

KÄHLER GEOMETRY ON TOTAL SPACES OF VECTOR BUNDLES OVER ELLIPTIC CURVES

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ABSTRACT. We study function theory and Kähler geometry on total spaces of vector bundles on an elliptic curve. For rank two vector bundles of degree zero, we show that any two total spaces are biholomorphic if and only if the corresponding vector bundles are isomorphic. We also construct complete Gauduchon Hermitian metrics with flat Chern-Ricci curvature on these total spaces. These metrics are natural in the sense that the corresponding spaces of holomorphic functions of polynomial growth coincide with ‘polynomials’ on these spaces. Moreover, we characterize all complete Kähler metrics with nonnegative bisectional curvature on total spaces of line bundles over an elliptic curve.

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1. INTRODUCTION

1.1. Vector bundles over elliptic curves: two examples. In this work, we are interested in the Kähler geometry of a special class of noncompact complex manifold, namely those which are total spaces of vector bundles over an elliptic curve. To motivate the questions guiding our investigation, we begin with two illustrative examples.

1.1.1. Line bundles with degree zero. It is well-known that line bundles over complex tori can be studied by theta functions. In particular, the theorem of Appel-Humbert (see [15, p.308] or [11, p.50] for example) states that any line bundle L over a complex torus $M = \mathbb{C}^n/\Lambda$ is isomorphic to a specific type of line bundle constructed by theta functions. For example, $\text{Pic}^0(M)$, which is the space of line bundles on a complex torus M with zero degree, can be identified with $\text{Hom}(\Lambda, S^1)$ with $S^1 = \{e^{2\pi i\theta} \mid \theta \in [0, 1)\}$.

Now we consider an elliptic curve

$$(1) \quad \Sigma = \mathbb{C}/\Gamma, \quad \Gamma = \mathbb{Z}\text{-span}\{1, \tau\} \text{ for } \tau \in \mathbb{C} \text{ with } \text{Im } \tau > 0.$$

Therefore, $\pi_1(\Sigma)$ is a free abelian group generated by

$$(2) \quad \gamma_1(t) = z + t, \gamma_2(t) = z + \tau t \text{ where } t \in [0, 1].$$

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Then the total space of any $L \in \text{Pic}^0(\Sigma)$ can be written as

$$(3) \quad X_\theta = \mathbb{C}^2 / \pi_1(\Sigma), \quad \text{with } \gamma_k(z, \xi) = (\gamma_k(z), (\rho(\gamma_k))\xi), \text{ for } k = 1, 2.$$

Here, we pick $\alpha \in \text{Hom}(\Lambda, S^1)$ as

$$(4) \quad \alpha(\gamma_k) = e^{2\pi i \theta_k} \quad \text{for some } \theta_k \in [0, 1) \quad \text{for } k \in \{1, 2\}.$$

It follows that the standard Euclidean metric on $\mathbb{C}^2$

$$\sqrt{-1}(dz \wedge d\bar{z} + d\xi \wedge d\bar{\xi})$$

descends to X_θ . Let ω be the corresponding Kähler form on X_θ for $\theta = (\theta_1, \theta_2)$. Then (X_θ, ω) is locally a metric product, and it splits globally only if $\theta = (0, 0)$. The corresponding function theory on X_θ depends on the arithmetic properties of θ . For example, there are no nonconstant holomorphic functions on X_θ for either of θ_1 and θ_2 in (4) is irrational.

1.1.2. An indecomposable rank 2 bundle of degree zero. We review a specific rank 2 bundle which appeared in Atiyah [2] and Demailly-Peternell-Schneider [12, Example 1.17]. In the latter, an example of a compact Kähler manifold with nef canonical line bundle is constructed. Moreover, it cannot admit any Kähler metric with nonnegative Ricci curvature. The example is the projective bundle $\mathbb{P}(E)$ for some rank 2 vector bundle E over an elliptic curve. It is the rank 2 vector bundle E that we would like to discuss.

Let Σ be the elliptic curve defined in (1). We consider a representation $\rho : \pi_1(\Sigma) \rightarrow \text{GL}(2, \mathbb{C})$ by

$$\rho(\gamma_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(\gamma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The vector bundle E which is associated to ρ is defined as

$$(5) \quad E = (\mathbb{C} \times \mathbb{C}^2) / \pi_1(\Sigma), \quad \text{with } \gamma_i(z, z_1, z_2) = (\gamma_i(z), \rho(\gamma_i)(z_1, z_2)) \text{ for } 1 \leq i \leq 2.$$

In fact, E is exactly the canonical rank 2 bundle F_2 of degree 0 in Atiyah [2, Theorem 5]. It is proved in [2] and [12] that E is indecomposable, and it is an extension of two trivial line bundles. It is further shown in [12] that E defined in (5) is numerically flat (i.e. both E and $\det(E^*)$ are nef), and can not be Griffiths nonnegative. We restate the first statement in the following.

Lemma 1.1 ([2], [12, Example 1.17]). *E defined in (5) is an extension of two trivial line bundles*

$$(6) \quad 0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{O} \rightarrow 0.$$

Moreover, $h^0(\Sigma, E) = 1$ and E does not split holomorphically.

In view of Lemma 1.1, we may wonder whether function theory on the total space of the bundle E can reflect the indecomposability of E in some sense. As an example, we may characterize holomorphic functions on its total space X .

Proposition 1.2 (a special case of Proposition 4.2). *Let X be the total space of E defined in (5). Then any $f \in \mathcal{O}(X)$ is the quotient of $f(z_2) \in \mathcal{O}(\mathbb{C})$ in the sense of (5). Here (z, z_1, z_2) denotes the global coordinates on $\mathbb{C}^3$ as in Proposition 4.2.*

1.1.3. Guiding questions. Vector bundles over compact Riemann surfaces have been extensively studied in the context of algebraic geometry. Given any vector bundle E of rank r over a compact Riemann surface M , we may write $E = \oplus_{i=1}^p E_i$ as a direct sum of indecomposable vector bundles. Then a classical result of Weil [38] states that E is associated to a representation $\rho : \pi_1(M) \rightarrow \text{GL}(r, \mathbb{C})$ if and only if $\deg(E_i) = 0$ for any $1 \leq i \leq p$. Moreover, indecomposable vector bundles over elliptic curves were classified by Atiyah [2], see also Oda [31]. Moduli spaces of stable (semistable) vector bundles on compact Riemann surfaces of $g \geq 2$ have been extensively studied since Narasimhan-Seshadri [28]. Tu [37] studied moduli problems on vector bundles on elliptic curves. In view of these algebraic results and two examples discussed in the above, we would like to study complex differential geometry and function theory on total spaces

of vector bundles on an elliptic curve (or a complex torus). In particular, our investigation is guided by the following question.

- Question 1.3.** (1) *On a compact Riemann surface M , Weil's result corresponds a representation $\pi_1(M) \rightarrow \mathrm{GL}(n, \mathbb{C})$ to a vector bundle of degree zero on M . However, this correspondence is not unique, see [29, p.128] and [13, p.578]. Can we characterize those representations which correspond to (holomorphically) isomorphic vector bundles?*
- (2) *Study the fine structure of the space of holomorphic functions on the total space of a vector bundle on an elliptic curve. Is it possible that the total spaces of two non-isomorphic vector bundles are biholomorphic complex manifolds?*
- (3) *What is the 'best' complete Kähler metric on the total space of a given vector bundle over an elliptic curve? Does the existence of a Kähler metric satisfying suitable curvature conditions on the total space relate to the algebraic structure of the vector bundle?*

1.2. Statement of main results. The goal of the paper is to give partial solutions to Question 1.3 when the underlying Riemann surface is an elliptic curve. Let Σ be the elliptic curve defined in (1). According to Appel-Humbert's theorem (see [27, Theorem on p.19] and [11, Theorem 5.17] for example), any line bundle over Σ is isomorphic to some $L(H, \alpha)$. Here H is a Hermitian bilinear form on $\mathbb{C}$ so that $\mathrm{Im} H$ is $\mathbb{Z}$ -valued when restricted onto $\Gamma \times \Gamma$, and $\alpha : \Gamma \rightarrow S^1$ satisfies (29). The line bundle $L(H, \alpha)$ over Σ is defined as a quotient of $\mathbb{C}^2$ under the action

$$(7) \quad (z, \xi) \rightarrow (z + \gamma, \alpha(\gamma) e^{\pi H z \bar{\gamma} + \frac{\pi}{2} H |\gamma|^2} \xi).$$

It is direct to check $\deg(L(H, \alpha)) = H \mathrm{Im} \tau$.

Motivated by the indecomposable vector bundle defined in (5) studied by [2] and [12], we consider a vector bundle E over Σ which is associated with a representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{GL}(2, \mathbb{C})$

$$(8) \quad \rho(\gamma_1) = e^{2\pi i \theta_1} \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, \quad \rho(\gamma_2) = e^{2\pi i \theta_2} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}.$$

Recall that $\pi_1(\Sigma)$ is generated by γ_1 and γ_2 in (2). Here we assume $\theta_i \in [0, 1)$ and $b_i \in \mathbb{C}$ for $i = \{1, 2\}$.

We consider a rank 2 vector bundle E of degree zero over Σ which belongs to one of the following three types.

- **Type I:** $E = L_1 \oplus L_2$ where $L_1, L_2 \in \mathrm{Pic}^0(\Sigma)$.
- **Type II:** $E = L_1 \oplus L_2$ where $\deg L_1 = -\deg L_2 > 0$.
- **Type III:** E is indecomposable and associated with a representation defined in (8).

We observe that any rank 2 vector bundle of degree zero over Σ belongs to exactly one of **Type I**, **Type II**, and **Type III**. Indeed, according to Weil [38], any indecomposable rank 2 bundle E of degree 0 must be associated with some representation ρ of $\pi_1(\Sigma)$ into $\mathrm{GL}(2, \mathbb{C})$. By a standard fact in linear algebra, two commutative matrices $\rho(\gamma_1)$ and $\rho(\gamma_2)$ are simultaneously similar to upper triangular matrices. Since E is indecomposable, we may further assume that at least one of $\rho(\gamma_1)$ and $\rho(\gamma_2)$ has multiple eigenvalues. In fact, Atiyah [2] shows that any such indecomposable bundle E admits an extension of some $L \in \mathrm{Pic}^0(\Sigma)$ by itself. Therefore, any indecomposable rank 2 bundle E of degree 0 corresponds to some upper triangular representation $\rho : \pi_1(\Sigma) \rightarrow \mathrm{GL}(2, \mathbb{C})$ defined in (8). Except for this aspect of Atiyah's result, our main results study vector bundles of **Type I**, **Type II**, and **Type III** directly and do not rely on the full power of the classification results in [2].

Let $\mathrm{Aut}(M)$ denote the group of biholomorphisms of a complex manifold M . By a delicate analysis of $\mathrm{Aut}(\mathbb{C}^n)$ under an invariant condition related to (7), we prove the following results which imply that the complex structures of total spaces are rigid enough to determine algebraic structures of underlying vector bundles.

Proposition 1.4. *Let E be any rank 2 vector bundle E of degree 0 over Σ and X the total space of E . Then*

- (1) E is of **Type III** if and only if it is associated to a representation in (8) with $b_2 \neq b_1\tau$. In particular, any two vector bundles E and $\tilde{E}$ of **Type III** are bundle isomorphic if the corresponding $\theta_k = \tilde{\theta}_k$ for $k = 1, 2$ in (8).
- (2) If E belongs to any of **Type I**, **Type II**, and **Type III**, we have a complete characterization of $\mathcal{O}(X)$, see (54), (55), and (56) for a detailed statement. In particular, if E is of **Type III**, then $\mathcal{O}(X)$ contains a nonconstant element if and only if both θ_1 and θ_2 in (8) are rational. Note that X is not holomorphically convex in this case.
- (3) X admits a complete flat Kähler metric if and only if E is of **Type I**.

Theorem 1.5. Let Σ (resp. $\tilde{\Sigma}$) be the elliptic curve defined by $\{1, \tau\}$ (resp. $\{1, \tilde{\tau}\}$) as in (1), and X (resp. Y) the total space of a rank 2 bundle E (resp. $\tilde{E}$) of degree 0 over Σ (resp. $\tilde{\Sigma}$). If X is biholomorphic to Y , then Σ and $\tilde{\Sigma}$ are isomorphic, and E is (holomorphically) bundle isomorphic to F .

Our next goal is to construct a graded linear structure of $\mathcal{O}(X)$. In view of Proposition 1.4 (and Proposition 4.2), a natural idea is to consider a notion of holomorphic functions of polynomial type on X . Precisely, we say that any given $f \in \mathcal{O}(X)$ is of *polynomial type* if the corresponding Taylor series in one of (54), (55), and (56) terminates after finite terms. One may check that this definition is independent of the choice of local coordinates in (54), or the choice of lifts in (55), and (56). In the meantime, we may introduce a natural Hermitian metric g on X and a corresponding subspace $\mathcal{O}_d(X, g) \subset \mathcal{O}(X)$. Let p be a fixed point on a complete Hermitian manifold (X, g) , and d_g the distance function with respect to g . For any real number $d > 0$, we define

$$(9) \quad \mathcal{O}_d(X, g) = \{f \in \mathcal{O}(X) \mid \exists C(d, f) > 0, |f(q)| \leq C(d_g(q, p) + 1)^d \text{ for all } q \in X\}.$$

We construct special Hermitian metrics on X such that the corresponding $\mathcal{O}_d(X, g)$ meshes well with holomorphic functions of polynomial type.

Theorem 1.6. Let X be the total space of any rank 2 bundle E of degree zero over an elliptic curve Σ defined in (1). There exists a complete Hermitian metric g on X which is of flat Chern-Ricci curvature and Gauduchon in the sense that $\sqrt{-1}\partial\bar{\partial}(\omega^2) = 0$. In case that E is of **Type III**, we get a family of such Hermitian metrics which are quasi-isometric to each other. Moreover, $\mathcal{O}_d(X, g)$ defined in (9) consists of all $f \in \mathcal{O}(X)$ whose corresponding power series in one of (54), (55), and (56) are finite and of highest degree $\leq [d]$. Here $[c]$ denotes the greatest integer $\leq c$. Moreover, g is Kähler if and only if E is of **Type I**.

It is known from From Calabi's work [4] that the total space of any Hermitian vector bundle over a compact Kähler manifold admits a complete Kähler metric. Let E be the indecomposable rank 2 bundle defined in (5). There is a natural choice of Hermitian metric on E introduced by Paun [32, p.53]. In Proposition 2.2, we show that any holomorphic function on its total space X grows with the Hadamard order ≥ 2 with respect to any corresponding Kähler metric of Calabi type on X . Therefore, Theorem 1.6 shows that function theory in Hermitian geometry might be more 'natural' than that in Kähler geometry in some cases. In the meantime, Hermitian metrics constructed in Theorem 1.6 could be viewed as non-Kähler Calabi-Yau metrics on a natural class of noncompact manifolds. We refer the reader to [36] for a nice introduction on compact non-Kähler Calabi-Yau manifolds.

From now on, we consider some characterization of complete Kähler metrics with nonnegative (holomorphic) bisectional curvature ($BI \geq 0$ for short) on the total space of a line bundle over an elliptic curve or a compact complex torus.

Theorem 1.7. (1) Let X be the total space of a line bundle $L(H, \alpha)$ (defined in (7)) over the elliptic curve Σ in (1). Then X admits a complete Kähler metric ω with $BI \geq 0$ if and only if $H = 0$. Moreover, assume that ω is non-flat. Let $\alpha(1) = e^{2\pi i\theta_1}$ and $\alpha(\tau) = e^{2\pi i\theta_2}$ for some $\theta_1, \theta_2 \in [0, 1)$. Then the following holds.

- (i) If either of θ_1 and θ_2 is irrational, then there exist
 - a constant $C > 0$.

- a smooth positive function $\tilde{u}(\xi)$ on $\mathbb{C}$ so that $\ln \tilde{u}$ is superharmonic.
 - $\tilde{u}(\xi)$ is rotationally symmetric as $\tilde{u}(\xi) = \tilde{u}(|\xi|)$ for any $\xi \in \mathbb{C}$
- such that

$$(10) \quad \omega = Cdz \wedge d\bar{z} + \tilde{u}(|\xi|)d\xi \wedge d\bar{\xi}.$$

- (ii) If both θ_1 and θ_2 are rational, let k denote the minimal positive integer so that $k\theta_1$ and $k\theta_2$ are integers. Then there exist
- a constant $C > 0$.
 - a holomorphic function $h(\xi) = \hat{h}(\xi^k)$ for an entire function $\hat{h}$ on $\mathbb{C}$.
 - a smooth positive function $\tilde{u}(\xi)$ on $\mathbb{C}$ so that $\ln \tilde{u}$ is superharmonic.
 - $\tilde{u}(\xi)$ is $\mathbb{Z}_k$ -invariant in the sense that $\tilde{u}(\xi) = \tilde{u}(e^{\frac{2\pi i}{k}} \xi)$ for any $\xi \in \mathbb{C}$
- such that

$$(11) \quad \omega = Cd(z + h(\xi)) \wedge d(\bar{z} + \overline{h(\xi)}) + \tilde{u}(\xi)d\xi \wedge d\bar{\xi}.$$

Here in both (10) and (11), (z, ξ) denote the corresponding (local) coordinates on X for those on $\mathbb{C}^2$ defined in (7).

- (2) Let $n \geq 3$ be an integer and $L(H, \alpha)$ a line bundle over a compact complex torus M of complex dimension $n - 1$, see (27) to (30) for a definition of $L(H, \alpha)$ and M . Assume that its total space X admits a complete Kähler metric g with $BI \geq 0$ so that the lifting metric $\tilde{g}$ on the universal covering $\tilde{X}$ satisfies $\tilde{g} = g_e + g_0$. Here g_e is the standard Euclidean metric on $\mathbb{C}^{n-1}$, and g_0 is a complete Kähler metric with nonnegative Gauss curvature on $\mathbb{C}$. Then $H = 0$.

Remark 1.8. Part (1) of Theorem 1.7 shows that both the geometric and arithmetic properties of $L(H, \alpha)$ impose strong constraints on complete Kähler metrics with $BI \geq 0$ on the total space X . In part (2) of Theorem 1.7, we assume that the universal covering of (X, g) splits with a complex Euclidean factor of dimension $n - 1$. It is possible that this assumption automatically holds in this setting. The intuition is that the rank of the fundamental group of the compact torus M might contribute to the dimension of flat Euclidean factor on $\tilde{X}$. For example, if we assume “nonnegative Riemann sectional curvature” instead of “ $BI \geq 0$ ”, the universal covering splits out a flat $\mathbb{C}^{n-1}$ by the proof of a result of Zheng [30, Corollary 5.1].

It is natural to consider Question 1.3 for vector bundles with nonzero degrees over an elliptic curve. What about function theory on the total spaces? Are there any complete Kähler metrics with $BI \geq 0$ on the total spaces? In this regard, we show (in Proposition 6.3) that the total space of some indecomposable rank n bundle of degree $-n < d < 0$ on an elliptic curve is holomorphically convex.

