f}^K(c_{i_0}), c_{i_0}) < \frac{\alpha_K}{2} + \frac{1}{M_K}$$

Now define $\hat{f}$ by

$$\hat{f}(x) = \begin{cases} \tilde{f}(x), & x \notin H^{i_{p-2}}_{m_K} \cup E^{i_{p-1}}_{m_K} \cup H^{i_{p-1}}_{m_K}, \\ c_{i_0}, & x \in E^{i_{p-1}}_{m_K}, \end{cases}$$

and interpolate linearly on

$$H^{i_{p-2}}_{m_K} \cup H^{i_{p-1}}_{m_K}$$

so that $\hat{f}$ remains continuous. (If $p=1$, the interval $H^{i_{p-2}}_{m_K}$ is omitted.)

By construction, we have $\|\tilde{f}-\hat{f}\|_\infty \leq \frac{\alpha_K}{2} + \frac{1}{M_K}$ on $E^{i_{p-1}}_{m_K}$. Since $H^{i_{p-1}}_{m_K}$ and $H^{i_{p-2}}_{m_K}$ each share an endpoint with $E^{i_{p-1}}_{m_K}$, and at their opposite endpoints $\tilde{f}$ and $\hat{f}$ coincide, the linear interpolation on these intervals preserves the same bound. Therefore, the estimate holds on $E^{i_{p-1}}_{m_K} \cup H^{i_{p-2}}_{m_K} \cup H^{i_{p-1}}_{m_K}$, and hence on the whole interval $[0,1]$, since $\hat{f} = \tilde{f}$ elsewhere.

Observe that the above estimation, together with (8) and (6), implies

$$\|f-\hat{f}\|_\infty \leq \|f-\tilde{f}\|_\infty + \|\tilde{f}-\hat{f}\|_\infty \leq \frac{\alpha_K}{4} + \left(\frac{\alpha_K}{2} + \frac{1}{M_K}\right) \leq \alpha_K < \frac{\gamma}{2}.$$

The point c_{i_0} is a periodic point of $\hat{f}$ whose prime period p is the same as the prime period of y (in particular, p divides K), and therefore $c_{i_0} \in C_{m_K} \cap \mathrm{Fix}(\hat{f}^K)$. Hence, for every $y \in \mathrm{Fix}(f^K)$ we obtain a periodic point $x \in C_{m_K} \cap \mathrm{Fix}(\hat{f}^K)$ whose prime period coincides with that of y. Moreover, if two points of $\mathrm{Fix}(f^K)$ have disjoint f–orbits, then the corresponding periodic $\hat{f}$–orbits constructed in C_{m_K} are also disjoint.

At this stage, we have shown how to construct a perturbation $\hat{f}$ of any map $f \in \mathcal{F}_K$ for all $K \geq 1$. We now proceed to define the residual set of functions G.

For every $K \geq 1$ choose any $f \in \mathcal{F}_K$ and take

$$\rho_{f,K} < \min\{\eta_{m_K}, \frac{1}{2} \operatorname{diam} E^i_{m_K}\}. \tag{9}$$

Observe that by (5) and the fact that $\delta_K < \alpha_K < \frac{\gamma}{2}$ we have $\rho_{f,K} < \frac{\gamma}{4}$. Moreover, by the choice of $\rho_{f,K}$ the following holds: if $g \in B(\hat{f}, \rho_{f,K})$ and $c \in C_{m_K} \cap \operatorname{Fix}(\hat{f}^K) \cap E^{i_c}_{m_K}$ for some $i_c \in \{1, \ldots, M_K\}$ then

$$\operatorname{Fix}(g^K) \cap E^{i_c}_{m_K} \neq \emptyset. \tag{10}$$

In fact, we have that $g^K(E^{i_c}_{m_K}) \subset F^{i_c}_{m_K}$ for every such g.

Define

$$G_K = \bigcup_{f \in \mathcal{F}_K} B(\hat{f}, \rho_{f,K})$$

and

$$G = \bigcap_{K \geq 1} G_K.$$

Each set G_K is open by definition, it is dense in $C([0,1])$ because $\mathcal{F}_K$ is dense and

$$\lim_{j \to \infty} \|f_{j,K} - \hat{f}_{j,K}\|_\infty \leq \lim_{j \to \infty} \gamma_{j,K} = 0.$$

In particular, G as a countable intersection of dense open sets is a residual subset of $C([0,1])$.

