On the dib-chromatic number of a digraph

Juan José Montellano-Ballesteros[†] Christian Rubio-Montiel[‡]

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Abstract

An acyclic coloring of a digraph that maximizes the number of colors such that each color class has a vertex pointing to all other classes and a vertex pointing to it from all other classes is known as the dib-chromatic number of a digraph.

In this paper, we answer the question about the existence of the dib-chromatic number and study the dib-chromatic number of bipartite digraphs.

Keywords. Complete coloring, acyclic coloring, b-chromatic number, irreducible coloring.

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1 Introduction

Chromatic graph theory requires ways to optimize the search for its parameters; one way is through algorithms. For instance, we would like to perform an operation to reduce the number of colors in a graph with a given proper coloring. One way to achieve this is by recoloring its vertices, not necessarily in a greedy way, but by distributing the vertices of a single color to the rest of the color classes. Of course, such recoloring is not feasible if such a color class contains a vertex with neighbors to all other color classes; such a vertex is called a *b-vertex*. A *b-coloring* of a given graph is a coloring of the vertices such that in each color class there exists a b-vertex. The *b-chromatic number* of the graph G is defined as the worst case for the number of colors used by this coloring heuristic method, equivalently defined as the maximum k such that a proper b-coloring using k colors is accepted, and denoted by $b(G)$. Note that, if G is a graph with a proper coloring using $\chi(G)$ colors, such a coloring is a b-coloring.

The study of the b-chromatic number emerged in the late 1990s [7] where its complexity was determined. Since then, several papers have expanded its study, for example [9, 12, 13].

[†]Instituto de Matemáticas, Universidad Nacional Autónoma de México, CDMX, Mexico. juancho@im.unam.mx.

[‡]División de Matemáticas e Ingeniería, FES Acatlán, Universidad Nacional Autónoma de México, Naucalpan, Mexico. christian.rubio@acatlan.unam.mx.

It has been of interest to extend the chromatic number to the theory of digraphs [10, 11], as it has been for other coloring parameters [1, 2, 3, 16, 17].

For instance, *the dichromatic number* dc is a generalization of the chromatic number for a graph, where a proper coloring in graphs corresponds to an *acyclic coloring* in a digraph D and the goal is to find the minimum number of colors $\mathrm{dc}(D)$ used in this type of coloring introduced by Neumann-Lara [16] and by Erdős [4] when they showed that in a tournament T_n then $\mathrm{dc}(T_n) = O(\frac{n}{\lg n})$, also see [8].

In an acyclic coloring of D with $\mathrm{dc}(D)$ colors, there are arcs in both directions between each pair of color classes, giving rise to the extension of *complete coloring* of graphs to digraphs [5, 6]. In [1], *the achromatic number* $\mathrm{dac}(D)$ of a digraph D is defined as the maximum over the acyclic and complete colorings in D, showing results such as in a tournament T_n then $\mathrm{dac}(T_n) = \Omega(\frac{n}{\lg n})$.

To generalize the b-chromatic number from an algorithmic point of view, one could define a "b-vertex" with respect to an acyclic coloring of a digraph D as a u vertex colored i such that $u \cup C_j$ contains a directed cycle for each color class C_j, $i \neq j$. However, if we consider a directed cycle $\overrightarrow{C}_3$, with an acyclic coloring using two colors, the singular color class is a "b-vertex" but the other color class has no "b-vertices".

In [10], a vertex u is called a *b^+-vertex* with respect to a coloring Γ of a digraph D if u is incident to a vertex colored with each color different from the color of u. Analogously, a vertex v is a *b^--vertex* with respect to Γ if v is incident from a vertex colored with each color different from the color of v.

A *b-coloring* of D is a coloring such that each color class contains a b^+-vertex and a b^--vertex. The *dib-chromatic number* of D, denoted by $\mathrm{dib}(D)$, is the maximum k such that D admits an acyclic b-coloring with k colors. The question then arises as to the existence of the $\mathrm{dib}(D)$ for a given digraph D.

This paper is divided as follows. Section 2 presents the existence of the dib-chromatic number. Section 3 addresses the study of the dib-chromatic number in bipartite digraphs, including simple bipartite digraphs.

2 Existence

In this section, we will show that for any given digraph D, the dib-chromatic number $\mathrm{dib}(D)$ of D exists, by showing that $\mathrm{dc}(D) \leq \mathrm{dib}(D)$, where $\mathrm{dc}(D)$ is the dichromatic number of D. We consider finite, and loopless digraphs, i.e., digraphs where digons are permitted.

Let $D = (V(D), E(D))$ be a digraph and $r \geq 2$ be an integer. An acyclic coloring Γ of $V(D)$ using r colors will be called *b-irreducible* if there is no other acyclic coloring Γ' of $V(D)$ using $r-1$ colors such that, for each $i \in [r-1] := \{1, \ldots, r-1\}$, $\Gamma^{-1}(i) \subseteq \Gamma'^{-1}(i)$.

Given an acyclic coloring of $V(D)$ using r colors and $i \in [r]$, we will say that a vertex $x \in \Gamma^{-1}(i)$ is a *b-vertex* if it is a b^+-vertex and a b^--vertex, and a pair of vertices $\{x, y\} \subseteq \Gamma^{-1}(i)$ will be called a *b-pair* if x is a b^+-vertex and y is a b^--vertex.

Recall that the *out-neighborhood* of a vertex x of D is $N_D^+(x) = \{y \in V(D) \colon xy \in E(D)\}$, and its *in-neighborhood* is $N_D^-(x) = \{y \in V(D) \colon yx \in E(D)\}$.

Lemma 1. *Let D be a digraph and $\Gamma : V(D) \to [r]$ be an acyclic coloring of D using r colors. If Γ is b-irreducible, then Γ is a b-coloring of D.*

Proof. Let D be a digraph, $\Gamma : V(D) \to [r]$ be an acyclic b-irreducible coloring of D using r colors and let us suppose there is no b-vertex nor b-pair in $\Gamma^{-1}(r)$.

For each $x \in \Gamma^{-1}(r)$ let $V_b^-(x) = \{j \in [r-1] : N_D^-(x) \cap \Gamma^{-1}(j) = \emptyset\}$ and $V_b^+(x) = \{j \in [r-1] : N_D^+(x) \cap \Gamma^{-1}(j) = \emptyset\}$.

Observe that since there is no b-vertices in $\Gamma^{-1}(r)$, it follows that for every $x \in \Gamma^{-1}(r)$ we have that $V_b^+(x) \neq \emptyset$ or $V_b^-(x) \neq \emptyset$.

Claim 2. *Either for every $x \in \Gamma^{-1}(r)$ we have $V_b^+(x) \neq \emptyset$, or for every $x \in \Gamma^{-1}(r)$ we have $V_b^-(x) \neq \emptyset$.*

Suppose there is $x \in \Gamma^{-1}(r)$ where $V_b^+(x) = \emptyset$. Hence, for every $j \in [r-1]$ it follows that $N_D^+(x) \cap \Gamma^{-1}(j) \neq \emptyset$. That is, x is a b^+-vertex. Since there is no b-pairs in $\Gamma^{-1}(r)$ it follows that for each $y \in \Gamma^{-1}(r) \setminus \{x\}$ there is $i_y \in [r-1]$ such that $N_D^-(y) \cap \Gamma^{-1}(i_y) = \emptyset$. That is, for every $y \in \Gamma^{-1}(r) \setminus \{x\}$ we have $V_b^-(y) \neq \emptyset$. Finally, since x is not a b-vertex, there is $i \in [r-1]$ such that $N_D^-(x) \cap \Gamma^{-1}(i) = \emptyset$. Thus $V_b^-(x) \neq \emptyset$ and the claim follows.

