Full spectrum of Love numbers of Reissner-Nordström black hole in D-dimensions

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ABSTRACT

We present a comprehensive analysis of the full spectrum of tidal Love numbers for Reissner-Nordström (RN) black holes in general spacetime dimensions. By perturbing the Einstein-Maxwell theory around the D-dimensional RN background, we derive an effective two dimensional quadratic action encompassing tensor, vector, and scalar-type perturbation sectors. Through diagonalization, we obtain master equations governing each sector and extract the corresponding Love numbers from the asymptotic behavior of the solutions. Our results confirm that all Love numbers vanish for four-dimensional RN black holes. In higher dimensions, the tensor and vector Love numbers reproduce previously known results. For the previously unknown scalar-type Love numbers, we show also they vanish for integer valued effective multipolar indices and display logarithmic running behavior when the corresponding indices are half integers.

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Contents

1 Introduction

Tidal Love numbers describe the deformability of a black hole under the influence of external force. It has been demonstrated that conservative linear tidal Love numbers vanish for several families of asymptotically flat black holes in four dimensions, including Schwarzschild [1], Kerr [2–4] and Reissner-Nordström (RN) [5–7] black holes. It was also found that for Kerr black holes, the conservative tidal Love numbers remain vanishing at nonlinear level [8–11]. These results provide strong support to the no-hair theorem, showing that black holes in four-dimensional general relativity are remarkably rigid objects. On the other hand, Love numbers are generically nonvanishing for charged black holes coupled to charged scalar or fermion [12–15] and black holes in higher dimensions [16–19], in modified gravitational theories [20–26], in the presence of a cosmological constant [27, 28], or in astrophysical environments [29–33]. Its potential observability by future gravitational wave experiments makes Love numbers useful theoretical diagnostics.

In this work, we shall revisit the computation of Love numbers of RN black holes in diverse dimensions for several reasons. In the previous work [6], only tensor and vector Love numbers were investigated. However, to gain complete knowledge about how RN black holes respond to external sources, it is necessary to compute also the scalar type Love numbers. On the other hand, within the framework of the realistic Einstein-Maxwell theory, charged black holes represent the simplest generalizations of the vacuum Schwarzschild solution. It provides a natural testbed to study graviton-photon mixing [34, 35], which may lead to conversion of gravitational perturbations into their electromagnetic counterparts and vice versa. Although astrophysical black holes are usually assumed to be neutral, there are well known astrophysical processes by which black holes can acquire charge [36, 37], albeit typically small.

We carry out the analysis by expanding the D-dimensional fields in terms of harmonic functions on S^{D-2}, plugging them into the Einstein-Maxwell action, linearized about the background RN black hole to yield an effective 2-dimensional quadratic action for perturbative fields in $1+1$ dimensions. Upon diagonalizing the perturbations, one obtains the complete set of master equations generalizing previous results for Schwarzschild black hole [38–42]. Although the tensor and vector modes were previously analyzed using the Ishibashi-Kodama formalism [6], we rederive them here within the effective-action framework to ensure a unified and self-consistent treatment of all perturbations channels, which is also necessary for cross-checking known results.

When restricted to $D = 4$, our master equations match with results in [43–45]. In $D > 4$, the metric perturbations contain also an intrinsic tensor mode that propagates on its own without mixing with others. This constitutes the simplest perturbation channel. The other components of the metric perturbations inevitably mix with the electromagnetic counterparts. Due to the different methods used in diagonalizing the perturbations, our master equations do not look the same as those in [6]. However, the Love numbers in the tensor and vector sectors do agree with those obtained in [6]. In particular, we confirm that all the Love number vanishes for $4D$ RN black holes [6,7], a feature similar to the Schwarzschild black hole [1,2,40]. In $D > 4$, we also recover earlier results for the tensor and vector Love numbers [6]. So far, the scalar type Love numbers of RN black hole have not been studied. In D-dimensional spacetimes, the effective multipolar index $\tilde{\ell}$ is related to the original multipolar index ℓ by $\tilde{\ell} = \ell/(D-3)$. We recall that in the case of Schwarzschild black hole, when $\tilde{\ell}$ is integer valued, all scalar type Love numbers vanish. Hence, it should be interesting to see if similar features appear also in RN black hole. Our results show that this is indeed the case, which may imply the existence of certain accidental symmetry, as was found for other cases, including $4D$ Schwarzschild black hole [46–48] and RN black hole [7] and certain higher dimensional black branes [49] or black holes [50].

The gauge invariance of the Love number has been addressed in [40] by using the worldline effective field theory (WEFT) formalism in which the Love numbers correspond to gauge invariant Wilsonian coefficients in front of higher derivative operators. Furthermore, it was shown in [40] that for Schwarzschild black hole, the Love numbers defined in WEFT match with those extracted directly from the large distance expansion of the perturbative solutions.

Direct computation shows that that the graviton-photon mixing decays sufficiently fast near the spatial infinity. Consequently, the relation between the WEFT coefficients and the falloff coefficients of perturbations near spatial infinity takes the form as that for Schwarzschild black hole. Thus, in this work we shall extract the Love numbers of RN black holes in all sectors from the large distance expansion of perturbative solutions.

This paper is organized as follows. In section 2, we study the full linear perturbations of the RN black hole background in Einstein-Maxwell theory in arbitrary dimensions. We derive the quadratic perturbative quadratic action and use spherical harmonics to decompose it and obtain an effective two-dimensional theory. Note that although our goal is to compute the previously unknown scalar sector, it is necessary to analyse the full linear analysis including both the tensor and vector sectors as well. In Section (3), we further simplify the two-dimensional effective theory by introducing auxiliary fields, thereby isolating the true physical degrees of freedom. We diagonalize the graviton-photon mixing sectors and compute the Love numbers. Having properly analyse both the tensor and vector sectors, we focus our attention on the new scalar sector and compute the corresponding Love numbers in general dimensions. We conclude the paper in section 4. We present the explicit forms of the potentials of the scalar sector in the appendix.

2 Perturbations about the RN black hole

We begin with the Einstein-Maxwell theory in general D dimensions

$$S = \frac{1}{16\pi}\int d^Dx\sqrt{-g}\big(R - \frac{1}{4}F^2\big)\ . \tag{1}$$

Next we consider perturbing the metric field $g_{\mu\nu}$ and the Maxwell field A_μ around a background solution denoted as $\bar{g}_{\mu\nu}$ and $\bar{A}_\mu$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad A_\mu = \bar{A}_\mu + a_\mu\ , \tag{2}$$

where, $\bar{g}_{\mu\nu}$ and $\bar{A}_\mu$ satisfy the equations of motion

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} - \frac{1}{2}\bar{F}^2_{\mu\nu} + \frac{1}{8}\bar{g}_{\mu\nu}\bar{F}^2 = 0, \quad \nabla_\mu\bar{F}^{\mu\nu} = 0\ . \tag{3}$$

For later convenience, we introduce the following notations

$$h = \bar{g}^{\mu\nu}h_{\mu\nu}, \quad \bar{F}_{\mu\nu} = \nabla_\mu\bar{A}_\nu - \nabla_\nu\bar{A}_\mu, \quad f_{\mu\nu} = \nabla_\mu a_\nu - \nabla_\nu a_\mu\ , \tag{4}$$

and use $\bar{g}^{\mu\nu}$ and $\bar{g}_{\mu\nu}$ to raise and lower tensorial indices. Plugging (2) into (1) and expand up to quadratic order in perturbations, we obtain

$$S^{(2)} = \frac{1}{16\pi}\int d^Dx(\mathcal{L}_a + \mathcal{L}_h + \mathcal{L}_m)\ , \tag{5}$$

where

$$
\begin{aligned}
\mathcal{L}_a &= -\frac{1}{4}\sqrt{-\bar{g}} f_{\mu\nu} f^{\mu\nu} \ , \\
\mathcal{L}_h &= \sqrt{-\bar{g}}\Big[-\frac{1}{4}\nabla_\lambda h_{\mu\nu}\nabla^\lambda h^{\mu\nu} + \frac{1}{2}\nabla_\lambda h_{\mu\nu}\nabla^\nu h^{\mu\lambda} - \frac{1}{2}\nabla_\mu h \nabla_\nu h^{\mu\nu} + \frac{1}{4}\nabla_\mu h \nabla^\mu h \Big] \ , \\
\mathcal{L}_m &= \sqrt{-\bar{g}}\Big[f_{\mu\nu}\bar{F}^{\mu\lambda} h^\nu{}_\lambda - \frac{1}{4}\bar{F}^{\mu\nu}\bar{F}^{\rho\sigma} h_{\mu\rho} h_{\nu\sigma} - \frac{1}{4} f_{\mu\nu}\bar{F}^{\mu\nu} h - \frac{\bar{F}^2}{8(D-2)} h_{\mu\nu} h^{\mu\nu} \\
&\qquad + \frac{\bar{F}^2}{16(D-2)} h^2 \Big] \ .
\end{aligned} \tag{6}
$$

It is important to note that the expression for $\mathcal{L}_h$ differs from the given in [40,51] by an overall factor of 2. While this factor is not significant in the case of pure gravity, it becomes crucial in our case, when the graviton-photon mixing occurs.

While (5) holds for any perturbations about any background satisfying (1), here we choose the background to be the D-dimensions RN black hole

$$
\begin{aligned}
d\bar{s}_D^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{D-2}^2 \,, \qquad f = 1 - \frac{2\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}} \ , \\
\bar{A}_{(1)} &= \psi dt \,, \qquad \psi = \sqrt{\frac{2(D-2)}{D-3}} \frac{q}{r^{D-3}} \ .
\end{aligned} \tag{7}
$$

Following the procedure of [40], we expand the perturbations in terms of spherical harmonics on S^{D-2} and impose gauge conditions to remove the unphysical redundancies.

