LINEAR SATURATION FOR $\mathcal{N}$ VIA BUTTERFLIES

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Abstract

Given a finite poset $\mathcal{P}$, how small can a family $\mathcal{F}$ of subsets of $[n]$ be such that $\mathcal{F}$ does not contain an induced copy of $\mathcal{P}$, but $\mathcal{F} \cup \{X\}$ contains such a copy for all $X \in \mathcal{P}([n]) \setminus \mathcal{F}$? This is known as the induced saturation number of $\mathcal{P}$, denoted by $\text{sat}^*(n, \mathcal{P})$. The main conjecture in the area is that the induced saturation number for any poset is either bounded, or linear.

In this paper we establish linearity for the induced saturation number of the 4-point poset $\mathcal{N}$. Previously, it was known that $2\sqrt{n} \leq \text{sat}^*(n, \mathcal{N}) \leq 2n$. We show that $\text{sat}^*(n, \mathcal{N}) \geq \frac{n+6}{4}$. A crucial role in the proof is played by a structural feature of $\mathcal{N}$-saturated families, namely that if the family contains two antichains, one completely above the other, then it must also contain a 'middle' point – greater than one antichain and less than the other.

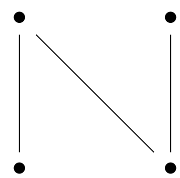
1 Introduction

We say that a poset $(\mathcal{Q}, \leq')$ contains an *induced copy* of a poset $(\mathcal{P}, \leq)$ if there exists an injective function $f : \mathcal{P} \to \mathcal{Q}$ such that $(\mathcal{P}, \leq)$ and $(f(\mathcal{P}), \leq')$ are isomorphic. For a fixed poset $\mathcal{P}$ we call a family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$ $\mathcal{P}$-*saturated* if $\mathcal{F}$ does not contain an induced copy of $\mathcal{P}$, but for every subset S of $[n]$ such that $S \notin \mathcal{F}$, the family $\mathcal{F} \cup \{S\}$ does contain such a copy. We denote by $\text{sat}^*(n, \mathcal{P})$ the size of the smallest $\mathcal{P}$-saturated family of subsets of $[n]$. In general we refer to $\text{sat}^*(n, \mathcal{P})$ as the *induced saturation number* of $\mathcal{P}$.

This notion, inspired by saturation for graphs, was introduced by Ferrara, Kay, Kramer, Martin, Reiniger, Smith and Sullivan [3]. Despite their similarity, graph saturation and poset saturation are vastly different, partially due to the rigidity of *induced* copies of posets. One of the most important conjectures in the area is that for every poset the saturation number is either bounded or linear [8]. The saturation number for posets has already been shown to display a dichotomy. Keszegh, Lemons, Martin, Pálvölgy and Patkós [8] showed that the saturation number is either bounded or at least $\log_2(n)$. This was later improved by Freschi, Piga, Sharifzadeh and Treglown [4] who showed that $\text{sat}^*(n, \mathcal{P})$ is either bounded or at least $2\sqrt{n}$. In the other direction, Bastide, Groenland, Ivan and Johnston [1] showed that the saturation number of any poset grows at most polynomially. Moreover, very recently Ivan and Jaffe [6] showed that for any poset, one can add at most 3 points to obtain a poset with saturation number at most linear.

Even more surprisingly, establishing the rate of growth even for simple posets is difficult, with only a handful of examples for which linearity or boundedness has been proven. Most notably, the following posets have linear saturation number: the butterfly (two maximal elements completely above two minimal elements) [5, 8], the antichain [2], and most recently the diamond (one maximal, one minimal and two other incomparable elements) [7], whose saturation number was stuck for a long time at the $\sqrt{n}$ lower bound.

In this paper we establish linearity for another key 4-poset, the $\mathcal{N}$, comprised of 2 minimal and 2 maximal elements, such that exactly one maximal element is comparable to both minimal elements, and exactly one minimal element is comparable to both minimal elements, as depicted below.



The Hasse diagram of the poset $\mathcal{N}$

The Hasse diagram of $\mathcal{N}$ is misleadingly simple. One of the main challenges when it comes to analysing this structure comes from the 'broken' symmetry of the $\mathcal{N}$ – some of its elements are comparable to exactly one other, and others are comparable to 2 – unlike for example the very symmetric butterfly poset.

It is not hard to check that the family $\mathcal{F} = \{\emptyset, [n], \{i\}, \{1, 2, \dots, i\} : 1 \leq i \leq n\}$ is $\mathcal{N}$-saturated for all $n \geq 3$, hence $sat^*(n, \mathcal{N}) \leq 2n$ [3]. However, the lower bound proved much more challenging. It was first shown that $\log_2 n \leq \text{sat}^*(n, \mathcal{N})$ [3], which was then improved to $\sqrt{n} \leq \text{sat}^*(n, \mathcal{N})$ [5], and most recently to $2\sqrt{n} \leq \text{sat}^*(n, \mathcal{N})$ [4]. We mention that these bounds were not specific to the $\mathcal{N}$ poset – they were in fact a consequence of the fact that the $\mathcal{N}$ is part of a larger class of special posets, namely posets that have the *unique cover twin property* (UCTP).

In this paper we show that the size of any $\mathcal{N}$-saturated family with ground set $[n]$ is at least $\frac{n+6}{4}$, therefore showing that $\text{sat}^*(n, \mathcal{N}) = \Theta(n)$. This also concludes the analysis of all posets discussed in the paper that launched the field [3], as the $\mathcal{N}$ was the final missing piece.

Our proof is heavily based on Lemma 3, which asserts that if $\mathcal{F}$ is an $\mathcal{N}$-saturated family containing sets $A_1, \dots, A_k$ and $B_1, \dots, B_l$ such that $B_i \subseteq A_j$ for all $i \leq l$ and $j \leq k$, then there exists $M \in \mathcal{F}$ in between them, i.e. $\cup_{i=1}^{l} B_i \subseteq M \subseteq \cap_{j=1}^{k} A_j$. In particular, if $\mathcal{F}$ contains a butterfly, it must also contain a 'middle' point, less than its maximal elements and bigger than its minimal elements. This is crucial to our analysis as the existence of this midpoint will serve as a pivot for identifying forbidden copies of $\mathcal{N}$.

The paper is organised as follows. Section 2 is dedicated to establishing the above mentioned connection between $\mathcal{N}$-saturated families and butterflies. In Section 3 we lay the groundwork for the main result by proving a few general lemmas that will be used frequently in the subsequent analysis. All these results come together in Section 4, Proposition 11, from which the linear lower bound for $\text{sat}^*(n, \mathcal{N})$ is then easily obtained.

2 $\mathcal{N}$-saturated families and butterflies

In this section we analyse the structure of an $\mathcal{N}$-saturated family. More precisely, we show that in such a family, if there exist two antichains one above the other, then there exists a set in the family that lies between them. This will motivate our analysis in later sections.

Let $\mathcal{F}$ be an $\mathcal{N}$-saturated family with ground set $[n]$. We observe that we must have $\emptyset, [n] \in \mathcal{F}$ as they are comparable to everything and therefore cannot form an $\mathcal{N}$ when added to the family.

Consider the Hasse diagram of $\mathcal{F} \setminus \{\emptyset, [n]\}$ as a graph, and let $\mathcal{G}$ be a connected component. We call $\mathcal{G}$ a *component of* $\mathcal{F}$ for simplicity.

We start with the following observation about maximal and minimal elements of $\mathcal{G}$.

Lemma 1. *Let T be a maximal element and S a minimal element of $\mathcal{G}$. Then $S \subseteq T$.*

Proof. If $S = T$, there is nothing to prove, so we may assume that $S \neq T$. Next, we add to the Hasse diagram of $\mathcal{G}$ all edges XY if $X \subset Y$. In this new graph, by connectedness, there exists a shortest path from S to T, say $S = X_1, X_2, \dots, X_l = T$, where $X_i \in \mathcal{F} \setminus \{\emptyset, [n]\}$ are pairwise distinct. If $l = 2$, we are done. If $l = 3$, we must have $S \subset X_2 \subset T$ since S is a minimal element and T is a maximal element, but then $S \subset T$ is a shorter path, a contradiction. Hence we may assume $l \geq 4$.