By the previous claim, without loss of generality, we suppose that for every $x \in \Gamma^{-1}(r)$ we have $V_b^+(x) \neq \emptyset$. Let $B_1 = \{x \in \Gamma^{-1}(r) : 1 \in V_b^+(x)\}$ and, for $1 < j \leq r-1$, let $B_j = \{x \in \Gamma^{-1}(r) : j \in V_b^+(x)\} \setminus \bigcup_{1 \leq i < j} B_i$.

Observe that since for every $x \in \Gamma^{-1}(r)$ we have $V_b^+(x) \neq \emptyset$, it follows that $\{B_1, \ldots, B_{r-1}\}$ is a set of pairwise disjoint sets whose union is $\Gamma^{-1}(r)$.

Let $\Gamma' : V(D) \to [r-1]$ be a coloring of D using $r-1$ colors defined as follows: Given $x \in V(D)$, $\Gamma'(x) = j$ whenever $\Gamma(x) = j$ or $x \in B_j$.

Claim 3. *Γ' is an acyclic coloring such that, for each $i \in [r-1]$, $\Gamma^{-1}[i] \subseteq \Gamma'^{-1}[i]$.*

By definition of Γ', it follows that, for each $i \in [r-1]$, we have $\Gamma^{-1}[i] \subseteq \Gamma'^{-1}[i]$. Let $j \in [r-1]$. By definition, $\Gamma'^{-1}(j) = \Gamma^{-1}(j) \cup B_j$ with $B_j \subseteq \Gamma^{-1}(r)$. Since Γ is acyclic, there is no cycle contained neither in $D[\Gamma^{-1}(j)]$ nor $D[B_j]$, and since for every $x \in B_j$, $N_D^+(x) \cap \Gamma^{-1}(j) = \emptyset$, it follows that there is no cycle in $D[\Gamma'^{-1}(j)]$.

From the last claim, it follows that Γ is not a b-irreducible coloring which is a contradiction. Thus, for every $i \in [r]$ there is either a b-vertex or a b-pair and the result follows. $\square$

Theorem 4. *Given a digraph D, $dc(D) \leq dib(D)$.*

Proof. Let D be a digraph and let Γ be an acyclic coloring of D with $\mathrm{dc}(D)$ colors. Clearly Γ is a b-irreducible acyclic coloring which, by Lemma 1, implies that Γ is a b-coloring and therefore $\mathrm{dc}(D) \leq \mathrm{dib}(D)$. $\square$

3 Bipartite digraphs

An *independent* set of vertices in a digraph D is a set of vertices without any arcs between them. A *bipartite* digraph is a digraph $D = (V(D), E(D))$ where V is partitioned into the independent sets A and B, for short denoted by $D = (A, B)$. In this section, we state some results on bipartite digraphs.

We recall that the *independence number* $\beta(D)$ of D is the maximum possible number of vertices of an independent set.

A digraph D is *weak* (or *weakly connected*) if for every pair $u, v \in V(D)$ there exists a uv-path (not necessarily a directed path), otherwise, D is called *disconnected.* The number of arcs in a shortest uv-path is denoted by $d(u, v)$, and if u and v belong to different components of a disconnected digraph, then $d(u, v) = \infty$.

Let δ_D^+ denote *the minimum out-degree*, δ_D^- denote *the minimum in-degree*, and $\delta_D = \min\{\delta_D^+, \delta_D^-\}$ denote the minimum degree of D, respectively.

Proposition 5. *Let $D = (A, B)$ be a bipartite digraph such that $\delta_D \geq 2$, then $dib(D) \geq 2$.*

Proof. Consider a directed uv-path P of maximum length, then vertex v has an out-neighbor in P thus D contains a directed cycle and the result follows. □

The following result is given in [10].

Corollary 6. *[10] For any digraph D of order n, then $dib(D) \leq n - \beta(D) + 1$.*

Lemma 7. *If $D = (A, B)$ is a connected bipartite digraph with $|A| = n \leq m = |B|$, then*

$$dib(D) \leq n + 1$$

Proof. By Corollary 6, we have $\text{dib}(D) \leq n + m - \beta(D) + 1 = n + m - max\{n, m\} + 1 = n + 1$. □

Lemma 8. *Let $D = (A, B)$ be a bipartite digraph. If D has a b-coloring with $k \geq 3$ colors, then a partition contains vertices of all k colors and the other partition contains vertices of at least $k - 1$ colors.*

Proof. Since $k \geq 3$, there are at least two b^+-vertices at some partition, say B. Hence A contains all the k colors in its vertices. If A contains a b^+-vertex then it is incident to $k - 1$ different colors in B, otherwise, all b^+-vertices belong to B, then B has vertices of all the k colors. □

The following two results are generalizations of a result given in [13], which provide sufficient and necessary conditions for bipartite digraphs with $\text{dib}(D) > 2$.

Theorem 9. *Let $D = (A, B)$ be a disconnected bipartite digraph such that $\delta_D \geq 2$ and it is the union of $D_1, \ldots, D_r$ weak bipartite subdigraphs. The component $D_i = (A_i, B_i)$ where $A = \bigcup_{i=1}^r A_i$ and $B = \bigcup_{i=1}^r B_i$. Then $dib(D) > 2$ if and only if*

1. $r \geq 3$; or

2. $r = 2$ and at least one of D_1, D_2 is not a complete symmetric bipartite digraph.

Proof. Suppose D has no b-colorings with at least 3 colors, by Proposition 5, $\mathrm{dib}(D) = 2$. Note that each component D_i has at least 4 vertices.

1. Case $r \geq 3$. We can easily color a vertex in D_i to be b-vertex of color c_i, for $i = 1, 2, 3$.

2. Case $r = 2$. Assume D_1 is not complete bipartite. Let $u \in A$ and $v \in B$ be, such that uv is not a digon in D_1. Color u and v by color c_1, the vertices of $A \setminus \{u\}$ by color c_2 and all vertices of $B \setminus \{v\}$ by color c_3. Because of the connectivity of D_1 there is a b-pair of color c_2 in A and a b-pair of color c_3 in B. Now, we can easily color a vertex in D_2 to be b-vertex of color c_1.

   On the other hand, if D_1 and D_2 are complete symmetric bipartite digraphs, its dichromatic and diachromatic number is 2, then $\mathrm{dib}(D) = 2$.

□

Let v be a vertex in D, let $\overline{N}^{\,+}(v) = \{u \in V(D) : vu \notin E(D), v \neq u\}$ and $\overline{N}^{\,-}(v) = \{w \in V(D) : wv \notin E(D), v \neq w\}$ denote the out-non-neighborhood and in-non-neighborhood of v, respectively. We define the sets $\overline{N}_{\cup}(v) = \overline{N}^{\,+}(v) \cup \overline{N}^{\,-}(v)$ and $\overline{N}_{\cap}(v) = \overline{N}^{\,+}(v) \cap \overline{N}^{\,-}(v)$.

