STACKING AND THE TRIVIALITY OF INVERTIBLE PHASES

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Abstract. We study the superselection sectors of two quantum lattice systems stacked onto each other in the operator algebraic framework. We show in particular that all irreducible sectors of a stacked system are unitarily equivalent to a product of irreducible sectors of the factors. This naturally leads to a faithful functor between the categories for each system and the category of the stacked system. We construct an intermediate 'product' category which we then show is equivalent to the stacked system category. As a consequence, the sectors associated with an invertible state are trivial, namely, invertible states support no anyonic quasi-particles.

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1. Introduction

In his famous conference [Kit13], Kitaev introduced and formalised the dichotomy between short-range and long-range entangled states of quantum lattice systems. Among many other aspects, he defined the notion of invertible states and put them within the class of states that do not have long-range entanglement. He argued in particular that gapped systems with an invertible ground state should not have *anyonic quasi-particle excitations*. They do not

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exhibit intrinsic topological order and the classification problem is reduced to understanding the role of symmetries: these are the symmetry protected topological orders.

Although the first mathematical description of anyons is often attributed to [LM77; GMS80; Wil82] in a differential geometric picture, braid statistics appeared simultaneously (in fact, shortly before) in the rigorous quantum field theory literature in [Frö76], see also the much more complete [Frö87]. The widespread use of *braided monoidal categories* is more recent although it also was suggested long ago in [FRS89; FG90], see also [FST94] and the references therein. It has been widely used in the physics literature since the seminal Appendix E of [Kit06]. The adaptation of the mathematical QFT setting to non-relativistic quantum lattice systems originated in [Naa11; Naa13] and was fully formalised by [Oga22]. In all of these works, the picture that arises is a classification of anyon types as equivalence classes of certain irreducible representations of the C*-algebra of observables, so the so-called superselection sectors.

A key concept in the definition of an invertible state is that of *stacking*. The stacking operation is the simple fact of considering two physical systems as two layers of one joint system. If the two quantum systems are described by quasi-local observable algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, we write $\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2$ for the observable algebra of the stacked system. Mathematically, it is given by the tensor product of C*-algebras, which is well-defined since our algebras are nuclear. For given states ω_1 and ω_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, the state $\omega_1 \otimes_{\mathrm{s}} \omega_2$ is defined in the obvious way. A state is then said to be invertible if after stacking with a suitable system, the resulting state is in the trivial phase, namely, it can be 'adiabatically deformed' to a product state.

In this work, we characterise, in full generality, the superselection sectors of stacked systems in terms of those of their components. Not surprisingly, we show that they are also obtained as a product of the components: a similar notion to the Deligne product in the language of category theory. Since product states correspond to a trivial category [NO22], this immediately reproduces a result by Ogata [Oga25], namely that product states are the neutral elements of the stacking product. However, our result also applies to stacking with systems in non-trivial phases. This implies a more interesting corollary of our result, which is nothing else than Kitaev's conclusion, that invertible states have trivial superselection sectors. That is, an invertible state does not support anyonic excitations, whether Abelian or not.

Our setup, which is explained in more detail in Section 2, is that of standard superselection sector theory for 2D quantum spin systems. The starting point is an irreducible reference representation π_0. In the applications that we have in mind, this π_0 would be obtained as the GNS representation of a pure ground state of a gapped quantum spin system. We assume that π_0 satisfies *approximate Haag duality* for cones and that the cone algebras $\pi_0(\Lambda)''$ are properly infinite (which follows for example if π_0 comes from a gapped ground state of a finite range interaction [Oga22]). In this setting, Haag duality was proven first for Abelian quantum double models [FN15], while the recent [OPR25] provides a proof of approximate Haag duality for the large class of string-net models, which include all quantum double ground states. Any model that can be adiabatically connected to these models, namely that are in the same phase, satisfy approximate Haag duality [Oga22].

The assumptions stated above allow us to define a braided C*-category $\Delta_{\mathfrak{A}}$ of representations satisfying the superselection criterion [Oga22]. Our main results relate $\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$ to $\Delta_{\mathfrak{A}_1}$

and $\Delta_{\mathfrak{A}_2}$. We state them in a slightly informal fashion here and refer to the main text for the details. The first one, Theorem 3.11, is rather simple: Stacking superselection sectors of the individual layers yield superselection sectors of the stacked system.

Theorem A. *If ρ_1 and ρ_2 are representations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ that satisfy the superselection criterion for π_1 and π_2, then $\rho_1 \otimes_{\mathrm{s}} \rho_2$ satisfies the superselection criterion for $\pi_1 \otimes_{\mathrm{s}} \pi_2$.*

In categorical terms, this naturally leads to a faithful functor $F : \Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2} \to \Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$, where $\boxtimes$ is an appropriate notion of the 'tensor product' of the categories of superselection sectors, which will be explained in Section 4.

The second result, Theorem 3.13 which is the technical heart of this work, is concerned with the converse of the above, namely that every irreducible representation in $\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$ is of this form. This requires a 'Type I' condition, namely that every factorial representation σ satisfying the superselection criterion for π_1 is such that $\sigma(\mathfrak{A}_1)''$ is a Type I factor.

Theorem B. *If the Type I condition holds, then any irreducible representation ρ which satisfies the superselection criterion for $\pi_1 \otimes_{\mathrm{s}} \pi_2$ is of the form $\rho \cong \rho_1 \otimes_{\mathrm{s}} \rho_2$ with irreducible ρ_1, ρ_2 satisfying the superselection criterion for $\pi_1 \otimes_{\mathrm{s}} \pi_2$.*

Together, these two results show that the sectors of the stacked system are exactly given by products of sectors of the individual layers. Not surprisingly, this can be lifted to the whole braiding structure, yielding the third result, Theorem 4.15 in Section 4:

Theorem C. *The braided tensor* W^**-categories $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ and $\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$ are equivalent.*

Since the superselection theory of a state in the trivial phase is trivial in the sense that the only sectors are direct sums of the reference representation, the above results immediately imply the following corollary, see Corollary 3.34 .

Corollary D. *The superselection structure of invertible phases is trivial.*

The Type I condition emphasizes a technical subtlety about fusion, see Remark 3.26: Without additional assumptions, there may be uncountably many irreducible factors in the decomposition of a product of representations, and they may not individually satisfy the superselection criterion.

We will show that the condition follows from two mild assumptions that are physically motivated. First of all, we assume that there are at most countably many irreducible superselection sectors, or anyon types. This is a common assumption, with the word 'anyon theory' in any axiomatisation usually defined to be equivalent to a unitary modular tensor category (UMTC), which implies even finitely many irreducible sectors. Our second assumption is the *approximate split property*, which is an assumption on the decay of correlations. In Proposition 3.25, whose proof draws on a similar result for conformal nets in [KLM01], we show that these assumptions imply the Type I property.. The arguments in Section 3.5 combine to prove that the approximate split property holds for all quantum phases having one generalized Levin-Wen tensor-network state representative. Indeed, these models satisfy a slightly stronger version of the approximate Haag duality, see again [OPR25], and they also satisfy a strict form of the area law for the entanglement entropy term. We show, see Theorem 3.29, that these two assumptions imply the approximate split property. It remains to appeal to [NO22] to deduce the approximate split property for all states in the phase.

We shall conclude this work with a few examples in Section 5. For the quantum double models, our results imply that the superselection sector theory of a model build from a direct product of groups is the stacking product of the sector theories for the models build on each factor. For Levin–Wen models, they imply that the sector theory is an invariant under Morita equivalence of the categories used to define the model. This does not require the a priori knowledge of the sector theory, but since Morita-equivalence is defined by the equivalence of the Drinfeld centres, it shows that the Drinfeld centre completely characterizes the sector theory. Our last example is that of the symmetry-enriched toric code model.

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2. Setting

2.1. **Algebras, states, and representations.** Let $\Gamma \subset \mathbb{R}^2$ be a countable set representing the sites of our quantum spin systems. In typical applications Γ will be for example a lattice, but for our purposes it is sufficient to assume that the number of sites contained in a ball of radius r does not grow too fast with r. This is true if Γ is a Delone set.