Specifically, we denote the line element on the $(D-2)$-dimensional sphere

$$
d\Omega_{D-2}^2 = \Omega_{AB} d\theta^A d\theta^B \ , \tag{8}
$$

where Ω_{AB} is the metric and the capital Latin letters $A, B, \cdots = 1, 2, \cdots D-2$ denote tensor indices on the sphere. The corresponding covariant derivative, D_A, satisfies the condition $D_A \Omega_{BC} = 0$. We the arrange different components of the perturbations $h_{\mu\nu}, a_\mu$ into three distinct categories according to their transformation properties under $SO(D-1)$

1. $SO(D-1)$ scalars: a_t, a_r, h_{tt}, h_{tr}, h_{rr}
2. $SO(D-1)$ vectors: a_A, h_{tA}, h_{rA}
3. $SO(D-1)$ tensors: h_{AB}

where the μ, ν indices have been divided into t, r, A. Thus, we need scalar, vector and tensor spherical harmonics in the harmonic expansion. Following [40], for notational simplicity, we omit the summation symbols over ℓ, m, and write the expansion as

$$
\begin{aligned}
a_t(t, r, \theta^A) &= a_t(t, r) Y_{\ell m} \ , \\
a_r(t, r, \theta^A) &= a_r(t, r) Y_{\ell m} \ ,
\end{aligned}
$$

$$
\begin{aligned}
a_A(t,r,\theta^A) &= a_L(t,r)D_AY_{\ell m} + a_T(t,r)Y^{(T)}_{A,\ell m}\ ,\\
h_{tt}(t,r,\theta^A) &= fH_0(t,r)Y_{\ell m}\ ,\\
h_{tr}(t,r,\theta^A) &= H_1(t,r)Y_{\ell m}\ ,\\
h_{rr}(t,r,\theta^A) &= f^{-1}H_2(t,r)Y_{\ell m}\ ,\\
h_{tA}(t,r,\theta^A) &= \mathcal{H}_0(t,r)D_AY_{\ell m} + h_0(t,r)Y^{(T)}_{A,\ell m},\\
h_{rA}(t,r,\theta^A) &= \mathcal{H}_1(t,r)D_AY_{\ell m} + h_1(t,r)Y^{(T)}_{A,\ell m}\ ,\\
h_{AB}(t,r,\theta^A) &= r^2\Big[\mathcal{K}(t,r)\Omega_{AB}Y_{\ell m} + G(t,r)D_{\{A}D_{B\}}Y_{\ell m}\\
&\quad + h_2(t,r)D_{(A}Y^{(T)}_{B),\ell m} + h_T(t,r)Y^{(TT)}_{AB,\ell m}\Big]\ .
\end{aligned}
\tag{9}
$$

Here $Y_{\ell m}$ is scalar harmonic function on S^{D-2}, $Y^{(T)}_{A,\ell m}$ denotes the transverse vector harmonics obeying $D^AY^{(T)}_{A,\ell m}=0$, and fianlly $Y^{(TT)}_{AB,\ell m}$ represents the symmetric transverse, traceless rank-2 tensor harmonics satisfying $Y^{(TT)}_{AB,\ell m}=Y^{(TT)}_{BA,\ell m}$, $D^AY^{(TT)}_{AB,\ell m}=0$, $\Omega^{AB}Y^{(TT)}_{AB,\ell m}=0$. The notation $\{AB\}$ means "symmetric and traceless".

By employing diffeomorphism and U(1) gauge transformations, we make the gauge choice

$$
a_L=0,\quad h_2=0,\quad \mathcal{H}_0=0,\quad \mathcal{K}=0,\quad G=0\ . \tag{10}
$$

When the background metric is given by RN black hole (7), the volume element factorizes

$$
\int d^Dx\sqrt{-\bar{g}} = \int dtdr\sqrt{-g_2}\int d^{D-2}\theta\sqrt{\det\Omega_{AB}},\quad \sqrt{-g_2}=r^{D-2}\ . \tag{11}
$$

After substituting the mode decomposition (9) into the quadratic action for perturbations (5), we perform the integration over S^{D-2} and obtain an effective 2-dimensional action

$$
S^{(2)} = \frac{1}{16\pi}\int dtdr(\widetilde{\mathcal{L}}_a + \widetilde{\mathcal{L}}_h + \widetilde{\mathcal{L}}_m)\ , \tag{12}
$$

where we have used the orthonormality conditions satisfied by various spherical harmonics.

Note that the scalar and vector spherical harmonics are both present in the expansion of the U(1) gauge field. Due to their mutual orthogonality, the corresponding perturbations decouple from each other, resulting in the effective action $\widetilde{\mathcal{L}}_a$ of the form

$$
\begin{aligned}
\widetilde{\mathcal{L}}_a &= \widetilde{\mathcal{L}}_a^{(s=1)} + \widetilde{\mathcal{L}}_a^{(s=0)}\ ,\\
\widetilde{\mathcal{L}}_a^{(s=1)} &= r^{D-4}\left[\frac{1}{2f}\dot{a}_T^2 - \frac{f}{2}a_T'^2 - \frac{(\ell+1)(\ell+D-4)}{2r^2}a_T^2\right]\ ,\\
\widetilde{\mathcal{L}}_a^{(s=0)} &= r^{D-4}\left[\frac{r^2}{2}(\dot{a}_r - a_t')^2 + \frac{\ell(\ell+D-3)}{2f}a_t^2 - \frac{\ell(\ell+D-3)}{2}fa_r^2\right]\ .
\end{aligned}
\tag{13}
$$

Since $\mathcal{L}_a$ in (6) does not contain any term involving the background electric field, the above results agree with those found in [40] for a U(1) gauge field perturbing around an arbitrary spherically symmetric black hole.

The action $\widetilde{\mathcal{L}}_h$ receives contributions from scalar, vector, and tensor modes, which do not mix with each other. We thus have

$$\widetilde{\mathcal{L}}_h = \widetilde{\mathcal{L}}_h^{(s=2)} + \widetilde{\mathcal{L}}_h^{(s=1)} + \widetilde{\mathcal{L}}_h^{(s=0)} \ , \tag{14}$$

where the effective action in each sector takes the form below, after proper integration by parts is done

$$\begin{aligned}
\widetilde{\mathcal{L}}_h^{(s=2)} &= \frac{r^{D-2}}{2}\left[\frac{1}{2f}\dot{h}_T^2 - \frac{f}{2}h_T'^2 + f\Big(\frac{D-3}{r^2} + \frac{6-2D-\ell(\ell+D-3)}{2fr^2} + \frac{f'}{rf}\Big)h_T^2\right] \ , \\
\widetilde{\mathcal{L}}_h^{(s=1)} &= \frac{r^{D-4}}{2}\Bigg[\Big(\dot{h}_1 + \frac{2}{r}h_0 - h_0'\Big)^2 + \frac{2(D-3)f + 2rf' - (\ell+1)(\ell+D-4)}{r^2}fh_1^2 \\
&\qquad + \frac{(\ell+1)(\ell+D-4) - 2(D-3)f - 2rf'}{r^2 f}h_0^2\Bigg] \ , \\
\widetilde{\mathcal{L}}_h^{(s=0)} &= \frac{\ell(\ell+D-3)r^{D-4}}{2}\Bigg[\dot{\mathcal{H}}_1^2 + \frac{2f}{r^2}\big((D-3)f + rf'\big)\mathcal{H}_1^2 + \frac{(D-2)\big((D-3)f + rf'\big)}{2\ell(\ell+D-3)}H_2^2 \\
&\quad + H_0\Bigg(\Big(f' + \frac{2(D-3)f}{r}\Big)\mathcal{H}_1 + 2f\mathcal{H}_1' - \Big(1 + \frac{(D-2)\big((D-3)f + rf'\big)}{\ell(\ell+D-3)}\Big)H_2 \\
&\quad - \frac{(D-2)rfH_2'}{\ell(\ell+D-3)}\Bigg) - \frac{2(D-3)f + rf'}{r}H_2\mathcal{H}_1 + H_1\Big(H_1 - 2\dot{\mathcal{H}}_1 + \frac{2(D-2)r}{\ell(\ell+D-3)}\dot{H}_2\Big)\Bigg] \\
&\quad + \frac{r^{D-3}}{8}\Big(rf'' + (D-2)f'\Big)\Big((H_0 + H_2)^2 - 4H_1^2\Big) \ .
\end{aligned} \tag{15}$$

It is worth noting that, compared to the result in [40], our expression for $\widetilde{\mathcal{L}}_h^{(s=0)}$ includes an additional term in the last line. This is because [40] used the background field equations to simplify the result. Moreover, for Schwarzschild black hole, this term vanishes. However, for RN black hole, this term does not vanish due to the presence of electric charge.

Finally, we present the explicit forms of $\widetilde{\mathcal{L}}_m$. According to the representations of perturbations under $SO(D-1)$, $\widetilde{\mathcal{L}}_m$ is separated into three pieces

$$\widetilde{\mathcal{L}}_m = \widetilde{\mathcal{L}}_m^{(s=2)} + \widetilde{\mathcal{L}}_m^{(s=1)} + \widetilde{\mathcal{L}}_m^{(s=0)} \ , \tag{16}$$

where

$$\begin{aligned}
\widetilde{\mathcal{L}}_m^{(s=2)} &= \frac{r^{D-2}\psi'^2}{4(D-2)}h_T^2, \\
\widetilde{\mathcal{L}}_m^{(s=1)} &= r^{D-4}\psi'\big(\dot{a}_T h_1 - a_T' h_0\big) + \frac{r^{D-4}\psi'^2}{2(D-2)}\big(fh_1^2 - f^{-1}h_0^2\big), \\
\widetilde{\mathcal{L}}_m^{(s=0)} &= \frac{r^{D-4}\psi'}{2}\Big[r^2(\dot{a}_r - a_t')(H_2 - H_0) - 2\ell(\ell+D-3)a_t\mathcal{H}_1\Big] \\
&+ \frac{r^{D-4}\psi'^2}{8(D-2)}\Big[4\ell(\ell+D-3)f\mathcal{H}_1^2 + r^2\big(H_0^2 + H_2^2 + 4(D-3)H_1^2 - 2(2D-5)H_0H_2\big)\Big] \ .
\end{aligned} \tag{17}$$

3 Love numbers in different sectors

We now proceed to solve for the perturbations in the static limit, from which we read off the Love numbers.

