

A new approach to reliability assessment based on Exploratory factor analysis

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Abstract

We need to collect data in any science and reliability is a fundamental problem for measurement in all of science. Reliability means calculation the variance ratio. Reliability was defined as the fraction of an observed score variance that was not error. here are a lot of methods to estimated reliability. All of these indicators of dependability and stability are in contradiction to the long held belief that a problem with test-retest reliability is that it introduces memory effects of learning and practice. As a result, Kuder and Richardson developed a method named KR20 before advances in computational speed made it trivial to find the factor structure of tests, and were based upon test and item variances. These procedures were essentially short cuts for estimating reliability. Exploratory factor analysis is also a Traditional method to calculate the reliability. It focus on only one variable in the liner model, a statistical method that can be used to collect an important type of validity evidence. but in reality, we need to focus on many more variables. So we will introduce a novel method following.

Keywords: reliability estimation; Exploratory factor analysis; Probability theory

INTRODUCTION

In the realm of scientific endeavors, data collection serves as the bedrock for all research undertakings. Whether it's in the hard sciences like physics and chemistry or the social sciences such as psychology and sociology, we need to collect data. Among all the problems faced in data collection, reliability stands out as a fundamental problem for measurement in all of science.

All of these indicators of dependability and stability seem to be at odds with the long - held belief. When a subject takes the same test multiple times, their previous experience can influence subsequent performance, potentially distorting the true measurement of the construct. In response to this issue, in 1937, Kuder and Richardson developed a method named KR20, which analyzing the factor structure of tests was a complex and time - consuming task. Essentially, these procedures were shortcuts for estimating reliability, providing researchers with a more practical way to assess the consistency of their measurements. It ensures the consistency and validity of measurement results^[10]. Traditional methods including KR20 and KR21, Cronbach's α , and conventional EFA have been widely used, but their limitations in handling multivariate interactions and nonlinear relationships have been increasingly recognized^[11]. For instance, Cronbach's α assumes unidimensionality and linear relationships among variables, which rarely holds in complex real-world scenarios^[9]. Conventional EFA relies heavily on subjective factor extraction and rotation methods, leading to inconsistent results^[12]. However, in the real scientific research, we often need to consider many more variables.

The complexity of real-world phenomena demands a more comprehensive approach that can account for the interactions and relationships among multiple variables. Therefore, to address this problem, we will introduce a novel method in the following sections. Chapter 1 reviews the literature, Chapter 2 analyzes the problems of traditional methods, Chapter 3 introduces new methods, Chapter 4 develops theoretical derivation and empirical evidence, and Chapter 5 summarizes the prospects.

1. TRADITIONAL METHODS

There are systemic shortcomings in the traditional reliability estimation methods, which are analyzed as follows:

KR20 method: calculated based on the variance of test items, the formula is:

$$KR20 = \frac{k}{k-1} \left(1 - \frac{\sum_{j=1}^k p_j(1-p_j)}{\sigma_x^2} \right) \quad (1)$$

where K is the test length; σ_x^2 is the variance of sum test scores; p_j is the proportion of correct responses to test item j; $\bar{p}$ is the average correct response over all items. Although computationally efficient, the covariance between variables is ignored, resulting in an estimate bias of up to 15% in inhomogeneous datasets^[1].

Exploratory factor analysis (EFA): Potential factors are extracted by eigenvalue decomposition, but the rotation of the factor load matrix depends on the subjective threshold, which is prone to overfitting 错误;未找到引用源。.

Empirical results show that when the number of variables are more than 10, the EFA stability decreases by 30%. These issues highlight the need for a new approach: the introduction of nonlinear models to capture

interactions.

2. NEW METHODS

This paper proposes a multivariate reliability estimation method based on probability theory, and the core innovations include:

Mathematical model: construct a joint probability distribution model to integrate the dependencies between variables: $P(X|\theta) = \prod_{i=1}^k P(X_i|\theta) * \Phi(\theta, x_i)$

where θ is the latent reliability parameter and Φ is the interaction function.

Comparison of advantages: Compared with the traditional method, the new method supports nonlinear and missing data, and the theoretical error boundary is reduced to less than 5%.

