MODERATE-TO-LARGE DEVIATION ASYMPTOTICS FOR REAL EIGENVALUES OF THE ELLIPTIC GINIBRE MATRICES

SUNG-SOO BYUN, JONAS JALOWY, YONG-WOO LEE, AND GRÉGORY SCHEHR

Abstract. We study the statistics of the number of real eigenvalues in the elliptic deformation of the real Ginibre ensemble. As the matrix dimension grows, the law of large numbers and the central limit theorem for the number of real eigenvalues are well understood, but the probabilities of rare events remain largely unexplored. Large deviation type results have been obtained only in extreme cases—when either a vanishingly small proportion of eigenvalues are real or almost all eigenvalues are real. Here, in both the strong and weak asymmetry regimes, we derive the probabilities of rare events in the moderate-to-large deviation regime, thereby providing a natural connection between the previously known regime of Gaussian fluctuations and the large deviation regime. Our results are new even for the classical real Ginibre ensemble.

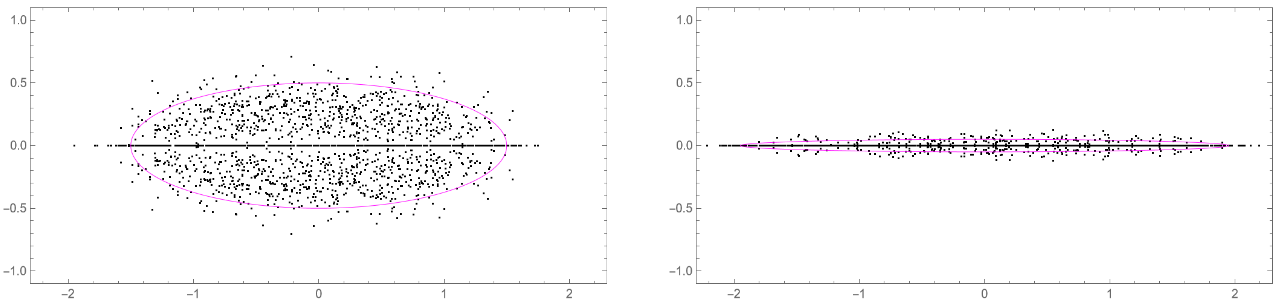
1. Introduction

Probabilistic limit theorems, which are central to modern probability theory and its applications—such as in statistical physics—provide a unifying framework for understanding fluctuations in complex systems across different scales. For a system of size n (for example, the sum of n independent random variables), the central limit theorem (CLT) typically yields Gaussian laws with fluctuations of order $n^{1/2}$, while the large deviation principle (LDP) characterises rare events of order n with exponential precision. Between these two regimes lies the *moderate deviation scale*, in which fluctuations exceed the CLT scale but remain sublinear in n. Results in this regime refine the Gaussian approximation and capture the onset of large deviation behaviour.

In random matrix theory, while central limit theorems for global statistics are well established and large deviation methods capture extreme fluctuations, systematic results in the intermediate regime—the moderate deviation principles—have received comparatively little attention. This intermediate scaling behaviour has recently been investigated in the context of the complex Ginibre ensemble, where it occurs in the statistics of the eigenvalue with the largest modulus [53] and in eigenvalue counting statistics [52, 54], i.e. the number of eigenvalues within a prescribed region of the complex plane, see also [2, 7, 17, 19–21, 41, 45, 46, 55, 57, 60, 72] and references therein for recent work on related variants. In this work, we address this regime for a fundamental observable in non-Hermitian random matrix theory: *the number of real eigenvalues in asymmetric random matrices*.

The statistics of real eigenvalues of non-Hermitian random matrices have been studied extensively in the literature. In particular, the number of real eigenvalues has been analysed in a variety of models, most notably the real Ginibre ensemble [24, 37, 50, 69] and its extensions. Prominent examples include the elliptic Ginibre ensembles [13, 14, 30, 38], spherical Ginibre matrices [24, 31, 36], the truncated orthogonal ensemble [33, 34, 44, 56], products of GinOE matrices [1, 26, 28, 32, 69], and asymmetric Wishart matrices [16].

Real eigenvalues of non-Hermitian random matrices, such as those in the real Ginibre ensemble, play a central role in physics, particularly in characterizing the spectral and dynamical properties of complex systems. Their number and distribution provide valuable insights into phase transitions, symmetry classes, and the onset of localization, thereby reflecting fundamental features of open and dissipative quantum systems (see, e.g. [9, 71]). Beyond their role as spectral diagnostics, real eigenvalues are also crucial for assessing the stability of complex systems. For example, they play a key role in determining the number of equilibrium points in non-relaxational dynamical systems [8, 39, 51], which generalize May's classical model of ecological networks [58]. Similarly, in neural networks, they correspond to non-oscillatory relaxation modes that govern the network's response to perturbations and overall dynamical stability [64, 70]. Finally, very interesting connections have been unveiled between the statistics of the real eigenvalues in the real Ginibre ensemble and annihilating Brownian particles on the line (see e.g. [43]).



(A) $\tau = 1/2$ (strong asymmetry). (B) $\tau = 19/20$ (weak asymmetry).

FIGURE 1. Eigenvalues of 100 realisations of the eGinOE with $n = 20$ for a given τ.

We now introduce our object of study. By definition, the real Ginibre ensemble (GinOE) is the ensemble of $n \times n$ matrices G with i.i.d. standard normal entries, see [11] for a review. Its elliptic extension, designed to interpolate between GinOE and the symmetric Gaussian orthogonal ensemble (GOE), is known as the elliptic GinOE (eGinOE), see Figure 1. It is defined by

$$X := \frac{\sqrt{1+\tau}}{2}(G + G^T) + \frac{\sqrt{1-\tau}}{2}(G - G^T),$$

where $\tau \in [0, 1)$ is the asymmetry parameter. In particular, $\tau = 0$ recovers GinOE, while the limit $\tau \to 1$ yields the classical Gaussian orthogonal ensemble (GOE) [27].

We begin with a brief overview of the known results on the statistics of the number of real eigenvalues in the eGinOE. In the study of the eGinOE, there are two distinct asymptotic regimes:

- **Strong Asymmetry**: the parameter $\tau \in [0, 1)$ is kept fixed as $n \to \infty$, see Figure 1 (A).
- **Weak Asymmetry** [40, 42]: the parameter $\tau \equiv \tau_n \to 1$ at a prescribed rate, see Figure 1 (B). We focus on the scaling
$$\tau = 1 - \frac{\alpha^2}{n}, \qquad \alpha \in [0, \infty).$$

Let $\mathcal{N}_n$ denote the number of real eigenvalues of the eGinOE matrix X. We denote by $\mathbb{P}$ and $\mathbb{E}$ the probability measure and expectation associated with $\mathcal{N}_n$, respectively. We first discuss the typical behaviour of the random variable $\mathcal{N}_n$. Here we emphasise that the results in most of the literature below (e.g. [13, 15, 38, 50]) were obtained for even dimensions n, see however [35, 49, 67] for the analysis for odd dimensions. Nevertheless, it is believed that the leading asymptotic behaviour remains the same for odd dimensions.

- For the strong asymmetry regime, where τ is fixed, it has been shown that both the mean and the variance of $\mathcal{N}_n$ grow at order $O(\sqrt{n})$. More precisely,
$$\lim_{n\to\infty} \frac{\mathbb{E}\mathcal{N}_n}{\sqrt{n}} = \sqrt{\frac{2}{\pi}\frac{1+\tau}{1-\tau}}, \qquad \lim_{n\to\infty} \frac{\operatorname{Var}\mathcal{N}_n}{\sqrt{n}} = (2-\sqrt{2})\sqrt{\frac{2}{\pi}\frac{1+\tau}{1-\tau}}. \tag{1.1}$$
These were established in [13,38], and for the special case $\tau = 0$ (the GinOE), these results had appeared in earlier works [24, 37].
- For the weak asymmetry regime, where $\tau = 1 - \alpha^2/n$, the growth is of a larger order $O(n)$. In this case, it was shown in [13] that
$$\lim_{n\to\infty} \frac{\mathbb{E}\mathcal{N}_n}{n} = c(\alpha), \qquad \lim_{n\to\infty} \frac{\operatorname{Var}\mathcal{N}_n}{n} = 2\Big(c(\alpha) - c(\sqrt{2}\alpha)\Big). \tag{1.2}$$
Here,
$$c(\alpha) := e^{-\frac{\alpha^2}{2}}\Big(I_0\big(\tfrac{\alpha^2}{2}\big) + I_1\big(\tfrac{\alpha^2}{2}\big)\Big), \tag{1.3}$$
where I_ν is the modified Bessel function of the first kind [62, Chapter 10].

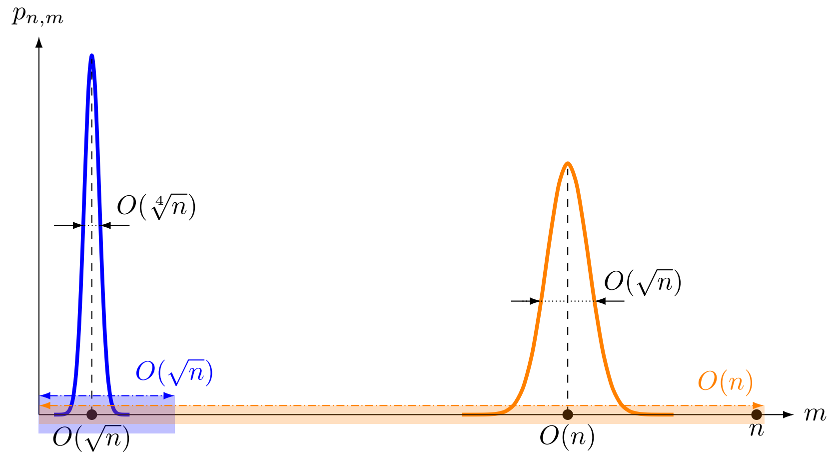


FIGURE 2. A schematic sketch of $m \mapsto p_{n,m}$. The graph on the left (blue) illustrates the case with strong asymmetry, while the graph on the right (orange) illustrates the case with weak asymmetry. The regimes studied in this paper are depicted on the x-axis as shaded regions.

Beyond the mean and variance of $\mathcal{N}_n$, a central limit theorem has also been established for both the strong and weak asymmetry regimes. More precisely, it was shown in [15,30,68] that

$$\frac{\mathcal{N}_n - \mathbb{E}\mathcal{N}_n}{\sqrt{\mathrm{Var}\mathcal{N}_n}} \to \mathrm{N}(0,1), \qquad \text{as } n \to \infty,$$

where $\mathrm{N}(0,1)$ denotes the standard normal distribution. See Figure 2 for an illustration of the typical Gaussian fluctuation regime.

In addition to the number of real eigenvalues, the averaged real eigenvalue density and its asymptotic behaviour were also derived in [24,38] for the strong asymmetry regime and in [13,25] for the weak asymmetry regime. The precise convergence rates, often called the finite-size corrections were further obtained in [14]. These asymptotic behaviours of the real eigenvalue density play an important role in the study of the complexity of random landscapes arising in non-gradient flows [8,39].

We now turn to the rare events and the asymptotic behaviour of their probabilities. For this, we denote

$$p_{n,m} := \mathbb{P}(\mathcal{N}_n = m) \tag{1.4}$$

for the probability that the eGinOE matrix has exactly m real eigenvalues. When discussing tail probabilities, it is important to distinguish between the left and right tails. The left tail corresponds to the probability of having an exceptionally small number of real eigenvalues compared to the typical value, while the right tail corresponds to the probability of having an exceptionally large number.

- We first consider the strong asymmetry regime with $\tau \in [0,1)$.
  - In the left tail, corresponding to the case $m = O(\sqrt{n}/\log n)$, it was shown in [15] that

    $$\lim_{n\to\infty} \frac{\log p_{n,m}}{\sqrt{n}} = -\sqrt{\frac{1+\tau}{1-\tau}}\frac{1}{\sqrt{2\pi}}\zeta(\tfrac{3}{2})\ , \tag{1.5}$$

    where ζ is the Riemann zeta function. The GinOE case ($\tau = 0$) was previously obtained in [50].
  - In the right tail, we consider $m = n - O(1)$. In this case, it was shown in [4] that

    $$\lim_{n\to\infty} \frac{\log p_{n,m}}{n^2} = -\frac{1}{4}\log\Big(\frac{2}{1+\tau}\Big). \tag{1.6}$$

    The special cases $m = n-2$ and $m = n$ can also be found in [5,23,38]. Moreover, [4] provides the precise correction terms beyond the leading order.