To each site $x \in \Gamma$ we assign a finite dimensional Hilbert space $\mathcal{H}_x$. We assume that $d_x := \dim \mathcal{H}_x$ is uniformly bounded. For $P_f(\Gamma)$ the set of finite subsets $\Lambda_f \subset \Gamma$, we can associate a composite Hilbert space $\mathcal{H}(\Lambda_f) := \bigotimes_{x\in\Lambda_f} \mathcal{H}_x$ with a corresponding algebra of observables $\mathfrak{A}(\Lambda_f) = B(\mathcal{H}(\Lambda_f)) = \bigotimes_{x\in\Lambda_f} M_{d_x}(\mathbb{C})$. With the natural inclusion of algebras ($\mathfrak{A}(\Lambda_1) \hookrightarrow \mathfrak{A}(\Lambda_2)$ when $\Lambda_1 \subset \Lambda_2 \subset \Gamma$) we can define a net of C*-algebras $\Lambda_f \mapsto \mathfrak{A}(\Lambda_f)$ which satisfies $[\mathfrak{A}(\Lambda), \mathfrak{A}(\widehat{\Lambda})] = \{0\}$ whenever $\Lambda \perp \widehat{\Lambda}$, where the orthogonality relation $\perp$ is given by disjointness. The local observables are then $\mathfrak{A}_{\mathrm{loc}} = \bigcup_{\Lambda_f\subset P(\Gamma)} \mathfrak{A}(\Lambda_f)$, from which we get a C*-algebra of observables as the inductive limit of the net: $\mathfrak{A} := \overline{\mathfrak{A}_{\mathrm{loc}}}^{\|\cdot\|}$. Note that $\mathfrak{A}$ is simple by [KR97, Prop. 11.4.2]. The local net $\Lambda \mapsto \mathfrak{A}(\Lambda)$ of C*-subalgebras with the orthogonality relation of disjointness is a (bosonic) quasi-local algebra [BR87, Def. 2.6.3], and we shall refer to any such algebra as a quantum spin system.

For a possibly infinite subset $\Lambda \subset \Gamma$, the operators localised in this region are given by $\mathfrak{A}(\Lambda) := \overline{\bigcup_{\Lambda_f\subset P_f(\Lambda)} \mathfrak{A}(\Lambda_f)}^{\|\cdot\|} \subset \mathfrak{A}$. In this paper, our only consideration is with such infinite regions we call cones, namely sets of the form

$$\Lambda_{\vec{z},\theta,\phi} = \Gamma \cap \{\vec{z} + r\vec{e}_\beta : r > 0, \beta \in (\theta - \phi, \theta + \phi)\}$$

for any $\vec{z} \in \mathbb{R}^2$, and angles $\theta \in \mathbb{R}$, $\phi \in (0, \pi)$, where $\vec{e}_\beta = (\cos\beta, \sin\beta)$. Note that in this definition a cone can have an opening angle greater than π, as long as it does not cover the whole space. The advantage of this definition is that the interior of the complement of a cone is again a cone.

The following is a physically relevant family of automorphisms, namely those that preserve locality sufficiently well. One may pick slightly different definitions, but it is important that the spectral flow [BMNS12], or quasi-adiabatic continuation [HW05], falls into the family one chooses. Denote by $d(\cdot,\cdot)$ the metric on Γ. Then for a finite set $\Lambda_f \in P_f(\Gamma)$, the diameter of Λ_f is defined as $\mathrm{D}(\Lambda_f) := \sup_{x,y\in\Lambda_f} d(x,y)$. An interaction is a family of maps $\Phi_t : \mathcal{P}_f(\Gamma) \to \mathfrak{A}$ such that $\Phi_t(\Lambda_f) = \Phi_t(\Lambda_f)^* \in \mathfrak{A}(\Lambda_f)$. We further assume that the interactions don't grow too fast, namely that

$$\sup_{t\in\mathbb{R}} \sup_{x\in\Gamma} \sum_{\Lambda_f \ni x} \frac{\|\Phi_t(\Lambda_f)\|}{\xi(\mathrm{D}(\Lambda_f)+1)} < \infty$$

for a positive non-increasing function ξ such that $\sup_{r\geq 1} r^n \xi(r)$ is finite for all $n \in \mathbb{N}$. The interaction Φ is defined to have *finite range* if there exists some $d^\Phi \geq 1$ such that $\Phi(\Lambda_f) = 0$ whenever $\mathrm{D}(\Lambda_f) > d^\Phi$. In this case, the only non-zero contributions to the energy arise from a finite clusters of neighbouring sites. The expression $\sum_{\Lambda_f \in \mathcal{P}_f(\Gamma)} \mathrm{i}[\Phi(\Lambda_f), A]$ is well-defined on $\mathfrak{A}_{\mathrm{loc}}$ and it can be extended to a densely defined, unbounded *-derivation δ^Φ of $\mathfrak{A}$. This derivation then generates a strongly continuous one-parameter group of automorphisms $\{\alpha_t^\Phi : t \in \mathbb{R}\}$ defined as the solution of

$$\frac{d}{dt}\alpha_t^\Phi(A) = \alpha_t^\Phi(\delta^\Phi(A)), \qquad \alpha_0^\Phi(A) = A,$$

for all A in the domain of δ^Φ. A locally generated automorphism (LGA) is an automorphism α such that $\alpha = \alpha_1^\Phi$ for an interaction Φ.

A state is a positive linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ such that $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and $\omega(\mathbb{I}) = 1$. The set of pure states of $\mathfrak{A}$ is denoted $\mathcal{P}(\mathfrak{A})$. For ω a state of C*-algebra $\mathfrak{A}$, there exists a cyclic representation $\pi_\omega : \mathfrak{A} \to B(\mathcal{H}_\omega)$ on a Hilbert space $\mathcal{H}_\omega$, called the GNS representation, such that $\omega(A) = \langle \Omega, \pi_\omega(A)\Omega\rangle$, where Ω is a cyclic vector in $\mathcal{H}_\omega$, i.e. $\pi_\omega(\mathfrak{A})\Omega$ is dense in $\mathcal{H}_\omega$. We shall refer to $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ as the GNS triple of ω. A state ω is a gapped ground state of an interaction Φ if there is $g > 0$ such that

$$-\mathrm{i}\omega(A^*\delta^\Phi(A)) \geq g\omega(A^*A)$$

for all $A \in \mathfrak{A}_{\mathrm{loc}}$ such that $\omega(A) = 0$.

Our goal is to classify certain physically relevant representations of $\mathfrak{A}$, which correspond to 'charged states' or anyons. These are given by the following superselection criterion, see [BF82; Oga22].

Definition 2.1. Let $(\mathcal{H}_0, \pi_0)$ be an irreducible representation of $\mathfrak{A}$. A representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ satisfies the superselection criterion (SSC) with respect to π_0 if

$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

for all cones Λ, where Λ^c is the complement of Λ in Γ. In this case, we say that π satisfies $\mathrm{SSC}(\pi_0)$.

Note that here and below $\cong$ stands for unitary equivalence of representations.

Remark 2.2. We will always assume that our reference representation π_0 is irreducible. Since we consider quantum spin systems, the quasi-local algebras we consider are UHF algebras, and hence this automatically implies that π_0 is defined on a separable Hilbert space, since every vector in the representation space is cyclic. Hence every representation π satisfying $\mathrm{SSC}(\pi_0)$ is also defined on a separable Hilbert space as the superselection criterion provides a unitary map between the two Hilbert spaces.

Unitary equivalence splits the set of all representations satisfying SSC(π_0) in disjoint classes called superselection sectors, or simply sectors. A representation which is equivalent to π_0 is in the trivial sector. We say that a sector is irreducible if its representatives are irreducible representations.

The aim of superselection sector theory is to show that the set of sectors has a very rich structure, namely that of a braided tensor category. To ensure that we can take direct sums and subobjects (that is, restrict to an invariant subspace of some representation) as well, we need to make the following technical assumption.

Assumption 2.3. *The reference representation π_0 is such that for all cones Λ the algebra $\pi_0(\mathfrak{A}(\Lambda))''$ is a properly infinite factor.*

If π_0 is irreducible, as will always be the case for us, the algebra $\pi_(\mathfrak{A}(\Lambda))''$ is automatically a factor. The properly infiniteness assumption is satisfied, for example, if π_0 is the GNS representation of a gapped ground state ω_0 of a finite range interaction Φ, see [Oga22, Lemma 5.3]. One can always apply a stabilisation procedure to get properly infinite factors [Oga25].

For our construction, we require the reference representations to satisfy a notion of Haag duality – in this case, approximate Haag duality. For a cone $\Lambda = \Lambda_{\vec{z},\theta,\phi}$, we denote $\mathbf{e}_\Lambda := \mathbf{e}_\theta$ and $\Lambda_\varepsilon := \Lambda_{\vec{z},\theta,\phi+\varepsilon}$. In particular, the cone $\Lambda_\varepsilon - t\mathbf{e}_\Lambda$ is a (backwards) translation and widening of the original cone Λ. Clearly, $\Lambda \subset \Lambda_\varepsilon - t\mathbf{e}_\Lambda$.