1.3. Further discussions. For any fixed Kähler manifold (M, g) , we define its Ricci rank as the maximum rank of the Ricci curvature over M . Wu-Zheng asks the following question in [41, Conjecture on p.307]: Let (M^n, g) be a complete Kähler manifold with $BI \geq 0$ and its Ricci rank $n - k$. Then its universal covering $(\tilde{M}, \tilde{g})$ is holomorphically isometric to $(N^{n-k} \times \mathbb{C}^k, h + g_e)$ where h is a complete Kähler metric on N with $BI \geq 0$, and g_e is the Euclidean metric on $\mathbb{C}^k$. In case that the curvature of (M, g) is bounded, a stronger structure theorem on its universal cover was obtained by Cao-Chen-Zhu [5, Theorem 2.8] using the Kähler-Ricci flow.

Motivated by structural theorems of Kähler manifolds with $BI \geq 0$ due to Howard-Smyth-Wu [18], Mok [24], Takayama [34, Theorem 1.6], Wu-Zheng [41], Zheng [30, Corollary 5.1], Ni-Tam [30, Theorem 5.1], Cao-Chen-Zhu [5], and Chau-Tam [8, Corollary 3.1], we propose the following question.

Question 1.9. Let (M^n, g) be a complete Kähler manifold with $BI(g) \geq 0$ and its Ricci rank be $n - k$ where $1 \leq k \leq n$. Then there exists a finite covering $(\widehat{M}, \widehat{g})$ with a locally trivial holomorphic fibration $\rho : \widehat{M} \rightarrow M_1$ where M_1 is complete flat Kähler manifold with its dimension k . Moreover, each fiber of ρ with the induced metric from $\widehat{g}$ is a simply-connected holomorphically isometric product manifold M_2 with each factor being one of the following:

- (1) a simply connected complete noncompact Kähler manifold (N_1, g_1) so that N_1 admits another complete Kähler metric with $BI > 0$. According to Yau's uniformization conjecture [42], N_1 is expected to be biholomorphic to a complex Euclidean space.
- (2) A product of complex projective spaces each of which is endowed with a Kähler metric with $BI \geq 0$.
- (3) A product of compact Hermitian symmetric spaces of rank ≥ 2 each of which is endowed with its canonical Kähler-Einstein metric.

Remark 1.10. Theorem 1.7 provides partial support for Question 1.9. Notably, the factor (N_1, g_1) in (1) of Question 1.9 might not satisfy $BI \geq 0$. Indeed, we construct an explicit example (see Example 3.4) so that it can be non-positively curved! This phenomenon contrasts with the compact case studied in [18] and [24] where only (2) and (3) in Question 1.9 might appear and the corresponding induced metric satisfies $BI \geq 0$.

The complete flat Kähler manifold M_1 in Question 1.9 can be written as $\mathbb{C}^k/G$ where G is a fixed-point-free discrete subgroup of $I_h(\mathbb{C}^k)$. Here $I_h(\mathbb{C}^k)$ denotes the holomorphic isometry group of $\mathbb{C}^k$. When G is not trivial, we have the following three cases. Recall that a finite cover has been taken into account in Question 1.9.

- (i) If G is a lattice of real rank $2k$, then M_1 is a compact complex torus $T_{\mathbb{C}}^k$.
- (ii) If G is a lattice of real rank $1 \leq l < 2k$, By a theorem of Remmert and Morimoto [26, Theorem 3.2], M_1 is holomorphically isomorphic to a product of $\mathbb{C}^{l_1}$, $(\mathbb{C}^*)^{l_2}$, and a toroidal group. Recall a toroidal group is an abelian complex Lie group on which every holomorphic function is constant. In general, any toroidal group is a holomorphic fiber bundle onto a complex torus with its fiber being a product of copies of $\mathbb{C}^*$, see [1] for a general introduction of toroidal groups. Cousin [10] constructed the first example of toroidal groups. For example, let Γ be a lattice in $\mathbb{C}^2$ generated by $(1, 0)$, $(0, 1)$, and (i, ia) where a is irrational. We may check by power series that $\mathbb{C}^2/\Gamma$ is a toroidal group; see [25, p.262] for further constructions of toroidal groups.
- (iii) If G is a semi-direct product associated with a lattice Γ and some representation $\rho : \Gamma \rightarrow U(r)$, M_1 is related to a holomorphic fiber bundle over $\mathbb{C}^{k-r}/\Gamma$ with fiber being a product of $\mathbb{C}^{r_1}$ and $(\mathbb{C}^*)^{r_2}$ for two integers r_1 and r_2 with $r_1 + r_2 = r$.

However, it is unclear to the authors whether Case (iii) has been fully classified. In view of this, we prove the following result.

Proposition 1.11. *Let $E = \bigoplus_{p=1}^k L_p(H_p, \alpha_p)$ where each L_p is a line bundle over a compact complex torus $M = \mathbb{C}^{n-k}/\Gamma$. Let X be the total space of E . Then X admits a complete flat Kähler metric if and only if $H_p = 0$ for $p = 1, \dots, k$. In particular, if $k = 1$, there exists a characterization (see (47)) of any complete flat Kähler metrics on X in terms of (z, ξ) which are the corresponding coordinates descended from those on $\mathbb{C}^2$ defined in (7).*

There has been important progress on the uniformization of complete Kähler manifolds with $BI \geq 0$ and Euclidean volume growth in recent years. We refer the reader to Chau-Tam [6], Liu [22], Lee-Tam [21] for more information, as well as Chau-Tam [7], Chen-Zhu [9] and Lott [23] for further interesting results without assuming Euclidean volume growth.

The structure of the paper is the following. In Section 2 we apply Calabi's construction to study the growth of holomorphic functions on the total space of a vector bundle, proving Proposition 2.2. In Section 3 we study Kähler geometry of total spaces of line bundles over an elliptic curve, Theorem 1.7 and Proposition 1.11 are proved. In Sections 4 and 5 we show a biholomorphic classification of total spaces of rank 2 bundles with degree 0 and studying their Hermitian geometry, proving Proposition 1.4, Theorem 1.5, and Theorem 1.6. Finally, we discuss the function theory on total spaces of certain bundles with nonzero degree in Section 6.

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2. FUNCTION THEORY VIA THE CALABI METHOD

In this section, we review Calabi's theorem that the total space of any holomorphic vector bundle over a compact Kähler manifold admits a complete Kähler metric, and apply it to study function theory on X which is the total space of the vector bundle E defined in (5). The main result is Proposition 2.2 which gives a sharp estimate on the growth of holomorphic functions on X .

Let X be the total space of $\pi : (E, h) \rightarrow M$ which is a Hermitian vector bundle over a complete Kähler manifold (M, g) . Calabi considered the Kähler metric $\tilde{g}$ on X in the form of

$$(12) \quad \tilde{\omega} = \pi^* \omega_g + \sqrt{-1} \partial \bar{\partial} u.$$

Here for any point $(p, v^\mu e_\mu) \in X$, u is a smooth single-variable function on $t = h(v^\mu e_\mu, \overline{v^\mu e_\mu})$ which is to be determined. For simplicity, we call any Kähler metric is *of the Calabi type* if it is in the form of (12).

Theorem 2.1 (Calabi [4, Theorem 3.2 on p.275]). *Let X be the total space of E which is a holomorphic vector bundle over a complete Kähler manifold (M, g) . Assume E admits a Hermitian metric h so that its Chern curvature is bounded from above, i.e. there exists some constant C so that*

$$(13) \quad R(S, \bar{T}, v^\alpha e_\alpha, \overline{w^\beta e_\beta}) \leq C g(S, \bar{T}) h(v^\alpha e_\alpha, \overline{w^\beta e_\beta})$$

for any $p \in Y$, $S, T \in T_p^{1,0} M$, and $v^\alpha e_\alpha, w^\beta e_\beta \in \pi^{-1}(p)$. Then X admits a complete Kähler metric of the Calabi type.

It is also mentioned on [4, p.272] that any Kähler metric in the form of (12) satisfies that each fiber is totally geodesic. We also refer to [19, Proposition 2.2 on p.2297] for a proof. For the sake of completeness, we review this fact in the special case of a rank 2 bundle over an elliptic curve Σ . We also discuss the choice of the function u in (12). These two ingredients are crucial in Calabi's proof of Theorem 2.1.

For any $p \in \Sigma$, choose a holomorphic normal coordinate (z, v^1, v^2) center at p such that $z(p) = 0$ and $\partial_z h(e_\alpha, \bar{e}_\beta) = 0$ at p . Recall $t = h(v, \bar{v}) = v^\alpha \bar{v}^\beta h(e_\alpha, \bar{e}_\beta)$.

$$(14) \quad \partial \bar{\partial} u(t) = u'(t) \partial \bar{\partial} t + u''(t) \partial t \wedge \bar{\partial} t.$$

Hence on $\pi^{-1}(p)$, the local components of $\tilde{g}$ are

$$(15) \quad \tilde{g} = \begin{pmatrix} g_{z\bar{z}} + u'(t) t_{z\bar{z}} & 0 & 0 \\ 0 & u' + u'' |v^1|^2 & u'' \bar{v}^1 v^2 \\ 0 & u'' v^1 \bar{v}^2 & u' + u'' |v^2|^2 \end{pmatrix}.$$

It follows that $\Gamma_{\alpha\beta}^w = \tilde{g}^{w\bar{l}} \partial_\alpha \tilde{g}_{\beta\bar{l}} = 0$ holds along $\pi^{-1}(p)$. Therefore, the fiber $\pi^{-1}(p)$ is totally geodesic.

Next, we consider the growth of geodesics along a fixed fiber. Let $\tilde{\omega}_p = \tilde{\omega}|_{\pi^{-1}(p)}$ the induced metric along $\pi^{-1}(p)$. Then for $v = (v^1, 0) \in \pi^{-1}(p)$, we may estimate the distance of v to the zero section Σ by

$$(16) \quad d_{\tilde{\omega}_p}(v, 0) = \int_0^{|v|} \sqrt{u'(t) + t u''(t)} \frac{1}{2\sqrt{t}} dt.$$

Therefore, $\tilde{\omega}$ is a complete Kähler metric if $u'(t) + t u''(t) > 0$ and the integral in (16) is divergent. In [4], the following form of $u(t)$ is considered

$$(17) \quad u(t) = A \log(C + t) - B \log \log(C + t),$$

where $C > 0$ is a large constant and A, B are constants to be determined.

$$\begin{aligned} u'(t) &= \frac{1}{C+t} \left(A - \frac{B}{\log(C+t)} \right), \\ u''(t) &= -\frac{1}{(C+t)^2} \left(A - \frac{B}{\log(C+t)} - \frac{B}{(\log(C+t))^2} \right), \\ u'(t) + tu''(t) &= \frac{1}{(C+t)^2} \left(\frac{Bt}{(\log(C+t))^2} + AC - \frac{BC}{\log(C+t)} \right). \end{aligned}$$

Let $A > \frac{B}{\log C}$. Then $u'(t) > 0$ for $t > 0$. Moreover, we have

$$(18) \quad \sqrt{u'(t) + tu''(t)} \frac{1}{2\sqrt{t}} \geq \frac{B}{2(C+t)\log(C+t)}.$$

In view of (16), we conclude that the distance function along each fiber goes to infinity. To sum up, we show that the Kähler metric in the form of (12) with u of the form (17) is complete.

As an application of the Calabi method, we study the growth of holomorphic functions of the vector bundle E defined in (5). For simplicity, we assume $\tau = \sqrt{-1}$ in (5), and consider the following hermitian metric on the trivial rank 2 bundle over $\mathbb{C}$,

$$(19) \quad |(v^1, v^2)|_h^2 = |v^1 - (\operatorname{Im} z)v^2|^2 + k|v^2|^2 \text{ for any } k > 0.$$

Such a family of Hermitian (fiber) metrics was introduced by Paun [32, p.53]. Note that (19) is invariant under the action of (Γ, ρ) defined in (5). Let h denote the corresponding Hermitian metric on E over the elliptic curve $\Sigma = \mathbb{C}/\Gamma$. According to the Calabi method, we may study Kähler metrics on X in the form of (12) where $\omega_0 = \sqrt{-1}dz \wedge d\bar{z}$ is a flat metric on Σ . Let $z = x + \sqrt{-1}y$ and $t = h(v, \bar{v}) = |v^1 - yv^2|^2 + k|v^2|^2$. Then:

$$\begin{aligned} h_{\alpha\bar{\beta}} &= \begin{pmatrix} 1 & -y \\ -y & y^2 + k \end{pmatrix}, \quad h^{\alpha\bar{\beta}} = \frac{1}{k} \begin{pmatrix} y^2 + k & y \\ y & 1 \end{pmatrix}. \\ \partial_z h_{\alpha\bar{\beta}} &= \begin{pmatrix} 0 & \frac{\sqrt{-1}}{2} \\ \frac{\sqrt{-1}}{2} & \frac{y}{\sqrt{-1}} \end{pmatrix}, \quad \Gamma_{\alpha z}^{\beta} = h^{\beta\bar{\delta}} \partial_z h_{\alpha\bar{\delta}} = \frac{\sqrt{-1}}{2k} \begin{pmatrix} y & 1 \\ k - y^2 & -y \end{pmatrix}. \\ \partial_{\bar{z}} \Gamma_{z\alpha}^{\beta} &= \frac{1}{4k} \begin{pmatrix} 1 & 0 \\ -2y & -1 \end{pmatrix}, \quad R_{z\bar{z}\alpha\bar{\beta}}^E = -\partial_{\bar{z}} \Gamma_{\alpha z}^{\gamma} h_{\gamma\bar{\beta}} = -\frac{1}{4k} \begin{pmatrix} 1 & -y \\ -y & y^2 - k \end{pmatrix}. \end{aligned}$$

According to [4, p.274], we may rewrite (12) as

$$(20) \quad \omega_g = \sqrt{-1} \left((1 - u'(t) R_{z\bar{z}\alpha\bar{\beta}}^E v^{\alpha} \bar{v}^{\beta}) dz \wedge d\bar{z} + (u''(t) h_{\alpha\bar{\xi}} \bar{v}^{\xi} h_{\eta\bar{\beta}} v^{\eta} + u'(t) h_{\alpha\bar{\beta}}) \nabla v^{\alpha} \wedge \nabla \bar{v}^{\beta} \right)$$

where

$$(21) \quad \nabla v^{\alpha} = dv^{\alpha} + (v^1 \Gamma_{1z}^{\alpha} + v^2 \Gamma_{2z}^{\alpha}) dz.$$

With the choice of Hermitian metric h on E in the form of (19), we may solve

$$\begin{aligned} \nabla v^1 &= dv^1 + (v^1 \Gamma_{1z}^1 + v^2 \Gamma_{2z}^1) dz = dv^1 + \frac{\sqrt{-1}}{2k} (yv^1 + (k - y^2)v^2) dz. \\ \nabla v^2 &= dv^2 + (v^1 \Gamma_{1z}^2 + v^2 \Gamma_{2z}^2) dz = dv^2 + \frac{\sqrt{-1}}{2k} (v^1 - yv^2) dz. \end{aligned}$$

$$\begin{aligned} R_{z\bar{z}\alpha\bar{\beta}}^E v^{\alpha} \bar{v}^{\beta} &= -\frac{1}{4k} (|v^1|^2 - y(v^1 \bar{v}^2 + v^2 \bar{v}^1) + (y^2 - k)|v^2|^2) = -\frac{1}{4k} (|v^1 - yv^2|^2 - k|v^2|^2). \\ h_{\alpha\bar{\xi}} \bar{v}^{\xi} h_{\eta\bar{\beta}} v^{\eta} &= \begin{pmatrix} |v^1 - yv^2|^2 & (\bar{v}^1 - y\bar{v}^2)(-yv^1 + (y^2 + k)v^2) \\ (v^1 - yv^2)(-y\bar{v}^1 + (y^2 + k)\bar{v}^2) & |-yv^1 + (y^2 + k)v^2|^2 \end{pmatrix}. \end{aligned}$$

If we introduce

$$\psi^1 = \nabla v^1 - y \nabla v^2, \quad \psi^2 = \nabla v^2, \quad a = v^1 - yv^2, \quad b = kv^2.$$

and

$$A_{\alpha\bar{\beta}} = \begin{pmatrix} |a|^2 & b\bar{a} \\ a\bar{b} & |b|^2 \end{pmatrix}, \quad B_{\alpha\bar{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}.$$

Then (20) can be further reduced to

$$(22) \quad \omega_g = \sqrt{-1} \left[\left(1 + \frac{1}{4k} u'(t) (|v^1 - yv^2|^2 - k|v^2|^2) \right) dz \wedge d\bar{z} + (u'' A_{\alpha\bar{\beta}} + u' B_{\alpha\bar{\beta}}) \psi^\alpha \wedge \overline{\psi^\beta} \right].$$

Let p be a fixed point on a complete Kähler manifold (X, g) , and d_g the distance function from p . We define the order of growth of a holomorphic function $f \in \mathcal{O}(X)$ (in the sense of Hadamard) as

$$(23) \quad d_{H,f} = \inf \{ \rho > 0 \mid \text{there exist } \rho, C_1, C_2 > 0 \text{ so that } |f(q)| \leq C_1 e^{C_2(d_g(q,p)+1)^\rho} \text{ for any } q \in X \}.$$

Then we introduce a notion of the minimal H -(Hadamard) order on (X, g) as

$$(24) \quad d_{Hmin}(X, g) = \inf \{ d_{H,f} \mid f \in \mathcal{O}(X, g) \}.$$

Proposition 2.2. *Let E be the indecomposable rank 2 vector bundle of degree zero defined in (5) endowed with the Hermitian metric on E defined in (19). Let X be the total space of E . Then for any complete Kähler metric g on X of the Calabi type, $d_{Hmin}(X, g) \geq 2$. Moreover, there exists a complete Kähler metric of the Calabi type so that the corresponding $d_{Hmin} = 2$.*

Proof of Proposition 2.2. In our case $t = |v^1 - yv^2|^2 + k|v^2|^2$. We observe that $|v^1 - yv^2|^2 - k|v^2|^2 = -t$ when $v^1 = yv^2$. We conclude that $tu'(t) \leq C$ for some constant C as ω_g is positive-definite. The integral in (16) can be bounded from the above. Indeed, for $x > 0$ large enough, we have

$$(25) \quad \int_1^x \sqrt{tu''(t) + u'(t)} \frac{1}{\sqrt{t}} dt \leq \left(\int_1^x (tu'(t))' dt \right)^{\frac{1}{2}} \left(\int_1^x \frac{1}{t} dt \right)^{\frac{1}{2}} \leq C \sqrt{\ln x}.$$

Now we consider the growth of any $f \in \mathcal{O}(X)$ with respect to the distance d induced by a Calabi-type metric g . According to Proposition 1.2, it suffices to estimate the growth of z_2 along each fiber. As $\ln t \geq \frac{1}{C^2} d^2$ by (25), we conclude $d_{H,f} \geq 2$.

