

MINIMAL CROSSING DIAGRAMS OF SPATIAL GRAPHS

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ABSTRACT. We prove that all 1-vertex spatial graphs with adequate diagrams have minimal crossing number, and that spatial graph diagrams obtained by replacing vertices and edges of a planar embedded graph by minimal crossing link or spatial graph diagrams have minimal crossing number. Finally, we give an example in answer to a question of Adams et al. about minimal crossing diagrams of rigid vertex graphs.

1. TAIT’S CONJECTURE AND ITS GENERALIZATIONS

The study of minimal crossing diagrams of links began in 1898 when P. G. Tait made the following conjecture, which was independently proved by Thistlethwaite [14], Kauffman [6], and Murasugi [10] in 1987.

Tait’s Conjecture. [14, 6, 10] *A reduced alternating diagram of a link L has minimal crossing number, and if L is prime then no non-alternating diagram has minimal crossing number.*

Lickorish and Thistlethwaite [8] generalized Tait’s Conjecture to Montesinos links (i.e., the numerator closure of a sum of rational tangles) with specific types of diagrams. In particular, a rational tangle in standard alternating form can be expressed as a sum $Q + H$ where Q is either trivial or an alternating rational tangle containing at least two crossings, including a final crossing between the bottom strings (as shown on the left side of Figure 1), and H is a row of (possibly zero) horizontal twists. Furthermore, if we have a tangle of the form $Q_1 + H_1 + \cdots + Q_m + H_m$, then all of the H_i can be put together into a single row of horizontal twists H at the right. Lickorish and Thistlethwaite defined a Montesinos link diagram written as the numerator closure of a tangle $Q_1 + \cdots + Q_m + H$ where the Q_i and H are as above to be *reduced* if either the diagram is alternating or H is empty.

Lickorish and Thistlethwaite’s Theorem. [8] *A reduced Montesinos link diagram has minimal crossing number.*

Thistlethwaite also proved a generalization of Tait’s Conjecture to adequate links, which we define below.

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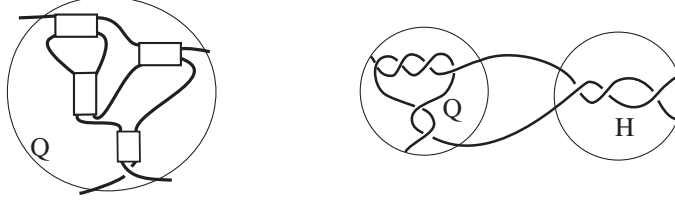


FIGURE 1. A rational tangle can be written as $Q + H$ where Q is trivial or is alternating and in the form on the left and H is a (possibly trivial) row of horizontal twists.

Definition 1.1. Let G be a diagram of a link. If we resolve all of the crossings as in the center of Figure 2, we obtain a collection of circles called the **all- A resolution** of G . If we resolve all of the crossings as on the right, we obtain a collection of circles called the **all- A^{-1} resolution** of G .

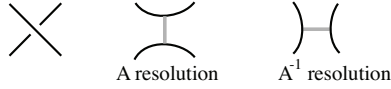


FIGURE 2. A - and A^{-1} -resolutions of a crossing and their associated grey segments.

Definition 1.2. We add grey segments to the collection of circles in the all- A or all- A^{-1} resolution of a link diagram G in place of crossings as in Figure 2. We say G is **A -adequate** or **A^{-1} -adequate** if every grey segment has its endpoints on distinct circles of the all- A resolution or all- A^{-1} resolution, respectively. If G is both A -adequate and A^{-1} -adequate, then we say G is **adequate**.

Thistlethwaite's Theorem. [15] *An adequate diagram of a link has minimal crossing number.*

In addition to the above results, Tait's Conjecture has been generalized to torus links by Murasugi [11], to cable knots by Kalfagianni and McConkey [5], to composite knots by Lackenby [7], as well as to other families of links (see for example [2, 4, 13]).

Sawollek generalized Tait's Conjecture to reduced alternating 4-valent spatial graphs using the definition below.

Definition 1.3. [12] *Let G be a diagram of a 4-valent spatial graph. We say G is **reduced and alternating**, if every smoothing of the vertices of G in the plane of diagram yields a reduced alternating diagram of a link.*

Sawollek's Theorem. [12] *Let G be a reduced alternating diagram of a 4-valent spatial graph Γ . Then G has minimal crossing number.*

In the next section, we generalize Sawollek's result to 1-vertex graphs with an arbitrary number of edges where *reduced alternating* is replaced by the weaker condition of *adequate*. Then, in Section 3 we generalize it to spatial graph diagrams made up of pieces with minimal crossing number lying on a planar framework. Finally, in Section 4, we present Adams et al.'s generalization of Tait's Conjecture to rigid vertex spatial graphs [1] and give an example in answer to a question they raise.