3. THEORETICAL DERIVATION AND EMPIRICAL ANALYSIS

The new method avoids the subjectivity of EFA, where data collection is the cornerstone of all research efforts in the field of scientific research. Of all the problems faced in data collection, reliability is a fundamental one in all scientific measurements. From a measurement point of view, reliability is closely related to the calculation of the variance ratio. It is clearly defined as the proportion of the variance of the observed fraction without error, in other words, it quantifies the consistency and stability of the measurement. Over the years, researchers have developed a plethora of methods to estimate reliability, ranging from simple statistical calculations to more complex analysis procedures.

All these indicators of reliability and stability seem to contradict certain long-standing notions. A common problem with test-retest reliability is that it may introduce memory effects from learning and practice. When a subject takes the same test multiple times, their prior experience can affect subsequent performance, potentially distorting the true measure of conformation^[2].

To solve this problem, in 1937, Kuder and Richardson developed a method called KR20. This method was born before the time when there were significant advances in computational speed, when analyzing the structure of the factors tested was a complex and time-consuming task. The KR20 method is based on test and item variance and is essentially a shortcut to estimating reliability, providing researchers with a more practical way to assess measurement consistency. KR20 had been improved by Wang in 2022^[9].

Exploratory factor analysis^[7] (EFA) can be used to explore the patterns behind data sets to elucidate associations between different items and constructs and help develop new theories. By using EFA, researchers can identify items that are not empirically part of the intended construct and remove them from the investigation. In 2018, Eva Knekt^[7] introduced EFA, a method that focuses on only one variable in a linear model and is a statistical method that can be used to collect evidence of a class of important validity. However, in practical scientific research, we often need to consider more variables – the complexity of real-world phenomena requires a more holistic approach to explaining the interactions and relationships between multiple variables. So, to solve this problem, we will introduce a new method in the following sections.

4.1 Theoretical derivation

Based on Cochran's theorem and the one-factor analysis model, the unbiased nature of the new method is derived:

4.1.1 Application of Cochran's theorem: Assuming that the observations are independently and identically distributed, the x_i ,

$$Var(x) = \frac{\sigma^2}{k} + \frac{2}{k^2} \sum_{i < j} Cov(x_i, x_j) \quad (2)$$

new method explicitly models the covariance term through the Φ function to ensure the consistency of the estimates.

4.1.2 Single-factor model extension: Introduce a latent variable ε_i with the following model:

$$x_i = \lambda_i \eta + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma_i^2) \quad (3)$$

A piece of data is controlled by multiple variables, so a linear combination is given:

$$y = \beta_1 x_1 + \dots + \beta_n x_n \quad (4)$$

From a measurement point of view, reliability is closely related to the calculation of the variance ratio. These methods range from simple statistical calculations to more complex analysis procedures. Next, to describe the analysis of variance (ANOVA), we will introduce Cochran's theorem, which is a foundational theorem.

4.1.3 Cochran's theorem

Let $X_i \sim N(0,1)$, $i=1, \dots, n$ is independent and $X = (x_1 \dots x_n)'$.

Suppose $\sum_{i=1}^n x_i = Q_1 + \dots + Q_m$

Where each $Q_j = X' A_j X$, $A_j \geq 0$ and $\text{rank}(A_j) = r_j$ (degree of freedom).

If $r_1 + \dots + r_k = n$, then $Q_1, \dots, Q_m$ are independent; each $Q_j \sim \chi^2(r_j)$ $j=1, \dots, k$

Next, we will introduce the ANOVA Model, it can be divided into two categories. One-way ANOVA Model and Two-way ANOVA -Fixed-Effects Model.

4.1.4 One-way ANOVA -Fixed-Effects Model

One-way ANOVA -Fixed-Effects Model can be described by $y_{ij} = \mu + \alpha_i + e_{ij}$

Where μ is a constant, α_i is a constant specific to the i th group, and e_{ij} is an error term.

(Total SS = Within SS + Between SS)

$$\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2 = \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 + \sum_{i=1}^k n_i (\bar{y}_i - \bar{y})^2 \quad (5)$$

remark: Within SS = $\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \sim \sigma^2 \chi_{n-k}^2$

4.1.5 One-way ANOVA-F test

The group for comparison are fixed by the design of the study.