We regard this path as a directed one, with an arrow from X_i to X_{i+1} if $X_i \subset X_{i+1}$, and from X_{i+1} to X_i otherwise. By minimality of the path, the edges must alternate in direction, otherwise by transitivity there are some non-adjacent elements X_i, X_j comparable, and replacing the edges $X_iX_{i+1}, \dots, X_{j-1}X_j$ by X_iX_j gives a shorter path. Thus we have $S \subset X_2, X_2 \supset X_3, X_3 \subset X_4$, where S is incomparable to both X_3 and X_4, and X_2 is incomparable to X_4. Therefore S, X_2, X_3, X_4 form an $\mathcal{N}$ in $\mathcal{F}$, a contradiction. □

The above lemma tells us that the maximal and minimal elements of a component $\mathcal{G}$ form a complete bipartite poset. The next lemma unifies these elements by identifying a set in $\mathcal{F}$ comparable to all maximal and all minimal elements of $\mathcal{G}$. The next result is an intermediary step towards that.

Lemma 2. *Let $A_1, \dots, A_k$ and $B_1, \dots, B_l$ be elements of $\mathcal{F}$, where $k, l \geq 1$. If $\cap_{i=1}^k A_i \notin \mathcal{F}$, then $\cap_{i=1}^k A_i$ must be a minimal element of any $\mathcal{N}$ formed when added to $\mathcal{F}$. Similarly, if $\cup_{i=1}^l B_i \notin \mathcal{F}$, then $\cup_{i=1}^l B_i$ must be a maximal element of any $\mathcal{N}$ formed when added to $\mathcal{F}$. Furthermore, $\cap_{i=1}^k A_i$ can be assumed to be the minimal element of some $\mathcal{N}$, that is comparable to both maximal elements, and $\cup_{i=1}^l B_i$ can be assumed to be the maximal element of some $\mathcal{N}$, that is comparable to both minimal elements, as illustrated below.*

Proof. By the symmetry of both the statements and the structure of $\mathcal{N}$ (under taking complements), it is enough to prove the results for $\cup_{i=1}^l B_i$.

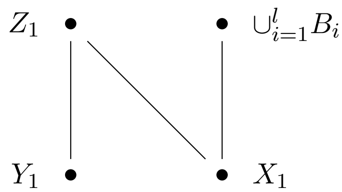
Suppose that $\cup_{i=1}^l B_i$ is a minimal element of a $\mathcal{N}$ formed with 3 other elements of $\mathcal{F}$. Then it is either the minimal element comparable with both maximal elements (figure (i) below), or it is the minimal element comparable to exactly one maximal element (figure (ii) below). Let X denote the maximal element comparable with both minimal elements.

(i) (ii)

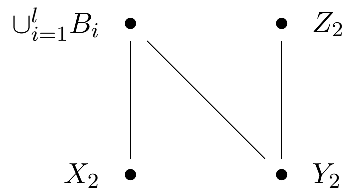
In the first case, since $\cup_{i=1}^{l} B_i$ is incomparable to Z, $Z \not\subseteq B_i$ for all $1 \leq i \leq l$. Moreover, there exists $B_j \not\subset Z$ for some $1 \leq j \leq l$. This implies that Z, X, Y and B_j form an $\mathcal{N}$ in $\mathcal{F}$, a contradiction.

In the second case, since $\cup_{i=1}^{l} B_i$ is incomparable to Y', then $Y' \not\subseteq B_i$ for all $1 \leq i \leq l$, and there exists B_s such that $B_s \not\subset Y'$ for some $1 \leq s \leq l$. Since B_s and Y' are incomparable, $B_s \not\subseteq Z'$, and since Z' and $\cup_{i=1}^{l} B_i$ are incomparable, $Z' \not\subseteq B_s$. This means that Z' and B_s are incomparable, which yields a copy of $\mathcal{N}$ in $\mathcal{F}$, given by B_s, X, Z' and Y'. This finishes the first part of the lemma.

Therefore, if $\cup_{i=1}^{l} B_i \notin \mathcal{F}$, it can only be a maximal element of any $\mathcal{N}$ formed when added to the family, as shown in below.



(iii)



(iv)

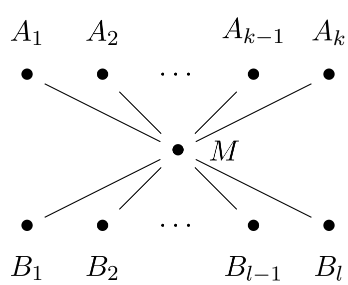
To finish the claim, we show that in fact, we may always assume that $\cup_{i=1}^{l} B_i$ is the maximal element comparable to both minimal elements (figure (iv)).

Suppose that it is not, and therefore $\cup_{i=1}^{l} B_i$ forms an $\mathcal{N}$ with $X_1, Y_1, Z_1 \in \mathcal{F}$, as shown in figure (iii). Since $\cup_{i=1}^{l} B_i$ is incomparable to Z_1, as before, there exists B_j incomparable to Z_1, for some $1 \leq j \leq l$. Furthermore, since $Y_1 \subset Z_1$, $B_j \not\subseteq Y_1$, and since $\cup_{i=1}^{l} B_i$ is incomparable to Y_1 we also get that $Y_1 \not\subset B_j$. Therefore B_j and Y_1 are incomparable. We also note that $B_j \not\subseteq X_1 \subset Z_1$, as B_j is incomparable to Z_1.

To avoid B_j, X_1, Y_1, Z_1 forming an $\mathcal{N}$ in $\mathcal{F}$, we must have $X_1 \not\subset B_j$. Therefore, B_j and X_1 are incomparable. This implies that $X_1, Z_1, \cup_{i=1}^{l} B_i, B_j$ form an $\mathcal{N}$ where $\cup_{i=1}^{l} B_i$ is the maximal comparable to both minimal elements (as shown in figure (iv)). This finishes the claim. $\square$

We are now ready to prove the main result of this section.

Lemma 3. *Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ be sets in $\mathcal{F}$ such that $\cup_{i=1}^{l} B_i \subseteq \cap_{i=1}^{k} A_i$, where $k, l \geq 1$. Then there exists $M \in \mathcal{F}$ such that $\cup_{i=1}^{l} B_i \subseteq M \subseteq \cap_{i=1}^{k} A_i$, as illustrated below.*



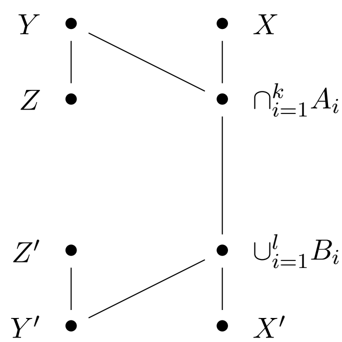
Proof. We observe first that if $k = 1$ or $l = 1$, we are trivially done, so we may assume that $k, l \geq 2$.

Suppose the statement is false. Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ be a configuration as described above, for which there exists no such M, and $|\cap_{i=1}^{k} A_i| - |\cup_{i=1}^{l} B_i|$ is minimal among such configurations.

We begin by observing that if $A_i \subseteq A_j$ for some $i \neq j$, then we can simply omit A_j. Therefore, without loss of generality, we may assume the A_i's are pairwise incomparable, and similarly, that the B_i's are pairwise incomparable.

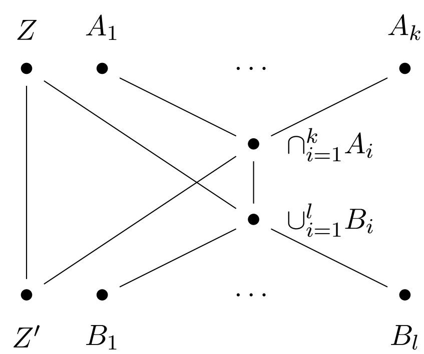
By assumption, there is no $M \in \mathcal{F}$ such that $\cup_{i=1}^{l} B_i \subseteq M \subseteq \cap_{i=1}^{k} A_i$. In particular, this means that $\cup_{i=1}^{l} B_i, \cap_{i=1}^{k} A_i \notin \mathcal{F}$. Therefore, each must form an $\mathcal{N}$ with three other elements of $\mathcal{F}$.

By Lemma 2, $\cup_{i=1}^{l} B_i \neq \cap_{i=1}^{k} A_i$ since $\cap_{i=1}^{k} A_i$ can only be a minimal element of such an $\mathcal{N}$, and $\cup_{i=1}^{l} B_i$ can only be a maximal element of such an $\mathcal{N}$. Thus $\cup_{i=1}^{l} B_i \subset \cap_{i=1}^{k} A_i$. Moreover, putting everything together, Lemma 2 gives us the following diagram for some $X, Y, Z, X', Y', Z' \in \mathcal{F}$.