Note that the $O(n^2)$ decay in the right tail is significantly faster than the $O(\sqrt{n})$ decay in the left tail. This is intuitively natural, since the typical mean of m is of order $O(\sqrt{n})$, which indicates that the distribution is more heavily skewed to the left, see Figure 2.

- Next, we consider the weak asymmetry regime with $\tau = 1 - \alpha^2/n$.
  - In the left tail, corresponding to the case $m = O(n/\log n)$, it was shown in [15] that

$$\lim_{n\to\infty} \frac{\log p_{n,m}}{n} \leq \frac{2}{\pi} \int_0^1 \log(1 - e^{-\alpha^2 s^2}) \sqrt{1-s^2}\, ds\,, \tag{1.7}$$

    independently of m. It is conjectured that this inequality is in fact an equality, but due to a technical issue in the proof of [15], only the inequality was established.
  - In the right tail, when $m = n - O(1)$, it was shown in [4] that

$$\lim_{n\to\infty} \frac{\log p_{n,m}}{n} = -\frac{\alpha^2}{8}. \tag{1.8}$$

    The precise corrections and asymptotic expansions were also obtained in [4].

  Note that, in contrast to the strong asymmetry regime, both the left and right tails here exhibit decay of order $O(n)$.

From the literature discussed so far, the typical value and its Gaussian fluctuations are well understood, and the probabilities of observing atypically small numbers of real eigenvalues, such as $m = O(\mathbb{E}\mathcal{N}_n/\log n)$, or extremely large ones, $m = n - O(1)$, have also been investigated. In addition, in the strong asymmetry regime—most notably for the real Ginibre ensemble ($\tau = 0$)—even the macroscopic regime $m = O(n)$ has been partially analysed using Coulomb gas techniques [61], see also [6] and Remark 2.

In contrast, two regimes remain largely unexplored: the intermediate regime $m = O(\sqrt{n})$ in the strongly asymmetric case, and the macroscopic regime $m = O(n)$ in the weakly asymmetric setting. In this work, we address both of these regimes, thereby extending the CLT scale and establishing a connection with the extreme large deviation regimes.

2. Main results and discussions

2.1. Main results. In this section, we present our main results concerning the asymptotic behaviour of $p_{n,m}$. We begin by recalling that, for $s \in \mathbb{C}$, the polylogarithm $\mathrm{Li}_s(z)$ is defined by

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \tag{2.1}$$

where the series converges absolutely for $|z| < 1$. The function $\mathrm{Li}_s(z)$ admits an analytic continuation to the complex z-plane, with a branch cut typically taken along $[1, \infty)$, see [62, Section 25.12] for details.

We are now ready to state our main results.

Theorem 2.1 (Moderate and large deviation probabilities).

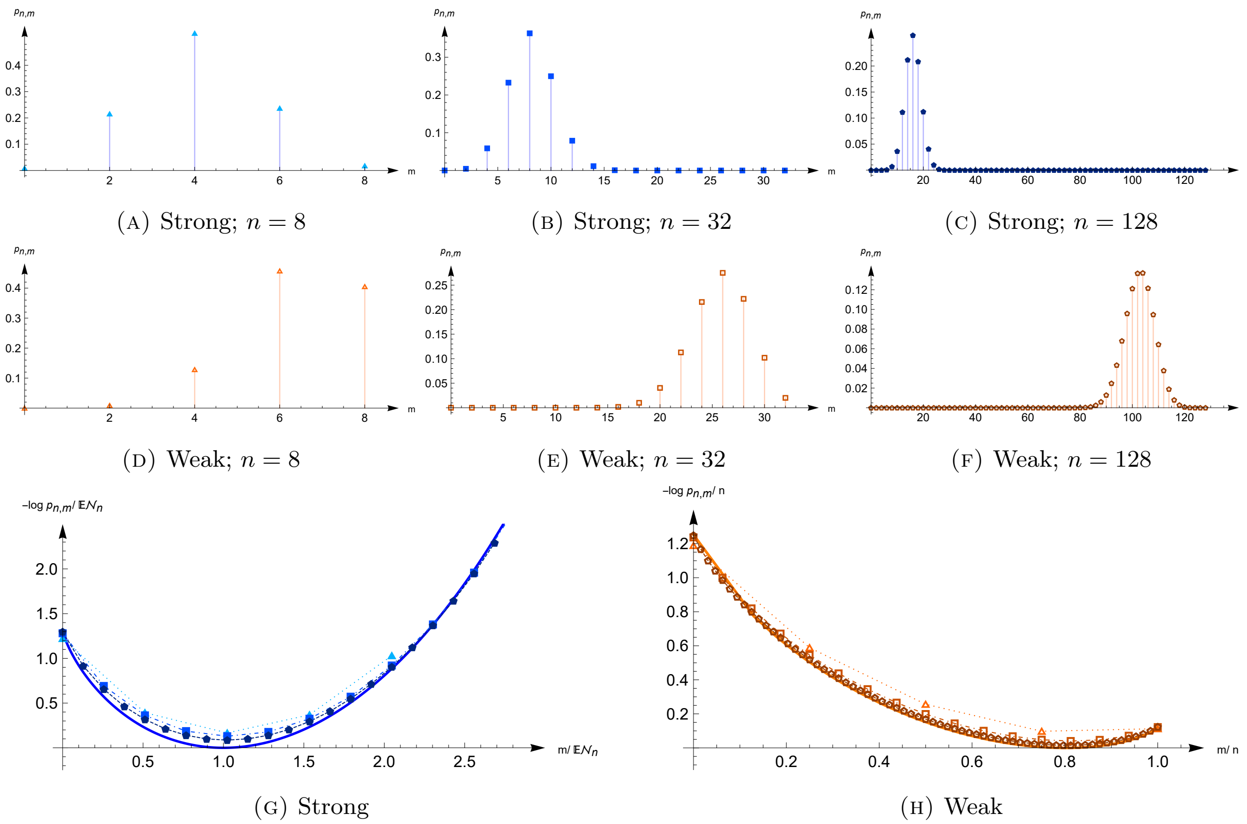
(i) **(Strong asymmetry regime)** *Let $\tau \in [0, 1)$ be fixed. Suppose that $2m/\mathbb{E}\mathcal{N}_{2n} \to x \in (0, \infty)$ as $n \to \infty$. Then we have*

$$\lim_{n\to\infty} \frac{\log p_{2n,2m}}{\mathbb{E}\mathcal{N}_{2n}} = -\frac{1}{2} \sup_{u\in\mathbb{R}} \Big\{ xu + \mathrm{Li}_{3/2}(1 - e^u) \Big\}. \tag{2.2}$$

Here, Li *is the polylogarithm* (2.1) *and recall that* $\mathbb{E}\mathcal{N}_{2n} \sim \sqrt{2n}\sqrt{\frac{2}{\pi}\frac{1+\tau}{1-\tau}}$ *by* (1.1).

(ii) **(Weak asymmetry regime)** *Let $\tau = 1 - \frac{\alpha^2}{2n}$ with fixed $\alpha \in (0, \infty)$. Suppose that $m/n \to x \in (0, 1)$ as $n \to \infty$. Then, we have*

$$\lim_{n\to\infty} \frac{\log p_{2n,2m}}{2n} = -\frac{1}{2} \sup_{u\in\mathbb{R}} \Big\{ xu - \frac{4}{\pi} \int_0^1 \log\Big(1 - (1 - e^u) e^{-\alpha^2 s^2}\Big) \sqrt{1-s^2}\, ds \Big\}. \tag{2.3}$$



(A) Strong; $n = 8$ (B) Strong; $n = 32$ (C) Strong; $n = 128$

(D) Weak; $n = 8$ (E) Weak; $n = 32$ (F) Weak; $n = 128$

(G) Strong (H) Weak

FIGURE 3. Figures (A)–(C) show the numerical plots of $m \mapsto p_{n,m}$ in the strong asymmetry regime for $n = 8, 32, 128$, respectively. Figures (D)–(F) display the corresponding plots in the weak asymmetry regime. Figures (G) and (H) present the graphs of $m \mapsto -\log p_{n,m}$ for both regimes (dotted line), together with their comparison to $m \mapsto \mathbb{E}\mathcal{N}_n\, \phi_{\rm s}(m/\mathbb{E}\mathcal{N}_n)$ and $m \mapsto n\, \phi_{\rm w}(m/n)$ (full solid line). Here, we set $\tau = 1/2$ for the strong asymmetry (blue) and $\alpha = 1$ for the weak asymmetry (orange). The formula (3.1) is used for the numerical evaluations.

See Figure 3 for the numerical verification of Theorem 2.1.

It is convenient to restate Theorem 2.1 for later discussions. For this, we write

$$\Psi_{\rm s}(z) \equiv \Psi_{\rm s}(z;\tau) := -\sqrt{\frac{1+\tau}{1-\tau}\frac{1}{2\pi}}\, \mathrm{Li}_{3/2}(1-z), \tag{2.4}$$

$$\Psi_{\rm w}(z) \equiv \Psi_{\rm w}(z;\alpha) := \frac{2}{\pi}\int_0^1 \log\Big(1-(1-z)e^{-\alpha^2 s^2}\Big)\sqrt{1-s^2}\, ds \tag{2.5}$$

and define

$$\phi_{\rm s}(x) \equiv \phi_{\rm s}(x;\tau) := \sup_{u\in\mathbb{R}}\Big\{\frac{x}{2}u - \Psi_{\rm s}(e^u)\Big\}, \qquad \phi_{\rm w}(x) \equiv \phi_{\rm w}(x;\alpha) := \sup_{u\in\mathbb{R}}\Big\{\frac{x}{2}u - \Psi_{\rm w}(e^u)\Big\}. \tag{2.6}$$

Then it follows from the inverse Legendre transform that

$$\phi_{\rm s}(x) = -\Psi_{\rm s}(\Phi_{\rm s}^{-1}(\tfrac{x}{2})) + \tfrac{x}{2}\log\Phi_{\rm s}^{-1}(\tfrac{x}{2}), \qquad \phi_{\rm w}(x) = -\Psi_{\rm w}(\Phi_{\rm w}^{-1}(\tfrac{x}{2})) + \tfrac{x}{2}\log\Phi_{\rm w}^{-1}(\tfrac{x}{2}), \tag{2.7}$$

where

$$\Phi_{\rm s}(z) \equiv \Phi_{\rm s}(z;\tau) := \sqrt{\frac{1+\tau}{1-\tau}\frac{1}{2\pi}}\,\frac{z\,\mathrm{Li}_{1/2}(1-z)}{1-z}, \tag{2.8}$$

$$\Phi_{\rm w}(z) \equiv \Phi_{\rm w}(z;\alpha) := \frac{2}{\pi}\int_0^1 \frac{z\, e^{-\alpha^2 s^2}}{1-(1-z)e^{-\alpha^2 s^2}}\sqrt{1-s^2}\, ds. \tag{2.9}$$

Now, Theorem 2.1 can be restated as follows: suppose $2m/\sqrt{2n} \to x \in (0,\infty)$ as $n \to \infty$ in the strong asymmetry regime, while $m/n \to x \in (0,1)$ as $n \to \infty$ in the weak asymmetry regime. Then

$$\lim_{n\to\infty} \frac{\log p_{2n,2m}}{\sqrt{2n}} = -\phi_{\rm s}(x) \quad \text{for the strong asymmetry}, \tag{2.10}$$

$$\lim_{n\to\infty} \frac{\log p_{2n,2m}}{2n} = -\phi_{\rm w}(x) \quad \text{for the weak asymmetry}. \tag{2.11}$$

The asymptotic behaviours of the rate functions $\phi_{\rm s}$ and $\phi_{\rm w}$ characterise how the deviation probabilities interpolate between the Gaussian fluctuation regime and the extreme large deviation regime, thereby clarifying the crossover structure between them. The asymptotic behaviours of $\phi_{\rm s}$ as $x \to 0$ and $x \to \infty$ are given by (2.13) and (2.14), respectively. Similarly, the asymptotic behaviours of $\phi_{\rm w}$ as $x \to 0$ and $x \to 1$ are given by (2.16) and (2.17), respectively. These asymptotics provide natural matchings with the previous findings discussed in the previous section.