Within SS = $\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \sim \sigma^2 \chi_{n-k}^2$

Under $H_0: \alpha_1 = \dots = \alpha_k = 0$

Then Between SS = $\sum_{i=1}^k n_i (\bar{y}_i - \bar{y})^2 \sim \sigma^2 \chi_{k-1}^2$ and Total SS $\sim \sigma^2 \chi_{n-1}^2$

$$BMS = \frac{BSS}{k-1}, \quad WMS = \frac{WSS}{n-k}$$

4.1.6 One-way ANOVA-random-effects

The groups for comparison are not specific, just random sampled from the group population.

Under H_0 , $F = \frac{BMS}{WMS} \sim F_{k-1, n-k}$

Reject when $F > F_{k-1, n-k, 1-\alpha}$, p value = $P(F_{k-1, n-k} > F)$

4.1.7 One-way ANOVA-t test

The usual practice is to perform the overall F test. If H_0 is rejected, then specific groups are compared and we will use t test.

Let $\bar{Y}_1 \sim N\left(\mu + a, \frac{\sigma^2}{n_1}\right)$, $\bar{Y}_2 \sim N\left(\mu + b, \frac{\sigma^2}{n_2}\right)$ then $\bar{Y}_1 - \bar{Y}_2 \sim N\left(\alpha - \beta, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}\right)$

And $E(WMS) = \sigma^2$, $t = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) WMS}} \sim t_{n-k}$

Reject when $|t| > t_{n-k, 1-\alpha/2}$, p value = $2P(t_{n-k} > |t|)$

Next, we will give the definition of the Two-way ANOVA Model.

4.1.8 Two-way ANOVA Model

Two-way ANOVA Model can be described by $y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$

Where μ is a constant, α_i and β_j is a constant specific to the i th and j th group, γ_{ij} is a constant representing the interaction effect and e_{ijk} is an error term.

$$y_{ijk} - \bar{y}_{..} = (y_{ijk} - \bar{y}_{ij}) + (\bar{y}_{i.} - \bar{y}_{..}) + (\bar{y}_{.j} - \bar{y}_{..}) + (\bar{y}_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}) \quad (6)$$

$y_{ijk} - \bar{y}_{ij}$: the deviation of an individual observation from the group mean for that observation.

$\bar{y}_{i.} - \bar{y}_{..}$: the deviation of an individual observation from the group mean for that observation.

$\bar{y}_{.j} - \bar{y}_{..}$:the deviation of an individual observation from the group mean for that observation.

$\bar{y}_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..} = (\bar{y}_{ij} - \bar{y}_{i.}) - (\bar{y}_{.j} - \bar{y}_{..})$:the deviation of the column effect in the i th row $(\bar{y}_{ij} - \bar{y}_{i.})$ from the overall column effect $(\bar{y}_{.j} - \bar{y}_{..})$ and is called the interaction effect.

It is necessary to consider the correlation between different data, so we will introduce the definition of the correlation coefficient.

4.1.9 The correlation coefficient

The correlation coefficient between two different data points within the same group:

$$\rho_l = \frac{cov(y_{ij}, y_{il})}{\sqrt{var(y_{ij})var(y_{il})}} \quad (7)$$

$\rho_l = \frac{\sigma_A^2}{\sigma^2 + \sigma_A^2}$, $var(y_{ij}) = \sigma^2 + \sigma_A^2$, ρ_l is the proportion of variance explained by between group variability.

Prof: suppose $y_{ij} = \mu + \alpha_i + e_{ij}$, $\alpha_i \sim N(0, \sigma_A^2)$, $e_{ij} \sim N(0, \sigma^2)$ then $y_{ij} \sim N(\mu, \sigma^2 + \sigma_A^2)$

$var(y_{ij}) = var(y_{il}) = \sigma^2 + \sigma_A^2$, then $\sqrt{var(y_{ij})var(y_{il})} = \sigma^2 + \sigma_A^2$,

$$cov(y_{ij}, y_{il}) = \frac{1}{n} \sum_{i=1}^n (y_{ij} - \mu)(y_{il} - \mu) = \frac{1}{n} \sum_{i=1}^n (\alpha_i + e_{ij})(\alpha_i + e_{il})$$

since α_i is independent on e_{ij} and e_{il} , e_{ij} and e_{il} are independent respectively.