First, we observe that all sets in the diagram are pairwise distinct. Indeed $\cap_{i=1}^{l} A_i, \cup_{i=1}^{l} B_i \notin \mathcal{F}$ are distinct from $X, X', Y, Y', Z, Z' \in \mathcal{F}$. Moreover, $\cup_{i=1}^{l} B_i \subset \cap_{i=1}^{k} A_i \subset X, Y$, thus X and Y are distinct from X', Y', Z'. Similarly X' and Y' are distinct from X, Y, Z. We are left to show that $Z \neq Z'$. However, if $Z = Z'$, then X, Y, Z, X' would form an $\mathcal{N}$ in $\mathcal{F}$, a contradiction.

Next, since by Lemma 2 $\cap_{i=1}^{k} A_i$ cannot be the maximal element of an $\mathcal{N}$, we get that Z', Y', X' and $\cap_{i=1}^{k} A_i$ cannot form an $\mathcal{N}$, thus $Z' \subset \cap_{i=1}^{k} A_i$. Similarly, we also must have $\cup_{i=1}^{l} B_i \subset Z$. We further observe that Z cannot be a subset of Z' as Z' is a subset of $\cap_{i=1}^{k} A_i$, and if Z and Z' were incomparable, then X, Y, Z, Z' would for an $\mathcal{N}$ in $\mathcal{F}$, a contradiction. Therefore we must have $Z' \subset Z$, and consequently $Z' \bigcup (\cup_{i=1}^{l} B_i) \subseteq Z \bigcap (\cap_{i=1}^{k} A_i)$, as depicted in the diagram below.



Using the fact that Z and $\cap_{i=1}^{k} A_i$ are incomparable, and that Z' and $\cup_{i=1}^{l} B_i$ are incomparable, we get that $\cup_{i=1}^{l} B_i \subset Z' \bigcup (\cup_{i=1}^{l} B_i) \subseteq Z \bigcap (\cap_{i=1}^{k} A_i) \subset \cap_{i=1}^{k} A_i$. Therefore $(Z \bigcap (\cap_{i=1}^{k} A_i)) \setminus (Z' \bigcup (\cup_{i=1}^{l} B_i)) \subset \cap_{i=1}^{k} A_i \setminus \cup_{i=1}^{l} B_i$.

By the minimality of $|\cap_{i=1}^{k} A_i| - |\cup_{i=1}^{l} B_i|$, the claim must hold for the configuration $Z, A_1, \ldots, A_k$, $Z', B_1, \ldots, B_l$. Thus, there exists $M \in \mathcal{F}$ such that $Z' \bigcup (\cup_{i=1}^{l} B_i) \subseteq M \subseteq Z \bigcap (\cap_{i=1}^{k} A_i)$. This however implies that $\cup_{i=1}^{l} B_i \subseteq M \subseteq \cap_{i=1}^{k} A_i$, a contradiction, finishing the proof of the lemma. $\square$

The above lemma will be used in the paper in the case where $k = l = 2$, or in other words, whenever we have butterflies in our $\mathcal{N}$-saturated family.

Corollary 4. *Let $\mathcal{F}$ be an $\mathcal{N}$-saturated family, and $A, B, C, D \in \mathcal{F}$ such that they form a butterfly with maximal elements A, B. Then there exists some $M \in \mathcal{F}$ such that $C \cup D \subseteq M \subseteq A \cap B$.*

Let $\mathcal{G}$ be a component of $\mathcal{F}$. Lemma 1 and 3 tell us that $\mathcal{G}$ has a 'midpoint' between its maximal elements and its minimal elements. It turns out that at least one of these 'midpoints' is a 'one-way valve' in the sense that everything in the component is comparable to it.

Lemma 5. *Let $\mathcal{G}$ be a component of $\mathcal{F}$. Let $B_1, \dots, B_l$ be the minimal elements, and $A_1, \dots, A_k$ be the maximal elements of $\mathcal{G}$. Let $M \in \mathcal{F}$ be such that $\cup_{i=1}^{l} B_i \subseteq M \subseteq \cap_{i=1}^{k} A_i$ and M minimal with this property. Then every $X \in \mathcal{G}$ is comparable to M.*

Proof. Suppose there exists $X \in \mathcal{G}$ such that M and X are incomparable. Since $\cup_{i=1}^{l} B_i \subseteq M$, then either $\cup_{i=1}^{l} B_i \subseteq X$, or $\cup_{i=1}^{l} B_i$ and X are incomparable. In the former case, by Lemma 3, there exists M' such that $B_1, \dots, B_l$ are subsets of M', and M' is a subset of both X and M, contradicting the minimality of M. In the latter case, there exists B_j such that B_j and X are incomparable for some $1 \leq j \leq l$. Furthermore, since $X \in \mathcal{G}$, there exists a minimal element comparable to it. Without loss of generality, let $B_1 \subset X$, which also implies that $j \neq 1$. Then B_1, X, M, B_j form an $\mathcal{N}$, a contradiction. $\square$

By symmetry, we of course have the following.

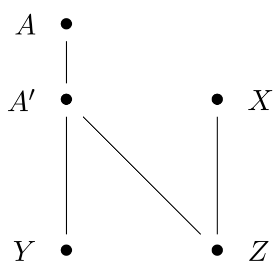
Lemma 6. *Let $\mathcal{G}$ be a component of $\mathcal{F}$. Let $B_1, \dots, B_l$ be the minimal elements, and $A_1, \dots, A_k$ be the maximal elements of $\mathcal{G}$. Let $M \in \mathcal{F}$ be such that $\cup_{i=1}^{l} B_i \subseteq M \subseteq \cap_{i=1}^{k} A_i$ and M maximal with this property. Then every $X \in \mathcal{G}$ is comparable to M.*

3 The setup and preliminary lemmas

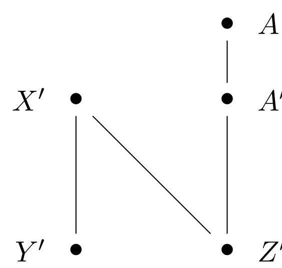
Let $\mathcal{F}$ be an $\mathcal{N}$-saturated family with ground set $[n]$, and $\mathcal{G}$ a component of $\mathcal{F}$. Let $M \in \mathcal{F}$ be a set less than all maximal elements and greater than all minimal elements of $\mathcal{G}$, which exists by Lemma 3, and is minimal with this property. In this section we explore some properties of $\mathcal{F}$ that will be heavily used later in the paper.

Lemma 7. *Suppose $A' \notin \mathcal{F}$ is a set for which there exists $A \in \mathcal{F}$ such that $A' \subset A$, and A' is a maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{A'\}$. Then we may assume that the other maximal element is also a subset of A.*

Proof. Since $A' \notin \mathcal{F}$ is a maximal element of some $\mathcal{N}$ in $\mathcal{F} \cup \{A'\}$, then we either have figure (i), or figure (ii) depicted below, for some $X, Y, Z, X', Y', Z' \in \mathcal{F}$.



(i)



(ii)

In the first case (figure (i)), A, X, Y, Z cannot form an $\mathcal{N}$, as they are all elements of $\mathcal{F}$. Thus A and X must be comparable. Since $A' \subset A$ and A' and X are incomparable, we must have $X \subset A$, as desired.

In the second case (figure (ii)), in order for A, X', Y', Z' to not form an $\mathcal{N}$ in $\mathcal{F}$, A must be comparable to X' or Y'. If X' and A are comparable, as before, we must have $X' \subset A$. If X' and A are incomparable, then Y' and A are comparable, and consequently $Y' \subset A$. This produces a butterfly in $\mathcal{F}$, namely X', Y', Z', A. By Corollary 4, there exists $M' \in \mathcal{F}$ such that $Y', Z' \subset M' \subset X', A$. We observe now that M' is incomparable to A'. Indeed, if $M' \subseteq A'$, then $Y' \subset A'$, a contradiction, and if $A' \subset M'$, then $A' \subset X'$, a contradiction. Therefore M', A', Y', Z' form an $\mathcal{N}$ in which the other maximal element is M', a subset of A, as desired. $\square$

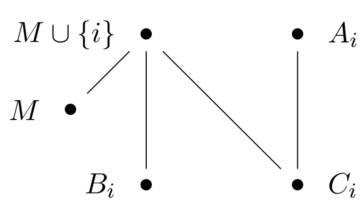
By symmetry, we also have the following lemma.