Definition 2.4 (Approximate Haag duality, [Oga22])**.** Let $(\mathcal{H}, \pi_0)$ be an irreducible representation of $\mathfrak{A}$. Then π_0 satisfies *approximate Haag duality* if for any $\varphi \in (0, 2\pi)$ and $\varepsilon > 0$ with $\varphi + 4\varepsilon < 2\pi$, there is some $R > 0$ and decreasing functions $f_\delta(t)$ with $\lim_{t\to\infty} f_\delta(t) = 0$ such that:

(i) for any cone $\Lambda \in \Gamma$ with $|\arg\Lambda| = \varphi$, there is a unitary $U : \mathcal{H} \to \mathcal{H}$ satisfying

$$\pi_0(\mathfrak{A}(\Lambda^c))' \subset \mathrm{Ad}(U)\left(\pi_0(\mathfrak{A}(\Lambda_\epsilon - R\mathbf{e}_\Lambda))''\right),$$

and

(ii) for any $\delta > 0$ and $t \geq 0$ there is a unitary $\tilde{U} \in \pi_0(\mathfrak{A}(\Lambda_{\varepsilon+\delta} - t\mathbf{e}_\Lambda))''$ satisfying

$$||U - \tilde{U}|| \leq f_\delta(t).$$

Note that (ii) should be understood as the fact that the unitary U in (i) is approximately localized in the cone Λ.

Assumption 2.5. *The reference representation π_0 satisfies approximate Haag duality.*

Haag duality has been shown for Abelian quantum double models [FN15]. As pointed out in the introduction, approximate Haag duality has recently been proven [OPR25] to hold for all string net states, and by stability [Oga22, Proposition 1.3] to all states in the topological phases that have a string net representative. This is believed to cover all non-chiral quantum phases.

We may identify $\pi(\mathfrak{A}(\Lambda))$ with $\mathfrak{A}(\Lambda)$ whenever it is clear from the context which representation is used. Note that we can do this because $\mathfrak{A}$ is simple and hence π automatically is injective. Going forward, we will always assume that our reference representations are irreducible, and satisfy Assumptions 2.3 and 2.5. This is enough to guarantee the existence of a braided C*-category of superselection sectors [Oga22].

2.2. **Stacking.** Given two quantum spin systems $\mathfrak{A}_1, \mathfrak{A}_2$, we can form a new 'stacked' system denoted $\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2$. By this, we mean the quasi-local algebra constructed from the local net

$$P_f(\Gamma) \ni \Lambda_f \mapsto (\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)(\Lambda_f) := \mathfrak{A}_1(\Lambda_f) \otimes \mathfrak{A}_2(\Lambda_f),$$

where the product on the right hand side is the standard tensor product of finite-dimensional matrices. This defines the product $\otimes_{\mathrm{s}}$, which must be well differentiated from the 'local' product used in the definition of a quasi-local algebra. That is, we reserve $\otimes$ for tensor products within a layer, whereas $\otimes_{\mathrm{s}}$ is the tensor product of algebras (or states) between layers. Here and in the following, we refer to the individual components of a stacked system as its 'layers' since this corresponds to the physical picture of putting two systems on top of each other.

Remark 2.6. Without loss of generality we may assume that both quantum spin systems are defined on the same set of underlying sites. Indeed, write $\Gamma := \Gamma_{\mathfrak{A}_1} \cup \Gamma_{\mathfrak{A}_2}$, where $\Gamma_{\mathfrak{A}_1}$ (resp. $\Gamma_{\mathfrak{A}_2}$) is the underlying set of sites for the quasi-local algebra $\mathfrak{A}_1$ (resp. $\mathfrak{A}_2$). Then for the first stacked layer, we can place a trivial Hilbert space $\mathbb{C}$ on each site in $\Gamma_{\mathfrak{A}_2} \setminus \Gamma_{\mathfrak{A}_1}$, and leave the sites in $\Gamma_{\mathfrak{A}_1}$ as before. The quasi-local algebra $\mathfrak{A}_1^{\Gamma}$ built on this new set is clearly isomorphic to the original $\mathfrak{A}_1$. We can do the same for the second layer.

We denote by the same symbol the stacking product of states $\omega_1 \otimes_{\mathrm{s}} \omega_2$ as well as that of representations $\pi_1 \otimes_{\mathrm{s}} \pi_2$ of $\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2$. Both are defined on simple products by $(\pi_1 \otimes_{\mathrm{s}} \pi_2)(A_1 \otimes_{\mathrm{s}} A_2) = \pi_1(A_1) \otimes_{\mathrm{s}} \pi_2(A_2)$, and extended by continuity to all of $\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2$.

2.3. **Invertible states.** While stacking is a general operation on any spin systems, it is central in Kitaev's definition of the class of invertible states [Kit13]. As we shall see, an immediate corollary of our result on the superselection sectors of stacked systems is that invertible states have trivial sectors.

Definition 2.7 ([NO22])**.** A state ω of a quantum spin system $\mathfrak{A}$ is called a product state for the cone Λ if it is of the form

$$\omega = \omega_\Lambda \otimes \omega_{\Lambda^c}.$$

where ω_Λ (resp. ω_{Λ^c}) is a state of $\mathfrak{A}(\Lambda)$ (resp. $\mathfrak{A}(\Lambda^c)$).

Note that the factors ω_Λ and ω_{Λ^c} uniquely determine the product state ω.

An important result regarding product states for a cone is that they have trivial superselection theory [NO22, Theorem 4.5].

Theorem 2.8. *Let ω_0 be a pure state such that its GNS representation π_0 is quasi-equivalent to $\pi_\Lambda \otimes \pi_{\Lambda^c}$. Then the corresponding sector theory is trivial, in that every representation π satisfying SSC(π_0) is quasi-equivalent to π_0.*

Since the superselection theory is stable under application of an LGA, the same is true for any state that can be connected to a product state (for cones) using such an automorphism, and such states do not host anyons. The application of this to invertible states is direct, as below.

Definition 2.9. A state ω of a quantum spin system $\mathfrak{A}$ is called invertible if there exists a quantum spin system $\bar{\mathfrak{A}}$ and a state $\bar{\omega}$ of $\bar{\mathfrak{A}}$, an LGA α of $\mathfrak{A} \otimes_{\mathrm{s}} \bar{\mathfrak{A}}$, and a cone Λ such that $(\omega \otimes_{\mathrm{s}} \bar{\omega}) \circ \alpha$ is a product state for the cone Λ.

We shall say that $\bar{\omega}$ is an inverse of ω. Explicitly, this reads

$$(\omega \otimes_{\mathrm{s}} \bar{\omega}) \circ \alpha = \phi_\Lambda \otimes \phi_{\Lambda^c}$$

where ϕ_Λ (resp. ϕ_{Λ^c}) is a state of $(\mathfrak{A} \otimes_{\mathrm{s}} \bar{\mathfrak{A}})(\Lambda)$ (resp. $(\mathfrak{A} \otimes_{\mathrm{s}} \bar{\mathfrak{A}})(\Lambda^c)$). We emphasise the difference between the tensor products of the left and right hand side of the equation: the left-hand side refers to the stacking of two layers, where the right-hand side corresponds to a geometric partition into a cone and its complement.

Remark 2.10. Note that we only need to assume that $(\omega \otimes_s \overline{\omega}) \circ \alpha$ is a product state with respect to a *single* cone Λ. This is sufficient because the superselection criterion is already a very strong condition, and allows us to localize the representation in the cone Λ.

3. Superselection sectors of stacked systems

We now are in a position to discuss the main results of our paper, namely the analysis of the superselection sector theory of stacked systems. Recall that an important class of examples we want to apply the theory to is that of ground states of gapped finite range interactions. Observe that if ω_1, ω_2 are pure gapped ground states of finite range interactions Φ_1, Φ_2, then $\omega = \omega_1 \otimes_{\mathrm{s}} \omega_2$ is a pure state and is the gapped ground state of the finite range interaction defined by $\Phi(\Lambda) = \Phi_1(\Lambda) \otimes_{\mathrm{s}} \mathbb{I} + \mathbb{I} \otimes_{\mathrm{s}} \Phi_2(\Lambda)$. In particular, its GNS representation π is irreducible.

3.1. Well-definedness of the sector theory for stacked systems.

In order to be able to discuss the superselection sectors of stacked systems, we first show that the stacked system again satisfies our assumptions. In particular, approximate Haag duality is inherited from the individual layers.

Used frequently below is a result regarding the commutants of tensor products of C*-algebras, which we state here for convenience of the reader. The proof can be found in [Tak02, Prop IV.4.13].

Lemma 3.1. *Let $\mathcal{A}_i \subset B(\mathcal{H}_i)$ be a unital C*-algebra for $i = 1, 2$ and $\overline{\otimes}$ the usual von Neumann-algebraic tensor product. Then*

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)'' = \mathcal{A}_1'' \,\overline{\otimes}\, \mathcal{A}_2''.$$

Equivalently,

$$(\mathcal{A}_1 \otimes \mathcal{A}_2)' = (\mathcal{A}_1'' \,\overline{\otimes}\, \mathcal{A}_2'')'.$$

As expected, approximate Haag duality is preserved under stacking.