3.1 Tensor Love numbers in diverse dimensions

Let us first focus on the tensor sector which is the simplest one to study. By "tensor", we mean the corresponding perturbations are expanded in terms of rank-2 symmetric traceless harmonics on S^{D-2}. Compared with the tensor sector action for Schwarzschild black hole [40, 51], the tensor mode action obtained here receives contributions from nontrivial background electric field. Specifically, the action takes the form

$$\begin{aligned}\mathcal{L}^{(s=2)} &= \widetilde{\mathcal{L}}_h^{(s=2)} + \widetilde{\mathcal{L}}_m^{(s=2)} \\ &= \frac{r^{D-2}}{2}\left[\frac{1}{2f}\dot{h}_T^2 - \frac{f}{2}h_T'^2 + f\Big(\frac{D-3}{r^2} + \frac{6-2D-\ell(\ell+D-3)}{2fr^2} + \frac{f'}{rf} + \frac{\psi'^2}{2(D-2)f}\Big)h_T^2\right] \end{aligned} \tag{18}$$

where the last term encodes the contribution from the background electric field. To rewrite the action in standard form, we redefine the field Ψ_T

$$\Psi_T(t,r) = \frac{1}{\sqrt{2}} r^{\frac{D-2}{2}} h_T \ , \tag{19}$$

and perform the necessary integration by parts which leads to

$$\begin{aligned}\mathcal{L}^{(s=2)} &= \frac{1}{2f}\dot{\Psi}_T^2 - \frac{f}{2}\Psi_T'^2 - \frac{1}{2}V_T(r)\Psi_T^2 \ , \\ V_T(r) &= \frac{\ell(\ell+D-3)+2(D-3)}{r^2} + \frac{D^2-14D+32}{4r^2}f + \frac{D-6}{2r}f' - \frac{\psi'^2}{D-2} \ .\end{aligned} \tag{20}$$

As demonstrated in [40], the tensor Love number is determined from the transverse-traceless component of $h_{\mu\nu}$ on the S^{D-2} direction. In particular, Ψ_T is related to the gauge-invariant tensor h_{AB}^{TT} by

$$h_{AB}^{TT} = \sqrt{2} r^{\frac{2-D}{2}} \Psi_T Y_{AB,\ell m}^{(TT)} \ . \tag{21}$$

$$\Psi_T(t,r) = \frac{1}{\sqrt{2}} r^{\frac{D-2}{2}} e^{-i\omega t} R_\ell(r) \ , \tag{22}$$

and adopting new variables below

$$x = \frac{r^{D-3} - \tilde{r}_+}{\tilde{r}_+ - \tilde{r}_-}, \quad \tilde{r}_\pm = r_\pm^{D-3} = \mu \pm \sqrt{\mu^2 - q^2}, \quad \tilde{\ell} = \frac{\ell}{D-3} \ , \tag{23}$$

we obtain

$$x(1+x)R_\ell''(x) + (2x+1)R_\ell'(x) - \left[\tilde{\ell}(\tilde{\ell}+1) - \frac{\big(\tilde{r}_+(1+x) - \tilde{r}_- x\big)^{\frac{2(D-2)}{D-3}}}{(D-3)^2(\tilde{r}_+ - \tilde{r}_-)^2 x(1+x)}\omega^2\right] R_\ell(x) = 0 \ . \tag{24}$$

In the static limit $\omega = 0$, the equation above reduces to that of a scalar perturbation in the background of Schwarzschild black hole [4,40]. Similarly, the solution that remains regular on the horizon $x = 0$ is given by

$$R_\ell = {}_2F_1(\tilde{\ell}+1, -\tilde{\ell}; 1; -x) \ . \tag{25}$$

Not surprisingly, this tensor Love numbers take similar form as those of a scalar perturbation in the background of Schwarzschild black hole [40].

To compare with [6], we can also perform a coordinate transformation $x \to z = \left(\frac{r_+}{r}\right)^{D-3}$ on (24) to convert it into the standard Fuchsian form

$$R_\ell''(z) + \frac{P(z)}{z} R_\ell'(z) + \frac{Q(z)}{z^2} R_\ell(z) = 0 \ . \tag{26}$$

The black hole horizon and spatial infinity reside at $z = 1$ and $z = 0$ respectively. The indices at singular point $z = 0$ are $\tilde{\ell}+1$ and $-\tilde{\ell}$. Therefore, we consider the following cases separately.

1. $2\tilde{\ell}+1 \in \mathbb{N}$: In a neighborhood of $z = 0$, the general solution has the form

$$R_\ell(z) = A z^{\tilde{\ell}+1} \Phi_{\text{resp}}(z) + B \left(z^{-\tilde{\ell}} \Phi_{\text{tidal}}(z) + R z^{\tilde{\ell}+1} \Phi_{\text{resp}}(z) \ln z \right) \ . \tag{27}$$

In this way of parameterization, Φ_{resp} and Φ_{tidal} are analytic functions in a neighborhood of $z = 0$. The ratio between A and B is fixed, once demanding regularity of the solution at the horizon. Hence the solution takes the form

$$R_\ell(z) = B z^{\tilde{\ell}+1} \left(k \Phi_{\text{resp}}(z) + z^{-2\tilde{\ell}-1} \Phi_{\text{tidal}}(z) + R \Phi_{\text{resp}}(z) \ln z \right) \ . \tag{28}$$

As discussed in [6], the presence of the logarithmic term implies that the exact value of k is ambiguous. However, the precise value of R can be obtained by substituting $z^{-\tilde{\ell}} \Phi_{\text{tidal}} + R z^{\tilde{\ell}+1} \Phi_{\text{resp}} \ln z$ into (26) and solve the equation order by order in small z expansion [6]. In terms of the radial coordinate z, we have

$$R = \frac{(-1)^{2\tilde{\ell}} \Gamma(\tilde{\ell}+1)^2}{(2\tilde{\ell})!(2\tilde{\ell}+1)!\Gamma(-\tilde{\ell})^2} (1-\sigma)^{2\tilde{\ell}+1}, \quad \sigma := \tilde{r}_-/\tilde{r}_+ \ . \tag{29}$$

When $\tilde{\ell}$ is an integer, the coefficient R vanish. Meanwhile, the solution regular at the horizon consists of only Φ_{tidal} which is a polynomial of degree $< 2\tilde{\ell}+1$. Therefore, when $\tilde{\ell}$ is an integer, tensor Love number always vanishes.

2. $2\tilde{\ell}+1 \notin \mathbb{N}$: In this case, from the asymptotic expansion of the solution (25) at infinity, we can read off the tensor Love number directly,

$$k_T = \frac{\Gamma(-2\tilde{\ell}-1)\Gamma(\tilde{\ell}+1)^2}{\Gamma(2\tilde{\ell}+1)\Gamma(-\tilde{\ell})^2} (1-\sigma)^{2\tilde{\ell}+1} \quad \text{(in terms of } z\text{)} \ , \tag{30}$$

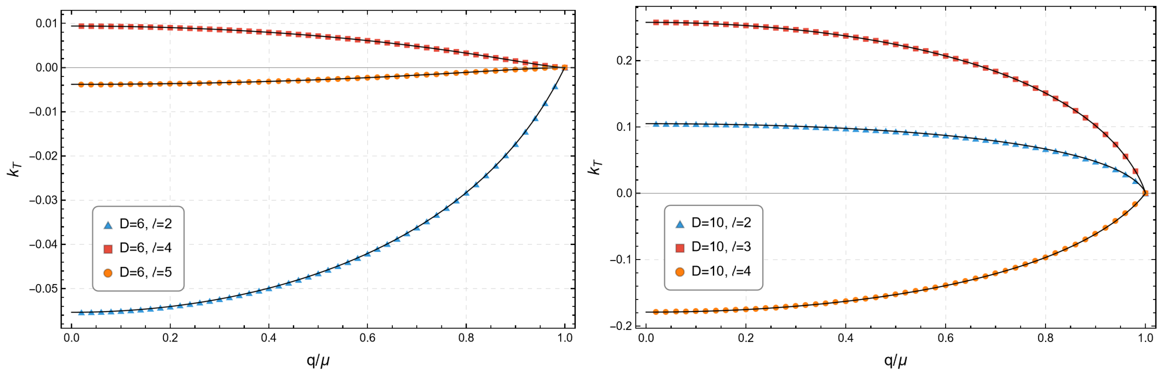

Figure 1: These plots show in $D = 6, 10$, and $2\tilde{\ell}+1 \notin \mathbb{N}$, the dependence of tensor Love numbers k_T on q/μ. Solid lines correspond to analytical values and numerical values are denoted by different markers.

which allows us to test the numerical approach, which will later be applied to the vector and scalar sectors in which analytical expressions of the Love numbers are currently beyond the scope. After imposing regularity at the horizon, the general solution of Eq.(26) takes the form

$$R_\ell(z) = B\left(k_T z^{\tilde{\ell}+1}\Phi_{\rm resp}(z) + z^{-\tilde{\ell}}\Phi_{\rm tidal}(z)\right). \tag{31}$$

The numerical value of k_T is determined by matching the solutions near the horizon ($z = 1$) and at infinity ($z = 0$). The regularity at the horizon serves as the key physical boundary condition. This requirement dictates that only the regular solution branch is selected near the horizon. By performing a series expansion in this region, the branch with $\log(1-z)$ is discarded. The regular solution at $z = 1$ is solved as a series expansion in powers of $(z-1)$ up to order $(z-1)^{15}$. We then evaluate the series and its derivative at $z = 1 - 10^{-8}$ which are used as initial values in the numerical integration of the differential equation towards $z = 0$. The resulting numerical solution is matched with the asymptotic behavior near $z = 0$ (31) to determine the value of k_T. In Fig.1, we plot both the analytical (solid line) and numerical (points) results for the tensor Love numbers in several cases obeying $2\tilde{\ell}+1 \notin \mathbb{N}$. The excellent agreement between the numerical and theoretical values demonstrates the validity of our method.

It is important to note that in $D = 4$, the intrinsic tensor harmonics on S^2 is absent [1]. Consequently, in $D = 4$, the tidal deformations of an RN black hole are characterized by the gravito-electric response and the gravito-magnetic response coefficients, which are associated with the scalar and vector-type perturbations respectively. It is well known that for Schwarzschild black hole, both of these coefficients vanish in $D = 4$ [40]. Later, we will compute gravito-electric response and the gravito-magnetic response coefficients for RN black holes in dimensions $D \geq 4$.

3.2 Effective action of vector type perturbations

Unlike the tensor sector (18), the Regge-Wheeler mode in the gravitational perturbation and the magnetic mode in the Maxwell field are both vector-type perturbations, and thus can mix with each other. As a result, the action in the vector sector is more involved than that in the tensor sector (18). In this section, we will show how to diagonalize the vector-type perturbations by introducing an auxiliary field.

Since both the gravitational and Maxwell field perturbations contain vector modes, they are coupled to each other. As a result, the effective Lagrangian in this sector is of the form

$$\mathcal{L}^{(s=1)} = \widetilde{\mathcal{L}}_h^{(s=1)} + \widetilde{\mathcal{L}}_a^{(s=1)} + \widetilde{\mathcal{L}}_m^{(s=1)}. \tag{32}$$

After performing the appropriate integration by parts, the action for the vector sector can be written as

$$\begin{aligned}\mathcal{L}^{(s=1)} &= r^{D-4}\Bigg[\frac{1}{2}\Big(\dot{h}_1 + \frac{2}{r}h_0 - h_0' - a_T\psi'\Big)^2 + \Big(\psi'' + \frac{D-2}{r}\psi'\Big)a_T h_0 \\ &\quad + (fh_1^2 - f^{-1}h_0^2)\Big(\frac{(D-3)f + rf'}{r^2} + \frac{\psi'^2}{2(D-2)} - \frac{(\ell+1)(\ell+D-4)}{2r^2}\Big) \\ &\quad + \frac{1}{2f}\dot{a}_T^2 - \frac{f}{2}a_T'^2 - \frac{1}{2}\Big(\frac{(\ell+1)(\ell+D-4)}{r^2} + \psi'^2\Big)a_T^2\Bigg] \,.\end{aligned} \tag{33}$$