Lemma 8. *Suppose $B' \notin \mathcal{F}$ is a set for which there exists $B \in \mathcal{F}$ such that $B \subset B'$, and B' is a minimal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{B'\}$. Then we may assume that the other minimal element also contains B as a subset.*

Lemma 9. *Suppose $i \notin M$ and $M \cup \{i\} \notin \mathcal{F}$. If $M \cup \{i\}$ is a maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{M \cup \{i\}\}$, then we may assume that it is the maximal comparable to both minimal elements, and that one of the minimal elements is M. Moreover, there are at most $|\mathcal{F}| - 2$ such singletons.*

Proof. Since $M \cup \{i\} \notin \mathcal{F}$, then it must form an $\mathcal{N}$ when added to $\mathcal{F}$. If it is a maximal element of such an $\mathcal{N}$ then one of the two cases below holds.

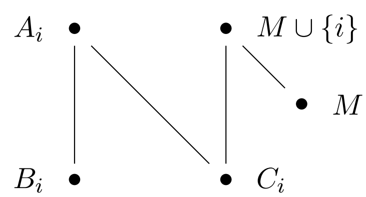
Case 1. $M \cup \{i\}$ is comparable to both minimal elements of $\mathcal{N}$, as illustrated below, where $A_i, B_i, C_i \in \mathcal{F}$.



If $C_i = M$, or $B_i = M$, then we are done. Therefore, we may assume that $B_i \neq M$ and $C_i \neq M$. Now, if both A_i and C_i are incomparable to M, then $M \cup \{i\}, M, A_i, C_i$ form an $\mathcal{N}$ where one of the minimal elements is M, as desired. Therefore, we may assume that at least one of A_i or C_i is comparable to M, which implies that A_i, C_i, M are in the same component of $\mathcal{F}$. By Lemma 5, both A_i, C_i are comparable to M. Since A_i and $M \cup \{i\}$ are incomparable, we must have $M \subset A_i$. Also, we cannot have $M \subseteq C_i$ as $C_i \subset M \cup \{i\}$ and $C_i \neq M$. Thus $C_i \subset M \subset A_i$.

Finally, we observe that B_i and M are incomparable. Indeed, by cardinality we cannot have $M \subset B_i$, and if $B_i \subset M$ then $B_i \subset A_i$, a contradiction. Therefore $M \cup \{i\}, M, A_i, B_i$ form the desired $\mathcal{N}$.

Case 2. $M \cup \{i\}$ is comparable to exactly one minimal element of $\mathcal{N}$, as illustrated below, where $A_i, B_i, C_i \in \mathcal{F}$.



If $C_i = M$, then A_i, B_i, M are in the same component of $\mathcal{F}$. However, B_i is incomparable to $C_i = M$, which contradicts Lemma 5. Therefore $C_i \neq M$.

If both A_i and C_i are incomparable to M, then $M \cup \{i\}, M, A_i, C_i$ form the desired $\mathcal{N}$, where $M \cup \{i\}$ is the maximal comparable to both minimal elements, and M is one of the minimal elements. Therefore, we may assume that at least one of A_i or C_i is comparable to M. This now means that A_i, B_i, C_i, M are in the same component of $\mathcal{F}$, thus, by Lemma 5, A_i, B_i, C_i are all comparable to M. Since B_i is incomparable to $M \cup \{i\}$, we must have $M \subset B_i$. Moreover, since $C_i \subset M \cup \{i\}$ and $C_i \neq M$, we must have $C_i \subset M$. But this implies that $C_i \subset M \subset B_i$, a contradiction.

For the second part of the lemma, pick X_i to be the other minimal element (distinct from M) of the $\mathcal{N}$ in $\mathcal{F} \cup \{M \cup \{i\}\}$ established earlier. We therefore have $X_i \subset M \cup \{i\}$, and M and X_i are incomparable, which implies that $i \in X_i$. Thus, $X_i \neq X_j$ if $i \neq j$, as otherwise $i \in X_i = X_j$, which implies that $i \in M \cup \{j\}$, and so $i \in M$, a contradiction. Moreover, $X_i \notin \{\emptyset, [n]\}$ as it is incomparable to M. Putting everything together we get that there are at most $|\mathcal{F}| - 2$ singletons i such that $i \notin M$, and $M \cup \{i\} \notin \mathcal{F}$ is a maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{M \cup \{i\}\}$. $\square$

The following lemma allows us to assume that the set M has relatively small cardinality.

Lemma 10. *There exists an $\mathcal{N}$-saturated family $\mathcal{F}'$ with ground set $[n]$ such that $|\mathcal{F}| = |\mathcal{F}'|$, and at least one of its component $\mathcal{G}'$ has a set of size at most $n/2$ that is comparable to all elements of the component.*

Proof. Let $\mathcal{G}$ be a component of $\mathcal{F}$. If the minimal set M that is between the maximal and the minimal elements of $\mathcal{G}$ satisfies $|M| \leq n/2$, we are done. Otherwise, let $\mathcal{F}' = \{[n] \setminus X : X \in \mathcal{F}\}$. Since $\mathcal{N}$ is invariant under taking complements, $\mathcal{F}'$ is an $\mathcal{N}$-saturated family with $\mathcal{G}' = \{[n] \setminus X : X \in \mathcal{G}\}$ as one of its components. Moreover, $|\mathcal{F}| = |\mathcal{F}'|$. Let M' be a minimal set comparable to all sets of $\mathcal{G}'$, which exists by Lemma 5. Therefore $M' \subseteq [n] \setminus M$, thus $|M'| \leq |[n] \setminus M| < n/2$, as desired. $\square$

4 Proof of the main result

We are now ready to prove that the saturation number of $\mathcal{N}$ is at least linear. The main part of the proof is in fact Proposition 11, which easily implies the linear lower bound in Theorem 12.

Proposition 11. *Let $\mathcal{F}$ be an $\mathcal{N}$-saturated family with ground set $[n]$. Let $\mathcal{G}$ be a component of $\mathcal{F}$ such that there exists $M \in \mathcal{F}$ minimal with the property that it is comparable to all sets of $\mathcal{G}$, and $|M| \le n/2$. Then, for at least $n/2 - |\mathcal{F}| + 2$ singletons $i \in [n]$, there exists S_i such that $i \notin S_i$ and $S_i, S_i \cup \{i\} \in \mathcal{F}$.*

Proof. Since $|M| \le n/2$, then exist at least $n/2$ singletons i such that $i \notin M$. By Lemma 9, there are at least $n/2 - (|\mathcal{F}| - 2)$ of singletons i such that $i \notin M$, and either $M \cup \{i\} \in \mathcal{F}$, or $M \cup \{i\} \notin \mathcal{F}$ and it can only be a minimal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{M \cup \{i\}\}$. We will show that for every such i, there exists S_i such that $i \notin S_i$ and $S_i, S_i \cup \{i\} \in \mathcal{F}$, which will finish the proof.

Suppose for a contradiction that there exists such a singleton i for which we cannot find two sets in $\mathcal{F}$ that differ exactly by i. This implies that $M \cup \{i\} \notin \mathcal{F}$, and by assumption, it can only be a minimal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{M \cup \{i\}\}$, as illustrated below, where A is the maximal element comparable to both minimal elements, and B is the other minimal element of the $\mathcal{N}$.

(1) (2)

Among all the possible $\mathcal{N}$ formed with $M \cup \{i\}$, we pick the one having $|B|$ maximal, and then under this choice of B, choose $|A|$ minimal. Since $M \subset M \cup \{i\}$, we have that A, B, C are in the same component of $\mathcal{F}$ as M, therefore they are all comparable to M by Lemma 5. This tells us that $M \subset B$. Since $M \cup \{i\}$ and B are incomparable, we must have that $i \notin B$. Moreover, in case (1), $B \subset C$, and since $M \cup \{i\}$ and C are incomparable, we also get that $i \notin C$. By our assumption, $B \cup \{i\} \notin \mathcal{F}$, thus it must form an $\mathcal{N}$ when added to $\mathcal{F}$.

Claim A. *$B \cup \{i\}$ cannot be a minimal element of any $\mathcal{N}$ in $\mathcal{F} \cup \{B \cup \{i\}\}$.*

Proof. Suppose that $B \cup \{i\}$ is one of the minimal elements of an $\mathcal{N}$. Then one of the two cases depicted below holds, where $X, Y, Z, X', Y', Z' \in \mathcal{F}$.