2.2. **Discussions.** We now discuss several aspects and implications of Theorem 2.1.

Remark 1 (Universal form in the strong asymmetry regime, cf. [31]). Notice that the right-hand side of (2.2) is independent of the choice of $\tau \in [0,1)$. This observation suggests a possible universality, which indeed extends beyond the present model. In a recent work by Forrester [31, Proposition 3], the analogous result for the spherical ensemble was established, where the same form as in (2.2) arises. In order to compare (2.2) and [31, Proposition 3], one needs the integral identity

$$\mathrm{Li}_{3/2}(z) = -\frac{2}{\sqrt{\pi}}\int_0^\infty \log\Big(1 - ze^{-t^2}\Big)\, dt, \tag{2.12}$$

which follows from the power series expansion of the logarithm on the right hand side for $|z|<1$, as well as the analytic continuation for $z \in \mathbb{C}\setminus[1,\infty)$. For the case $x=0$ in (2.2), corresponding to the extremal probability that there are no real eigenvalues, the possibility of such a universal form was discussed in [15, Remark 1.4] and [31, Section 3.3].

We conclude this remark on universality by noting that the same large deviation function $\phi_{\rm s}(x)$ in (2.6) describes the current fluctuations in the simple symmetric exclusion process (SSEP), first computed in [18], see Eq. (2) there with $\rho_a = 0$ and $\rho_b = 1$. It is also interesting to note that a similar connection was found between the counting statistics in the complex Ginibre ensemble [52] and the current fluctuations for a Brownian gas of non-interacting particles. This points out to further connections between Ginibre ensembles and the statistical mechanics particle systems, as previously noticed in [43] in the context of annihilating particles systems on the line.

Next, we discuss various limiting cases and interconnections between our main results and the existing literature. A summary is provided in Figure 4.

Remark 2 (Extremal case; matching to the left and right tail large deviations in the strong asymmetry regime). Note that since $\mathrm{Li}_s(1) = \zeta(s)$, we have

$$\lim_{x\to 0+} \phi_{\rm s}(x) = -\Psi_{\rm s}(0) = \sqrt{\frac{1-\tau}{1+\tau}\frac{1}{2\pi}}\zeta(\tfrac{3}{2}). \tag{2.13}$$

Thus, by taking the limit $x \to 0$ in our result (2.10), we can recover the previous finding (1.5).

On the other hand, the opposite limit $x \to \infty$ in (2.10) is more delicate. Clearly, one cannot expect to directly recover (1.6), due to the different scalings $O(\sqrt{n})$ and $O(n)$. In the special case $\tau = 0$, however, the Coulomb gas approach was used in [61] to show that, for $\gamma \in (0,1)$,

$$\lim_{n\to\infty} \frac{\log p_{n,\gamma n}}{n^2} = -\mathcal{I}[\mu_\gamma],$$

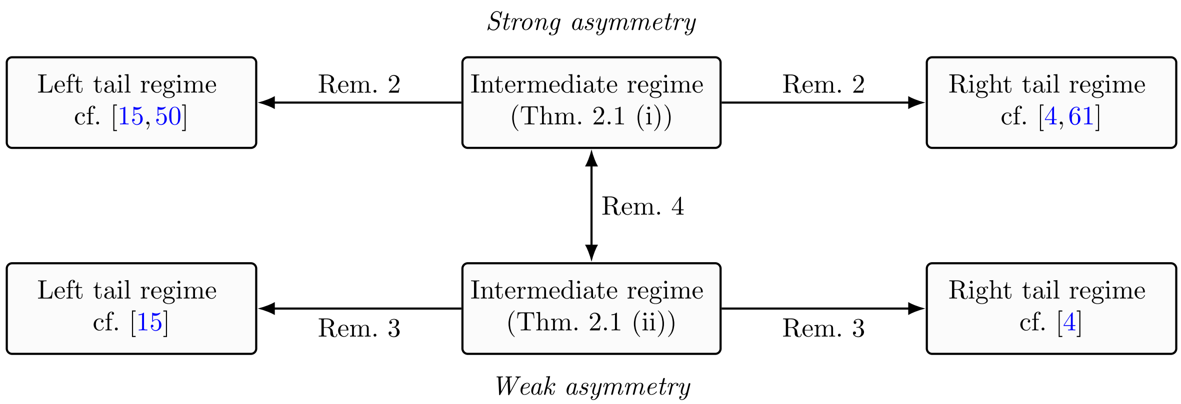


FIGURE 4. The diagram illustrates the interrelations between our main results and the existing literature, the details of which can be found in Remarks 2, 3, and 4.

where μ_γ is the minimiser of the energy functional

$$\mathcal{I}[\mu] := \frac{1}{2}\Big(\int |z|^2 \, d\mu(z) - \iint \log|z-w| \, d\mu(z) \, d\mu(w) \Big) - \frac{3}{8}$$

among all probability measures that place mass $(1-\gamma)$ on $\mathbb{C} \setminus \mathbb{R}$, and mass γ on $\mathbb{R}$. See also [31] for recent progress on this electrostatic approach; cf. [12].

Unlike in [31], where the energy functional is explicitly evaluated for the spherical ensemble, no closed-form expression is available in the GinOE case. Consequently, a quantitative analysis of $\mathcal{I}[\mu_\gamma]$ remains challenging. Nevertheless, one may still expect a relation between the limit $x \to \infty$ in our formulation and the limit $\gamma \to 0$ in $\mathcal{I}[\mu_\gamma]$. Indeed, direct computations from (2.6) show that

$$\phi_{\rm s}(x) \sim \frac{1-\tau}{1+\tau} \frac{\pi^2}{48} x^3, \qquad \text{as } x \to \infty, \tag{2.14}$$

which implies

$$\log p_{2n,2m} \sim -\frac{1+\tau}{1-\tau} \frac{\pi^2}{48} x^3 \sqrt{2n}, \qquad \text{as } \frac{2m}{\sqrt{2n}} = x \to \infty. \tag{2.15}$$

Here, we used the asymptotic of the polylogarithm $\mathrm{Li}_s(\pm e^u) = -u^s/\Gamma(s+1) + o(u^s)$, as $\operatorname{Re} u \to \infty$. For the spherical ensemble, a similar cubic divergence of the rate function reproduces the $\gamma \to 0$ limit of $\mathcal{I}[\mu_\gamma]$, see [31, Eq. (3.12)]. We also mention that the opposite limit $\gamma \to 1$ should be read off from (1.6), leading to $\mathcal{I}[\mu_{\gamma=1}] = \frac{1}{4} \log(\frac{2}{1+\tau})$.

Remark 3 (Extremal case; matching to the left and right tail large deviations in the weak asymmetry regime). In contrast to the strong asymmetry regime, in the weak asymmetry regime both extremal limits $x \to 0$ and $x \to 1$ in (2.11) can be easily connected with previous findings. For the left-tail limit $x \to 0$, we have

$$\lim_{x \to 0+} \phi_{\rm w}(x) = -\Psi_{\rm w}(0) = -\frac{2}{\pi} \int_0^1 \log\Big(1 - e^{-\alpha^2 s^2}\Big) \sqrt{1-s^2} \, ds. \tag{2.16}$$

The right-hand side coincides with (1.7). By continuity, this provides stronger conjectural evidence that (1.7) in fact holds as an equality. For the right-tail limit $x \to 1$, we obtain

$$\lim_{x \to 1-} \phi_{\rm w}(x) = \sup_{u \in \mathbb{R}} \Big\{ \frac{u}{2} - \Psi_{\rm w}(e^u) \Big\} = \frac{\alpha^2}{8}, \tag{2.17}$$

which coincides with (1.8). The last equality follows from the observation that

$$\frac{d}{du}\Big(\frac{u}{2} - \Psi_{\rm w}(e^u)\Big) = \frac{1}{2} - \frac{2}{\pi} \int_0^1 \frac{e^{u-\alpha^2 s^2}}{e^{u-\alpha^2 s^2} + 1 - e^{-\alpha^2 s^2}} \sqrt{1-s^2} \, ds > \frac{1}{2} - \frac{2}{\pi} \int_0^1 \sqrt{1-s^2} \, ds = 0,$$

which leads to

$$\sup_{u\in\mathbb{R}}\Big\{\frac{u}{2}-\Psi_{\rm w}(e^u)\Big\}=\lim_{u\to\infty}\Big(\frac{u}{2}-\Psi_{\rm w}(e^u)\Big)=\frac{u}{2}-\frac{2}{\pi}\int_0^1(u-\alpha^2s^2)\sqrt{1-s^2}\,ds=\frac{\alpha^2}{8}.$$

By definition, the rate functions in the moderate deviation probabilities are closely connected with the typical expected value. Specifically, the expected value is recovered from the minimiser of the rate function, while the second-order correction at this point, the curvature of the rate function at the critical point, is related to the variance. Using our closed formulas, we can verify these facts, which we formulate in the following proposition.

Proposition 2.2 (Minimiser and curvature of the rate functions). *Let $\phi_{\rm s}$ and $\phi_{\rm w}$ denote the rate functions in* (2.6).

- *The minima of the rate functions $\phi_{\rm s}$ and $\phi_{\rm w}$ occur at the limiting expected value of $\mathcal{N}_{2n}$ (normalised by the typical scale). In other words, if $x_{\rm s}$ and $x_{\rm w}$ denote the unique minimisers of $\phi_{\rm s}$ and $\phi_{\rm w}$, then*

$$x_{\rm s}=\sqrt{\frac{1+\tau}{1-\tau}\frac{2}{\pi}}=\lim_{n\to\infty}\frac{\mathbb{E}\mathcal{N}_{2n}}{\sqrt{2n}},\qquad x_{\rm w}=c(\alpha)=\lim_{n\to\infty}\frac{\mathbb{E}\mathcal{N}_{2n}}{2n},\tag{2.18}$$

- *The curvatures of the rate functions $\phi_{\rm s}$ and $\phi_{\rm w}$ at their minima are the reciprocal of the variance of $\mathcal{N}_{2n}$ (normalised by the typical scale). More precisely,*

$$\phi_{\rm s}''(x_{\rm s})=\frac{1}{(2-\sqrt{2})\sqrt{\frac{1+\tau}{1-\tau}\frac{2}{\pi}}}=\lim_{n\to\infty}\frac{\sqrt{2n}}{\mathrm{Var}\mathcal{N}_{2n}},\tag{2.19}$$

$$\phi_{\rm w}''(x_{\rm w})=\frac{1}{2(c(\alpha)-c(\sqrt{2}\alpha))}=\lim_{n\to\infty}\frac{2n}{\mathrm{Var}\mathcal{N}_{2n}}.\tag{2.20}$$

This proposition will be shown in Subsection 3.2.

Remark 4 (Interpolating properties of the rate functions). A notable feature of the weak asymmetry regime is that it connects naturally with the strong asymmetry regime when the parameters are chosen appropriately. This relationship has been studied extensively in the literature (see, e.g. [13, Section 2.1] and [15, Remark 1.4]), and it also holds in our present formulation. Specifically, suppose $m/\mathbb{E}\mathcal{N}_{2n}\to y$ as $n\to\infty$. By (1.1), (1.2), (2.18) and Theorem 2.1, we have

$$\lim_{n\to\infty}\frac{\log p_{2n,2m}}{\mathbb{E}\mathcal{N}_{2n}}=\begin{cases}-\dfrac{\phi_{\rm s}(x_{\rm s}y)}{x_{\rm s}} & \text{for the strong asymmetry,}\\[2ex] -\dfrac{\phi_{\rm w}(x_{\rm w}y)}{x_{\rm w}} & \text{for the weak asymmetry.}\end{cases}\tag{2.21}$$

Then, assuming $1\ll\alpha\ll\sqrt{2n}$ as $n\to\infty$, the (renormalised) rate function in (2.21) for the weak asymmetry coincide with that for the strong asymmetry in the limit $\alpha\to\infty$. Indeed, this is a consequence of the following observation.

