BOTT MANIFOLDS OF BOTT–SAMELSON TYPE AND ASSEMBLIES OF ORDERED PARTITIONS

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Abstract. A Bott manifold is a smooth projective toric variety having an iterated $\mathbb{C}P^1$-bundle structure. A certain family of Bott manifolds is used to understand the structure of Bott–Samelson varieties (or Bott–Samelson–Demazure–Hansen varieties), which provide desingularizations of Schubert varieties. Indeed, each Bott–Samelson variety is diffeomorphic to a Bott manifold. However, not all Bott manifolds originate from Bott–Samelson varieties. Those that do are specifically referred to as Bott manifolds of *Bott–Samelson type*. In this paper, we provide a characterization of Bott manifolds of Bott–Samelson type by exploring their relationship with combinatorial objects called assemblies of ordered partitions. Using this relationship, we enumerate Bott manifolds of Bott–Samelson type and describe isomorphic Bott manifolds of Bott–Samelson type in terms of assemblies of ordered partitions.

1. Introduction

A *Bott manifold* is a smooth projective toric variety that arises as the total space of an iterated sequence of $\mathbb{C}P^1$-bundles. This iterated sequence of Bott manifolds is called a *Bott tower*, a notion introduced by Grossberg and Karshon [8]. The original motivation of Bott towers stems from their connection to another interesting family of smooth projective varieties, called *Bott–Samelson varieties*[1]. These varieties provide desingularizations of Schubert varieties. Although a Bott–Samelson variety is not toric in general, Grossberg and Karshon [8] showed that it is always diffeomorphic to a Bott manifold. Moreover, Pasquier [13] proved that there exists a toric degeneration in which the generic fiber is a Bott–Samelson variety and the special fiber is the corresponding Bott manifold considered in [13]. Because of this relation, Bott manifolds provide an interesting connection between the geometry, topology, combinatorics, and representation theory (see, for instance, [8, 13, 4]).

We note that not all Bott manifolds arise from Bott–Samelson varieties. Indeed, in each dimension m, there exist infinitely many Bott manifolds, whereas there are only finitely many Bott–Samelson varieties. A Bott manifold coming from a Bott–Samelson variety is called a Bott manifold of *Bott–Samelson type* (or of *BS type*). This family includes, in particular, toric Schubert varieties in the full flag variety (cf. [1, Exercise 18.2.7] and [12, Theorem 4.23]).

In this paper, we investigate Bott manifolds of BS type by associating them with certain combinatorial objects called *assemblies of ordered partitions*. In order to state our main theorem, we first introduce some notation and terminology. See Section 3 for more details.

Let $[n] := \{1, \dots, n\}$. Given a sequence $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$ of integers between 1 and n, we associate to it a Bott manifold $\mathcal{B}_{\boldsymbol{i}}$ of BS type by considering an iterated sequence of $\mathbb{C}P^1$-fibrations, where the fibration structures are determined by the sequence $\boldsymbol{i}$ (see Definition 2.3 for a precise description). Thus, the family of Bott manifolds of BS type can be parameterized by the sequences in $[n]^m$.

An *assembly of ordered partition* of $[m]$ with bound n is a sequence $\sigma = (\sigma_1, \dots, \sigma_\ell)$ of ordered partitions $\sigma_a = (\sigma_a^1, \dots, \sigma_a^{r_a})$ such that $\min(\sigma_1) < \cdots < \min(\sigma_\ell)$, $r_1 + \cdots + r_\ell + (\ell - 1) \leq n$, and

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[1] The construction of the Bott–Samelson varieties is due to Bott and Samelson [2] in the framework of compact Lie groups, and was later developed in the algebro-geometric setting by Hansen [9] and Demazure [7]. For this reason, Bott–Samelson varieties are also referred to as Bott–Samelson–Demazure–Hansen varieties.

$\{\sigma_a^b : a \in [\ell], b \in [r_a]\}$ is a partition of $[m]$ (see Definition 3.2 for more details). Let $\mathsf{AOP}(n,m)$ denote the set of such assemblies of ordered partitions. For an ordered partition $\eta = (\eta^1, \dots, \eta^r)$, we write $\overline{\eta} = (\eta^r, \dots, \eta^1)$. We define the equivalence relation $\sim$ on $\mathsf{AOP}(n,m)$ as follows: for $\sigma = (\sigma_1, \dots, \sigma_\ell), \tau = (\tau_1, \dots, \tau_{\ell'}) \in \mathsf{AOP}(n,m)$, we have $\sigma \sim \tau$ if $\ell = \ell'$ and for each $a \in [\ell]$ either $\tau_a = \sigma_a$ or $\tau_a = \overline{\sigma}_a$. We now state our first main result.

Theorem 1.1. *Let n and m be positive integers. There is a bijection between the set of Bott manifolds of BS type and the set of equivalence classes of $\mathsf{AOP}(n,m)$ under the relation $\sim$:*

$$\{\mathcal{B}_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^m\} \overset{1:1}{\longleftrightarrow} \mathsf{AOP}(n,m)/\sim .$$

To obtain this theorem, we analyze the *Bott matrices* corresponding to Bott manifolds of BS type, which are lower triangular matrices that encode the iterated $\mathbb{C}P^1$-bundle structures. By classifying all such Bott matrices, we enumerate the Bott manifolds of BS type.

Corollary 1.2. *The generating function for the number $b(n,m)$ of Bott manifolds of BS type determined by the sequences in $[n]^m$ is given by*

$$\sum_{n,m\geq 1} \frac{b(n,m)}{m!} x^m y^n = \frac{1}{y(1-y)} \left(\exp\left(\frac{y}{2} \left(\frac{1}{1-y(e^x-1)} + y(e^x-1) - 1 \right) \right) - 1 \right).$$

The set $\mathsf{AOP}(n,m)$ of assemblies of ordered partitions can also be used to distinguish Bott manifolds of BS type up to isomorphisms as toric varieties. Let $\sigma = (\sigma_1, \dots, \sigma_\ell) \in \mathsf{AOP}(n,m)$. For a simple transposition $s_i = (i, i+1) \in \mathfrak{S}_m$, we say that s_i is an *admissible transposition* for σ if i and $i+1$ are not neighbors in σ. Here, c and d are *neighbors* in σ if they belong to the same block of some ordered partition σ_a, or if they appear consecutively within the same block of some ordered partition σ_a (see Definition 3.5 for the precise definition). We define another equivalence relation $\approx$ on $\mathsf{AOP}(n,m)$ as follows: for $\sigma, \tau \in \mathsf{AOP}(n,m)$, we have $\sigma \approx \tau$ if there exist $\tilde{\sigma}, \tilde{\tau} \in \mathsf{AOP}(n,m)$ with $\tilde{\sigma} \sim \sigma$ and $\tilde{\tau} \sim \tau$ such that there is a sequence of admissible transpositions sending $\tilde{\sigma}$ to $\tilde{\tau}$. With this terminology, we state our second main theorem.

Theorem 1.3. *Let n and m be positive integers. There is a bijection between the set of isomorphism classes of Bott manifolds of BS type (as toric varieties) and the set of equivalence classes of $\mathsf{AOP}(n,m)$ under the relation $\approx$:*

$$\{\mathcal{B}_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^m\}/(isom) \overset{1:1}{\longleftrightarrow} \mathsf{AOP}(n,m)/\approx .$$

We say that a Bott manifold is *decomposable* if it is isomorphic to a product of lower-dimensional Bott manifolds, and *indecomposable* otherwise. Using the fact that any indecomposable Bott manifold of BS type corresponds to an assembly consisting of a single ordered partition (see Proposition 5.10), we analyze the isomorphism classes of indecomposable Bott manifolds of BS type in terms of the corresponding sequences (see Theorem 5.13).

We note that sequences with pairwise distinct integers correspond to *toric* Schubert varieties. More precisely, if $\boldsymbol{i} = (i_1, \dots, i_m)$ consists of distinct integers, then the Schubert variety $X_{s_{i_1} \cdots s_{i_m}}$ in the full flag variety is isomorphic to the Bott manifold $\mathcal{B}_{\boldsymbol{i}}$ of BS type. The classification of isomorphism classes of toric Schubert varieties was studied in [12]. Using Theorem 1.3, we give another proof of their result in type A (see Corollary 5.14).

The structure of this paper is as follows. In Section 2, we recall the definition of Bott manifolds and describe their fans, with a focus on the special case of BS type. In Section 3, we introduce assemblies of ordered partitions and consider their connection with Bott manifolds of BS type. We use this correspondence to enumerate Bott manifolds of BS type, proving Theorem 1.1 and Corollary 1.2 in Section 4. In Section 5, we analyze isomorphism classes of Bott manifolds of BS type via admissible transpositions on assemblies, and prove Theorem 1.3.