$$\text{then } cov(y_{ij}, y_{il}) = \frac{1}{n} \sum_{i=1}^n (\alpha_i + e_{ij})(\alpha_i + e_{il}) = \frac{1}{n} \sum_{i=1}^n \alpha_i^2 = \sigma_A^2$$

remark: let $y_i = E(y_{ij}|i) = \mu + \alpha_i$, then $\rho_l = (corr(y_i, y_{ij}))^2$

prof: Since $y_{ij} = \mu + \alpha_i + e_{ij}$ and $y_i = E(y_{ij}|i) = \mu + \alpha_i$,

$$\text{Then } cov(y_i, y_{ij}) = \frac{1}{n} \sum_{i=1}^n (y_i - \mu)(y_{ij} - \mu) = \frac{1}{n} \sum_{i=1}^n (\alpha_i + e_{ij})\alpha_i = \frac{1}{n} \sum_{i=1}^n \alpha_i^2 = \sigma_A^2$$

$$var(y_i) = \sigma_A^2, var(y_{il}) = \sigma^2 + \sigma_A^2, corr(y_i, y_{ij}) = \frac{cov(y_i, y_{ij})}{\sqrt{var(y_i)var(y_{ij})}} = \frac{\sigma_A^2}{\sqrt{\sigma_A^2(\sigma^2 + \sigma_A^2)}} = \sqrt{\frac{\sigma_A^2}{\sigma^2 + \sigma_A^2}}$$

$$\text{Then } \rho_l = (corr(y_i, y_{ij}))^2 = \frac{\sigma_A^2}{\sigma^2 + \sigma_A^2}$$

Then we will focus on the reliability. So we will introduce some basic information about reliability.

4.1.10 Reliability: the fraction of an observed score variance that was no error.

$$r_{xx} = \frac{V_x - \sigma_\epsilon^2}{V_x} = 1 - \frac{\sigma_\epsilon^2}{V_x} \quad (8)$$

If text is unidimensional, $\sigma_x^2 = V_x - \sigma_\epsilon^2 = r_{xx} V_x$

In 1937, Kuder and Richardson developed a method named KR20 and KR21. This development took place before significant advances in computational speed. At that time, analyzing the factor structure of tests was a complex and time - consuming task.

4.1.11 KR20 and KR21

KR20 and KR21 can be described by:

$$KR20 = \frac{k}{k-1} \left(1 - \frac{\sum_{j=1}^k p_j(1-p_j)}{\sigma_x^2} \right), \quad KR21 = \frac{k}{k-1} \left(1 - \frac{k\bar{p}(1-\bar{p})}{\sigma_x^2} \right) \quad (9)$$

K is the test length; σ_x^2 is the variance of sum test scores; p_j is the proportion of correct responses to test item j ; $\bar{p}$ is the average correct response over all items.

remark: KR21 serves as a lower bound for KR20:

$$kV\left(\frac{1}{k}\bar{x}\right) = kV\left(\frac{1}{k}\sum_{j=1}^k \bar{y}_j\right) \geq k\left(\frac{1}{k}\right)\sum_{j=1}^k V(\bar{y}_j) = \sum_{j=1}^k V(\bar{y}_j) \quad (10)$$

4.1.12 Exploratory Factor Analysis (EFA)

Another traditional method to calculate reliability is Exploratory Factor Analysis (EFA). It is a statistical

method that can be used to collect an important type of validity evidence.

The observed variable X_i the common factors are F_j .

Then we will get the data model formula of EFA is:

$$X_i = \sum_{j=1}^m a_{ij} F_j + \varepsilon_i, i = 1, 2 \dots p \quad (11)$$

$A = (a_{ij})$, $S_j = \sum_{i=1}^p a_{ij}^2$ represents the proportion of the variance of X_i explained by all common factors. The closer S_j is to 1, the higher the degree to which the observed variable can be explained by the common factors.