Theorem 3.2. *Let π_1, π_2 be irreducible representations of $\mathfrak{A}_1, \mathfrak{A}_2$ on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively, and assume they both satisfy approximate Haag duality. Then $\pi_1 \otimes_{\mathrm{s}} \pi_2$ also satisfies approximate Haag duality.*

Proof. Each π_i is assumed to satisfy approximate Haag duality, and so let $R_i, f_{\delta_i}(t),$ and $U_i : \mathcal{H}_i \to \mathcal{H}_i$ be the corresponding values, decreasing functions, and unitaries, see Definition 2.4.

Set $R = \max R_i$. Then we have the following: for any cone Λ,

$$\begin{aligned}
(\pi_1 \otimes_{\mathrm{s}} \pi_2\left((\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)(\Lambda^c)\right))' &= (\pi_1(\mathfrak{A}_1(\Lambda^c)) \otimes_{\mathrm{s}} \pi_2(\mathfrak{A}_2(\Lambda^c)))' \\
&= (\pi_1(\mathfrak{A}_1(\Lambda^c))'' \,\overline{\otimes}\, \pi_2(\mathfrak{A}_2(\Lambda^c))'')' \\
&= (\pi_1(\mathfrak{A}_1(\Lambda^c))' \,\overline{\otimes}\, \pi_2(\mathfrak{A}_2(\Lambda^c))')'' \\
&\subset (\mathrm{Ad}(U_1)\pi_1(\mathfrak{A}_1(\Lambda - R\mathbf{e}_\Lambda))'' \,\overline{\otimes}\, \mathrm{Ad}(U_2)\pi_2(\mathfrak{A}_2(\Lambda - R\mathbf{e}_\Lambda)'')'' \\
&= \mathrm{Ad}(U_1 \otimes_{\mathrm{s}} U_2)(\pi_1 \otimes_{\mathrm{s}} \pi_2((\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)(\Lambda - R\mathbf{e}_\Lambda)))''.
\end{aligned}$$

We first used Lemma 3.1, then the commutation theorem for tensor products of von Neumann algebras (see [KR97, Theorem 11.2.16]), while the inclusion is the definition of approximate Haag duality. The last equality follows again from the lemma. This shows condition (i) of the duality.

We turn to condition (ii). Let $\widetilde{U}_1, \widetilde{U}_2$ be the unitaries given since π_1, π_2 satisfy approximate Haag duality. Let $\delta = \max \delta_i$. Then $\widetilde{U}_1 \otimes_{\mathrm{s}} \widetilde{U}_2 \in (\pi_1 \otimes_{\mathrm{s}} \pi_2((\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)(\Lambda_{\varepsilon+\delta} - t\mathbf{e}_\Lambda)))''$, again by Lemma 3.1. Moreover,

$$\begin{aligned}
&||(U_1 \otimes_{\mathrm{s}} U_2) - (\widetilde{U}_1 \otimes_{\mathrm{s}} \widetilde{U}_2)|| \\
&\leq ||(U_1 \otimes_{\mathrm{s}} U_2) - (\widetilde{U}_1 \otimes_{\mathrm{s}} U_2)|| + ||(\widetilde{U}_1 \otimes_{\mathrm{s}} U_2) - (\widetilde{U}_1 \otimes_{\mathrm{s}} \widetilde{U}_2)|| \\
&\leq ||(U_1 - \widetilde{U}_1)|| \cdot ||U_2|| + ||\widetilde{U}_1|| \cdot ||(U_2 - \widetilde{U}_2)|| < f_{\delta_1}(t) + f_{\delta_2}(t)
\end{aligned}$$

and the function $f_\delta(t) = f_{\delta_1}(t) + f_{\delta_2}(t)$ is a decreasing function with $\lim_{t\to\infty} f_\delta(t) = 0$. □

With this, we conclude that the sector theory for the stacked system is well-defined if it is well-defined for each layer:

Proposition 3.3. *Assume that $\pi_j, j = 1, 2$, satisfy Assumptions 2.3 and 2.5. Then $\pi_1 \otimes_{\mathrm{s}} \pi_2$ satisfies Assumptions 2.3 and 2.5.*

Proof. We have just proved that the stacked representation satisfies Assumption 2.5. The properly infinite factor Assumption 2.3 follows immediately from Lemma 3.1. □

3.2. **Tensor decomposition of the stacked sectors.** Our first main result says that for every sector of the stacked system, we get sectors for each of the layers by restricting the representation.

We first recall a standard result.

Proposition 3.4. *For $j = 1, 2$, let the GNS representation of the state ω_j of $\mathfrak{A}_j$ be given as $(\mathcal{H}_j, \pi_j, \Omega_j)$. Then $(\mathcal{H}_1 \otimes \mathcal{H}_2, \pi_1 \otimes_{\mathrm{s}} \pi_2, \Omega_1 \otimes \Omega_2)$ is the GNS representation of the product state $\omega_1 \otimes_{\mathrm{s}} \omega_2$.*

Proof. It suffices to check that $\omega_1 \otimes_{\mathrm{s}} \omega_2(\cdot) = \langle \Omega_1 \otimes \Omega_1, (\pi_1 \otimes_{\mathrm{s}} \pi_2)(\cdot)(\Omega_1 \otimes \Omega_2)\rangle$ and that $\Omega_1 \otimes \Omega_2$ is cyclic under the action of $\pi_1 \otimes_{\mathrm{s}} \pi_2(\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)$. Indeed,

$$\begin{aligned}
\langle \Omega_1 \otimes \Omega_2, &(\pi_1 \otimes_{\mathrm{s}} \pi_2)(A_1 \otimes A_2)(\Omega_1 \otimes \Omega_2)\rangle = \langle \Omega_1 \otimes \Omega_2, \pi_1(A_1)\Omega_1 \otimes \pi_2(A_2)\Omega_2\rangle \\
&= \langle \Omega_1, \pi_1(A_1)\Omega_1\rangle \langle \Omega_1, \pi_2(A_2)\Omega_2\rangle = \omega_1(A_1)\omega_2(A_2) = (\omega_1 \otimes_{\mathrm{s}} \omega_2)(A_1 \otimes A_2)
\end{aligned}$$

for all $A_i \in \mathfrak{A}_j, j = 1, 2$. Moreover, since $\mathcal{H}_j = \overline{\pi_j(\mathfrak{A}_j)\Omega_j}$, it is clear that the Hilbert space $\overline{\pi_1 \otimes_{\mathrm{s}} \pi_2(\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)(\Omega_1 \otimes \Omega_2)}$ is the closed subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ that is generated by the pure tensors; namely, all of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Hence, $\Omega_1 \otimes \Omega_2$ is cyclic. □

Remark 3.5. Note that unlike in the case of representations of groups, the tensor product of irreducible representations of C*-algebras is again irreducible. The reason for this is that a representation is irreducible if and only if the von Neumann algebra generated by its image is the entire collection of bounded operators. Thus, for irreducible representations $\rho_i : \mathfrak{B}_i \to B(\mathcal{K}_i)$, $i = 1, 2$, we have

$$(\rho_1(\mathfrak{B}_1) \otimes \rho_2(\mathfrak{B}_2))'' = \rho_1(\mathfrak{B}_1)'' \overline{\otimes} \rho_2(\mathfrak{B}_2)'' = B(\mathcal{K}_1) \overline{\otimes} B(\mathcal{K}_2) = B(\mathcal{K}_1 \otimes \mathcal{K}_2),$$

where the first equality follows from Lemma 3.1 and the last follows from the commutation theorem for tensor products of von Neumann algebras.

An operator $A_1 \in \mathfrak{A}_1$ can be extended to an operator of the stacked system by stacking with the identity: $A_1 \otimes_{\mathrm{s}} \mathbb{I} \in \mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2$. In a similar fashion, we can restrict a representation of the stacked system to a representation of one layer. The natural question is then whether this new restricted representation satisfies the SSC w.r.t. the corresponding tensor product factor.

Theorem 3.6. *Let σ be an irreducible representation satisfying SSC(π), where again $\pi = \pi_1 \otimes_{\mathrm{s}} \pi_2$, and let $\sigma_1 := \sigma \restriction \mathfrak{A}_1$, namely $\sigma_1(A) = \sigma(A \otimes_{\mathrm{s}} \mathbb{I})$ for all $A \in \mathfrak{A}_1$. Then σ_1 satisfies SSC(π_1). Similarly, we get a representation $\sigma_2 := \sigma \restriction \mathfrak{A}_2$ satisfying SSC(π_2).*

Before we give the proof of this theorem we will note some useful properties of this restricted representation. Note that we will be in the setting of Theorem 3.11 for Lemmas 3.7 through 3.10. Clearly, analogous statements hold for σ_2.