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2. Bott manifolds of Bott–Samelson type

In this section, we recall Bott manifolds from [8], which are smooth projective toric varieties. Moreover, we consider a special family of Bott manifolds consisting of *Bott manifolds of Bott–Samelson type* (or of *BS type*).

Definition 2.1 ([8, §2.1])**.** A *Bott tower* $\mathcal{B}_\bullet$ is an iterated $\mathbb{C}P^1$-bundle starting with a point:

$$\begin{array}{ccccccccc} \mathcal{B}_m & \xrightarrow{\pi_m} & \mathcal{B}_{m-1} & \xrightarrow{\pi_{m-1}} & \cdots & \xrightarrow{\pi_2} & \mathcal{B}_1 & \xrightarrow{\pi_1} & \mathcal{B}_0, \\ \| & & & & & & \| & & \| \\ P(\underline{\mathbb{C}} \oplus \xi_m) & & & & & & \mathbb{C}P^1 & & \{\text{a point}\} \end{array}$$

where each $\mathcal{B}_j$ is the complex projectivization of the Whitney sum of a holomorphic line bundle ξ_j and the trivial line bundle $\underline{\mathbb{C}}$ over $\mathcal{B}_{j-1}$. The total space $\mathcal{B}_m$ is called a *Bott manifold*.

A Bott manifold $\mathcal{B}_m$ is a smooth projective toric variety by the above construction. Indeed, the induced projective bundles of a sum of holomorphic line bundles over a smooth projective toric variety is again a smooth projective toric variety (cf. [6, §7.3]). By considering the first Chern classes of the holomorphic line bundles ξ_j used to construct $\mathcal{B}_m$, we obtain the ray generators of the fan of $\mathcal{B}_m$. Because of the fibration structure, the Picard group of $\mathcal{B}_{j-1}$ is isomorphic to the free abelian group of rank $j-1$ (cf. [10, Exercise II.7.9]). Indeed, the Picard group of $\mathcal{B}_{j-1}$ is generated by the line bundles $\gamma_{j-1,k}$ for $1 \le k < j$. Here, $\gamma_{j,k}$ is the pullback of the tautological line bundle over $\mathcal{B}_k$ via the composition $\pi_{k+1} \circ \pi_{k+2} \circ \cdots \circ \pi_j \colon \mathcal{B}_j \to \mathcal{B}_k$ for $k < j$. Therefore, there exist integers $b_{j,k} \in \mathbb{Z}$ for $1 \le k < j$ such that

$$\xi_j = \bigotimes_{1 \le k < j} \gamma_{j-1,k}^{\otimes b_{j,k}},$$

where, $\gamma_{j-1,k}^{\otimes(-1)}$ means the dual of $\gamma_{j-1,k}$. Hence, the set $\{b_{j,k}\}_{1 \le k < j \le m}$ of integers determines the Bott manifold $\mathcal{B}_m$.

Let $[B_{j,k}]$ be the $m \times m$ lower triangular matrix given by $B_{j,k} = b_{j,k}$ and $B_{k,j} = 0$ for $1 \le k < j \le m$, and $B_{j,j} = -1$ for $j \in [m]$. We call such a matrix a *Bott matrix*. A Bott manifold determines a Bott matrix and vice versa. Indeed, as explained in [8, §2.3], there is a bijection between the set of Bott manifolds of dimension m and the set of $m \times m$ lower triangular integer matrices with diagonal entries -1.

The fan $\Sigma \subseteq \mathbb{R}^m$ of $\mathcal{B}_m$ can be expressed in terms of the integers $\{b_{j,k}\}_{1 \le k < j \le m}$ that determine $\mathcal{B}_m$. For the Bott manifold $\mathcal{B}_m$ determined by $\{b_{j,k}\}_{1 \le k < j \le m}$, the primitive generators of the rays in the fan $\Sigma \subseteq \mathbb{R}^m$ of $\mathcal{B}_m$ are the column vectors of the following matrix

$$[\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_m\ \mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_m] := \begin{bmatrix} 1 & 0 & \cdots & & 0 & -1 & 0 & \cdots & & 0 \\ 0 & 1 & \ddots & & & b_{2,1} & -1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & & \vdots & b_{3,1} & b_{3,2} & -1 & & \\ & & & \ddots & 0 & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & & \cdots & 0 & 1 & b_{m,1} & b_{m,2} & \dots & b_{m,m-1} & -1 \end{bmatrix}. \tag{2.1}$$

Note that $\mathbf{e}_1, \dots, \mathbf{e}_m$ are the standard basis vectors of $\mathbb{R}^m$, and the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_m]$ is the Bott matrix associated to $\mathcal{B}_m$. For any set S of m ray generators, S forms a maximal cone of Σ if and only if

$$\{\mathbf{e}_i, \mathbf{v}_i\} \not\subseteq S \quad \text{for all } i \in [m].$$

Because of this description, we can see that the fan Σ is the normal fan of a polytope that is combinatorially equivalent to the m-dimensional cube $[0,1]^m$.

Example 2.2. For $m = 2$, the primitive ray generators of the fan Σ of $\mathcal{B}_2$ are the column vectors of the following matrix.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & b_{2,1} & -1 \end{bmatrix} = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{v}_1\ \mathbf{v}_2].$$

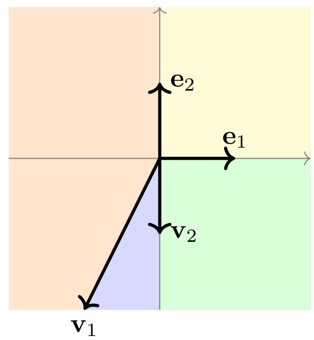

FIGURE 1. The fan of a Bott manifold $\mathcal{B}_2$ for $b_{2,1} = -2$.

There are four maximal cones

$$\mathrm{Cone}(\{\mathbf{e}_1, \mathbf{e}_2\}), \mathrm{Cone}(\{\mathbf{e}_1, \mathbf{v}_2\}), \mathrm{Cone}(\{\mathbf{v}_1, \mathbf{e}_2\}), \mathrm{Cone}(\{\mathbf{v}_1, \mathbf{v}_2\}),$$

and we depict the fan of $\mathcal{B}_2$ when $b_{2,1} = -2$ in Figure 1.

There is a special family of Bott manifolds called Bott manifolds of BS type. Each of these is associated with a sequence of integers.

Definition 2.3. Let $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$. We define the Bott matrix $B_{\boldsymbol{i}} = [b_{j,k}]$ as follows: For $1 \le k < j \le m$, we set

$$b_{j,k} = \begin{cases} -2 & \text{if } |i_j - i_k| = 0, \\ 1 & \text{if } |i_j - i_k| = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad b_{k,j} = 0,$$

and set $b_{j,j} = -1$ for $j \in [m]$. We denote by $\mathcal{B}_{\boldsymbol{i}}$ the Bott manifold determined by the Bott matrix $B_{\boldsymbol{i}}$.

We say that a Bott matrix B is *of BS type* if $B = B_{\boldsymbol{i}}$ for some $\boldsymbol{i}$. We also say that a Bott manifold is *of BS type* if the corresponding Bott matrix is of BS type.

Example 2.4. Let $\boldsymbol{i} = (1, 2, 1, 3)$. Then the Bott matrix $B_{\boldsymbol{i}}$ is

$$B_{\boldsymbol{i}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Definition 2.5. For positive integers n and m, we define $B(n, m)$ to be the set of Bott matrices of BS type that can be obtained from the sequences in $[n]^m$:

$$B(n, m) := \{B_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^m\}.$$

We also define $b(n, m)$ to be the number of elements in $B(n, m)$.

By the definition of $B(n, m)$, we have $B(n_1, m) \subseteq B(n_2, m)$ whenever $n_1 \le n_2$. Note that every element in $B(n, m)$ is an $m \times m$ lower triangular matrix such that the diagonal entries are all -1 and the entries below the main diagonal are -2, 1, or 0. Hence, $b(n, m) \le 3^{(m-1)m/2}$ for all n, and the weakly increasing sequence $(b(n, m))_{n \in \mathbb{Z}_{>0}}$ converges. The following proposition shows that the sequence $(b(n, m))_{n \in \mathbb{Z}_{>0}}$ has a constant value for $n \ge 2m - 1$.

Proposition 2.6. *For any positive integers m and n with $n \ge 2m - 1$, we have $B(n, m) = B(2m - 1, m)$.*

Proof. Let $n \ge 2m - 1$. We claim that $B(n, m) = B(n + 1, m)$. Since we have the inclusion $B(n, m) \subseteq B(n + 1, m)$, it suffices to prove that $B(n + 1, m) \subseteq B(n, m)$.