It remains to construct a Calabi-type metric such that $d_{Hmin} = 2$. We consider a smooth positive function $h(t)$ on $[0, +\infty)$ so that $h(t) = t^2$ on $[0, \alpha]$ and

$$h(t) = \frac{1}{t \ln t (\ln \ln t)^2} \text{ for } t \in [\alpha + 1, +\infty).$$

Here α is a sufficiently large constant such that $\ln \ln t > 2$ when $t > \alpha$. Then $\int_0^{+\infty} h(t) dt$ converges to a constant C . Let

$$(26) \quad (tu'(t))' = \frac{2k}{C} h(t) \text{ for } t \geq 0.$$

Then the coefficient of the $\sqrt{-1} dz \wedge d\bar{z}$ term in (22) is positive. Note that $t \leq |a|^2 + |b|^2 \leq kt$ and $tu''(t) + u'(t) > 0$. Hence $u''(t)A + u'(t)B$ is positive definite. We further estimate the geodesic distance by (16). For sufficiently large $x > 0$,

$$\int_0^x \sqrt{tu''(t) + u'(t)} \frac{1}{\sqrt{t}} dt \sim \int_{\alpha+1}^x \frac{1}{t \sqrt{\ln t \ln \ln t}} dt = \infty.$$

Hence (22) defines a complete Calabi-type metric. Finally, $d_{Hmin} = 2$ follows from an elementary identity

$$\lim_{x \rightarrow +\infty} \frac{\ln \ln x}{\ln \int_{\alpha+1}^x \frac{1}{t \sqrt{\ln t \ln \ln t}} dt} = 2.$$

To sum up, (26) is an alternative choice of u in the proof of Theorem 2.1 under which holomorphic functions grow more ‘slowly’ than in (17). $\square$

3. LINE BUNDLES WHOSE TOTAL SPACES ADMIT KÄHLER METRICS WITH $BI \geq 0$

In this section, we study complete Kähler metrics with $BI \geq 0$ on the total space of a line bundle over a complex torus, proving Theorem 1.7 and Proposition 1.11. We begin with a statement of Appel-Humbert's theorem ([27] and [11]) which characterizes any line bundle over a compact complex torus.

Let V be a complex vector space of dimension n and $V = \mathbb{C}\text{-span}\{e_1, \dots, e_n\}$. Let H be a Hermitian bilinear form on $V \times V$. For any $z \in V$ with $z = z^i e_i$, we define

$$(27) \quad h_{i\bar{j}} = H(e_i, e_j) = \operatorname{Re} H(e_i, e_j) + \sqrt{-1} \operatorname{Im} H(e_i, e_j).$$

Then $\operatorname{Re} H$ is symmetric while $\operatorname{Im} H$ is skew-symmetric, and both are i -invariant in the sense of $(\cdot, \cdot) = (i\cdot, i\cdot)$. We may also consider the corresponding real $(1, 1)$ form

$$(28) \quad H_{form} = \sqrt{-1} h_{i\bar{j}} dz^i \wedge \overline{dz^j}.$$

Next we pick $\Gamma \subset V$ which is a lattice of real rank $2n$. Let $V_{\mathbb{R}}$ denote the underlying real vector space of V . Then we have $\Gamma \otimes_{\mathbb{Z}} \mathbb{R} = V_{\mathbb{R}}$. It is possible to define an endomorphism J with $J^2 = -id$ on $V_{\mathbb{R}}$ (and its complexification $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$) so that V is exactly the $\sqrt{-1}$ eigenspace. On the compact complex torus $M = V/\Gamma$, the real $(1, 1)$ form H_{form} defined in (28) lies in $H^2(M, \mathbb{Z})$ if and only if $\operatorname{Im} H$ is $\mathbb{Z}$ -valued on $\Gamma \times \Gamma$. We refer the reader to [33, Lemma 3.7] for a nice geometric explanation of this fact.

Let H be a Hermitian bilinear form on V so that $\operatorname{Im} H$ is $\mathbb{Z}$ -valued when restricted onto $\Gamma \times \Gamma$. Let $\alpha : \Gamma \rightarrow S^1$ so that

$$(29) \quad \alpha(\gamma_1 + \gamma_2) = e^{i\pi \operatorname{Im} H(\gamma_1, \gamma_2)} \alpha(\gamma_1) \alpha(\gamma_2).$$

Then we may define a line bundle $L(H, \alpha)$ over M as a quotient of $V \times \mathbb{C}$ under the action

$$(30) \quad (z, \xi) \rightarrow (z + \gamma, \alpha(\gamma) e^{\pi H(z, \gamma) + \frac{\pi}{2} H(\gamma, \gamma)} \xi).$$

Obviously, (30) reduces to (7) when M is an elliptic curve. Appel-Humbert's theorem (see [27, Theorem on p.19] and [11, Theorem 5.17 on p.50]) states that any line bundle over M is isomorphic to some $L(H, \alpha)$. Note that $h(z, \xi) = e^{-\pi H(z, z)} |\xi|^2$ is invariant under the action (30) and descends to a Hermitian metric on $L(H, \alpha)$. Moreover, its curvature form $\Theta(L) = \pi H_{form}$ and a representative of $c_1(L)$ is $-\operatorname{Im} H$.

We state an observation on a class of subharmonic functions and recall a simple fact in linear algebra.

Lemma 3.1. *Given a fixed integer $n \geq 1$. Let $f(z)$ be a holomorphic function on $\mathbb{C}^n$ which is not identically zero, and u a positive smooth function with $\Delta \ln u \leq 0$ (or $\Delta \ln u \geq 0$) everywhere on $\mathbb{C}^n$. We assume either of the following*

- (1) $n = 1$ and $u(z)|f(z)|^2$ is bounded between two positive constants on $\mathbb{C}$.
- (2) $u(z)|f(z)|^2$ is doubly periodic (invariant under $z \rightarrow z + \gamma$ for any $\gamma \in \Gamma$ for a lattice Γ of real rank $2n$ in $\mathbb{C}^n$).

Then $\ln u$ is harmonic and $u(z)|f(z)|^2$ is constant.

Proof of Lemma 3.1. (1) Note that f has no zeros and $\Delta \ln(h|f|^2) = \Delta \ln u \leq 0$. Then $-\ln(h(z)|f(z)|^2)$ is a bounded subharmonic function. Therefore, $h(z)|f(z)|^2$ is constant. Note that any smooth bounded subharmonic function with sub-logarithmic growth on $\mathbb{C}$ is constant. It follows that $\Delta \ln h = -\Delta \ln |f|^2 = 0$.

(2) Let $Z(f)$ denote the zero set of an entire function f . Note that $Z(f)$ is a union of analytic hypersurfaces (with multiplicity) in $\mathbb{C}^n$. By the Poincaré-Lelong formula ([15, p.388]), we have

$$\sqrt{-1} \partial \bar{\partial} \ln(u|f|^2) = \sqrt{-1} \partial \bar{\partial} \ln u + \sqrt{-1} \partial \bar{\partial} \ln(|f|^2) = \sqrt{-1} \partial \bar{\partial} \ln u, \quad z \in \mathbb{C}^n \setminus Z(f).$$

According to our assumption, $\sqrt{-1} \partial \bar{\partial} \ln u$ is doubly-periodic in $\mathbb{C}^n \setminus Z(f)$. As it is smooth, we observe that it is in fact doubly-periodic on $\mathbb{C}^n$. In view of

$$\sqrt{-1} \partial \bar{\partial} \ln u \wedge \omega_0^{n-1} = \frac{n}{4} (\Delta \ln u) \omega_0^n$$

where ω_0 the standard Kähler form on $\mathbb{C}^n$, $\Delta \ln u$ is also doubly periodic. But then $\Delta \ln u = 0$ everywhere as the integral of $\Delta \ln u$ is zero in a fundamental parallelogram (corresponding to a closed torus). We conclude that $\ln(u(z)|f(z)|^2)$ is subharmonic, and it attains the maximum at some point. Therefore, $u(z)|f(z)|^2$ is constant. $\square$

Lemma 3.2. *Let $a_1, a_2, \dots, a_n$ be n -dimensional column vectors, and $A = a_1 \bar{a}_1^T + \dots + a_n \bar{a}_n^T$. Then $A = B \bar{B}^T$ where $B = (a_1, a_2, \dots, a_n)$. In particular, $\det A = |\det B|^2$.*

Proof of Proposition 1.11. If each $H_p = 0$, it follows from (30) that

$$\omega = \sqrt{-1} \left(\sum_{q=1}^{n-k} dz_q \wedge d\bar{z}_q + \sum_{p=1}^k d\xi_p \wedge d\bar{\xi}_p \right)$$

is a well-defined complete flat Kähler metric on X .

Conversely, assume that X admits a complete flat Kähler metric g . There exists $F \in \text{Aut}(\mathbb{C}^n)$ with $F(z, \xi) = (f_1(z, \xi), f_2(z, \xi), \dots, f_n(z, \xi))$ such that

$$\omega_g = \sqrt{-1} \sum_{j=1}^n df_j \wedge d\bar{f}_j,$$

is invariant with respect to

$$(31) \quad (z, \xi_1, \dots, \xi_k) \rightarrow (z + \gamma, \alpha_1(\gamma) e^{\pi H_1(z, \gamma) + \frac{\pi}{2} H_1(\gamma, \gamma)} \xi_1, \dots, \alpha_k(\gamma) e^{\pi H_k(z, \gamma) + \frac{\pi}{2} H_k(\gamma, \gamma)} \xi_k).$$

Along the zero section of E which is $\{\xi_1 = \dots = \xi_k = 0\}$, we have

$$df_j = f_{j,1}(z) dz_1 + \dots + f_{j,n-k}(z) dz_{n-k} + f_{j,n-k+1}(z) d\xi_1 + \dots + f_{j,n}(z) d\xi_k$$

for $j = 1, 2, \dots, n$. It follows from Lemma 3.2 that there exists a holomorphic function $h(z)$ such that $\det(g) = |h(z)|^2 > 0$. The volume form

$$(32) \quad \det(g) dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_{n-k} \wedge d\bar{z}_{n-k} \wedge \dots \wedge d\xi_k \wedge d\bar{\xi}_k$$

is invariant with respect to (31). Therefore, for any $\gamma \in \Lambda$,

$$(33) \quad |h(z)|^2 = |h(z + \gamma)|^2 e^{\pi H_1(z, \gamma) + \pi H_1(\gamma, z) + \pi H_1(\gamma, \gamma)} \dots e^{\pi H_k(z, \gamma) + \pi H_k(\gamma, z) + \pi H_k(\gamma, \gamma)}.$$

Then

$$(34) \quad |h(z)|^2 e^{\pi(H_1(z, z) + \dots + H_k(z, z))}$$

is invariant under $z \rightarrow z + \gamma$, therefore bounded between two positive constants. For any fixed $z_0 \in \mathbb{C}^{n-k}$, consider a holomorphic function of a single variable t as

$$|h(tz_0)|^2 e^{\pi(H_1(z_0, z_0) + \dots + H_k(z_0, z_0))|t|^2}.$$

It follows from part (1) of Lemma 3.1 that

$$(35) \quad H_1(z_0, z_0) + \dots + H_k(z_0, z_0) = 0.$$

If there exists some $1 \leq p \leq k$ so that $H_p(z_0, z_0) > 0$, then we compare the coefficients in front of $d\xi_p \wedge d\bar{\xi}_p$ and get

$$\sum_{j=1}^n (|f_{j,n-k+p}(z)|^2) e^{\pi H_p(z, z)}$$

is invariant under $z \rightarrow z + \gamma$. Similarly, we get

$$\sum_{j=1}^n (|f_{j,n-k+p}(tz_0)|^2) e^{\pi H_k(z_0, z_0)|t|^2}$$

is bounded. Taking $|t| \rightarrow +\infty$, we conclude that $f_{j,n-k+p}(tz_0) = 0$ for each j by the maximum principle. It's a contradiction as the Jacobian of F is nondegenerate. Hence $H_p = 0$ for all $p = 1, 2, \dots, k$. $\square$

We restate part (2) of Theorem 1.7 separately in the following.

Lemma 3.3. *Let $L(H, \alpha)$ be a line bundle over a compact complex torus $\mathbb{C}^{n-1}/\Gamma$ and X the total space. Assume that X admits a complete Kähler metric g with*

$$(36) \quad \omega_g = \sqrt{-1}(u df_1 \wedge d\bar{f}_1 + \sum_{j=2}^n df_j \wedge d\bar{f}_j)$$

Here $F(z, \xi) = (f_1(z, \xi), f_2(z, \xi), \dots, f_n(z, \xi)) \in \text{Aut}(\mathbb{C}^n)$ and $u(z, \xi) := u \circ f_1(z, \xi)$ such that $u(w)|dw|^2$ is a complete Kähler metric with nonnegative Gauss curvature on $\mathbb{C}$. Then $H = 0$.

Proof of Lemma 3.3. Throughout the proof, we focus on the zero section $\{\xi = 0\}$. Let

$$df_j = f_{j,1}(z)dz_1 + \dots + f_{j,n-1}(z)dz_{n-1} + f_{j,n}(z)d\xi$$

for $j = 1, 2, \dots, n$. It follows from Lemma 3.2 that there exists a holomorphic function $h(z)$ such that $\det(g) = u(z)|h(z)|^2 > 0$. Considering the volume form,

$$u(z)|h(z)|^2 e^{\pi H(z, z)}$$

is invariant under $z \rightarrow z + \gamma$. For any fixed $z_0 = (z_{0,1}, \dots, z_{0,n-1})$,

$$(37) \quad \Delta_t \log \left(u(tz_0)|h(tz_0)|^2 e^{\pi H(z_0, z_0)|t|^2} \right) = \frac{1}{4} \left(\Delta \ln u \Big| \frac{\partial f_1}{\partial z_k} z_{0,k} \Big|^2 + 2\pi H(z_0, z_0) \right).$$

Note that $\Delta \log u(z) \leq 0$. It follows from part (1) of Lemma 3.1 that

$$H(z_0, z_0) \geq 0.$$

Suppose that there exists z_0 with $H(z_0, z_0) > 0$. We compare the coefficients in front of $d\xi \wedge d\bar{\xi}$ on $\{tz_0 | t \in \mathbb{C}\}$. One has

$$(38) \quad (u(tz_0)|f_{1,n}(tz_0)|^2 + |f_{2,n}(tz_0)|^2 + \dots + |f_{n,n}(tz_0)|^2) e^{\pi H(z_0, z_0)|t|^2}$$

is bounded. We will show that $u(z)|f_{1,n}(z)|^2$ is constant, and then we get a contradiction as before.

Case 1: If $f_{1,p}(z)$ is identically zero for any $1 \leq p \leq n-1$, then $f_1(z, 0)$ is constant, and so is $u(z, 0) = u(f_1(z, 0))$. Note that $H(z_0, z_0) > 0$. It follows from (38) that $f_{1,n}(tz_0) = \dots = f_{n,n}(tz_0) = 0$. This is a contradiction, as the determinant of the complex Jacobian of F is nowhere vanishing.

Case 2: We assume that there exists some $1 \leq p \leq n-1$ such that $f_{1,p}(z)$ is not identically zero. Comparing the coefficients in front of $dz_p \wedge d\bar{z}_p$, we get that

$$u(z)|f_{1,p}(z)|^2 + |f_{2,p}(z)|^2 + \dots + |f_{n,p}(z)|^2$$

is doubly periodic. As each $f_{k,p}$ with $2 \leq k \leq n$ has to be constant, $u(z)|f_{1,p}(z)|^2$ is doubly periodic. By part (2) of Lemma 3.1, $u(z)|f_{1,p}(z)|^2$ must be constant. Then $f_{1,p}$ has no zeros and $u(z) = \frac{C}{|f_{1,p}(z)|^2}$. By (38), we get

$$C \left| \frac{f_{1,n}(tz_0)}{f_{1,1}(tz_0)} \right|^2 + |f_{2,n}(tz_0)|^2 + \dots + |f_{n,n}(tz_0)|^2 \rightarrow 0.$$

However, it follows that $f_{k,n}(tz_0) = 0$ for any $1 \leq k \leq n$. Contradiction again!

To sum up, we have proved $H \equiv 0$. □

Proof of Theorem 1.7. Step 1: We prove the first part of Theorem 1.7. Let X be the total space of a line bundle $L(H, \alpha)$ over an elliptic curve Σ . We show that X admits a complete Kähler metric with nonnegative bisectional curvature if and only if $H = 0$.

Assume X admits a complete Kähler metric g with $BI \geq 0$. We show that the corresponding universal covering manifold $(\tilde{X}, \tilde{g})$ is holomorphically isometric to $\mathbb{C} \times N$ where (N, h) is a complete Kähler metric with nonnegative Gauss curvature on $\mathbb{C}$. Once it holds, $H = 0$ follows from Lemma 3.3.

To see it, we use a remarkable result of Ni-Tam [30]. For any fixed point $p \in X$, we consider the Busemann function $B_\sigma(x) = \lim_{s \rightarrow \infty} (s - d(x, \sigma(s)))$ associated to a geodesic ray σ from p . Then we define

$$(39) \quad \mathcal{B}(x) = \sup_{\sigma} B_\sigma(x).$$

According to Wu [39, Theorem A on p.58], $\mathcal{B}$ is a Lipschitz continuous (with Lipschitz constant 1) plurisubharmonic (PSH for short) function on X . In [30] the authors study the heat flow of $\mathcal{B}$ with respect to the metric g

$$(40) \quad \frac{\partial}{\partial t} v - \Delta_g v = 0, \quad v(x, 0) = \mathcal{B}(x).$$

For any fixed $t > 0$, $v(x, t)$ is a smooth PSH function with $\sup_{x \in X} |\nabla v(x, t)| \leq 1$. However, v can not be strictly PSH as the restriction of v onto the zero section of L is constant. According to [30, Lemma 4.1], there are two cases.

Case 1: $\mathcal{B}$ is pluriharmonic on X . If so, $\mathcal{B}$ is a smooth harmonic function with linear growth. Following the proof of [30, Theorem 4.1], we see that $\nabla \mathcal{B}$ is parallel. Then $\nabla \mathcal{B}$ generates a geodesic line with $\text{Ric}(\nabla \mathcal{B}, \nabla \mathcal{B}) = 0$. Moreover, $J\nabla \mathcal{B}$ is also parallel and its lifting to $\tilde{X}$ generates a geodesic line. In this case $(\tilde{X}, \tilde{g})$ splits as $(\mathbb{C} \times N, g_e + h)$.