Lemma 3.7. *The representation σ_1 satisfies the following properties:*

(i) σ_1 is a factor representation.
(ii) For any cone Λ, $\sigma_1 \restriction \mathfrak{A}_1(\Lambda)$ is a factor representation.

Proof. To prove (i), note that $\sigma(\mathbb{I} \otimes_{\mathrm{s}} \mathfrak{A}_2) \subset \sigma_1(\mathfrak{A}_1)'$ since observables in different layers commute. Hence we have

$$\sigma_1(\mathfrak{A}_1) \vee \sigma_1(\mathfrak{A}_1)' \supset \sigma(\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathbb{I}) \vee \sigma(\mathbb{I} \otimes_{\mathrm{s}} \mathfrak{A}_2) = \sigma(\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)'' = B(\mathcal{H}),$$

where $\sigma_1(\mathfrak{A}_1) \vee \sigma_1(\mathfrak{A}_1)'$ denotes the smallest von Neumann algebra generated by $\sigma_1(\mathfrak{A}_1)$ and $\sigma_1(\mathfrak{A}_1)'$. Thus, $\sigma_1(\mathfrak{A}_1)'' \vee \sigma_1(\mathfrak{A}_1)' = B(\mathcal{H})$, or equivalently, $\sigma_1(\mathfrak{A}_1)'' \cap \sigma_1(\mathfrak{A}_1)' = \mathbb{C}\mathbb{I}$.

The proof of (ii) is similar, using $\mathfrak{A}_1 = \mathfrak{A}_1(\Lambda) \otimes \mathfrak{A}_1(\Lambda^c)$, and $\sigma_1(\mathfrak{A}_1(\Lambda^c)) \subset \sigma_1(\mathfrak{A}_1(\Lambda))'$. □

Lemma 3.8. *For any cone Λ, we have that $\sigma_1 \restriction \mathfrak{A}_1(\Lambda^c) \simeq_{q.e.} \pi_1 \restriction \mathfrak{A}_1(\Lambda^c)$, where $\simeq_{q.e.}$ denotes quasi-equivalence of representations.*

Proof. Note that $\pi_1 \restriction \mathfrak{A}(\Lambda^c)$ is quasi-equivalent to $\mathfrak{A}(\Lambda^c) \ni A \mapsto \pi_1(A) \otimes_{\mathrm{s}} \mathbb{I}$, since the latter is equivalent to a direct sum of countably many copies of $\pi_1 \restriction \mathfrak{A}(\Lambda^c)$. By the superselection criterion, there is some unitary U_Λ such that $U_\Lambda \sigma(A) U_\Lambda^* = \pi(A)$ for each $A \in \mathfrak{A}(\Lambda^c) = \mathfrak{A}_1(\Lambda^c) \otimes_{\mathrm{s}} \mathfrak{A}_2(\Lambda^c)$. In particular, for $A \in \mathfrak{A}_1(\Lambda^c)$, we have

$$U_\Lambda \sigma_1(A) U_\Lambda^* = U_\Lambda \sigma(A \otimes_{\mathrm{s}} \mathbb{I}) U_\Lambda^* = \pi(A \otimes_{\mathrm{s}} \mathbb{I}) = \pi_1(A) \otimes_{\mathrm{s}} \mathbb{I}.$$

Hence, σ_1 is unitarily equivalent to $\mathfrak{A}_1(\Lambda^c) \ni A \mapsto \pi_1(A) \otimes_{\mathrm{s}} \mathbb{I}$, and so $\sigma_1 \restriction \mathfrak{A}_1(\Lambda^c) \simeq_{q.e.} \pi_1 \restriction \mathfrak{A}_1(\Lambda^c)$. □

Corollary 3.9. *Every subrepresentation of $\pi_1 \restriction \mathfrak{A}_1(\Lambda^c)$ is quasi-equivalent to every subrepresentation of $\sigma_1 \restriction \mathfrak{A}_1(\Lambda^c)$.*

Proof. Immediate from Proposition 10.3.12 of [KR97]. □

Lemma 3.10. *The von Neumann algebras $\sigma_1(\mathfrak{A}_1(\Lambda^c))'$ and $\pi_1(\mathfrak{A}_1(\Lambda^c))'$ are properly infinite.*

Proof. The von Neumann algebra $\pi_2(\mathfrak{A}_2(\Lambda^c))''$ is properly infinite by assumption, and hence so is $\mathbb{I} \otimes_{\mathrm{s}} \pi_2(\mathfrak{A}_2(\Lambda^c))''$. Let U_Λ be the unitary from Theorem 3.11, i.e. for which $U_\Lambda \sigma(A) U_\Lambda^* = \pi(A)$ for all $A \in \mathfrak{A}(\Lambda^c)$. It follows that $U_\Lambda^*(\mathbb{I} \otimes_{\mathrm{s}} \pi_2(\mathfrak{A}_2(\Lambda^c)))'' U_\Lambda$ is also properly infinite. Then for $A_1 \in \mathfrak{A}_1(\Lambda^c), A_2 \in \mathfrak{A}_2(\Lambda^c)$,

$$\begin{aligned}\sigma_1(A_1) U_\Lambda^*(\mathbb{I} \otimes_{\mathrm{s}} \pi_2(A_2)) U_\Lambda &= \sigma(A_1 \otimes_{\mathrm{s}} \mathbb{I}) \sigma(\mathbb{I} \otimes_{\mathrm{s}} A_2) \\ &= \sigma(\mathbb{I} \otimes_{\mathrm{s}} A_2) \sigma(A_1 \otimes_{\mathrm{s}} \mathbb{I}) = U_\Lambda^*(\mathbb{I} \otimes_{\mathrm{s}} \pi_2(A_2)) U_\Lambda \sigma_1(A_1),\end{aligned}$$

and so $U_\Lambda^*(\mathbb{I} \otimes_{\mathrm{s}} \pi_2(\mathfrak{A}_2(\Lambda^c))) U_\Lambda \subset \sigma_1(\mathfrak{A}_1(\Lambda^c))'$. Taking double commutants of both sides yields that $\sigma_1(\mathfrak{A}_1(\Lambda^c))'$ contains a properly infinite subalgebra, and therefore is properly infinite.

Now choose a cone $\widehat{\Lambda}$ for which $\Lambda^c \cap \widehat{\Lambda} = \emptyset$. By locality, $\pi_1(\mathfrak{A}_1(\Lambda^c)) \subset \pi_1(\mathfrak{A}_1(\widehat{\Lambda}))'$, and so $\pi_1(\mathfrak{A}_1(\widehat{\Lambda}))'' \subset \pi_1(\mathfrak{A}_1(\Lambda^c))'$. Since $\pi_1(\mathfrak{A}_1(\widehat{\Lambda}))''$ is properly infinite by assumption, $\pi_1(\mathfrak{A}_1(\Lambda^c))'$ must be as well. □

These preliminary results are enough to see that the restriction of a representation to one layer of the stacked system does indeed satisfy the SSC(π_1).

Proof of Theorem 3.6. Let Λ be any cone. By Lemma 3.8, it follows that $\sigma_1 \restriction \mathfrak{A}_1(\Lambda^c) \simeq_{q.e.} \pi_1 \restriction \mathfrak{A}_1(\Lambda^c)$. There is therefore a $*$-isomorphism

$$\tau_\Lambda : \sigma_1(\mathfrak{A}_1(\Lambda^c))'' \to \pi_1(\mathfrak{A}(\Lambda^c))''$$

such that $\tau_\Lambda(\sigma_1(A)) = \pi_1(A)$ for all $A \in \mathfrak{A}_1(\Lambda^c)$. But since $\sigma_1(\mathfrak{A}_1(\Lambda^c))'$ and $\pi_1(\mathfrak{A}_1(\Lambda^c))'$ are both properly infinite, τ_Λ is unitarily implemented by [SZ19, Corollary 8.12] (see also Exercise 9.6.32 of [KR97]), namely $\pi_1 \restriction \mathfrak{A}_1(\Lambda^c) \cong \sigma_1 \restriction \mathfrak{A}_1(\Lambda^c)$. This concludes the proof of the theorem since this holds for all cones. □

3.3. **The superselection sectors of a stacked system.** We are now in a position to prove, by utilising the results above, our first two main theorems. The first theorem says that we can construct superselection sectors of the stacked system from sectors of the individual layers as expected.