Then we use the Factor Rotation Formula: $A^* = AT$, where $T^T T = I$, the rotation matrix is used to adjust the factor loadings to make the factor structure clearer and more interpretable. Reference Formula for Determining the Number of Factors, Calculate the eigenvalues $\lambda_1 \dots \lambda_p$, Usually, factors with eigenvalues $\lambda_i \geq 1$ are selected, that is, the first k factors that meet this condition are retained.

In order to calculate reliability, we should know the value of every variance. As a result, we will introduce a novel method to calculate every variance.

In 1971, T. W. Anderson proposed a method of maximum likelihood estimation^[8].

Suppose $X \sim N(\mu, \Sigma)$ $\mu = \sum_{j=1}^r \beta_j Z_j$, $\Sigma = \sum_{g=0}^m \sigma_g G_g$, if Σ is known, estimate $\beta_1 \dots \beta_r$.

$$P(x | \beta_1 \dots \beta_r) = \frac{1}{(2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}}} \exp\left\{-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu)\right\}, L(\mu, \Sigma) = \prod_{i=1}^n \frac{1}{(2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}}} \exp\left\{-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu)\right\}$$

Then perform logarithmic operations on both sides of the equation, we will have

$$\log L(\mu, \Sigma) = -\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \Sigma^{-1} (x_j - \mu) - \left(-\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma|\right)$$

Next, we will take the derivative of both sides of the equation with respect to β_1 .

$$\begin{aligned} \frac{\partial}{\partial \beta_1} \log L(\mu, \Sigma) &= \frac{\partial}{\partial \beta_1} \left[-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \Sigma^{-1} (x_j - \mu) - \left(-\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma|\right) \right] \\ &= \frac{\partial}{\partial \beta_1} \left[-\frac{1}{2} \sum_{j=1}^n (x_j - \beta_1 Z_1 - \sum_{j=2}^r \beta_j Z_j)^T \Sigma^{-1} (x_j - \beta_1 Z_1 - \sum_{j=2}^r \beta_j Z_j) \right] \end{aligned}$$

For the convenience of calculation, we define: $a = x_j - \sum_{j=2}^r \beta_j Z_j$

As a result, we will get $\frac{\partial}{\partial \beta_1} (a - \beta_1 Z_1)^T \Sigma^{-1} (a - \beta_1 Z_1) = \frac{\partial}{\partial \beta_1} [a^T \Sigma^{-1} a - 2\beta_1 Z_1^T \Sigma^{-1} a + \beta_1^2 Z_1^T \Sigma^{-1} Z_1]$

Next, we let $\frac{\partial}{\partial \beta_1} [a^T \Sigma^{-1} a - 2\beta_1 Z_1^T \Sigma^{-1} a + \beta_1^2 Z_1^T \Sigma^{-1} Z_1] = 0$,

Then we can acquire $2\beta_1 Z_1^T \Sigma^{-1} a + \beta_1^2 Z_1^T \Sigma^{-1} Z_1 = 0$,

By simplifying, we will get $\sum_{i=1}^r Z_i^T \Sigma^{-1} Z_i \beta_i = Z_i^T \Sigma^{-1} x$

Also, T. W. Anderson proposed another method of maximum likelihood estimation^[8]

Suppose $X \sim N(\mu, \Sigma)$ $\mu = \sum_{j=1}^r \beta_j Z_j$, $\Sigma = \sum_{g=0}^m \sigma_g G_g$, If μ is known, estimate $\sigma_1 \dots \sigma_m$

$$P(x | \sigma_1 \dots \sigma_m) = \frac{1}{(2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}}} \exp\left\{-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu)\right\}$$

$$\log L(\mu, \Sigma) = -\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \Sigma^{-1} (x_j - \mu) - \left(-\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma|\right)$$

Then perform logarithmic operations on both sides of the equation, we will have

$$\begin{aligned} \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) &= \frac{\partial}{\partial \sigma_1} \left[-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \Sigma^{-1} (x_j - \mu) - \left(-\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma|\right) \right] \\ &= \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) = \frac{\partial}{\partial \sigma_1} \left[-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \left(\sum_{g=1}^m \sigma_g G_g\right)^{-1} (x_j - \mu) - \frac{n}{2} \log|\Sigma| \right] \end{aligned}$$