Case 2: For any $t > 0$,

$$\mathcal{K}(x, t) = \{w \in T_x^{1,0}(X) \mid v_{\alpha\bar{\beta}} w^\alpha = 0, \quad \forall 1 \leq \beta \leq n\}$$

is a nontrivial parallel distribution on X . Now we fix some $t_0 > 0$ and write $v(x)$ as $v(x, t_0)$. Let $\rho : \tilde{X} \rightarrow X$ and $\rho^* v$ the lifting of v onto $\tilde{X}$. The universal covering $(\tilde{X}, \tilde{g})$ is holomorphically isometric to as $N_1 \times N_2$ so that $T^{1,0} N_1 = \mathcal{K}(x, t_0)$. In particular, the complex Hessian of $\rho^* v$ vanishes along $T^{1,0} N_1$. We observe that N_1 is holomorphically isometric to $\mathbb{C}$. Indeed, for any fixed $w \in N_2$, both $\nabla \rho^* v(\cdot, w)$ and $J\nabla \rho^* v(\cdot, w)$ are parallel along N_1 .

Step 2: We give a complete characterization of complete Kähler metrics with $BI \geq 0$ on X , proving (10) and (11).

Now we have $H = 0$ by Lemma 3.3. Let $\alpha(1) = e^{2\pi i \theta_1}$ and $\alpha(\tau) = e^{2\pi i \theta_2}$ for $\theta_1, \theta_2 \in [0, 1)$. We study the Kähler metric $\omega = \sqrt{-1}(udf_1 \wedge d\bar{f}_1 + df_2 \wedge d\bar{f}_2)$ at any point $(z, \xi) \in \mathbb{C}^2$. One has

$$(41) \quad df_1 = f_{1,1}(z, \xi)dz + f_{1,2}(z, \xi)d\xi, \quad df_2 = f_{2,1}(z, \xi)dz + f_{2,2}(z, \xi)d\xi.$$

The local components of ω read

$$\begin{pmatrix} g_{z\bar{z}} & g_{z\bar{\xi}} \\ g_{\xi\bar{z}} & g_{\xi\bar{\xi}} \end{pmatrix} = \begin{pmatrix} u(z, \xi)|f_{1,1}(z, \xi)|^2 + |f_{2,1}(z, \xi)|^2 & u(z, \xi)f_{1,1}(z, \xi)\overline{f_{1,2}(z, \xi)} + f_{2,1}(z, \xi)\overline{f_{2,2}(z, \xi)} \\ u(z, \xi)f_{1,2}(z, \xi)\overline{f_{1,1}(z, \xi)} + f_{2,2}(z, \xi)\overline{f_{2,1}(z, \xi)} & u(z, \xi)|f_{1,2}(z, \xi)|^2 + |f_{2,2}(z, \xi)|^2 \end{pmatrix}.$$

It follows from Lemma 3.2 that there exists a holomorphic function $h(z, \xi)$ such that

$$(42) \quad \det g(z, \xi) = u(z, \xi)|h(z, \xi)|^2 > 0.$$

Note that for any fixed ξ , $\det g$ takes values in a compact set $K \times S^1 \subset \mathbb{C}^2$ where K can be chosen as the closed fundamental parallelogram in $\mathbb{C}$. By part (1) of Lemma 3.1, there exists some constant $C(\xi) > 0$ such that

$$(43) \quad u(z, \xi) = \frac{C(\xi)}{|h(z, \xi)|^2}.$$

For any fixed ξ , we know that $g_{z\bar{z}}$ is bounded, and $f_{2,1}(z, \xi)$ is bounded. Hence $f_{2,1}(z, \xi) = f_{2,1}(\xi)$ which is independent of z . Moreover, for a fixed ξ ,

$$(44) \quad u(z, \xi)|f_{1,1}(z, \xi)|^2 = C(\xi) \left| \frac{f_{1,1}(z, \xi)}{h(z, \xi)} \right|^2$$

is bounded. Hence there exists $C_1(\xi)$ with $f_{1,1}(z, \xi) = C_1(\xi)h(z, \xi)$. By the same argument, $f_{2,2}(z, \xi) = f_{2,2}(\xi)$ and $f_{1,2}(z, \xi) = C_2(\xi)h(z, \xi)$. By (41), we have $\frac{\partial f_2}{\partial z} = f_{2,1}(\xi)$ and $\frac{\partial f_2}{\partial \xi} = f_{2,2}(\xi)$. Therefore, $f_{2,1}(\xi)$ must be constant. It follows that

$$(45) \quad g = \begin{pmatrix} C(\xi)|C_1(\xi)|^2 + |f_{2,1}|^2 & C(\xi)C_1(\xi)\overline{C_2(\xi)} + f_{2,1}\overline{f_{2,2}(\xi)} \\ C(\xi)C_2(\xi)\overline{C_1(\xi)} + f_{2,2}(\xi)\overline{f_{2,1}} & C(\xi)|C_2(\xi)|^2 + |f_{2,2}(\xi)|^2 \end{pmatrix}.$$

Therefore $C(\xi)|C_1(\xi)|^2$ must be constant due to the Kähler condition $\frac{\partial g_{z\bar{z}}}{\partial \xi} = \frac{\partial g_{\xi\bar{\xi}}}{\partial z}$.

Case 1: $C_1(\xi)$ is not identically zero. In this case, $C = C(\xi)|C_1(\xi)|^2 > 0$ and $C_1(\xi)$ is nowhere vanishing. Let $\lambda(\xi) = \frac{C_2(\xi)}{C_1(\xi)}$. (45) can be simplified as

$$(46) \quad \omega = \sqrt{-1} \left[C(dz + \lambda(\xi)d\xi) \wedge \overline{(dz + \lambda(\xi)d\xi)} + (f_{2,1}dz + f_{2,2}(\xi)d\xi) \wedge \overline{(f_{2,1}dz + f_{2,2}(\xi)d\xi)} \right].$$

We observe that

$$(a\eta_1 + b\eta_2) \wedge (\bar{a}\bar{\eta}_1 + \bar{b}\bar{\eta}_2) + (\bar{b}\eta_1 - \bar{a}\eta_2) \wedge (b\bar{\eta}_1 - a\bar{\eta}_2) = (|a|^2 + |b|^2)(\eta_1 \wedge \bar{\eta}_1 + \eta_2 \wedge \bar{\eta}_2).$$

Let $\eta_1 = \sqrt{C}(dz + \lambda(\xi)d\xi)$ and $\eta_2 = f_{2,1}dz + f_{2,2}(\xi)d\xi$ while $a = \sqrt{C}$ and $b = \overline{f_{2,1}}$. We conclude that (46) can be reduced to

$$(47) \quad \omega = \sqrt{-1} \left[(C + |f_{2,1}|^2)d(z + h_1(\xi)) \wedge \overline{d(z + h_1(\xi))} + |h_2(\xi)|^2 d\xi \wedge \overline{d\xi} \right]$$

where

$$(48) \quad h'_1(\xi) = \frac{C\lambda(\xi) + \overline{f_{2,1}}f_{2,2}(\xi)}{C + |f_{2,1}|^2}, \quad h_2(\xi) = \frac{\sqrt{C}}{\sqrt{C + |f_{2,1}|^2}}(f_{2,1}\lambda(\xi) - f_{2,2}(\xi)).$$

Note that both h_1 and h_2 are holomorphic on $\mathbb{C}$ and h_2 is nowhere vanishing. It follows that (47) corresponds to a flat Kähler metric on $\mathbb{C}^2$. Moreover, the induced metric along the ξ direction does not need to be flat.

Case 2: $C_1(\xi) \equiv 0$. In this case, $f_{1,1}(z, \xi) \equiv 0$ and f_1 depends only on ξ . Then we can rewrite $u(f_1(\xi))df_1 \wedge \overline{df_1}$ as $u(f_1(\xi))|\frac{\partial f_1}{\partial \xi}|^2 d\xi \wedge \overline{d\xi}$. Moreover, $f_{2,1}$ is a nonzero constant and we may simplify (45) as

$$(49) \quad \omega = \tilde{u}(\xi)d\xi \wedge \overline{d\xi} + C(dz + dh(\xi)) \wedge (\overline{dz} + \overline{dh(\xi)}),$$

where

$$(50) \quad \tilde{u}(\xi) = u(f_1(\xi))|\frac{\partial f_1}{\partial \xi}|^2, \quad h'(\xi) = \frac{f_{2,2}(\xi)}{f_{2,1}}.$$

Recall ω is invariant under the action

$$(51) \quad (z, \xi) \rightarrow (z + 1, e^{2\pi i \theta_1} \xi), \quad \text{and} \quad (z, \xi) \rightarrow (z + \tau, e^{2\pi i \theta_2} \xi).$$

Therefore, $\tilde{u}(\xi)$ satisfies $\tilde{u}(e^{2\pi i \theta_1} \xi) = \tilde{u}(e^{2\pi i \theta_2} \xi) = \tilde{u}(\xi)$. Moreover, $\Delta \ln \tilde{u} \leq 0$ and $h(\xi)$ is a holomorphic function with the invariant property $h(e^{2\pi i \theta_1} \xi) = h(e^{2\pi i \theta_2} \xi) = h(\xi)$. In particular, if one of θ_1 and θ_2 is irrational, $h(\xi)$ is forced to be constant and (49) is reduced to

$$(52) \quad \omega = Cdz \wedge \overline{dz} + \tilde{u}(|\xi|)d\xi \wedge \overline{d\xi}.$$

If both θ_1 and θ_2 are rational, there exists a minimal positive integer k so that $k\theta_1$ and $k\theta_2$ are integers. $\tilde{u}(\xi)$ is $\mathbb{Z}_k$ -invariant in the sense that $\tilde{u}(\xi) = \tilde{u}(e^{\frac{2\pi i}{k}} \xi)$ for any $\xi \in \mathbb{C}$. Similarly, $h(\xi)$ is an entire function of ξ^k . Then (49) is reduced to

$$(53) \quad \omega = Cdz \wedge \overline{dz} + \tilde{u}(\xi)d\xi \wedge \overline{d\xi} + \tilde{u}(\xi)d\xi \wedge \overline{d\xi}.$$

□

It follows from (47) and (49) that the zero section of X (which corresponds to $\{\xi = 0\}$) is totally geodesic. However, the fiber is not totally geodesic in general. We give an example to show that each fiber with the induced metric has nonpositive Gauss curvature.

Example 3.4. It follows from (49) that any nonflat complete Kähler metric g on X with $BI \geq 0$ must be of the form

$$g = \begin{pmatrix} C & \overline{\tilde{h}(\xi)} \\ \tilde{h}(\xi) & \tilde{u}(\xi) + |\tilde{h}(\xi)|^2 \end{pmatrix}.$$

Here $\tilde{h}(\xi) = h'(\xi)$ for h defined in (49). Let $\tilde{u}(\xi) = \frac{1}{1+|\xi|^2}$ and $\tilde{h}(\xi) = \xi$. Note that we choose $\tilde{u}(\xi)|d\xi|^2$ as Hamilton's cigar soliton on $\mathbb{C}$ in [17]. We may choose either $\theta_1 = \theta_2 = 1$ or $\theta_1 = \theta_2 = \frac{1}{2}$. Then either the corresponding X is globally trivial or a finite cover of X is so. In either case, the induced metric on the fiber $(|\xi|^2 + \frac{1}{1+|\xi|^2})|d\xi|^2$ has nonpositive Gauss curvature everywhere. To see it, we consider $f(t) = t + \frac{1}{1+t}$ on $[0, \infty)$. Then

$$t(\ln f)' = \frac{2t^2 + t}{t^2 + t + 1} - \frac{t}{1+t}, \quad (t(\ln f)')' = \frac{t(4t^2 + 7t + 4)}{(1+t)^2(t^2 + t + 1)^2} \geq 0.$$

4. FUNCTION THEORY ON TOTAL SPACES OF RANK 2 BUNDLES OF DEGREE ZERO

In this section, we study the function theory on total spaces of rank 2 bundles of degree zero over an elliptic curve Σ defined by (1). Our main goal is to prove Theorem 1.5. During the course of the proof, we develop several function-theoretic and geometric results on the total spaces, including Proposition 4.2 and Proposition 4.3.

First of all, we observe an explicit holomorphic bundle isomorphism between two vector bundles which are associated with representations (8) with the same value of $b_2 - b_1\tau$.

Lemma 4.1. Let X (resp. Y) be the total space of E (resp. $\tilde{E}$) over the elliptic curve Σ defined by (1). Suppose E is associated with the representation $\rho_E(\gamma_k) = e^{2\pi i \theta_k} \begin{pmatrix} 1 & b_k \\ 0 & 1 \end{pmatrix}$ and $\tilde{E}$ is associated with $\rho_{\tilde{E}}(\gamma_k) = e^{2\pi i \theta_k} \begin{pmatrix} 1 & \tilde{b}_k \\ 0 & 1 \end{pmatrix}$ with $k = \{1, 2\}$.

If $b_2 - b_1\tau = \tilde{b}_2 - \tilde{b}_1\tau$, then $F(z, z_1, z_2) = (z, z_1 - b_1 z z_2 + \tilde{b}_1 z z_2, z_2)$ defines a holomorphic bundle isomorphism between E and $\tilde{E}$. In particular, if E is indecomposable, then $b_2 - b_1\tau \neq 0$.

Proof of Lemma 4.1. It is a direct computation to show that:

$$\begin{aligned} F(z + 1, e^{2\pi i \theta_1} \begin{pmatrix} z_1 + b_1 z_2 \\ z_2 \end{pmatrix}) &= (z + 1, e^{2\pi i \theta_1} \begin{pmatrix} 1 & \tilde{b}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 - b_1 z z_2 + \tilde{b}_1 z z_2 \\ z_2 \end{pmatrix}), \\ F(z + \tau, e^{2\pi i \theta_2} \begin{pmatrix} z_1 + b_2 z_2 \\ z_2 \end{pmatrix}) &= (z + \tau, e^{2\pi i \theta_2} \begin{pmatrix} 1 & \tilde{b}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 - b_1 z z_2 + \tilde{b}_1 z z_2 \\ z_2 \end{pmatrix}). \end{aligned}$$

Therefore F descends to a bundle isomorphism between E and $\tilde{E}$. $\square$

The following result gives a complete characterization of $\mathcal{O}(X)$ when X is the total space of any rank-2 vector bundle of degree 0. It is part (2) of Proposition 1.4.

Proposition 4.2. Let E be a rank 2 vector bundle of degree 0 over the elliptic curve Σ defined in (1), and X the total space of E . Then exactly one of the following conclusions holds.

- (A) $E = L_1 \oplus L_2$ for two line bundles L_1 and L_2 with $\deg(L_1) + \deg(L_2) = 0$ and $L_1 \neq L_2$. We may choose ξ_1 and ξ_2 as local coordinates of L_1 and L_2 along the fiber directions respectively. Then any $f \in \mathcal{O}(X)$ has a Taylor series

$$(54) \quad f = \sum_{k=0}^{\infty} \sum_{p+q=k} a_{p,q} \xi_1^p \xi_2^q,$$

where $a_{p,q}$ is the local coefficient of any element in $H^0(\Sigma, L_1^{-p} \otimes L_2^{-q})$ for any integers $p, q \geq 0$. In particular, there are two subcases.

- (I) If $L_1 \neq L_2$ and $L_1, L_2 \in \text{Pic}^0(\Sigma)$, $a_{p,q}$ is nonzero (a constant) if and only if $L_1^{-p} \otimes L_2^{-q} = \mathcal{O}$.

- (II) If $\deg(L_1) = -\deg(L_2) > 0$, then $a_{p,q} = 0$ if $p > q$. Moreover, $a_{p,p} = 0$ if $L_2 \neq L_1^{-1}$, and it can be any constant if $L_2 = L_1^{-1}$. If $p < q$, $a_{p,q}$ is determined by $H^0(\Sigma, L_1^{-p} \otimes L_2^{-q})$ with $\dim H^0(\Sigma, L_1^{-p} \otimes L_2^{-q}) = (q-p)\deg(L_1)$.
- (B) E is associated with a representation defined by (8). For any given $f \in \mathcal{O}(X)$, we consider its lift (still denoted by f for simplicity) in $\mathcal{O}(\mathbb{C}^3)$. Then we have
- (I) If θ_1 or θ_2 is irrational, then $\mathcal{O}(X) = \{\text{constant}\}$.
- (II) If both θ_1 and θ_2 are rational, we may assume that m is the smallest positive integer such that $m\theta_1, m\theta_2 \in \mathbb{Z}$. Then we consider two subcases.
- (1) If $b_2 \neq b_1\tau$, then any lift of $f \in \mathcal{O}(X)$ admits a Taylor series

$$(55) \quad f(z, z_1, z_2) = \sum_{k \geq 0, m|k}^{\infty} c_k z_2^k, \text{ where } c_k \in \mathbb{C}.$$

(2) If $b_2 = b_1\tau$, then any lift of $f \in \mathcal{O}(X)$ admits a Taylor series

$$(56) \quad f(z, z_1, z_2) = \sum_{k \geq 0, m|k}^{\infty} \sum_{p+q=k} c_{p,q} (z_1 - b_1 z z_2)^p z_2^q, \text{ where } c_{p,q} \in \mathbb{C}, \text{ for integers } p, q \geq 0.$$

In Case (B), E is decomposable (hence isomorphic to $L \oplus L$ for some $L \in \text{Pic}^0(\Sigma)$) by Lemma 4.1) if and only if $b_2 = b_1\tau$.

Proof of Proposition 4.2. If $E = L_1 \oplus L_2$, then any holomorphic function on its total space X has a nice form of Taylor series. We refer the reader to [40, Corollary 3.5] for the corresponding Taylor series in the case of a line bundle. In our case, let ξ_1 and ξ_2 denote local coordinates of L_1 and L_2 along the fiber directions respectively. We may check

$$(57) \quad a_{p,q} \xi_1^p \xi_2^q \mid p+q=n, \text{ where } a_{p,q} \text{ are local coefficients of some } \sigma \in H^0(\Sigma, L_1^{-p} \otimes L_2^{-q}),$$

are globally defined on X . In general, for any given $f \in \mathcal{O}(X)$ we have the Taylor series as in (54).

Regarding the coefficients $a_{p,q}$, we recall a basic result on $\dim H^0(\Sigma, L)$ for any line bundle L over the elliptic curve Σ . It states that $H^0(\Sigma, L)$ is nonzero if and only if $L = \mathcal{O}$. In this case $H^0(\Sigma, L)$ is spanned by constant sections. In the case of $c_1(L) > 0$ and $\dim H^0(\Sigma, L) = \deg(L)$, we refer to [11, Theorem 6.44 on p.55] for a proof. For example, if we assume $\deg(L_1) = \deg(L_2) = 0$ and $L_1 \neq L_2$, by Appel-Humbert's theorem, we may assume $L_1 = L(0, \alpha)$ and $L_2 = L(0, \beta)$ where $\alpha, \beta : \Gamma \rightarrow S^1$ are two characters. In this case $H^0(\Sigma, L_1^{-p} \otimes L_2^{-q})$ is nonzero if and only if $\alpha(p\gamma)\beta(q\gamma) = 1$ for any $\gamma \in \mathbb{Z}\text{-span}\{1, \tau\}$.