Suppose $B \in B(n+1, m)$. Then $B = B_{\boldsymbol{i}}$ for some $\boldsymbol{i} = (i_1, \dots, i_m) \in [n+1]^m$. If $n+1$ is not an element of $\boldsymbol{i}$, then $\boldsymbol{i} \in [n]^m$. Hence, $B_{\boldsymbol{i}} \in B(n, m)$. If 1 is not an element of $\boldsymbol{i}$, then $B_{\boldsymbol{i}} = B_{\boldsymbol{i}'}$, where $\boldsymbol{i}' = (i_1 - 1, \dots, i_m - 1) \in [n]^m$. Thus, we also have $B_{\boldsymbol{i}} \in B(n, m)$.

Now suppose that $\boldsymbol{i}$ contains both 1 and $n+1$. Let $(a_1, \dots, a_m)$ be the weakly increasing sequence obtained by sorting the elements in $\boldsymbol{i}$. Since $a_1 = 1$, $a_m = n+1$, and $\sum_{j=1}^{m-1}(a_{j+1} - a_j) = n \geq 2m - 1$, there exists $j \in [m-1]$ such that $a_{j+1} - a_j > 2$. Let $\boldsymbol{i}' = (i'_1, \dots, i'_m) \in [n]^m$ be the sequence given by

$$i'_k = \begin{cases} i_k & \text{if } i_k \leq a_j, \\ i_k - 1 & \text{if } i_k \geq a_{j+1}. \end{cases}$$

By Definition 2.3 and the inequality $a_{j+1} - a_j > 2$, we have $B_{\boldsymbol{i}} = B_{\boldsymbol{i}'}$. Therefore, $B_{\boldsymbol{i}} \in B(n, m)$. Accordingly, the claim $B(n+1, m) \subseteq B(n, m)$ holds and we are done. □

Remark 2.7. Let $G = \mathrm{GL}_{n+1}(\mathbb{C})$ and let B be a Borel subgroup of G. For a sequence $\boldsymbol{i}$, we have another geometric object, called a *Bott–Samelson variety* $Z_{\boldsymbol{i}}$, which is defined by the quotient space

$$Z_{\boldsymbol{i}} := (P_{i_1} \times P_{i_2} \times \cdots \times P_{i_m})/B^m.$$

Here, for $i \in [n]$, P_i is the minimal parabolic subgroup of G containing B, and the action of B^m on the product of minimal parabolic subgroups is defined by

$$(b_1, b_2, \dots, b_m) \cdot (p_1, p_2, \dots, p_m) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{m-1}^{-1} p_m b_m)$$

for $(b_1, b_2, \dots, b_m) \in B^m$ and $(p_1, p_2, \dots, p_m) \in P_{i_1} \times P_{i_2} \times \cdots \times P_{i_m}$. A Bott–Samelson variety is not necessarily toric, but it is diffeomorphic to the Bott manifold $\mathcal{B}_{\boldsymbol{i}}$ as shown in [8]. Moreover, the Bott manifold $\mathcal{B}_{\boldsymbol{i}}$ can be obtained as a toric degeneration of the Bott–Samelson variety $Z_{\boldsymbol{i}}$ (see [13]). This is the reason why we call such Bott manifolds of *Bott–Samelson type*.

Remark 2.8. A family of Bott manifolds of BS type includes *toric Schubert varieties* in a full flag variety of Lie type A. For a simple algebraic group G and a Borel subgroup B of G, the homogeneous space G/B is a smooth projective algebraic variety, called a *full flag variety*. The Schubert varieties are subvarieties of G/B parameterized by the Weyl group W of G. The maximal torus T in B acts on G/B via the left multiplication, and the set of fixed points of this action can be parameterized by the Weyl group W. For $w \in W$, we denote by wB the corresponding fixed point in G/B. The Schubert variety X_w is defined by the closure $\overline{BwB/B}$ of the B-orbit of wB in G/B. Any Schubert variety admits T-action induced by that on G/B and we say that X_w is *toric* if X_w is a toric variety with respect to the action of a quotient of T. For a toric Schubert variety X_w in G/B, suppose that $\boldsymbol{i}$ is a reduced decomposition of w. Then the toric Schubert variety X_w is T-equivariantly isomorphic to the Bott–Samelson variety $Z_{\boldsymbol{i}}$ (see [1, Chapter 18]). Moreover, in this case, X_w is isomorphic to $\mathcal{B}_{\boldsymbol{i}}$.

3. Assemblies of ordered partitions

In this section, we introduce assemblies of ordered partitions. We then find a surjection from the set of assemblies of ordered partitions to the set of Bott matrices of BS type (see Proposition 3.7). This map will be used to enumerate the number $b(n, m)$ of Bott manifolds of BS type (see Proposition 4.3 and Theorem 1.1).

Definition 3.1. An *ordered partition* is a sequence $\tau = (\tau^1, \dots, \tau^r)$ of mutually disjoint nonempty sets. We say that $\{\tau^1, \dots, \tau^r\}$ is the *underlying partition* of τ. We define

$$U(\tau) = \tau^1 \cup \cdots \cup \tau^r, \qquad \min(\tau) = \min(U(\tau)).$$

Note that the underlying partition of an ordered partition τ is a partition of $U(\tau)$.

Definition 3.2. For a positive integer m, an *assembly of ordered partitions* of $[m]$ is a sequence $\sigma = (\sigma_1, \dots, \sigma_\ell)$ of ordered partitions satisfying the following conditions:

(1) $\{U(\sigma_1), \dots, U(\sigma_\ell)\}$ is a partition of $[m]$,
(2) $\min(\sigma_1) < \cdots < \min(\sigma_\ell)$.

Let $\sigma_a = (\sigma_a^1, \dots, \sigma_a^{r_a})$ for $a \in [\ell]$. We say that the assembly $\sigma = (\sigma_1, \dots, \sigma_\ell)$ of ordered partitions has *bound* n if

$$r_1 + \cdots + r_\ell + (\ell - 1) \le n. \tag{3.1}$$

We denote by $\mathsf{AOP}(n, m)$ the set of assemblies of ordered partitions of $[m]$ with bound n.

For example, $((\{2,4,6\},\{1,5\}),(\{3\},\{7,9\}),(\{8\}))$ is an assembly of the ordered partition of [9]. For convenience, we use "|" to distinguish each block in an ordered partition, that is,

$$((\{2,4,6\},\{1,5\}),(\{3\},\{7,9\}),(\{8\})) = ((2\,4\,6 \,|\, 1\,5),(3 \,|\, 7\,9 \,|\, 8)).$$

Note that, for $\sigma = (\sigma_1, \dots, \sigma_\ell) \in \mathsf{AOP}(n, m)$, since $\min(\sigma_1) < \cdots < \min(\sigma_\ell)$, we can identify σ with the set $\{\sigma_1, \dots, \sigma_\ell\}$ of ordered partitions.

Let $\beta : [n]^m \to B(n, m)$ be the map defined by $\beta(\boldsymbol{i}) = B_{\boldsymbol{i}}$, which is surjective by definition. In what follows, we define two maps $\alpha\colon [n]^m \to \mathsf{AOP}(n, m)$ and $F\colon \mathsf{AOP}(n, m) \to B(n, m)$ such that the following diagram commutes.

$$\begin{tikzcd} {[n]^m} \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & \mathsf{AOP}(n,m) \arrow[d, "F"] \\ & B(n,m) \end{tikzcd} \tag{3.2}$$

We define the map $\alpha\colon [n]^m \to \mathsf{AOP}(n, m)$ as follows.

Definition 3.3. Consider a sequence $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$. Let $a_1, \dots, a_t$ be the integers appearing in $\boldsymbol{i}$, where $a_1 < \cdots < a_t$. For each $j \in [t]$, let A_j be the set of indices of the entries equal to a_j's in $\boldsymbol{i}$, that is, $A_j = \{k \in [m] : i_k = a_j\}$. We define the ordered partitions $\sigma_1, \dots, \sigma_\ell$ as follows. First, σ_1 is the ordered partition $(A_1, \dots, A_{r_1})$, where r_1 is the largest integer such that $a_1, \dots, a_{r_1}$ are consecutive integers. Next, σ_2 is the ordered partition $(A_{r_1+1}, \dots, A_{r_1+r_2})$, where r_2 is the largest integer such that $a_{r_1+1}, \dots, a_{r_1+r_2}$ are consecutive integers. Continuing in this way, the ordered partitions $\sigma_1, \dots, \sigma_\ell$ are defined until we have $r_1 + \cdots + r_\ell = t$. Finally, we define $\alpha(\boldsymbol{i})$ to be $\sigma = (\sigma_1, \dots, \sigma_\ell) \in \mathsf{AOP}(n, m)$.