To diagonalize all the perturbations, we introduce an auxiliary field $Q_{\rm aux}(t,r)$ as in [40], and reexpress the action (33) as

$$\begin{aligned}\mathcal{L}_{\rm aux}^{(s=1)} &= r^{D-4}\Bigg[Q_{\rm aux}\Big(\dot{h}_1 + \frac{2}{r}h_0 - h_0' - a_T\psi'\Big) - \frac{1}{2}Q_{\rm aux}^2 + \Big(\psi'' + \frac{D-2}{r}\psi'\Big)a_T h_0 \\ &\quad + (fh_1^2 - f^{-1}h_0^2)\Big(\frac{(D-3)f + rf'}{r^2} + \frac{\psi'^2}{2(D-2)} - \frac{(\ell+1)(\ell+D-4)}{2r^2}\Big) \\ &\quad + \frac{1}{2f}\dot{a}_T^2 - \frac{f}{2}a_T'^2 - \frac{1}{2}\Big(\frac{(\ell+1)(\ell+D-4)}{r^2} + \psi'^2\Big)a_T^2\Bigg] \,.\end{aligned} \tag{34}$$

Upon solving for $Q_{\rm aux}$ from its equation of motion and substituting back, the action above (34) recovers the original action (33) without the auxiliary field. On the other hand, one can first solve for h_0 and h_1 in terms of $Q_{\rm aux}$ as

$$h_0 = -\frac{rf}{(\ell-1)(\ell+D-2)}\big[(D-2)Q_{\rm aux} + rQ_{\rm aux}'\big] \,, \quad h_1 = -\frac{r^2}{(\ell-1)(\ell+D-2)f}\dot{Q}_{\rm aux} \,. \tag{35}$$

Here, the expressions above have been simplified using the fact that the background RN black hole solution (7) satisfies

$$f'' + \frac{D-2}{r}f' - \frac{D-3}{D-2}\psi'^2 = 0, \quad \psi'' + \frac{D-2}{r}\psi' = 0 \ . \tag{36}$$

Therefore, the vector mode from the metric has only one independent degree of freedom. Next, we introduce $\Psi_{\rm RW}$ and Ψ_V below

$$\Psi_{\rm RW} = \sqrt{\frac{r^{D-2}}{(\ell-1)(\ell+D-2)}}Q_{\rm aux}, \quad \Psi_V = \sqrt{(\ell-1)(\ell+D-2)r^{D-4}}a_T \ , \tag{37}$$

in terms of which the action (34) can be transformed into

$$\begin{aligned} \mathcal{L}^{(s=1)} &= \frac{1}{2f}\dot{\Psi}_{\rm RW}^2 - \frac{f}{2}\Psi_{\rm RW}'^2 - \frac{1}{2}V_{\rm RW}(r)\Psi_{\rm RW}^2 \\ &\quad + \frac{1}{(\ell-1)(\ell+D-2)}\Big(\frac{1}{2f}\dot{\Psi}_V^2 - \frac{f}{2}\Psi_V'^2 - \frac{1}{2}V_V(r)\Psi_V^2\Big) - \frac{\psi'}{r}\Psi_{\rm RW}\Psi_V \ , \\ V_{\rm RW}(r) &= \frac{(\ell+1)(\ell+D-4)}{r^2} + f\frac{(D-6)(D-4)}{4r^2} - f'\frac{D+2}{2r} - \frac{\psi'^2}{D-2} \ , \\ V_V(r) &= \frac{(\ell+1)(\ell+D-4)}{r^2} + f\frac{(D-6)(D-4)}{4r^2} + f'\frac{D-4}{2r} + \psi'^2 \ . \end{aligned} \tag{38}$$

From the expression of the action, we see that the Regge-Wheeler mode is coupled to the magnetic vector mode due to the nontrivial background electric field. Specific to $D=4$, we find that the equations of motion for $\Psi_{\rm RW}$ and Ψ_V match with those previous obtained in [43][1]

The gauge-invariant vector-type Love numbers are encoded in the magnetic components of the Maxwell field strength and linearized Weyl tensor,

$$F_{AB} = \partial_A a_B - \partial_B a_A, \quad C_{trAB} = 2r^2\partial_r(r^{-2}D_{[A}h_{B]t}) \ . \tag{39}$$

Plugging (9), one obtains [40]

$$F_{AB} = r^{\frac{4-D}{2}}\Psi_V D_{[A}Y^{(T)}_{B],\ell m}, \quad C_{trAB} \xrightarrow{r\to\infty} r^{\frac{2-D}{2}}\Psi_{\rm RW}D_{[A}Y^{(T)}_{B],\ell m} \ , \tag{40}$$

which can be matched to the results derived using WEFT. The matching procedure has been carried out for Schwarzschild black hole in [40], confirming the agreement between Love numbers extracted from the large distance expansions of Ψ_V , $\Psi_{\rm RW}$ and those defined using wilsonian coefficients on the WEFT. The Love numbers derived from Ψ_V and $\Psi_{\rm RW}$ are also known as the magnetic response coefficient and the gravito-magnetic response coefficient.

To diagonalizing (38), we follow the strategy of [43] by first rewriting $\Psi_{\rm RW}$ and Ψ_V as linear combinations of $\mathcal{W}_\pm(t,r)$

$$\Psi_{\rm RW} = \alpha_1\mathcal{W}_+ + \alpha_2\mathcal{W}_-, \quad \Psi_V = \alpha_3\mathcal{W}_+ + \alpha_4\mathcal{W}_- \ . \tag{41}$$

[1]Here, $\Psi_{\rm RW}$ differs from π_g in [43] by an overall constant , namely $\Psi_{\rm RW} = \frac{\pi_g}{\ell(\ell-1)(\ell+1)(\ell+2)}$.

By demanding that $\mathcal{W}_{\pm}(t,r)$ decouple from each other, we can solve for the coefficients α_is which turn out to be constant in this case [2]

$$\begin{aligned}
\alpha_1 &= \Big[1+\frac{c_1^2}{(\ell-1)(\ell+D-2)}\Big]^{-\frac{1}{2}}\,, & \alpha_2 &= \Big[1+\frac{(\ell-1)(\ell+D-2)}{c_1^2}\Big]^{-\frac{1}{2}}\,,\\
\alpha_3 &= c_1\alpha_1\,, & \alpha_4 &= -\frac{(\ell-1)(\ell+D-2)}{c_1}\alpha_2\,,
\end{aligned} \tag{42}$$

where the constant c_1 is given by

$$c_1=\frac{\sqrt{2}\sqrt{D-2}(\ell-1)(\ell+D-2)q}{(D-1)\sqrt{D-3}\mu+\sqrt{(D-3)(D-1)^2\mu^2+2(D-2)(\ell-1)(\ell+D-2)q^2}}\,. \tag{43}$$

In terms of $\mathcal{W}_{\pm}(t,r)$, the Lagrangian (38) becomes

$$\begin{aligned}
&\mathcal{L}^{(s=1)}=\mathcal{L}_{\mathcal{W}+}+\mathcal{L}_{\mathcal{W}-}\,,\\
&\mathcal{L}_{\mathcal{W}+}=\frac{1}{2f}\dot{\mathcal{W}}_+^2-\frac{f}{2}\mathcal{W}_+'^2-\frac{1}{2}V_{\mathcal{W}+}(r)\mathcal{W}_+^2\,,\\
&\mathcal{L}_{\mathcal{W}-}=\frac{1}{2f}\dot{\mathcal{W}}_-^2-\frac{f}{2}\mathcal{W}_-'^2-\frac{1}{2}V_{\mathcal{W}-}(r)\mathcal{W}_-^2\,,\\
&V_{\mathcal{W}+}=\frac{(D-6)(D-4)f+4(\ell+1)(\ell+D-4)}{4r^2}\\
&+\frac{1}{2(D-2)\big(c_1^2+(\ell-1)(\ell+D-2)\big)r}\bigg[(D-2)f'\big(c_1^2(D-4)-(D+2)(\ell-1)(\ell+D-2)\big)\\
&+4c_1(D-2)(\ell-1)(\ell+D-2)\psi'+2r\big(c_1^2(D-2)-(\ell-1)(\ell+D-2)\big)\psi'^2\bigg]\,,\\
&V_{\mathcal{W}-}=\frac{(D-6)(D-4)f+4(\ell+1)(\ell+D-4)}{4r^2}\\
&-\frac{1}{2(D-2)\big(c_1^2+(\ell-1)(\ell+D-2)\big)r}\bigg[(D-2)f'\big[c_1^2(D+2)-(D-4)(\ell-1)(\ell+D-2)\big]\\
&+4c_1(D-2)(\ell-1)(\ell+D-2)\psi'+2r\big(c_1^2-(D-2)(\ell-1)(\ell+D-2)\big)\psi'^2\bigg]\,.
\end{aligned} \tag{44}$$

For Schwarzschild black hole with $q=0$, the combination coefficients in (41) reduce to

$$\alpha_1=1,\quad \alpha_2=\alpha_3=0,\quad \alpha_4=-\sqrt{(\ell-1)(\ell+D-2)}. \tag{45}$$

3.3 Vector-type Love numbers in diverse dimensions

Now we proceed to solve for $\mathcal{W}_{\pm}(t,r)$. Plugging the ansatz below into the equations of motion from (44)

$$\mathcal{W}_{\pm}(t,r)=e^{-i\omega t}r^{\frac{D-2}{2}}R_{\mathcal{W}\pm}(r)\,, \tag{46}$$

[2]In general, they can be functions of the radial coordinate r.

we obtain

$$x(1+x)R''_{\mathcal{W}\pm}+(1+2x)R'_{\mathcal{W}\pm}+\frac{\omega^2 r_+^2\big((1-\sigma)x+1\big)^{\frac{2(D-2)}{D-3}}}{(D-3)^2(1-\sigma)^2x(x+1)}R_{\mathcal{W}\pm}$$
$$+\Big[\frac{(D^2-4D+5)(1+\sigma)\pm c_{\mathcal{W}}}{2(D-3)^2\big((1-\sigma)x+1\big)}-\frac{(D-2)(2D-5)\sigma}{(D-3)^2\big((1-\sigma)x+1\big)^2}-\tilde{\ell}(\tilde{\ell}+1)\Big]R_{\mathcal{W}\pm}=0\ , \quad (47)$$

where $\sigma := \tilde{r}_-/\tilde{r}_+$ (see (23) for the definition of $\tilde{r}_\pm$) and the new constant $c_{\mathcal{W}}$ is defined as

$$c_{\mathcal{W}}=\sqrt{(D-3)^2(D-1)^2(1+\sigma)^2-8(D-2)(D-3)\big(D-2-\tilde{\ell}(\tilde{\ell}+1)(D-3)^2\big)\sigma}\ . \quad (48)$$

In the static case $\omega = 0$, solutions to the equations above can be expressed in terms of the Heun function. After imposing regular boundary condition at the outer horizon $x = 0$, the solution is

$$\begin{aligned}
R_{\mathcal{W}\pm} &= \big((1-\sigma)x+1\big)^{-\frac{D-2}{D-3}}\mathrm{HeunG}\Big[\frac{1}{1-\sigma},q_\pm,\alpha,\beta,1,1,-x\Big]\ ,\\
q_\pm &= -\frac{\mp c_{\mathcal{W}}+2\big((D-3)^2\tilde{\ell}(\tilde{\ell}+1)-D+2\big)+(D-3)(D-1)(1+\sigma)}{2(D-3)^2(1-\sigma)}\ ,\\
\alpha &= -\frac{D-2}{D-3}-\tilde{\ell},\quad \beta=\tilde{\ell}-\frac{1}{D-3}\ .
\end{aligned} \quad (49)$$

Had we known the connection for Heun function, the solution above would then be expressed as a linear combination of two other solutions with definite and distinct fall-offs near infinity, from which we read off the Love numbers. However, due to the lacking of the connection formula for Heun function, one still needs to employ numerical method. We thus adopt the same approach as in the previous section which applies to all Fuchsian type equations.

