SPORADIC POINTS ON $X_0(N)$

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Abstract. We determine all integers N for which the modular curve $X_0(N)$ admits a sporadic CM point (of any degree), as well as all N for which $X_0(N)$ admits a sporadic point, whether CM or non-CM. In a sense, our results generalize the classification of isogenies of elliptic curves over $\mathbb{Q}$ due to Mazur and Kenku: their work determines the $X_0(N)$ with degree 1 sporadic points, whereas we classify all $X_0(N)$ that have a sporadic point of arbitrary degree.

1. Introduction

Following [VV24], for a nice[1] curve X over a field k, we define the *minimum density degree* (over k) $\delta(X/k)$ [2] to be the smallest integer n such that X has infinitely many points of degree n over k. We only consider curves over $\mathbb{Q}$ in this paper, so we write simply $\delta(X) := \delta(X/\mathbb{Q})$. A *sporadic point* $x \in X(\overline{\mathbb{Q}})$ is a point such that $\deg x < \delta(X)$. To avoid repetition, when speaking of sporadic points on modular curves, we will always mean (unless stated otherwise) *non-cuspidal* sporadic points.

Isolated and sporadic points on modular curves have been a subject of considerable recent interest. For a definition of isolated points, see [BEL$^+$19]; the set of sporadic points is a subset of the isolated points on a curve. In general, one does not expect a curve to have any sporadic points at all. In [Naj16] the second author found the first example of a sporadic point on $X_1(N)$ (and the one of lowest possible degree, 3), after which many have been found by van Hoeij [vH]. In [BEL$^+$19] Bourdon, Ejder, Liu, Odumodu and Viray proved that, assuming Serre's uniformity conjecture, there are only finitely many j-invariants $j(x) \in \mathbb{Q}$, with x a sporadic point on $X_1(N)$; this is proved unconditionally for N a prime power by Bourdon and Ejder [BE25, Ejd22].

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[1] smooth, projective, geometrically integral

[2] We follow the notation of [CGPS22] here; note that this is denoted $\min \delta(X/k)$ in [VV24].

Mazur proved in [Maz77] that there are no sporadic points of degree 1 on $X_1(N)$; the same was proved for degree 2 by Kamienny, Kenku and Momose [KF88, Kam92]. In [DEvH$^+$21] Derickx, Etropolski, van Hoeij, Morrow and Zureick-Brown proved that the sporadic points on $X_1(21)$ found in [Naj16] are the only degree 3 sporadic points on $X_1(N)$. The authors of this paper proved in [DN24a] that there are no degree 4 sporadic points on $X_1(N)$ and ask whether there exist sporadic points on $X_1(N)$ of every degree $d > 4$. van Hoeij's examples from [vH] give sporadic points of degree $5 \leq d \leq 13$ (see [DN24a, Appendix A]). Clark, Genao, Pollack and Saia showed in [CGPS22] that all $X_1(N)$ for $N \geq 721$ have sporadic points. For further results on isolated and sporadic points on $X_1(N)$ and $X_0(N)$, see [BGRW24, BHK$^+$25, BN21, Ter24].

The aim of this paper is to study sporadic points on the modular curves $X_0(N)$. In [DN24a, Appendix A], the authors showed that there exist sporadic points on $X_0(N)$ (for varying N) of all degrees $d \leq 2166$, and conjectured that sporadic points exist on $X_0(N)$ for every degree d. In this work, we investigate for which values of N the curve $X_0(N)$ admits a sporadic point (of any degree).

Elliptic curves with complex multiplication (CM) are a common source of sporadic points on $X_0(N)$. In [CGPS22, Theorem 8.2.] the authors find 50 values of N with no sporadic CM points on $X_0(N)$, 106 values for which they could not determine whether there are CM sporadic points, and prove that there are sporadic CM points for all other N. The first of the main results of this paper determines for the remaining 106 values of N whether $X_0(N)$ has a sporadic CM point.

Theorem 1.1. *The modular curve $X_0(N)$ has a sporadic CM point if and only if*

$$N \notin S_{CM} := \{1-13, 15-18, 20-26, 30-33, 35-37, 39-41, 46-50, \\ 53, 59-61, 65, 70-72, 79, 80, 83, 87, 89, 94, 96, 101, 131, 144\}$$

If a curve has a sporadic CM point, it obviously has a sporadic point. However, even for curves without a sporadic CM point, it is still possible that they have a sporadic non-CM point. Our second main result determines all modular curves $X_0(N)$ with sporadic points, whether CM or non-CM.

Theorem 1.2. *The modular curve $X_0(N)$ has a sporadic point if and only if*

$$N \notin (S_{CM} \backslash \{15, 17, 21, 37\}) .$$

Our results can, in a sense, be viewed as a generalization of the classification of isogenies of elliptic curves over $\mathbb{Q}$ by Mazur and Kenku (see [Ken81, Maz78]): in those papers, the modular curves $X_0(N)$ with degree-1 sporadic points are determined, whereas here we classify the curves $X_0(N)$ that have a sporadic point of arbitrary degree.

We split the proof of Theorem 1.1 by the least degree of a CM point on $X_0(N)$; the proofs are given in Section 4. The reason that the authors of

[CGPS22] were unable to determine whether certain $X_0(N)$ have sporadic CM points is that the values of $\delta(X_0(N))$ were not known, and the available bounds on them were not good enough to determine whether CM points produce sporadic points. In fact, this turns out to be quite difficult. One of the tools we use here are new bounds on $\delta(X)$ that we obtain using recent results of Kadets and Vogt [KV25] on low-degree points on curves. These results say that for $\delta(X) = d$ to be relatively small, the curve has to have either a degree d map to $\mathbb{P}^1$ or a positive rank elliptic curve, or a very small degree map to a curve of very small genus (see Theorem 3.1). To rule out the existence of maps of degree d to elliptic curves of positive rank we apply recent methods of [DO24]. To rule out degree 2 maps from $X_0(N)$ to small genus curves, we describe involutions of $X_0(N)$ in detail in Section 3.1.