By [26, Theorem 4], there is a residual set E in $C([0,1])$ such that every $f \in E$ has the shadowing property. Intersection of residual sets remains residual, hence replacing G with the set $G \cap E$ when necessary, we may assume that every map in G has the shadowing property. By Coven and Hedlund [15] or Sharkovsky [35], we have, for any interval map, $\overline{\operatorname{Per}(f)} = \overline{\operatorname{Rec}(f)}$. By Oprocha [29], $\overline{\operatorname{Rec}(f)} = \Omega(f)$ whenever f has the shadowing property. Summing the above facts together, we have

$$\Omega(g) = \overline{\operatorname{Per}(g)} \quad \text{for every } g \in G. \tag{11}$$

Fix any $g \in G$ and any $K \geq 1$. By the definition of G, there exists a map $f_{j,K} \in \mathcal{F}_K$ with $j \geq 1$ such that

$$g \in B(\hat{f}_{j,K}, \rho_{f_{j,K}}).$$

Then the following set is nonempty and contains at least one element from each $\mathcal{F}_K$:

$$\mathcal{B}_g = \left\{ f \in \bigcup_{K \geq 1} \mathcal{F}_K : \ g \in B(\hat{f}, \rho_{f,K}) \right\}.$$

For every $f \in \mathcal{B}_g$ there exists some $K_f \geq 1$ such that $f \in \mathcal{F}_{K_f}$, and the corresponding m_{K_f} is chosen as in the above construction so that the partition

$$\{E^i_{m_{K_f}}\}_{i=1}^{M_{K_f}} \ \cup \ \{H^i_{m_{K_f}}\}_{i=1}^{M_{K_f}}$$

of $[0,1]$ satisfies conditions (5)–(7).

Since $g \in G$, for every $K \geq 1$ there exists some $f \in \mathcal{B}_g$ such that $K_f > K$. We now define

$$Z(g) = \bigcap_{K \geq 1} \ \bigcup_{\{f \in \mathcal{B}_g : \, K_f \geq K\}} E_{m_{K_f}}, \qquad A(g) = \overline{\bigcup_{x \in Z(g)} \omega(x)}.$$

The set $Z(g)$ is obtained as the intersection of unions of the sets $E_{m_{K_f}}$ at arbitrarily fine levels. Since $\lambda(\bigcup_{j \geq 0} E_{m_j}) = 1$ for every increasing sequence m_j, it follows that $Z(g)$ has full Lebesgue measure proving (a).

Since $g(E_m) \subset F_m$ and $\lambda(F_m) \to 0$, we have $\lambda(g(Z(g))) = 0$. Replacing $Z(g)$ by $Z'(g) = Z(g) \setminus g(Z(g))$, we still have $\lambda(Z'(g)) = 1$, and

$$g^{-1}(Z'(g)) \subset [0,1] \setminus Z(g),$$

so $\lambda(g^{-1}(Z'(g))) = 0$ because $\lambda([0,1] \setminus Z(g)) = 0$, proving (b).

To see that $\omega(x)$ is a solenoid and that $B(\omega(x))$ has zero Lebesgue measure for every $x \in Z(g)$, one can follow the lines of the proof of [26, Theorem 1]. Indeed, for each $x \in Z(g)$ the orbit $\mathrm{Orb}_g(x)$ is contained in a nested sequence of sets F_{m_j}, $j \geq 1$, and by selecting the subsequence $\{I_{i,j_s}\}_{s\geq 0}$ of components visited infinitely often and ordered so that $g(I_{i,j_s}) \subset \mathrm{Int}\, I_{i,j_{s+1}}$, one obtains

$$\omega(x) = \bigcap_{s\geq 1} \bigcup_{i\geq 1} I_{i,j_s}.$$

This shows that $\omega(x)$ is a solenoid for every $x \in Z(g)$ proving (c).

To prove property (d), fix any $x \in Z(g)$. Observe that $\omega(x)$ is fully contained either in $\bigcap_{s\geq 1} D_{j_s}$ or in $\bigcap_{s\geq 1} G_{j_s}$, for the sequence $\{j_s\}_{s\geq 0}$ chosen above. This implies that either $B(\omega(x)) \subset [0,1] \setminus L_{j_s}$ or, respectively, $B(\omega(x)) \subset [0,1] \setminus R_{j_s}$ for every $s \geq 1$, since points from L_{j_s} and R_{j_s} have disjoint itineraries and their orbits cannot converge to the same solenoid. Assume, on the contrary, that $\lambda(B(\omega(x))) > 0$. By the Lebesgue density theorem, there exists a nondegenerate interval $[a,b]$ such that

$$\frac{\lambda\left([a,b] \cap B(\omega(x))\right)}{(b-a)} > \frac{7}{8}.$$

Since for large m the sets L_m and R_m both meet every interval of positive measure, contributing approximately the same proportion of its measure, and $\lambda(F_m \cup H_m) \to 0$, we have

$$\lambda(L_m \cap B(\omega(x))) > \frac{b-a}{8} \quad \text{and} \quad \lambda(R_m \cap B(\omega(x))) > \frac{b-a}{8}.$$

Since ω-limit sets of points from R_m and L_m are disjoint, it is a contradiction.

To prove (e), note that for each m we have

$$A(g) \subset \bigcup_{i=1}^{M} F_m^i, \qquad M = 2^m, \qquad \mathrm{diam}(F_m^i) = 2\eta_m = 2^{\,1-2m(m+1)}.$$

Thus, for any $s > 0$,

$$\mathcal{H}^s_\infty(A(g)) \leq \sum_{i=1}^{M} (\mathrm{diam}\, F_m^i)^s = M \cdot (2^{\,1-2m(m+1)})^s = 2^{\,m+s-2s\,m(m+1)} \xrightarrow[m\to\infty]{} 0.$$

Hence $\mathcal{H}^s(A(g)) = 0$ for every $s > 0$, and therefore $\dim_H(A(g)) = 0$.