Theorem 10. *Let $D = (A, B)$ be a weakly connected bipartite digraph such that $\delta_D \geq 2$.*

1. If $dib(D) > 2$ then $A \subseteq \bigcup_{v \in B} \overline{N}_{\cup}(v)$ or $B \subseteq \bigcup_{v \in A} \overline{N}_{\cup}(v)$; and

2. If $A \subseteq \bigcup_{v \in B} \overline{N}_{\cap}(v)$ or $B \subseteq \bigcup_{v \in A} \overline{N}_{\cap}(v)$ then $dib(D) > 2$.

Proof. 1. We assume there exists a b-coloring using $k \geq 3$ colors of D. By Lemma 8, the set A or B has vertices of all k colors. Without loss of generality, A contains vertices v_i colored c_i for $i \in \{1, \ldots, k\}$. If there is $w \in B \setminus \bigcup_{i=1}^{k} \overline{N}_{\cup}(v_i)$, then $A \cup \{w\}$ are the vertices of a symmetric star and the coloring is not acyclic, hence, we have $B \subseteq \bigcup_{i=1}^{k} \overline{N}_{\cup}(v_i) \subseteq \bigcup_{v \in A_1} \overline{N}_{\cup}(v)$.

2. Now, suppose D has no b-colorings with at least 3 colors, by Proposition 5, $\mathrm{dib}(D) = 2$. Without loss of generality, suppose $B \subseteq \cup_{v \in A} \overline{N}_{\cap}(v)$.

   Let w_1 a vertex of B. By assumption, there exists a vertex $v_1 \in A$ such that $w_1 \in \overline{N}_{\cap}(v_1)$, then there are no arc between v_1 and w_1, so color v_1 and $\overline{N}_{\cap}(v_1) \cap B$ by c_1. Since $\delta(v_1) \geq 2$, there exists at least one uncolored vertex $w_2 \in B$ and a vertex $v_2 \in A \setminus \{v_1\}$ such that $w_2 \in \overline{N}_{\cap}(v_2)$, then $v_2 w_2 \notin E(D)$ and $w_2 v_2 \notin E(D)$. Color v_2, w_2 and the uncolored vertices of $\overline{N}_{\cap}(v_2) \cap B$ by c_2. Since $\delta(w_1) \geq 2$, there exists at least one uncolored vertex $v \in A$. We continue with this procedure until all vertices of B are colored. If there are uncolored vertices in A, color them all with an additional color c_i, with $i \geq 3$. We use $k \geq 3$ colors and at least $k - 1$ color classes have non-empty intersection with the sets A and B by construction. Suppose that the coloring is not a b-coloring, then there is a color class without a b-vertex or a b-pair. Recolor the vertices of this color class obtaining an acyclic coloring with $k-1$ colors. Again at least $k-2$ color classes have non-empty intersection with the sets A and B. Continue this process until a b-coloring is obtained. Since a b-coloring using 2 colors is not possible, a contradiction to the assumption. Hence the result is a b-coloring with at least 3 colors. $\square$

3.1 On simple bipartite digraphs

Recall a simple digraph is a digraph without digons, that is, an oriented graph.

Theorem 11. *Let D be a simple bipartite digraph. If $\delta(D) \geq 2$ then $dib(D) \geq 3$.*

Proof. Let $D = (A, B)$ be a simple bipartite digraph, with $\delta(D) \geq 2$ and suppose $\mathrm{dib}(D) \leq 2$. A (non-directed) path $P = (x_1, \ldots, x_k)$ in D will be called *bad* if every $x_{2i} \in V(P)$, with $1 \leq 2i \leq k$, x_{2i} is a sink in P (respectively, x_{2i} is a source in P); and every $x_{2i-1} \in V(P)$, with $1 \leq 2i-1 \leq k$, x_{2i-1} is a source in P (respectively, x_{2i-1} is a sink in P).

Let $P = (x_1, \ldots, x_k)$ be a bad path in D of maximum order. Without loss of generality, we will assume that every $x_{2i} \in V(P)$, with $1 \leq 2i \leq k$, x_{2i} is a sink in P; every $x_{2i-1} \in V(P)$, with $1 \leq 2i-1 \leq k$, x_{2i-1} is a source in P and that the sources are in A and the sinks are in B.

Claim 12. *$k \leq 7$.*

Proof. Suppose there is a bad path $(x_1, x_2, \ldots, x_k)$ in D with $k \geq 8$. Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_1) = \Gamma(x_4) = \Gamma(x_7) = c_1$; $\Gamma(x_2) = \Gamma(x_5) = \Gamma(x_8) = c_2$ and $\Gamma(x_3) = \Gamma(x_6) = c_3$. Also, for every $x \in A \setminus \{x_1, x_3, x_5, x_7\}$, $\Gamma(x) = c_1$ and for every $x \in B \setminus \{x_2, x_4, x_6, x_8\}$, $\Gamma(x) = c_3$. It is not hard to see that this is a b-coloring of D with 3 colors, where x_2, x_4 and x_6 are b^--vertices and x_3, x_5 and x_7 are b^+-vertices, which is not possible. $\square$

Claim 13. *$k \leq 6$.*

Proof. Suppose there is a bad path $P = (x_1, x_2, \ldots, x_7)$ in D. By Claim 12 there is no bad path of order 8, thus $N^+(x_7) \subseteq V(P)$ and since $\delta(D) \geq 2$, either $x_7x_2 \in E(D)$ or $x_7x_4 \in E(D)$. If $x_7x_2 \in E(D)$ then in D there is the undirected cycle $C_6 = (x_2, x_3, x_4, x_5, x_6, x_7)$. Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_3) = \Gamma(x_6) = c_1$; $\Gamma(x_4) = \Gamma(x_7) = c_2$ and $\Gamma(x_2) = \Gamma(x_5) = c_3$. Also, for every $x \in A \setminus \{x_3, x_5, x_7\}$, $\Gamma(x) = c_1$ and for every $x \in B \setminus \{x_2, x_4, x_6\}$, $\Gamma(x) = c_3$. It is not hard to see that this is a b-coloring of D with 3 colors, where x_2, x_4 and x_6 are b^--vertices and x_3, x_5 and x_7 are b^+-vertices, which is not possible.