2. ADEQUATE 1-VERTEX GRAPHS

Below we extend the definitions of reduced, alternating, and adequate from links to spatial graphs with arbitrary even valence.

Definition 2.1. *Let G be a diagram of an even valent spatial graph. A **smoothing** of G is a link diagram obtained by replacing neighborhoods of vertices by non-intersecting planar segments joining the endpoints of edges.*

Definition 2.2. *Let G be a diagram of an even valent spatial graph. We say G is **reduced**, **alternating**, or **adequate** if every smoothing of the vertices of G yields a link diagram which is reduced, alternating, or adequate, respectively.*

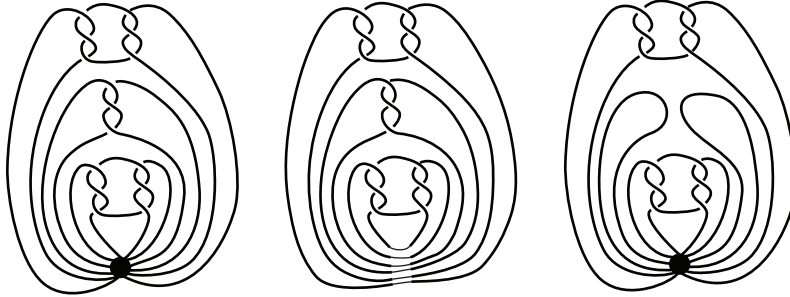


FIGURE 3. The smoothing in the center image is not reduced. On the right is a diagram with fewer crossings than the original.

Observe that the diagram of the spatial graph on the left in Figure 3 has the property that connecting adjacent endpoints would give us a reduced alternating link. However, the graph is not reduced because there is a different smoothing of the vertex (illustrated in the center) which is not reduced. Furthermore, the diagram on the left in Figure 3 does not have minimal crossing number, since we can twist around the vertex to get a diagram with fewer crossings, illustrated on the right.

It is tempting to think that if a spatial graph diagram G can be isotoped to a diagram G' , then a link obtained by a smoothing of the vertices of G will be isotopic to a link obtained by a smoothing of the vertices of G' . However, the spatial graph diagram G on the left in Figure 4 is isotopic G' ,

but neither smoothing of G is isotopic to a smoothing of G' . In particular, both smoothings of G are non-trivial knots, but neither smoothing of G' is a non-trivial knot.

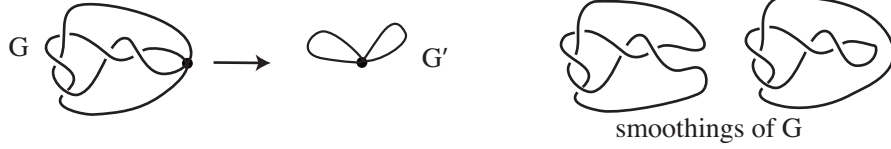


FIGURE 4. G is isotopic to G' , but neither smoothing of G is isotopic to a smoothing of G' .

Thus instead of using smoothings of a spatial graph diagram to obtain a link we use n -string tangles to associate a different link to the spatial graph.

Definition 2.1. An n -string tangle diagram is a diagram of a disk B together with a set T of n arcs where the endpoints of the arcs on ∂B are numbered $1, \dots, 2n$ going clockwise. Such a tangle diagram is said to be **adequate** if no matter how we join the endpoints of T by disjoint planar arcs we get an adequate link diagram.

Definition 2.2. Let (B, T) be an n -string tangle diagram, and let (B^*, T^*) be the mirror image of (B, T) so that the endpoints of T^* are numbered $1, \dots, 2n$ going counterclockwise. Let L_T be the link diagram obtained by gluing B and B^* together so that each endpoint j of T is attached to the endpoint j of T^* .

Lemma 2.1. Let (B, T) be an adequate n -string tangle diagram. Then the link diagram L_T is adequate.

Proof. Suppose without loss of generality, that the all- A resolution of L_T has a grey segment $S \subseteq B$ with both endpoints on a single circle C .