Next, we are going to show that for an arbitrary $g \in G$ the attractor $A(g)$ coincides with the closure of its periodic points, that is, $A(g) = \overline{\mathrm{Per}(g)}$ and that (f) holds. Fix $\varepsilon \in (0, \frac{1}{4})$. For any $g \in G$ and any $z_0 \in \mathrm{Per}(g)$ with prime period p, choose $K_0 > p$, a multiple of p, such that $\frac{1}{K_0} < \frac{\varepsilon}{2}$. Since $M_{K_0} > K_0$, we also have $\frac{1}{M_{K_0}} < \frac{\varepsilon}{2}$. Finally, there exists some $f \in \mathcal{F}_{K_0} \cap \mathcal{B}_g$ such that $g \in B(\hat{f}, \rho_{f,K_0})$.

Since $\|g - \hat{f}\|_\infty < \rho_{f,K_0} < \frac{\gamma}{4}$ and $\|\hat{f} - f\|_\infty \leq \alpha_{K_0} < \frac{\gamma}{2}$, we obtain $\|g - f\|_\infty < \gamma$. As $z_0 \in \mathrm{Fix}(g^{K_0})$, by (4) there exists a point

$$x \in \mathrm{Fix}(f^{K_0}) \quad \text{with} \quad d(x, z_0) < \tfrac{1}{4K_0}.$$

Moreover, there exists an index $i_{K_0} \in \{1, \ldots, M_{K_0}\}$ and

$$\hat{x}_0 \in \mathrm{Fix}(\hat{f}^{K_0}) \cap E_{m_{K_0}}^{i_{K_0}}$$

such that

$$d(\hat{x}_0, z_0) < \delta_{K_0} + \tfrac{1}{4K_0}.$$

Note that by the definition of $\hat{f}$ we have $\hat{x}_0 \in C_{m_{K_0}}$. By (10) there exists

$$z_1 \in \operatorname{Fix}(g^{K_0}) \cap \operatorname{Int} E^{i_{K_0}}_{m_{K_0}},$$

and therefore

$$\begin{aligned} d(z_0, z_1) &< d(z_0, \hat{x}_0) + d(\hat{x}_0, z_1) < \delta_{K_0} + \tfrac{1}{4K_0} + \operatorname{diam} E^{i_{K_0}}_{m_{K_0}} \\ &< \delta_{K_0} + \tfrac{1}{M_{K_0}} + \tfrac{1}{4K_0} < \tfrac{3}{K_0}. \end{aligned}$$

Now we repeat the reasoning for the point z_1 in place of z_0, choosing $K_1 > K_0$, a multiple of K_0, such that $\frac{1}{K_1} < \frac{\varepsilon}{4}$.

By the same arguments, for the partition determined by m_{K_1} we find and index $i_{K_1} \in \{1, \dots, M_{K_1}\}$ and a subinterval $E^{i_{K_1}}_{m_{K_1}} \subset E^{i_{K_0}}_{m_{K_0}}$ containing a point

$$\hat{x}_1 \in \operatorname{Fix}(\hat{f}^{K_1}) \cap E^{i_{K_1}}_{m_{K_1}} \cap C_{m_{K_1}}$$

and consequently, there also exists

$$z_2 \in \operatorname{Fix}(g^{K_1}) \cap \operatorname{Int} E^{i_{K_1}}_{m_{K_1}}$$

with $d(z_1, z_2) < \frac{3}{K_1}$. Clearly, we can also require that $\frac{1}{M_{K_1}} < \frac{\varepsilon}{4}$.

Iterating this process, we obtain a sequence of periodic points $\{z_j\}_{j \geq 0}$ with increasing periods, contained in a descending sequence of intervals with shrinking diameters (namely $z_j \in E^{i_{K_{j-1}}}_{m_{K_{j-1}}}$ for $j \geq 1$), and satisfying

$$d(z_j, z_{j+1}) < \tfrac{3}{K_j}, \qquad \text{with } \tfrac{1}{K_j} < \tfrac{\varepsilon}{2^{j+1}}.$$

Thus $\{z_j\}_{j \geq 0}$ converges to some

$$z = \lim_{j \to \infty} z_j.$$

By the construction, since each $E^{i_{K_j}}_{m_{K_j}}$ is closed, we have $z \in E^{i_{K_j}}_{m_{K_j}}$ for all $j \geq 1$, hence

$$z \in \bigcup_{\{f \in \mathcal{B}_g \colon K_f \geq K_0\}} E_{m_{K_f}}.$$

Together with $K_j \to \infty$, this implies $z \in Z(g)$.

Moreover,

$$d(z_0, z) \leq \sum_{j=1}^{\infty} \tfrac{3}{K_j} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $z_0 \in Z(g)$. Since z_0 was an arbitrary point of $\operatorname{Fix}(g^{K_0})$, we conclude $\operatorname{Per}(g) \subset Z(g)$. Consequently, $\operatorname{Per}(g) \subset A(g)$, and since $A(g)$ is closed, we have $\overline{\operatorname{Per}(g)} \subset A(g)$.