For the strong asymmetry, it follows from straightforward computations using the definition (2.6) that

$$\frac{\Psi_{\rm s}(x)}{x_{\rm s}}=-\frac{\mathrm{Li}_{3/2}(1-x)}{2},\qquad\frac{\Phi_{\rm s}(x)}{x_{\rm s}}=\frac{x\,\mathrm{Li}_{1/2}(1-x)}{2(1-x)}.\tag{2.22}$$

For the weak asymmetry, we suppose $1\ll\alpha\ll\sqrt{2n}$ as $n\to\infty$. Then the leading contribution of the integral in (2.5) comes from small $s\ll1$. Thus, we have

$$\Psi_{\rm w}(x)=\frac{2}{\pi}\int_0^1\log\Big(1-(1-x)e^{-\alpha^2s^2}\Big)\,ds\sim\frac{2}{\pi\alpha}\int_0^\infty\log\Big(1-(1-x)e^{-t^2}\Big)\,dt=-\frac{1}{\sqrt{\pi\alpha^2}}\,\mathrm{Li}_{3/2}(1-x),$$

as $\alpha\to\infty$. Similar computations give rise to

$$\Phi_{\rm w}(x)\sim\frac{2}{\pi\alpha}\int_0^\infty\frac{xe^{-t^2}}{1-(1-x)e^{-t^2}}\,dt=\frac{1}{\sqrt{\pi\alpha^2}}\frac{x\,\mathrm{Li}_{1/2}(1-x)}{1-x}.\tag{2.23}$$

On the other hand, using $I_\nu(x) = e^x/\sqrt{2\pi x}\,(1+o(1))$ as $x \to \infty$, we have $c(\alpha) = 2/\sqrt{\pi\alpha^2}\,(1+o(1))$ as $\alpha \to \infty$. Hence, putting the above asymptotics together, we obtain

$$\lim_{\alpha\to\infty} \frac{\Psi_{\rm w}(x)}{x_{\rm w}} = -\frac{\mathrm{Li}_{3/2}(1-x)}{2}, \qquad \lim_{\alpha\to\infty} \frac{\Phi_{\rm w}(x)}{x_{\rm w}} = \frac{x\,\mathrm{Li}_{1/2}(1-x)}{2(1-x)}. \tag{2.24}$$

These coincide with (2.22), implying that the renormalised rate function in (2.21) for the weak asymmetry regime matches that for the strong asymmetry regime in the limit $\alpha \to \infty$.

Our next result concerns the asymptotic behaviour of the generating function of $p_{2n,2k}$ and the cumulants of $\mathcal{N}_{2n}$. The following proposition is indeed a key step in the proofs of our main results, while also being of independent interest. To state it, we recall that the Stirling number of the second kind $S(n,k)$ is defined by

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n,$$

see [62, Section 26.8]. We also recall that for a random variable X, the ℓ-th cumulant $\kappa_\ell(X)$ is defined by

$$\kappa_\ell(X) = \frac{d^\ell}{dt^\ell}\log \mathbb{E}[e^{tX}]\Big|_{t=0}.$$

Proposition 2.3 (**Asymptotic behaviour of the generating function and cumulants**).

(i) **(Strong asymmetry regime)** *Let $\tau \in [0,1)$ be fixed. Then we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n}} \log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) = \Psi_{\rm s}(z), \tag{2.25}$$

locally uniformly in $\mathbb{C}\setminus(-\infty,0]$, where $\Psi_{\rm s}$ is given by (2.4). Furthermore for any $\ell \in \mathbb{N}_0$, we have

$$\lim_{n\to\infty} \frac{\kappa_\ell(\mathcal{N}_{2n})}{\sqrt{2n}} = 2^\ell \sum_{m=1}^{\ell}(-1)^{m+1}(m-1)!\,S(\ell,m)\sqrt{\frac{1+\tau}{1-\tau}\frac{1}{2\pi m}}. \tag{2.26}$$

(ii) **(Weak asymmetry regime)** *Let $\tau = 1-\frac{\alpha^2}{2n}$ with fixed $\alpha \in (0,\infty)$. Then we have*

$$\lim_{n\to\infty} \frac{1}{2n} \log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) = \Psi_{\rm w}(z), \tag{2.27}$$

locally uniformly in $\mathbb{C}\setminus(-\infty,0]$, where $\Psi_{\rm w}$ is given by (2.5). Furthermore for any $\ell \in \mathbb{N}_0$, we have

$$\lim_{n\to\infty} \frac{\kappa_\ell(\mathcal{N}_{2n})}{2n} = 2^\ell \sum_{m=1}^{\ell}(-1)^{m+1}(m-1)!\,S(\ell,m)\frac{c(\sqrt{m}\alpha)}{2}, \tag{2.28}$$

where $c(\alpha)$ is given by (1.3).

This proposition will also be shown in Subsection 3.2. In particular, Proposition 2.3 (i) provides a slight extension of [15, Theorem 1.5].

Notice also that the first two cumulants, $\ell = 1,2$ in (2.26) and (2.28) reproduce the mean and variance given in (1.1) and (1.2). For $\ell = 3$, we have

$$\lim_{n\to\infty} \frac{\kappa_3(\mathcal{N}_{2n})}{\sqrt{2n}} = \sqrt{\frac{1+\tau}{1-\tau}\frac{1}{\pi}}\,\frac{4}{3}\,(3\sqrt{2}-9+2\sqrt{6})$$

for the strong asymmetry and

$$\lim_{n\to\infty} \frac{\kappa_3(\mathcal{N}_{2n})}{2n} = 4e^{-\frac{\alpha^2}{2}}\Big(I_0\big(\tfrac{\alpha^2}{2}\big)+I_1\big(\tfrac{\alpha^2}{2}\big)\Big) - 12e^{-\alpha^2}\Big(I_0\big(\alpha^2\big)+I_1\big(\alpha^2\big)\Big) + 8e^{-\frac{3\alpha^2}{2}}\Big(I_0\big(\tfrac{3\alpha^2}{2}\big)+I_1\big(\tfrac{3\alpha^2}{2}\big)\Big)$$

for the weak asymmetry.

2.3. **Idea of the proof.** We conclude this section by outlining heuristics for how Theorem 2.1 can be derived from Proposition 2.3, together with the actual proof strategy that makes these heuristics rigorous. Indeed, the underlying heuristic argument captures main ingredients, but establishing it rigorously requires essential mathematical foundations, which will be addressed in Theorem 4.1.

We first note that the Laplace transform of $\mathcal{N}_{2n}$ is the generating function of $p_{2n,2k}$, namely

$$\mathbb{E}e^{s\mathcal{N}_{2n}} = \sum_{k=0}^{n} e^{2sk} p_{2n,2k} = \sum_{k=0}^{n} z^k p_{2n,2k}\Big|_{z=e^{2s}}. \tag{2.29}$$

The first key step in the proof of Theorem 2.1 is Proposition 2.3, which provides

$$\sum_{k=0}^{n} e^{2sk} p_{2n,2k} \sim \begin{cases} \exp\Big[\sqrt{2n}\,\Psi_{\rm s}(e^{2s})\Big] & \text{for the strong asymmetry,} \\ \exp\Big[2n\,\Psi_{\rm w}(e^{2s})\Big] & \text{for the weak asymmetry,} \end{cases} \tag{2.30}$$

as $n\to\infty$. Since our interest is in $p_{2n,2k}$ for moderate deviation ranges of k, the natural question is:

If we know the asymptotic behaviour of the polynomials with increasing degree,
how can we derive the asymptotics of their coefficients?

To this end, we first assume the existence of a convex limiting *exponential profile* for $p_{2n,2k}$; that is, there exists a convex function $\phi:[0,\infty)\to\mathbb{R}$ such that

$$p_{2n,2k} \sim \begin{cases} \exp\Big[-\sqrt{2n}\,\phi_{\rm s}\Big(\frac{2k}{\sqrt{2n}}\Big)\Big] & \text{for the strong asymmetry,} \\ \exp\Big[-2n\,\phi_{\rm w}\Big(\frac{k}{n}\Big)\Big] & \text{for the weak asymmetry.} \end{cases} \tag{2.31}$$

From this ansatz and Laplace's method, Ψ can be identified as the Legendre transform of ϕ. More precisely, in the strong asymmetry case,

$$\mathbb{E}e^{s\mathcal{N}_{2n}} \sim \sum_{k=0}^{n} \exp\Big[\sqrt{2n}\Big(s\frac{2k}{\sqrt{2n}} - \phi\Big(\frac{2k}{\sqrt{2n}}\Big)\Big)\Big] \sim \exp\Big[\sqrt{2n}\sup_{x\in[0,\infty)}\{sx-\phi(x)\}\Big]$$

and an analogous expression holds in the weak asymmetry case. Therefore, by (2.30), we have $\Psi(e^{2s}) = \phi^*(s)$, where f^* denotes the Legendre transform of a function f. Since the Legendre transform is involutive for convex functions, we can in turn recover ϕ as the Legendre transform of $\Psi(e^{2s})$.

The main step required to make this argument rigorous is the verification of the ansatz (2.31). For this, we rely on a recent result of [47]. Roughly speaking, the existence of a limiting generating function implies the existence of a limiting exponential profile for the coefficients. Such a statement has been rigorously established when the leading asymptotics of the exponential profile match the order of the polynomial, namely $O(n)$. This is precisely the case in the weak asymmetry regime, where we can directly apply [47, Theorem 5.1]. As a consequence, the convexity of the ansatz (2.31) also follows.

In contrast, in the strong asymmetry regime, the leading asymptotic is only $O(\sqrt{n})$. Hence, an extension of [47, Theorem 5.1] is required to handle the case where only a small portion of the polynomial genuinely contributes to the leading-order asymptotic behaviour. This extension constitutes one of the main steps of our analysis and is formulated in Theorem 4.1.

We refer the reader to Table 1 for the summary of the key ingredients of the proofs presented in this paper.

	Strong Asymmetry	Weak Asymmetry
Limit of generating function	Proposition 2.3 (i)	Proposition 2.3 (ii)
Zeros and exponential profiles of polynomials	Theorem 4.1	[47, Theorem 5.1]

TABLE 1. Strategy and main ingredients of the proofs presented in this paper.

3. The generating function of the number of real eigenvalues

In this section, we compile some preliminaries and show Propositions 2.2 and 2.3.

3.1. **Preliminaries on the generating functions.**

In this subsection, we collect known results on the generating function of $p_{2n,2k}$. For the case $\tau = 0$, these results can be found in [50], while the general case $\tau \in [0,1)$ is treated in [15].

The first main observation regarding the generating function arises from the underlying algebraic structure of the eGinOE, namely the fact that it forms a Pfaffian point process [37, 38]. Moreover, in even dimensions, the Pfaffian reduces to a determinant, leading to the following formula:

$$\sum_{k=0}^{n} z^k p_{2n,2k} = \det\Big[\mathrm{I}_n + (z-1)M_n\Big], \tag{3.1}$$

where I_n is the $n \times n$ identity matrix and M_n is an $n \times n$ symmetric matrix with entries

$$[M_n]_{j,k} = \frac{1}{\sqrt{2\pi}} \frac{(\tau/2)^{j+k-2}}{\sqrt{\Gamma(2j-1)\Gamma(2k-1)}} \int_{\mathbb{R}} e^{-\frac{x^2}{1+\tau}} H_{2j-2}\Big(\frac{x}{\sqrt{2\tau}}\Big) H_{2k-2}\Big(\frac{x}{\sqrt{2\tau}}\Big)\, dx.$$

Here, H_k is the k-th Hermite polynomial. Furthermore, the spectrum of M_n is contained in the interval $(0,1)$. These results were established in [15, Proposition 2.1 and Lemma 2.3]. For the GinOE case, analogous determinantal formulas can also be found in [5, 29, 50].