For example, for $\boldsymbol{i} = (6, 3, 8, 3, 6, 2, 9) \in [9]^7$, we have $a_1 = 2$, $a_2 = 3$, $a_3 = 6$, $a_4 = 8$, $a_5 = 9$, and $A_1 = \{6\}$, $A_2 = \{2, 4\}$, $A_3 = \{1, 5\}$, $A_4 = \{3\}$, $A_5 = \{7\}$; hence,

$$\sigma(\boldsymbol{i}) = ((6 \,|\, 2\,4), (1\,5), (3 \,|\, 7)).$$

The next lemma follows immediately from the definition of the map α.

Lemma 3.4. *Let $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$. For any positive integer $k < \min\{i_1, \dots, i_m\}$, we have $\sigma(\boldsymbol{i}) = \sigma(\boldsymbol{i} - k)$, where $\boldsymbol{i} - k = (i_1 - k, i_2 - k, \dots, i_m - k)$.*

Note that, using the notation in the construction of $\alpha(\boldsymbol{i})$ above, the largest integer appearing in $\boldsymbol{i}$ is at least $r_1 + \cdots + r_\ell + (\ell - 1)$. Hence, for $\boldsymbol{i} \in [n]^m$, the inequality (3.1) holds, which implies that $\sigma(\boldsymbol{i}) \in \mathsf{AOP}(n, m)$.

Definition 3.5. Let $\sigma = (\sigma_1, \dots, \sigma_\ell) \in \mathsf{AOP}(n, m)$, where $\sigma_i = (\sigma_i^1 \,|\, \cdots \,|\, \sigma_i^{r_i})$. For integers $1 \le k < j \le m$, we say that k and j are *neighbors* in σ if they are in the same block of an ordered partition σ_a, or if they appear in consecutive blocks of σ_a. In other words, k and j are neighbors in σ if $k \in \sigma_a^b$ and $j \in \sigma_a^{b-1} \cup \sigma_a^b \cup \sigma_a^{b+1}$ for some a and b.

Now we define the map $F\colon \mathsf{AOP}(n, m) \to B(n, m)$.

Definition 3.6. Consider

$$\sigma = ((\sigma_1^1 \,|\, \cdots \,|\, \sigma_1^{r_1}), \dots, (\sigma_\ell^1 \,|\, \cdots \,|\, \sigma_\ell^{r_\ell})) \in \mathsf{AOP}(n, m).$$

For $1 \le k < j \le m$, let

$$B_{j,k} = \begin{cases} -2 & \text{if } j, k \in \sigma_a^b \text{ for some } a \text{ and } b, \\ 1 & \text{if } j \in \sigma_a^b \text{ and } k \in \sigma_a^{b-1} \cup \sigma_a^{b+1} \text{ for some } a \text{ and } b, \\ 0 & \text{otherwise.} \end{cases}$$

We also set $B_{j,j} = -1$ for $j \in [m]$ and $B_{j,k} = 0$ for $1 \leq j < k \leq m$. Then we define $F(\sigma)$ to be the $m \times m$ Bott matrix $[B_{j,k}]$.

Note that for $1 \leq k < j \leq m$, we have $B_{j,k} \neq 0$ if and only if j and k are neighbors in σ. The following proposition shows that the diagram in (3.2) commutes, and consequently, the map $F\colon \mathsf{AOP}(n,m) \to B(n,m)$ is surjective.

Proposition 3.7. *For any $\boldsymbol{i} \in [n]^m$, we have $F(\alpha(\boldsymbol{i})) = \beta(\boldsymbol{i})$. Moreover, the map $F : \mathsf{AOP}(n,m) \to B(n,m)$ is surjective.*

Proof. Let $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$, and let

$$\alpha(\boldsymbol{i}) = \sigma = ((\sigma_1^1 \,|\, \cdots \,|\, \sigma_1^{r_1}), \dots, (\sigma_\ell^1 \,|\, \cdots \,|\, \sigma_\ell^{r_\ell})).$$

By the definition of the map α, for $1 \leq k < j \leq m$, we have

(1) $i_j = i_k$ if and only if $j, k \in \sigma_a^b$ for some a and b;
(2) $|i_j - i_k| = 1$ if and only if $j \in \sigma_a^b$ and $k \in \sigma_a^{b-1} \cup \sigma_a^{b+1}$ for some a and b;
(3) $|i_j - i_k| > 1$ if and only if j and k are not neighbors in σ.

Thus, the definition of the matrix $F(\sigma) = [B_{j,k}]$ in Definition 3.6 is identical to that of $\beta(\boldsymbol{i}) = B_{\boldsymbol{i}}$ in Definition 2.3. Accordingly, we obtain $F(\alpha(\boldsymbol{i})) = \beta(\boldsymbol{i})$.

Since $\beta = F \circ \alpha$ and $\beta : [n]^m \to B(n,m)$ is surjective, the map $F : \mathsf{AOP}(n,m) \to B(n,m)$ is also surjective. $\square$

4. Enumeration of Bott manifolds of Bott–Samelson type

In this section, we prove Theorem 1.1, which relates Bott manifolds of BS type and assemblies of ordered partitions. As an application, we give a proof of Corollary 1.2, which enumerates the number $b(n,m)$ of Bott manifolds of BS type of dimension m determined by $\boldsymbol{i} \in [n]^m$.

We begin with some definitions. For an ordered partition $\eta = (\eta^1, \dots, \eta^r)$, we define $\overline{\eta} = (\eta^r, \dots, \eta^1)$. We define an equivalence relation $\sim$ on $\mathsf{AOP}(n,m)$ as follows.

Definition 4.1. Let $\sigma = (\sigma_1, \dots, \sigma_\ell)$ and $\tau = (\tau_1, \dots, \tau_{\ell'})$ be elements in $\mathsf{AOP}(n,m)$. We define $\sigma \sim \tau$ if $\ell = \ell'$ and $\sigma_a = \tau_a$ or $\sigma_a = \bar{\tau}_a$ for all $a = 1, \dots, \ell$.

Example 4.2. There are 14 equivalence classes in $\mathsf{AOP}(5,3)/\sim$ as follows:

$\{(1,2,3)\}$,
$\{(1\,2\,|\,3)\} \sim \{(3\,|\,1\,2)\}$, $\{(1\,3\,|\,2)\} \sim \{(2\,|\,1\,3)\}$, $\{(2\,3\,|\,1)\} \sim \{(1\,|\,2\,3)\}$,
$\{(1\,|\,2\,|\,3)\} \sim \{(3\,|\,2\,|\,1)\}$, $\{(1\,|\,3\,|\,2)\} \sim \{(2\,|\,3\,|\,1)\}$, $\{(2\,|\,1\,|\,3)\} \sim \{(3\,|\,1\,|\,2)\}$,
$\{(1\,2),(3)\}$, $\{(1\,3),(2)\}$, $\{(2\,3),(1)\}$,
$\{(1\,|\,2),(3)\} \sim \{(2\,|\,1),(3)\}$, $\{(1\,|\,3),(2)\} \sim \{(3\,|\,1),(2)\}$, $\{(2\,|\,3),(1)\} \sim \{(3\,|\,2),(1)\}$,
$\{(1),(2),(3)\}$.

The following proposition gives a connection between Bott matrices of BS type and assemblies of ordered partitions.

Proposition 4.3. *There is a bijection between the set $B(n,m)$ of Bott matrices of BS type and the set $\mathsf{AOP}(n,m)/\sim$ of equivalence classes of assemblies of ordered partitions.*

Proof. Recall from Proposition 3.7 that the map $F\colon \mathsf{AOP}(n,m) \to B(n,m)$ is surjective. Hence, it suffices to prove that for $\sigma, \tau \in \mathsf{AOP}(n,m)$, we have $F(\sigma) = F(\tau)$ if and only if $\sigma \sim \tau$.

Suppose that $\sigma \sim \tau$. By the definition of $\sim$, the neighborhood relations in σ and τ are the same: for $1 \leq k < j \leq m$, we have that j and k are neighbors in σ if and only if they are neighbors in τ. Moreover, k and j are in the same block in σ if and only if they are also in the same block in τ. Hence, by the definition of the map F in Definition 3.6, we have $F(\sigma) = F(\tau)$.

Suppose now that $F(\sigma) = F(\tau) = [B_{j,k}]$. Considering the pairs (j,k) with $B_{j,k} = -2$, we obtain

$$\{\sigma_1^1, \dots, \sigma_1^{r_1}, \dots, \sigma_\ell^1, \dots, \sigma_\ell^{r_\ell}\} = \{\tau_1^1, \dots, \tau_1^{r_1'}, \dots, \tau_{\ell'}^1, \dots, \tau_{\ell'}^{r_{\ell'}'}\}.$$

We call a sequence $(c_1, \dots, c_r)$ of integers a *path* if the following condition holds:

(1) $B_{c_i,c_j} \neq -2$ for all $i \neq j$,

(2) $B_{c_i,c_{i+1}} = 1$ for all $i \in [r-1]$.