In Section 5, we prove Theorem 1.2. To do so, we must show that every $X_0(N)$ without sporadic CM points also has no sporadic non-CM points. This is achieved essentially by a Mordell-Weil sieve. Although the broad underlying idea is simple (and standard), the implementation is technically very challenging. In particular, it required determining all the degree ≤ 5 points on $X_0(144)$, a feat that, to the best of our knowledge, has not been accomplished for $X_0(N)$ in degree greater than 3. Indeed, more than half of the total effort on this paper was spent on various attempts to prove Theorem 5.3 and Theorem 5.5. Interestingly, the 4 modular curves $X_0(N)$ that have sporadic non-CM points, but no sporadic CM points, have *rational* non-CM sporadic points.

1.1. **Data availability and reproducibility.** All the code for the computations in this article is available at:

`https://github.com/nt-lib/X0N-sporadic`

We used `magma` [BCP97] and `SageMath` [The23].

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2. Previous work

Let U be the set of integers N for which it was not determined in [CGPS22, Theorem 8.2] whether there exists a sporadic CM point $X_0(N)$. Following [CGPS22] define $d_{CM}(X_0(N))$ to be the least degree of a CM point on $X_0(N)$.

We split the set as

$$U = U_4 \cup U_6 \cup U_8 \cup U_{12},$$

where U_d is the subset of U consisting of N such that $d_{CM}(X_0(N)) = d$. The sets U_d are as follows:

$$\begin{aligned}
U_4 := \{&60, 70, 72, 80, 87, 90, 94, 96, 105, 108, 110, 120, 126, 132, 138, 150, 152,\\
&160, 168, 174, 188, 190, 192, 195, 204, 208, 210, 216, 230, 231, 234, 236,\\
&238, 240, 248, 255, 261, 264, 270, 272, 276, 282, 285, 286, 287, 303, 304,\\
&310, 312, 315, 318, 320, 324, 332, 334, 336, 350, 357, 376, 380, 384, 392,\\
&395, 400, 413, 416, 429, 430, 435, 447, 455, 476, 483, 496, 501, 519, 524,\\
&535, 591, 623, 635\},\\
U_6 := \{&140, 144, 180, 220, 252, 280, 288, 300, 348, 426, 432, 468, 472, 558, 560,\\
&572, 576\},\\
U_8 := \{&360, 420, 440, 504, 528, 600, 672\},\\
U_{12} := \{&720\}.
\end{aligned}$$

The values of N where the authors of [CGPS22] determine there are no sporadic CM-points (see [CGPS22, Table 2]) are

$$\begin{aligned}
\{&1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 28, 29,\\
&30, 31, 32, 33, 35, 36, 37, 39, 40, 41, 46, 47, 48, 49, 50, 53, 59, 61, 65, 71, 79, 83,\\
&89, 101, 131\}.
\end{aligned}$$

For all these values except

$$N \in \{15, 17, 21, 37\} \tag{2.1}$$

we also know there are no sporadic points without CM on these curves, since these all either have genus 0, or have infinitely many degree 2 points by [Bar99, Theorem 4.3] and no rational non-cuspidal points (see [Maz78, Ken81] and the references therein). The curves $X_0(N)$ for $N \in \{15, 17, 21, 37\}$ all have rational non-cuspidal non-CM points (and finitely many of them!), hence do have sporadic non-cuspidal points.

3. Tools

In this section we recall some results from other work that we will use. One of the main tools we use for the proof of Theorem 1.1 is the following result of Kadets and Vogt [KV25].

Theorem 3.1 ([KV25, Theorem 1.3.])**.** *Suppose X/k is a nice curve of genus g and $\delta(X) = d$. Let $m := \lceil d/2 \rceil - 1$ and let $\epsilon := 3d - 1 - 6m$. Then one of the following holds:*

1. *There exists a nonconstant morphism of curves $\phi : X \to Y$ of degree at least 2 such that $d = \delta(Y/k) \cdot \deg \phi$;*
2. *The genus of X is bounded by*

$$g \leq \max\left(\frac{d(d-1)}{2} + 1, 3m(m-1) + m\epsilon\right).$$

In (2) we get $g \leq 7, 16, 33$ for $d = 4, 6, 8$, respectively.

We will need *Abramovich's bound* [Abr96, Theorem 0.1] (see [DN24a, Theorem 4.1] for the constant $\frac{325}{2^{15}}$): for a modular curve X_H corresponding to a modular group $H \leq \mathrm{PSL}_2(\mathbb{Z})$ we have

$$\operatorname{gon}_{\mathbb{C}} X_H > \frac{325}{2^{15}}[\mathrm{PSL}_2(\mathbb{Z}) : H]. \tag{3.1}$$

By [Fre94, Proposition 2], it follows that for every $x \in X(\mathbb{Q})$ that satisfies

$$\deg_k x < \frac{1}{2} \operatorname{gon}_{\mathbb{C}} X_H.$$

is sporadic. Note that $\operatorname{gon}_{\mathbb{C}} X \leq \operatorname{gon}_{\mathbb{Q}} X$ for any curve $X/\mathbb{Q}$.

Recall that the modular degree of an elliptic curve E of conductor M is the minimal degree of a morphism $X_0(M) \to E$. Furthermore, if N is an integer for which there exists a nonconstant map $X_0(N) \to E$, then the conductor of E has to divide N.