Theorem 3.11 (Theorem A)**.** *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be quantum spin systems on Γ and suppose that we have irreducible reference representations π_1 and π_2 satisfying Assumptions 2.3 and 2.5. Then $\pi_1 \otimes_{\mathrm{s}} \pi_2$ also satisfies the assumptions. If ρ_1 and ρ_2 satisfy the superselection criterion for π_1 and π_2 respectively, then $\rho_1 \otimes_{\mathrm{s}} \rho_2$ satisfies the superselection criterion for $\pi_1 \otimes_{\mathrm{s}} \pi_2$.*

Proof. The first claim is Proposition 3.3. Let Λ be a cone and U_i be a unitary implementing the equivalence from the superselection criterion, i.e.

$$U_i \rho_i(A_i) = \pi_i(A_i) U_i$$

for all $A_i \in \mathfrak{A}_i(\Lambda^c)$ and $i = 1, 2$. Then clearly

$$U_1 \otimes U_2(\rho_1 \otimes \rho_2(A_1 \otimes_{\mathrm{s}} A_2)) = (\pi_1 \otimes \pi_2(A_1 \otimes_{\mathrm{s}} A_2)) U_1 \otimes U_2$$

for all $A_1 \otimes_{\mathrm{s}} A_2 \in (\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)(\Lambda^c)$, so $\rho_1 \otimes \rho_2$ satisfies SSC($\pi_1 \otimes_{\mathrm{s}} \pi_2$). □

Our second main result concerns the inverse of this construction. In particular, a natural question is which sectors of the stacked system come from stacking sectors in the individual layers. Note that Theorem 3.6 gives (representatives of) sectors σ_1 and σ_2 for each of the layers, but in general $\sigma \ncong \sigma_1 \otimes_{\mathrm{s}} \sigma_2$.[1] However, if σ is irreducible, this does turn out to be the case, provided the Type I condition holds.

The following proposition is seen as Example 11.2.5 in [KR97], and hence we point the reader towards the source and omit the proof.

Proposition 3.12. *A factor $\mathcal{R} \subset B(\mathcal{K})$ is Type I if and only if there exist Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ and a unitary $U : \mathcal{K} \to \mathcal{K}_1 \otimes \mathcal{K}_2$ such that $U\mathcal{R}U^* = B(\mathcal{K}_1) \otimes \mathbb{I}_{\mathcal{K}_2}$.*

Theorem 3.13 (Theorem B). *Let π_1 and π_2 be as above and assume that every factor representation σ_1 satisfying the superselection criterion for π_1 is such that $\sigma_1(\mathfrak{A}_1)''$ is a Type I factor. Let $\pi = \pi_1 \otimes_{\mathrm{s}} \pi_2$. If ρ is an irreducible representation that satisfies SSC(π), then $\rho \cong \rho_1 \otimes_{\mathrm{s}} \rho_2$ where ρ_1, ρ_2 are irreducible representations satisfying SSC(π_1), SSC(π_2) respectively.*

Proof. Let $\rho : \mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2 \to B(\mathcal{H})$ be an irreducible representation satisfying SSC(π). For $i = 1, 2$, let $\sigma_i = \rho \restriction \mathfrak{A}_i$, namely $\sigma_1(A_1) = \rho(A_1 \otimes_{\mathrm{s}} \mathbb{I})$ and $\sigma_2(A_2) = \rho(\mathbb{I} \otimes_{\mathrm{s}} A_2)$ for all $A_i \in \mathfrak{A}_i$. By Theorem 3.6 and Lemma 3.7, σ_1 is a factor representation which satisfies SSC(π_1), and thus by our assumption we have that $\sigma_1(\mathfrak{A}_1)''$ is a Type I factor. Therefore there are Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ and a unitary $U : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2$ such that

$$U\sigma_1(\mathfrak{A}_1)'' = (B(\mathcal{K}_1) \otimes \mathbb{I}_{\mathcal{K}_2})U.$$

Since $\sigma_2(\mathfrak{A}_2) \subset \sigma_1(\mathfrak{A}_1)'$, we have

$$U\sigma_2(\mathfrak{A}_2)U^* \subset U\sigma_1(\mathfrak{A}_1)'U^* = (U\sigma_1(\mathfrak{A}_1)''U^*)' = (B(\mathcal{K}_1) \otimes \mathbb{I}_{\mathcal{K}_1})' = \mathbb{I}_{\mathcal{K}_1} \otimes B(\mathcal{K}_2),$$

where the last equality follows from the commutation theorem for tensor products of von Neumann algebras.

For $A_1 \in \mathfrak{A}_1$ and $A_2 \in \mathfrak{A}_2$, this implies that there exist $B_1 \in B(\mathcal{K}_1)$ and $B_2 \in B(\mathcal{K}_2)$ such that $U\sigma_1(A_1)U^* = B_1 \otimes \mathbb{I}_{\mathcal{K}_2}$ and $U\sigma_2(A_2)U^* = \mathbb{I}_{\mathcal{K}_1} \otimes B_2$. Thus, we define $\rho_i : \mathfrak{A}_i \to B(\mathcal{K}_i)$ as

$$\rho_1(A_1) = B_1 \text{ and } \rho_2(A_2) = B_2.$$

It is easy to see that ρ_1 and ρ_2 are indeed representations, and

$$U\rho(A_1 \otimes_{\mathrm{s}} A_2)U^* = U\sigma_1(A_1)U^*U\sigma_2(A_2)U^* = B_1 \otimes_{\mathrm{s}} B_2 = \rho_1(A_1) \otimes_{\mathrm{s}} \rho_2(A_2),$$

so $\rho \cong \rho_1 \otimes_{\mathrm{s}} \rho_2$. Moreover, since ρ is irreducible and for a $T \in \rho_1(\mathfrak{A}_1)'$ we have $U^*(T \otimes_{\mathrm{s}} \mathbb{I}_{\mathcal{K}_2})U \in \rho(\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2)'$, then ρ_1 must also be irreducible and similarly for ρ_2.

Finally, for any cone Λ, Theorem 3.6 yields a unitary V_Λ such that $V_\Lambda^* \sigma_1(A) V_\Lambda = \pi_1(A)$ for all $A \in \mathfrak{A}_1(\Lambda^c)$, and by the above,

$$(UV_\Lambda)\pi_1(A)(UV_\Lambda)^* = \rho_1(A) \otimes \mathbb{I}_{\mathcal{K}_2},$$

namely, $\pi_1 \restriction \mathfrak{A}_1(\Lambda^c)$ and $\rho_1 \restriction \mathfrak{A}_1(\Lambda^c)$ are quasi-equivalent. We want to show that this is in fact a unitary equivalence. By quasi-equivalence, there exists a $*$-isomorphism $\tau : \rho_1(\mathfrak{A}_1(\Lambda^c))'' \to \pi_1(\mathfrak{A}_1(\Lambda^c))''$. By replacing Λ^c with Λ, it also follows that $\rho_1(\mathfrak{A}_1(\Lambda))''$ and $\pi_1(\mathfrak{A}_1(\Lambda))''$ are $*$-isomorphic. Since the latter is properly infinite by assumption, so is $\rho_1(\mathfrak{A}_1(\Lambda))''$. From

[1] This is analogous to the observation that a quantum state on a bipartite system in general cannot be recovered from its reduced density matrices.

locality it follows that $\rho_1(\mathfrak{A}_1(\Lambda))'' \subset \rho_1(\mathfrak{A}_1(\Lambda^c))'$, hence the right-hand side of the inclusion is also a properly infinite von Neumann algebra. By Lemma 3.10, $\pi_1(\mathfrak{A}_1(\Lambda^c))'$ is properly infinite, and hence τ is unitarily implemented, again by [SZ19, Corollary 8.12]. Therefore, $\pi_1 \upharpoonright \mathfrak{A}_1(\Lambda^c) \cong \rho_1 \upharpoonright \mathfrak{A}_1(\Lambda^c)$ and ρ_1 satisfies SSC(π_1). $\square$

3.4. **Type I property.** In this section we show that the assumption made in Theorem B are true under natural conditions. We will prove that if

(i) π_1 satisfies the *approximate split property*, Definition 3.14, and
(ii) π_1 has at most countably many super-selection sectors,

then every factor representation ρ_1 satisfying SSC(π_1) is Type I. By this, we mean the von Neumann algebra $\rho_1(\mathfrak{A}_1)''$ is of Type I. These conditions are in fact satisfied by a wide variety of physical models of interest. For example, (i) is satisfied for every quantum double model and is stable under the application of any locally generated automorphism (see Section 3.5 for the validity of the split property and its relation with the area law for the entanglement entropy), and (ii) must be satisfied for any physically realistic model, since we would not expect a physical system to give rise to uncountably many particle types (and indeed, in axiomatizations of anyon theories, the number of anyon types is usually assumed to be finite, see e.g. [Kit06]).