Lemma 3.1. *For any $z \in \mathbb{C}$ with $|z-1| < 1$, we have*

$$\log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) = -\sum_{k=1}^{\infty} \frac{1}{k}(1-z)^k \operatorname{Tr}(M_n^k). \tag{3.2}$$

Proof. Using (3.1) and the matrix identity $\log\det A = \operatorname{Tr}\log A$, we have

$$\log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) = \log\det\Big[\mathrm{I}_n + (z-1)M_n\Big] = \operatorname{Tr}\log\Big[\mathrm{I}_n + (z-1)M_n\Big]. \tag{3.3}$$

Since all the eigenvalue of M_n lie in the interval $(0,1)$, for given n and $|z-1| < 1$, we have Taylor expansion of matrix-valued logarithm

$$\log\Big[\mathrm{I}_n + (z-1)M_n\Big] = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}(z-1)^k M_n^k.$$

This gives rise to the desired result (3.2). ☐

The main step in [15] for the asymptotic analysis is the study of the moments $\operatorname{Tr}(M_n^k)$. In particular, it was shown in [15, Propositions 2.5 and 2.6] that for any fixed $k \in \mathbb{N}$, the following limits hold:

- For the strong asymmetry regime,

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n}} \operatorname{Tr}(M_n^k) = \sqrt{\frac{1+\tau}{1-\tau}\frac{1}{2\pi k}}. \tag{3.4}$$

- For the weak asymmetry regime,

$$\lim_{n\to\infty} \frac{1}{2n} \operatorname{Tr}(M_n^k) = \frac{e^{-k\alpha^2/2}}{2}\Big[I_0\Big(\frac{k\alpha^2}{2}\Big) + I_1\Big(\frac{k\alpha^2}{2}\Big)\Big]. \tag{3.5}$$

These asymptotic behaviours play a central role in the subsequent analysis.

3.2. **Proof of Propositions 2.2 and 2.3.** In this subsection, we show Propositions 2.2 and 2.3.

Proof of Proposition 2.2. Let us consider the rate function $\phi_{\rm s}$ given in (2.6) for a constant $\tau \in [0,1)$. Defining $\psi_{\rm s}(x) := \Psi_{\rm s}(e^x)$, we write $\phi_{\rm s} = \psi_{\rm s}^*$, where f^* denotes the Legendre transform of a function f. It follows from direct computations that

$$\psi_{\rm s}'(x) = \sqrt{\frac{1+\tau}{1-\tau}\frac{1}{2\pi}}\,\frac{e^x}{1-e^x}\,\mathrm{Li}_{3/2}(1-e^x), \tag{3.6}$$

$$\psi_{\rm s}''(x) = \sqrt{\frac{1+\tau}{1-\tau}\frac{1}{2\pi}}\,\frac{e^x}{(1-e^x)^2}\Big[\mathrm{Li}_{1/2}(1-e^x) - e^x\,\mathrm{Li}_{-1/2}(1-e^x)\Big]. \tag{3.7}$$

Given a twice-differentiable function f, we have identities

$$(f^*)'(x) = (f')^{-1}(x), \qquad (f^*)''(x) = \frac{1}{f''((f')^{-1}(x))}.$$

Let us denote by $x_{\rm s}$ the minima of $\phi_{\rm s}$. Then, we have $0 = \phi_{\rm s}'(x_{\rm s}) = \frac{1}{2}(\psi_{\rm s}^*)'(\frac{x_{\rm s}}{2}) = \frac{1}{2}(\psi_{\rm s}')^{-1}(\frac{x_{\rm s}}{2})$. Therefore,

$$x_{\rm s} = 2\psi_{\rm s}'(0) = \sqrt{\frac{1+\tau}{1-\tau}\frac{2}{\pi}} = \lim_{n\to\infty}\frac{\mathbb{E}\mathcal{N}_{2n}}{\sqrt{2n}}.$$

Furthermore, we have

$$\phi_{\rm s}''(x_{\rm s}) = \frac{1}{4}(\psi_{\rm s}^*)''(\frac{x_{\rm s}}{2}) = \frac{1}{4\psi_{\rm s}''((\psi_{\rm s}')^{-1}(\frac{x_{\rm s}}{2}))} = \frac{1}{4\psi_{\rm s}''(0)} = \frac{1}{(2-\sqrt{2})\sqrt{\frac{1+\tau}{1-\tau}\frac{2}{\pi}}} = \lim_{n\to\infty}\frac{\sqrt{2n}}{\mathrm{Var}\mathcal{N}_{2n}}.$$

Similarly, let us denote by $x_{\rm w}$ the minima of the rate function $\phi_{\rm w}$ and define $\psi_{\rm w}(x) := \Psi_{\rm w}(e^x)$. Then, we have

$$\psi_{\rm s}'(x) = \frac{2}{\pi}\int_0^1 \frac{e^x}{e^x + e^{\alpha^2 s^2} - 1}\sqrt{1-s^2}\,ds, \tag{3.8}$$

$$\psi_{\rm s}''(x) = \frac{2}{\pi}\int_0^1 \frac{e^x(e^{\alpha^2 s^2}-1)}{(e^x + e^{\alpha^2 s^2} - 1)^2}\sqrt{1-s^2}\,ds. \tag{3.9}$$

Then, using $0 = \phi_{\rm w}'(x_{\rm w}) = \frac{1}{2}(\psi_{\rm w}^*)'(\frac{x_{\rm w}}{2}) = \frac{1}{2}(\psi_{\rm w}')^{-1}(\frac{x_{\rm w}}{2})$, we obtain

$$x_{\rm w} = 2\psi_{\rm w}(0) = c(\alpha) = \lim_{n\to\infty}\frac{\mathbb{E}\mathcal{N}_{2n}}{2n},$$

and

$$\phi_{\rm w}''(x_{\rm w}) = \frac{1}{4}\frac{1}{\psi''(0)} = \frac{1}{2(c(\alpha) - c(\sqrt{2}\alpha))} = \lim_{n\to\infty}\frac{2n}{\mathrm{Var}\mathcal{N}_{2n}}.$$

This completes the proof. ∎

Proof of Proposition 2.3. We first prove the locally uniform convergence (2.25) for $z \in \mathbb{C}\setminus(-\infty,0]$. Suppose $|z+1| > \epsilon$ for some $\epsilon > 0$. We observe from (3.3) that

$$\log\Big(\sum_{k=0}^n z^k p_{2n,2k}\Big) = \mathrm{Tr}\log\Big[\mathrm{I}_n + (z-1)M_n\Big] \le c_\epsilon |z-1|\,\mathrm{Tr}\,M_n,$$

where $c_\epsilon > 0$ is a constant such that $|\log(1+w)| \le c_\epsilon|w|$ for any $w \in \mathbb{C}\setminus(-\infty,0]$ satisfying $|w+1| > \epsilon$. Here, we also used the fact that M_n is positive-definite. Together with (3.4), the above bound implies

$$\Big\{\frac{1}{\sqrt{2n}}\log\Big(\sum_{k=0}^n z^k p_{2n,2k}\Big)\Big\}_{n=1}^\infty$$

is a sequence of locally bounded holomorphic functions on $\mathbb{C}\setminus(-\infty,0]$. Hence, by Montel's theorem, it is a normal family. On the other hand, by [15, Theorem 1.5], the sequence converges to $\Psi_{\rm s}$ on the disk $\{z \in \mathbb{C} : |z-1| < 1\}$. Thus, applying the identity theorem, the assertion (2.25) follows.

Recall that by definition, for a random variable X we have

$$\log \mathbb{E}e^{tX} = \sum_{\ell=1}^{\infty} \frac{t^\ell}{\ell!}\kappa_\ell(X), \tag{3.10}$$

where $\kappa_\ell(X)$ is the ℓ-th cumulant of X. Then, (2.25) in turn implies that

$$\log \mathbb{E}e^{t\mathcal{N}_{2n}} = -\sqrt{2n}\sqrt{\frac{1+\tau}{1-\tau}\frac{1}{2\pi}}\,\mathrm{Li}_{3/2}(1-e^{2t}) + o(\sqrt{n}),$$

as $n\to\infty$. Substituting the series expansion of the polylogarithm

$$\mathrm{Li}_{3/2}(1-e^{2t}) = \sum_{\ell=1}^{\infty}\frac{(2t)^\ell}{\ell!}\sum_{m=1}^{\ell}\frac{(-1)^m}{\sqrt{m}}(m-1)!S(\ell,m),$$

which can be obtained along the lines presented in the supplementary material of [63]. Comparing the coefficients using (3.10), we obtain the desired result (2.26).

Next, we prove the second claim (ii). Since the eigenvalues of M_n lie in the interval $(0,1)$, the power series (3.2) is absolutely convergent for sufficiently large n whenever $|z-1|<1$. Hence it suffices to establish (2.27) for $z\in(0,1)$, and the general case then follows from the identity theorem.

We compute the upper bound for the limiting generating function in (2.27). Assume $z\in(0,1)$. Since $(1-z)^k\operatorname{Tr}M_n^k>0$, we deduce from (3.2) that for any $k_0\in\mathbb{N}$,

$$\log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) < -\sum_{k=1}^{k_0}\frac{1}{k}(1-z)^k\operatorname{Tr}M_n^k.$$

Thus, together with (3.5), we have

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{2n}\log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) &\le -\lim_{n\to\infty}\sum_{k=1}^{k_0}\frac{1}{k}(1-z)^k\frac{1}{2n}\operatorname{Tr}M_n^k\\ &= -\sum_{k=1}^{k_0}\frac{1}{2k}(1-z)^k e^{-k\alpha^2/2}\Big[I_0\Big(\frac{k\alpha^2}{2}\Big)+I_1\Big(\frac{k\alpha^2}{2}\Big)\Big].\end{aligned}$$

Letting $k_0\to\infty$, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{2n}\log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) &\le -\sum_{k=1}^{\infty}\frac{1}{2k}(1-z)^k e^{-k\alpha^2/2}\Big[I_0\Big(\frac{k\alpha^2}{2}\Big)+I_1\Big(\frac{k\alpha^2}{2}\Big)\Big]\\ = -\frac{2}{\pi}\int_0^1\sum_{k=1}^{\infty}\frac{1}{k}(1-z)^k e^{-k\alpha^2s^2}\sqrt{1-s^2}\,ds &= \frac{2}{\pi}\int_0^1\log\Big(1-(1-z)e^{-\alpha^2s^2}\Big)\sqrt{1-s^2}\,ds.\end{aligned} \tag{3.11}$$

For the first equality, we use the integral identity (see e.g. [15, Proof of Lemma A.1])

$$\int_0^1 e^{-xs^2}\sqrt{1-s^2}ds = \frac{\pi}{4}e^{-\frac{x}{2}}\Big[I_0\Big(\frac{x}{2}\Big)+I_1\Big(\frac{x}{2}\Big)\Big], \tag{3.12}$$

while for the second equality, we use the Taylor expansion of $\log(1-x)$.

We show the lower bound for the limiting generating function in (2.27). Let

$$\log(1-x) = -\sum_{j=1}^{k}\frac{1}{j}x^j + R_k(x), \qquad R_k(x) := -\int_0^x\Big(\frac{x-t}{1-t}\Big)^k dt.$$

Notice that for $x\in[0,1)$, we have $R_k(x)\ge -x^{k+1}/(1-x)$. Furthermore, since the spectrum of M_n is contained in the interval $(0,1)$, we have $\operatorname{Tr}M_n^k<n$ for any $k\in\mathbb{N}$. Therefore, for any $k_0\in\mathbb{N}$, we have

$$\frac{1}{2n}\log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) > -\sum_{k=1}^{k_0}\frac{(1-z)^k}{k}\frac{1}{2n}\operatorname{Tr}M_n^k - \sum_{k=k_0+1}^{\infty}\frac{(1-z)^k}{2k}$$

$$= -\sum_{k=1}^{k_0} \frac{1}{k}(1-z)^k \frac{1}{2n} \operatorname{Tr} M_n^k + \frac{R_{k_0}(1-z)}{2} \geq -\sum_{k=1}^{k_0} \frac{1}{k}(1-z)^k \frac{1}{2n} \operatorname{Tr} M_n^k - \frac{(1-z)^{k_0+1}}{2z}.$$

Taking the limit $n \to \infty$, this yields

$$\liminf_{n\to\infty} \frac{1}{2n} \log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) \geq -\sum_{k=1}^{k_0} \frac{1}{2k}(1-z)^k c(\sqrt{k}\alpha) - \frac{(1-z)^{k_0+1}}{2z}.$$

Since $z \in (0,1)$, by taking limit $k_0 \to \infty$, we obtain

$$\liminf_{n\to\infty} \frac{1}{2n} \log\Big(\sum_{k=0}^{n} z^k p_{2n,2k}\Big) \geq \frac{2}{\pi}\int_0^1 \log\Big(1-(1-z)e^{-\alpha^2 s^2}\Big)\sqrt{1-s^2}\, ds. \tag{3.13}$$

Combining (3.11) and (3.13), we obtain (2.27) for $z \in (0,1)$. This completes the proof of (2.27).