A path is *maximal* if it cannot be extended to a longer path.

Let $(c_1, \dots, c_r)$ be a maximal path containing 1. Since $\min(\sigma_1) < \cdots < \min(\sigma_\ell)$, we have $1 \in \sigma_1^1 \cup \cdots \cup \sigma_1^{r_1}$. Hence, by the construction of $F(\sigma) = [B_{j,k}]$, we obtain $r = r_1$, and $c_i \in \sigma_1^i$ for all $i \in [r_1]$ or $c_i \in \sigma_1^{r_1+1-i}$ for all $i \in [r_1]$. Similarly, considering $F(\tau) = [B_{j,k}]$, we obtain $r = r_1'$, and $c_i \in \tau_1^i$ for all $i \in [r_1']$ or $c_i \in \tau_1^{r_1'+1-i}$ for all $i \in [r_1']$. This shows that $r_1 = r_1'$, and $\sigma_1 = \tau_1$ or $\sigma_1 = \overline{\tau}_1$. Now, let $(d_1, \dots, d_s)$ be a maximal path containing $\min([m] \setminus (\sigma_1^1 \cup \cdots \cup \sigma_1^{r_1}))$. By the same argument, we obtain that $r_2 = r_2'$, and $\sigma_2 = \tau_2$ or $\sigma_2 = \overline{\tau}_2$. Continuing in this way, we obtain $\sigma \sim \tau$, which completes the proof. □

Since there is a bijection between $\{\mathcal{B}_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^m\}$ and $B(n,m)$, we obtain Theorem 1.1 immediately from Proposition 4.3. We now prove Corollary 1.2 using the argument of the exponential formula (see [14, Corollary 5.1.6]).

Proof of Corollary 1.2. By Theorem 1.1, we have $b(n,m) = |\mathsf{AOP}(n,m)/\sim|$. Recall that $\mathsf{AOP}(n,m)$ is the set of assemblies $\sigma = (\sigma_1, \dots, \sigma_\ell)$ of ordered partitions $\sigma_i = (\sigma_i^1, \dots, \sigma_i^{r_i})$ such that $\{\sigma_i^j : i \in [\ell], j \in [r_i]\}$ is a partition of $[m]$ and

$$(r_1 + 1) + \cdots + (r_\ell + 1) \le n + 1.$$

Let $\mathsf{AOP}'(n,m)$ be the set of such assemblies of ordered partitions satisfying

$$(r_1 + 1) + \cdots + (r_\ell + 1) = n + 1, \tag{4.1}$$

and let $b'(n,m) = |\mathsf{AOP}'(n,m)/\sim|$. Observe that $\mathsf{AOP}(n,m) = \bigcup_{i=1}^n \mathsf{AOP}'(i,m)$ holds, and for any $\sigma, \tau \in \mathsf{AOP}(n,m)$, if $\sigma \in \mathsf{AOP}'(i,m)$ and $\sigma \sim \tau$, then $\tau \in \mathsf{AOP}'(i,m)$. Thus, for any $m \ge 1$, we have

$$\sum_{n\ge1} b(n,m) y^n = \sum_{n\ge1} \left(\sum_{i=1}^n b'(i,m) \right) y^n = \sum_{i\ge1} b'(i,m) y^i \sum_{n\ge i} y^{n-i} = \frac{1}{1-y} \sum_{i\ge1} b'(i,m) y^i.$$

Hence, it suffices to prove that

$$\sum_{m\ge1} \left(\sum_{n\ge1} b'(n,m) y^{n+1} \right) \frac{x^m}{m!} = \exp\left(\frac{y}{2} \left(\frac{1}{1 - y(e^x - 1)} + y(e^x - 1) - 1 \right) \right) - 1. \tag{4.2}$$

For two ordered partitions π and ρ, we define $\pi \sim \rho$ if $\pi = \rho$ or $\pi = \overline{\rho}$. Let $f(m,k)$ be the number of ordered partitions of $[m]$ with k blocks. Then, for any positive integer k,

$$\sum_{m\ge1} f(m,k) \frac{x^m}{m!} = \sum_{m\ge1} \left(\sum_{\substack{i_1,\dots,i_k \ge 1 \\ i_1 + \cdots + i_k = m}} \frac{m!}{i_1! \cdots i_k!} \right) \frac{x^m}{m!} = (e^x - 1)^k. \tag{4.3}$$

Let $\tilde{f}(m,k)$ be the number of equivalence classes of ordered partitions of $[m]$ with k blocks under the relation $\sim$. By definition, we have $\tilde{f}(m,k) = f(m,k)/2$ if $k \ge 2$, and $\tilde{f}(m,k) = f(m,k)$ if $k = 1$. Hence, by (4.3), we have

$$\begin{aligned} \sum_{m,k\ge1} \tilde{f}(m,k) \frac{x^m}{m!} y^{k+1} &= (e^x - 1) y^2 + \sum_{k\ge2} \frac{1}{2} (e^x - 1)^k y^{k+1} \\ &= \frac{y}{2} \left(\frac{1}{1 - y(e^x - 1)} - y(e^x - 1) - 1 \right) + y^2 (e^x - 1) \\ &= \frac{y}{2} \left(\frac{1}{1 - y(e^x - 1)} + y(e^x - 1) - 1 \right). \end{aligned} \tag{4.4}$$

Now consider $\sigma = (\sigma_1, \dots, \sigma_\ell) \in \mathsf{AOP}'(n,m)$. Observe that the equivalence class $[\sigma] \in \mathsf{AOP}'(n,m)/\sim$ can be identified with the set $\{[\sigma_1], \dots, [\sigma_\ell]\}$ of the equivalence classes $[\sigma_i]$ of ordered partitions under the relation $\sim$. Such a set is constructed as follows.

$m \backslash n$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	3	3	3	3
3	1	4	10	13	14	14	14
4	1	8	33	63	84	90	91

TABLE 1. The first few terms of the number $b(n,m)$ of Bott matrices in $B(n,m)$.

(1) Partition $[m]$ into nonempty blocks $A_1, \dots, A_\ell$ for some $\ell \geq 1$, where $\min(A_1) < \cdots < \min(A_\ell)$. The number of ways to do this is

$$\sum_{\ell \geq 1} \sum_{a_1+\cdots+a_\ell=m} \frac{1}{\ell!}\binom{m}{a_1,\dots,a_\ell}.$$

(2) For each $i \in [\ell]$, create an equivalence class $[\sigma_i]$ of ordered partitions, where σ_i is an ordered partition of A_i with r_i nonempty blocks, such that (4.1) holds. The number of ways to do this is

$$\sum_{\substack{r_1,\dots,r_\ell \geq 1 \\ (r_1+1)+\cdots+(r_\ell+1)=n+1}} \tilde{f}(a_1,r_1)\cdots\tilde{f}(a_\ell,r_\ell).$$

This shows that for any $m \geq 1$,

$$\sum_{n\geq 1} b'(n,m)y^{n+1} = \sum_{\ell \geq 1} \sum_{a_1+\cdots+a_\ell=m} \frac{1}{\ell!}\binom{m}{a_1,\dots,a_\ell} \prod_{i=1}^{\ell}\left(\sum_{r_i\geq 1}\tilde{f}(a_i,r_i)y^{r_i+1}\right).$$

Therefore, by (4.4), we have

$$\begin{aligned}\sum_{m\geq 1}\left(\sum_{n\geq 1} b'(n,m)y^{n+1}\right)\frac{x^m}{m!} &= \sum_{\ell\geq 1}\frac{1}{\ell!}\prod_{i=1}^{\ell}\left(\sum_{a_i,r_i\geq 1}\tilde{f}(a_i,r_i)\frac{x^{a_i}}{a_i!}y^{r_i+1}\right)\\ &= \exp\left(\frac{y}{2}\left(\frac{1}{1-y(e^x-1)}+y(e^x-1)-1\right)\right)-1.\end{aligned}$$

Hence, we obtain (4.2), which completes the proof. □

In Table 1, we display the first few terms of $b(n,m)$.

5. Isomorphism classes of Bott manifolds of Bott–Samelson type

In this section, we reinterpret the two operations on Bott matrices introduced in [3], called **Op. 1** and **Op. 2**, in terms of assemblies of ordered partitions. Using this interpretation, we prove Theorem 1.3.