The key technical step in showing this result is, following an approach by [KLM01], a decomposition result of representations satisfying the superselection criterion. For any irreducible representation ρ of $\mathfrak{A}$ that satisfies SSC(π), the restriction $\rho_1 = \rho \upharpoonright \mathfrak{A}_1$ again satisfies SSC(π_1), see Theorem 3.6. The representation ρ_1 will certainly not be irreducible, since its image has a non-trivial commutant, however it can be decomposed into a direct integral of irreducible representations,

$$\rho_1 \cong \int_X^{\oplus} \rho_x d\mu(x)$$

for some measure space (X, Σ, μ), see e.g. [Tak02, Thm. IV.8.32]. The question remains whether each ρ_x (or at least all ρ_x for x outside of some zero-measure set) satisfies SSC(π_1). Theorem 3.15 below shows that when π_1 satisfies the approximate split property, this is indeed the case.

Definition 3.14 (Approximate split property)**.** Let $\mathfrak{B}$ be a quasi-local algebra on a planar lattice Γ, and let σ be a representation. Let $d > 0$. For two cones $\Lambda_1 \subset \Lambda_2$, we shall write $\Lambda_1 \Subset \Lambda_2$ if Λ_2 opens at a wider angle than Λ_1 and $\mathrm{dist}(\Lambda_1, \Lambda_2^c) > d$. We say σ satisfies the *approximate split property* if for any pair of cones $\Lambda_1 \Subset \Lambda_2$, there exists a Type I factor $\mathcal{N}$ satisfying

$$\sigma(\mathfrak{B}(\Lambda_1))'' \subset \mathcal{N} \subset \sigma(\mathfrak{B}(\Lambda_2))''.$$

Note that the definition implicitly depends on d, which will be model-dependent. It holds for example for the ground state representation of abelian quantum double models with $d = 1$ [FN15]. In the next section we show that it holds for a wide class of models of interest.

Theorem 3.15. *Suppose φ is an irreducible representation of $\mathfrak{B}$ satisfying the approximate split property, $\varphi(\mathfrak{B}(\Lambda))''$ is properly infinite for every cone Λ, and $\rho : \mathfrak{B} \to B(\mathcal{H})$ is a representation satisfying SSC(φ). Let (X, Σ, μ) be a measure space and $\mathcal{H}_x$ a Hilbert space* *for each $x \in X$ so that*

$$\mathcal{H} \cong \int_X^{\oplus} \mathcal{H}_x d\mu(x)$$

is a direct integral decomposition of $\mathcal{H}$ for which

$$\rho \cong \int_X^{\oplus} \rho_x d\mu(x)$$

is a decomposition of ρ into irreducible representations $\rho_x : \mathfrak{B} \to B(\mathcal{H}_x)$. Then ρ_x satisfies SSC(φ) for μ-almost every $x \in X$.

Before proving this result, we make some comments and show some helpful lemmas. To simplify notation, we assume without loss of generality that φ and ρ are represented on the same Hilbert space $\varphi, \rho : \mathfrak{B} \to B(\mathcal{H})$.[2] We also identify $\mathfrak{B}$ with its image under φ, so that $\mathfrak{B} \equiv \varphi(\mathfrak{B})$ and $\mathfrak{B}(X) \equiv \varphi(\mathfrak{B}(X))$ for every $X \subset \Gamma$. This means that studying representations of $\mathfrak{B}$ satisfying SSC(φ) is equivalent to studying $*$-homomorphisms $\mathfrak{B} \to B(\mathcal{H})$ that can be unitarily implemented when restricted to any cone.

We begin with some preliminaries. Choose a cone Λ_a and define the *auxiliary algebra* as

$$\mathfrak{B}^{\Lambda_a} = \overline{\bigcup_{x \in \mathbb{R}^2} \mathfrak{B}(\Lambda_a^c + x)''}^{\|\cdot\|}. \tag{3.1}$$

There is an equivalent way of describing the auxiliary algebra by specifying a family of 'admissible cones'. Denote by $\mathcal{C}$ the set of all cones in $\mathbb{R}^2$. For a choice of cone Λ_a, define the family of admissible cones with respect to this choice by

$$\mathcal{C}^{\Lambda_a} = \{\Lambda \in \mathcal{C} \mid \exists x \in \mathbb{R}^2 \text{ such that } (\Lambda + x) \cap \Lambda_a = \emptyset\}. \tag{3.2}$$

Then $\mathfrak{B}^{\Lambda_a}$ is the C*-algebra generated by the sub-algebras $\mathfrak{B}(\Lambda)''$ as Λ ranges over $\mathcal{C}^{\Lambda_a}$.

If ρ is a representation of $\mathfrak{B}$ satisfying SSC(φ), then by definition for any cone $\Lambda \in \mathcal{C}$, $\rho_\Lambda := \rho|_{\mathfrak{B}(\Lambda)}$ is unitarily implemented, and therefore extends to a normal (namely, σ-weakly continuous) representation of $\mathfrak{B}(\Lambda)''$. By letting Λ range over every cone in $\mathcal{C}^{\Lambda_a}$ and completing with respect to the norm we obtain a representation of $\mathfrak{B}^{\Lambda_a}$ that restricts to a normal representation on $\mathfrak{B}(\Lambda)''$ for every $\Lambda \in \mathcal{C}^{\Lambda_a}$. The representation obtained from ρ in this way is referred to as the representation of $\mathfrak{B}^{\Lambda_a}$ generated by ρ. If some representation of $\mathfrak{B}^{\Lambda_a}$ restricts to a representation of $\mathfrak{B}$ that satisfies SSC(φ), we call it localizable. It is easy to see that the localizable representations of $\mathfrak{B}^{\Lambda_a}$ are precisely those that are generated by representations of $\mathfrak{B}$ satisfying SSC(φ).

Definition 3.16. (i) Let $S \subset \mathcal{C}$ be a set of cones in Γ. A family of representations $\rho_\Lambda : \mathfrak{B}(\Lambda)'' \to B(\mathcal{H})$, for $\Lambda \in S$, is a *locally normal family* of representations with respect to S if

- ρ_Λ is normal for each $\Lambda \in S$
- $\rho_\Lambda = \rho_{\tilde{\Lambda}}|_{\mathfrak{B}(\Lambda)''}$ whenever $\Lambda \subset \tilde{\Lambda}$

(ii) A representation of the auxiliary algebra $\mathfrak{B}^{\Lambda_a}$ is a *locally normal representation* if it restricts on cones in $\mathcal{C}^{\Lambda_a}$ to a locally normal family of representations with respect to $\mathcal{C}^{\Lambda_a}$.

[2]This can be done because our representations satisfy the superselection criterion.

We immediately note the relation between the two notions just introduced. Every locally normal family of representations with respect to $\mathcal{C}^{\Lambda_a}$ generates a locally normal representation of $\mathfrak{B}^{\Lambda_a}$ by ranging over the cones in $\mathcal{C}^{\Lambda_a}$ and completing with respect to the norm. Reciprocally, in fact by definition, every locally normal representation of $\mathfrak{B}^{\Lambda_a}$ restricts on cones in $\mathcal{C}^{\Lambda_a}$ to a locally normal family of representations with respect to $\mathcal{C}^{\Lambda_a}$. Thus, the notions of a locally normal representation and a locally normal family of representations with respect to $\mathcal{C}^{\Lambda_a}$ are interchangeable. In fact, any representation of $\mathfrak{B}^{\Lambda_a}$ is locally normal provided it is on a sparable Hilbert space, see [Tak02, Thm. V.5.1].

For the purpose of this section, the interest of locally normal families is that they are unitarily implemented.

Lemma 3.17. *For any cone Λ and separable Hilbert space $\mathcal{H}$, a representation $\mathfrak{B}(\Lambda)'' \to B(\mathcal{H})$ is normal if and only if it is unitarily implemented. In particular, a locally normal family $\{\rho_\Lambda : \Lambda \in \mathcal{C}\}$ with respect to $\mathcal{C}$ gives rise to a representation of $\mathfrak{B}$ satisfying the SSC.*

Proof. We claim that $\mathfrak{B}(\Lambda)'$ is properly infinite for any $\Lambda \in \mathcal{C}$. Indeed, $\mathfrak{B}(\Lambda_1) \subset \mathfrak{B}(\Lambda)'$ for any disjoint cones Λ_1, Λ. Taking double commutants yields $\mathfrak{B}(\Lambda_1)'' \subset \mathfrak{B}(\Lambda)'$ and since $\mathfrak{B}(\Lambda_1)''$ is a properly infinite factor by assumption, so is $\mathfrak{B}(\Lambda)'$. With this, the claim is an immediate consequence of [Tak02, Proposition V.3.1], where σ-finiteness follows from separability of $\mathcal{H}$ and faithfulness follows from the fact that the kernel must be generated by a central projection, since it is a σ-weakly closed two sided ideal. $\square$

Remark 3.18. While every localizable representation of $\mathfrak{B}^{\Lambda_a}$, meaning it is generated by a representation of $\mathfrak{B}$ satisfying the SSC, is locally normal, the converse may in principle not be true as it may fail to be unitarily implementable on cones not in $\mathcal{C}^{\Lambda_a}$.