Finally, we show (2.28). We have

$$\log \mathbb{E} e^{t\mathcal{N}_{2n}} = \frac{4n}{\pi}\int_0^1 \log\Big(1-(1-e^{2t})e^{-\alpha^2 s^2}\Big)\sqrt{1-s^2}\, ds + o(1),$$

as $n \to \infty$. By series expansions, we have

$$\log\Big(1-(1-e^{2t})e^{-\alpha^2 s^2}\Big) = \sum_{\ell=1}^{\infty} \frac{t^\ell}{\ell!} 2^\ell \sum_{m=1}^{\ell} (-1)^{m-1}(m-1)! S(\ell,m) e^{-m\alpha^2 s^2}.$$

Thus, using Fubini's theorem and (3.12), we have

$$\log \mathbb{E} e^{t\mathcal{N}_{2n}} = n\sum_{\ell=1}^{\infty} \frac{t^\ell}{\ell!} 2^\ell \sum_{m=1}^{\ell} (-1)^{m-1}(m-1)! S(\ell,m) e^{-m\alpha^2/2}\Big[I_0\Big(\frac{m\alpha^2}{2}\Big) + I_1\Big(\frac{m\alpha^2}{2}\Big)\Big] + o(n),$$

as $n \to \infty$. Then, the conclusion follows by a comparison of coefficients using (3.10). ☐

4. Exponential profiles of the generating functions

In this section, we will make the ansatz (2.31) rigorous and complete the proof of Theorem 2.1.

4.1. **A local Large Deviation Principle.** By the Gärtner-Ellis Theorem, convergence of the log-moment generating function as in Proposition 2.3 implies a Large Deviation Principle (LDP) with speed n or $\sqrt{n}$, respectively (see e.g. [22]). From this viewpoint, (2.31) corresponds to the following local LDP that holds under a convexity assumption of the limit log-Laplace transform $x \mapsto \Psi_{\mathrm{s/w}}(e^x)$.

The following theorem gives a rigorous justification of the ansatz (2.31).

Theorem 4.1 (**Exponential profile of coefficients**). *Consider a sequence of polynomials $\{P_n(z)\}_{n\in\mathbb{N}}$ of degree n with non-positive real roots and $P_n(1) = 1$, so that we can denote it by*

$$P_n(z) = \sum_{k=0}^{n} a_{n,k} z^k = \prod_{j=1}^{n} \frac{z+\lambda_{n,j}}{1+\lambda_{n,j}}, \qquad \lambda_{n,1},\dots,\lambda_{n,n} \geq 0. \tag{4.1}$$

Suppose there is a speed $c_n \to \infty$ as $n \to \infty$, such that

$$\Psi(z) := \lim_{n\to\infty} \frac{1}{c_n} \log P_n(z), \quad z \in (0,\infty) \tag{4.2}$$

exists (which is automatically smooth) and $\Phi(z) := z\Psi'(z)$ is strictly increasing. Define $\underline{m} := \lim_{x\to 0+} \Phi(x)$, $\overline{m} := \lim_{x\to\infty} \Phi(x)$ and the exponential profile

$$g(x) := \Psi(\Phi^{-1}(x)) - x \log \Phi^{-1}(x) = -\sup_{t\in\mathbb{R}}\{tx - \Psi(e^t)\}, \tag{4.3}$$

which is a smooth, strictly concave function on $(\underline{m},\overline{m})$. Then, for any $[a,b] \subset (\underline{m},\overline{m})$, we have

$$\sup_{k\in\{0,\dots,n\}\cap[ac_n,bc_n]} \left| \frac{\log a_{n,k}}{c_n} - g\Big(\frac{k}{c_n}\Big)\right| \to 0, \qquad \text{as } n \to \infty. \tag{4.4}$$

Remark 5.

(1) Note that Φ is strictly increasing, if and only if $\Psi(e^x)$ is convex. By the Nevanlinna-Pick representation (see e.g. Theorem 10 in [59, Chapter 3]), Φ is the Stieltjes transform of some (not probability) measure μ_∞ with mass $\overline{m}$ on $\mathbb{R}$. It is the vague limit of the empirical zero measure

$$\mu_n := \frac{1}{c_n}\sum_{j=1}^n \delta_{-\lambda_{n,j}} \longrightarrow \mu_\infty \quad \text{as } n\to\infty. \tag{4.5}$$

Hence, strict monotonicity of Φ is equivalent to $\mu_\infty \neq c\delta_0$. Similarly, Ψ can be viewed as the logarithmic potential of μ_∞.

(2) It is implicit that $c_n = O(n)$. Indeed, if $c_n \gg n$, then $\Psi'(z) = 0$ for any $z > 0$. Thus strict monotonicity of Φ cannot be achieved.

Remark 6. The proof of Theorem 4.1 follows the approach of [47, Proof of Theorem 5.1], wherein the real rootedness of P_n is crucial in order to represent P_n as the generating function of sums of Bernoulli random variables. Let us comment on how Theorem 4.1 may be of independent interest as a generalization of the main tool in [47,48]. Therein, it is shown that the following two statements are equivalent:

(a) The coefficients $a_{n,k}$ of P_n have an exponential profile g, in the sense that $\frac{1}{n}\log a_{n,k} - g(k/n) \to 0$ as $n\to\infty$, uniformly in $(\underline{m}+\varepsilon)n \le k \le (\overline{m}-\varepsilon)n$ as in (4.4).

(b) The empirical zero distribution of the polynomial P_n converges weakly to some distribution μ on $[-\infty, 0]$, such that $\Phi(z) = \int_{(-\infty,0]} \frac{z}{z-y}\mu(dy)$ for $z > 0$ is the inverse of $\alpha \mapsto e^{-g'(\alpha)}$.

This corresponds to $c_n = n$ in Theorem 4.1, which extends to the case where also a negligible part $c_n = o(n)$ of the coefficients may be considered with a different scaling. This corresponds to zooming into the empirical distribution around 0, while forgetting the majority of zeros that drift away to $-\infty$. Indeed, if the above μ is not $\delta_{-\infty}$, then the conditions of Theorem 4.1 fail for slow speeds $c_n = o(n)$. Thus, we can obtain moderate deviation asymptotics for these coefficients, *because* the zero distribution of P_n drifts to $-\infty$ reasonably fast.

Note that this is the case for the characteristic polynomial (3.1), whose zeros are $1 - 1/\lambda_j$ for $\lambda_j \in (0,1)$ being the eigenvalues of M_n. Thus, it is crucial that most eigenvalues of M_n are close to 0.

Proof of Theorem 4.1. Let us first show that Ψ is holomorphic on $D = \{z\in\mathbb{C} : \operatorname{Re} z > 0\}$. Since P_n has only non-positive real roots and using the empirical measure (4.5), we write

$$\Psi_n(z) := \frac{1}{c_n}\log P_n(z) = \int_{-\infty}^0 \log\Big(\frac{z-y}{1-y}\Big)\mu_n(dy) \to \Psi(z)$$

for any $z\in(0,\infty)$, as $n\to\infty$ and where Ψ_n is holomorphic on D. Note that the convergence $\Psi_n \to \Psi$ of convergent monotonic functions $\Psi_n(z)$ is locally uniform in $(0,\infty)$, but also

$$\Psi_n'(z) = \frac{P_n'(z)}{c_nP_n(z)} = \int_{-\infty}^0 \frac{1}{z-y}\mu_n(dy)$$

are monotonic in $z\in(0,\infty)$. Hence for any $z\in D_\delta := \{z\in\mathbb{C} : \operatorname{Re} z \ge \delta\}$, $\delta > 0$, we have a uniform bound

$$|\Psi_n'(z)| \le \Psi_n'(\delta) \le \frac{1}{\delta/2}\int_{\delta/2}^{\delta} \Psi_n'(z)dz \to \frac{2}{\delta}(\Psi(\delta) - \Psi(\delta/2)) < \infty.$$

This implies a locally uniform bound for Ψ_n on compact subsets of D_δ by

$$|\Psi_n(z)| \le |\Psi_n(\delta)| + \int_\delta^z |\Psi_n'(z)|dz < c_\delta(1 + |z-\delta|) < \infty.$$

Therefore, Montel's theorem (or, Vitali-Porter) implies that Ψ_n converges locally uniformly to a holomorphic limit on D_δ that coincides with Ψ on (δ,∞) for all $\delta > 0$, hence we call it Ψ. Similarly, we have locally uniform convergence of convergent (completely) monotonic functions: for $z\in(0,\infty)$,

$$\Phi_n(z) := z\Psi_n'(z) = \frac{z}{c_n}\frac{P_n'(z)}{P_n(z)} = \int_{-\infty}^0 \frac{z}{z-y}\mu_n(dy) \to \Phi(z) := z\Psi'(z),$$

as $n \to \infty$. By strict monotonicity and continuity of Φ, the inverse $\Phi^{-1} : (\underline{m}, \overline{m}) \to (0, \infty)$ is well-defined and continuous. Again by locally uniform convergence of convergent monotonic functions, the convergence $\Phi_n^{-1} \to \Phi^{-1}$ is locally uniform in $(\underline{m}, \overline{m})$. Since $-g$ is the Legendre transform of the strict convex function $\Psi(e^x)$ on $x \in \mathbb{R}$, it follows that g is automatically strictly concave and smooth.

We are now ready to establish the local LDP. Let $S_n := \sum_{k=1}^n B_{n,k}$, where $B_{n,k} \overset{d}{=} \mathrm{Ber}(\frac{1}{1+\lambda_{n,k}})$ are independent Bernoulli random variables for $k, n \in \mathbb{N}$ with $k \le n$. By the representation (4.1), we have $P_n(z) = \mathbb{E}z^{S_n}$ and $a_{n,k} = \mathbb{P}(S_n = k)$.

First, we tilt the probability measure $\mathbb{P}$. More specifically, for a parameter $\theta > 0$, we define a probability measure $\widetilde{\mathbb{P}}_\theta$ induced from $\mathbb{P}$ by

$$\widetilde{\mathbb{E}}_\theta h(B_{n,1}, \dots, B_{n,n}) = \frac{\mathbb{E}[h(B_{n,1}, \dots, B_{n,n})\theta^{S_n}]}{P_n(\theta)} ,$$

for any bounded function $h : \{0,1\}^n \to \mathbb{R}$. Substituting $h(x_1, \dots, x_n) = \mathbf{1}_{\{x_1+\cdots+x_n=k\}}$, we have

$$\mathbb{P}(S_n = k) = P_n(\theta)\theta^{-k}\widetilde{\mathbb{P}}_\theta(S_n = k).$$

Equivalently, we have

$$\frac{1}{c_n} \log \mathbb{P}(S_n = k) = -\frac{k}{c_n} \log \theta + \frac{1}{c_n} \log P_n(\theta) + \frac{1}{c_n} \log \widetilde{\mathbb{P}}_\theta(S_n = k). \tag{4.6}$$

Our goal is to tune the parameter $\theta \approx \Phi^{-1}(k/c_n)$ so that the first two terms of (4.6) give the desired result $g(x) = \Psi(\Phi^{-1}(x)) - x \log \Phi^{-1}(x)$ and the last term will be negligible.