Since $g \in G$, by (11) we have $\Omega(g) = \overline{\operatorname{Per}(g)}$. For every $x \in Z(g)$ we thus obtain $\omega(x) \subset \Omega(g) = \overline{\operatorname{Per}(g)}$, hence $A(g) = \overline{\bigcup_{x \in Z(g)} \omega(x)} \subset \overline{\operatorname{Per}(g)}$. Therefore $A(g) = \overline{\operatorname{Per}(g)}$, establishing (f). Since $\Omega(g) = A(g)$, every orbit satisfies $\omega(x) \subset A(g)$, and thus the basin of attraction is the whole interval: $B(A(g)) = [0, 1]$.

To complete the proof of the theorem, fix any open set U such that $U \cap A(g) \neq \emptyset$. Pick a point $z \in U \cap A(g)$. Since $\operatorname{diam}(F^i_m) \to 0$ as $m \to \infty$, there exists m large enough and an index i such that

$$z \in F^i_m \subset U.$$

By the construction of $\hat{f}$, such F^i_m contains a K–periodic orbit. Because $g \in G$ is a small perturbation of $\hat{f}$, relation (10) guarantees that g also has a K–periodic point in F^i_m. Therefore, any orbit that enters F^i_m returns to F^i_m infinitely many times, with gaps bounded by K. Since $F^i_m \subset U$ and for every $x \in F^i_m$ we have $\omega(x) \cap F^i_m \neq \emptyset$, it follows that the set $\{x \in [0,1] : \omega(x) \cap U \neq \emptyset\}$ contains F^i_m and, therefore, has positive Lebesgue measure.

Hence, if we take $\varepsilon = 1/2K$ and $\delta = \lambda(F^i_m)$, then along any sequence $k_n \to \infty$ with $k_0 > 4K$,

$$\lambda\left(\left\{x \in [0,1] : \frac{1}{k_n}\sum_{j=0}^{k_n-1} \chi_U(g^j(x)) > \varepsilon\right\}\right) \geq \lambda(F^i_m) > \delta.$$

This shows that U is an essential set, proving (g) and completing the proof of the theorem. □

The next theorem shows that, in a typical scenario, the situation opposite to Conjecture 1.2 occurs. There are no regular attractors, since the global attractor consisting of solenoids attracts a full-measure set. On the other hand, the global attractor for any f has zero Lebesgue measure, and thus no local attractor can support any absolutely continuous measure with respect to the Lebesgue measure.

Theorem 3.2. *There exists a residual family of functions $G \subset C([0,1])$ such that, for any $g \in G$, the global Milnor, statistical, and physical attractors of g all coincide with the nonwandering set $\Omega(g)$, which is a Cantor set. The attractor $A(g) = \Omega(g)$ is robust but not Lyapunov stable, and the only asymptotically stable sets of g are invariant, nondegenerate periodic intervals, including $[0,1]$ itself. Moreover, no map $g \in G$ has a Palis attractor.*

Proof. Let G be the set from Theorem 3.1. For any $g \in G$, let $A_{\text{phys}}(g)$, $A_{\text{stat}}(g)$ and $\Lambda(g)$ denote the physical, statistical, and global Milnor attractor of g, respectively. We first claim that $A_{\text{phys}}(g) = A_{\text{stat}}(g) = \Lambda(g) = A(g) = \Omega(g)$. By Lemma 2.9 we have $A_{\text{phys}}(g) \subseteq A_{\text{stat}}(g) \subseteq \Lambda(g) \subseteq \Omega(g)$ and so by Theorem 3.1((f)), it is enough to show that $A(g) \subseteq A_{\text{phys}}(g)$. Suppose instead that $A(g) \setminus A_{\text{phys}}(g) \neq \emptyset$. Then there exists an open set U with $U \cap A(g) \neq \emptyset$ and $U \cap A_{\text{phys}}(g) = \emptyset$. By Theorem 3.1 ((g)), U is essential, contradicting the definition of $A_{\text{phys}}(g)$. Indeed, $A(g) \subseteq A_{\text{phys}}(g)$ and so the claim holds.

It is well known that a set A is asymptotically stable for g if there exists a neighborhood $U \supset A$ with $g(\overline{U}) \subset U$ and $A = \bigcap_{n\geq 0} g^n(\overline{U})$. For interval maps, this implies that A is always a finite union of (possibly degenerate) closed intervals, since one can start with a finite trapping neighborhood and the number of components cannot increase under iteration. In our case, when $g \in G$, every periodic point is approximated by periodic points of arbitrarily high period (see [26]), so A cannot contain isolated points, which shows that $\Omega(g)$ is a Cantor set. In particular, $\Omega(g)$ is not asymptotically stable.