Through a coordinate transformation $x \to z = \left(\frac{r_+}{r}\right)^{D-3} = 1/((1-\sigma)x+1)$, we can convert (47) into the standard Fuchsian form. For equations of this type, the indices near the singular point $z = 0$ are $-\tilde{\ell}$ and $\tilde{\ell}+1$ and the analysis can be divided into the two cases:

1. $2\tilde{\ell}+1 \in \mathbb{N}$: Similarly, by imposing the regularity condition of the solution at the horizon, we fix the ratio of the two independent solutions, and the solution takes the form near $z = 0$

   $$R_{\mathcal{W}\pm}(z)=B_\pm\left(k_\pm z^{\tilde{\ell}+1}\Psi_{\mathrm{resp}\pm}(z)+z^{-\tilde{\ell}}\Psi_{\mathrm{tidal}\pm}(z)+R_\pm z^{\tilde{\ell}+1}\Psi_{\mathrm{resp}\pm}(z)\ln z\right)\ , \quad (50)$$

   where Φ_{resp} and Φ_{tidal} are analytic functions in a neighborhood of $z = 0$.

   The exact value of $R_\pm$ can be obtained using the method described in Sec.3.1. As an example, when $\tilde{\ell} = 1$, we have

   $$R_\pm = -\frac{1}{96(D-3)^6}\Big[\big(D^2(\sigma+1)-8D(\sigma+1)\mp c_{\mathcal{W}}+13(\sigma+1)\big)$$

$$
\begin{aligned}
&\left(D^4\left(\sigma^2+34\sigma+1\right)-12D^3\left(\sigma^2+30\sigma+1\right)+2D^2\left(25\sigma^2+706\sigma+25\right)\right.\\
&-4D\left(\pm c_{\mathcal{W}}\sigma\pm c_{\mathcal{W}}+23\sigma^2+610\sigma+23\right)-c_{\mathcal{W}}^2\\
&\left.\pm 8c_{\mathcal{W}}(\sigma+1)+65\sigma^2+1570\sigma+65\right)\Big],
\end{aligned} \tag{51}
$$

It can be seen that Ψ_{resp} can be completely absorbed order by order into Ψ_{tidal}, yielding

$$
R_{\mathcal{W}\pm}(z)=B_\pm\left(z^{-\tilde{\ell}}\Psi'_{\text{tidal}\pm}(z)+R_\pm z^{\tilde{\ell}+1}\Psi_{\text{resp}\pm}(z)\ln z\right). \tag{52}
$$

Near spatial infinity $(z\to 0)$, the asymptotic behaviors of Ψ_{RW} and Ψ_V is then

$$
\begin{aligned}
r^{\frac{2-D}{2}}\Psi_{\text{RW}}|_{z\to 0}&=z^{\tilde{\ell}+1}\left(\alpha_1B_+R_++\alpha_2B_-R_-\right)\ln z+z^{-\tilde{\ell}}(\alpha_1B_++\alpha_2B_-)\ , &(53)\\
r^{\frac{2-D}{2}}\Psi_V|_{z\to 0}&=z^{\tilde{\ell}+1}\left(\alpha_3B_+R_++\alpha_4B_-R_-\right)\ln z+z^{-\tilde{\ell}}(\alpha_3B_++\alpha_4B_-)\ . &(54)
\end{aligned}
$$

Therefore, the Love numbers are defined as (coefficient of the logarithmic term over coefficient of the source term):

$$
k_{\text{RW}}=\frac{\alpha_1B_+R_++\alpha_2B_-R_-}{\alpha_1B_++\alpha_2B_-}\ln z\ ,\quad k_V=\frac{\alpha_3B_+R_++\alpha_4B_-R_-}{\alpha_3B_++\alpha_4B_-}\ln z\ , \tag{55}
$$

which clearly exhibits gravito-photon mixing. Thus in general, the Love number depends on the ratio of the amplitudes on the horizon. Following [6], when computing the Love number for a specific mode, we set the tidal field of the other mode to zero, thereby fixing the relative amplitude between the two modes. Thus, when calculating k_{RW} (k_V), we set the coefficient of the $z^{-\tilde{\ell}}$ term in $\Psi_V|_{z\to 0}$ ($\Psi_{\text{RW}}|_{z\to 0}$) to zero, fixing the ratio of the amplitudes to be $B_+/B_-=-\alpha_4/\alpha_3$ ($B_+/B_-=-\alpha_2/\alpha_1$). This allows us to derive k_{RW} and k_V analytically. A special case happens in $D=4$, as one finds $R_\pm$ vanishes. We verify that the tidal field $\Phi_{\text{tidal}\pm}$ is given by polynomials of degree less than $2\tilde{\ell}+1$ and thus $k_{\text{RW}}=k_V=0$. In $D>4$, $R_\pm$ is in general nonvanishing. For $D=5$, we list values of the Love numbers (logarithm omitted) for various ℓ in Fig.2.

2. $2\tilde{\ell}+1\notin\mathbb{N}$: In this case, logarithmic term is absent in (50). Near the boundary at $z=0$, the general solution is given by

$$
R_{\mathcal{W}\pm}(z)=B_\pm\left(k_\pm z^{\tilde{\ell}+1}\Psi_{\text{resp}\pm}(z)+z^{-\tilde{\ell}}\Psi_{\text{tidal}\pm}(z)\right)\ , \tag{56}
$$

where Φ_{resp} and Φ_{tidal} are analytic functions in a neighborhood of $z=0$. Regularity at the horizon constrains the relative normalization between the two solutions. As discussed above, the Love number for one mode is computed by setting the tidal field of the other mode to zero. Thus, we obtain

$$
k_{\text{RW}}=\frac{\alpha_2\alpha_3k_--\alpha_1\alpha_4k_+}{\alpha_2\alpha_3-\alpha_1\alpha_4}\ ,\quad k_V=\frac{\alpha_1\alpha_4k_--\alpha_2\alpha_3k_+}{\alpha_1\alpha_4-\alpha_2\alpha_3}\ . \tag{57}
$$

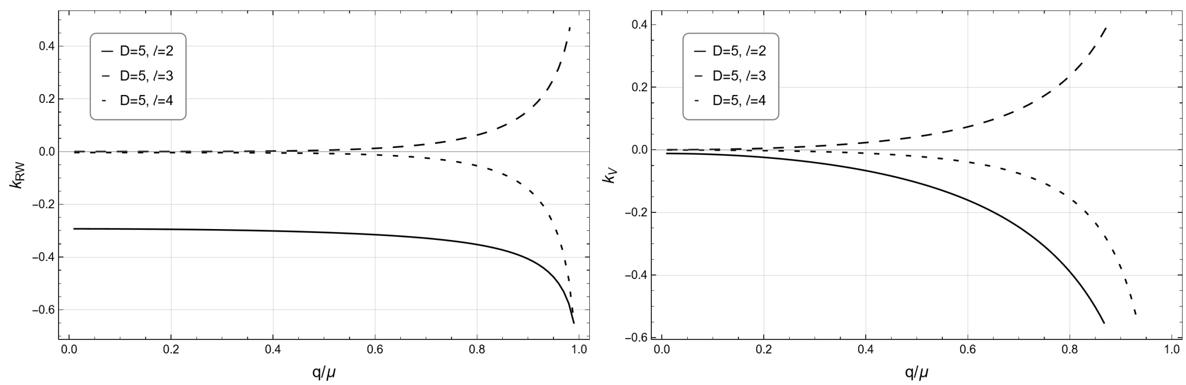

Figure 2: These plots show how vector type Love numbers in $D = 5$ behave as we change q/μ and ℓ. $\ln z$ is omitted in the plots.

For given D and ℓ, the values of $k_{\rm RW}$ and k_V are computed numerically as before. Their dependence on q/μ is shown in Fig.3 for $D = 10, \ell = 2, 3, 4$. Our results are consistent with those given in [6]. When $q = 0$, our results agree with the previously known behavior for Schwarzschild black hole [40]. On the other hand, different from the neutral case, non-zero vector Love numbers are obtained for $D > 5$ even when $\ell = n(D-3) \pm 1$, where $n \in \mathbb{N}^+$ [6]. We show a few examples in this case in Fig.4.

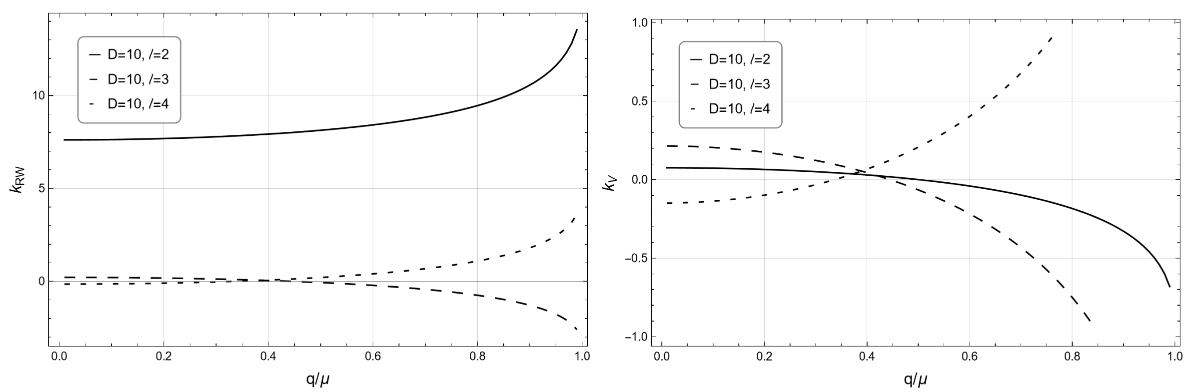

Figure 3: These plots show how vector type Love numbers in $D = 10$ behave as we change q/μ and ℓ.