Let $m > 0$ be a fixed integer, and let B be a Bott matrix of size $m \times m$, with column vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$. For $I \subseteq [m]$, consider the $m \times m$ lower triangular matrix L_I whose jth column vector $\mathbf{c}_j$ is defined by

$$\mathbf{c}_j := \begin{cases} \mathbf{e}_j & \text{if } j \in I, \\ \mathbf{v}_j & \text{if } j \notin I. \end{cases}$$

By [3, Proposition 3.1], the matrix $B_I := L_I^{-1}L_{I^c}$ is again a Bott matrix. We call the map that sends B to B_I for some $I \subseteq [m]$ the *first operation* **Op. 1**.

For $i \in [m-1]$, we denote by $\dot{s}_i$ the permutation matrix of the simple transposition $s_i = (i, i+1) \in \mathfrak{S}_m$, that is, the matrix whose $(s_i(j), j)$-entry is 1 for $j = 1, \dots, m$, and all other entries are 0. If $\dot{s}_i B \dot{s}_i$ is again a Bott matrix, then we call the map that sends B to $\dot{s}_i B \dot{s}_i$ for some $i \in [m-1]$ the *second operation* **Op. 2**. One can study the isomorphism classes of Bott manifolds using these two operations:

Proposition 5.1 ([3, Proposition 3.4])**.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be Bott manifolds with the corresponding Bott matrices B and B', respectively. Then $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic (as toric varieties) if and only if B can be obtained from B' by applying a sequence of the two operations **Op. 1** and **Op. 2**.*

We recall a brief idea of the proof of Proposition 5.1 for the reader's convenience. Note that the fan of a Bott manifold of dimension m is combinatorially equivalent to the normal fan of a cube $[0,1]^m$, and each pair $\{\mathbf{e}_j, \mathbf{v}_j\}$ of ray generators corresponds to two rays that do not form a cone (cf. Section 2). Since a Bott manifold is smooth, the set of ray generators forming a maximal cone constitutes a $\mathbb{Z}$-basis of the lattice. The first operation **Op. 1** corresponds to the choice of a $\mathbb{Z}$-basis, and the second operation **Op. 2** corresponds to the choice of ordering of that basis.

We will consider operations on the set of Bott matrices of BS type to study the isomorphism classes.

Example 5.2. Let $m = 3$ and $\boldsymbol{i} = (3,1,2)$. Then the matrix $B = B_{\boldsymbol{i}}$ is given by

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

If $I = \emptyset$, we have $L_I = L_\emptyset = B$ and $L_{I^c} = L_{\{1,2,3\}} = [\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3]$. Accordingly, we obtain

$$B_\emptyset = B^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix}.$$

Considering all subsets I of $[3]$, we obtain the following table.

I	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$L_I^{-1}L_{I^c}$	B^{-1}	B^{-1}	B^{-1}	B	B^{-1}	B	B	B

Observe that in Example 5.2, the subsets $\emptyset, \{1\}, \{2\}$, and $\{1,2\}$ produce the Bott matrix B^{-1}, which is *not* contained in $B(n,m)$ since the $(3,1)$-entry is -1. On the other hand, the other subsets provide the same matrix as B. Hence, in this example, the map $B \mapsto B_I$ produces a matrix in $B(n,m)$ if and only if $3 \in I$. Moreover, $\{3\}$ is the set of row indices with nonzero entries below the main diagonal of B. The following lemma shows that this phenomenon holds in general.

Lemma 5.3. *Let $B = [b_{j,k}] \in B(n,m)$, and let*

$$J_B := \{j \in [m] \mid \textit{there exists } k < j \textit{ such that } b_{j,k} \neq 0\}.$$

(1) *If $J_B \not\subseteq I$, we have $B_I \notin B(n,m)$.*
(2) *If $J_B \subseteq I$, we have $B_I = B \in B(n,m)$.*

Proof. For the first statement, suppose that $J_B \not\subseteq I$. Let $p = \min(J_B \setminus I)$ and $q = \min\{k \in [m] : b_{p,k} \neq 0\}$. Note that since $p \in J_B$, we have $q < p$. We claim that the (p,q)-entry of B_I is $-b_{p,q}$. This implies $B_I \notin B(n,m)$ because its entry $-b_{p,q}$, which is either 2 or -1, is below the diagonal and is not an element of $\{0,1,-2\}$. Hence, it suffices to prove the claim.

Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be the column vectors of B, and let $\mathbf{r}_1, \dots, \mathbf{r}_m$ be the row vectors of L_I^{-1}. Then the (p,q)-entry of $B_I = L_I^{-1}L_{I^c}$ is

$$(B_I)_{p,q} = \begin{cases} \mathbf{r}_p \cdot \mathbf{v}_q & \text{if } q \in I, \\ \mathbf{r}_p \cdot \mathbf{e}_q & \text{if } q \notin I. \end{cases} \tag{5.1}$$

Since $L_I^{-1}L_I$ is the identity matrix, for every $t \in [m]$, we have

$$\delta_{p,t} = (L_I^{-1}L_I)_{p,t} = \begin{cases} \mathbf{r}_p \cdot \mathbf{e}_t & \text{if } t \in I, \\ \mathbf{r}_p \cdot \mathbf{v}_t & \text{if } t \notin I. \end{cases} \tag{5.2}$$

By adding (5.1) and (5.2) with $t = q$, we obtain

$$(B_I)_{p,q} = \mathbf{r}_p \cdot (\mathbf{e}_q + \mathbf{v}_q). \tag{5.3}$$

Since the column vectors of L_I form a basis of $\mathbb{R}^m$, there exist $c_i, d_i \in \mathbb{R}$ such that

$$\mathbf{e}_q + \mathbf{v}_q = \sum_{i \notin I} c_i \mathbf{v}_i + \sum_{i \in I} d_i \mathbf{e}_i. \tag{5.4}$$

By (5.2) and the fact that $p \notin I$, taking the dot product with $\mathbf{r}_p$ on both sides of (5.4) gives

$$\mathbf{r}_p \cdot (\mathbf{e}_q + \mathbf{v}_q) = c_p. \tag{5.5}$$

On the other hand, since $q < p$ and $p \notin I$, taking the dot product with $\mathbf{e}_p$ on both sides of (5.4) gives

$$\mathbf{e}_p \cdot \mathbf{v}_q = \sum_{i \in I^c} c_i \mathbf{e}_p \cdot \mathbf{v}_i = \sum_{i \in I^c \cap J_B} c_i \mathbf{e}_p \cdot \mathbf{v}_i + \sum_{i \in I^c \setminus J_B} c_i \mathbf{e}_p \cdot \mathbf{v}_i. \tag{5.6}$$

Note that $\mathbf{e}_j \cdot \mathbf{v}_k = b_{j,k}$. If $i \in I^c \cap J_B = J_B \setminus I$, then $i \geq p = \min(J_B \setminus I)$. Since B is lower triangular, $\mathbf{e}_p \cdot \mathbf{v}_i = b_{p,i} = -\delta_{i,p}$. If $i \in I^c \setminus J_B$, then since $i \notin J_B$, we have $b_{j,i} = 0$ for all $j \neq i$. Since $i \neq p$, we have $\mathbf{e}_p \cdot \mathbf{v}_i = b_{p,i} = 0$. Thus, we obtain from (5.6) that

$$b_{p,q} = \mathbf{e}_p \cdot \mathbf{v}_q = -c_p. \tag{5.7}$$

Combining (5.3), (5.5), and (5.7), we obtain $(B_I)_{p,q} = -b_{p,q}$, as claimed.

For the second statement, suppose $J_B \subseteq I$. We claim that $L_I^2 = E_m$. Let $\mathbf{u}_1, \dots, \mathbf{u}_m$ be the column vectors of L_I^2. By the definition of L_I, we have $\mathbf{u}_k = \mathbf{e}_k$ for all $k \in I$. Suppose that $k \notin I$. Note that for two $m \times m$ matrices A and B, the kth column of AB is the linear combination of the columns of A with coefficients given by the kth column vector of B. Therefore, we have

$$\mathbf{u}_k = \sum_{j \in I} (\mathbf{e}_j \cdot \mathbf{v}_k) \mathbf{e}_j + \sum_{j \notin I} (\mathbf{e}_j \cdot \mathbf{v}_k) \mathbf{v}_j.$$

Since $J_B \subseteq I$, we have $\mathbf{e}_j \cdot \mathbf{v}_k = b_{j,k} = -\delta_{j,k}$ for all $j \notin I$. Hence, by the above equation, we obtain

$$\mathbf{u}_k = \sum_{j \neq k} (\mathbf{e}_j \cdot \mathbf{v}_k) \mathbf{e}_j - \mathbf{v}_k = (\mathbf{v}_k + \mathbf{e}_k) - \mathbf{v}_k = \mathbf{e}_k.$$

Thus, $\mathbf{u}_k = \mathbf{e}_k$ for all $k \in [m]$, and we obtain $L_I^2 = E_m$, as claimed. Equivalently, $L_I^{-1} = L_I$.