We now prove the robustness of the attractor $\Lambda(g)$. Let $\Theta(f) = \bigcup_{x\in X} \omega(x)$ denote the union of all ω–limit sets of a function f, and let $\mathrm{CR}(f)$ be its chain recurrent set. Every point of an ω–limit set is chain recurrent, hence $\Theta(f) \subseteq \mathrm{CR}(f)$, and since $\mathrm{CR}(f)$ is closed we also have $\overline{\Theta(f)} \subseteq \mathrm{CR}(f)$. It follows that $\Lambda(f) \subseteq \overline{\Theta(f)} \subseteq \mathrm{CR}(f)$. By Theorem 3.1, for any $g \in G$ we have $\Lambda(g) = \Omega(g)$ and clearly also $\Omega(f) = \mathrm{CR}(f) = \mathrm{Rec}(f)$ since we know that g has the shadowing property (e.g. see the proof of Theorem 3.4.2 in [5], cf. [29]). Now let $g_n \to g$ in the uniform topology on $C([0,1])$. Since the chain recurrent set varies upper semi–continuously under uniform convergence (see Akin [2, Theorem 7.23]),

$$\limsup_{n\to\infty} \Lambda(g_n) \subseteq \limsup_{n\to\infty} \mathrm{CR}(g_n) \subseteq \mathrm{CR}(g) = \Lambda(g).$$

By Lemma 2.14 we also have lower semi–continuity, namely $\Lambda(g) \subseteq \liminf_{n\to\infty} \Lambda(g_n)$. Therefore $\lim_{n\to\infty} d_H(\Lambda(g_n), \Lambda(g)) = 0$, which proves that $\Lambda(g)$ is robust.

Finally, if D is an invariant set such that $g|_D$ is transitive, then $D = \omega(x)$ for some $x \in [0,1]$. If $x \in Z(g)$, then by Theorem 3.1(d) the basin $B(\omega(x))$ has zero Lebesgue measure. Suppose instead that $B(\omega(x))$ has positive measure. Then, by the construction, there exist two disjoint cycles of intervals for g, each intersecting $B(\omega(x))$ in a set of positive measure. But then the orbit of x would eventually intersect both cycles, which is impossible. Hence $B(\omega(x))$ has zero Lebesgue measure for every $x \in [0,1]$. Therefore, for any invariant ergodic measure μ, we have $\lambda(B(\mathrm{supp}(\mu))) = 0$, and thus no subset of $\Lambda(g)$ can satisfy the definition of a Palis attractor. □

Proof of Theorem A and Theorem B. It is an immediate consequence of Theorem 3.2. □

Remark 3.3. *By Theorem 3.1(a) and (b), the map g is totally singular:*

$$\lambda(Z(g)) = 1 \quad \textit{and} \quad \lambda(g^{-1}(Z(g))) = 0.$$

Moreover, by Theorem 3.1(d), the basin $B(\omega(x))$ has zero Lebesgue measure for every $x \in Z(g)$, hence g admits no SRB (physical) invariant probability measure. Therefore g is weird *in the terminology of* [1].

Let $f : M \to M$ be a continuous map on a compact metric space M. A compact invariant set $\Lambda \subset M$ is called a *quasi-attractor* if there exists a decreasing sequence of attracting regions $(U_n)_{n\geq 0}$, i.e.

$$f(\overline{U_n}) \subset \mathrm{int}(U_n) \quad \text{for all } n,$$

such that

$$\Lambda = \bigcap_{n\geq 0} \bigcap_{k\geq 0} f^k(U_n) = \bigcap_{n\geq 0} f^n(U_n).$$

Equivalently, Λ is a quasi-attractor if it is compact, invariant, *chain transitive*, and there exists an attracting region U with $\Lambda \subset \bigcap_{n\geq 0} f^n(U)$.

Remark 3.4. *Suppose that Λ is a set with attracting region U for a typical map $g \in G$. Then by Theorem 3.1 there are $x, y \in \Lambda$ such that $\omega(x), \omega(y)$ are distinct solenoids. Points in different solenoids have incompatible itineraries, so they belong to different chain recurrent classes. Thus Λ is not chain transitive, and consequently is not a quasi-attractor.*

Example 3.5. *To illustrate the existence of local asymptotically stable sets in G, consider an interval map f with an attracting periodic point p. Then there exists an open neighborhood U of $\mathrm{Orb}_f(p)$ such that $f(\overline{U}) \subset U$. Now take the set G from Theorem 3.1. By density, there is $g \in G$ sufficiently close to f such that $g(\overline{U}) \subset U$, and hence $A = \bigcap_{n\geq 0} g^n(\overline{U})$ is asymptotically stable. This shows that, although the global attractor $\Lambda(g) = \Omega(g)$ is never asymptotically stable, maps $g \in G$ can still have local asymptotically stable sets. In fact, by repeating this construction near several attracting periodic points, we see that there is no uniform upper bound on the number of disjoint asymptotically stable sets that a map $g \in G$ can possess.*

4. Interval map satisfying $A_{\text{PHYS}} \subsetneq A_{\text{STAT}} \subsetneq \Lambda$

In the following section, we present the example of an interval map whose physical attractor is not equal to the statistical one. It is based on the construction of a carefully chosen full measure subset of sequences X contained in the full shift, whose image under a suitable map is a full λ-measure subset of the interval. As the final result obtained for the interval is strictly dependent on the combinatorial and dynamical behavior of the elements in X, below we recall some basic definitions and properties from symbolic dynamics.