Lemma 3.2. *Let $N < 778$ and $f : X_0(N) \to E$ be a morphism to an elliptic curve E of positive rank over $\mathbb{Q}$. Then the conductor of E divides N and the modular degree of E divides the degree of f.*

Proof. If E has conductor N, this follows from the definition of the modular degree. If the conductor of E is smaller than N, this follows from [DHJO25, Proposition 3.6 and Proposition 3.7] which were proven using the techniques of [DO24]. □

3.1. **Computing genera of quotients of $X_0(N)$.** The proofs that CM points on $X_0(N)$ are sporadic in Theorem 4.5, Theorem 4.8, and Theorem 4.10 rely on the fact that, for $N \in U_6$, U_8, or U_{12}, the curves $X_0(N)$ admit no quotient by an involution of small genus. This condition is necessary in order to rule out the possibility that we are in case (2) of Theorem 3.1, or, more precisely, in a specialization of that theorem to the relevant degrees, as stated in Theorem 4.4, Theorem 4.7, and Theorem 4.9.

For a curve X over the complex numbers we have $g(X) = 2 \dim H_1(X, \mathbb{Z})$. If $\iota \in \operatorname{Aut} X$ is an automorphism, then the genus of X/ι can be computed in terms of the action of ι on $\dim H_1(X, \mathbb{Q})$ as follows:

$$g(X/\iota) = 2 \dim H_1(X/\iota, \mathbb{Q}) = 2 \dim H_1(X, \mathbb{Q})^{\iota}.$$

By the universal coefficient theorem and since $H_1(X, \mathbb{Z})$ is torsion-free for complex curves, one has $H_1(X, \mathbb{Q}) \cong H_1(X, \mathbb{Z}) \otimes \mathbb{Q}$. If $X = X_0(N)(\mathbb{C})$, then $H_1(X, \mathbb{Z})$ can be described in terms of the space of modular symbols $\mathbb{S}_2(\Gamma_0(N))$, as originally proved in [Man73, Theorem 1.9]. Algorithms for computing a basis of $\mathbb{S}_2(\Gamma_0(N))$ are described in [Ste07, §3], whose notation we follow. Moreover, if the automorphism ι is induced by an element $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ in the normalizer of $\Gamma_0(N)$, then the action of γ on a basis of $\mathbb{S}_2(\Gamma_0(N); \mathbb{Z})$ can be computed as described in [Ste07, §3.2].

Fortunately, with a few exceptions, the automorphisms of $X_0(N)$ are induced by elements $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. More precisely, let $\mathrm{Norm}_{\mathrm{GL}_2^+(\mathbb{Q})}(\Gamma_0(N))$ denote the normaliser of $\Gamma_0(N)$ in $\mathrm{GL}_2(\mathbb{Q})$ and set

$$B_0(N) := \mathrm{Norm}_{\mathrm{GL}_2^+(\mathbb{Q})}(\Gamma_0(N))/\Gamma_0(N).$$

By [KM88, Theorem 0.1] (see also [Har11] for a correction in the case $N = 108$) we have $\mathrm{Aut}\, X_0(N)(\mathbb{C}) \simeq B_0(N)$ for $N \neq 37, 63, 108$.

In [Bar08, Theorem 11] an explicit set of generators is given for $B_0(N)$[3]. We compute the genera of the quotients of $X_0(N)$ by computing the action of $\mathrm{Aut}\, X_0(N)(\mathbb{C})$ on $\mathbb{S}_2(\Gamma_0(N))$ and taking invariants. The code for this is available at `involutions.ipynb` in our repository.

Remark 3.3. An explicit description of $B_0(N)$ in terms of generators and relations was claimed in [AL70, Theorem 8] without proof, as a correction of [LN64, Theorem 3]. However, as observed in [AS90, Bar08], Theorem 8 of Atkin and Lehner is incorrect, since some of the relations given there do not hold. The two papers [AS90, Bar08] independently provide corrected descriptions of $B_0(N)$. Nevertheless, [AS90] itself contains an inaccuracy that causes the result there to be incorrect in the subcase where $v_2(N) = 8$; see [Bar08, p. 2170].

The results of [Bar08] are not complete, as mentioned in Observation 17 of that article. In particular, when $v_2(N) \geq 5$ the set of relations listed is not guaranteed to be complete.

Remark 3.4. As noted above, there are multiple incorrect statements in the literature regarding the structure of the automorphism group of $X_0(N)$. So we briefly explain why the main results of this article are nonetheless valid. Our reasoning relies on [Har11] (unpublished) having addressed all errors in [Ken77] and on the correctness of [Bar08, Theorem 11]. We only need the aforementioned results for the finite list of values N in $(U_6 \setminus \{144\}) \cup U_8 \cup U_{12}$.

Should further mistakes in the description of $\mathrm{Aut}\, X_0(N)(\mathbb{C})$ for these values of N come to light, the main results of this paper could be readily adjusted by recomputing the relevant cases with the corrected automorphism group.

Remark 3.3 shows that there are at least four incorrect statements in the literature concerning the structure of $\mathrm{Aut}\, X_0(N)(\mathbb{C})$. These statements appear in the papers [AS90, AL70, KM88, LN64], and all of these inaccuracies are such that they affect the final description of the automorphism group. Furthermore, to our knowledge, corrections to the issues in [AS90, KM88] have not yet appeared in print.

Therefore, it would be valuable to formalise the results on the structure of $\mathrm{Aut}\, X_0(N)(\mathbb{C})$ using a proof assistant. Such a formalisation would finally

[3]Bars defines $B_0(N)$ as $\mathrm{Norm}_{\mathrm{SL}_2^+(\mathbb{R})}(\Gamma_0(N))/\Gamma_0(N)$ instead; however, up to scalars all those matrices in $\mathrm{Norm}_{\mathrm{SL}_2^+(\mathbb{R})}(\Gamma_0(N))$ lie in $\mathrm{GL}_2^+(\mathbb{Q})$ and all scalar matrices induce the identity automorphism of $X_0(N)$, so there is no harm in this difference.

provide a complete and consistent account, which, at the time of writing, does not appear to be available.

4. Sporadic CM points

4.1. **Degree 4.** We first deal with the easiest case, $N \in U_4$.