By Lemma 8 we may assume that $B \subset Z$ in the first case, and $B \subset Z'$ in the second case. Since $B \cup \{i\}$ incomparable to Z (or Z'), we must have that $i \notin Z$ (or $i \notin Z'$). Since $M \subset B \subset Z$ (or $M \subset B \subset Z'$), we get that $M \cup \{i\}$ is incomparable to Z (or Z'). This now implies that

$M \cup \{i\}, X, Y, Z$ (or $M \cup \{i\}, X', Y', Z'$) form an $\mathcal{N}$ with $M \cup \{i\}, Z$ (or Z') being the minimal elements, contradicting the maximality of $|B|$. $\square$

Claim B. *There exists $D \in \mathcal{F}$ such that $M \cup \{i\} \subset D \subset B \cup \{i\}$.*

Proof. By Claim A, $B \cup \{i\}$ is always a maximal element of any $\mathcal{N}$ in $\mathcal{F} \cup \{B \cup \{i\}\}$. We therefore have two cases, depicted below, for some $X, Y, Z, X', Y', Z' \in \mathcal{F}$.

(i) (ii)

In case (i), if $i \notin Y \cup Z$, then $Y, Z \subseteq B$, and $B \not\subseteq X$ as Y is incomparable to X, so $Z \neq B$. Moreover, since X is incomparable to $B \cup \{i\}$, $X \not\subseteq B$. Therefore X and B are incomparable. If $Y \neq B$, then B, X, Y, Z form an $\mathcal{N}$ in $\mathcal{F}$, a contradiction. If $Y = B$, then Z is incomparable to B but $Z \subset B \cup \{i\}$, so $i \in Z$, a contradiction. Hence $i \in Y$ or $i \in Z$, and let $D = Y$ or $D = Z$ such that $i \in D$.

Since $Y, Z \subset A$ and $M \subset A$, all of X, Y, Z, A, M are in the same component $\mathcal{G}$ of $\mathcal{F}$, thus comparable to M by Lemma 5. Since $i \notin M$, we can only have $M \subset D$. This immediately implies that $M \cup \{i\} \subset D \subset B \cup \{i\}$ as desired.

In case (ii), we begin by showing that $i \in Z'$. Suppose that $i \notin Z'$, then $Z' \subseteq B$. We also observe that Y' is incomparable to B, since Y' is incomparable to both Z' and $B \cup \{i\}$. Next, X', Y', Z', B cannot form an $\mathcal{N}$ in $\mathcal{F}$, so we must have $B \subset X'$ or $B = Z'$. In either case we get $M \subset B \subset X'$. Since X' is incomparable to $B \cup \{i\}$, we get that $i \notin X'$, and consequently X' is incomparable to $M \cup \{i\}$. Furthermore, by Lemma 7, we may assume that $X' \subset A$.

Putting everything together we get that X', Y', Z' are all in the component $\mathcal{G}$ of $\mathcal{F}$, and hence comparable to M. In particular, this means that $M \subset Y'$, as otherwise $Y' \subseteq M \subset B \cup \{i\}$, a contradiction.

We recall that $M \cup \{i\}$ is a minimal element of an $\mathcal{N}$ as depicted in the first picture of this proof. We consider the cases (1) and (2) separately.

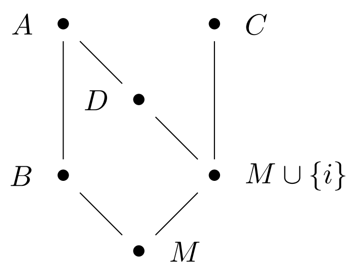
In the first case corresponding to figure (1), we see that $Y' \subset X' \subset A$. Since A, B, C, Y' cannot form an $\mathcal{N}$, we must have $Y' \subset C$. But now A, B, C, Y' form a butterfly, and so, by Corollary 4, there exists $M' \in \mathcal{F}$ such that $Y', B \subset M' \subset A, C$. We observe that $M \subset C$ (to avoid A, M, B, C from forming an $\mathcal{N}$), thus $i \notin C$. Since $M' \subset C$, we also get that $i \notin M'$. Moreover, since $M \subset Y' \subset M'$, we get that M' is incomparable to $M \cup \{i\}$. Finally, $M \cup \{i\}, A, M', C$ form an $\mathcal{N}$, and $B \subset M'$, contradicting the maximality of $|B|$.

In the second case corresponding to figure (2), we first note that X' is incomparable to C. Indeed, if $X' \subset C$ then $B \subset C$, a contradiction, and since $i \in C$, we cannot have $C \subseteq X'$. Therefore $M \cup \{i\}, A, X', C$ form an $\mathcal{N}$, which contradicts the maximality of $|B|$ as $B \subset X'$.

Therefore we must have that $i \in Z'$. Let $D = Z' \subset B \cup \{i\}$. Since $D \subset A$, we get that $D \in \mathcal{G}$, thus D is comparable to M. This gives $M \subset D$ since $i \in D$. Finally, we get that $M \cup \{i\} \subset D \subset B \cup \{i\}$ as claimed. $\square$

We now observe now that Claim B implies that case (1) cannot happen, i.e. $M \cup \{i\}$ cannot be the minimal element of an $\mathcal{N}$ that is comparable to exactly one maximal (as depicted in (1)). Indeed, if that were the case, we have that $D \subset B \cup \{i\} \subset A$, and C and D are incomparable (as $i \notin C$, $i \in D$, and $C \not\subset D$ otherwise $C \subset A$). This now implies that A, B, C, D form an $\mathcal{N}$ in $\mathcal{F}$, a contradiction.

Therefore $M \cup \{i\}$ can only be the minimal element of an $\mathcal{N}$ that is comparable to both maximal elements. In that case, B and D are incomparable ($i \in D$, so $D \nsubseteq B$, and $B \not\subset D$ since $D \subset B \cup \{i\}$), which in turn implies that D and C are incomparable (if $D \subset C$, then A, B, D, C form an $\mathcal{N}$ in $\mathcal{F}$, and if $C \subseteq D$, then $C \subset A$). Threfore we have the following Hasse diagram.



The next claim tells us that, by our choice of A and B, there is not much 'room' between them.

Claim C. *Let $T, T' \in \mathcal{F}$. If $B \subseteq T \subseteq A$, then $T = B$ or $T = A$. Also, if $B \subset T'$, then $A \subseteq T'$.*

Proof. Suppose that $T \neq A$ and $T \neq B$. Then $M \subset B \subset T \subset A$. Since C is incomparable to both A and B, it is therefore incomparable to T. If $i \notin T$, then $A, T, C, M \cup \{i\}$ form an $\mathcal{N}$, contradicting the maximality of $|B|$. If $i \in T$, then $T, B, C, M \cup \{i\}$ form an $\mathcal{N}$, contradicting the minimality of $|A|$. Thus, $T = A$ or $T = B$.

Next, suppose that $B \subset T'$ but $A \nsubseteq T'$. By the first part of the claim, $T' \not\subset A$, hence T must be incomparable to A. For T, B, A, D to not form an $\mathcal{N}$ in $\mathcal{F}$, we must have have $D \subset T$. However, now T, B, A, D form a butterfly in $\mathcal{F}$. By Corollary 4, there exists $M' \in \mathcal{F}$ such that $B, D \subset M' \subset A, T$, contradicting the first part of the claim. $\square$

We now look at $A \setminus \{i\}$. By assumption, $A \setminus \{i\} \notin \mathcal{F}$, thus it must form an $\mathcal{N}$ when added to $\mathcal{F}$.

Claim D. *$A \setminus \{i\}$ cannot be a minimal element of any $\mathcal{N}$ in $\mathcal{F} \cup \{A \setminus \{i\}\}$.*

Proof. By assumption, $A \setminus \{i\} \notin \mathcal{F}$, thus it must form an $\mathcal{N}$ when added to $\mathcal{F}$. Suppose it is one of the minimal elements of such an $\mathcal{N}$. We therefore have the two cases illustrated below, where $X, Y, Z, X', Y', Z' \in \mathcal{F}$.

By Lemma 8, we may assume that $B \subset Z$ in the first case, and $B \subset Z'$ in the second case. Consequently, this gives by Claim C that $A \subseteq Z$ (or Z' in the second case), thus $A \setminus \{i\} \subset Z$ (or Z' in the second case), a contradiction. $\square$

We are now going to inductively define sets $A_m, B_m \in \mathcal{F}$ for all integers $m \geq 0$ as follows. We set $A_0 = A$ and $B_0 = B$. We know that $A_0 \setminus \{i\} \notin \mathcal{F}$ can only be a maximal element of any $\mathcal{N}$ in $\mathcal{F} \cup \{A_0 \setminus \{i\}\}$. Among all such possible configurations, we define B_1 to be the minimal element comparable to both maximal elements of such an $\mathcal{N}$, of maximum cardinality. Having chosen B_1, we then define A_1 to be the other maximal element of such an $\mathcal{N}$ with minimal cardinality. This is illustrated below, where $K_1, K_1' \in \mathcal{F}$.