4.1. **Symbolic dynamics.** For the alphabet $\mathcal{A} = \{0, 1\}$ let

$$\Sigma = \{x = (x_i)_{i\geq 0} : x_i \in \mathcal{A}, i \geq 0\}$$

denote the full shift over $\mathcal{A}$. Let $T : \Sigma \ni (x_i)_{i\geq 0} \to (x_{i+1})_{i\geq 0} \in \Sigma$ denote the shift map. The finite sequences of symbols over $\mathcal{A}$ are called *words* or *blocks*. The *length* of a word is the number of symbols in that word. For any $n \geq 1$ let $\mathcal{L}_n = \{x_0 \dots x_{n-1} : x_i \in \mathcal{A}, i = 0, \dots, n-1\}$ be the set of all words of length n over the alphabet $\mathcal{A}$ and let $\mathcal{L} = \bigcup_{n\geq 1} \mathcal{L}_n$ be the set of all finite words over $\mathcal{A}$. For any $w \in \mathcal{L}$ a *cylinder set of the word* w is the set $[w] = \{wx : x \in \Sigma\}$, that is, the set of all sequences in Σ that begin with w. For any $n \geq 1$ and any letter $a \in \mathcal{A}$ we denote by a^n the word of length n consisting of n repetitions of letter a.

We introduce a natural order on $\mathcal{L}$ as follows:

$$w_1 \prec w_2 \Leftrightarrow (|w_1| < |w_2|) \vee (|w_1| = |w_2| \wedge w_1 <_2 w_2),$$

where $<_2$ denotes the usual order of binary numbers. The first elements in this order are

$$0 \prec 1 \prec 00 \prec 01 \prec 10 \prec 11 \prec 000 \prec \dots$$

Throughout the paper, we will use the notation $\mathcal{L} = \{w_n : n \geq 0\}$, assuming that $w_0 = 0$ and $w_{n+1} = \min\{w \in \mathcal{L} : w_n \prec w\}$ for every $n \geq 0$.

For $x \in \Sigma$ and integers $0 \leq k < l$, we use two related notations:

$$x_{[k,l]} = x_k x_{k+1} \dots x_l, \qquad x_{[k,l)} = x_k x_{k+1} \dots x_{l-1}.$$

Both denote finite subsequences of x, called *blocks* (or *subwords*) of x. Such a subsequence is called *block of* x or *subword of* x. For any $l > 0$, a *prefix of length* l of the word x (finite or infinite) is $x_{[0,l)}$. Denote $\mathbf{0} = 0^\infty$ and $\mathbf{1} = 1^\infty$.

Example 4.1. *We will show that, for the example constructed below,*

$$A_{phys} = \{0\} \subsetneq A_{stat} = \{0, 1\} \subsetneq \Lambda = [0, 1].$$

These inclusions will be established by Lemmas 4.2, 4.4, and 4.5.

Construction: We start with the definition of a particular family of infinite sequences $\{x_c\}_{c\in\Sigma} \subset \Sigma$. Each element x_c is constructed by repetitively combining the following three types of blocks, whose lengths depend on the sequence $\{k_n\}_{n\geq 0}$, where $k_n = 2^{2^n}$ and $n \geq 0$:

$$\begin{aligned} &A_n = 0^{k_{2n}}, n \geq 0, \\ &B_n^c = \begin{cases} 1^{k_{2n+1}-|w_n|} & \text{if } w_n = c_{[0,|w_n|)} \\ 0^{k_{2n+1}-|w_n|} & \text{if } w_n \neq c_{[0,|w_n|)}, \end{cases} \\ &w_n \in \mathcal{L}. \end{aligned}$$

In particular, $|A_n| = k_{2n}$ and $|B_n^c w_n| = k_{2n+1}$. For any $c \in \Sigma$ sequence x_c is defined as follows:

$$x_c = A_0 B_0^c w_0 A_1 B_1^c w_1 A_2 B_2^c w_2 A_3 B_3^c w_3 A_4 B_4^c w_4 A_5 B_5^c w_5 A_6 B_6^c w_6 \dots$$

For technical reasons, we introduce two additional sequences of lengths, namely $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$:

$$\begin{aligned}
a_0 &= k_0, \\
a_n &= k_0 + \cdots + k_{2n} = \sum_{i=0}^{2n} 2^{2^i} \text{ for } n \geq 1, \\
b_0 &= k_0 + k_1, \\
b_n &= k_0 + \cdots + k_{2n} + k_{2n+1} = \sum_{i=0}^{2n+1} 2^{2^i} \text{ for } n \geq 0.
\end{aligned}$$

For $n \geq 0$ and every $c \in \Sigma$, the value a_n indicates the position in x_c where the block A_n ends, while b_n indicates the position where the block w_n ends.

We summarize below several asymptotic relations between the sequences a_n, b_n, and k_n, which will be used repeatedly in the subsequent proofs.

$$\frac{a_n}{k_{2n}} \longrightarrow 1, \qquad \frac{b_n}{k_{2n+1}} \longrightarrow 1. \tag{12}$$

Indeed, since $k_m = 2^{2^m}$ and $k_{m+1} = k_m^2$, each term k_{m+1} dominates all preceding ones. Hence $a_n = k_{2n} + \sum_{i=0}^{2n-1} k_i \leq k_{2n} + 2k_{2n-1}$ and $b_n = k_{2n+1} + \sum_{i=0}^{2n} k_i \leq k_{2n+1} + 2k_{2n}$, giving $1 \leq \frac{a_n}{k_{2n}} \leq 1 + \frac{2}{k_{2n-1}}$ and $1 \leq \frac{b_n}{k_{2n+1}} \leq 1 + \frac{2}{k_{2n}}$, which yields the desired limits.