Let $b > a > 0$ such that $[a, b] \subset (\underline{m}, \overline{m})$ is some compact interval, then for sufficiently large n,

$$\theta_*(n, k) := \Phi_n^{-1}\Big(\frac{k}{c_n}\Big)$$

is well-defined for any $k \in \{0, \dots, n\} \cap [ac_n, bc_n]$. By locally uniform convergence of $\Phi_n^{-1} \to \Phi^{-1}$, we have

$$\sup_{k\in\{0,\dots,n\}\cap[ac_n,bc_n]} \left| \theta_*(n, k) - \Phi^{-1}(\tfrac{k}{c_n}) \right| \xrightarrow{n\to\infty} 0. \tag{4.7}$$

It follows from locally uniform convergence of $\Psi_n \to \Psi$ that

$$\sup_{k\in\{0,\dots,n\}\cap[ac_n,bc_n]} \left| \frac{1}{c_n} \log P_n(\theta_*(n, k)) - \Psi\Big(\Phi^{-1}(\tfrac{k}{c_n})\Big) \right| \xrightarrow{n\to\infty} 0, \tag{4.8}$$

and similarly,

$$\sup_{k\in\{0,\dots,n\}\cap[ac_n,bc_n]} \left| \frac{k}{c_n} \log \theta_*(n, k) - \frac{k}{c_n} \log \Phi^{-1}(\tfrac{k}{c_n}) \right| \xrightarrow{n\to\infty} 0. \tag{4.9}$$

In particular, the first two terms of (4.6) approximate $g(k/c_n)$.

By definition, $\widetilde{\mathbb{P}}_\theta(B_{n,j} = 1) = \theta/(\theta + \lambda_{n,j})$. Thus for $\theta_* \equiv \theta_*(n, k)$, we have

$$\widetilde{\sigma}_{n,k}^2 := \widetilde{\mathrm{Var}}_{\theta_*}(S_n) = \sum_{j=1}^n \frac{\lambda_{n,j}\theta_*}{(\theta_* + \lambda_{n,j})^2} = c_n\theta_* \int_{-\infty}^0 \frac{-y}{(\theta_* - y)^2} \mu_n(dy) = c_n\theta_*\Phi_n'(\theta_*).$$

We claim that

$$0 < \liminf_{n\to\infty} \inf_k \frac{1}{c_n} \widetilde{\sigma}_{n,k}^2 \le \limsup_{n\to\infty} \sup_k \frac{1}{c_n} \widetilde{\sigma}_{n,k}^2 < \infty, \tag{4.10}$$

where again $k \in \{0, \dots, n\} \cap [ac_n, bc_n]$ for $[a, b] \subset (\underline{m}, \overline{m})$. The claim follows from the following argument.

(1) By locally uniform convergence of $\Phi_n^{-1} \to \Phi^{-1}$, we have $\theta_*(n, k) \in \Phi^{-1}([a/2, 2b])$ for sufficiently large $n \in \mathbb{N}$. Thus, for sufficiently large n, the value $\theta_*(n, k)$ is bounded away from 0 and ∞ for all $k \in \{0, \dots, n\} \cap [ac_n, bc_n]$.

(2) We now show the upper bound in (4.10). An elementary bound for integrands yields

$$\int_{-\infty}^{0} \frac{-y}{(\theta_* - y)^2} \mu_n(dy) \le \int_{-\infty}^{-1} \frac{1}{\theta_* - y} \mu_n(dy) + \frac{1}{\theta_*} \int_{-1}^{0} \frac{1}{\theta_* - y} \mu_n(dy) \le \max\left\{1, \frac{1}{\theta_*}\right\} \Phi_n(\theta_*).$$

Then the upper bound follows from the convergence of $\Phi_n \to \Phi$ and (4.7).

(3) We show the lower bound in (4.10). Since $\Phi_n(z)$ and $\Phi(z)$ are strictly increasing in $z > 0$ and $\Phi_n' \to \Phi'$ locally uniformly, we observe that

$$\liminf_{n\to\infty} \inf_k \Phi_n'(\theta_*) \ge \inf_{x \in [a/2, 2b]} \Phi'(x) > 0.$$

These prove the claim (4.10).

Since $c_n \to \infty$ as $n \to \infty$, the inequalities (4.10) implies $\tilde{\sigma}_{n,k} \to \infty$ for $k \in \{0, \dots, n\} \cap [ac_n, bc_n]$ as $n \to \infty$. Due to the divergence, we can apply the local limit theorem [10, Theorem 2] to $\widetilde{P}_{\theta_*(n,k)}(x) := \widetilde{\mathbb{E}}_{\theta_*(n,k)} x^{S_n}$, hence

$$\sup_{x\in\mathbb{R}} \left| \widetilde{\sigma}_{n,k} \widetilde{\mathbb{P}}_{\theta_*(n,k)} \left[S_n = \lfloor \widetilde{\sigma}_{n,k} x \rfloor + k \right] - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| \xrightarrow{n\to\infty} 0$$

for $x \in \mathbb{R}$. Substituting $x = 0$, we obtain

$$\widetilde{\mathbb{P}}_{\theta_*(n,k)}[S_n = k] \sim \frac{1}{\sqrt{2\pi}\widetilde{\sigma}_{n,k}}.$$

Thus, we conclude that

$$\sup_{k\in\{0,\dots,n\}\cap[ac_n, bc_n]} \left| \frac{1}{c_n} \log \widetilde{\mathbb{P}}_{\theta_*(n,k)}(S_n = k) \right| \xrightarrow{n\to\infty} 0. \tag{4.11}$$

Putting (4.7), (4.8), and (4.11) into (4.6), we obtain (4.4), since $a_{n,k} = \mathbb{P}(S_n = k)$. ☐

4.2. **Proof of Theorem 2.1.** We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We first prove Theorem 2.1 (i), the strong asymmetry regime. Choose $c_n = \sqrt{2n}$ and let

$$\Psi_n(z) := \frac{1}{c_n} \log\Big(\sum_{k=0}^{n} z^k p_{2n,2k} \Big), \qquad \Phi_n(z) := z\Psi_n'(z).$$

By Proposition 2.3, the sequences $\Psi_n(z)$ converge to $\Psi_{\mathrm{s}}(z)$ in $\mathbb{C} \setminus (-\infty, 0]$. Using the definitions (2.4) and (2.8), it follows that Ψ_{s} and Φ_{s} satisfy the conditions in Theorem 4.1 with $\underline{m} = 0$ and $\overline{m} = \infty$. Indeed, by twice differentiation of (2.12) it follows analogously to (3.9) that

$$\Big(\frac{d}{du}\Big)^2 \int_0^\infty \log\Big(1 - (1 - e^u)e^{-t^2}\Big)\, dt = \int_0^\infty \frac{e^u e^{-t^2}(1 - e^{-t^2})}{(1 - (1 - e^u)e^{-t^2})^2}\, dt > 0,$$

hence $\Psi_{\mathrm{s}}(e^u)$ is strictly convex on $(0, \infty)$ and Φ is strictly monotonous. Then, the conclusion of Theorem 2.1 (i) directly follows from Theorem 4.1 with $\phi_{\mathrm{s}} = -g$.

For the proof of the weak asymmetry regime, Theorem 2.1 (ii), we choose $c_n = 2n$. Then, the above arguments again work after replacing Ψ_{s}, Φ_{s}, and $\overline{m} = \infty$ by Ψ_{w}, Φ_{w}, and $\overline{m} = 1$, respectively and recalling that convexity of $\Psi_{\mathrm{w}}(e^z)$ follows from (3.9). This finishes the proof. ☐

5. Conclusion

In this paper, we have investigated the statistics of the total number of real eigenvalues of random matrices belonging to the elliptic real Ginibre ensemble of size $n \times n$. We have analyzed both the weak asymmetry regime (corresponding to $1 - \tau = O(1/n)$) and the strong asymmetry regime (for fixed $\tau \in [0, 1)$). In both cases, we derived the asymptotic form of the probability $p_{n,m}$ that a matrix possesses exactly m real eigenvalues in the large n limit.

In the weak asymmetry regime, we focused on the scaling $m = O(n)$ and showed that

$$p_{n,m} \sim \exp[-n\, \phi_{\mathrm{w}}(m/n; \alpha)]\,,$$

where the rate function $\phi_{\mathrm{w}}(x; \alpha)$ was computed explicitly. This describes the full large deviation behaviour in this regime and, in particular, contains as a special case the Gaussian fluctuations corresponding to the minimum of $\phi_{\rm w}(x;\alpha)$. In the strong asymmetry regime, we instead considered the scaling $m = O(\sqrt{n})$ and obtained

$$p_{n,m} \sim \exp[-\sqrt{n}\,\phi_{\rm s}(m/\sqrt{n};\tau)]\,,$$

as stated in Theorem 2.1(i). Here again, the minimum of the rate function $\phi_{\rm s}(x;\tau)$ corresponds to typical Gaussian fluctuations. However, in contrast to the strongly asymmetric case, this regime coexists with an additional large deviation regime $m = O(n)$, previously studied in [61] using a Coulomb gas approach. Understanding precisely how these two regimes match and interpolate remains a challenging open problem.

Our results naturally give rise to several further questions. First, while we have focused on the total number of real eigenvalues along the entire real axis, it would be interesting to study the statistics of the number of real eigenvalues contained in a finite interval $[a,b] \subset \mathbb{R}$. One could, for example, examine how the cumulants of this local counting variable depend on the length or position of the interval; see [3, 29] for a related discussion in the symmetric interval case. Second, it would be natural to explore the joint statistics of real and complex eigenvalues—for instance, the number of complex eigenvalues lying in a region of the complex plane that intersects the real axis.

Finally, our analysis relied on extending the recent techniques developed in [47,48] to extract large deviation form from the analysis of associated generating functions. It would be interesting to explore whether similar methods can be applied to study large deviations and rate functions in other models of integrable probability. For instance, large-deviation forms akin to (2.31) were conjectured for the distribution of the number of real roots of real random polynomials [63, 65, 66]. Notably, several parallels have been observed [66] between the real roots of the so-called real Weyl polynomials and the real eigenvalues of the real Ginibre ensemble analyzed here. It would therefore be natural to investigate whether the methods developed in the present work can be extended to the study of real random polynomials and related problems [63].

Acknowledgements. SSB was supported by the National Research Foundation of Korea grants (RS-2023-00301976, RS-2025-00516909). JJ was supported by the DFG priority program SPP 2265 Random Geometric Systems. GS was supported by ANR Grant No. ANR-23-CE30-0020-01 EDIPS. We thank Peter Forrester and Mihail Poplavskyi for useful discussions.

References

[1] G. Akemann and S.-S. Byun, *The Product of m real $N \times N$ Ginibre matrices: Real eigenvalues in the critical regime $m = O(N)$*, Constr. Approx. **59** (2024), 31–59.

[2] G. Akemann, S.-S. Byun and M. Ebke, *Universality of the number variance in rotational invariant two-dimensional Coulomb gases*, J. Stat. Phys. **190** (2023), 1–34.

[3] G. Akemann, S.-S. Byun, M. Ebke and G. Schehr, *Universality in the number variance and counting statistics of the real and symplectic Ginibre ensemble*, J. Phys. A **56** (2023), 495202.

[4] G. Akemann, S.-S. Byun and Y.-W. Lee, *The probability of almost all eigenvalues being real for the elliptic real Ginibre ensemble*, Nonlinearity **38** (2025), 105015.

[5] G. Akemann and E. Kanzieper, *Integrable structure of Ginibre's ensemble of real random matrices and a Pfaffian integration theorem*, J. Stat. Phys. **129** (2007), 1159–1231.

[6] R. Allez, J. Touboul and G. Wainrib, *Index distribution of the Ginibre ensemble*, J. Phys. A **47** (2014), 042001.

[7] Y. Ameur, C. Charlier, J. Cronvall and J. Lenells, *Disk counting statistics near hard edges of random normal matrices: the multi-component regime*, Adv. Math. **441** (2024), 109549.