Next we consider the case that E is associated with a representation defined by (8). For any given $f \in \mathcal{O}(X)$, we consider its lift $f \in \mathcal{O}(\mathbb{C}^3)$. For any $(z, z_1, z_2) \in \mathbb{C}^3$, f satisfies

$$(58) \quad f(z, z_1, z_2) = f(z+1, e^{2\pi i \theta_1}(z_1 + b_1 z_2), e^{2\pi i \theta_1} z_2) = f(z+\tau, e^{2\pi i \theta_2}(z_1 + b_2 z_2), e^{2\pi i \theta_2} z_2).$$

We consider its Taylor expansion

$$(59) \quad f(z, z_1, z_2) = f_0(z) + \sum_{k=1}^{+\infty} \sum_{t=0}^k f_{k-t,t}(z) z_1^{k-t} z_2^t.$$

It follows from (58) that

$$\begin{aligned} f(z, z_1, z_2) &= f_0(z+1) + \sum_{k=1}^{+\infty} e^{2\pi i \theta_1 k} \sum_{t=0}^k f_{k-t,t}(z+1) (z_1 + b_1 z_2)^{k-t} z_2^t, \\ f(z, z_1, z_2) &= f_0(z+\tau) + \sum_{k=1}^{+\infty} e^{2\pi i \theta_2 k} \sum_{t=0}^k f_{k-t,t}(z+\tau) (z_1 + b_2 z_2)^{k-t} z_2^t. \end{aligned}$$

We proceed to compare the coefficients of the homogeneous monomials of z_1 and z_2 . First of all, $f_0(z)$ must be constant since it is doubly periodic. For $k \geq 1$,

$$(60) \quad f_{k,0}(z) = e^{2\pi i \theta_1 k} f_{k,0}(z+1) = e^{2\pi i \theta_2 k} f_{k,0}(z+\tau).$$

It follows that $f_{k,0}$ is bounded, hence constant. Whenever one of $e^{2\pi i\theta_1 k}$ and $e^{2\pi i\theta_2 k}$ differs from 1, we have $f_{k,0} = 0$. We may further show $f_{k-1,1} = \dots = f_{0,k} = 0$ by the same argument.

If both θ_1 and θ_2 are rational, then we consider any positive integer k so that $e^{2\pi i\theta_1 k} = e^{2\pi i\theta_2 k} = 1$. Now that $f_{k,0}(z) = c_{k,0}$ which is a constant. Recall the binomial identity states that $(z_1 + b_1 z_2)^k = \sum_{p=0}^k C_k^p b_1^p z_1^{k-p} z_2^p$. We consider the coefficient of $z_1^{k-1} z_2$ in (59) and get

$$f_{k-1,1}(z) = f_{k-1,1}(z+1) + c_{k,0} k b_1 = f_{k-1,1}(z+\tau) + c_{k,0} k b_2.$$

Then $f_{k-1,1}(z)$ is of linear growth and we may assume $f_{k-1,1}(z) = Az + B$. Then $A + c_{k,0} k b_1 = A\tau + c_{k,0} k b_2 = 0$. It follows that $f_{k,0} = c_{k,0} \neq 0$ only if $b_2 = b_1 \tau$.

To sum up, when $b_2 \neq b_1 \tau$ and $k \geq 1$, we have $f_{k,0} = 0$ and $f_{k-1,1} = B$. If $k \geq 2$, we may apply the above argument on $f_{k-2,2}$. It follows that

$$f_{k-2,2}(z) = f_{k-2,2}(z+1) + B(k-1)b_1 = f_{k-2,2}(z+\tau) + B(k-1)b_2.$$

Similarly, we have $f_{k-2,2}(z)$ is of linear growth. Hence $f_{k-1,1} = 0$ and $f_{k-2,2}(z) = \text{constant}$. In the end, we have $f_{k,0} = f_{k-1,1} = \dots = f_{1,k-1} = 0$ and $f_{0,k}$ is constant. Hence f only depends on z_2 and the Taylor series (55) holds.

When $b_2 = b_1 \tau$ and $f_{k,0}(z) = c_{k,0} \neq 0$, we have

$$f_{k-1,1}(z) = -c_{k,0} b_1 k z + c_{k-1,1}.$$

$$f_{k-2,2}(z) = c_{k,0} (-1)^2 \frac{k(k-1)}{2} b_1^2 z^2 + c_{k-1,1} (k-1) (-b_1 z) + c_{k-2,2}.$$

...

$$f_{k-t,t}(z) = \sum_{s=0}^t c_{k-s,s} C_{k-s}^{t-s} (-b_1 z)^{t-s}.$$

Here $c_{k-t,t}$ are constants determined by $f_{k-t,t}(z)$. Hence

$$(61) \quad \sum_{t=0}^k f_{k-t,t}(z) z_1^{k-t} z_2^t = \sum_{t=0}^k c_{k-t,t} (z_1 - b_1 z z_2)^{k-t} z_2^t.$$

To sum up, we have proved (56). Finally, we consider any vector bundle E in Cases (B) and prove that it is decomposable if and only if $b_2 = b_1 \tau$. Obviously, the ‘if’ direction follows from Lemma 4.1. Conversely, by Lemma 4.1 it suffices to show E is indecomposable under the assumption of $b_1 = 0$ and $b_2 \neq 0$. We observe that any vector bundle E in Cases (B) is isomorphic to $L \otimes \widehat{E}$ where $L \in \text{Pic}^0(\Sigma)$ and $\widehat{E}$ is associated with the representation (8) with parameters $\theta_1 = \theta_2 = b_1 = 0$ and $b_2 \neq 0$. But $\widehat{E}$ is indecomposable by Lemma 1.1, so is $E = L \otimes \widehat{E}$. □

Our next result determines which total spaces of rank 2 vector bundles of degree 0 admit complete flat Kähler metrics. It is exactly part (3) of Proposition 1.4.

Proposition 4.3. *Let E be a rank 2 vector bundle of degree 0 over the elliptic curve Σ defined in (1), and X the total space of E . Then X admits a complete flat Kähler metric if and only if E is isomorphic to $L_1 \oplus L_2$ for any given $L_1, L_2 \in \text{Pic}^0(\Sigma)$.*

Proof of Proposition 4.3. Let $\rho : \widetilde{X} \rightarrow X$ be the universal covering. If X admits a complete flat Kähler metric ω_0 , then there exists a biholomorphic map $F = (f_0, f_1, f_2) \in \text{Aut}(\mathbb{C}^3)$ such that

$$(62) \quad \rho^* \omega_0 = \sqrt{-1} (df_0 \wedge \overline{df_0} + df_1 \wedge \overline{df_1} + df_2 \wedge \overline{df_2}).$$

At any point in the preimage of the zero section of E under ρ (i.e. $\{z_1 = z_2 = 0\}$), one has:

$$(63) \quad df_i(z, z_1, z_2) = \lambda_i(z) dz + \xi_i(z) dz_1 + \eta_i(z) dz_2, \text{ for } 0 \leq i \leq 2.$$

As in the proof of Proposition 1.11, the idea is to analyze the invariance property of $\rho^* \omega_0$ along the zero section. It remains to deal with the case that E is associated with a representation (8).

Note that $\rho^*\omega_0$ is preserved under the action defined by (8). On $\{z_1 = z_2 = 0\}$, we compare the coefficients in front of $dz \wedge d\bar{z}$ and get that $\Lambda(z) = |\lambda_0(z)|^2 + |\lambda_1(z)|^2 + |\lambda_2(z)|^2$ is doubly periodic. As Λ is a subharmonic function while attains its maximum in the interior, $\Lambda(z)$ is constant and so is each of $\lambda_0, \lambda_1, \lambda_2$. Similarly, ξ_0, ξ_1, ξ_2 are constant after comparing the coefficients in front of $dz_1 \wedge d\bar{z}_1$. Now (63) is reduced to

$$df_i(z, z_1, z_2) = \lambda_i dz + \xi_i dz_1 + \eta_i(z) dz_2, \quad \text{for } 0 \leq i \leq 2.$$

Next we compare the coefficients in front of $dz_2 \wedge d\bar{z}_1$. Then

$$\begin{aligned} (64) \quad & \eta_0(z)\bar{\xi}_0 + \eta_1(z)\bar{\xi}_1 + \eta_2(z)\bar{\xi}_2 \\ &= \eta_0(z+1)\bar{\xi}_0 + b_1|\xi_0|^2 + \eta_1(z+1)\bar{\xi}_1 + b_1|\xi_1|^2 + \eta_2(z+1)\bar{\xi}_2 + b_1|\xi_2|^2 \\ &= \eta_0(z+\tau)\bar{\xi}_0 + b_2|\xi_0|^2 + \eta_1(z+\tau)\bar{\xi}_1 + b_2|\xi_1|^2 + \eta_2(z+\tau)\bar{\xi}_2 + b_2|\xi_2|^2. \end{aligned}$$

As $h(z) = \eta_0(z)\bar{\xi}_0 + \eta_1(z)\bar{\xi}_1 + \eta_2(z)\bar{\xi}_2$ is holomorphic, by Liouville's theorem $h(z) = (c_0 + c_1 + c_2)z + B$, where $c_k z$ is the first order term of $\eta_k(z)\bar{\xi}_k$. We plug the expression of $h(z)$ into equation (64) and get

$$\begin{aligned} c_0 + c_2 + c_2 &= -b_1(|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2), \\ (c_0 + c_2 + c_2)\tau &= -b_2(|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2). \end{aligned}$$

It follows that $b_2 = b_1\tau$. By Lemma 4.1, E is isomorphic to $L \oplus L$ for some $L \in \text{Pic}^0(\Sigma)$. □

Recall the fundamental domain $\mathcal{F}$ of modular group acting on the upper half plane $\mathbb{H}$:

$$\begin{aligned} (65) \quad & \mathcal{F} = \{z \in \mathbb{H} \mid |z| > 1, |\text{Re } z| < \frac{1}{2}\}, \\ & \tilde{\mathcal{F}} = \mathcal{F} \cup \{z \in \mathbb{H} \mid |z| = 1, -\frac{1}{2} < |\text{Re } z| \leq 0, \text{ or } |z| \geq 1, \text{Re } z = -\frac{1}{2}\}. \end{aligned}$$

The following result completes the biholomorphic classification on total spaces of any rank two vector bundles of degree zero over an elliptic curve. It is a restatement of Theorem 1.5.

Theorem 4.4. *Let Σ (resp. $\tilde{\Sigma}$) be the elliptic curve generated by $\{1, \tau\}$ (resp. $\{1, \tilde{\tau}\}$), and X (resp. $\tilde{X}$) the total space of a vector bundle E (resp. $\tilde{E}$) of rank 2 and degree 0 over Σ (resp. $\tilde{\Sigma}$). We may further assume that $\tau, \tilde{\tau} \in \tilde{\mathcal{F}}$ defined in (65). If X is biholomorphic to $\tilde{X}$, then $\tau = \tilde{\tau}$ and E is bundle isomorphic to $\tilde{E}$.*

Proof of Theorem 4.4. Recall that any rank 2 vector bundle of degree zero over an elliptic curve must belong to **Type I**, **Type II**, and **Type III**. We consider the lifting biholomorphism $F \in \text{Aut}(\mathbb{C}^3)$ defined in (66). Now we analyze the biholomorphic mapping F case by case.

Case 1: E is of **Type I** and $\tilde{E}$ is of **Type II** or **Type III**.

In this case, there are no biholomorphisms between X and $\tilde{X}$, as a direct consequence of Proposition 4.3.

Case 2: Both E and $\tilde{E}$ are of **Type III**. In this case E (resp. $\tilde{E}$) is associated with a representation (8) where the parameters of ρ (resp. $\tilde{\rho}$) are θ_1, b_1, θ_2 , and b_2 (resp. $\tilde{\theta}_1, \tilde{b}_1, \tilde{\theta}_2$, and $\tilde{b}_2$). According to Lemma 4.1, we may assume that $b_1 = \tilde{b}_1 = 0$ after a bundle isomorphism. Note that b_2 (resp. $\tilde{b}_2$) represents the previous $b_2 - b_1\tau$ (resp. $\tilde{b}_2 - \tilde{b}_1\tilde{\tau}$), which is nonzero for an indecomposable bundle.

Step 1: If $f : X \rightarrow \tilde{X}$ is a biholomorphic map, then f can be lifted to some $F \in \text{Aut}(\mathbb{C}^3)$. Let

$$(66) \quad F(z, z_1, z_2) = (F_0(z, z_1, z_2), F_1(z, z_1, z_2), F_2(z, z_1, z_2)).$$

Then there exists $p, q, r, s \in \mathbb{Z}$, such that

$$(67) \quad F(z+1, \rho(\gamma_1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = (F_0(z, z_1, z_2) + p + q\tilde{\tau}, (\tilde{\rho}(\tilde{\gamma}_1))^p (\tilde{\rho}(\tilde{\gamma}_2))^q \begin{pmatrix} F_1(z, z_1, z_2) \\ F_2(z, z_1, z_2) \end{pmatrix}).$$

$$(68) \quad F(z+\tau, \rho(\gamma_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = (F_0(z, z_1, z_2) + r + s\tilde{\tau}, (\tilde{\rho}(\tilde{\gamma}_1))^r (\tilde{\rho}(\tilde{\gamma}_2))^s \begin{pmatrix} F_1(z, z_1, z_2) \\ F_2(z, z_1, z_2) \end{pmatrix}).$$

We expand three components of F into Taylor series with respect to z_1 and z_2 .

$$(69) \quad F_0(z, z_1, z_2) = \xi_0(z) + \xi_1(z)z_1 + \xi_2(z)z_2 + \sum_{i+j \geq 2} \xi_{ij}(z)z_1^i z_2^j.$$

$$(70) \quad F_1(z, z_1, z_2) = \lambda_0(z) + \lambda_1(z)z_1 + \lambda_2(z)z_2 + \sum_{i+j \geq 2} \lambda_{ij}(z)z_1^i z_2^j.$$

$$(71) \quad F_2(z, z_1, z_2) = \eta_0(z) + \eta_1(z)z_1 + \eta_2(z)z_2 + \sum_{i+j \geq 2} \eta_{ij}(z)z_1^i z_2^j.$$

Then we rewrite Equations (67) and (68) as follows.

$$(72) \quad F_0(z+1, e^{2\pi i \theta_1} z_1, e^{2\pi i \theta_1} z_2) = F_0(z, z_1, z_2) + p + q\tilde{\tau}.$$

$$(73) \quad F_0(z+\tau, e^{2\pi i \theta_2} (z_1 + b_2 z_2), e^{2\pi i \theta_2} z_2) = F_0(z, z_1, z_2) + r + s\tilde{\tau}.$$

$$(74) \quad F_1(z+1, e^{2\pi i \theta_1} z_1, e^{2\pi i \theta_1} z_2) = e^{2\pi i (p\tilde{\theta}_1 + q\tilde{\theta}_2)} (F_1(z, z_1, z_2) + q\tilde{b}_2 F_2(z, z_1, z_2)).$$

$$(75) \quad F_1(z+\tau, e^{2\pi i \theta_2} (z_1 + b_2 z_2), e^{2\pi i \theta_2} z_2) = e^{2\pi i (r\tilde{\theta}_1 + s\tilde{\theta}_2)} (F_1(z, z_1, z_2) + s\tilde{b}_2 F_2(z, z_1, z_2)).$$

$$(76) \quad F_2(z+1, e^{2\pi i \theta_1} z_1, e^{2\pi i \theta_1} z_2) = e^{2\pi i (p\tilde{\theta}_1 + q\tilde{\theta}_2)} F_2(z, z_1, z_2).$$

$$(77) \quad F_2(z+\tau, e^{2\pi i \theta_2} (z_1 + b_2 z_2), e^{2\pi i \theta_2} z_2) = e^{2\pi i (r\tilde{\theta}_1 + s\tilde{\theta}_2)} F_2(z, z_1, z_2).$$

Now we may further determine $\xi_0(z)$. Indeed, it follows from (72) and (73) that

$$\begin{aligned} \xi_0(z+1) &= \xi_0(z) + p + q\tilde{\tau}; \\ \xi_0(z+\tau) &= \xi_0(z) + r + s\tilde{\tau}. \end{aligned}$$

Hence $\xi_0(z) = Az + B$ where $A = p + q\tilde{\tau}$, $A\tau = r + s\tilde{\tau}$, and B is constant. We see $A \neq 0$ as $F(z, 0, 0)$ is injective with respect to z . We arrive at

$$(78) \quad \tau = \frac{r + s\tilde{\tau}}{p + q\tilde{\tau}}, \text{ where } \begin{pmatrix} s & r \\ q & p \end{pmatrix} \in \text{GL}(2, \mathbb{Z}).$$

Step 2: We claim that if such a biholomorphism exists, then

$$(79) \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{pmatrix} - \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \in \mathbb{Z}^2.$$

To prove the claim, we consider $\eta_0(z), \eta_1(z), \eta_2(z)$. Let $z_1 = z_2 = 0$. we get $\eta_0(z+1) = e^{2\pi i (p\tilde{\theta}_1 + q\tilde{\theta}_2)} \eta_0(z)$ and $\eta_0(z+\tau) = e^{2\pi i (r\tilde{\theta}_1 + s\tilde{\theta}_2)} \eta_0(z)$ from (76) and (77). Then η_0 is bounded, hence constant. It also follows from (76) and (77) that

$$\begin{aligned} e^{2\pi i \theta_1} (\eta_1(z+1)z_1 + \eta_2(z+1)z_2) &= e^{2\pi i (p\tilde{\theta}_1 + q\tilde{\theta}_2)} (\eta_1(z)z_1 + \eta_2(z)z_2); \\ e^{2\pi i \theta_2} (\eta_1(z+\tau)(z_1 + b_2 z_2) + \eta_2(z+\tau)z_2) &= e^{2\pi i (r\tilde{\theta}_1 + s\tilde{\theta}_2)} (\eta_1(z)z_1 + \eta_2(z)z_2). \end{aligned}$$

Comparing the coefficients of z_1 term, we see that η_1 is constant. If (79) does not hold, we have $\eta_1(z) = 0$. By a similar argument, $\eta_2(z) = 0$. But then the Jacobian of F vanishes along $z_1 = z_2 = 0$. This is a contraction as F is a biholomorphism. Therefore we prove (79).

Step 3: Show that Σ is biholomorphic to $\tilde{\Sigma}$.

If at least of one of θ_1 and θ_2 is nonzero, then $\eta_0 = 0$ in view of (79). Comparing the constant terms in (74) and (75), $\lambda_0 = 0$. Hence F maps the zero section of X to that of $\tilde{X}$.