3.4 Effective action of scalar type perturbations

Analogously to the case of Schwarzschild black hole, the metric perturbation $h_{\mu\nu}$ and the Maxwell field perturbation a_μ both contribute to the scalar-type perturbations. From (13), (15) and (17), we collect the relevant terms in the effective action of scalar perturbations

$$\mathcal{L}^{(s=0)} = \widetilde{\mathcal{L}}_h^{(s=0)} + \widetilde{\mathcal{L}}_a^{(s=0)} + \widetilde{\mathcal{L}}_m^{(s=0)} \ . \tag{58}$$

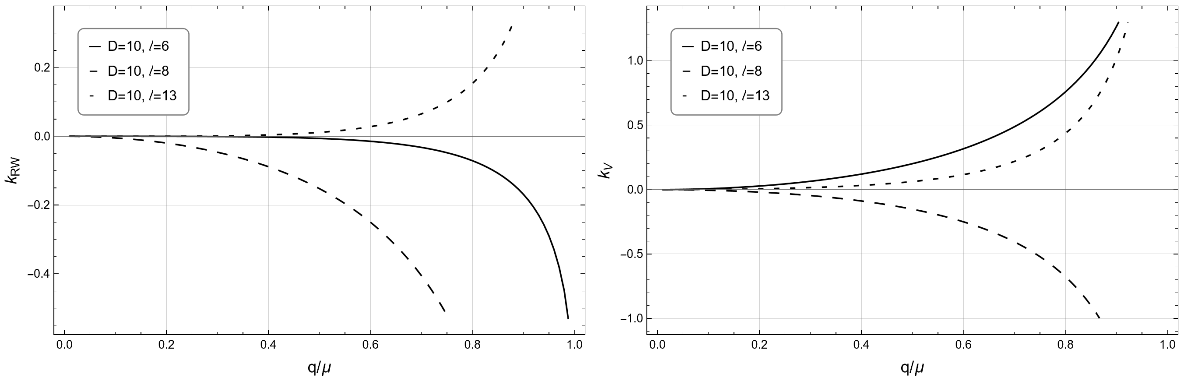

Figure 4: These plots show that vector Love numbers for RN black hole are generically non-zero in $D > 5$ even when $\ell = n(D-3) \pm 1$, where $n \in \mathbb{N}^+$, which is different from Schwarzschild black hole.

In order to eliminate the redundant degrees of freedom, we introduce the auxiliary field $P_{\rm aux}$ and replace part of the action in as follows

$$\left(\dot{a}_r - a_t' + \frac{1}{2}\psi'(H_2 - H_0)\right)^2 \longrightarrow \left(2P_{\rm aux}\left(\dot{a}_r - a_t' + \frac{1}{2}\psi'(H_2 - H_0)\right) - P_{\rm aux}^2\right) . \tag{59}$$

Next, rather than solving for $P_{\rm aux}$, we solve for a_t and a_r using their equations of motion in terms of $P_{\rm aux}$

$$\begin{aligned} a_t &= f\mathcal{H}_1\psi' - \frac{rf}{\ell(\ell+D-3)}\left[(D-2)P_{\rm aux} + rP_{\rm aux}'\right], \\ a_r &= -\frac{r^2}{\ell(\ell+D-3)f}\dot{P}_{\rm aux} \ , \end{aligned} \tag{60}$$

and substitute the expressions above back to (58). Since H_1 obeys an algebraic equation, we can solve it easily

$$H_1 = \dot{\mathcal{H}}_1 - \frac{r(D-2)}{\ell(\ell+D-3)}\dot{H}_2 \ , \tag{61}$$

which will again be plugged in back to the action. We further trade $\mathcal{H}_1$ for another variable $\mathcal{V}$

$$\mathcal{V} = \mathcal{H}_1 - \frac{r(D-2)}{2\ell(\ell+D-3)}H_2 \ , \tag{62}$$

so that we can solve H_2 in terms of $\mathcal{V}$ from H_0 equation of motion

$$\begin{aligned} H_2 = &-\frac{2\ell(\ell+D-3)}{2\ell(\ell+D-3) - 2(D-2)f + (D-2)rf'} \\ &\times\left(\frac{r^2\psi'}{\ell(\ell+D-3)}P_{\rm aux} - f'\mathcal{V} - \frac{2f}{r}\left((D-3)\mathcal{V} + r\mathcal{V}'\right)\right) . \end{aligned} \tag{63}$$

Eventually, the scalar sector action reduces to one involving only two variables, $\mathcal{V}$ and $P_{\rm aux}$, which are coupled together. To transform the action into standard form, we define $\{\Psi_{\rm Z}, \Psi_S\}$

$$\begin{aligned}
\Psi_S &= \frac{r^{\frac{D}{2}}}{\sqrt{\ell(\ell+D-3)}} P_{\rm aux} \ , \\
\Psi_{\rm Z} &= \frac{2\sqrt{\ell(D-2)(\ell+D-3)\mathcal{F}}\, r^{\frac{D-4}{2}} f}{\mathcal{G}} \mathcal{V} \ ,
\end{aligned} \tag{64}$$

where the function $\mathcal{F}$ and $\mathcal{G}$ are

$$\begin{aligned}
\mathcal{F} &= -2(D-2)rf' - 2(D-3)(D-2)f + (D-3)\big(2\ell(\ell+D-3) + r^2\psi'^2\big) \ , \\
\mathcal{G} &= (D-2)rf' - 2(D-2)f + 2\ell(\ell+D-3)
\end{aligned} \tag{65}$$

Now the scalar sector action becomes

$$\begin{aligned}
\mathcal{L}^{(s=0)} &= \mathcal{L}_{\rm Z} + \mathcal{L}_S + \mathcal{L}_{\rm mix} \ , \\
\mathcal{L}_{\rm Z} &= \frac{1}{2f}\dot{\Psi}_{\rm Z}^2 - \frac{f}{2}\Psi_{\rm Z}'^2 - \frac{1}{2}V_{\rm Z}(r)\Psi_{\rm Z}^2 \\
\mathcal{L}_S &= \frac{1}{2f}\dot{\Psi}_S^2 - \frac{f}{2}\Psi_S'^2 - \frac{1}{2}V_S(r)\Psi_S^2 \ , \\
\mathcal{L}_m &= -\frac{2\sqrt{D-2}\, r\psi'}{\sqrt{\mathcal{F}}}\Big(\frac{1}{2f}\dot{\Psi}_S\dot{\Psi}_{\rm Z} - \frac{f}{2}\Psi_S'\Psi_{\rm Z}' - \frac{1}{2}V_{m1}(r)\Psi_S\Psi_{\rm Z}' - \frac{1}{2}V_{m2}(r)\Psi_S\Psi_{\rm Z}\Big).
\end{aligned} \tag{66}$$

The explicit forms of $V_{\rm Z}, V_S, V_{m1}, V_{m2}$ are complicated and we postpone them to the appendix (82). In the vector sector (38), the two modes entangle with each other via a mixed mass term. In contrast, here the mixing terms contain derivatives of fields, making the decoupling process more complicated.

The gauge-invariant scalar type Love numbers are determined from the C_{trtr} component of Weyl tensor and the temporal component of the Maxwell field A_t. We find that they are related to $\Psi_{\rm Z}$ and Ψ_S via [40]

$$A_t = \frac{D+2\ell-4}{2\sqrt{\ell(\ell+D-3)}} r^{\frac{2-D}{2}}\Psi_S Y_{\ell m}, \quad C_{trtr} \xrightarrow{r\to\infty} 2(D-3)\ell(\ell+D-3) r^{\frac{D}{2}-5}\Psi_{\rm Z} Y_{\ell m} \ . \tag{67}$$

The Love numbers obtained from Ψ_S and $\Psi_{\rm Z}$ are commonly referred to as the electric response coefficient and the gravito-electric response coefficient.

We now move on to diagonalize the scalar type perturbations. We make the ansatz

$$\Psi_S = a_1(r)\mathcal{U}_+ + a_2(r)\mathcal{U}_-, \quad \Psi_{\rm Z} = a_3(r)\mathcal{U}_+ + a_4(r)\mathcal{U}_- \ , \tag{68}$$

and try to find $a_1, \cdots a_4$ so that there is no mixing between $\mathcal{U}_\pm$. Different from the vector sector (41), here $a_1, \cdots a_4$ are functions of r rather than constants. Their explicit forms are

$$\begin{aligned}
a_1 &= -\frac{c_2(D-2)q + (\ell-1)(\ell+D-2)r^{D-3}}{r^{D-3}\sqrt{(\ell-1)(\ell+D-2)}\sqrt{c_2^2+(\ell-1)(\ell+D-2)}} \ , \\
a_3 &= \frac{c_2\sqrt{\mathcal{F}}}{\sqrt{2(D-3)}\sqrt{(\ell-1)(\ell+D-2)}\sqrt{c_2^2+(\ell-1)(\ell+D-2)}} \ , \\
a_2 &= \frac{c_2 r^{D-3} + (2-D)q}{r^{D-3}\sqrt{c_2^2+(\ell-1)(\ell+D-2)}}, \quad a_4 = \frac{\sqrt{\mathcal{F}}}{\sqrt{2(D-3)}\sqrt{c_2^2+(\ell-1)(\ell+D-2)}} \ ,
\end{aligned} \tag{69}$$

where the constant c_2 is

$$c_2 = -\frac{2(\ell-1)(\ell+D-2)q}{(D-1)\mu + \sqrt{(D-1)^2\mu^2 + 4(\ell-1)(\ell+D-2)q^2}} \ . \tag{70}$$

and then we decouple the action (66) successfully

$$\begin{aligned}
\mathcal{L}^{(s=0)} &= \mathcal{L}_{\mathcal{U}+} + \mathcal{L}_{\mathcal{U}-} \ , \\
\mathcal{L}_{\mathcal{U}+} &= \frac{1}{2f}\dot{\mathcal{U}}_+^2 - \frac{f}{2}\mathcal{U}_+'^2 - \frac{1}{2}V_{\mathcal{U}+}(r)\mathcal{U}_+^2 \ , \\
\mathcal{L}_{\mathcal{U}-} &= \frac{1}{2f}\dot{\mathcal{U}}_-^2 - \frac{f}{2}\mathcal{U}_-'^2 - \frac{1}{2}V_{\mathcal{U}-}(r)\mathcal{U}_-^2 \ .
\end{aligned} \tag{71}$$

where the potentials in equations above are postponed to the appendix (83) and (83) . For $D = 4$, we find that the equations of motion for $\mathcal{U}_\pm$ agree with those in [44] where the corresponding functions are denoted by $R_\pm$. For Schwarzschild black hole ($q = 0$) the coefficients in (69) reduce to

$$a_1 = -1, \quad a_2 = a_3 = 0, \quad a_4 = 1 \ . \tag{72}$$

3.5 Scalar-type Love numbers in diverse dimensions

We now proceed to solve the decoupled scalar modes (71). After applying separation of variables

$$\mathcal{U}_\pm(t,r) = e^{-i\omega t} r^{\frac{D-2}{2}} R_{\mathcal{U}\pm}(r) \ , \tag{73}$$

and taking the static limit $\omega \to 0$, the equations for the radial functions become

$$R_{\mathcal{U}\pm}'' + \frac{((D-2)f + rf')}{rf}R_{\mathcal{U}\pm}' + \frac{(D^2-6D+8)f + 2(D-2)rf' - 4r^2V_{\mathcal{U}\pm}}{4r^2 f}R_{\mathcal{U}\pm} = 0 \ . \tag{74}$$