We will show that $i \in A_1$ and that $A_1 \setminus \{i\} \notin \mathcal{F}$ can only be the maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{A \setminus \{i\}\}$. After that, we repeat the process, where we define $B_2 \subset A_2$ from $B_1 \subset A_1$, the same way we defined $B_1 \subset A_1$ from $B_0 \subset A_0$. In order to show that the process can always continue, we will show, by induction on m the following properties.

Claim 1. *$A_m \subset A_{m-1}$, and A_m is incomparable to B_{m-1}. Moreover, we either have B_m and B_{m-1} incomparable, or $B_m \subset B_{m-1}$.*

Claim 2. *If $B_m \subset T$, where $T \in \mathcal{F}$, then T comparable to A_{m-1}.*

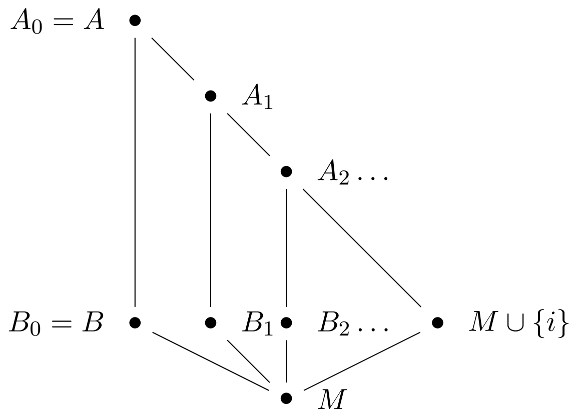
Claim 3. *$B_m \cup \{i\}$ can only be a maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{B_m \cup \{i\}\}$.*

Claim 4. *There exists $D_m \in \mathcal{F}$ such that $M \cup \{i\} \subset D_m \subset B_m \cup \{i\}$.*

Claim 5. *Let $T \in \mathcal{F}$. If $B_m \subseteq T \subseteq A_m$, then $T = A_m$ or $T = B_m$. If $B_m \subset T$, then $A_m \subseteq T$.*

Claim 6. *$A_m \setminus \{i\}$ can only be a maximal element in an $\mathcal{N}$ in $\mathcal{F} \cup \{A_m \setminus \{i\}\}$.*

Note that since $B_m \subset A_{m-1} \setminus \{i\}$, we have by construction that $i \notin B_m$ for all $m \geq 0$. Moreover, since $A_m \subset A_{m-1}$, but A_m and $A_{m-1} \setminus \{i\}$ are incomparable, we must have $i \in A_m$ for all $m \geq 0$. These observations, together with Claim 4, give us the following picture.

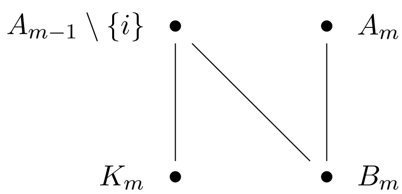


Once these claims have been established, a contradiction is immediately reached as the process must terminate between A_0 and $M \cup \{i\}$, hence in finite number of steps.

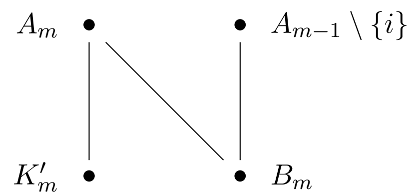
We prove these claims by induction on m. When $m = 0$, Claims 1 and 2 are vacuously true, and Claims 3, 4, 5 and 6 are Claims A, B, C and D from above. Let now $m \geq 1$, and assume that they are true for all $m' < m$.

Claim 1. *$A_m \subset A_{m-1}$, and A_m is incomparable to B_{m-1}. Moreover, we either have B_m and B_{m-1} incomparable, or $B_m \subset B_{m-1}$.*

Proof. By construction, A_m and $A_{m-1} \setminus \{i\}$ are part of an $\mathcal{N}$, and we have one of the two following cases illustrated below, where $K_m, K'_m \in \mathcal{F}$.



(1.1)



(1.2)

We show first that we always have $A_m \subset A_{m-1}$. Indeed, this must be the case in the picture on the left, as otherwise A_{m-1}, A_m, B_m, K_m would form an $\mathcal{N}$ in $\mathcal{F}$. In the picture on the right, if $A_m \not\subset A_{m-1}$, they are incomparable as we cannot have $A_{m-1} \setminus \{i\} \subset A_{m-1} \subseteq A_m$. Thus, in order for A_{m-1}, A_m, B_m, K'_m to not form an $\mathcal{N}$ in $\mathcal{F}$, we must have $K'_m \subset A_{m-1}$. This forms a butterfly, namely A_{m-1}, A_m, B_m, K'_m. By Corollary 4, there exists $M' \in \mathcal{F}$ such that $K'_m, B_m \subset M' \subset A_m, A_{m-1}$. Since $K'_m \subset A_{m-1}$, but it is incomparable to $A_{m-1} \setminus \{i\}$, we have that $i \in K'_m$, and so $i \in M'$. Since $M' \subset A_{m-1}$, we get that M' is incomparable to $A_{m-1} \setminus \{i\}$. Therefore $M', K'_m, B_m, A_{m-1} \setminus \{i\}$ form an $\mathcal{N}$, contradicting the minimality of $|A_m|$. Therefore $A_m \subset A_{m-1}$. This also implies that $i \in A_m$.

Next, suppose that A_m and B_{m-1} are comparable. If $B_{m-1} \subseteq A_m \subset A_{m-1}$, we get by Claim 5 for $m-1$ that $B_{m-1} = A_m$, a contradiction as $i \notin B_{m-1}$. For the same reason, we also cannot have $A_m \subset B_{m-1}$. Therefore, A_m and B_{m-1} are incomparable. Finally, we cannot have $B_{m-1} \subseteq B_m$, otherwise $B_{m-1} \subseteq B_m \subset A_m$, a contradiction. $\square$

We now make a couple of useful observations that will help in future analysis.

If $A_{m-1}\setminus\{i\}$ is the maximal comparable to both minimal elements of the $\mathcal{N}$, and K_m is the other minimal (figure 1.1), if $B_m \subset B_{m-1}$, then B_{m-1} and K_m are incomparable. Indeed, $B_{m-1} \not\subseteq K_m$, otherwise $B_m \subset K_m$. Moreover, $K_m \not\subset B_{m-1}$, otherwise B_{m-1}, K_m, B_m, A_m form an $\mathcal{N}$ in $\mathcal{F}$.

If $A_{m-1} \setminus \{i\}$ is only comparable to B_m, and K'_m is the other minimal of the $\mathcal{N}$ (figure 1.2), then B_{m-1} and B_m are incomparable. Indeed, if $B_m \subset B_{m-1}$, then $K'_m \subset B_{m-1}$, otherwise A_m, K'_m, B_m, B_{m-1} would form an $\mathcal{N}$ in $\mathcal{F}$. But then $K'_m \subset A_{m-1} \setminus \{i\}$, a contradiction.

Claim 2. *Let $T \in \mathcal{F}$. If $B_m \subset T$, then T is comparable to A_{m-1}.*

Proof. Suppose that there exists $T \in \mathcal{F}$ such that $B_m \subset T$, but T and A_{m-1} are incomparable, in particular $T \not\subseteq B_{m-1}$. By Claim 5 for $m-1$, we cannot have $B_{m-1} \subset T$, so T and B_{m-1} are incomparable.

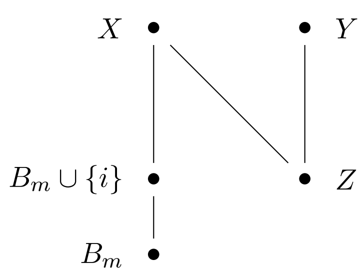
If B_m and B_{m-1} are incomparable, then A_{m-1}, T, B_{m-1}, B_m form an $\mathcal{N}$ in $\mathcal{F}$, a contradiction.

If $B_m \subset B_{m-1}$, then we are in case (1.1) ($A_{m-1} \setminus \{i\}$ is comparable to both minimal elements). Since A_{m-1}, K_m, B_m, T cannot form an $\mathcal{N}$, we must have $K_m \subset T$. Since B_{m-1} is incomparable to K_m, we have that A_{m-1}, B_{m-1}, K_m, T form an $\mathcal{N}$, a contradiction. $\square$

Claim 3. *$B_m \cup \{i\}$ can only be a maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{B_m \cup \{i\}\}$.*

Proof. Suppose that $B_m \cup \{i\} \notin \mathcal{F}$ is a minimal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{i\}$. We will find an $\mathcal{N}$ formed by $A_{m-1} \setminus \{i\}$ which contradicts the maximality of $|B_m|$. We split the analysis into the following natural cases.