Otherwise if $\theta_1 = \theta_2 = 0$, by (74) and (75) we have

$$\lambda_0(z+1) = \lambda_0(z) + q\tilde{b}_2 \eta_0, \quad \lambda_0(z+\tau) = \lambda_0(z) + s\tilde{b}_2 \eta_0.$$

It follows that $\lambda_0(z) = Cz + D$ where $C = q\tilde{b}_2\eta_0$ and $C\tau = s\tilde{b}_2\eta_0$. We derive that C has to be zero as $\text{Im } \tau > 0$. Then $\lambda_0(z) = \lambda_0$ which is a constant. Suppose that $F(z, 0, 0) = (\xi_0(z), c_1, c_2)$. One can check that $G(z, z_1, z_2) = (F_0(z, z_1, z_2), F_1(z, z_1, z_2) - c_1, F_2(z, z_1, z_2) - c_2)$ is also a well-defined biholomorphism.

To sum up, we can always get a biholomorphism which maps the zero section of X to the zero section of $\tilde{X}$. Hence Σ is biholomorphic to $\tilde{\Sigma}$. Moreover, by a standard fact on elliptic curves, the matrix $\begin{pmatrix} s & r \\ q & p \end{pmatrix}$ in (78) indeed lies in $\text{SL}(2, \mathbb{Z})$. If $\tau, \tilde{\tau} \in \tilde{\mathcal{F}}$ defined in (65), then $\tau = \tilde{\tau}$ and there exists $\begin{pmatrix} s & r \\ q & p \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})$ such that $\tau = \frac{s\tau + r}{q\tau + p}$.

Step 4: We may check that

$$(80) \quad F(z, z_1, z_2) = ((p + q\tau)z, c_1 z_1 + q\tilde{b}_2 z z_2, z_2), \quad \text{where } c_1 = \frac{\tilde{b}_2(s - q\tau)}{b_2},$$

satisfies (72)-(77) and thus induces a bundle isomorphism between E and $\tilde{E}$. Moreover, if $\tau \neq i, -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}$, $\begin{pmatrix} s & r \\ q & p \end{pmatrix}$ has to be the identity matrix or its negative. Then (79) reduces to $\theta_k = \tilde{\theta}_k$ or $\theta_k = -\tilde{\theta}_k$. Consequently, (80) has a simple expression $F(z, z_1, z_2) = (z, \frac{\tilde{b}_2}{b_2} z_1, z_2)$ or $(-z, -\frac{\tilde{b}_2}{b_2} z_1, z_2)$.

Case 3: Both E and $\tilde{E}$ are of **Type I**. We assume $E = L_1 \oplus L_2$ with $L_1 = L(0, \alpha)$ and $L_2 = L(0, \beta)$, and $\tilde{E} = \tilde{L}_1 \oplus \tilde{L}_2$ with $\tilde{L}_1 = L(0, \tilde{\alpha})$ and $\tilde{L}_2 = L(0, \tilde{\beta})$.

We proceed as the proof of **Case 2**. Namely, if X and $\tilde{X}$ are biholomorphic, then Σ and $\tilde{\Sigma}$ are isomorphic. There exists some $\begin{pmatrix} s & r \\ q & p \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$ so that $\tau = \frac{r + s\tilde{\tau}}{p + q\tilde{\tau}}$. Moreover, α and β satisfy

$$\alpha(1) = \tilde{\alpha}(p + q\tilde{\tau}), \quad \alpha(\tau) = \tilde{\alpha}(r + s\tilde{\tau}); \quad \beta(1) = \tilde{\beta}(p + q\tilde{\tau}), \quad \beta(\tau) = \tilde{\beta}(r + s\tilde{\tau}).$$

For simplicity, we assume two elliptic curves have the same $\tau = \tilde{\tau}$. One can check that

$$(81) \quad F(z, z_1, z_2) = ((p + q\tau)z, z_1, z_2)$$

is a well-defined bundle isomorphism.

Case 4: E is of **Type II** and $\tilde{E}$ is of **Type III**. We may assume $L_1 = L(H, \alpha)$ and $L_2 = L(-H, \beta)$ with $H > 0$. We will show there are no biholomorphisms between X and $\tilde{X}$.

Suppose that there exists a biholomorphic map F . As the proof of **Case 2**, we have

$$(82) \quad \begin{aligned} & F(z + 1, \alpha(1)e^{\pi H z + \pi \frac{H}{2}} z_1, \beta(1)e^{-\pi H z - \pi \frac{H}{2}} z_2) \\ &= (F_0(z, z_1, z_2) + p + q\tilde{\tau}, (\tilde{\rho}(\tilde{\gamma}_1))^p (\tilde{\rho}(\tilde{\gamma}_2))^q \begin{pmatrix} F_1(z, z_1, z_2) \\ F_2(z, z_1, z_2) \end{pmatrix}). \end{aligned}$$

$$(83) \quad \begin{aligned} & F(z + \tau, \alpha(\tau)e^{\pi H z \tilde{\tau} + \pi \frac{H}{2} |\tau|^2} z_1, \beta(\tau)e^{-\pi H z \tilde{\tau} - \pi \frac{H}{2} |\tau|^2} z_2) \\ &= (F_0(z, z_1, z_2) + r + s\tilde{\tau}, (\tilde{\rho}(\tilde{\gamma}_1))^r (\tilde{\rho}(\tilde{\gamma}_2))^s \begin{pmatrix} F_1(z, z_1, z_2) \\ F_2(z, z_1, z_2) \end{pmatrix}). \end{aligned}$$

for some $p, q, r, s \in \mathbb{Z}$. Hence we get

$$(84) \quad F_2(z + 1, \alpha(1)e^{\pi H z + \pi \frac{H}{2}} z_1, \beta(1)e^{-\pi H z - \pi \frac{H}{2}} z_2) = e^{2\pi i(p\tilde{\theta}_1 + q\tilde{\theta}_2)} F_2(z, z_1, z_2).$$

$$(85) \quad F_2(z + \tau, \alpha(\tau)e^{\pi H z \tilde{\tau} + \pi \frac{H}{2} |\tau|^2} z_1, \beta(\tau)e^{-\pi H z \tilde{\tau} - \pi \frac{H}{2} |\tau|^2} z_2) = e^{2\pi i(r\tilde{\theta}_1 + s\tilde{\theta}_2)} F_2(z, z_1, z_2).$$

Recall that the Taylor series of F are given in (69), (70), and (71). Now we study the coefficient of z_1 in (84) and (85).

$$\eta_1(z + 1)\alpha(1)e^{\pi H z + \pi \frac{H}{2}} = e^{2\pi i(p\tilde{\theta}_1 + q\tilde{\theta}_2)} \eta_1(z).$$

$$\eta_1(z + \tau)\alpha(\tau)e^{\pi H z \tilde{\tau} + \pi \frac{H}{2} |\tau|^2} = e^{2\pi i(r\tilde{\theta}_1 + s\tilde{\theta}_2)} \eta_1(z).$$

It follows that $|\eta_1(z)|^2 e^{\pi H|z|^2}$ is invariant under $z \rightarrow z+1$ and $z \rightarrow z+\tau$. Then $|\eta_1(z)|^2 e^{\pi H|z|^2}$ is bounded and $\lim_{|z| \rightarrow \infty} |\eta_1(z)|^2 = 0$ as $H > 0$. Hence η_1 must be zero. Similarly, we may show that ξ_1 and λ_1 are zero. It is a contradiction, as the Jacobian of F is nondegenerate.

Case 5: Both E and $\tilde{E}$ are of **Type II**. We assume $E = L_1 \oplus L_2$ with $L_1 = L(H, \alpha)$ and $L_2 = L(-H, \beta)$, and $\tilde{E} = \tilde{L}_1 \oplus \tilde{L}_2$ with $\tilde{L}_1 = L(\tilde{H}, \tilde{\alpha})$ and $\tilde{L}_2 = L(-\tilde{H}, \tilde{\beta})$. As X and $\tilde{X}$ are biholomorphic, we consider the lifting biholomorphism $F = (F_0, F_1, F_2) \in \text{Aut}(\mathbb{C}^3)$.

Step 1: Study F_0 . In this case, there exists $p, q, r, s \in \mathbb{Z}$, such that

$$(86) \quad F_0(z+1, \alpha(1)e^{\pi H z + \pi \frac{H}{2}} z_1, \beta(1)e^{-\pi H z - \pi \frac{H}{2}} z_2) = F_0(z, z_1, z_2) + p + q\tilde{\tau}.$$

$$(87) \quad F_0(z+\tau, \alpha(\tau)e^{\pi H z \bar{\tau} + \pi \frac{H}{2}|\tau|^2} z_1, \beta(\tau)e^{-\pi H z \bar{\tau} - \pi \frac{H}{2}|\tau|^2} z_2) = F_0(z, z_1, z_2) + r + s\tilde{\tau}.$$

Let $z_1 = z_2 = 0$. Then we have $\xi_0(z) = Az + B$ for some constant B as before. Here $A = p + q\tilde{\tau}$ and $A\tau = r + s\tilde{\tau}$. Note that $|A|^2 = (p + q \text{Re } \tau)^2 + q^2(\text{Im } \tau)^2 \geq 1$.

Step 2: Without loss of generality, we assume that $\tilde{H} \geq H > 0$. We will show that $H = \tilde{H}$.

$$(88) \quad \begin{aligned} & F_2(z+1, \alpha(1)e^{\pi H z + \pi \frac{H}{2}} z_1, \beta(1)e^{-\pi H z - \pi \frac{H}{2}} z_2) \\ &= \tilde{\beta}(A)e^{-\pi \tilde{H} F_0 \bar{A} - \frac{\pi}{2} \tilde{H} |A|^2} F_2(z, z_1, z_2). \end{aligned}$$

$$(89) \quad \begin{aligned} & F_2(z+\tau, \alpha(\tau)e^{\pi H z \bar{\tau} + \pi \frac{H}{2}|\tau|^2} z_1, \beta(\tau)e^{-\pi H z \bar{\tau} - \pi \frac{H}{2}|\tau|^2} z_2) \\ &= \tilde{\beta}(A\tau)e^{-\pi \tilde{H} F_0 \bar{A} \bar{\tau} - \frac{\pi}{2} \tilde{H} |A|^2 |\tau|^2} F_2(z, z_1, z_2). \end{aligned}$$

Hence $\eta_0(z)$ satisfies:

$$\begin{aligned} \eta_0(z+1) &= \tilde{\beta}(A)e^{-\pi \tilde{H}(Az+B)\bar{A} - \frac{\pi}{2} \tilde{H} |A|^2} \eta_0(z). \\ \eta_0(z+\tau) &= \tilde{\beta}(A\tau)e^{-\pi \tilde{H}(Az+B)\bar{A} \bar{\tau} - \frac{\pi}{2} \tilde{H} |A|^2 |\tau|^2} \eta_0(z). \end{aligned}$$

It follows that

$$(90) \quad |\eta_0(z)|^2 e^{\pi \tilde{H} |A|^2 |z|^2 + \pi \tilde{H} (A\bar{B}z + B\bar{A}\bar{z})}$$

is double periodic under $z \rightarrow z+1$ and $z \rightarrow z+\tau$. Taking $|z| \rightarrow \infty$, one has $\eta_0(z) = 0$ by the maximum principle. The same argument can be used to prove that

$$(91) \quad |\eta_1(z)|^2 e^{\pi(\tilde{H}|A|^2 + H)|z|^2 + \pi \tilde{H}(A\bar{B}z + B\bar{A}\bar{z})}, \quad |\eta_2(z)|^2 e^{\pi(\tilde{H}|A|^2 - H)|z|^2 + \pi \tilde{H}(A\bar{B}z + B\bar{A}\bar{z})}$$

are both double periodic. Then $\eta_1(z) = 0$. If $\tilde{H} > H$ or $|A| > 1$, we have $\eta_2(z) = 0$. This is a contradiction as $F \in \text{Aut}(\mathbb{C}^3)$ and $\frac{\partial F_2}{\partial z}, \frac{\partial F_2}{\partial z_1}$, and $\frac{\partial F_2}{\partial z_2}$ can not vanish simultaneously. Hence $H = \tilde{H}$ and $|A| = |p + q\tilde{\tau}| = 1$.

Step 3: Show $\tau = \tilde{\tau}$. Recall there exists $p, q, r, s \in \mathbb{Z}$ such that: $r + s\tilde{\tau} = \tau(p + q\tilde{\tau})$. We discuss $|p + q\tilde{\tau}| = 1$ case by case.

Subcase 1: $q = 0$. Then $p = \pm 1$ and $\pm\tau = r + s\tilde{\tau}$. Since $\tau, \tilde{\tau} \in \tilde{\mathcal{F}}$, we have $\text{Im } \tau = \text{Im } \tilde{\tau}$ or $\text{Im } \tau \geq 2\text{Im } \tilde{\tau} > 1$. If $\text{Im } \tau \geq 2\text{Im } \tilde{\tau}$, we consider the inverse mapping F^{-1} and run Step 1. Then there exist $r', s' \in \mathbb{Z}$ such that $\tilde{\tau} = r' + s'\tau$ and $\text{Im } \tilde{\tau} \geq \text{Im } \tau$, which is a contradiction. Hence $\text{Im } \tau = \text{Im } \tilde{\tau}$ and $\tau = \tilde{\tau}$. Note that $A = \pm 1$ in this case.

Subcase 2: $p = 0, q = \pm 1$ and $|\tilde{\tau}| = 1$. Then $\pm\tau\tilde{\tau} = r + s\tilde{\tau}$. Let $\tilde{\tau} = e^{i\theta}$ with $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. Then $\pm\tau = s + re^{-i\theta}$. Since $\tau \in \tilde{\mathcal{F}}$, a simple analysis shows $s = 0$ and $r = \pm 1$, and $\tau = \tilde{\tau} = i$.

Subcase 3: $p = q = 1$ (or $p = q = -1$) and $\tilde{\tau} = e^{\frac{2\pi}{3}i}$. Obviously $\tau = se^{\frac{\pi}{3}i} + re^{-\frac{\pi}{3}i}$. Similarly, we have $\tau = e^{\frac{2\pi}{3}i}$.

Step 4: Determine the relation between β and $\tilde{\beta}$. It follows from (91) that

$$(92) \quad |\eta_2(z)|^2 e^{\pi \tilde{H} (A\bar{B}z + B\bar{A}\bar{z})}$$

is double periodic. By Liouville's theorem, $\eta_2(z) = Ce^{-\pi \tilde{H} A\bar{B}z}$.

$$Ce^{-\pi \tilde{H} A\bar{B}(z+1)} \beta(1)e^{-\pi H z - \frac{H}{2}} = Ce^{-\pi \tilde{H} A\bar{B}z} \tilde{\beta}(A)e^{-\pi \tilde{H} (Az+B)\bar{A} - \frac{\pi}{2} \tilde{H} |A|^2}.$$

$$Ce^{-\pi \tilde{H} A\bar{B}(z+\tau)} \beta(\tau)e^{-\pi H z \bar{\tau} - \frac{H}{2}|\tau|^2} = Ce^{-\pi \tilde{H} A\bar{B}z} \tilde{\beta}(A\tau)e^{-\pi \tilde{H} (Az+B)\bar{A} \bar{\tau} - \frac{\pi}{2} \tilde{H} |A|^2 |\tau|^2}.$$

Therefore,

$$(93) \quad \beta(1)\tilde{\beta}(A)^{-1} = e^{\pi H(\overline{AB}-B\overline{A})}, \quad \beta(\tau)\tilde{\beta}(A\tau)^{-1} = e^{\pi H(\overline{AB}\tau-B\overline{A}\tau)}.$$

Step 5: We determine the relation between α and $\tilde{\alpha}$.

Note that $\xi_1(z) = 0$ and $\eta_1(z) = 0$ as in Step 1 and 2. Then $\lambda_1(z)$ is nowhere vanishing. By the same argument in Step 4, one has

$$(94) \quad \lambda_1(z) = Ce^{\pi \tilde{H} \overline{AB} z}.$$

Hence

$$(95) \quad \alpha(1)\tilde{\alpha}(A)^{-1} = e^{-\pi H(\overline{AB}-B\overline{A})}, \quad \alpha(\tau)\tilde{\alpha}(A\tau)^{-1} = e^{-\pi H(\overline{AB}\tau-B\overline{A}\tau)}.$$

Combining (93) and (95), we get

$$(96) \quad \alpha(1)\beta(1) = \tilde{\alpha}(A)\tilde{\beta}(A), \quad \alpha(\tau)\beta(\tau) = \tilde{\alpha}(A\tau)\tilde{\beta}(A\tau).$$

Step 6: Finally, it is straightforward to check

$$(97) \quad \Phi(z, z_1, z_2) = (Az + B, e^{\pi H \overline{AB} z} z_1, e^{-\pi H \overline{AB} z} z_2)$$

is a desired bundle isomorphism between E and $\tilde{E}$.

In fact, the above proof shows a stronger statement. Namely, given two vector bundles $E = L_1 \oplus L_2$ and $\tilde{E} = \tilde{L}_1 \oplus \tilde{L}_2$ over an elliptic curve defined by the lattice $\mathbb{Z}$ -span $\{1, \tau\}$ where $\tau \in \tilde{\mathcal{F}}$ defined in (65). Here we assume $H > 0$ and $L_1 = L(H, \alpha)$, $L_2 = L(-H, \beta)$, $\tilde{L}_1 = L(\tilde{H}, \tilde{\alpha})$, and $\tilde{L}_2 = L(-\tilde{H}, \tilde{\beta})$. Then we introduce A as

$$(98) \quad A = \begin{cases} \pm 1 & \text{if } \tau \neq i, e^{\frac{2\pi i}{3}} \\ \pm 1 \text{ or } \pm i & \text{if } \tau = i \\ \pm 1 \text{ or } \pm e^{\frac{\pi i}{3}} \text{ or } \pm e^{\frac{2\pi i}{3}} & \text{if } \tau = e^{\frac{2\pi i}{3}}. \end{cases}$$

Assume (96) holds and

$$\beta(1)\tilde{\beta}(A)^{-1} = e^{2\pi i \theta} \text{ and } \beta(\tau)\tilde{\beta}(A\tau)^{-1} = e^{2\pi i \phi}$$

hold for some $\theta, \phi \in [0, 1)$. Then E and $\tilde{E}$ are bundle isomorphic by (97). Here B in (97) is determined in the following way. Assume $\overline{AB} = u + vi$ and $\tau = a + bi$. Then $\text{Im } \overline{AB} = v$, $\text{Im } \overline{AB}\tau = va + ub$. For any θ, ϕ , $Hv = \theta$, $H(va + ub) = \phi$ has a unique solution $v = \frac{\theta}{H}$, $u = \frac{\phi - \theta a}{Hb}$. Then we solve $B = \frac{1}{A}(u + vi)$. $\square$

5. HERMITIAN GEOMETRY ON TOTAL SPACES OF RANK 2 BUNDLES OF DEGREE ZERO

Let E be a rank 2 bundle of degree zero over an elliptic curve and X its total space. In this section we construct complete Hermitian metrics on X and prove Theorem 1.6. If E is of **Type I**, the standard Euclidean metric descends to X as the Kähler metric g in Proposition 4.3. It remains to construct the desired Hermitian metric g if E is of **Type II** and **Type III** respectively.