Hence, $x_7x_4 \in E(D)$ and, by an analogous argument than above, $x_1x_4 \in E(D)$. Thus, $\{x_1, x_3, x_5, x_7\} \subseteq N^-(x_4)$ and since $\delta(D) \geq 2$, there is a pair $\{z_1, z_2\} \subseteq N^+(x_4)$ in $A \setminus \{x_1, x_3, x_5, x_7\}$. Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_1) = \Gamma(x_4) = \Gamma(x_5) = c_1$; $\Gamma(x_2) = \Gamma(x_7) = \Gamma(z_1) = c_2$ and $\Gamma(x_3) = \Gamma(x_6) = \Gamma(z_2) = c_3$. Also, for every $x \in A \setminus \{z_1, z_2, x_1, x_3, x_5, x_7\}$, $\Gamma(x) = c_3$ and for every $x \in B \setminus \{x_2, x_4, x_6\}$, $\Gamma(x) = c_2$. It is not hard to see that this is a b-coloring of D with 3 colors, where x_4 is a b-vertex, x_2 and x_6 are b^--vertices; and x_3 and x_7 are b^+-vertices, which is not possible. $\square$

Claim 14. *$k \leq 5$.*

Proof. Suppose there is a bad path $P = (x_1, x_2, \ldots, x_6)$ in D. By Claim 13 there is no bad path of order 7, thus $N^+(x_1) \subseteq V(P)$ and since $\delta(D) \geq 2$, either $x_1x_6 \in E(D)$ or $x_1x_4 \in E(D)$. If $x_1x_6 \in E(D)$ then in D there is the undirected cycle $C_6 = (x_1, x_2, x_3, x_4, x_5, x_6)$. As in Claim 13 we see that this is not possible. Thus $x_1x_4 \in E(D)$. In an analogous way we see that $x_3x_6 \in E(D)$. Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_1) = \Gamma(x_6) = c_1$; $\Gamma(x_3) = \Gamma(x_4) = c_2$ and $\Gamma(x_2) = \Gamma(x_5) = c_3$. Also, for every $x \in A \setminus \{x_1, x_3, x_5\}$, $\Gamma(x) = c_1$ and for every $x \in B \setminus \{x_2, x_4, x_6\}$, $\Gamma(x) = c_3$. It is not hard to see that this is a b-coloring of D with 3 colors, where x_1, x_3 and x_5 are b^+-vertices; and x_2, x_4 and x_6 are b^--vertices, which is not possible. $\square$

Claim 15. *$k \leq 4$.*

Proof. Suppose there is a bad path $P = (x_1, x_2, \ldots, x_5)$ in D. By Claim 14 there is no bad path of order 6, thus $N^+(x_1) \subseteq V(P)$ and since $\delta(D) \geq 2$, $x_1x_4 \in E(D)$. In an analogous way, we see that $x_5x_2 \in E(D)$. Moreover, $d^+(x_1) = d^+(x_3) = d^+(x_5) = 2$. Thus, $\{x_1, x_3, x_5\} \subseteq N^-(x_2)$ and $\{x_1, x_3, x_5\} \subseteq N^-(x_4)$ and therefore $N^+(x_2), N^+(x_4) \subseteq A \setminus \{x_1, x_3, x_5\}$. Let $\{z_1, z_2\} \subseteq N^+(x_2)$ and $\{w_1, w_2\} \subseteq N^+(x_4)$.

Case 1. $z_1 = w_1$ and $z_2 = w_2$.

Since $\delta(D) \geq 2$ and $\{x_2, x_4\} \subseteq N^-(z_1)$, there is a pair $\{y_1, y_2\} \subseteq N^+(z_1)$ such that $\{y_1, y_2\} \subseteq B \setminus \{x_2, x_4\}$. Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_2) = \Gamma(x_5) = \Gamma(z_1) = c_1$; $\Gamma(x_1) = \Gamma(x_4) = \Gamma(y_2) = c_2$ and $\Gamma(x_3) = \Gamma(y_1) = \Gamma(z_2) = c_3$. Also, for every $x \in A \setminus \{z_1, z_2, x_1, x_3, x_5\}$, $\Gamma(x) = c_1$ and for every $x \in B \setminus \{x_2, x_4, y_1, y_2\}$, $\Gamma(x) = c_2$. It is not hard to see that this is a b-coloring of D with 3 colors, where x_4 is a b-vertex, x_3 and z_1 are b^+-vertices; and x_2 and z_2 are b^--vertices, which is not possible.

Case 2. $z_1 = w_1$ and $z_2 \neq w_2$.

Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_1) = \Gamma(x_4) = \Gamma(z_2) = c_1$; $\Gamma(x_5) = \Gamma(z_1) = c_2$ and $\Gamma(x_2) = \Gamma(x_3) = \Gamma(w_2) = c_3$. Also, for every $x \in A \setminus \{z_1, z_2, w_2, x_1, x_3, x_5\}$, $\Gamma(x) = c_3$ and for every $x \in B \setminus \{x_2, x_4\}$, $\Gamma(x) = c_1$. It is not hard to see that this is a b-coloring of D with 3 colors, where x_2 and x_4 are b-vertex, and x_5 and z_1 is a b^+-vertex and a b^--vertex, respectively; which is not possible.

Case 3. $\{z_1, z_2\} \cap \{w_1, w_2\} = \emptyset$.

Since $w_1 \notin N^+(x_2)$ and $\delta(D) \geq 2$ there is $y_1 \in N^-(w_1) \cap (B \setminus \{x_2, x_4\})$. Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_1) = \Gamma(x_4) = \Gamma(z_1) = c_1$; $\Gamma(x_2) = \Gamma(x_3) = \Gamma(y_1) = \Gamma(w_2) = c_2$ and $\Gamma(x_5) = \Gamma(w_1) = \Gamma(z_2) = c_3$. Also, for every $x \in A \setminus \{z_1, z_2, w_1, w_2, x_1, x_3, x_5\}$, $\Gamma(x) = c_1$ and for every $x \in B \setminus \{x_2, x_4, y_1\}$, $\Gamma(x) = c_2$. It is not hard to see that the chromatic classes of colors c_1 and c_3 are acyclic. For the chromatic class of color c_2, observe that the only vertices of color c_2 in A are x_3 and w_2. Thus, any directed cycle of color c_2 is a square and contains the vertices $\{x_3, w_2\}$. Since $d^+(x_3) = 2$ and $\Gamma(x_4) = c_1$, any directed cycle of color c_2 must contain the path (x_3, x_2, w_2) which is not possible since $x_2w_2 \notin E(D)$. Hence, Γ is a b-coloring of D with 3 colors, where x_2 and x_4 are b-vertices, and x_5 and w_1 is a b^+-vertex and a b^--vertex, respectively; which is not possible. □

Claim 16. *D is 2-regular.*

Proof. Let $x \in V(D)$ with $d^+(x) = r \geq 2$ and let $N^+(x) = \{y_1, \dots, y_r\}$. By Claim 15 it follows that for some $z \in V(D) \setminus \{x, y_1, \dots, y_r\}$, $\bigcap_{y \in N^+(x)} N^-(y) = \{x, z\}$, which again, by Claim 15, implies that $r = 2$ (in other case, there is a bad path of order 5 where x_1, x_3 and x_5 are sinks, and x_2, x_4 are sources). Thus, for every $x \in V(D)$, $d^+(x) = 2$ and since $\delta(D) \geq 2$, the claim follows. □