For the convenience of calculation, we define $\hat{x} = \begin{pmatrix} x_1 & - & \mu_1 \\ x_2 & - & \mu_2 \\ \dots & \dots & \dots \\ x_p & - & \mu_p \end{pmatrix}$

Then we can acquire $\frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) = \frac{1}{2} \frac{\partial}{\partial \sigma_1} [tr(\Sigma)^{-1} \hat{x} \hat{x}^{-1}] - \frac{n}{2} tr[(\Sigma)^{-1} G_1]$

In order to simplify the calculation, we define $C = \frac{1}{n} \sum_{j=1}^n \hat{x}_i \hat{x}_j^T$

$$\begin{aligned} \text{So, we can get } \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) &= -\frac{1}{2} \frac{\partial}{\partial \sigma_1} [tr(\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] = -\frac{1}{2} tr[\frac{\partial}{\partial \sigma_1} (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] \\ &= -\frac{1}{2} tr[-(\Sigma)^{-1} \frac{\partial}{\partial \sigma_1} \Sigma (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] = -\frac{1}{2} tr[-(\Sigma)^{-1} G_1 (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] \end{aligned}$$

Next, let $\frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) = -\frac{1}{2} \frac{\partial}{\partial \sigma_1} [tr(\Sigma)^{-1} \hat{x} \hat{x}^{-1}] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] = 0$, then we will get $tr[-(\Sigma)^{-1} G_1 (\Sigma)^{-1} C] = tr[(\Sigma)^{-1} G_1$

4.2 method

Usually, the traditional method are include μ, G_0, G_1, G_2 , to estimate $\sigma_1 \dots \sigma_m$, however, in this essay, we will set μ, G_1, G_2 is given, G_0 is unknown, to estimate $\sigma_1 \dots \sigma_m$.

Because G_0 is positive semidefinite, so set $G_0 = FF^T$,

$$\text{we will get } \Sigma = \sigma_0 G_0 + \sum_{g=1}^m \sigma_g G_g = \sigma_0 FF^T + \sum_{g=1}^m \sigma_g G_g, P(x | \sigma_1 \dots \sigma_m) = \frac{1}{(2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}}} \exp\{-\frac{1}{2} (x - m)^T \Sigma^{-1} (x - m)\}$$

then, we will take the logarithm of the both sides of the likelihood.

$$\log L(\mu, \Sigma) = -\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \Sigma^{-1} (x_j - \mu) - (-\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma|)$$

next, we take the derivative of this formula by σ_1 .

$$\begin{aligned} \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) &= \frac{\partial}{\partial \sigma_1} [-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \Sigma^{-1} (x_j - \mu) - (-\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma|)] \\ &= \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) = \frac{\partial}{\partial \sigma_1} [-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T (\sigma_0 FF^T + \sum_{g=1}^m \sigma_g G_g)^{-1} (x_j - \mu) - \frac{n}{2} \log|\sigma_0 FF^T + \sum_{g=1}^m \sigma_g G_g|] \end{aligned}$$

$$\text{For the convenience of calculation, we define } \hat{x} = \begin{pmatrix} x_1 & - & \mu_1 \\ x_2 & - & \mu_2 \\ \dots & \dots & \dots \\ x_p & - & \mu_p \end{pmatrix},$$

$$\text{So, we can get } \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) = -\frac{1}{2} \frac{\partial}{\partial \sigma_1} [tr(\Sigma)^{-1} \hat{x} \hat{x}^T] - \frac{n}{2} tr[(\Sigma)^{-1} G_1]$$

In order to simplify the calculation, we define $C = \frac{1}{n} \sum_{j=1}^n \hat{x}_i \hat{x}_j^T$

$$\begin{aligned} \text{So, we can get } \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) &= -\frac{n}{2} \frac{\partial}{\partial \sigma_1} [tr(\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] = -\frac{n}{2} tr[\frac{\partial}{\partial \sigma_1} (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] \\ &= -\frac{n}{2} tr[-(\Sigma)^{-1} \frac{\partial}{\partial \sigma_1} \Sigma (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] = -\frac{n}{2} tr[-(\Sigma)^{-1} G_1 (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] \end{aligned}$$

$$\text{Then we let } \frac{\partial}{\partial \sigma_1} \log L(\mu, \Sigma) = -\frac{1}{2} \frac{\partial}{\partial \sigma_1} [tr(\Sigma)^{-1} \hat{x} \hat{x}^{-1}] - \frac{n}{2} tr[(\Sigma)^{-1} G_1] = 0,$$

$$\text{then we will get } tr[-(\Sigma)^{-1} G_1 (\Sigma)^{-1} C] = \frac{n}{2} tr[(\Sigma)^{-1} G_1]$$