Let R denote the set of arcs of the all- A resolution of L_T which are contained in B^* and have their endpoints on ∂B . Let J be the link diagram $T \cup R$. The all- A resolution of J intersected with B is identical to the all- A resolution of L intersected with B . Since the grey segment $S \subseteq B$, either the circle C is contained in B or C is the union of arcs in B and arcs in R . In either case, C is a circle in the all- A resolution of J .

But since T is an adequate tangle and R is a set of disjoint planar arcs that join the endpoints of T , by definition J must be adequate. As this is impossible, it follows that L_T is adequate. $\square$

Definition 2.3. Let G be a spatial graph diagram G with a single vertex v . Define the **tangle diagram associated to G** to be (B, T) where $B = \text{cl}(S^2 - N(v))$ and $T = G \cap B$.

Lemma 2.2. Let G be a spatial graph diagram with a single vertex and suppose G' is isotopic to G . Let T and T' be the tangle diagrams associated to G and G' , respectively. Then the link diagrams L_T and $L_{T'}$ are isotopic.

Proof. Suppose that one of the five Reidemeister moves for spatial graphs takes G to G' . We see below that there is an isotopy taking L_T to $L_{T'}$.

Let $B = \text{cl}(S^2 - N(v))$, $T = G \cap B$, and (B^*, T^*) be the mirror image of (B, T) . A planar isotopy or an R1, R2, or R3 move taking G to G' induces an analogous move on (B, T) . On L_G , this does an R1, R2, or R3 move on T and T^* , giving us an isotopy from L_T to $L_{T'}$.

An R4 move taking G to G' pulls an arc over or under the vertex v . This means we are pulling an arc of T over or under $N(v)$. On L_T , this pulls an arc of T along the inside of ∂B in (B, T) and along the inside of ∂B^* in (B^*, T^*) . (In Figure 5, the arc being pulled is turquoise.) This gives us an isotopy from L_T to $L_{T'}$.

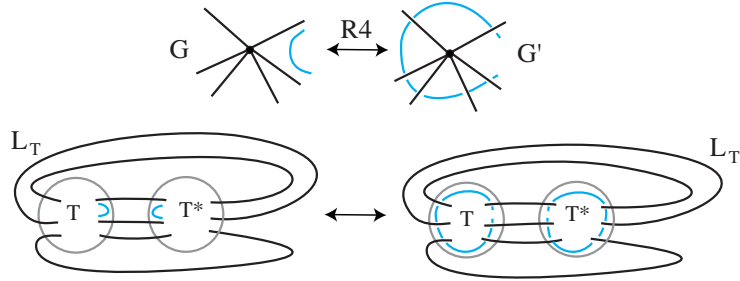


FIGURE 5. An R4 move taking G to G' induces an isotopy taking L_T to $L_{T'}$.

Finally, an R5 move taking G to G' , twists or untwists a pair of adjacent arcs around v . On L_T , this adds or removes a crossing to (B, T) near ∂B and adds or removes its mirror image to (B^*, T^*) near ∂B^* . As we see in Figure 6, this gives us an R2 move taking L_T to $L_{T'}$. $\square$

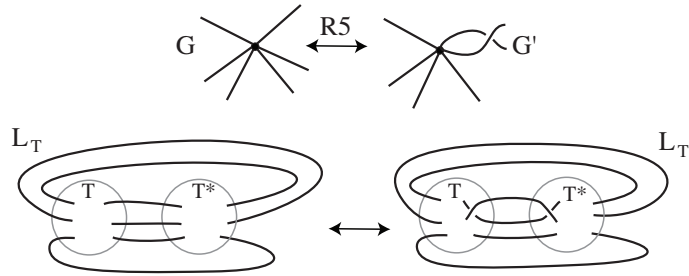


FIGURE 6. An R5 move taking G to G' induces an isotopy taking L_T to $L_{T'}$.

Lemma 2.3. *Let G be a spatial graph diagram with a single vertex with associated tangle diagram T . Suppose that L_T is adequate. Then G has minimal crossing number.*

Proof. Suppose there is an isotopy taking G to G' , and T' is the associated tangle diagram of G' . Then by Lemma 2.2, there is an isotopy taking L_T to $L_{T'}$. But L_T and $L_{T'}$ have two times the number of crossings of G and G' , respectively. Since L_T is adequate, Thistlethwaite's Theorem [15] implies that it has minimal crossing number. Thus G' cannot have fewer crossings than G . $\square$

Theorem 2.3. *Let G be an adequate spatial graph diagram with a single vertex. Then G has minimal crossing number.*

Proof. Since G is adequate, the tangle diagram T associated to G is adequate. Now it follows from Lemma 2.1 that the link diagram L_T is adequate. Thus the result follows by Lemma 2.3. $\square$

Observe that a reduced alternating graph is adequate, and hence Theorem 2.3 generalizes Sawollek's Theorem to 1-vertex graphs with any valence.