[8] G. Ben Arous, Y. V. Fyodorov and B. A. Khoruzhenko. *Counting equilibria of large complex systems by instability index*, Proc. Natl. Acad. Sci. USA **118** (2021), e2023719118.

[9] C. M. Bender and S. Boettcher, *Real spectra in non-Hermitian Hamiltonians having PT symmetry*, Phys. Rev. Lett. **80** (1998), 5243.

[10] E. Bender, *Central and local limit theorems applied to asymptotic enumeration*, J. Combinatorial Theory Ser. A **15** (1973), 91–111.

[11] S.-S. Byun and P. J. Forrester, *Progress on the study of the Ginibre ensembles*, Springer Singapore, KIAS Springer Series in Mathematics (2025).

[12] S.-S. Byun and P. J. Forrester, *Electrostatic computations for statistical mechanics and random matrix applications*, arXiv:2510.14334.

[13] S.-S. Byun, N.-G. Kang, J. O. Lee and J. Lee, *Real eigenvalues of elliptic random matrices*, Int. Math. Res. Not. **2023** (2023), 2243–2280.

[14] S.-S. Byun and Y.-W. Lee, *Finite size corrections for real eigenvalues of the elliptic Ginibre matrices*, Random Matrices Theory Appl. **13** (2024), 2450005.
[15] S.-S. Byun, L. D. Molag and N. Simm, *Large deviations and fluctuations of real eigenvalues of elliptic random matrices*, Electron. J. Probab. **30** (2025), 40 pp.
[16] S.-S. Byun and K. Noda, *Real eigenvalues of asymmetric Wishart matrices: Expected number, global density and integrable structure*, arXiv:2503.14942.
[17] S.-S. Byun and S. Park, *Large gap probabilities of complex and symplectic spherical ensembles with point charges*, J. Funct. Anal. **290** (2026), 111260.
[18] B. Derrida and A. Gerschenfeld, *Current fluctuations of the one dimensional symmetric simple exclusion process with step initial condition*, J. Stat. Phys. **136** (2009), 1.
[19] C. Charlier, *Asymptotics of determinants with a rotation-invariant weight and discontinuities along circles*, Adv. Math. **408** (2022), 108600.
[20] C. Charlier, *Large gap asymptotics on annuli in the random normal matrix model*, Math. Ann. **388** (2024), 3529–3587.
[21] G. Cipolloni, L. Erdős, D. Schröder and Y. Xu, *Directional extremal statistics for Ginibre eigenvalues*, J. Math. Phys. **63** (2022), 103303.
[22] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, corrected reprint of the second edition, Springer, Berlin (2010).
[23] A. Edelman, *The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law*, J. Multivariate Anal. **60** (1997), 203–232.
[24] A. Edelman, E. Kostlan and M. Shub, *How many eigenvalues of a random matrix are real?* J. Amer. Math. Soc. **7** (1994), 247–267.
[25] K. B. Efetov, *Directed quantum chaos*, Phys. Rev. Lett. **79** (1997), 491.
[26] W. FitzGerald and N. Simm, *Fluctuations and correlations for products of real asymmetric random matrices*, Ann. Inst. Henri Poincaré Probab. Stat. **59** (2023), 2308–2342.
[27] P. J. Forrester, *Log-gases and random matrices*, Princeton University Press, Princeton, NJ (2010).
[28] P. J. Forrester, *Probability of all eigenvalues real for products of standard Gaussian matrices*, J. Phys. A **47** (2014), 065202.
[29] P. J. Forrester, *Diffusion processes and the asymptotic bulk gap probability for the real Ginibre ensemble*, J. Phys. A **48** (2015), 324001.
[30] P. J. Forrester, *Local central limit theorem for real eigenvalue fluctuations of elliptic GinOE matrices*, Electron. Commun. Probab. **29** (2024), 1–11.
[31] P. J. Forrester, *Asymptotics of the real eigenvalue distribution for the real spherical ensemble*, arXiv:2508.04139.
[32] P. J. Forrester and J. R. Ipsen, *Real eigenvalue statistics for products of asymmetric real Gaussian matrices*, Linear Algebra Appl. **510** (2016), 259–290.
[33] P. J. Forrester, J. R. Ipsen and S. Kumar, *How many eigenvalues of a product of truncated orthogonal matrices are real?*, Exp. Math. **29** (2020), 276–290.
[34] P. J. Forrester and S. Kumar, *The probability that all eigenvalues are real for products of truncated real orthogonal random matrices*, J. Theoret. Probab. **31** (2018) 2056–2071.
[35] P. J. Forrester and A. Mays, *A method to calculate correlation functions for $\beta = 1$ random matrices of odd size*, J. Stat. Phys. **134** (2009), 443–462.
[36] P. J. Forrester and A. Mays, *Pfaffian point processes for the Gaussian real generalised eigenvalue problem*, Prob. Theory and Rel. Fields **154** (2012), 1–47.
[37] P. J. Forrester and T. Nagao, *Eigenvalue statistics of the real Ginibre ensemble*, Phys. Rev. Lett. **99** (2007), 050603.
[38] P. J. Forrester and T. Nagao, *Skew orthogonal polynomials and the partly symmetric real Ginibre ensemble*, J. Phys. A **41** (2008), 375003.
[39] Y. V. Fyodorov, *Topology trivialization transition in random nongradient autonomous ODEs on a sphere*, J. Stat. Mech. **2016** (2016), 124003.
[40] Y. V. Fyodorov, B. A. Khoruzhenko and H.-J. Sommers, *Almost-Hermitian random matrices: crossover from Wigner-Dyson to Ginibre eigenvalue statistics*, Phys. Rev. Lett. **79** (1997), 557–560.
[41] Y. V. Fyodorov, B. A. Khoruzhenko and T. Prellberg, *Zeros of conditional Gaussian analytic functions, random sub-unitary matrices and q-series*, arXiv:2412.06086.
[42] Y. V. Fyodorov, B. A. Khoruzhenko and H.-J. Sommers, *Universality in the random matrix spectra in the regime of weak non-Hermiticity*, Ann. Inst. H. Poincaré Phys. Théor. **68** (1998), 449–489.
[43] B. Garrod, M. Poplavskyi, R. Tribe and O. Zaboronski, *Examples of interacting particle systems on Z as Pfaffian point processes: annihilating and coalescing random walks*, Ann. Henri Poincaré **19** (2018), 3635–3662.
[44] M. Gebert and M. Poplavskyi, *On pure complex spectrum for truncations of random orthogonal matrices and Kac polynomials*, arXiv:1905.03154.
[45] A. Goel, P. Lopatto and X. Xie, *Central limit theorem for the complex eigenvalues of Gaussian random matrices*, Electron. Commun. Probab. **29** (2024), Paper No. 16, 13 pp.
[46] X. Hu and Y. Ma, *Convergence rate of extreme eigenvalue of Ginibre ensembles to Gumbel distribution*, arXiv:2506.04560.
[47] J. Jalowy, Z. Kabluchko and A. Marynych, *Zeros and exponential profiles of polynomials I: Limit distributions, finite free convolutions and repeated differentiation*, arXiv:2504.11593.
[48] J. Jalowy, Z. Kabluchko and A. Marynych, *Zeros and exponential profiles of polynomials II: Examples*, arXiv:2509.11248.

[49] E. Kanzieper and G. Akemann, *Statistics of real eigenvalues in Ginibre's ensemble of random real matrices*, Phys. Rev. Lett. **95** (2005), 230201.

[50] E. Kanzieper, M. Poplavskyi, C. Timm, R. Tribe and O. Zaboronski, *What is the probability that a large random matrix has no real eigenvalues?*, Ann. Appl. Probab. **26** (2016), 2733–2753.

[51] P. Kivimae, *Concentration of equilibria and relative instability in disordered non-relaxational dynamics*, Comm. Math. Phys. **405**, 289 (2024).

[52] B. Lacroix-A-Chez-Toine, J. A. M. Garzón, C. S. H. Calva, I. P. Castillo, A. Kundu, S. N. Majumdar and G. Schehr, *Intermediate deviation regime for the full eigenvalue statistics in the complex Ginibre ensemble*, Phys. Rev. E **100** (2019), 012137.

[53] B. Lacroix-A-Chez-Toine, A. Grabsch, S. N. Majumdar and G. Schehr, *Extremes of 2d Coulomb gas: universal intermediate deviation regime*, J. Stat. Mech. Theory Exp. **2018** (2018), 013203.

[54] B. Lacroix-A-Chez-Toine, S. N. Majumdar and G. Schehr, *Rotating trapped fermions in two dimensions and the complex Ginibre ensemble: Exact results for the entanglement entropy and number variance*, Phys. Rev. A **99** (2019), 021602.

[55] P. Le Doussal and G. Schehr, *Linear statistics at the microscopic scale for the 2D Coulomb gas*, J. Phys. A (Online), arXiv:2503.18586.

[56] A. Little, F. Mezzadri and N. Simm, *On the number of real eigenvalues of a product of truncated orthogonal random matrices*, Electron. J. Probab. **27** (2022), 32 pp.

[57] Y. Ma and X. Meng, *Precise convergence rate of spectral radius of product of complex Ginibre*, arXiv:2510.07942.

[58] R. M. May, *Will a large complex system be stable?*, Nature **238** (1972), 413.

[59] J. A. Mingo and R. Speicher. *Free probability and random matrices.* Vol. 35. New York: Springer (2017).

[60] J. Marzo, L. D. Molag and J. Ortega-Cerdà, *Universality for fluctuations of counting statistics of random normal matrices*, arXiv:2508.04386.

[61] L. C. G. del Molino, K. Pakdaman, J. Touboul and G. Wainrib, *The real Ginibre ensemble with $k = O(n)$ real eigenvalues*, J. Stat. Phys. **163** (2016), 303–323.

[62] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, eds. *NIST Handbook of Mathematical Functions*, Cambridge: Cambridge University Press (2010).

[63] M. Poplavskyi and G. Schehr, *Exact persistence exponent for the 2d-diffusion equation and related Kac polynomials*, Phys. Rev. Lett. **121** (2018), 150601, see Supp. Mat. in https://arxiv.org/abs/1806.11275

[64] K. Rajan and L. F. Abbott, *Eigenvalue spectra of random matrices for neural networks*, Phys. Rev. Lett. **97** (2006), 188104.

[65] G. Schehr and S. N. Majumdar, *Statistics of the number of zero crossings: from random polynomials to the diffusion equation*, Phys. Rev. Lett. **99** (2007), 060603.

[66] G. Schehr and S. N. Majumdar, *Real roots of random polynomials and zero crossing properties of diffusion equation*, J. Stat. Phys. **132** (2008), 235.

[67] C. D. Sinclair, *Averages over Ginibre's ensemble of random real matrices*, Int. Math. Res. Not. **2007** (2007), rnm015, 15 pp.

[68] N. Simm, *Central limit theorems for the real eigenvalues of large Gaussian random matrices*, Random Matrices Theory Appl. **6** (2017), 1750002.

[69] N. Simm, *On the real spectrum of a product of Gaussian matrices*, Electron. Commun. Probab. **22** (2017), 11.

[70] H. J. Sommers, A. Crisanti, H. Sompolinsky and Y. Stein, *Spectrum of large random asymmetric matrices*, Phys. Rev. Lett. **60** (1988), 1895.

[71] Z. Xiao, K. Kawabata, X. Luo, T. Ohtsuki and R. Shindou, *Level statistics of real eigenvalues in non-Hermitian systems*, Phys. Rev. Res. **4** (2022), 043196.

[72] Y. Xu and Q. Zeng, *Large deviations for the extremal eigenvalues of Ginibre ensembles*, preprint.

Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Republic of Korea
Email address: `sungsoobyun@snu.ac.kr`

Institute of Mathematics, Paderborn University, Warburger Str. 100, 33098 Paderborn, Germany
Email address: `jjalowy@math.upb.de`

Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Republic of Korea
Email address: `hellowoo@snu.ac.kr`

Sorbonne Université, Laboratoire de Physique Théorique et Hautes Energies, CNRS UMR 7589, 4 Place Jussieu, 75252 Paris Cedex 05, France
Email address: `schehr@lpthe.jussieu.fr`