Claim 17. *There are no bad paths of order 4.*

Proof. Let $P = (x_1, x_2, x_3, x_4)$ be a bad path in D. By Claim 15 it follows that $x_1x_4 \in E(D)$. By Claim 16, $\delta(D) = 2$ and therefore we see that $N^+(x_2) = \{z_1, z_2\}$ and $N^+(x_4) = \{w_1, w_2\}$ where $N^+(x_2), N^+(x_4) \subseteq A \setminus \{x_1, x_3\}$. Again, by Claim 15, either $w_1 = z_1$ and $z_2 = w_2$; or $\{z_1, z_2\} \cap \{w_1, w_2\} = \emptyset$.
Case 1. $\{z_1, z_2\} \cap \{w_1, w_2\} = \emptyset$.
By Claim 15 there is a pair $\{y_1, y_2\} \subseteq B \setminus \{x_2, x_4\}$ such that $N^+(y_1) = \{z_1, z_2\}$ and $N^+(y_2) = \{w_1, w_2\}$. Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_1) = \Gamma(y_2) = \Gamma(z_1) = \Gamma(w_1) = c_1$; $\Gamma(x_2) = \Gamma(x_3) = \Gamma(w_2) = c_2$ and $\Gamma(x_4) = \Gamma(y_1) = \Gamma(z_2) = c_3$. Also, for every $x \in A \setminus \{z_1, z_2, w_1, w_2, x_1, x_3, x_5\}$, $\Gamma(x) = c_2$ and for every $x \in B \setminus \{x_2, x_4, y_1, y_2\}$, $\Gamma(x) = c_3$. It is not hard to see that Γ is a b-coloring of D with 3 colors, where x_4 is a b-vertex; x_1 and x_2 are b^+-vertices; and z_1 and w_2 are b^--vertices, which is not possible.
Case 2. $w_1 = z_1$ and $z_2 = w_2$.
Since $\delta(D) = 2$, $N^+(z_1) = \{y_1, y_2\}$ and $N^-(x_1) = \{y_3, y_4\}$ where $N^+(z_1), N^-(x_1) \subseteq B \setminus \{x_2, x_4\}$.

Let $\Gamma : V(D) \to \{c_1, c_2, c_3\}$ defined as: $\Gamma(x_4) = c_1$; $\Gamma(x_1) = \Gamma(x_2) = \Gamma(z_1) = c_2$ and $\Gamma(x_3) = \Gamma(z_2) = c_3$. For the vertices $\{y_1, y_2, y_3, y_4\}$, it is possible that either $\{y_1, y_2\} = \{y_3, y_4\}$; or $\{y_1, y_2\} \cap \{y_3, y_4\}\emptyset$ or $y_1 = y_3$ and $y_2 \neq y_4$. In any case, color the vertices with colors c_1 and c_3 in a way that both colors are present in $N^-(x_1)$ an $N^+(z_1)$.

Also, for every $x \in A \setminus \{z_1, z_2, x_1, x_3, x_5\}$, $\Gamma(x) = c_2$ and for every $x \in B \setminus \{x_2, x_4, y_1, y_2, y_3, y_4\}$, $\Gamma(x) = c_1$. It is not hard to see that Γ is a b-coloring of D with 3 colors, where x_4 is a b-vertex; x_3 and z_1 are b^+-vertices; and x_1 and z_2 are b^--vertices, which is not possible. □

From Claim 17 we see that if $dib(D) \leq 2$, then there is no bad path of order 4. Since $\delta(D) \geq 2$, this is not possible, and the result follows.

□

Corollary 18. *Let D be a simple bipartite 2-regular digraph. Then $dib(D) = 3$.*

Proof. By Theorem 11 we see that $dib(D) \geq 3$, and it is not hard to see that $dib(D) \leq \max_{x \in V(D)} \{d^+_D(x), d^-_D(x)\} + 1 = 3$ and the result follows. □

From lemmas 7 and 8 we obtain the following upper bound.

Corollary 19. *Let $D = (A, B)$ be a simple bipartite digraph with $|A| = n \leq m = |B|$. If there is no source or there is no sink in B, $dib(D) \leq n$.*

Proof. By Lemma 7, $\text{dib}(D) \leq n + 1$. If $\text{dib}(D) = n + 1$, by Lemma 8, there are no vertices of some color c_{n+1} in A, and b^+-vertices and b^--vertices of $c_1 \dots, c_n$ belong to A. Hence, each vertex in A is a b-vertex, and the b^+-vertex and b^--vertex of color c_{n+1} are a source and a sink in B, respectively. $\square$

Now we present some lower bounds. For this, given a digraph $D = (A, B)$, let $\delta_A = \min\{d^+_D(x), d^-_D(x) : x \in A\}$ and $\delta_B = \min\{d^+_D(x), d^-_D(x) : x \in B\}$.

Theorem 20. *Let $D = (A, B)$ be a simple bipartite digraph, with $|A| = n \leq m = |B|$. If*

$$2n^2 < \Big(\frac{m}{m - \delta_A}\Big)^{\lfloor \frac{m}{n} \rfloor}$$

then $dib(D) \geq n$. Moreover, there is a b-coloring of D with n colors such that every vertex in A is a b-vertex.

Proof. Let $D = (A, B)$ be a simple bipartite digraph, with $|A| = n \leq m = |B|$, and assume that $B = [m]$. Let $P^*_n(m)$ be the set of n-partitions of $[m]$ such that each part has cardinality either $\lfloor \frac{m}{n} \rfloor$ or $\lceil \frac{m}{n} \rceil$ and let $re(m, n)$ be the residue of m modulo n. Observe that for every n-partition in $P^*_n(m)$, the number of parts of order $\lfloor \frac{m}{n} \rfloor$ is $n - re(m, n)$. First we will assume that $re(n, m) \neq 0$.

It follows that $\binom{m}{\lceil \frac{m}{n} \rceil}|P^*_{n-1}(m - \lceil \frac{m}{n} \rceil)| = re(m, n)|P^*_n(m)|$ and that $\binom{m}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m - \lfloor \frac{m}{n} \rfloor)| = (n - re(m, n))|P^*_n(m)|$. Thus

$$n|P^*_n(m)| = \binom{m}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m - \lfloor \frac{m}{n} \rfloor)| + \binom{m}{\lceil \frac{m}{n} \rceil}|P^*_{n-1}(m - \lceil \frac{m}{n} \rceil)|. \quad (1)$$

On the one hand, observe that $\frac{\binom{m}{\lfloor \frac{m}{n} \rfloor}}{\binom{m-\delta_A}{\lfloor \frac{m}{n} \rfloor}} = \frac{m!(m-\delta_A-\lfloor \frac{m}{n} \rfloor)!}{(m-\lfloor \frac{m}{n} \rfloor)!(m-\delta_A)!} \geq (\frac{m}{m-\delta_A})^{\lfloor \frac{m}{n} \rfloor}$ and, similarly, $\frac{\binom{m}{\lceil \frac{m}{n} \rceil}}{\binom{m-\delta_A}{\lceil \frac{m}{n} \rceil}} \geq (\frac{m}{m-\delta_A})^{\lceil \frac{m}{n} \rceil}$. If $2n^2 < (\frac{m}{m-\delta_A})^{\lfloor \frac{m}{n} \rfloor}$ then also $2n^2 < (\frac{m}{m-\delta_A})^{\lceil \frac{m}{n} \rceil}$ and therefore $2n^2 < min\{\frac{\binom{m}{\lfloor \frac{m}{n} \rfloor}}{\binom{m-\delta_A}{\lfloor \frac{m}{n} \rfloor}}, \frac{\binom{m}{\lceil \frac{m}{n} \rceil}}{\binom{m-\delta_A}{\lceil \frac{m}{n} \rceil}}\}$. Thus, by (1) we see that

$$n|P^*_n(m)| > 2n^2\binom{m-\delta_A}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m - \lfloor \frac{m}{n} \rfloor)| + 2n^2\binom{m-\delta_A}{\lceil \frac{m}{n} \rceil}|P^*_{n-1}(m - \lceil \frac{m}{n} \rceil)|. \quad (2)$$