Now let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be the columns of $B_I = L_I^{-1} L_{I^c}$. By the claim, we have $L_I L_{I^c} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_m]$. If $k \notin I$, the kth columns of L_I and L_{I^c} are $\mathbf{v}_k$ and $\mathbf{e}_k$, respectively; hence $\mathbf{w}_k = \mathbf{v}_k$. Suppose that $k \in I$. Then, by the same arguments above, we obtain

$$\mathbf{w}_k = \sum_{j \in I} (\mathbf{e}_j \cdot \mathbf{v}_k) \mathbf{e}_j + \sum_{j \notin I} (\mathbf{e}_j \cdot \mathbf{v}_k) \mathbf{v}_j = \mathbf{v}_k.$$

Therefore, $\mathbf{w}_k = \mathbf{v}_k$ for all $k \in [m]$, and we obtain the second statement $B_I = B$. □

By Lemma 5.3, we obtain the following proposition.

Proposition 5.4. *Let $B \in B(n,m)$ be a Bott matrix of BS type. For a subset $I \subseteq [m]$, after applying the first operation* **Op. 1***, we have $B_I = B$ or $B_I \notin B(n,m)$.*

Proposition 5.4 tells us that the first operation acts **Op. 1** on the set of Bott matrices of BS type trivially. To understand the second operation **Op. 2**, we introduce some definitions.

Definition 5.5. Let $\sigma \in \mathsf{AOP}(n,m)$. For a simple transposition $s_j = (j, j+1) \in \mathfrak{S}_m$, we denote by $s_j \cdot \sigma$ the assembly of an ordered partition obtained from σ by swapping the two integers j and $j+1$. We say that s_j is an *admissible* transposition of σ if j and $j+1$ are not neighbors in σ.

Definition 5.6. For $\sigma, \tau \in \mathsf{AOP}(n,m)$, we define $\sigma \approx \tau$ if there exist $\rho^{(0)}, \rho^{(1)}, \dots, \rho^{(k)} \in \mathsf{AOP}(n,m)$ such that $\rho^{(0)} \sim \sigma$, $\rho^{(k)} \sim \tau$, and for each $i \in [k]$, there is an admissible transposition s_{j_i} of $\rho^{(i-1)}$ such that $s_{j_i} \cdot \rho^{(i-1)} = \rho^{(i)}$.

For example, we have $((1\,2),(3)) \approx ((1\,3),(2)) \approx ((2\,3),(1))$. Recall the map $F\colon \mathsf{AOP}(n,m) \to B(n,m)$ given in Definition 3.6. The following proposition describes the case where the second operation **Op. 2** can be applied to $F(\sigma)$.

Proposition 5.7. *Let $\sigma \in \mathsf{AOP}(n,m)$ and $i \in [m-1]$. Then $\dot{s}_i F(\sigma)\dot{s}_i \in B(n,m)$ if and only if s_i is an admissible transposition of σ.*

Proof. Let $F(\sigma) = [b_{j,k}]$. Since $\dot{s}_i F(\sigma)\dot{s}_i \in B(n,m)$ is the matrix obtained from $[b_{j,k}]$ by swapping the ith row and the $(i+1)$st row and swapping the ith column and the $(i+1)$st column, we have $\dot{s}_i F(\sigma)\dot{s}_i \in B(n,m)$ if and only if $b_{i+1,i} = 0$. By the definition of the map F, we have $b_{i+1,i} = 0$ if and only if i and $i+1$ are not neighbors. This proves the proposition. □

We are now ready to prove Theorem 1.3, which states that there is a bijection from $\mathsf{AOP}(n,m)/\approx$ to $\{\mathcal{B}_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^m\}/(\text{isom})$.

Proof of Theorem 1.3. For a Bott matrix $B \in B(n,m)$, let $\mathcal{B}(B)$ denote the Bott manifold corresponding to B. Define $\phi : (\mathsf{AOP}(n,m)/\approx) \to \{\mathcal{B}_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^m\}/(\text{isom})$ by $\phi([\sigma]) = [\mathcal{B}(F(\sigma))]$, where $[\sigma] = \{\tau \in \mathsf{AOP}(n,m) : \tau \approx \sigma\}$, and $[\mathcal{B}(F(\sigma))]$ is the isomorphism class containing $\mathcal{B}(F(\sigma))$. By Propositions 5.4 and 5.7, the map ϕ is well defined. By Proposition 3.7, the map $F\colon \mathsf{AOP}(n,m) \to B(n,m)$ is surjective, and hence, ϕ is also surjective.

Suppose that $\phi([\sigma]) = \phi([\tau])$. Then the Bott manifolds $\mathcal{B}(F(\sigma))$ and $\mathcal{B}(F(\tau))$ are isomorphic. Thus, by Proposition 5.1, the Bott matrix $F(\sigma)$ is obtained from $F(\tau)$ by applying a sequence of the operations **Op. 1** and **Op. 2**. By Propositions 5.4 and 5.7, this implies $\sigma \approx \tau$, that is $[\sigma] = [\tau]$. Therefore, ϕ is injective, which completes the proof. □

Now we introduce the notion of the indecomposability of Bott manifolds. We note that in some literature, indecomposability has different meanings (cf. [5]).

Definition 5.8. We say that a Bott manifold is *indecomposable* if it is not isomorphic to a product of lower-dimensional Bott manifolds. Otherwise, we say that it is *decomposable*. Similarly, we say that a Bott matrix is *indecomposable* or *decomposable* if the corresponding Bott manifold is indecomposable or decomposable, respectively.

Recall from [6, Proposition 3.1.14] that one can describe the fan of a product of two toric varieties using those of toric varieties. Let $\Sigma_1 \subseteq (N_1)_{\mathbb{R}}$ and $\Sigma_2 \subseteq (N_2)_{\mathbb{R}}$ be fans, where N_1 and N_2 are lattices. Then

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 : \sigma_i \in \Sigma_i\}$$

is a fan in $(N_1)_{\mathbb{R}} \times (N_2)_{\mathbb{R}} = (N_1 \times N_2)_{\mathbb{R}}$ and

$$X_{\Sigma_1 \times \Sigma_2} \cong X_{\Sigma_1} \times X_{\Sigma_2}.$$

Considering Bott manifolds, we can see that a Bott manifold $\mathcal{B}(B)$ is decomposable if and only if after applying a certain sequence of two operations **Op. 1** and **Op. 2**, the Bott matrix B can be expressed by a block diagonal matrix whose diagonal blocks are lower-dimensional Bott matrices. Hence, we have the following lemma.

Lemma 5.9. *A Bott matrix B of size $m \times m$ is decomposable if and only if there is a sequence of operations **Op. 1** and **Op. 2** whose application to B yields a block diagonal matrix whose diagonal blocks are Bott matrices of size less than $m \times m$.*

Let $\sigma = (\sigma_1, \dots, \sigma_\ell) \in \mathsf{AOP}(n,m)$. Then each σ_a is an ordered partition whose underlying partition is a partition of $\{b+1, \dots, b+r\}$ for some $b, r \geq 1$. Let $\sigma_a - b$ denote the ordered partition obtained from σ_a by replacing each element i by $i - b$. For simplicity, we write (σ_a) to mean the assembly $(\sigma_a - b)$ of ordered partitions consisting of a single ordered partition $\sigma_a - b$. We also write $F(\sigma_a)$ in place of $F((\sigma_a))$.

Proposition 5.10. *Let $\boldsymbol{i} \in [n]^m$ and $\alpha(\boldsymbol{i}) = (\sigma_1, \dots, \sigma_\ell)$. Then we have*

$$\mathcal{B}_{\boldsymbol{i}} \cong \mathcal{B}(F(\sigma_1)) \times \cdots \times \mathcal{B}(F(\sigma_\ell)).$$

Moreover, each $\mathcal{B}(F(\sigma_a))$ is indecomposable.

Proof. Recall that for $1 \leq k < j \leq m$, the (j,k)-entry of $B_{\boldsymbol{i}}$ is zero if and only if j and k are not neighbors in $\alpha(\boldsymbol{i})$. Accordingly, if $j \in \sigma_{a_1}$ and $k \in \sigma_{a_2}$ with $a_1 \neq a_2$, then the (j,k)-entry of $B_{\boldsymbol{i}}$ is 0. Therefore, there is a sequence of operations **Op. 1** and **Op. 2** whose application to $F(\alpha(\boldsymbol{i}))$

yields a block diagonal matrix whose diagonal blocks are $F(\sigma_1), \dots, F(\sigma_\ell)$. Then by Lemma 5.9, the Bott manifold $\mathcal{B}_{\boldsymbol{i}}$ is the product of $\mathcal{B}(F(\sigma_1)), \dots, \mathcal{B}(F(\sigma_\ell))$.