Proposition 4.1. *For the values $N = 90$ and all $N \in U_4$ such that $N \geq 105$, $X_0(N)$ has a sporadic CM point.*

Proof. By [DO24, Theorem 1.8], all the $X_0(N)$ in question have $\delta(X_0(N)) > 4$. □

Proposition 4.2. *For all*

$$N \in \{60, 70, 72, 80, 87, 94, 96\}$$

there are no sporadic CM points on $X_0(N)$.

Proof. For all of these values, we have $\delta(X_0(N)) \leq \operatorname{gon}_{\mathbb{Q}} X_0(N) = 4$ by [NO24, Theorem 1.2]. □

Remark 4.3. Note that we will prove a stronger result, that there are no sporadic points on these curves, of any kind, in Section 5.

4.2. **Degree 6.** Specializing Theorem 3.1 to $d = 6$ gives a good way to rule the existence of infinitely many degree 6 points [4].

Corollary 4.4. *If $C/\mathbb{Q}$ is a nice curve such that $C(\mathbb{Q}) \neq \emptyset$, $\delta(C) = 6$, and $g(C) \geq 17$, then one of the following holds:*

(1) C admits a $\mathbb{Q}$-rational morphism of degree 6 to either $\mathbb{P}^1$ or an elliptic curve E of positive rank over $\mathbb{Q}$.

(2) C admits a $\mathbb{Q}$-rational morphism of degree 2 to a genus 4 curve with infinitely many cubic points over $\mathbb{Q}$.

Proposition 4.5. *For all $N \in U_6 \backslash \{144\}$ we have $\delta(X_0(N)) > 6$ and hence the curve $X_0(N)$ has a sporadic CM point. The modular curve $X_0(144)$ does not have a sporadic CM point.*

Remark 4.6. In fact, it turns out that $X_0(144)$ does not have any sporadic points, as we will prove in Section 5.1. It turns out that proving non-CM points cannot be sporadic is considerably more difficult than proving the same for CM points.

Proof. None of them have $\delta(C) \leq 5$ by [DHJO25, Theorem 1.9], so it suffices to show $\delta(C) \neq 6$. We will show $\delta(C) \neq 6$ using Theorem 4.4. Note that all of the curves have $g(X_0(N)) \geq 19$ and none of them are of gonality ≤ 6 over $\mathbb{Q}$ (see [NO24, Theorems 1.2-1.4]). So in order to show $\delta(C) \neq 6$, it suffices to rule out the possibility of a map of degree 6 to a positive rank elliptic curve, and a map of degree 2 to a genus 4 curve.

[4] this specialization to $d = 6$ can also be found in [DO25, Corollary 5.9])

An LMFDB search yields a list of all elliptic curves which have positive rank, modular degree ≤ 6 and whose conductor divides one of the integers in $U_6 \setminus \{144\}$; by Theorem 3.2 these are the only possible $E/\mathbb{Q}$ of positive rank to which $X_0(N)$ can have a degree 6 map. The results of this search have been summarized in Table 1.

N	LMFDB Label	Modular degree	Quadratic form
348	58.a1	4	$8(4x_0^2-4x_0x_1+4x_1^2-6x_0x_2+3x_1x_2+4x_2^2+3x_0x_3-6x_1x_3-4x_2x_3+4x_3^2)$
426	142.a1	4	$8(2x_0^2-x_0x_1+2x_1^2)$
472	118.a1	4	$8(2x_0^2-2x_0x_1+2x_1^2+x_0x_2-2x_1x_2+2x_2^2)$
472	236.a1	6	$12(x_0^2+x_1^2)$
572	143.a1	4	$24(x_0^2+x_1^2-x_0x_2+x_2^2)$

TABLE 1. Elliptic curves whose conductor divides some $N \in U_6\setminus\{144\}$ having positive rank and modular degree ≤ 6.

Using the methods of [DO24] we compute the quadratic forms in Table 1 such that the possible degrees of $X_0(N) \to E$ are the values of the quadratic forms in the table. We see that $X_0(472)$ cannot have a degree 6 map to a positive rank elliptic curve. This proves that case (1) of Theorem 4.4 is impossible.

Case (2) in Theorem 4.4 implies that the curve has $\operatorname{gon}_{\mathbb{C}} X_0(N) \leq 6$, as for a curve Y of genus $g(Y) = 4$, we have $\operatorname{gon}_{\mathbb{C}} Y \leq 3$ (see e.g. [Poo07, Proposition A.1. (v)]). For all $N \in U_6$ with $N \geq 300$, Abramovich's bound gives $\operatorname{gon}_{\mathbb{C}} X_0(N) \geq 8$, so this shows that we cannot be in case (2). In the remaining cases we check that for every N and every involution ι of $X_0(N)$, all of which we can explicitly obtain using the methods of Section 3.1, we have $g(X_0(N)/\iota) \geq 7$, hence the case (2) in Theorem 4.4 is not possible. The code for this is available at `involutions.ipynb` in our repository.

In the case $N = 144$ we have $\delta(X_0(N)) \leq \operatorname{gon}_{\mathbb{Q}} X_0(N) = 6$ by [NO24, Theorem 1.4] □

4.3. **Degree 8.** We have the following Corollary of Theorem 3.1 for degree 12.

Corollary 4.7. *If $C/\mathbb{Q}$ is a nice curve such that $C(\mathbb{Q}) \neq \emptyset$, $\delta(C) = 8$, and $g(C) \geq 34$, then one of the following holds:*

(1) C admits a $\mathbb{Q}$-rational morphism of degree 8 to either $\mathbb{P}^1$ or an elliptic curve E of positive rank over $\mathbb{Q}$.

(2) C admits a $\mathbb{Q}$-rational morphism of degree 2 to a curve with infinitely many quartic points over $\mathbb{Q}$ of genus ≤ 7.

Proof. Since $g(C) \geq 34$, case (2) of Theorem 3.1 cannot occur. This means that there exists a non-constant morphism of curves $\phi\colon C \to Y$ of degree at least 2 over $\mathbb{Q}$ with the property that $8 = \delta(Y) \cdot \deg \phi$.