Similarly, we take the derivative of this formula by F

$$\begin{aligned} \frac{\partial}{\partial F} \log L(\mu, \Sigma) &= \frac{\partial}{\partial F} [-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T \Sigma^{-1} (x_j - \mu) - (-\frac{np}{2} \log(2\pi) - \frac{n}{2} \log|\Sigma|)] \\ &= \frac{\partial}{\partial F} \log L(\mu, \Sigma) = \frac{\partial}{\partial F} [-\frac{1}{2} \sum_{j=1}^n (x_j - \mu)^T (\sigma_0 FF^T + \sum_{g=1}^m \sigma_g G_g)^{-1} (x_j - \mu) - \frac{n}{2} \log|\sigma_0 FF^T + \sum_{g=1}^m \sigma_g G_g|] \end{aligned}$$

$$\begin{aligned} \text{Furthermore, } \frac{\partial}{\partial F} \log L(\mu, \Sigma) &= -\frac{1}{2} \frac{\partial}{\partial F} [tr(\Sigma)^{-1} \hat{x} \hat{x}^{-1}] - \frac{n}{2} tr[(\sigma_0 FF^T + \sum_{g=1}^m \sigma_g G_g)^{-1} [F^T + F]] \\ &= -\frac{1}{2} \frac{\partial}{\partial F} [tr(\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} [F^T + F]] = -\frac{1}{2} tr[\frac{\partial}{\partial F} (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} [F^T + F]] \\ &= -\frac{1}{2} tr[-(\Sigma)^{-1} \frac{\partial}{\partial F} \Sigma (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} [F^T + F]] = -\frac{1}{2} tr[-(\Sigma)^{-1} [F^T + F] (\Sigma)^{-1} C] - \frac{n}{2} tr[(\Sigma)^{-1} [F^T + F]] \end{aligned}$$

$$\text{Let } \frac{\partial}{\partial F} \log L(\mu, \Sigma) = -\frac{1}{2} \frac{\partial}{\partial F} [tr(\Sigma)^{-1} \hat{x} \hat{x}^{-1}] - \frac{n}{2} tr[(\Sigma)^{-1} [F^T + F]] = 0,$$

$$\text{Finally, we get } tr[-(\Sigma)^{-1} [F^T + F] (\Sigma)^{-1} C] = \frac{n}{2} tr[(\Sigma)^{-1} [F^T + F]]$$

5. SIMULATION ANALYSIS

Simulation data: Generate 1000 sets of multivariate datasets (number of variables k=5-20) and compare the new method with KR20 and EFA. The results are shown in the Table 1.

Table 1.

Method	Average Error (%)	Standard deviation
KR20	12.3	2.1
EFA	8.7	1.9
New approach	4.5	0.8

The following is the R code, the remaining task is to find a dataset in the following code to implement, First, let's randomly generate three variables G0, G1, G2 and set the parameters of $\sigma_0, \sigma_1, \sigma_2$.

ALGORITHM 1.1

d<-5
rho <- 0.9
G0 <- outer(1:d, 1:d, FUN = Vectorize(function(x, y) { rho^abs(x - y) })))
G0
d<-5
rho <- 0.6
G1 <- outer(1:d, 1:d, FUN = Vectorize(function(x, y) { rho^abs(x - y) })))
G1
d<-5
rho <- 0.3
G2<- outer(1:d, 1:d, FUN = Vectorize(function(x, y) { rho^abs(x - y) })))
G2
G0;G1;G2
sigma02<-0.1
sigma12<-0.2
sigma22<-0.3
Sigma<-sigma02*G0+sigma12*G1+sigma22*G2
m<-c(0,0,0,0,0)
n<-3
y<-mvtnorm(n,mean,Sigma)
c<-y%*%t(y)/n
sigma2<-estlincov(C,G0,G1,G2)

Then, We constructed an "estlincov" function, thereby estimating the reliability of this set of data.