Now we prove that each $\mathcal{B}(F(\sigma_a))$ is indecomposable. Suppose that the underlying partition of σ_a is a partition of $\{b+1, \dots, b+r\}$. Then $(\sigma_a) = (\sigma_a - b) \in \mathsf{AOP}(n, r)$. Since (σ_a) contains only one ordered partition σ_a, each $j \in [r]$ has a neighbor k in (σ_a). This shows that there is no sequence of admissible transpositions that turns the Bott matrix $F(\sigma_1)$ into a block diagonal matrix whose diagonal blocks are Bott matrices of smaller size. Thus, by Proposition 5.4, Proposition 5.7, and Lemma 5.9, we obtain that $F(\sigma_a)$ is indecomposable. □

Definition 5.11. Let $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$. For $j \in [m-1]$, we define $s_j \cdot \boldsymbol{i}$ to be the sequence obtained from $\boldsymbol{i}$ by exchanging i_j and i_{j+1}. If $|i_j - i_{j+1}| > 1$, the operation that sends $\boldsymbol{i}$ to $s_j \cdot \boldsymbol{i}$ is called a *2-move*.

Recall that j and $j+1$ are not neighbors in $\alpha(\boldsymbol{i})$ if and only if $|i_j - i_{j+1}| > 1$. Thus, we obtain the following lemma.

Lemma 5.12. *Let $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$ and $j \in [m-1]$. Then s_j is an admissible transposition of $\alpha(\boldsymbol{i})$ if and only if $|i_j - i_{j+1}| > 1$. Moreover, if s_j is an admissible transposition of $\alpha(\boldsymbol{i})$, then $s_j \cdot \alpha(\boldsymbol{i}) = \alpha(s_j \cdot \boldsymbol{i})$.*

For $\boldsymbol{i} = (i_1, \dots, i_m) \in [n]^m$, the *standardization* $\mathrm{st}(\boldsymbol{i})$ of $\boldsymbol{i}$ is the sequence defined by

$$\mathrm{st}(\boldsymbol{i}) = (i_1 - k + 1, \dots, i_m - k + 1),$$

where k is the smallest integer in $\boldsymbol{i}$. We also define $\iota_n(\boldsymbol{i})$ to be the sequence obtained from $\boldsymbol{i}$ by replacing every integer $i \in [n]$ with $n+1-i$. By the definition of the map α, we have $\alpha(\mathrm{st}(\boldsymbol{i})) = \alpha(\boldsymbol{i})$ and $\alpha(\iota_n(\boldsymbol{i})) \sim \alpha(\boldsymbol{i})$.

Theorem 5.13. *Let $\boldsymbol{i}, \boldsymbol{i}' \in [n]^m$. Suppose that the Bott manifolds $\mathcal{B}_{\boldsymbol{i}}$ and $\mathcal{B}_{\boldsymbol{i}'}$ are indecomposable. Then $\mathcal{B}_{\boldsymbol{i}}$ and $\mathcal{B}_{\boldsymbol{i}'}$ are isomorphic (as toric varieties) if and only if either $\mathrm{st}(\boldsymbol{i}')$ or $\mathrm{st}(\iota_n(\boldsymbol{i}'))$ is obtained from $\mathrm{st}(\boldsymbol{i})$ via a sequence of 2-moves.*

Proof. The "if" statement follows from Theorem 1.3, Lemma 5.12, and the fact that $\alpha(\mathrm{st}(\boldsymbol{i})) = \alpha(\boldsymbol{i})$ and $\alpha(\iota_n(\boldsymbol{i})) \sim \alpha(\boldsymbol{i})$.

For the "only if" statement, suppose that $\mathcal{B}_{\boldsymbol{i}}$ and $\mathcal{B}_{\boldsymbol{i}'}$ are isomorphic. By Proposition 5.10, each of $\alpha(\boldsymbol{i})$ and $\alpha(\boldsymbol{i}')$ consists of one ordered partition, i.e., we can write $\alpha(\boldsymbol{i}) = \alpha(\mathrm{st}(\boldsymbol{i})) = (\sigma)$ and $\alpha(\boldsymbol{i}') = \alpha(\mathrm{st}(\boldsymbol{i}')) = (\sigma')$. By Theorem 1.3, there exist $\rho^{(0)}, \rho^{(1)}, \dots, \rho^{(k)} \in \mathsf{AOP}(n, m)$ such that $\rho^{(0)} \sim (\sigma)$, $\rho^{(k)} \sim (\sigma')$, and for each $i \in [k]$, there is an admissible transposition s_{j_i} of $\rho^{(i-1)}$ such that $s_{j_i} \cdot \rho^{(i-1)} = \rho^{(i)}$. Since (σ) has only one ordered partition, $\rho^{(0)} \sim (\sigma)$ implies that $\rho^{(0)} = (\sigma^{(0)})$ for some ordered partition $\sigma^{(0)}$. Similarly, for each $i \in [k]$, we have $\rho^{(i)} = (\sigma^{(i)})$ for some ordered partition $\sigma^{(i)}$. Since $\rho^{(0)} = (\sigma^{(0)}) \sim (\sigma)$, we have $\sigma^{(0)} = \sigma$ or $\sigma^{(0)} = \overline{\sigma}$. By replacing every $\sigma^{(i)}$ by $\overline{\sigma^{(i)}}$ if necessary, we may assume that $\rho^{(0)} = (\sigma) = \alpha(\mathrm{st}(\boldsymbol{i}))$. On the other hand, $\rho^{(k)}$ is either $\alpha(\mathrm{st}(\boldsymbol{i}'))$ or $\alpha(\mathrm{st}(\iota_n(\boldsymbol{i}')))$.

Since s_{j_i} is an admissible transposition of $\rho^{(i-1)}$ such that $s_{j_i} \cdot \rho^{(i-1)} = \rho^{(i)}$, by Lemma 5.12, we have $\rho^{(i)} = \alpha(\boldsymbol{i}^{(i)})$, where $\boldsymbol{i}^{(0)} = \mathrm{st}(\boldsymbol{i})$ and $\boldsymbol{i}^{(i+1)} = s_{j_i} \cdot \boldsymbol{i}^{(i)}$. Thus, $\rho^{(k)} = \alpha(\boldsymbol{i}^{(k)})$, which is equal to either $\alpha(\mathrm{st}(\boldsymbol{i}'))$ or $\alpha(\mathrm{st}(\iota_n(\boldsymbol{i}')))$. Since each of $\boldsymbol{i}^{(k)}$, $\mathrm{st}(\boldsymbol{i}')$, and $\mathrm{st}(\iota_n(\boldsymbol{i}'))$ has the smallest integer 1, and their images under α have only one ordered partition, we obtain that $\boldsymbol{i}^{(k)}$ is equal to either $\mathrm{st}(\boldsymbol{i}')$ or $\mathrm{st}(\iota_n(\boldsymbol{i}'))$. This means that $\mathrm{st}(\boldsymbol{i}')$ or $\mathrm{st}(\iota_n(\boldsymbol{i}'))$ is obtained from $\mathrm{st}(\boldsymbol{i})$ via a sequence of 2-moves. □

Recall from [11] that a Schubert variety X_w is toric if and only if any reduced decomposition $\boldsymbol{i} = (i_1, \dots, i_m)$ of w consists of distinct integers. Moreover, in this case, a toric Schubert variety X_w is isomorphic to the Bott manifold $\mathcal{B}_{\boldsymbol{i}}$ of BS type corresponding to $\boldsymbol{i}$ (see Remark 2.8). Applying Theorem 5.13 to toric Schubert varieties in type A_m of dimension m gives another proof of the result in [12, Proposition 4.1] for the case of type A.

Corollary 5.14. *Let X_w and $X_{w'}$ be toric Schubert varieties in type A_m of dimension m. Then X_w and $X_{w'}$ are isomorphic (as toric varieties) if and only if either $w' = w$ or $w' = w_0 w w_0$. Here, w_0 is the longest element in the Weyl group.*

Proof. We first note that if a toric Schubert variety X_w in type A_m has dimension m, then for any reduced decomposition $\boldsymbol{i} = (i_1, \dots, i_m)$ of w, we have $\{i_1, \dots, i_m\} = [m]$. Hence, X_w is indecomposable. Moreover, w_0ww_0 can be obtained from w by replacing every simple reflection s_i in a reduced decomposition with s_{m+1-i}. Since a sequence of 2-moves on a sequence does not change the corresponding permutation, we obtain the result by Theorem 5.13. □

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