For $n \geq 1$, using $k_m = 2^{2^m}$ we have $k_{2n} = (k_{2n-1})^2$, so $\frac{2k_{2n-1}}{k_{2n}} = \frac{2}{k_{2n-1}} \to 0$ as $n \to \infty$. Since $a_n = \sum_{i=0}^{2n} k_i \geq k_{2n}$ and $\sum_{i=0}^{2n-1} k_i \leq 2k_{2n-1}$, we obtain

$$\frac{\sum_{i=0}^{n-1} k_{2i+1}}{a_n} \leq \frac{2k_{2n-1}}{k_{2n}} = \frac{2}{k_{2n-1}} \longrightarrow 0. \tag{13}$$

Denote $Q = \{x_c : c \in \Sigma\} \subset \Sigma$ and define $X = \bigcup_{n\in\mathbb{N}} T^n(Q)$. Consider the tent map φ on the interval $[0,1]$. The full shift (Σ, T) introduced above is an almost one-to-one extension of $([0,1], \varphi)$. Let $\Phi : \Sigma \to [0,1]$ be the natural coding map, which is one-to-one almost everywhere and two-to-one only on the set

$$\{w01^\infty : w \in \mathcal{L}\} \cup \{w10^\infty : w \in \mathcal{L}\}.$$

In particular, Φ is injective on X.

The image $\Phi(Q) \subset [0,1]$ is a Cantor set, and

$$\Phi(X) = \bigcup_{n=0}^{\infty} \varphi^n(\Phi(Q))$$

is a dense countable union of Cantor sets in $[0,1]$. By the Oxtoby–Ulam theorem, there exists an order-preserving homeomorphism $h : [0,1] \to [0,1]$ such that $\lambda(\tilde{X}) = 1$, where $\tilde{X} = h(\Phi(X))$. Observe that the endpoints of the interval correspond to the constant sequences:

$$(h \circ \Phi)(\mathbf{0}) = 0, \qquad (h \circ \Phi)(\mathbf{1}) = 1.$$

Define a measure $\mu = h^\star\lambda$ on $[0,1]$. By construction, $\mu(\Phi(Q)) > 0$ (as ensured by the Oxtoby–Ulam theorem), and μ is not invariant under φ. Next, define the map

$$f = h \circ \varphi \circ h^{-1}$$

and the non-atomic measure $\nu = \Phi^\star\mu$ on Σ, where for any measurable set $A \subset \Sigma$ we write $\Phi^\star\mu(A) = \mu(\Phi(A))$. Since Φ is injective on X and $\mu(\Phi(X)) = 1$, it follows that $\nu(X) = 1$, that is, ν is fully supported on X.

The relations among the maps defined above are summarized in the following commutative diagram:

$$\begin{tikzcd}
(\Sigma,\nu) \arrow[r, "\Phi"] \arrow[d, "T"] & ([0,1],\mu) \arrow[r, "h"] \arrow[d, "\varphi"] & ([0,1],\lambda) \arrow[d, "f"] \\
(\Sigma,\nu) \arrow[r, "\Phi"] & ([0,1],\mu) \arrow[r, "h"] & ([0,1],\lambda)
\end{tikzcd}$$

Lemma 4.2. *For the interval map f defined above, we have $A_{stat}(f) = \{0,1\}$.*

Proof. Fix $c \in \Sigma$. For any $K \geq 1$, let $U_0 = [0^K]$ and $U_1 = [1^K]$. For each $n \geq 0$, consider the block $A_n = 0^{k_{2n}}$ in x_c. All iterates $T^j(x_c)$ with $a_n \leq j < a_n + k_{2n} - K$ lie entirely inside U_0. Hence, by (12), the frequency of visits to U_0 up to time a_n satisfies

$$\frac{1}{a_n}\sum_{j=0}^{a_n-1}\chi_{U_0}(T^j x_c) \;\geq\; \frac{k_{2n}-K}{a_n} \;\longrightarrow\; 1,$$

which means that the point $\mathbf{0}$ belongs to the $\sigma_T(x_c)$.

There are infinitely many n for which $B_n^c = 1^{k_{2n+1}-|w_n|}$, so for those n the orbit segment $\{T^j(x_c) : a_n \leq j < a_n + k_{2n+1} - |w_n| - K\}$ lies in U_1. Thus, by (12),

$$\frac{1}{b_n}\sum_{j=0}^{b_n-1}\chi_{U_1}(T^j x_c) \geq \frac{k_{2n+1}-|w_n|-K}{b_n} \longrightarrow 1,$$

and $\mathbf{1} \in \sigma_T(x_c)$.