Proposition 2.1. *Let G be a spatial graph diagram with a single vertex and an edge E with no crossings such that $G - E$ is adequate. Then G is adequate, and hence achieves its minimal crossing number.*

Proof. Suppose G is not adequate. Then there is a smoothing of the vertex v which produces a non-adequate link diagram L . Without loss of generality, the all- A resolution of L has a grey segment S with both endpoints on a single circle of the resolution. Let R be the set of arcs with endpoints in ∂B obtained by intersecting the all- A resolution of L with B .

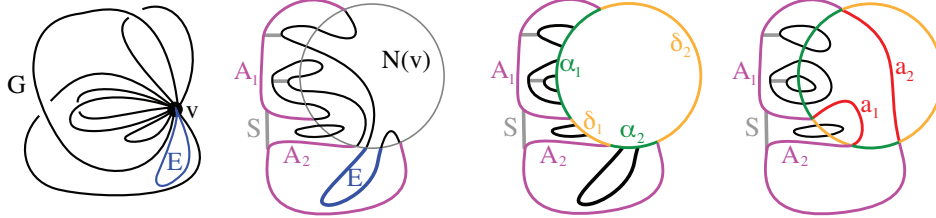


FIGURE 7. The second picture is the all- A resolution of L , and the fourth picture is a smoothing of v for $G - E$ which joins the endpoints of A_1 and A_2 .

Since E has no crossings, S must come from the resolution of a crossing in $G - E$. If S has both endpoints on a single arc in R or circle of the all- A resolution contained in $B = \text{cl}(S^2 - N(v))$, then S would have both endpoints on a circle in the all- A resolution of any link obtained by smoothing v for $G - E$. This is impossible, since $G - E$ is adequate. Thus S must go between distinct arcs A_1 and A_2 in R (see A_1 , A_2 , and S in the second picture of Figure 7).

Now since A_1 and A_2 are disjoint, there are disjoint arcs α_1 and α_2 in ∂B such that both endpoints of α_i match those of A_i (see the green arcs in the third picture of Figure 7). Let δ_1 and δ_2 denote the arcs of $\partial B - (\alpha_1 \cup \alpha_2)$.

Since $S \cup A_1 \cup A_2$ separates B and is disjoint from the other arcs of R , both endpoints of each arc of $R - (A_1 \cup A_2)$ are in the interior of a single α_i or δ_i . (In the third picture of Figure 7, the arcs of $R - (A_1 \cup A_2)$ are black.)

Now draw arcs a_1 and a_2 in $N(v)$ parallel to δ_1 and δ_2 respectively, which join the endpoints of A_1 with those of A_2 . (In the fourth picture of Figure 7 a_1 and a_2 are red.) Then choose disjoint arcs in $N(v) - (a_1 \cup a_2)$ joining the endpoints of each arc in $R - (E \cup A_1 \cup A_2)$ into a circle. This set of arcs together with a_1 and a_2 gives us a smoothing of v for $G - E$, which produces a link whose all- A resolution includes the circle $A_1 \cup a_1 \cup A_2 \cup a_2$. As S has both endpoints on this circle, this contradicts our assumption that $G - E$ is adequate. $\square$

Observe that the conclusion of Proposition 2.1 is false if we replace adequate by reduced and alternating. In particular, the diagram on the left in Figure 8 is not alternating, as we see from the smoothing on the right. But if we delete the edge E we obtain a reduced alternating diagram.

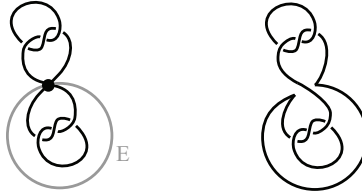


FIGURE 8. This diagram is not alternating, but if we delete E it becomes alternating.

3. SPATIAL GRAPH DIAGRAMS ON PLANAR FRAMEWORKS

In this section, we prove that spatial graph diagrams obtained by replacing vertices and edges of a planar embedded graph by minimal crossing link or spatial graph diagrams have minimal crossing number.