5.1. The Type II case. In this subsection, we prove Theorem 1.6 when E is of **Type II**, i.e. $E = L_1 \oplus L_2$ where $\deg L_1 = -\deg L_2 > 0$. First of all, we consider X be the total space of a line bundle of the form $L(H, \alpha)$ and introduce

$$(99) \quad \omega_L = \sqrt{-1}(dz \wedge d\bar{z} + e^{\pi H|z|^2} \partial(e^{-\pi H|z|^2} \xi) \wedge \bar{\partial}(e^{-\pi H|z|^2} \bar{\xi})).$$

Lemma 5.1. ω_L defined in (99) is a complete Hermitian metric on X .

Proof of Lemma 5.1. It is direct to check that ω is a well-defined Hermitian metric on X with its local components

$$(100) \quad \begin{pmatrix} 1 + e^{-\pi H|z|^2} (\pi H)^2 |z|^2 |\xi|^2 & e^{-\pi H|z|^2} (-\pi H \bar{z}) \xi \\ e^{-\pi H|z|^2} (-\pi H z) \bar{\xi} & e^{-\pi H|z|^2} \end{pmatrix}.$$

Indeed, we may rewrite (100) as

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} |a|^2 & \bar{a} \\ a & 1 \end{pmatrix}, \quad \text{where } c = e^{-\pi H|z|^2} \text{ and } a = -\pi H z \bar{\xi}.$$

We consider ω as the lift metric on the universal cover of X . For a fixed point $p = (0, 0)$, let $\gamma(t)$ with $t \in [t_0, t_1]$ denote any C^1 curve connecting p and any point $q = (z_*, \xi_*)$. Let $\gamma(t) = (z(t), \xi(t))$. Then $g(\gamma'(t), \gamma'(t)) = |z'(t)|^2 + c|\bar{a}z'(t) + \xi'(t)|^2$. In below, we estimate the length of γ as q tends to infinity.

Case 1: If $|z_*| \rightarrow \infty$, we provide the length of γ .

$$\int_{t_0}^{t_1} \sqrt{g(\gamma'(t), \gamma'(t))} dt \geq \int_{t_0}^{t_1} |z'(t)| dt = |z_*|.$$

Case 2: If $|z_*|$ is bounded and $|\xi_*| \rightarrow \infty$. Let $M = \sup_{t \in [t_1, t_2]} |z(t)|$.

Subcase 2a: We assume $\max(e^{\pi H M^2}, (\pi H M)^2) \geq \ln |\xi_*|$. Then we may apply Case 1 on a part of $\gamma(t)$ to conclude that

$$\int_{t_0}^{t_1} \sqrt{g(\gamma'(t), \gamma'(t))} dt \geq M \geq \begin{cases} \frac{1}{\pi H} \sqrt{\ln(\ln |\xi_*|)}, & \text{if } H > 0, \\ \frac{1}{\pi |H|} \sqrt{\ln |\xi_*|}, & \text{if } H \leq 0. \end{cases}$$

Subcase 2b: We assume $\max(e^{\pi H M^2}, (\pi H M)^2) \leq \ln |\xi_*|$.

$$\begin{aligned} g(\gamma'(t), \gamma'(t)) &= (1 + c|a|^2)|z'(t)|^2 + 2\operatorname{Re}(c\bar{a}z'(t)\bar{\xi}'(t)) + c|\xi'(t)|^2 \\ &= (1 + c|a|^2) \left| z'(t) + \frac{ca\xi'(t)}{1 + c|a|^2} \right|^2 + \frac{c|\xi'(t)|^2}{1 + c|a|^2} \\ &\geq \frac{|\xi'(t)|^2}{e^{\pi H|z(t)|^2} + (\pi H)^2|z(t)|^2|\xi(t)|^2}. \end{aligned}$$

As we assume that $|z|$ is bounded along $\gamma(t)$, there exists some constant $C > 0$ such that

$$\int_{t_0}^{t_1} \sqrt{g(\gamma'(t), \gamma'(t))} dt \geq \frac{1}{\sqrt{\ln |\xi_*|}} \int_{|\xi(t)| \geq 1} \frac{|\xi'(t)|}{2|\xi|} dt \geq \frac{1}{2} \sqrt{\ln |\xi_*|}.$$

Therefore, we have shown that the lifting metric ω on the universal cover of X (hence ω on X) is complete. $\square$

Motivated by (99), we introduce a Hermitian metric on the total space X of $E = L_1(H, \alpha) \oplus L_2(-H, \beta)$ as follows.

$$(101) \quad \begin{aligned} \omega = & \sqrt{-1} \left(dz \wedge d\bar{z} + e^{\pi H|z|^2} \partial(e^{-\pi H|z|^2} \xi_1) \wedge \bar{\partial}(e^{-\pi H|z|^2} \bar{\xi}_1) \right. \\ & \left. + e^{-\pi H|z|^2} \partial(e^{\pi H|z|^2} \xi_2) \wedge \bar{\partial}(e^{\pi H|z|^2} \bar{\xi}_2) \right). \end{aligned}$$

Lemma 5.2. ω defined in (101) is a complete Gauduchon metric on X with zero Chern-Ricci curvature.

Proof of Lemma 5.2. First of all, we may check that ω is invariant under the action

$$(z, \xi_1, \xi_2) \rightarrow (z + \gamma, \alpha(\gamma)e^{\pi H z \bar{\gamma} + \frac{\pi}{2} H |\gamma|^2} \xi_1, \beta(\gamma)e^{-\pi H z \bar{\gamma} - \frac{\pi}{2} H |\gamma|^2} \xi_2).$$

Hence ω descends onto X and its local components read

$$(102) \quad \begin{pmatrix} 1 + e^{-\pi H|z|^2} (\pi H)^2 |z|^2 |\xi_1|^2 + e^{\pi H|z|^2} (\pi H)^2 |z|^2 |\xi_2|^2 & e^{-\pi H|z|^2} (-\pi H \bar{z}) \xi_1 & e^{\pi H|z|^2} (\pi H \bar{z}) \xi_2 \\ e^{-\pi H|z|^2} (-\pi H z) \bar{\xi}_1 & e^{-\pi H|z|^2} & 0 \\ e^{\pi H|z|^2} (\pi H z) \bar{\xi}_2 & 0 & e^{\pi H|z|^2} \end{pmatrix}.$$

After a comparison between (100) and (102), we see that the completeness of ω can be proved similarly as in Lemma 5.1. It follows from Lemma 3.2 or a direct computation that $\det(g) = 1$. Then the first Chern-Ricci curvature is equal to zero. It is straightforward to check that ω is Gauduchon. Namely, $\sqrt{-1} \partial \bar{\partial}(\omega^2) = 0$ in the sense of [14]. $\square$

*Proof of Theorem 1.6 when E is of **Type II**.* In view of Lemma 5.2, it suffices to determine $\mathcal{O}_d(X, g)$ which is defined in (9). Consider the complete Hermitian metric (X, g) defined in (101). Let Σ denote the zero section of the vector bundle $E = L_1(H, \alpha) \oplus L_2(-H, \beta)$, and $\tilde{d}(q)$ the distance function (with respect to g) of any point $q \in X$ to Σ . Equivalently, we need to characterize all $f \in \mathcal{O}(X)$ so that there exists some $C(d, f) > 0$

$$(103) \quad |f(q)| \leq C(\tilde{d}(q) + 1)^d \text{ for all } q \in X.$$

With a slight abuse of notations, let $\pi : E \rightarrow \Sigma$ be the projection and ξ_1 and ξ_2 denote local coordinates of L_1 and L_2 along the fiber directions respectively. It follows from (54) that any $f \in \mathcal{O}(X)$ can be expressed as $f = \sum_{n=p+q \geq 0}^\infty a_{p,q} \xi_1^p \xi_2^q$, where $a_{p,q}$ is the local coefficient of any element in $H^0(\Sigma, L_1^{-p} \otimes L_2^{-q})$ for any integers $p, q \geq 0$. In order to characterize (103), it suffices to show both ξ_1 and ξ_2 grow linearly with respect to $\tilde{d}$ on $\pi^{-1}(U)$ for a small open set $U \subset \Sigma$. We argue by three steps.

Step 1: We observe that a special fiber $\pi^{-1}(0)$ of the point $z = 0$ on Σ is totally geodesic with respect to the Riemannian connection of g . Indeed, by (102), $\frac{\partial}{\partial z}$ is the normal direction along $\pi^{-1}(0)$. The Riemannian Christoffel symbols (in terms of real coordinates) satisfies $\Gamma_{ab}^c = 0$ where a, b is from the real and imaginary coordinates of ξ_1, ξ_2 , and c is from those of z .

Step 2: We show any fiber $\pi^{-1}(B)$ for some point $z = B$ on Σ is totally geodesic. To that end, we consider the bundle automorphism Φ defined in (97) in the case of $A = 1$. Namely

$$(104) \quad \Phi(z, \xi_1, \xi_2) = (z + B, e^{\pi H \bar{B} z} \xi_1, e^{-\pi H \bar{B} z} \xi_2).$$

It is easy to check the pull back metric $\Phi^*(\omega)$ has the corresponding local components

$$(105) \quad \begin{pmatrix} 1 + e^{-\pi H |z|^2 + \pi H |B|^2} (\pi H)^2 |z|^2 |\xi_1|^2 + e^{\pi H |z|^2 - \pi H |B|^2} (\pi H)^2 |z|^2 |\xi_2|^2 & A \\ B & C \end{pmatrix}.$$

Here we set

$$A = \begin{pmatrix} e^{-\pi H |z|^2 + \pi H |B|^2} (-\pi H \bar{z}) \xi_1 & e^{\pi H |z|^2 - \pi H |B|^2} (\pi H \bar{z}) \xi_2 \end{pmatrix},$$

$$B = \begin{pmatrix} e^{-\pi H |z|^2 + \pi H |B|^2} (-\pi H z) \bar{\xi}_1 \\ e^{\pi H |z|^2 - \pi H |B|^2} (\pi H z) \bar{\xi}_2 \end{pmatrix}, \quad C = \begin{pmatrix} e^{-\pi H |z|^2 + \pi H |B|^2} & 0 \\ 0 & e^{\pi H |z|^2 - \pi H |B|^2} \end{pmatrix}.$$

By a similar argument as in Step 1, we get that the fiber of $z = 0$ is totally geodesic with respect to $\Phi^*\omega$. It implies that the fiber of $z = B$ is totally geodesic with respect to ω . Equivalently, we find a coordinate change near the fiber $\pi^{-1}(B)$ so that ω is reduced to a specific diagonal form as that near the fiber $\pi^{-1}(0)$.

Step 3: We estimate the growth of ξ_i ($i = 1, 2$) with respect to $\tilde{d}$. Now that we have each fiber $\pi^{-1}(z)$ is totally geodesic. Moreover, any geodesic of X which begins with $z \in \Sigma$ and meet Σ orthogonally must stay inside the fiber $\pi^{-1}(z)$. In other words, for any $q \in X$, $\tilde{d}$ is realized by a geodesic inside the fiber containing q . Therefore, we are at the similar situation as in the Calabi method (Theorem 2.1). Since the induced metric on each fiber is Euclidean, we conclude that both ξ_1 and ξ_2 grow linearly with respect to $\tilde{d}$. $\square$

5.2. The Type III case. Assume that E is an indecomposable rank 2 bundle of degree zero which is associated with a representation defined in (8). Now we introduce a Hermitian metric on the total space of E . We begin with the following $(1, 1)$ form on $\mathbb{C}^3$. Recall that $z = x + y\sqrt{-1}$ and Σ is the elliptic curve defined in (1).

$$(106) \quad \omega = \sqrt{-1} \left[dz \wedge d\bar{z} + \partial \left(z_1 - (b_1 x + \frac{b_2 - b_1 \operatorname{Re} \tau}{\operatorname{Im} \tau} y) z_2 \right) \wedge \bar{\partial} \left(\bar{z}_1 - (\bar{b}_1 x + \frac{\bar{b}_2 - \bar{b}_1 \operatorname{Re} \tau}{\operatorname{Im} \tau} y) \bar{z}_2 \right) \right. \\ \left. + dz_2 \wedge d\bar{z}_2 \right].$$

Now we review some special cases of (106). If the representation (8) associated to E satisfies $b_2 = b_1\tau$, then (106) is reduced to

$$\omega = \sqrt{-1} \left(dz \wedge d\bar{z} + \partial(z_1 - b_1 z z_2) \wedge \bar{\partial}(\bar{z}_1 - \overline{b_1 z z_2}) + dz_2 \wedge d\bar{z}_2 \right).$$

By Lemma 4.1, we may choose $F(z_1, z_2, z_3) = (z, z_1 + b_1 z z_2, z_2) \in \text{Aut}(\mathbb{C}^3)$ which descends to an bundle isomorphism between $L \oplus L$ and E for some $L \in \text{Pic}^0(\Sigma)$. Then $F^*\omega$ becomes the descended Euclidean metric on X .

If we choose E as (5) and the corresponding $\tau = \sqrt{-1}$, then (106) is reduced to

$$\omega = \sqrt{-1} (dz \wedge d\bar{z} + \partial(z_1 - y z_2) \wedge \bar{\partial}(\bar{z}_1 - \bar{y} \bar{z}_2) + dz_2 \wedge d\bar{z}_2).$$

We note that both $z_1 - y z_2$ and z_2 terms appear in Paun's Hermitian fiber metric defined in (19). In this sense, (106) is a natural 'Hermitian' generalization of Calabi's Kähler metric defined in (12). By a direct calculation, we may show the following result.

Lemma 5.3. ω defined by (106) descends to a Hermitian metric on X such that

- (1) ω is Gauduchon.
- (2) The first Chern-Ricci curvature of ω is zero.
- (3) ω is a Kähler metric if and only if $b_2 = b_1\tau$.

Now we focus on the case of $b_2 \neq b_1\tau$ which corresponds to all indecomposable bundles by Proposition 4.2. It suffices to consider any fixed pair of (b_1, b_2) with $b_2 \neq b_1\tau$ as the resulting total space with the same value of (θ_1, θ_2) must be biholomorphic. After we choose $b_1 = 1$ and $b_2 = \bar{\tau}$, (106) is further reduced to

$$(107) \quad \omega = \sqrt{-1} \left[dz \wedge d\bar{z} + dz_1 \wedge d\bar{z}_1 - z dz_1 \wedge d\bar{z}_2 - \bar{z} dz_2 \wedge d\bar{z}_1 + (1 + |z|^2) dz_2 \wedge d\bar{z}_2 \right].$$

Lemma 5.4. ω defined in (107) is complete.

Proof of Lemma 5.4. We may follow the proof of Lemma 5.1. Consider the lifting metric ω on the universal covering $\tilde{X}$. For a fixed point $p = (0, 0, 0)$, let $\gamma(t) = (z(t), \lambda(t), \eta(t))$ be any C^1 curve connecting p and an arbitrary point $q = (z_*, z_{1*}, z_{2*})$. Once we write $\gamma'(t) = (z'(t), \lambda'(t), \eta'(t))$, then

$$(108) \quad g(\gamma'(t), \gamma'(t)) = |z'(t)|^2 + |\lambda'(t) - \bar{z}\eta'(t)|^2 + |\eta'(t)|^2.$$

If either $|z_*|$ or $|z_{1*}|$ goes to ∞ , we may estimate $L(\gamma(t)) \rightarrow \infty$. Otherwise, we rewrite the above equation as

$$g(\gamma'(t), \gamma'(t)) = |z'(t)|^2 + (1 + |z|^2) \left| \frac{\lambda'(t)z}{1 + |z|^2} - \eta'(t) \right|^2 + \frac{|\lambda'(t)|^2}{1 + |z|^2}.$$

The remaining argument follows exactly as in Case 2 of the proof of Lemma 5.1. □

Proof of Theorem 1.6 when E is of Type III. As E is of Type III, we have $b_2 \neq b_1\tau$ in the representation (8). By Proposition 4.2, $\mathcal{O}(X)$ is nonempty if and only if both θ_1 and θ_2 are rational. Let m is the smallest positive integer such that $m\theta_1, m\theta_2 \in \mathbb{Z}$. By (55), we need to estimate the growth of z_2^m with respect to d_g on X .

Let $\rho : \tilde{X} \rightarrow X$ denote the universal covering and $\pi : E \rightarrow \Sigma$ the projection map. Unlike the Type II case, the fiber $\pi^{-1}(0)$ for the point $z = 0$ on Σ is no longer totally geodesic. Given a fixed point $p = \rho((0, 0, 0))$ and any point $q = \rho((z, z_1, z_2)) \in X$. Let $d_g(q)$ denote the distance from p to q . Thanks to (108), we have $d_g(q) \geq |z_2(q)|$ on $\tilde{X}$. It descends to X as $|z_2^m(q)| \leq (d_g(q))^m$. Moreover, it attains an equality along the sequence $\rho((0, 0, z_2))$ as $|z_2| \rightarrow \infty$. Hence, the desired characterization of $\mathcal{O}_d(X, g)$ follows. □

Remark 5.5. We may view ω defined in (106) as a family of Hermitian metrics on the same total space X . Moreover, they are quasi-isometric to each other. First of all, we may check the bundle isomorphism defined in Lemma 4.1 is an isometry with respect to ω . Therefore, it suffices to consider a vector bundle E of **Type III** with $b_1 = 0$ and $b_2 \neq 0$. As b_2 varies, it is direct to check the bundle isomorphism defined in (80) is a quasi-isometry.

6. FUNCTION THEORY ON TOTAL SPACES OF SOME VECTOR BUNDLES OF NONZERO DEGREES

In this section, we consider a class of vector bundles of nonzero degrees over an elliptic curve, and prove Proposition 6.3.

Following Grothendieck's exposition of Weil's result ([16]), Narasimhan-Seshadri [28, Proposition 6.2 on p.551] proved that any indecomposable vector bundle of rank n and degree d with $(-n < d \leq 0)$ over a compact Riemann surface M of genus ≥ 1 can be associated to a representation of some Fuchsian group constructed from $\pi_1(M)$. We begin with a lemma regarding the existence of a hyperbolic orbifold structure on any compact Riemann surface with genus ≥ 1 .