If v is any non-constant finite word, visits of the orbit to the cylinder $[v]$ occur only inside the short blocks w_n or within a $|v|$–neighborhood of their boundaries. Since $\sum_{i=0}^{n}(|w_i| + 2|v|)/b_n \to 0$, we get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\chi_{[v]}(T^j x_c) = 0.$$

Therefore $\sigma_T(x_c) = \{\mathbf{0},\mathbf{1}\}$ for all $c \in \Sigma$. As $X = \bigcup_{n\geq 0} T^n(Q)$ and $\nu(X) = 1$, it follows that $\sigma_T(x) = \{\mathbf{0},\mathbf{1}\}$ for ν–almost every $x \in \Sigma$.

The map $h \circ \Phi$ is one-to-one on X and sends $\mathbf{0},\mathbf{1}$ to $0,1$. Since $\lambda(\tilde{X}) = \lambda(h(\Phi(X))) = 1$, we have that for λ–a.e. $\tilde{x} \in [0,1]$ there exists a unique $x \in X$ such that $\tilde{x} = (h\circ\Phi)(x)$. Time averages for f along $\tilde{x}$ coincide with those for T along x, hence $\sigma_f(\tilde{x}) = (h\circ\Phi)(\sigma_T(x)) = \{0,1\}$ for λ–a.e. $\tilde{x}$.

By Theorem 2.6 we have that $A_{\text{stat}}(f)$ is the smallest closed set containing the σ–limit sets of almost all orbits, therefore we obtain $A_{\text{stat}}(f) = \{0,1\}$. □

Lemma 4.3. *The cylinder* [1] *is not essential for the system* (Σ, T, ν).

Proof. Fix $U = [1]$. Take any $x_c \in Q$ and recall that x_c is built from blocks $A_n = 0^{k_{2n}}$, B_n^c, and w_n with $|A_n| = k_{2n}$ and $|B_n^c w_n| = k_{2n+1}$. During each block A_n, all symbols are 0, so the orbit of x_c does not visit $U = [1]$ for k_{2n} consecutive steps. Up to time a_n, the number of positions belonging to U is bounded by the total length of all previous $B_i^c w_i$ blocks:

$$\#\{0 \leq j < a_n : T^j(x_c) \in U\} \;\leq\; \sum_{i=0}^{n-1} k_{2i+1}.$$

Therefore,

$$\frac{1}{a_n}\sum_{j=0}^{a_n-1}\chi_U(T^j(x_c)) \;\leq\; \frac{\sum_{i=0}^{n-1} k_{2i+1}}{a_n}.$$

By (13), it follows that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{j=0}^{a_n-1} \chi_U(T^j(x_c)) = 0.$$

Since the same argument applies to all shifts $T^m(Q)$, this limit holds for all $x \in X = \bigcup_{m\geq 0} T^m(Q)$. As ν is supported on X, the convergence also holds for ν–almost every $x \in \Sigma$. Thus [1] is not essential. □

Lemma 4.4. $A_{phys}(f) = \{0\}$.

Proof. By definition, $A_{\text{phys}}(f) \subseteq A_{\text{stat}}(f)$, and from Lemma 4.2 we know that $A_{\text{stat}}(f) = \{0, 1\}$. To prove the claim, it suffices to show that $1 \notin A_{\text{phys}}(f)$.

Suppose, for contradiction, that $1 \in A_{\text{phys}}(f)$. Then every neighborhood of 1 would be essential. In particular, there would exist an essential open set $U \subset [0, 1]$ with

$$U \subseteq (h \circ \Phi)([1]).$$

As $(h \circ \Phi)$ is a homeomorphism on X and ν is the pullback measure, this would imply that $[1] \subset \Sigma$ is essential for (Σ, T, ν). This contradicts Lemma 4.3. Hence $1 \notin A_{\text{phys}}(f)$, and consequently $A_{\text{phys}}(f) = \{0\}$. □

Lemma 4.5. $\Lambda(f) = [0, 1]$.

Proof. By construction, for every $c \in \Sigma$ the orbit of x_c under T is dense in Σ. Since Φ semi-conjugates T with the tent map φ and h is a homeomorphism of $[0, 1]$, the orbit of $h(\Phi(x_c))$ under $f = h \circ \varphi \circ h^{-1}$ is dense in $[0, 1]$. Hence $\Lambda = [0, 1]$. □

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(M. Foryś-Krawiec) AGH University of Krakow, Faculty of Applied Mathematics, al. Mickiewicza 30, 30-059 Kraków, Poland.
Email address: maforys@agh.edu.pl

(J. Hantáková) Silesian University in Opava, Mathematical Institute, Na Rybníčku 626/1, 746 01 Opava, Czech Republic
Email address: jana.hantakova@math.slu.cz

(M. Kowalewski) AGH University of Krakow, Faculty of Applied Mathematics, al. Mickiewicza 30, 30-059 Kraków, Poland.
Email address: kowalewski@agh.edu.pl

(P. Oprocha) Centre of Excellence IT4Innovations - Institute for Research and Applications of Fuzzy Modeling, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic – & – AGH University of Krakow, Faculty of Applied Mathematics, al. Mickiewicza 30, 30-059 Kraków, Poland.
Email address: piotr.oprocha@osu.cz