Definition 3.1. *Let P be a planar embedded graph (i.e., a diagram with no crossings).*

- An **edge replacement** of P is obtained by replacing an edge E_i by a link or spatial graph diagram G_i containing the vertices of E_i such that G_i lies in a neighborhood $N(E_i)$.
- A **vertex replacement** of P is obtained by replacing a vertex V_j together with its incident edges $E_{j1}, \dots, E_{jn}$ by a link or spatial graph diagram G_j containing each vertex V_i of $E_{ji} - V_j$ such that G_j lies in a neighborhood $N(V_j \cup E_{j1} \cup \dots \cup E_{jn})$.
- If G is obtained from P by some number of edge and vertex replacements, we say that G **has framework P** .

Figure 9 shows an example where edges E_1 and E_2 and vertices v_1 and v_2 are replaced by link or spatial graph diagrams of the corresponding color. The center image shows the neighborhoods of the edges and vertices that are being replaced. Observe that G does not have even valence.

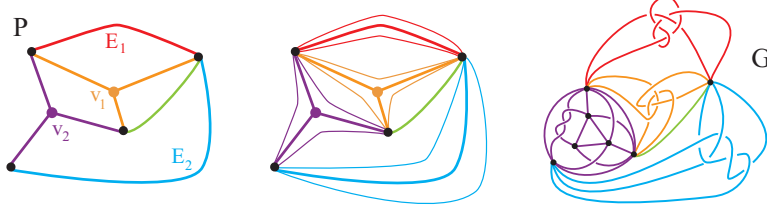


FIGURE 9. On the left we illustrate P , in the center we show the neighborhoods of edges and vertices that we will replace, and on the right we have G .

Theorem 3.1. *Let G be a spatial graph diagram with a planar graph framework P , such that all of the link or spatial graph diagrams replacing edges and vertices have minimal crossing number. Then G has minimal crossing number.*

Proof. Let $G_1, \dots, G_m$ denote the link or spatial graph diagrams that replace edges and vertices of P . Then each G_i has minimal crossing number. Since G is on a planar graph framework and $G_1, \dots, G_m$ are pairwise disjoint, all of the crossings of G contained in $G_1, \dots, G_m$.

Suppose that G is isotopic to a spatial graph diagram G' . Then each G_i is isotopic to some G'_i in G' , and G'_i has at least as many crossings as G_i . In addition, G' may have crossings which are not contained in one of the G'_i . Thus G' has at least as many crossings as G . $\square$

To see that we can apply this theorem to the spatial graph diagram in Figure 9, observe that the orange and red links are reduced and alternating. The blue is the Kinoshita θ -curve which is known to have minimal crossing number 5 [9]. Finally, the purple is an embedding of the complete graph K_6 which was shown to have minimal crossing number in [3]. Thus G has minimal crossing number by Theorem 3.1.

The following corollary is immediate by using Thistlethwaite's Theorem [15]

Corollary 1. *Let G be a spatial graph diagram with a planar graph framework P , such that all of replacements for edges and vertices are adequate link diagrams. Then G has minimal crossing number.*

Let G be a spatial graph diagram like the one in the Corollary which has even valence. While there is a smoothing that gives us a pairwise disjoint collection of adequate links, not every smoothing necessarily produces an adequate link. Thus G is not necessarily adequate.

Open Question. *Let G be an adequate diagram of an even valent spatial graph. Does G necessarily have minimal crossing number?*

4. RIGID VERTEX SPATIAL GRAPHS

A *rigid vertex spatial graph* is one which has a small planar disk around each vertex. A *rigid vertex isotopy* is an isotopy which preserves this set of disks. Such an isotopy can turn over a disk, but the edges coming out of a disk cannot be permuted. As a result, a diagram can have minimal crossing number as a rigid vertex graph, but not as a spatial graph. For example Figure 10 shows a spatial graph diagram that has minimal crossing number 1 as a rigid vertex spatial graph, but would have minimal crossing number 0 as an ordinary spatial graph.

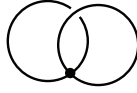


FIGURE 10. This rigid vertex spatial graph has minimal crossing number 1.

Adams et al. [1] defined reduced and alternating spatial graph diagrams as follows.

Definition 4.1. [1] *Let G be a spatial graph diagram, which may or may not be a rigid vertex spatial graph.*

- (1) G is **reduced** if there is no circle C in the plane of the diagram such that C intersects G transversely in a single crossing.
- (2) G is **alternating** if
 - crossings alternate along each edge.
 - given be a planar neighborhood N of vertices and uncrossed edges, first crossings along edges exiting any component of N alternate as we go around the boundary of the component.