Case 1. $B_m \cup \{i\}$ is the minimal element comparable to exactly one maximal element, as illustrated below, where $X, Y, Z \in \mathcal{F}$.



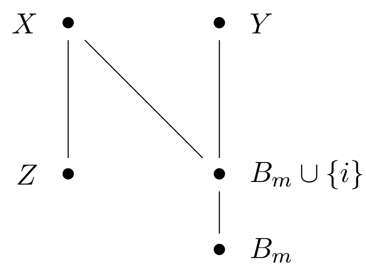
By Lemma 8, we may assume that $B_m \subset Z \subset Y$. Since Z and Y are incomparable to $B_m \cup \{i\}$, we have that $i \notin Z$ and $i \notin Y$. Since $B_m \subset X, Y, Z$, by Claim 2 above, X, Y, Z are all comparable to A_{m-1}. Since $i \in A_{m-1}$, we get that $Y, Z \subset A_{m-1} \setminus \{i\} \subset A_{m-1}$. Since X and Y are incomparable, we must also have $X \subset A_{m-1}$. Since $i \in X$, this implies that X and $A_{m-1} \setminus \{i\}$ are incomparable.

By Claim 5 for $m-1$, we have that $B_{m-1} \not\subset X$, otherwise X lies strictly between B_{m-1}, A_{m-1}. Also, since $B_{m-1} \subset A_{m-1} \setminus \{i\}$, we cannot have $X \subseteq B_{m-1}$. Therefore X and B_{m-1} are incomparable. Consequently $B_{m-1} \not\subseteq Z$, so Z is either incomparable to B_{m-1}, or $Z \subset B_{m-1}$.

If B_{m-1} and Z are incomparable, then $A_{m-1} \setminus \{i\}, B_{m-1}, Z, X$ form an $\mathcal{N}$, contradicting the maximality of $|B_m|$. Therefore $Z \subset B_{m-1}$, which implies that $B_m \subset B_{m-1}$. This can only happen in the case where $A_{m-1} \setminus \{i\}$ is the maximal element comparable to both minimal elements (figure 1.1). As remarked above, in this case, B_{m-1} and K_m are incomparable, so $K_m \not\subseteq Z$. Moreover, $Z \not\subset K_m$, as $B_m \subset Z$ and B_m and K_m are incomparable. Therefore, Z and K_m are incomparable.

Since X, Z, B_{m-1}, K_m cannot form an $\mathcal{N}$, we must have K_m and X incomparable ($X \not\subseteq K_m$ as B_m and K_m are incomparable). But then $A_{m-1} \setminus \{i\}, K_m, Z, X$ form an $\mathcal{N}$, contradicting the maximality of $|B_m|$.

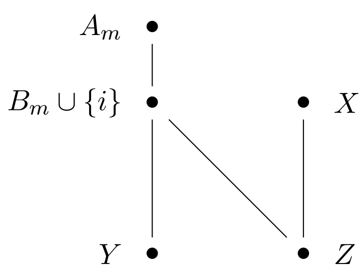
Case 2. $B_m \cup \{i\}$ is the minimal element comparable to both maximal elements, as illustrated below, where $X', Y', Z' \in \mathcal{F}$.



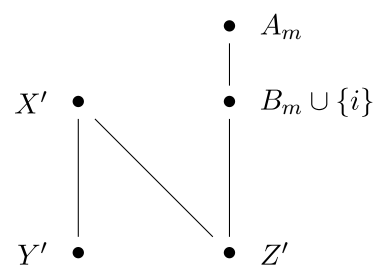
Again, we may assume that $B_m \subset X, Y, Z$, which by Claim 2 gives that X, Y, Z are all comparable to A_{m-1}. Since $B_m \subset Z$, but $B_m \cup \{i\} \not\subset Z$, we have that $i \notin Z$, thus $Z \subset A_{m-1} \setminus \{i\}$. Since Y is incomparable to Z, we must have that $Y \subset A_{m-1}$, and since X is incomparable to Y, we must have that $X \subset A_{m-1}$. Then, the exact argument as in Case 1 follows. □

Claim 4. *There exists $D_m \in \mathcal{F}$ such that $M \cup \{i\} \subset D_m \subset B_m \cup \{i\}$.*

Proof. By Claim 3, $B_m \cup \{i\}$ must be a maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{B_m \cup \{i\}\}$. We therefore have two cases, as illustrated below, where $X, Y, Z, X', Y', Z' \in \mathcal{F}$.



(b.1)



(b.2)

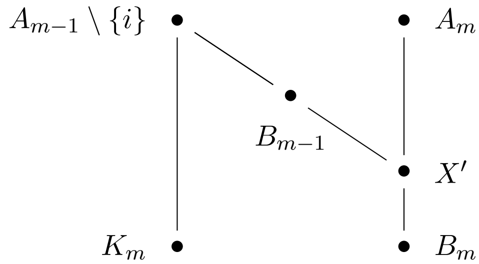
In the first case, diagram (b.1), if $i \notin Y \cup Z$, then $Y, Z \subseteq B_m$. Moreover, $B_m \not\subset X$ as Y is incomparable to X, so $Z \neq B_m$. Since X is incomparable to $B_m \cup \{i\}$, we cannot have $X \subseteq B_m$. Therefore X and B_m are incomparable, and so either $Y \neq B_m$, B_m, X, Y, Z form an $\mathcal{N}$ in $\mathcal{F}$, a contradiction; or $Y = B_m$, then Z is incomparable to B_m but $Z \subset B_m \cup \{i\}$, so $i \in Z$, a contradiction.

Therefore we must have $i \in Y$ or $i \in Z$, and we take $D_m = Y$ or $D_m = Z$ such that $i \in D_m$. Since $Y, Z \subset A_m \subset A$, all of X, Y, Z, A_m, A are in the same component $\mathcal{G}$, thus they must be comparable to M by Lemma 5. Since $i \notin M$, we must have $M \subset D_m$. This gives $M \cup \{i\} \subset D_m \subset B_m \cup \{i\}$.

In the second case, diagram (b.2), we have that $i \in Z'$.

Indeed, suppose that $i \notin Z'$. Then $Z' \subseteq B_m$. Furthermore, since Y' is incomparable to Z' and $B_m \cup \{i\}$, we must have that Y' and B_m are incomparable. In order for X', Y', Z', B_m to not form an $\mathcal{N}$, we must have $B_m \subset X'$. Since X' and $B_m \cup \{i\}$ are incomparable, we get that $i \notin X'$.

By Lemma 7, we may assume that $X' \subset A_m$. By Claim 1 we also have that $A_m \subset A_{m-1}$, thus $X' \subset A_{m-1} - \{i\}$. Since A_m and B_{m-1} are incomparable (Claim 1), X' cannot be incomparable to B_{m-1}, otherwise $A_{m-1} \setminus \{i\}, B_{m-1}, X', A_m$ would form an $\mathcal{N}$, contradicting the maximality of $|B_m|$. Since B_{m-1} and A_m are incomparable, we cannot have $B_{m-1} \subseteq X'$ otherwise $B_{m-1} \subset A_m$, thus $X' \subset B_{m-1}$. This implies that $B_m \subset B_{m-1}$, which means that $A_{m-1} \setminus \{i\}$ is the maximal element comparable to both minimal elements in an $\mathcal{N}$, i.e. we have the picture (1.1) in Claim 1. Putting everything together we get the following diagram.



We know that K_m and B_{m-1} are incomparable, and that B_{m-1} and A_m are incomparable. Since K_m is incomparable to both B_{m-1} and B_m, and $B_m \subset X' \subset B_{m-1}$, X' and K_m must be incomparable too. Therefore $A_{m-1} \setminus \{i\}, K_m, X', A_m$ form an $\mathcal{N}$, contradicting the maximality of $|B_m|$.

Thus, we have $i \in Z'$. Let $D_m = Z'$. Since $Z' \subset A_m \subset A$, we have $D_m \in \mathcal{G}$ and therefore it is comparable to M. As $i \notin M$, we get that $M \subset D_m$, thus $M \cup \{i\} \subset D_m \subset B_m \cup \{i\}$ as desired. □

By the above claim, we have that $D_m \subset A_m$, and since $i \notin B_m$, D_m and B_m are incomparable. Therefore A_m, B_m, D_m form a Λ_2, or a *chevron*. We will use this fact to prove the next claim.