If $\deg \phi = 8$, then Y has infinitely many rational points and hence it is either $\mathbb{P}^1$ or an elliptic curve E of positive rank over $\mathbb{Q}$ by Faltings' theorem.

If $\deg \phi = 4$, then Y has infinitely many quadratic points. Hence Y admits a degree 2 rational morphism to $\mathbb{P}^1$ or to an elliptic curve E of positive rank over $\mathbb{Q}$ by [HS91] and [KV25, Theorem 1.2 (1)]. Therefore, the desired map of degree 8 to $\mathbb{P}^1$ or E can be obtained as a composition $C \to Y \to (\mathbb{P}^1 \text{ or } E)$.

If $\deg \phi = 2$, then $\delta(Y) = 4$. If $g(Y) \leq 7$ there is nothing to prove. If $g(Y) \geq 8$, we apply the same reasoning as above but this time to the curve Y and $d = 4$. Then again case (2) of Theorem 3.1 cannot occur, and by a similar argument we get a map $\phi' : Y \to (\mathbb{P}^1 \text{ or } E)$ of degree 4. And hence $\phi' \circ \phi : C \to (\mathbb{P}^1 \text{ or } E)$ of degree 8. □

Proposition 4.8. *For all $N \in U_8$ we have $\delta(X_0(N)) > 8$ and hence the curve $X_0(N)$ has a sporadic CM point.*

Proof. We will show $\delta(X_0(N)) > 8$ for all $N \in U_8$ by applying Theorem 4.7. We have $g(X_0(N)) \geq 57$ for all $N \in U_8$. In all cases, Abramovich's bound gives us $\operatorname{gon}_{\mathbb{Q}} X_0(N) \geq \operatorname{gon}_{\mathbb{C}} X_0(N) > 8$.

We search for all elliptic curves of positive rank over $\mathbb{Q}$ in the LMFDB of conductor dividing N and modular degree ≤ 8; by Theorem 3.2 these are the only possible elliptic curves E of positive rank such that $X_0(N) \to E$ might be of degree 8. The curves are listed in Table 2. The quadratic form column gives the quadratic form whose values are the possible degrees $X_0(N) \to E$, and which is computed using the methods of [DO24].

N	LMFDB Label	Modular degree	Quadratic form
440	88.a1	8	$48(x_0^2 - x_0x_1 + x_1^2)$
528	88.a1	8	$32(2x_0^2 + 2x_1^2 - 3x_0x_2 + 2x_2^2 - 3x_1x_3 + 2x_3^2)$
528	176.a2	8	$16(2x_0^2 - x_0x_1 + 2x_1^2)$
600	200.b2	8	$32(x_0^2 - x_0x_1 + x_1^2)$
672	112.a2	8	$64(x_0^2 + x_1^2 - x_0x_2 + x_2^2 - x_1x_3 + x_3^2)$
672	224.a2	8	$32(x_0^2 - x_0x_1 + x_1^2)$

TABLE 2. Elliptic curves whose conductor divides some $N \in U_8$ having positive rank and modular degree ≤ 8.

We see that all coefficients of modular forms are divisible by 16, hence there are no maps $X_0(N) \to E$ of degree 8, showing that case Theorem 4.7 (1) cannot occur.

We note that a curve of genus ≤ 7 has $\mathbb{C}$-gonality ≤ 5, see [Poo07, Proposition A.1 (v)]. This means that $X_0(N)$ would have $\mathbb{C}$-gonality ≤ 10, see [Poo07, Proposition A.1 (vi)]. But for all $N \in U_8 \backslash \{360, 440\}$, Abramovich's

bound gives us $\operatorname{gon}_{\mathbb{C}} X_0(N) \geq 11$. For the cases $N = 360$ and 440, we obtain using the methods of Section 3.1 that for all involutions ι, we have $X_0(N)/\iota$ is of genus ≥ 25, so we cannot be in case (2). The code that checks this is available at `involutions.ipynb` in our repository.

□

4.4. **Degree 12.** We need to show that the known degree 12 CM point on $X_0(720)$ is sporadic. We have the following specialization of Theorem 3.1 for degree 12.

Corollary 4.9. *If $C/\mathbb{Q}$ is a nice curve such that $C(\mathbb{Q}) \neq \emptyset$, $\delta(C) = 12$, and $g(C) \geq 34$, then one of the following holds:*

1. *C admits a $\mathbb{Q}$-rational morphism of degree 12 to either $\mathbb{P}^1$ or an elliptic curve E of positive rank over $\mathbb{Q}$.*
2. *C admits a $\mathbb{Q}$-rational morphism of degree 2 to a curve with infinitely many sextic points over $\mathbb{Q}$ of genus ≤ 16.*
3. *C admits a $\mathbb{Q}$-rational morphism of degree 3 to a curve with infinitely many quartic points over $\mathbb{Q}$ of genus ≤ 7.*
4. *C admits a $\mathbb{Q}$-rational morphism of degree 4 to a curve with infinitely many cubic points over $\mathbb{Q}$ of genus ≤ 4.*

Proof. This is proved using similar arguments as Theorem 4.7. □

Proposition 4.10. *The curve $X_0(720)$ has a sporadic CM point.*

Proof. We need to show that $\delta(X_0(720)) > 12$. Abramovich's bound gives us

$$\operatorname{gon}_{\mathbb{Q}} X_0(720) \geq \operatorname{gon}_{\mathbb{C}} X_0(720) \geq 18,$$

hence it follows that

$$\delta(X_0(720)) \geq \frac{\operatorname{gon}_{\mathbb{Q}} X_0(720)}{2} \geq 9.$$

We eliminate the possibilities of $\delta(X_0(720)) = 9, 10, 11, 12$ by doing a case-by-case analysis.

$\boxed{\delta(X_0(720)) = 12}$ By Theorem 4.9, the curve $X_0(720)$ has a degree d map over $\mathbb{Q}$ to a curve Y with a curve with infinitely many degree $12/d$ points over $\mathbb{Q}$, where $d \in 2, 3, 4, 12$.