On the other hand, for every $x \in A$, the number of n-partitions $\{Y_1, \dots, Y_n\}$ of $P^*_n(m)$ such that there is a part Y_j of order $\lfloor \frac{m}{n} \rfloor$ such that either $N^+_D(x) \cap Y_j = \emptyset$ or $N^-_D(x) \cap Y_j = \emptyset$ is at most $\binom{|B \setminus N^+_D(x)|}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m - \lfloor \frac{m}{n} \rfloor)| + \binom{|B \setminus N^-_D(x)|}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m - \lfloor \frac{m}{n} \rfloor)| \leq 2\binom{m-\delta_A}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m - \lfloor \frac{m}{n} \rfloor)|$. In a similar way we see that the number of n-partitions $\{Y_1, \dots, Y_n\}$ of $P^*_n(m)$ such that there is a part Y_j of order $\lceil \frac{m}{n} \rceil$ such that either $N^+_D(x) \cap Y_j = \emptyset$ or $N^-_D(x) \cap Y_j = \emptyset$ is at most

$2\binom{m-\delta_A}{\lceil \frac{m}{n} \rceil}|P^*_{n-1}(m-\lceil \frac{m}{n}\rceil)|$. Hence, for each vertex $x \in A$, the number of elements $\{Y_1, \ldots, Y_n\}$ of $P^*_n(m)$ such that there is a part Y_j such that either $N^+_D(x) \cap Y_j = \emptyset$ or $N^-_D(x) \cap Y_j = \emptyset$ is at most $2\Big(\binom{m-\delta_A}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m-\lfloor \frac{m}{n}\rfloor)| + \binom{m-\delta_A}{\lceil \frac{m}{n} \rceil}|P^*_{n-1}(m-\lceil \frac{m}{n}\rceil)\Big)$.

By (2) we see that $|P^*_n(m)| > 2n\Big(\binom{m-\delta_A}{\lceil \frac{m}{n} \rceil}|P^*_{n-1}(m-\lceil \frac{m}{n}\rceil)| + \binom{m-\delta_A}{\lfloor \frac{m}{n} \rfloor}|P^*_{n-1}(m-\lfloor \frac{m}{n}\rfloor)\Big)$ which implies there is a partition $\{Y_1, \ldots, Y_n\}$ in $P^*_n(m)$ such that for every $x \in A$ we have that for every $j \in [n]$, $N^+_D(x) \cap Y_j \neq \emptyset$ and $N^-_D(x) \cap Y_j \neq \emptyset$.

Its is not hard to see that the coloring $\Gamma : V(D) \to [n]$ where each vertex in A receives different color; and for each vertex $y \in B$, y receives color $i \in [n]$ if and only if $y \in Y_i$, is a b-coloring of D where every vertex in A is a b-vertex.

The proof for the case when $re(m, n) = 0$ is similar, and the result follows. □

Corollary 21. *Let D be a simple bipartite digraph, with $|A| = n \leq m = |B|$. Let p be such that $\lfloor \frac{m}{n} \rfloor = p(1 + 2log_2(n))$. If $\delta_A > m\Big(1 - \frac{1}{2^{\frac{1}{p}}}\Big)$ then $dib(D) \geq n$.*

Proof. If $\delta_A > m\Big(1 - \frac{1}{2^{\frac{1}{p}}}\Big)$ then $\left(\frac{m}{m-\delta_A}\right)^{\lfloor \frac{m}{n} \rfloor} > \left(\frac{m}{m-m\left(1-\frac{1}{2^{\frac{1}{p}}}\right)}\right)^{\lfloor \frac{m}{n} \rfloor} = \left(2^{\frac{1}{p}}\right)^{\lfloor \frac{m}{n} \rfloor} = 2^{\frac{\lfloor \frac{m}{n} \rfloor}{p}} = 2^{1+2log_2(n)} = 2n^2$ and from Theorem 20 the result follows. □

Theorem 22. *Let $D = (A, B)$ be a simple digraph, with $|A| = n \leq m = |B|$. If $\delta_A \geq 2n(n-1)$, then $dib(D) \geq n$. Moreover, if D is an orientation of $K_{n,m}$, with $\delta_A \geq n^2$, then $dib(D) \geq n$.*

Proof. Let $A = \{x_1, \ldots, x_n\}$. First, assume that D is an orientation of $K_{n,m}$. Colored x_1 with color c_1 and x_2 with color c_2. Choose a vertex in $N^+(x_1)$ and another in $N^-(x_1)$ and assign them color c_2. Similarly, choose a vertex in $N^+(x_2)$ and another in $N^-(x_2)$ and assign them color c_1. Colored x_3 with color c_3. Since $\{N^+(x_3), N^-(x_3)\}$ is a partition of B, in $N^+(x_3) \cup N^-(x_3)$ there are already vertices with color c_1 and color c_2. Thus, we only need to choose at most one new vertex to be of color c_1 and another new vertex to be of color c_2 such that in $N^+(x_3)$ there are vertices of color c_1 and c_2 as well as in $N^-(x_3)$. Now, choose a new vertex in $N^+(x_1)$ and another in $N^-(x_1)$ and assign them color c_3. In this way, in $N^+(x_1)$ there are vertices of color c_2 and c_3 as well as in $N^-(x_1)$. Since $\{N^+(x_2), N^-(x_2)\}$ is a partition of B, either in $N^+(x_2)$ or in $N^-(x_2)$ there is already a vertex of color c_3. So, we only have to choose at most one new vertex to be of color c_3 so that in $N^+(x_2)$ there are vertices of color c_1 and c_3 as well as in $N^-(x_2)$. So far we have colored at most 9 vertices in B. Following this procedure, assume that for $3 \leq r < n$, for every $i \in [r]$ x_i receives color c_i; for each $i \in [r]$ there are vertices of colors $\{c_1, \ldots, c_r\} \setminus \{c_i\}$ in $N^+(x_i)$ and in $N^-(x_i)$; and that there are at most r^2 colored vertices in B. Recall that $n^2 \leq \delta_A$, hence for each $x_i \in A$, with $i \in [r]$, in $N^+(x_i)$ (as well as in $N^-(x_i)$) there are at least $n^2 - r^2$ vertices with no color.

Let colored x_{r+1} with color c_{r+1}. In $N^+(x_{r+1}) \cup N^-(x_{r+1})$ there are already vertices of color $c_1, \ldots, c_r$. Thus, for each color c_i, with $i \in [r]$, we only have to choose at most one new vertex in B and assign color c_i to it, so that there are vertices of colors $\{c_1, \ldots, c_r\}$ in $N^+(x_{r+1})$ and in $N^-(x_{r+1})$. Now choose any new vertex y in B and color it with color c_{r+1}.