Lemma 6.1 (see [28, p.549] for example). *Given any compact Riemann surface M with genus ≥ 1 , a point $x_0 \in X$, and an integer $N \geq 1$. There exists a simply-connected Riemann surface U (which is biholomorphic to the upper half plane $\mathbb{H}$), and a branched covering $p : U \rightarrow M$ with the following properties:*

- (i) *The branched locus on M is the single point x_0 .*
- (ii) *Let $\tilde{\pi}$ be the covering transformation group of p . For any $y \in p^{-1}(x_0)$, the stabilizer subgroup of $\tilde{\pi}$ at y which is denoted by π_y , is a finite cyclic group of order N . Moreover, N is independent of the choice of $y \in p^{-1}(x_0)$.*

Given an elliptic curve Σ with a fixed point $x_0 \in \Sigma$, and an integer $N \geq 1$. By Lemma 6.1, we have a branched covering $p : \mathbb{H} \rightarrow \Sigma$. The corresponding covering transformation group $\tilde{\pi}$ is also called the orbifold fundamental group of (Σ, x_0) . It can be expressed as an extension of $\pi_1(\Sigma)$ by π_y .

$$(109) \quad 1 \longrightarrow \pi_y \longrightarrow \tilde{\pi} \longrightarrow \pi_1(\Sigma) \longrightarrow 1.$$

Moreover, it has a presentation

$$(110) \quad \tilde{\pi} = \{ \langle \gamma_1, \gamma_2 \rangle \mid [\gamma_1, \gamma_2] \gamma_0 = \gamma_0^N = 1 \}.$$

By Lemma 6.1, $\tilde{\pi}$ can be realized as a Fuchsian group which acts discretely and cocompactly on $\mathbb{H}$, and it is of signature $(1; N)$. In other words, there exists a representation $\sigma : \tilde{\pi} \rightarrow \text{PSL}(2, \mathbb{R})$ whose image is a Fuchsian group of signature $(1; N)$. We refer to [3] and [20] for basic facts on Fuchsian groups. We write the left action of $\gamma \in \tilde{\pi}$ on $\mathbb{H}$ as $z \rightarrow \sigma(\gamma)(z)$ for any $\gamma \in \tilde{\pi}$. The underlying elliptic curve Σ admits a hyperbolic orbifold structure with a conical singularity (of angle $\frac{2\pi}{N}$) at x_0 . Recall that a character τ of π_y is defined as an element in $\text{Hom}(\pi_y, \mathbb{C}^*)$. Following [28, p.549], we call a representation $\rho : \tilde{\pi} \rightarrow \text{GL}(n, \mathbb{C})$ is of type τ if $\rho(\gamma) = \tau(\gamma)E$ for any $\gamma \in \pi_y$ and E the identity matrix. Given any representation ρ of type τ , we may define an action of $\tilde{\pi}$ on $\tilde{E} = \mathbb{H} \times \mathbb{C}^n$ by

$$(111) \quad (z, z_1, \dots, z_n) \rightarrow (\sigma(\gamma)(z), \rho(\gamma)(z_1, \dots, z_n)), \quad \gamma \in \tilde{\pi}.$$

Following [28, p.549], we call such a rank n trivial bundle $\tilde{E}$ with the action (111) a π -bundle. Let $p_*(\mathcal{O}(\tilde{E}))$ denote the direct image sheaf of $\mathcal{O}(\tilde{E})$ of $p : \mathbb{H} \rightarrow \Sigma$, and $p_*^{\tilde{\pi}}(\mathcal{O}(\tilde{E}))$ the subsheaf of $\tilde{\pi}$ -invariant sections. It can be shown that $p_*^{\tilde{\pi}}(\mathcal{O}(\tilde{E}))$ is a locally free sheaf of rank n . Let $p_*^{\tilde{\pi}}(\tilde{E})$ denote the corresponding vector bundle on Σ .

Lemma 6.2 ([28, Proposition 6.2]). *Let $n \geq 1$ and $-n < d \leq 0$ two integers. Consider the elliptic curve Σ defined in (1) and any fixed point $x_0 \in \Sigma$. Let $p : \mathbb{H} \rightarrow \Sigma$ be the branched covering in Lemma 6.1. Assume that γ_0 is the generator of the cyclic group π_y ($y \in p^{-1}(x_0)$) and choose the character $\tau(\gamma_0) = e^{-\frac{2\pi id}{n}}$. Then for any given indecomposable vector bundle E of rank n and degree d , there exists a representation $\rho : \tilde{\pi} \rightarrow \text{GL}(n, \mathbb{C})$ is of type τ so that E is isomorphic to $p_*^{\tilde{\pi}}(\tilde{E})$.*

Let $N = n = 2$ and $d = -1$ in Lemma 6.1 and Lemma 6.2. We define a *pointed elliptic curve* as an elliptic curve Σ with a choice of $x_0 \in \Sigma$. There exists a specific pointed elliptic curve whose corresponding covering transformation group $\tilde{\pi}$ (in Lemma 6.1) is an arithmetic Fuchsian group of signature $(1; 2)$. We refer the reader to [20, Chapter 5] for an introduction of arithmetic Fuchsian groups. The particular example we discussed is associated to a representation $\sigma : \tilde{\pi} \rightarrow \mathrm{SL}(2, \mathbb{R})$

$$(112) \quad \sigma(\gamma_1) = \alpha = \begin{pmatrix} \frac{\sqrt{6}+\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{6}-\sqrt{2}}{2} \end{pmatrix}, \quad \sigma(\gamma_2) = \beta = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}.$$

It follows from (112) that

$$\sigma(\gamma_0) = \delta = \begin{pmatrix} \sqrt{3} & -(\sqrt{2} + \sqrt{6}) \\ \sqrt{6} - \sqrt{2} & -\sqrt{3} \end{pmatrix}, \quad \text{and} \quad \delta^2 = -E.$$

The image of the unique fixed point of δ in $\mathbb{H}$ under the map $p : \mathbb{H} \rightarrow \Sigma$ is exactly our choice of $x_0 \in \Sigma$. Indeed, the particular choice of σ in (112) is from the list of the case $e = 2$ in [35, Theorem 4.1 on p.392].

Now we define a specific representation $\rho : \tilde{\pi} \rightarrow U(2)$ of type τ as follows

$$(113) \quad \rho(\gamma_1) = A = e^{2\pi i \theta_1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\gamma_2) = B = e^{2\pi i \theta_2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where } \theta_1, \theta_2 \in [0, 1).$$

It follows that $\rho(\gamma_0) = -E$, which implies that $\tau(\gamma_0) = -1$. According to Lemma 6.2, ρ defines a rank 2 bundle $p_*^{\tilde{\pi}}(\tilde{E})$ of degree -1 . In particular, it is pointed out by Donaldson [13, p.579] (see also [28, Proposition 9.1]) that if $\theta_1 = \theta_2 = 0$, ρ in (113) corresponds to the ‘canonical’ indecomposable vector bundle $E(2, -1)$ in [2, Theorem 10]. It follows from [2] that any indecomposable vector bundle E of rank 2 and degree -1 can be written as $E = E(2, -1) \otimes L$ where $L \in \mathrm{Pic}^0(M)$. Together with Lemma 6.2, (112) and (113) yield all indecomposable rank 2 bundles of degree -1 on (Σ, x_0) .

Proposition 6.3. *Let (Σ, p) be a pointed elliptic curve and $E = p_*^{\tilde{\pi}}(\tilde{E})$ with σ and ρ defined in (112) and (113). Let X denote the total space of E . Then $\mathcal{O}(X)$ can be determined. In particular, the total space of $E(2, -1)$ is holomorphically convex.*

Proof of Proposition 6.3. For any given $f \in \mathcal{O}(X)$, we consider its lift $f \in \mathcal{O}(\mathbb{H} \times \mathbb{C}^2)$. For any $(z, z_1, z_2) \in \mathbb{H} \times \mathbb{C}^2$, f satisfies

$$(114) \quad f(z, z_1, z_2) = f(\sigma(\gamma_1)(z), e^{2\pi i \theta_1} z_2, e^{2\pi i \theta_1} z_1) = f(\sigma(\gamma_2)(z), -e^{2\pi i \theta_2} z_1, e^{2\pi i \theta_2} z_2).$$

We consider its Taylor expansion

$$(115) \quad f(z, z_1, z_2) = f_0(z) + \sum_{k=1}^{+\infty} \sum_{t=0}^k f_{k-t,t}(z) z_1^{k-t} z_2^t.$$

By (114) and (112), It follows that for any $k \geq t \geq 0$ and $z \in \mathbb{H}$

$$(116) \quad f_{k-t,t}(z) = e^{2\pi i \theta_1 k i} f_{t,k-t}(\alpha(z)) = e^{2\pi i \theta_2 k i} (-1)^{k-t} f_{k-t,t}(\beta(z)),$$

$$(117) \quad f_{t,k-t}(z) = e^{2\pi i \theta_1 k i} f_{k-t,t}(\alpha(z)) = e^{2\pi i \theta_2 k i} (-1)^t f_{t,k-t}(\beta(z)).$$

Then $|f_{k-t,t}(z)|$ is invariant under the subgroup H generated by α^2 and β . We may consider the group homomorphism $\psi : \sigma(\tilde{\pi}) \rightarrow \mathbb{Z}_2$ defined as $\psi(\alpha) = 1$ and $\psi(\beta) = 0$. Obviously, the kernel of ψ is H and it is index 2 subgroup of $\sigma(\tilde{\pi})$. So H is also a cocompact Fuchsian group (of signature $(1; 2, 2)$). Therefore, $f_{k-t,t}(z) = c_{k-t,t}$ which is a constant by Liouville’s theorem. Moreover, $c_{t,k-t} = e^{2\theta_1 k} c_{k-t,t}$ and

$$c_{k-t,t} = \begin{cases} \neq 0, & \text{if all of } 2\theta_1 k, \theta_2 k + \frac{k-t}{2}, \text{ and } \theta_2 k + \frac{t}{2} \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $\theta_1 = \theta_2 = 0$, X is the total space of $E(2, -1)$ and any $f \in \mathcal{O}(X)$ satisfies

$$f(z, z_1, z_2) = c_0 + c_{2,0}(z_1^2 + z_2^2) + [c_{4,0}(z_1^4 + z_2^4) + c_{2,2}z_1^2 z_2^2] + \cdots.$$

It follows that X is holomorphically convex. □

Remark 6.4. *The proof of Proposition 6.3 relies on the observation that in a cocompact Fuchsian group of signature $(1; 2)$ generated by α and β , the index 2 subgroup H generated by α^2 and β is also a Fuchsian group which acts cocompactly. The exact forms of α and β are not essential. In general, given any indecomposable vector bundle of rank $n \geq 2$ and degree $-n < d < 0$ with n and d coprime, a representation ρ (as in (113)) which corresponds to $E(n, d)$ in [2, Theorem 10] is given on [13, p.579]. By analyzing some suitable finite index subgroups it is possible to determine $\mathcal{O}(X)$ for the corresponding total space X .*

However, it is unclear whether similar results to those in Proposition 6.3 hold without assuming $-n < d < 0$. Given any two integers n and d , any indecomposable vector bundle of rank n and degree d over an elliptic curve is stable if and only if n and d are coprime, see [2] and [37, Appendix A]. It seems natural to ask if there is any connection between the stability of vector bundles and the existence of nonconstant holomorphic functions on the corresponding total spaces. We end the paper with a special case of Question 1.3.

Question 6.5. *Consider any indecomposable vector bundle E of rank n and degree d over an elliptic curve with $n \geq 2$ and $(n, d) = 1$. Let X be the total space of E . Can we determine $\mathcal{O}(X)$? What is the ‘best’ complete Kähler (Hermitian) metric on X ?*

REFERENCES

- [1] Abe, Y., and Kopfermann, K. *Toroidal groups*, vol. 1759 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001. Line bundles, cohomology and quasi-abelian varieties. (cited on page 6).
- [2] Atiyah, M. F. Vector bundles over an elliptic curve. *Proc. London Math. Soc.* (3) 7 (1957), 414–452. (cited on pages 2, 3, 27, 28).
- [3] Beardon, A. F. *The geometry of discrete groups*, vol. 91 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original. (cited on page 26).
- [4] Calabi, E. Métriques kählériennes et fibrés holomorphes. *Ann. Sci. École Norm. Sup.* (4) 12, 2 (1979), 269–294. (cited on pages 4, 7, 8).
- [5] Cao, H.-D., Chen, B.-L., and Zhu, X.-P. Recent developments on Hamilton’s Ricci flow. In *Surveys in differential geometry. Vol. XII. Geometric flows*, vol. 12 of *Surv. Differ. Geom.* Int. Press, Somerville, MA, 2008, pp. 47–112. (cited on page 5).
- [6] Chau, A., and Tam, L.-F. On the complex structure of Kähler manifolds with nonnegative curvature. *J. Differential Geom.* 73, 3 (2006), 491–530. (cited on page 6).
- [7] Chau, A., and Tam, L.-F. On the Steinness of a class of Kähler manifolds. *J. Differential Geom.* 79, 2 (2008), 167–183. (cited on page 6).
- [8] Chau, A., and Tam, L.-F. On the simply connectedness of nonnegatively curved Kähler manifolds and applications. *Trans. Amer. Math. Soc.* 363, 12 (2011), 6291–6308. (cited on page 5).
- [9] Chen, B., and Zhu, X. Yau’s uniformization conjecture for manifolds with non-maximal volume growth. *Acta Math. Sci. Ser. B (Engl. Ed.)* 38, 5 (2018), 1468–1484. (cited on page 6).
- [10] Cousin, P. Sur les fonctions triplement périodiques de deux variables. *Acta Math.* 33, 1 (1910), 105–232. (cited on page 6).
- [11] Debarre, O. *Complex tori and abelian varieties*, french ed., vol. 11 of *SMF/AMS Texts and Monographs*. American Mathematical Society; Société Mathématique de France, 2005. (cited on pages 1, 3, 10, 16).
- [12] Demailly, J.-P., Peternell, T., and Schneider, M. Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.* 3, 2 (1994), 295–345. (cited on pages 2, 3).
- [13] Donaldson, S. Atiyah’s work on holomorphic vector bundles and gauge theories. *Bull. Amer. Math. Soc. (N.S.)* 58, 4 (2021), 567–610. (cited on pages 3, 27, 28).
- [14] Gauduchon, P. Le théorème de l’excentricité nulle. *C. R. Acad. Sci. Paris Sér. A-B* 285, 5 (1977), A387–A390. (cited on page 23).
- [15] Griffiths, P., and Harris, J. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original. (cited on pages 1, 10).
- [16] Grothendieck, A. Sur le mémoire de Weil. Généralisation des fonctions abéliennes. In *Séminaire Bourbaki : années 1956/57 - 1957/58, exposés 137-168*, no. 4 in *Séminaire Bourbaki*. Société mathématique de France, 1958, pp. 57–71. (cited on page 26).
- [17] Hamilton, R. S. The Ricci flow on surfaces. In *Mathematics and general relativity (Santa Cruz, CA, 1986)*, vol. 71 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1988, pp. 237–262. (cited on page 15).
- [18] Howard, A., Smyth, B., and Wu, H. On compact Kähler manifolds of nonnegative bisectional curvature. I. *Acta Math.* 147, 1-2 (1981), 51–56. (cited on pages 5, 6).

- [19] Hwang, A. D., and Singer, M. A. A momentum construction for circle-invariant Kähler metrics. *Trans. Amer. Math. Soc.* 354, 6 (2002), 2285–2325. (cited on page 7).
- [20] Katok, S. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. (cited on pages 26, 27).
- [21] Lee, M.-C., and Tam, L.-F. Chern-Ricci flows on noncompact complex manifolds. *J. Differential Geom.* 115, 3 (2020), 529–564. (cited on page 6).
- [22] Liu, G. On Yau’s uniformization conjecture. *Camb. J. Math.* 7, 1-2 (2019), 33–70. (cited on page 6).
- [23] Lott, J. Comparison geometry of holomorphic bisectional curvature for Kähler manifolds and limit spaces. *Duke Math. J.* 170, 14 (2021), 3039–3071. (cited on page 6).
- [24] Mok, N. The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. *J. Differential Geom.* 27, 2 (1988), 179–214. (cited on pages 5, 6).
- [25] Morimoto, A. Non-compact complex lie groups without non-constant holomorphic functions. In *Proceedings of the Conference on Complex Analysis* (Berlin, Heidelberg, 1965), A. Aeppli, E. Calabi, and H. Röhl, Eds., Springer Berlin Heidelberg, pp. 256–272. (cited on page 6).
- [26] Morimoto, A. On the classification of noncompact complex abelian Lie groups. *Trans. Amer. Math. Soc.* 123 (1966), 200–228. (cited on page 6).
- [27] Mumford, D. *Abelian varieties*, vol. 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition. (cited on pages 3, 10).
- [28] Narasimhan, M. S., and Seshadri, C. S. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math. (2)* 82 (1965), 540–567. (cited on pages 2, 26, 27).
- [29] Newstead, P. E. *Introduction to moduli problems and orbit spaces*, vol. 51 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research, Bombay; Narosa Publishing House, New Delhi, 1978. (cited on page 3).
- [30] Ni, L., and Tam, L.-F. Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature. *J. Differential Geom.* 64, 3 (2003), 457–524. (cited on pages 5, 13).
- [31] Oda, T. Vector bundles on an elliptic curve. *Nagoya Math. J.* 43 (1971), 41–72. (cited on page 2).
- [32] Paun, M. On the Albanese map of compact Kähler manifolds with numerically effective Ricci curvature. *Comm. Anal. Geom.* 9, 1 (2001), 35–60. (cited on pages 4, 8).
- [33] Schnell, C. Notes on a graduate course on abelian varieties. Available at <https://www.math.stonybrook.edu/~cschnell/pdf/notes/abelian-varieties.pdf>, 2025. (cited on page 10).
- [34] Takayama, S. The Levi problem and the structure theorem for non-negatively curved complete Kähler manifolds. *J. Reine Angew. Math.* 504 (1998), 139–157. (cited on page 5).
- [35] Takeuchi, K. Arithmetic Fuchsian groups with signature $(1; e)$. *J. Math. Soc. Japan* 35, 3 (1983), 381–407. (cited on page 27).
- [36] Tosatti, V. Non-Kähler Calabi-Yau manifolds. In *Analysis, complex geometry, and mathematical physics: in honor of Duong H. Phong*, vol. 644 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2015, pp. 261–277. (cited on page 4).
- [37] Tu, L. W. Semistable bundles over an elliptic curve. *Adv. Math.* 98, 1 (1993), 1–26. (cited on pages 2, 28).
- [38] Weil, A. Généralisation des fonctions abéliennes. *J. Math. Pures Appl. 9e série*, 17, 1-4 (1938), 47–87. (cited on pages 2, 3).
- [39] Wu, H. An elementary method in the study of nonnegative curvature. *Acta Math.* 142, 1-2 (1979), 57–78. (cited on page 13).
- [40] Wu, H., and Yang, B. Negatively curved Kähler metrics on total spaces of a class of vector bundles, 34 pages. arXiv:2508.16354. (cited on page 16).
- [41] Wu, H.-H., and Zheng, F. Kähler manifolds with slightly positive bisectional curvature. In *Explorations in complex and Riemannian geometry*, vol. 332 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2003, pp. 305–325. (cited on page 5).
- [42] Yau, S.-T. A review of complex differential geometry. In *Several complex variables and complex geometry, Part 2* (Santa Cruz, CA, 1989), vol. 52, Part 2 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1991, pp. 619–625. (cited on page 6).

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