The case $d = 2$: We eliminate this case by checking that for every involution w of $X_0(720)$ involutions using the methods of Section 3.1, we have $g(X_0(720)/w) \geq 57$. The code that checks this is available at `involutions.ipynb` in our repository.

The case $d = 3$: by Theorem 4.9 (3), Y would need to have $g(Y) \leq 7$. However, this would imply (see e.g. [Poo07, Proposition A.1. (v)]) that $\operatorname{gon}_{\mathbb{C}} Y \leq 5$, so it would follow that $\operatorname{gon}_{\mathbb{C}} X \leq 15$, which is a contradiction.

The case $d = 4$: by Theorem 4.9 (4), Y would need to have $g(Y) \leq 4$. However, this would imply (see e.g. [Poo07, Proposition A.1. (v)]) that $\operatorname{gon}_{\mathbb{C}} Y \leq 3$, so it would follow that $\operatorname{gon}_{\mathbb{C}} X \leq 12$, which is a contradiction.

THE CASE $d = 12$: by Abramovich's bound, the curve cannot have a map of degree 12 to $\mathbb{P}^1$. Hence, it would need to have a map of degree 12 to an elliptic curve with positive rank over $\mathbb{Q}$. A search in the LMFDB shows that there are no elliptic curves with positive rank over $\mathbb{Q}$ whose conductor divides 720 and modular degree < 16, from which we conclude by Theorem 3.2 that there are no degree 12 morphisms $X_0(720) \to E$ to an elliptic curve with positive rank over $\mathbb{Q}$.

$\boxed{\delta(X_0(720)) = 11}$ Applying *Theorem* 3.1 to $\delta = 11$ and $X_0(720)$, we get that the only possibility is a degree 11 map to $\mathbb{P}^1$ or a positive rank elliptic curve, both of which we have seen are impossible.

$\boxed{\delta(X_0(720)) = 10}$ Applying *Theorem* 3.1 to $\delta = 10$ and $X_0(720)$ tells us that we need to rule out a degree $d = 2, 5, 10$ maps to elliptic curves with infinitely many degree $10/d$ points. All of these degrees are easily eliminated using the same arguments as in the $\delta = 12$ case.

$\boxed{\delta(X_0(720))) = 9}$ This case is ruled out exactly the same as the previous cases.

□

5. Ruling out lower degree points

It remains to prove Theorem 1.2. To do this, we need to check whether there are any sporadic non-CM points on the modular curves $X_0(N)$ for

$$N \in \{60, 70, 72, 80, 87, 94, 96, 144\}.$$

For all $N \neq 144$, this amounts to showing that there are no non-cuspidal points on $X_0(N)$ of degree ≤ 3, while for $N = 144$ we need to rule out the possibility of points of degree ≤ 5.

We prove the nonexistence of degree 2 and 3 points on these curves in the following 2 propositions.

Proposition 5.1. *For all*

$$N \in \{60, 70, 72, 80, 87, 94, 96, 144\}$$

there are no quadratic points on $X_0(N)$.

Proof. The possibility of quadratic points on $X_0(N)$ has been ruled out for $N = 60, 94$ in [NV23], for $N \in \{70, 87, 96\}$ in [NN25], for $N = 72$ in [OS19], and for $N = 80$ in [AKMJ+24]. The fact that $X_0(72)$ has no non-cuspidal points immediately implies that neither does $X_0(144)$. □

Proposition 5.2. *There are no non-cuspidal points of degree* 3 *on* $X_0(N)$ *for*

$$N \in \{60, 70, 72, 80, 87, 94, 96, 144\}.$$

Proof. First note that $J_0(N)(\mathbb{Q})$ is finite for all the N; see [DEvH+21, Theorem 3.1]. In the cases $N = 60, 70, 80, 94, 96$, the curve $X_0(N/2)$ is a hyperelliptic curve of genus > 2, and hence has finitely many cubic points. Let

$n := N/2$. Let $C \in X_0(n)(\mathbb{Q})$ be a rational cusp. It follows that any rational effective divisor D of degree 3 lies in the Riemann-Roch space $\mathcal{L}(A + 3C)$ for some $A \in J_0(n)(\mathbb{Q})$. Since the dimension $l(A + 3C) \leq 1$ it is unique in its divisor class. By looping through all $A \in J_0(n)(\mathbb{Q})$ we find all the cubic points on $X_0(n)(\mathbb{Q})$. We check for each point whether it lifts to a cubic point on $X_0(n)$. None do, so we are done in these cases. The `magma` code that checks this can be found at `Prop_8_1.m`.

We now consider the cases $N = 72$ and 87. Let $J_C(N)$ denote the cuspidal subgroup of $J_0(N)$, i.e., the subgroup generated by divisors supported on cusps. We compute that $J_C(87)(\mathbb{Q}) \simeq \mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}/140\mathbb{Z}$ and by considering the reductions of $J_0(87)$ modulo $5, 7, 11$ and using the fact that reduction modulo an odd prime of a good reduction is injective on $J_0(N)_{\mathrm{tors}}$, we conclude $J_C(87)(\mathbb{Q}) \simeq J_0(87)(\mathbb{Q})$. The fact that $J_C(72)(\mathbb{Q}) \simeq J_0(72)(\mathbb{Q})$ has been proved in [Lup24a] [5], and we have

$$J_C(72)(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}.$$

Let D be a hypothetical rational noncuspidal effective divisor on $X_0(N)$ of degree 3, and let $C \in X_0(N)(\mathbb{Q})$ be a cusp. Since $X_0(N)$ is not trigonal over $\mathbb{F}_5$ (see [DN24b, Section 6]) and hence also not trigonal over $\mathbb{Q}$, the map $X_0(N)^{(3)}(\mathbb{Q}) \to J_0(N)(\mathbb{Q})$ sending D to $[D - 3C]$ is injective. Moreover, every effective degree 3 divisor B over $\mathbb{F}_5$ satisfies $l(B) = 1$.