For every $i \in [r]$, $y \in N^+(x_i)$ or $y \in N^-(x_i)$. Therefore, for each color c_i, with $i \in [r]$, we only have to choose at most one new vertex in B and assign to it color c_{r+1}, so that for each $i \in [r]$ there are vertices of color c_{r+1} in $N^+(x_i)$ and in $N^-(x_i)$. Observe that in this step we colored at most $2r+1$ new vertices in B. Thus, so far there are at most $(r+1)^2$ colored vertices in B.

Since $\delta_A \geq n^2$, we can proceed in this way until coloring the vertex x_n with color c_n and coloring some vertices in B in a way that, for each $i \in [n]$, x_i has color c_i, and such that there are vertices of colors $\{c_1, \dots, c_n\} \setminus \{c_i\}$ in $N^+(x_i)$ and in $N^-(x_i)$. If there are more vertices in B, let assign them color c_1. This coloring of D is a b-coloring with n colors, and the result follows.

For the general case, colored x_1 with color c_1 and x_2 with color c_2. Choose a vertex in $N^+(x_1)$ and other in $N^-(x_1)$ and assign them color c_2. Similarly, choose a vertex in $N^+(x_2)$ and other in $N^-(x_2)$ and assign them color c_1. So far we have colored at most 4 vertices in B. Assume that for some $2 \leq r < n$, for every $i \in [r]$ x_i receives color c_i; for each $i \in [r]$ there are vertices of colors $\{c_1, \dots, c_r\} \setminus \{c_i\}$ in $N^+(x_i)$ and in $N^-(x_i)$; and that there at most $2r(r-1)$ colored vertices in B. Since $\delta_A \geq 2n(n-1)$, for each $x_i \in A$, with $i \in [r]$, in $N^+(x_i)$ (as well as in $N^-(x_i)$) there are at least $2n(n-1) - 2r(r-1)$ vertices with no color. Let colored x_{r+1} with color c_{r+1}, and for each $i \in [r]$ choose a vertex in $N^+(x_{r+1})$ and another in $N^-(x_{r+1})$ and assign them color c_i. Also, for each $i \in [r]$, choose a vertex in $N^+(x_i)$ and another in $N^-(x_i)$ and assign them color c_{r+1}. Observe that in this step we colored at most $4r$ vertices in B, so there are at most $2(r+1)r$ colored vertices in B. From here, as in the case of the orientation of $K_{n,m}$, the result follows. □

For the next theorem, given a bipartite digraph $D = (A, B)$, and a pair of vertices $\{y_1, y_2\} \subseteq B$, let $c(y_1, y_2) = |N_D^+(y_1) \cap N_D^-(y_2)|$.

Theorem 23. *Let $D = (A, B)$ be a simple bipartite digraph, with $|A| = n \leq m = |B|$. If for a pair $\{y_1, y_2\} \subseteq B$,*

$$2c(y_1, y_2)^2 < \left(\frac{m-2}{m-2-(\delta_A - 1)}\right)^{\lfloor \frac{m-2}{c(y_1,y_2)} \rfloor}$$

then $dib(D) \geq c(y_1, y_2) + 1$.

Proof. Let $D = (A, B)$ be a simple bipartite digraph and $\{y_1, y_2\} \subseteq B$ be as in the statement. Let $C = N_D^+(y_1) \cap N_D^-(y_2) \subseteq A$ and let D' the subdigraph of D induced by $C \cup B \setminus \{y_1, y_2\}$. Observe that $|C| = c(y_1, y_2)$, $|B \setminus \{y_1, y_2\}| = m - 2$ and $\delta_C \geq \delta_A - 1$, thus, by Lemma 20, there is a b-coloring $\Gamma' : V(D') \to [c(y_1, y_2)]$ where the vertices in C receive the colors $\{1, \dots, c(y_1, y_2)\}$ and all of them are b-vertices.

Let $\Gamma : V(D) \to [c(y_1, y_2) + 1]$ defined as follows: $\Gamma(x) = \Gamma'(x)$ if $x \in V(D')$; and $\Gamma(x) = c(y_1, y_2) + 1$ in other case. Since $C = N_D^+(y_1) \cap N_D^-(y_2)$, all the vertices in C are b-vertices in Γ; and y_1 is a b^+-vertex and y_2 is a b^--vertex. All the chromatic classes of color $i \in [c(y_1, y_2)]$ are stars, thus are acyclic. For the chromatic class of color $c(y_1, y_2) + 1$, since the only vertices of color $c(y_1, y_2) + 1$ in B are $\{y_1, y_2\}$, any directed cycle of that color must be a square and contains the pair of vertices $\{y_1, y_2\}$, which is not possible since none of the vertices in $C = N_D^+(y_1) \cap N_D^-(y_2)$ receives color $c(y_1, y_2) + 1$. □

An orientation of the complete bipartite graph (A, B), with $|A| = n$ and $|B| = m$, such that each arc goes from A to B is denoted by $\overrightarrow{K_{n,m}}$.

The following theorem shows that $\mathrm{dib}(D)$ is an increasing function on $min\{n, m\}$, when D is an orientation of the complete bipartite graph $K_{n,m}$.

Theorem 24. *Let $D = (A, B)$ be an orientation of $K_{n,m}$, with $|A| = n \leq m = |B|$, and let r be an integer such that $n \geq r2^r(1 + \frac{1}{2^r})^r + 2r - 1$. Then*

$$r \leq dib(D).$$

Proof. Let $D = (A, B)$ be an orientation of $K_{n,m}$, with $|A| = n \leq m = |B|$, and let r be the greater integer such that there is a subdigraph $D' = (A', B')$ of D isomorphic to $\overrightarrow{K_{r,r}}$, with $A' = \{x_1, \dots, x_r\}$ and $B' = \{y_1, \dots, y_r\}$.

Let $\Gamma : V(D) \to \{c_1, \dots, c_r\}$ be a coloring where: for each $i \in [r]$, $\Gamma(x_i) = \Gamma(y_i) = c_i$; for each $x \in A \setminus A'$, $\Gamma(x) = c_1$, and for each $x \in B \setminus B'$, $\Gamma(x) = c_2$.

Each chromatic class is a star, thus Γ is an acyclic coloring, and for each $i \in [r]$, x_i is a b^+-vertex and y_i is a b^--vertex. Thus, Γ is a b-coloring and $r \leq \mathrm{dib}(D)$. Therefore, to end the proof we just have to show that if $n \geq r2^r(1 + \frac{1}{2^r})^r + 2r - 1$ then there is a copy of $\overrightarrow{K_{r,r}}$ in D. Without loss of generality, let us suppose that $\sum\limits_{x\in A} d_D^+(x) \geq \sum\limits_{x\in A} d_D^-(x)$.

Let $H = (X, Y)$ be the bipartite graph where $X = A$ and Y is the set of r-subsets of vertices of B. Given $z \in X$ and $S \in Y$, zS is an edge of H if $S \subseteq N_D^+(z)$. If there is no copy of $\overrightarrow{K_{r,r}}$, it follows that for every $S \in Y$, $d_H(S) \leq r - 1$ which implies that $\sum\limits_{z\in X} \binom{d_D^+(z)}{r} \leq (r-1)\binom{m}{r}$.