Adams et al.'s Theorem. [1] *Let Γ be an even-valent rigid vertex spatial graph whose diagram G is reduced and alternating. If G contains no uncrossed cycle, then G has minimal crossing number.*

Adams et al. shows that the above theorem for rigid-vertex graphs cannot be extended to graphs with at least one vertex of valence 3, but asks whether it could be extended to graphs with at least one odd-valence vertex but none of valence 3. In particular, they ask for such an example which does not have a circle passing transversely through a single crossing and a single edge. The spatial graph diagram in Figure 11 has such a circle (illustrated in red). If there is such a circle going through a crossing and an edge e , then we can eliminate the crossing by a rigid isotopy that twists the graph around e (see Figure 11).

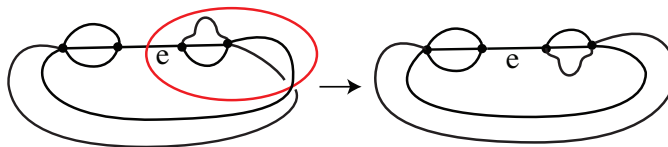


FIGURE 11. A rigid isotopy eliminates the crossing intersected by the red circle.

The diagram on the left in Figure 12 provides a rigid vertex spatial graph diagram which answers the question raised by Adams et al.. In particular, the neighborhood N of vertices and uncrossed edges is illustrated in yellow on the right, with edges exiting N numbered consecutively so that it is easy to check that first crossings alternate as we go along ∂N . Observe that there are no vertices of valence 3, no uncrossed cycles, and no circle passing transversely through a single crossing and a single edge.

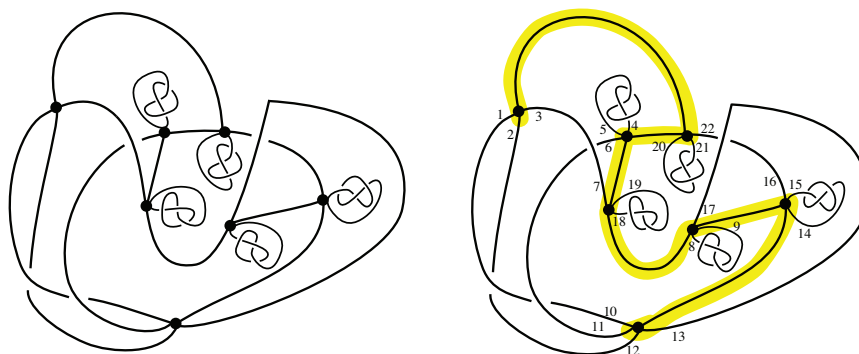


FIGURE 12. A spatial graph diagram which answers the question of Adams et al.. The neighborhood of the vertices and uncrossed edges is illustrated in yellow on the right.

Figure 13 illustrates a rigid vertex isotopy which transforms the reduced alternating diagram on the left to a non-alternating diagram with fewer crossings. In the first step we lift the grey arc, turning over the red and purple disks as indicated above the figure. In the second step we turn over the turquoise disk and eliminate a crossing. Note that while the initial diagram is alternating, neither the second nor the third diagram in Figure 13 is alternating. In particular, two consecutive edges exiting a neighborhood of the orange disk are overcrossings.

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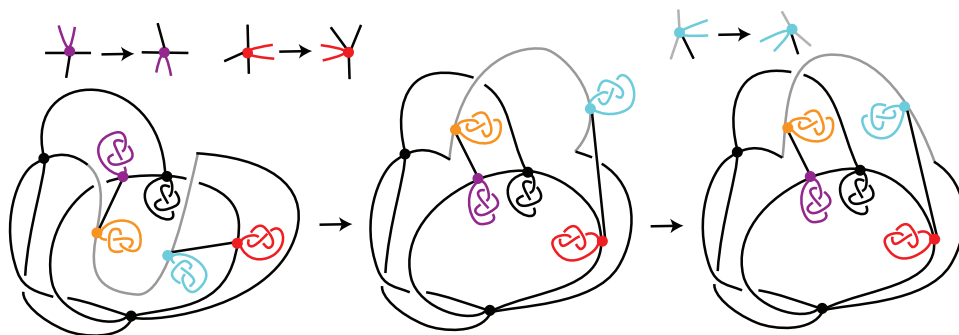


FIGURE 13. A rigid-vertex isotopy which turns over the red, purple, and turquoise disks reduces the number of crossings.

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