Again, in terms of the variable $z = \left(\frac{r_+}{r}\right)^{D-3}$, near spatial infinity at $z = 0$, the solutions are characterized by exponents $-\tilde{\ell}$ and $\tilde{\ell}+1$. Similar to the vector case, we consider two subcases:

1. $2\tilde{\ell} + 1 \in \mathbb{N}$: The general solution in this case is

$$R_{\mathcal{U}\pm}(z) = B_\pm\left(k_\pm z^{\tilde{\ell}+1}\Psi_{\text{resp}\pm} + z^{-\tilde{\ell}}\Psi_{\text{tidal}\pm} + R_\pm z^{\tilde{\ell}+1}\Psi_{\text{resp}\pm}\ln z\right) . \tag{75}$$

The unambiguous Love number is usually defined using coefficients $R_\pm$ (when it is non-vanishing) which can be computed analytically by substituting the expansion into the equations. Accordingly, the asymptotic behaviors of Ψ_S and $\Psi_{\rm Z}$ near the boundary are given by

$$\Psi_S|_{z\to 0} = (a_1 B_+ + a_2 B_-) z^{-\tilde{\ell}} + (a_1 B_+ R_+ + a_2 B_- R_-) z^{\tilde{\ell}+1} \ln z \ , \tag{76}$$

$$\Psi_{\rm Z}|_{z\to 0} = (a_3 B_+ + a_4 B_-) z^{-\tilde{\ell}} + (a_3 B_+ R_+ + a_4 B_- R_-) z^{\tilde{\ell}+1} \ln z \ . \tag{77}$$

We then define the Love number for one mode by turning off the tidal field of the other mode and obtain

$$k_S = \frac{a_1 \lambda_S R_+ + a_2 R_-}{a_1 \lambda_S + a_2} \ln z = \frac{c_2^2 R_- + (\ell-1)(D+\ell-2) R_+}{c_2^2 + (\ell-1)(D+\ell-2)} \ln z \ , \tag{78}$$

$$k_{\rm Z} = \frac{a_3 \lambda_Z R_+ + a_4 R_-}{a_3 \lambda_Z + a_4} \ln z = \frac{c_2^2 R_+ + (\ell-1)(D+\ell-2) R_-}{c_2^2 + (\ell-1)(D+\ell-2)} \ln z \ . \tag{79}$$

Here, $\lambda_S = -a_4/a_3$ and $\lambda_{\rm Z} = -a_2/a_1$ are the relative amplitudes determined by setting the one of the source term to zero, and c_2 is defined in (70).

We find that for $\tilde{\ell} \in \mathbb{N}$, $R_\pm$ vanishes. However, unlike the tensor and vector cases, the scalar tidal field $\Phi_{\text{tidal}_\pm}$ becomes an infinite polynomial rather than a polynomial of degree $< 2\tilde{\ell}+1$. To isolate the tidal and possible response terms, we shift $\tilde{\ell}$ away from integer by a small amount ϵ. Via numerical calculations, we find that as ϵ approaches zero, $k_\pm$ in front of the response term also tends to zero. We exhibit the results in Fig.5 for $D = 4, 5, 6$. In particular, we confirm that in $D = 4$, the solution reduces to pure tidal field term and the scalar type Love numbers vanish [7]. For $\tilde{\ell} \in \frac{1}{2} + \mathbb{N}$, $R_\pm$ takes non-zero real value and the Love numbers exhibit logarithmic running behaviors. We show the results in Fig. 6 for the the case of $D = 5$ and $\ell = 3, 5, 7$. When switching off the electric charge, our results agree with known results for Schwarzschild black hole [40].

2. $2\tilde{\ell} + 1 \notin \mathbb{N}$: In this case, the two branches are linearly independent, therefore $R_\pm = 0$. The general solution is of the form

$$R_{\mathcal{U}\pm}(z) = B_\pm \left(k_\pm z^{\tilde{\ell}+1} \Psi_{\text{resp}\pm} + z^{-\tilde{\ell}} \Psi_{\text{tidal}\pm} \right) . \tag{80}$$

The love numbers are defined as

$$k_S = \frac{c_2^2 k_- + (\ell-1)(D+\ell-2) k_+}{c_2^2 + (\ell-1)(D+\ell-2)}, \quad k_{\rm Z} = \frac{c_2^2 k_+ + (\ell-1)(D+\ell-2) k_-}{c_2^2 + (\ell-1)(D+\ell-2)} \ . \tag{81}$$

The values of k_S and $k_{\rm Z}$ are computed via numerical method as before for given D and ℓ. Examples corresponding to $D = 10$ and $\ell = 2, 3, 4$ are shown in Fig.7. In the neutral limit $q \to 0$, the numerical results agree with the one in [40].

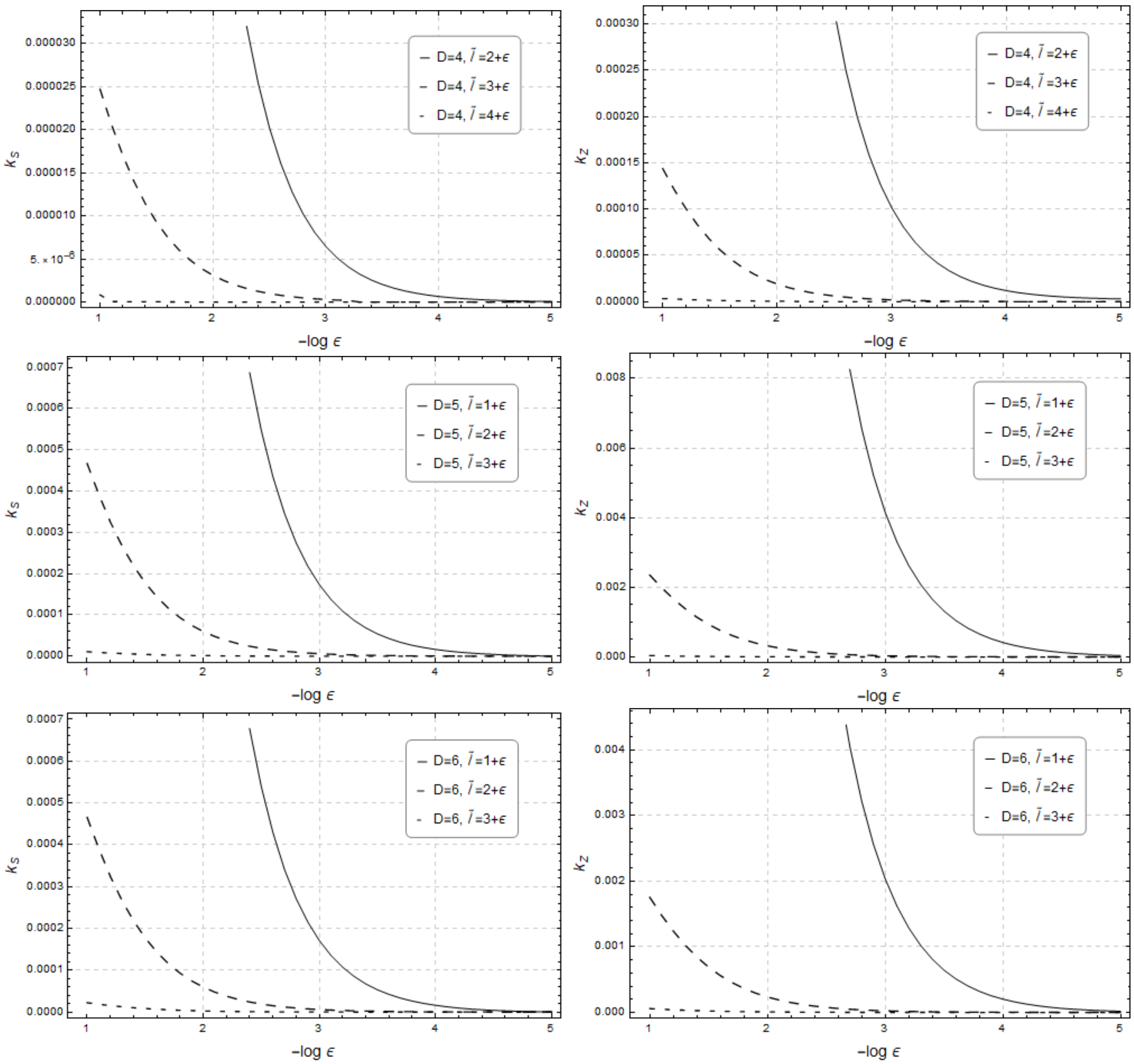

Figure 5: In these plots, we show that when $\tilde{\ell} \in \mathbb{N}, q/\mu = 0.1$, the scalar type Love numbers of RN black hole vanish. Examples above correspond to $D = 4, 5, 6$ and ϵ ranges from 10^{-1} to 10^{-5}.

4 Conclusions

In this paper, we computed the tidal Love numbers of D dimensional RN black holes in all perturbation channels. We obtained the effective action for the perturbations and successfully diagonalized them to arrive at a set of master equations. We showed that tensor perturbations are governed by hypergeometric functions and the vector perturbations satisfy Heun type equations in general. The scalar perturbations obey more complicated equations which might be reduced to simpler equations using the trick employed in $D = 4$, which deserves further study in the future.

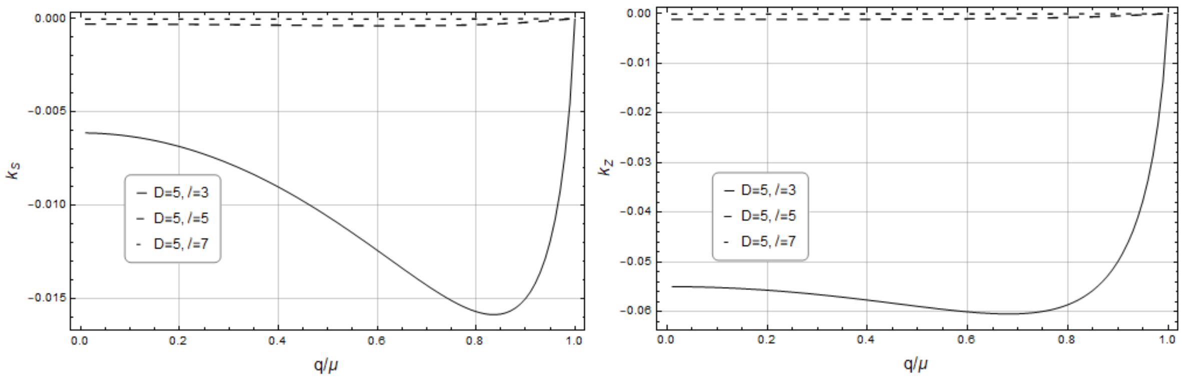

Figure 6: Scalar Love numbers for RN black hole in $D = 5$ with $\ell = 3, 5, 7$, where $\ln z$ is omitted.