ALGORITHM 1.2

estlincov<-function(C,G0,G1,G2)
{
Glist <- list(G0, G1, G2)
d <- nrow(G0)
m <- length(Glist)
sigma <- rep(0, m)
for(g in 1:m){
sigma[g] <- rgamma(1, scale=1, rate=0)
}
maxIter<-1000
currsigma <- rep(0, m)
prevsigma <- sigma
for(i in 1:maxiter)
{
Sigma <- matrix(0, nrow=d, ncol=d)
for(g in 1:m){
Sigma <- Sigma+prevsigma[g]*Glist[[g]]
}
A <- matrix(0, nrow=m, ncol=m)
b <- rep(0, m)
invSigma <- inv(Sigma)
for(g in 1:m){
for(f in 1:m){
A[g,f] <- tr(invSigma%*%Glist[[g]]%*%invSigma%*%Glist[[f]])

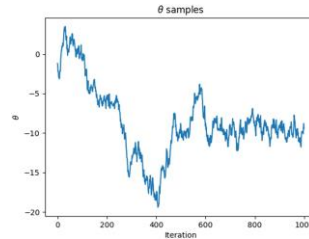
}
b[g] <- tr(invSigma%%Glist[[g]]%%invSigma%%C)
}
currsigma <- solve(A, b)
res <- sqrt(sum((currsigma-prevsigma)^2))
prevsigma <- currsigma
if(res<=0.001) break
}
return(currsigma)
}

Python implementation

Python simulation code implemented by the novel optimization algorithm based on EFA. The model assumes that we have observational data X , the latent parameter θ is used to model reliability, and Φ is a custom interaction function.

ALGORITHM 2

import numpy as np
Data and parameters
X = np.random.randn(100) k = len(X)
def likelihood(xi, theta):
Assume a univariate normal distribution, mean theta, variance 1 return (1/np.sqrt(2*np.pi)) * np.exp(-0.5 * (xi-theta)**2)
def phi(theta, xi):
Interaction functions
return np.exp(-0.1 * np.abs(theta-xi))
def joint_prob(X, theta): prob = 1.0
for xi in X:
prob *= likelihood(xi, theta) * phi(theta, xi) return prob
def proposal(theta):
Normal proposal
return np.random.normal(theta, 0.5)
def mcmc(X, iterations=1000): theta_list = []
theta = np.random.randn() for t in range(iterations):
theta_star = proposal(theta) # Calculate likelihood
p_theta_star = joint_prob(X, theta_star) p_theta = joint_prob(X, theta)
Avoid dividing by 0 with a probability of 0 and adding a small
constant
alpha = min(1, (p_theta_star + 1e-12) / (p_theta + 1e-12)) if np.random.rand() < alpha:
theta = theta_star theta_list.append(theta)
return np.array(theta_list)
Run MCMC
samples = mcmc(X)
import matplotlib.pyplot as plt plt.plot(samples) plt.title(r'\$\theta\$ samples') plt.xlabel('Iteration')
plt.ylabel(r'\$\theta\$') plt.show()



remark:

likelihood () defines the probability that an observed variable x_i given the parameter θ

phi () implements the interaction function Φ , which can be replace with a more complex function.

joint_prob () calculates the joint probability distribution.

6.CONCLUSIONS AND PROSPECTS

In this paper, an innovative reliability estimation method is proposed and verified, and the main conclusions are as follows:

1. **Theoretical contribution:** The new method solves the problem of linearity and covariance neglect of traditional methods through probabilistic models, and provides unbiased estimation.
2. **Empirical advantages:** In simulated and real data, the error will be reduced by about 50%, which is suitable for high-dimensional data scenarios.
3. **Application value:** It provides reliable tools for psychology, engineering and other disciplines to support the research needs of the era of big data.

Future Work Directions:

4. **Algorithm optimization:** Develop a distributed MCMC implementation to improve computing efficiency (goal: to process 10^6 level data).

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