Claim 5. *Let $T, T' \in \mathcal{F}$. If $B_m \subseteq T \subseteq A_m$, then $T = A_m$ or $T = B_m$. If $B_m \subset T'$, then $A_m \subseteq T'$.*

Proof. Suppose that $B_m \subset T \subset A_m$.

If $A_{m-1} \setminus \{i\}$ is the maximal comparable to both minimal elements (figure (1.1) in Claim 1), since K_m is incomparable to both A_m and B_m, we must have that T is incomparable to K_m. If $i \in T$, then $A_{m-1} \setminus \{i\}$ and T are incomparable, and so $A_{m-1} \setminus \{i\}, K_m, B_m, T$ form an $\mathcal{N}$, contradicting the minimality of $|A_m|$. If $i \notin T$, then $T \subset A_m \subset A_{m-1}$, thus $T \subset A_{m-1} \setminus \{i\}$. Therefore $A_{m-1} \setminus \{i\}, K_m, A_m, T$ form an $\mathcal{N}$, contradicting the maximality of $|B_m|$.

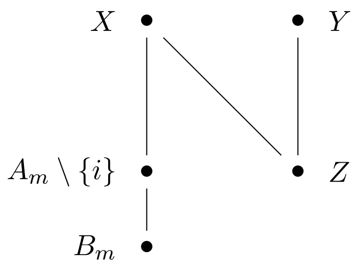
Now suppose that $A_{m-1} \setminus \{i\}$ is the maximal comparable to exactly one minimal element (figure 1.2 in Claim 1). If $i \notin T$, then $T \subset A_{m-1} \setminus \{i\}$ is also incomparable to K'_m. This is because $T \not\subseteq K'_m$ as K'_m and B_m are incomparable, and $K'_m \not\subset T$, otherwise $K'_m \subset T \subset A_{m-1} \setminus \{i\}$. Therefore $A_{m-1} \setminus \{i\}, K'_m, A_m, T$ form an $\mathcal{N}$, contradicting the maximality of $|B_m|$. If $i \in T$, then $T \not\subseteq B_{m-1}$ as $i \notin B_{m-1}$. Also, since $T \subset A_{m-1}$, T and $A_{m-1} \setminus \{i\}$ are incomparable. Moreover, $T \subset A_m \subset A_{m-1}$, so by Claim 5 with $m-1$, we cannot have $B_{m-1} \subset T$. Therefore B_{m-1} and T are incomparable. As remarked at the end of Claim 1, in this case we have that B_{m-1} and B_m are incomparable. Therefore $A_{m-1} \setminus \{i\}, B_{m-1}, T, B_m$ form an $\mathcal{N}$, contradicting the minimality of $|A_m|$.

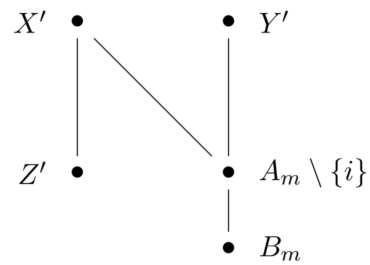
For the second part, suppose that $B_m \subset T'$, but $A_m \not\subseteq T'$. By the first part of the claim, $T' \not\subset A_m$, therefore T' and A_m are incomparable.

We now consider the chevron A_m, B_m, D_m, which must not form an $\mathcal{N}$ with T', thus $D_m \subset T'$. But now A_m, D_m, B_m, T' form a butterfly, so by Corollary 4, there exists $M' \in \mathcal{F}$ strictly between A_m and B_m, contradicting the first part of the claim. □

Claim 6. *$A_m \setminus \{i\} \notin \mathcal{F}$ can only be a maximal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{A_m \setminus \{i\}\}$.*

Proof. Suppose that $A_m \setminus \{i\} \notin \mathcal{F}$ is a minimal element of an $\mathcal{N}$ in $\mathcal{F} \cup \{A_m \setminus \{i\}\}$. Depending on the type of minimal element, we have the following two cases, where $X, Y, Z, X', Y', Z' \in \mathcal{F}$.





By Lemma 8, we may assume that $B_m \subset Z$ in the first case, and $B_m \subset Z'$ in the second case. By Claim 5, we get that $A_m \subseteq Z$, thus $A_m \setminus \{i\} \subset Z$, a contradiction. □

This finishes the inductive step. Therefore, this produces an infinite sequence of sets $A_0 = A \supset A_1 \supset \cdots \supset M \cup \{i\}$. However, A_0 is finite, so the process must terminate, a contradiction.

Therefore, there must exist $S \in \mathcal{F}$ such that $i \notin S$ and $S \cup \{i\} \in \mathcal{F}$, which finishes the proof. □

Theorem 12. *Let $\mathcal{F}$ be an $\mathcal{N}$-saturated family with ground set $[n]$. Then $|\mathcal{F}| \geq \frac{n+6}{4}$.*

Proof. Since we are only interested in the size of $\mathcal{F}$, we may assume by Lemma 10 that $\mathcal{F}$ has a component $\mathcal{G}$ that contains a set M comparable to all elements of $\mathcal{G}$ such that $|M| \leq n/2$. Then, by Proposition 11, there are at least $n/2 - |\mathcal{F}| + 2$ singletons $i \in [n]$ such that there exist $S_i, S_i \cup \{i\} \in \mathcal{F}, i \notin S_i$. For each such i, we fix a choice of S_i with this property. We now construct a graph G with vertex set $\mathcal{F}$, and edges only between the pairs $(S_i, S_i \cup \{i\})$, for all such singletons. By construction, G has at least $n/2 - |\mathcal{F}| + 2$ edges.

We observe that G is acyclic. Indeed, suppose that there exists a cycle $X_1, \ldots, X_{k-1}, X_k \in G$ with $X_k = X_1$, and $(X_i, X_{i+1}) \in E(G)$ for $1 \leq i < k$. Let j_i be the singleton corresponding to the edge (X_i, X_{i+1}), and since the edges are distinct, the j_i's are also distinct. By construction, $j_1 \in X_1$ if and only if $j_1 \notin X_2$, if and only if $j_1 \notin X_3 \ldots$ if and only if $j_1 \notin X_k = X_1$, a contradiction.

Therefore G is indeed acyclic, and so the number of edges is at most $|\mathcal{F}| - 1$, which implies that $n/2 - |\mathcal{F}| + 2 \leq |\mathcal{F}| - 1$, hence $|\mathcal{F}| \geq \frac{n+6}{4}$. □

Theorem 12, together with the fact that $\text{sat}^*(n, \mathcal{N}) \leq 2n$ shows that the saturation number to the $\mathcal{N}$ poset has indeed linear growth.

Theorem 13. *$sat^*(n, \mathcal{N}) = \Theta(n)$.*

5 Concluding remarks and further work

While the saturation number for the $\mathcal{N}$ poset is linear, the precise structure of an $\mathcal{N}$-saturated family remains elusive. Our proof sheds a bit of light on this question via Lemma 3, where we show that in any such family, not just minimal, two antichains, one completely above the other, must have a 'midpoint' between them that lies in the family. Moreover, Lemma 1 tells us that in every connected component of the Hasse diagram of an $\mathcal{N}$-saturated family, there exists a point comparable to every other set of the component. Of course, these features are far from sufficient to characterise $\mathcal{N}$-saturated families, since for example any chain has both of these properties. We therefore ask the following.

Question 14. *Does there exist a structural classification of an $\mathcal{N}$-saturated family? In other words, how does the Hasse diagram of such a family look like?*

Since our proof worked with an arbitrary $\mathcal{N}$-saturated family, it is plausible that if one were to work with a *minimal* $\mathcal{N}$-saturated family, i.e. one such that $|\mathcal{F}| = \text{sat}^*(n, \mathcal{N})$, then more structural properties will emerge, and so we also ask the following.

Question 15. *Let $\mathcal{F}$ be an $\mathcal{N}$-saturated family such that $|\mathcal{F}| = sat^*(n, \mathcal{N})$. How does the Hasse diagram of $\mathcal{F}$ look like?*

It is worth mentioning that while finding the rate of growth of the saturation number is challenging in itself, characterising the saturated families is almost uncharted territory, with the only exception being the antichain, where saturated families are essentially unions of disjoint chains [2].

Finally, one can ask what is the leading constant for $\text{sat}^*(n, \mathcal{N})$. We believe that although our proof gives linearity, it does not give the optimal constant. In fact, we conjecture the following.

Conjecture 16. $sat^*(n, \mathcal{N}) = 2n$.

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