Since $\operatorname{rk} J_0(N)(\mathbb{Q}) = 0$ and $X_0(N)$ has good reduction at 5, the reduction mod 5 map $J_0(N)(\mathbb{Q}) \to J_0(N)(\mathbb{F}_5)$ is injective. Hence the composition map

$$f : X_0(N)^{(3)}(\mathbb{Q}) \to J_0(N)(\mathbb{F}_5), \quad B \mapsto [(B - 3C)_{\mathbb{F}_5}]$$

is injective as well. Thus, there is at most one effective rational divisor of degree 3 reducing to any fixed $\mathbb{F}_5$-divisor class.

Cusps of $X_0(N)(\overline{\mathbb{Q}})$ reduce injectively modulo any prime above 5 to $X_0(N)(\overline{\mathbb{F}}_5)$, so different effective rational cuspidal divisor of degree 3 map under f to different divisor classes in $J_0(N)(\mathbb{F}_5)$. For each $x \in J_0(\mathbb{F}_5)$, we choose representative divisor A_x such that $[A_x] = x$, and count the number of such x that have $l(A_x + 3C_{\mathbb{F}_5}) = 1$. If the number matches the number of such x equals the number of effective rational degree-3 cuspidal divisors, then no rational non-cuspidal degree-3 divisors can exist in $X_0(N)^{(3)}(\mathbb{Q})$.

The modular curve $X_0(87)_{\mathbb{Q}}$ has 4 cusps, all of which are defined over $\mathbb{Q}$. We compute the $l(A_x + 3C_{\mathbb{F}_5})$ for all $A_x \in J_C(87)(\mathbb{F}_5)$, and find exactly 20 divisors $x \in J_C(87)(\mathbb{F}_5)$ such that $l(A_x + 3C_{\mathbb{F}_5}) = 1$. This matches the number of rational effective degree 3 cuspidal divisors:

$$\binom{\#\text{cusps} + 3 - 1}{3} = \binom{6}{3} = 20.$$

Hence all effective degree-3 rational divisors in $X_0(87)^{(3)}(\mathbb{Q})$ are cuspidal. The code verifying this can be found in `87.m`.

[5] There is a gap in the proof of this in the published version, which has been corrected in the arxiv version `https://arxiv.org/abs/2205.13017`.

We do a similar search for $X_0(72)$ and find 152 divisors $A_x \in J_C(72)(\mathbb{F}_5)$ such that $l(A_x + 3C_{\mathbb{F}_5}) = 1$. The curve $X_0(72)_{\mathbb{Q}}$ has 8 rational cusps and 4 degree 2 cusps, giving

$$\binom{8+3-1}{3} + 4 \cdot 8 = 152$$

cuspidal divisors. Thus, every effective degree-3 divisor in $X_0(72)^{(3)}(\mathbb{F}_5)$, and hence in $X_0(72)^{(3)}(\mathbb{Q})$ is cuspidal. The code verifying this can be found in `72.m`.

Finally, since $X_0(72)$ has no degree ≤ 3 non-cuspidal points, neither does $X_0(144)$. □

5.1. **Degree 4 and 5 points on $X_0(144)$.** Proving that $X_0(144)$ has no sporadic points turns out to be by far the hardest.

Proposition 5.3. *The group $J_0(144)(\mathbb{Q})$ is generated by divisors supported on the cusps and is isomorphic to $\mathbb{Z}/4\mathbb{Z}^3 \times \mathbb{Z}/24\mathbb{Z}^3$.*

Proof. The Mordell-Weil rank of $J_0(144)(\mathbb{Q})$ is 0 by [DEvH+21, Theorem 3.1], so $J_0(144)(\mathbb{Q}) = J_0(144)(\mathbb{Q})_{\text{tors}}$ and hence we are reduced to computing the torsion subgroup of $J_0(144)$. As before, $J_C(144)$ denotes the cuspidal subgroup of $J_0(144)$, i.e., the subgroup generated by divisors supported on cusps. We compute $J_C(144)(\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z}^3 \times \mathbb{Z}/24\mathbb{Z}^3$ using the methods of [DEvH+21, Section 3], so it suffices to show $J_C(144)(\mathbb{Q}) = J_0(144)(\mathbb{Q})_{tors}$.

We tried to apply the methods of [DEvH+21, Section 3] and [Lup24b] in order to show $J_C(144)(\mathbb{Q}) = J_0(144)(\mathbb{Q})_{tors}$ directly but all of these failed. Our strategy will be to find a cover $X \to X_0(144)$ so that we can compute $J(X)(\mathbb{Q})_{tors}$ and use this cover to compute $J_0(144)(\mathbb{Q})_{tors}$.

To this end let $\Delta = \langle -1, 11 \rangle \subseteq (\mathbb{Z}/144\mathbb{Z})^\times$. Then $f := X_\Delta(144) \to X_0(144)$ is a map of degree 2. The diamond operator $\iota := \langle 5 \rangle$ is the involution of $X_\Delta(144)$ such that $X_0(144) = X_\Delta/\iota$. The genus of $X_\Delta(144)$ is 25, so, by the Riemann-Hurwitz formula, the map f is unramified and hence étale. In particular, X_Δ is a subcover of the Shimura cover $X_2(N) \to X_0(N)$ from [Maz77, Eq. II.11.5]. Let $f^* : J_0(144) \to J_\Delta(144)$, similarly to [Maz77, Prop. II.11.6] we get that $\ker f^*$ is isomorphic to the Cartier dual of $\mathbb{Z}/N\mathbb{Z}^\times/\Delta$, i.e. we have the following isomorphisms of group schemes over $\mathbb{Q}$:

$$\ker f^* \cong \mathrm{Hom}(\mathbb{Z}/N\mathbb{Z}^\times/\Delta, \mathbb{G}_m) \cong \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{G}_m) \cong \mathbb{Z}/2\mathbb{Z}.$$