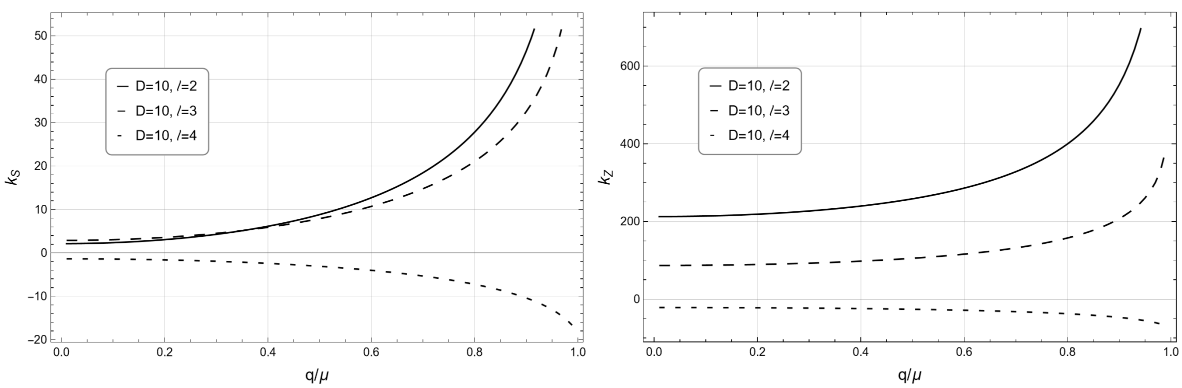

Figure 7: Scalar Love numbers for $D = 10$ RN black holes with $\ell = 2, 3, 4$.

From the master equations, we extracted Love numbers in various sector. In the tensor and vector sectors, we recovered known results in the literature. In the scalar sector, we observed that the Love numbers vanish when the effective multipolar index $\tilde{\ell} \in \mathbb{N}$, analogous to the Schawrzschild black hole. It is possible that the vanishing of tidal Love numbers in higher dimensions can be associated with certain accidental symmetry. As expected, when $\tilde{\ell} \in \mathbb{N} + \frac{1}{2}$, the scalar Love numbers exhibit logarithmic running behaviors and when $\tilde{\ell} \notin \mathbb{N}$, the scalar Love numbers are finite real values.

In this work, we focused on the purely electric RN black hole. It is conceivable that via electromagnetic duality, our results can be generalized to dyonically charged black hole. We would like to consider spherically symmetric black hole in Einstein-Maxwell-dilaton system, some of which do admit analytically solutions. Finally, it should be interesting to consider nonlinear tidal Love numbers for charged black holes as was done for the neutral ones [8].

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A Explicit forms of the potentials in the scalar sector

Here, we present the potential in (66)

$$
\begin{aligned}
V_S =\ & \frac{D\big((D-2)f+2rf'\big)}{4r^2} + \frac{1}{\mathcal{G}^2 r^2}\bigg[-4(D-2)^3 f^3 - (D-2)^3 r^3 f'^3 \\
& +4\ell^2(\ell+D-3)^2\big(\ell(\ell+D-3)+r^2\psi'^2\big) - (D-2)^2 r^2 f'^2\big(3\ell(\ell+D-3)+r^2\psi'^2\big) \\
& +4(D-2)^2 f^2\big(3\ell(\ell+D-3)-2(D-3)r^2\psi'^2\big) + (D-2)f\Big(2r^2\psi'^2\big((D-4)(D-2)rf' \\
& +2(2D-7)\ell(\ell+D-3)\big) + 3(D-2)^2 r^2 f'^2 + 2(D-3)r^4\psi'^4 - 12\ell^2(\ell+D-3)^2\Big)\bigg]\,, \\
V_{\rm Z} =\ & \frac{1}{4(D-2)\mathcal{G}^2 r^2 \mathcal{F}^2}\bigg[8(D-2)f\mathcal{F}^4 - 8(D-2)\mathcal{G}^4\big(2(D-2)^2 f+\mathcal{F}\big) + 8\mathcal{G}\mathcal{F}^2\Big(\\
& 4(D-1)(D-2)^3 f^2 + (D-2)^2 f\big(\mathcal{F}-4(D-1)\ell(D+\ell-3)\big) + 2\mathcal{F}\ell(D+\ell-3)\Big) \\
& +2(D-2)\mathcal{G}^3\Big(8(D-2)f\big(2(D-2)(D-1)\ell(D+\ell-3)-D\mathcal{F}\big) - 16(D-1)(D-2)^3 f^2 \\
& +\mathcal{F}\big(4(D+3)\ell(D+\ell-3)-\mathcal{F}\big)\Big) + \mathcal{G}^2\Big(8(D-2)^3 f^2\big(4(D-2)(D-1)^2\ell(D+\ell-3) \\
& -(D^2-1)\mathcal{F}\big) - 16(D-1)^2(D-2)^5 f^3 - (D-2)^2 f\big(-8(D-1)(D+5)\mathcal{F}\ell(D+\ell-3) \\
& -(D+20)\mathcal{F}^2 + 16(D-2)(D-1)^2\ell^2(D+\ell-3)^2\big) + 4\mathcal{F}\big(2(D-1)\mathcal{F}\ell(D+\ell-3) \\
& -8(D-2)(D-1)\ell^2(D+\ell-3)^2-\mathcal{F}^2\big)\Big)\bigg]\,, \\
V_{m1} =\ & -\frac{(D-3)f}{r\mathcal{F}}\Big(2(D-2)rf' + 2(D-3)\big((D-2)f-\ell(D+\ell-3)\big) + r^2\psi'^2\Big)\,, \\
V_{m2} =\ & \frac{1}{4(D-2)\mathcal{G}^2 r^2 \mathcal{F}^2}\bigg[8(D-2)f\mathcal{F}^4 - 8(D-2)\mathcal{G}^4\big(6(D-2)^2 f+\mathcal{F}\big) \\
& +8\mathcal{G}\mathcal{F}^2\Big(2(D-1)(D-2)^3 f^2 + (D-2)f\big((D-3)\mathcal{F}-2(D-2)(D-1)\ell(D+\ell-3)\big) \\
& +2\mathcal{F}\ell(D+\ell-3)\Big) - 2(D-2)\mathcal{G}^3\Big(48(D-1)(D-2)^3 f^2 - \mathcal{F}\big(4(D+1)\ell(D+\ell-3) \\
& -\mathcal{F}\big) - 12(D-2)f\big(4(D-2)(D-1)\ell(D+\ell-3)-D\mathcal{F}\big)\Big) + \mathcal{G}^2\Big((D-2)^2 f\big(D\mathcal{F}^2 \\
& +8(D-1)(2D+3)\mathcal{F}\ell(D+\ell-3) - 48(D-2)(D-1)^2\ell^2(D+\ell-3)^2\big) + 8(D-1)(D \\
& -2)^3 f^2\big(12(D-2)(D-1)\ell(D+\ell-3)-(2D+1)\mathcal{F}\big) - 48(D-1)^2(D-2)^5 f^3
\end{aligned}
$$

$$+4\mathcal{F}\Big(2(D-2)\mathcal{F}\ell(D+\ell-3)-4(D-2)(D-1)\ell^2(D+\ell-3)^2-\mathcal{F}^2\Big)\Big)\Bigg]. \tag{82}$$

After diagonalizing the perturbations in scalar sector, the potential terms in the decoupled action (71) are given by

$$\begin{aligned}
V_{\mathcal{U}+} = & \frac{\sqrt{D-2}\left(2ra_3\mathcal{F}a_1'\psi'+a_1\left(a_3\left(2\mathcal{F}\left(r\psi''+\psi'\right)-r\psi'\mathcal{F}'\right)-2r\mathcal{F}a_3'\psi'\right)\right)}{2\mathcal{F}^{3/2}}V_{m1} \\
& +\frac{\sqrt{D-2}ra_1a_3\psi'}{\sqrt{\mathcal{F}}}V_{m1}'-\frac{2\sqrt{D-2}ra_1a_3\psi'}{\sqrt{\mathcal{F}}}V_{m2}+a_1^2V_S+a_3^2V_Z \\
& +\frac{1}{2\mathcal{F}^{3/2}}\Bigg[a_3\Big(2\sqrt{D-2}\mathcal{F}\left(rfa_1'\psi''+\psi'\left(rfa_1''+a_1'\left(rf'+f\right)\right)\right) \\
& -\sqrt{D-2}rfa_1'\psi'\mathcal{F}'-2\mathcal{F}^{3/2}\left(fa_3''+a_3'f'\right)\Big)+a_1\Big(-2\mathcal{F}^{3/2}\left(fa_1''+a_1'f'\right) \\
& -\sqrt{D-2}rfa_3'\psi'\mathcal{F}'+2\sqrt{D-2}\mathcal{F}\left(rfa_3'\psi''+\psi'\left(rfa_3''+a_3'\left(rf'+f\right)\right)\right)\Big)\Bigg], \\
V_{\mathcal{U}-} = & \frac{\sqrt{D-2}\left(2ra_4\mathcal{F}a_2'\psi'+a_2\left(a_4\left(2\mathcal{F}\left(r\psi''+\psi'\right)-r\psi'\mathcal{F}'\right)-2r\mathcal{F}a_4'\psi'\right)\right)}{2\mathcal{F}^{3/2}}V_{m1} \\
& +\frac{\sqrt{D-2}ra_2a_4\psi'}{\sqrt{\mathcal{F}}}V_{m1}'-\frac{2\sqrt{D-2}ra_2a_4\psi'}{\sqrt{\mathcal{F}}}V_{m2}+a_2^2V_S+a_4^2V_Z \\
& +\frac{1}{2\mathcal{F}^{3/2}}\Bigg[a_4\Big(2\sqrt{D-2}\mathcal{F}\left(rfa_2'\psi''+\psi'\left(rfa_2''+a_2'\left(rf'+f\right)\right)\right) \\
& -\sqrt{D-2}rfa_2'\psi'\mathcal{F}'-2\mathcal{F}^{3/2}\left(fa_4''+a_4'f'\right)\Big)+a_2\Big(-2\mathcal{F}^{3/2}\left(fa_2''+a_2'f'\right) \\
& -\sqrt{D-2}rfa_4'\psi'\mathcal{F}'+2\sqrt{D-2}\mathcal{F}\left(rfa_4'\psi''+\psi'\left(rfa_4''+a_4'\left(rf'+f\right)\right)\right)\Big)\Bigg].
\end{aligned} \tag{83}$$

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