Since $\sum\limits_{x\in A} d_D^+(x) \geq \sum\limits_{x\in A} d_D^-(x)$, it follows that $\sum\limits_{x\in A} d_D^+(x) \geq \frac{nm}{2}$ and therefore $\sum\limits_{z\in X} \binom{d_D^+(z)}{r} \geq n\binom{\lfloor \frac{m}{2} \rfloor}{r}$. Hence

$$(r-1)\binom{m}{r} \geq n\binom{\lfloor \frac{m}{2} \rfloor}{r},$$

which implies that

$$\frac{\binom{m}{r}}{\binom{\lfloor \frac{m}{2} \rfloor}{r}} \geq \frac{n}{r-1}.$$

On the one hand, $\frac{\binom{m}{r}}{\binom{\lfloor \frac{m}{2} \rfloor}{r}} \leq \left(\frac{m-r+1}{\lfloor \frac{m}{2} \rfloor - r+1}\right)^r = 2^r\left(\frac{m-r+1}{2\lfloor \frac{m}{2} \rfloor - 2r+2}\right)^r \leq 2^r\left(\frac{m-r+1}{m-2r+1}\right)^r = 2^r\left(1 + \frac{r}{m-2r+1}\right)^r$, and since $m \geq n \geq r2^r(1 + \frac{1}{2^r})^r + 2r - 1$, it follows that $\frac{\binom{m}{r}}{\binom{\lfloor \frac{m}{2} \rfloor}{r}} \leq 2^r\left(1 + \frac{r}{r2^r(1+\frac{1}{2^r})^r}\right)^r = 2^r\left(1 + \frac{1}{2^r(1+\frac{1}{2^r})^r}\right)^r$.

On the other hand, since $n \geq r2^r(1 + \frac{1}{2^r})^r + 2r - 1$, it follows that $\frac{n}{r-1} > 2^r(1 + \frac{1}{2^r})^r$. Thus, $2^r\left(1 + \frac{1}{2^r(1+\frac{1}{2^r})^r}\right)^r > 2^r(1 + \frac{1}{2^r})^r$, which is not possible, and the result follows. $\square$

4 Proposed problems

To conclude, we would like to present a couple of problems that, in our opinion, might be of interest. The first is related to Theorems 9, 10, and 11, and the second is related to tournaments.

Problem 1. Weaken the condition of $\delta_D \geq 2$ when considering bipartite digraphs with $dib(D) \geq 3$, for example, require that $\delta_D^+ \geq 1$, $\delta_D^- \geq 1$ and $\delta_D^+ + \delta_D^- \geq 3$. In this case, Figure 1 shows an example.

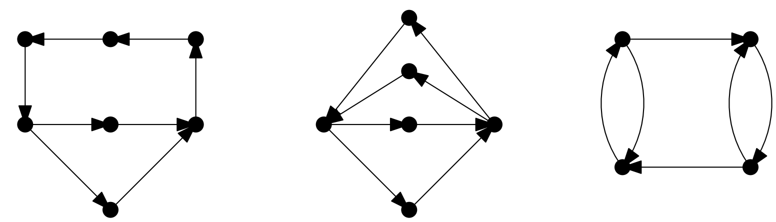

Figure 1: There are bipartite graphs with 3 components whose $\mathrm{dib}(D)$ is 2.

Problem 2. The behavior of the b-chromatic number is closely related to the Grundy number, for example [14, 15]. In the case of the digrundy number dG [2] and the dib-chromatic number, we believe that this behavior will hold. In particular, given a tournament T_n, it was shown in [1] that the dichromatic number $\mathrm{dac}(T_n) = \Omega(\frac{n}{\lg n})$, such bound is given using an order in the coloring, so the coloring turns out to be greedy, hence $\mathrm{dG}(T_n) = \Omega(\frac{n}{\lg n})$. The question is also whether $\mathrm{dib}(T_n) = \Omega(\frac{n}{\lg n})$.

5 Statements and Declarations

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References

[1] G. Araujo-Pardo, J. J. Montellano-Ballesteros, M. Olsen, and C. Rubio-Montiel. The diachromatic number of digraphs. *Electron. J. Combin.*, 25(3):Paper No. 3.51, 17, 2018.

[2] G. Araujo-Pardo, J. J. Montellano-Ballesteros, M. Olsen, and C. Rubio-Montiel. The digrundy number of digraphs. *Discrete Appl. Math.*, 317:117–123, 2022.

[3] K. J. Edwards. Harmonious chromatic number of directed graphs. *Discrete Appl. Math.*, 161(3):369–376, 2013.

[4] P. Erdős. Problems and results in number theory and graph theory. In *Proceedings of the Ninth Manitoba Conference on Numerical Mathematics and Computing (Univ. Manitoba, Winnipeg, Man., 1979)*, volume XXVII of *Congress. Numer.*, pages 3–21. Utilitas Math., Winnipeg, MB, 1980.

[5] S. M. Hegde and L. P. Castelino. Further results on harmonious colorings of digraphs. *AKCE Int. J. Graphs Comb.*, 8(2):151–159, 2011.

[6] S. M. Hegde and L. P. Castelino. Harmonious colorings of digraphs. *Ars Combin.*, 119:339–352, 2015.

[7] R. W. Irving and D. F. Manlove. The b-chromatic number of a graph. *Discrete Appl. Math.*, 91(1-3):127–141, 1999.

[8] H. Jacob and H. Meyniel. Extension of Turán's and Brooks' theorems and new notions of stability and coloring in digraphs. In *Combinatorial mathematics (Marseille-Luminy, 1981)*, volume 75 of *North-Holland Math. Stud.*, pages 365–370. North-Holland, Amsterdam, 1983.

[9] M. Jakovac and S. Klavžar. The b-chromatic number of cubic graphs. *Graphs Combin.*, 26(1):107–118, 2010.

[10] N. Javier-Nol, C. Rubio-Montiel, and I. Torres-Ramos. The dib-chromatic number of digraphs, 2024.

[11] J. Kok and N. K. Sudev. The b-chromatic number of certain graphs and digraphs. *J. Discrete Math. Sci. Cryptogr.*, 19(2):435–445, 2016.

[12] M. Kouider and M. Mahéo. Some bounds for the b-chromatic number of a graph. *Discrete Math.*, 256(1-2):267–277, 2002.

[13] J. Kratochvíl, Z. Tuza, and M. Voigt. On the b-chromatic number of graphs. In *Graph-theoretic concepts in computer science*, volume 2573 of *Lecture Notes in Comput. Sci.*, pages 310–320. Springer, Berlin, 2002.

[14] Zoya Masih and Manouchehr Zaker. On Grundy and b-chromatic number of some families of graphs: a comparative study. *Graphs Combin.*, 37(2):605–620, 2021.

[15] Zoya Masih and Manouchehr Zaker. Some comparative results concerning the Grundy and b-chromatic number of graphs. *Discrete Appl. Math.*, 306:1–6, 2022.

[16] V. Neumann-Lara. The dichromatic number of a digraph. *J. Combin. Theory Ser. B*, 33(3):265–270, 1982.

[17] É. Sopena. Complete oriented colourings and the oriented achromatic number. *Discrete Appl. Math.*, 173:102–112, 2014.