Using the methods of [DEvH+21, Section 3] we compute that $J_\Delta(144)(\mathbb{Q})_{tors}$ is generated by cuspidal divisors and that

$$J_\Delta(144)(\mathbb{Q})_{tors} \cong \mathbb{Z}/2\mathbb{Z}^2 \times \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}^4 \times \mathbb{Z}/48\mathbb{Z}.$$

Now $f^*(J_0(144)(\mathbb{Q}))$ is invariant under the action of $\langle 5 \rangle$ so most certainly $f^*(J_0(144)(\mathbb{Q})) \subseteq J_\Delta(144)(\mathbb{Q})^{\langle 5 \rangle}$. We compute that $J_\Delta(144)(\mathbb{Q})^{\langle 5 \rangle} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}^2 \times \mathbb{Z}/24\mathbb{Z}^3$. From this we get that $J_0(144)(\mathbb{Q})_{tors}$ fits in the exact

sequence

$$\mathbb{Z}/2\mathbb{Z} \to J_0(144)(\mathbb{Q})_{tors} \to J_\Delta(144)(\mathbb{Q})^{\langle 5\rangle} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}^2 \times \mathbb{Z}/24\mathbb{Z}^3.$$

However, since we already know that $J_0(144)(\mathbb{Q})_{tors}$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}^3 \times \mathbb{Z}/24\mathbb{Z}^3$, this implies that $J_0(144)(\mathbb{Q})_{tors}$ has to actually be isomorphic to $\mathbb{Z}/4\mathbb{Z}^3 \times \mathbb{Z}/24\mathbb{Z}^3$ for cardinality reasons.

Remark 5.4. While trying to prove the previous proposition, we noticed a curious phenomenon. The natural map $f : X_0(144) \to X_0(9)$ is of degree 12. Taking $c \in X_0(9)(\mathbb{Q})$, we get that $f^*(c)$ is a theta characteristic, i.e., $2f^*(c)$ is in the canonical class. The chance for this happening at random is $\frac{\#J_0(144)(\mathbb{Q})_{\text{tors}}}{\#J_0(144)(\mathbb{Q})[2]} = \frac{1}{1728}$. We note that the choice of $c \in X_0(9)(\mathbb{Q})$ as $X_0(9) \simeq \mathbb{P}^1$, so $f^*(c)$ will be linearly equivalent to $f^*(d)$ for some other $d \in X_0(9)(\mathbb{Q})$.

□

Proposition 5.5. *The modular curve $X_0(144)$ has no non-cuspidal points of degree 4 or 5.*

Proof. We will show that any $D \in X_0(144)^{(d)}(\mathbb{Q})$ is cuspidal for $d = 4, 5$. We have the following commutative diagram

$$\begin{array}{ccc} X_0(144)^{(d)}(\mathbb{Q}) & \xhookrightarrow[A]{} & J_0(144)(\mathbb{Q}) \\ \downarrow{\scriptstyle \text{red}_5} & & \downarrow{\scriptstyle \text{red}_{5,J}} \\ X_0(144)^{(d)}(\mathbb{F}_5) & \xhookrightarrow[A_5]{} & J_0(144)(\mathbb{F}_5) \end{array},$$

where A and A_5 are the Abel-Jacobi maps (with some fixed base point), and the vertical maps are reductions modulo 5. The maps A and A_5 are injective because $\text{gon}_{\mathbb{F}_5} X_0(144) = \text{gon}_{\mathbb{Q}} X_0(144) = 6 > d$ by [NO24, Theorem 1.4 and Proposition 5.15], while $\text{red}_{5,J}$ is injective by the injectivity of the torsion and $\text{rk}\, J_0(144)(\mathbb{Q}) = 0$. The injectivity of red_5 follows by the commutativity of the diagram.

Any $D \in X_0(144)^{(d)}(\mathbb{Q})$ has to satisfy

$$A_5 \circ \text{red}_5(D) \in \text{red}_{5,J}(J_0(144)(\mathbb{Q})).$$

We run through all $x \in X_0(144)^{(d)}(\mathbb{F}_5)$ and count the number of those satisfying $A_5(x) \in \text{red}_{5,J}(J_0(144)(\mathbb{Q}))$. We obtain that their number is exactly equal to the number of degree d cuspidal divisors in $X_0(144)^{(d)}(\mathbb{Q})$. The `magma` computations that were used to verify this can be found in `Prop_8_3.m`. From the injectivity of the maps in the diagram, it follows that D is cuspidal. □

Appendix A. Castelnuovo-Severi inequality with at least 3 maps

When considering how to bound the gonalities of the quotients of $X_0(N)$ by all involutions, we proved the following result, which can be used to

bound the gonality of a curve X using the quotient by an involution that does not lie in the center of $\operatorname{Aut} X$. We did not need this result in the end, but we believe it might be of use to others, and of independent interest.

Proposition A.1. *Let X be a curve of genus g and ι an automorphism of X which is not in the center of $\operatorname{Aut} X$. If*

$$g(X) > |\iota| \cdot g(X/\iota) + (|\iota| - 1)(d-1). \tag{A.1}$$

Then a map of degree d to $X \to \mathbb{P}^1$ factors through $X/\langle g \cdot \iota \cdot g^{-1} | g \in \operatorname{Aut} X \rangle$.

Proof. First note that if (A.1) is satisfied, then a map of degree d to $f : X \to \mathbb{P}^1$ factors through X/ι by the Castelnouovo-Severi inequality (see [Sti09, Theorem 3.11.3]). For an automorphism $g \in \operatorname{Aut} X$, the curves X/ι and $X/g \cdot \iota \cdot g^{-1}$ are isomorphic, so $X/g \cdot \iota \cdot g^{-1}$ also satisfies (A.1). Hence f also factors through $X/g \cdot \iota \cdot g^{-1}$. The proposition now follows. □

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