In order to proceed, we will need a suitable countable subset of the cones in $\mathcal{C}$. Note that the possible cones in the plane can be indexed by the points of the manifold

$$\mathcal{M} = \mathbb{R}^2 \times \mathbb{S}^1 \times (0, 2\pi),$$

where the locations of the vertex of each cone are given by $\mathbb{R}^2$, the orientations of the cones are labelled by $\mathbb{S}^1$, and the opening angle of the cones are in $(0, 2\pi)$. For any $p \in \mathcal{M}$, let Λ_p be its corresponding cone. Let $\mathcal{M}_\mathbb{Q} = \mathcal{M} \cap \mathbb{Q}^4$. Now define

$$\mathcal{C}_\mathbb{Q} = \{\Lambda_p \in \mathcal{C} \mid p \in \mathcal{M}_\mathbb{Q}\} \text{ and } \mathcal{C}_\mathbb{Q}^{\Lambda_a} = \{\Lambda_p \in \mathcal{C}^{\Lambda_a} \mid p \in \mathcal{M}_\mathbb{Q}\}.$$

Since a normal representation of the von Neumann algebra of a cone restricts to a normal representation of the von Neumann algebra of any sub-cone, it is easy to see that any locally normal family of representations with respect to $\mathcal{C}_\mathbb{Q}^{\Lambda_a}$ can be extended uniquely to a locally normal family of representations with respect to $\mathcal{C}^{\Lambda_a}$.

Definition 3.19. Let $\mathcal{N}$ be a Type I factor. The compact operators relative to $\mathcal{N}$ is the C*-subalgebra of $\mathcal{N}$ generated by the finite projections in $\mathcal{N}$.

Note that these are precisely the operators corresponding to the compact operators tensored with the identity under the unitary conjugation given in Proposition 3.12.

By the approximate split property, for each pair of cones $\Lambda_1 \Subset \Lambda_2$, let $\mathcal{N}(\Lambda_1, \Lambda_2)$ be a choice of Type I von Neumann factor satisfying

$$\mathfrak{B}(\Lambda_1)'' \subset \mathcal{N}(\Lambda_1, \Lambda_2) \subset \mathfrak{B}(\Lambda_2)''$$

and let $K(\Lambda_1, \Lambda_2)$ be the compact operators relative to $\mathcal{N}(\Lambda_1, \Lambda_2)$. For readers uncomfortable with the Axiom of Choice, we remark that the choice of $\mathcal{N}(\Lambda_1, \Lambda_2)$ can be made canonical, as shown in [DL84], however the exact choice is unimportant. We define a new unital C*-algebra $\mathfrak{K}^{\Lambda_a}$ to be generated by the identity and the algebras $K(\Lambda_1, \Lambda_2)$ for $\Lambda_1 \Subset \Lambda_2$ both belonging to $\mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$. Note that since $\mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ is countable and each $K(\Lambda_1, \Lambda_2)$ is separable, $\mathfrak{K}^{\Lambda_a}$ is separable as well.

Clearly, if $\Lambda_1 \Subset \Lambda_2 \subset \Lambda_3 \Subset \Lambda_4$, then $\mathcal{N}(\Lambda_1, \Lambda_2) \subset \mathcal{N}(\Lambda_3, \Lambda_4)$. However, we may not have that $K(\Lambda_1, \Lambda_2) \subset K(\Lambda_3, \Lambda_4)$. To remedy this issue, for $\Lambda_1 \Subset \Lambda_2$ we define

$$\mathfrak{L}(\Lambda_1, \Lambda_2) = \mathcal{N}(\Lambda_1, \Lambda_2) \cap \mathfrak{K}^{\Lambda_a}.$$

We therefore have the inclusions $K(\Lambda_1, \Lambda_2) \subset \mathfrak{L}(\Lambda_1, \Lambda_2) \subset \mathcal{N}(\Lambda_1, \Lambda_2)$, and also that $\mathfrak{K}^{\Lambda_a}$ is the C*-subalgebra of $\mathfrak{B}^{\Lambda_a}$ generated by the identity and the algebras $\mathfrak{L}(\Lambda_1, \Lambda_2)$ as $\Lambda_1 \Subset \Lambda_2$ run in $\mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$. Importantly, $\mathfrak{K}^{\Lambda_a}$ is not a C*-subalgebra of $\mathfrak{B}$, which is why it is necessary to carry out this analysis in the context of normal representations rather than restricting our attention to representations on $\mathfrak{B}$, even though it is the latter that we are really interested in.

Lemma 3.20. *Suppose* $\Lambda_1 \Subset \Lambda_2 \subset \Lambda_3 \Subset \Lambda_4$ *for* $\Lambda_1, ..., \Lambda_4 \in \mathcal{C}^{\Lambda_a}$. *Then*

$$\mathfrak{L}(\Lambda_1, \Lambda_2) \subset \mathfrak{L}(\Lambda_3, \Lambda_4).$$

Proof. Since $\mathfrak{L}(\Lambda_1, \Lambda_2) \subset \mathcal{N}(\Lambda_1, \Lambda_2) \subset \mathcal{N}(\Lambda_3, \Lambda_4)$, we have

$$\mathfrak{L}(\Lambda_1, \Lambda_2) = \mathcal{N}(\Lambda_1, \Lambda_2) \cap \mathfrak{K}^{\Lambda_a} \subset \mathcal{N}(\Lambda_3, \Lambda_4) \cap \mathfrak{K}^{\Lambda_a} = \mathfrak{L}(\Lambda_3, \Lambda_4).$$ □

The idea is to look at representations which are non-degenerate when restricted to the compact operators. The following lemma is Corollary 53 in [KLM01], where the proof can be found as well.

Lemma 3.21. *Let* $\mathcal{N}$ *be a Type I factor with separable predual,* $K \subset \mathcal{N}$ *the ideal of compact operators relative to* $\mathcal{N}$, *and* $\mathfrak{L}$ *a* C*-*algebra satisfying* $K \subset \mathfrak{L} \subset \mathcal{N}$. *If* ρ *is a representation of* $\mathfrak{L}$ *for which* $\rho|_K$ *is non-degenerate, then* ρ *is* σ-*weakly continuous and therefore extends uniquely to a normal representation of* $\mathcal{N}$.

We can now show that locally normal representations of $\mathfrak{B}^{\Lambda_a}$ are uniquely determined by there restrictions to $\mathfrak{K}^{\Lambda_a}$.

Lemma 3.22. *Let* ρ *be a locally normal representation of* $\mathfrak{B}^{\Lambda_a}$. *Then* ρ *restricts to a representation of* $\mathfrak{K}^{\Lambda_a}$ *and* $\rho|_{K(\Lambda_1,\Lambda_2)}$ *is non-degenerate for every pair* $\Lambda_1, \Lambda_2 \in \mathcal{C}^{\Lambda_a}$ *with* $\Lambda_1 \Subset \Lambda_2$.

Conversely, if σ *is a representation of* $\mathfrak{K}^{\Lambda_a}$ *such that* $\sigma|_{K(\Lambda_1,\Lambda_2)}$ *is non-degenerate for all* $\Lambda_1, \Lambda_2 \in \mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ *with* $\Lambda_1 \Subset \Lambda_2$, *then there exists a unique locally normal representation* $\widetilde{\sigma}$ *of* $\mathfrak{B}^{\Lambda_a}$ *that extends* σ.

Proof. We begin by showing that $\rho|_{K(\Lambda_1,\Lambda_2)}$ is non-degenerate. It is easy to see that the compact operators act non-degenerately on the Hilbert space on which they are defined, and so must the compact operators relative to any Type I factor. Thus, $K(\Lambda_1, \Lambda_1)$ must act non-degenerately on $\mathcal{H}$. Moreover, ρ is locally normal, and so as before, it follows from the assumption that our cone von Neumann algebras are properly infinite that ρ is implemented by conjugation with a unitary on $K(\Lambda_1, \Lambda_2)$. Since $K(\Lambda_1, \Lambda_)$ acts non-degenerately on $\mathcal{H}$, it must also do so after unitary conjugation. Hence, $\rho|_{K(\Lambda_1,\Lambda_2)}$ is non-degenerate.

We now show that a representation σ of $\mathfrak{K}^{\Lambda_a}$ extends to a locally normal representation $\tilde{\sigma}$ when $\sigma|_{K(\Lambda_1,\Lambda_2)}$ is non-degenerate for $\Lambda_1, \Lambda_2 \in \mathcal{C}^{\Lambda_a}$, $\Lambda_1 \Subset \Lambda_2$. For any such pair Λ_1, Λ_2, we denote by $\widetilde{\sigma}_{\Lambda_1,\Lambda_2}$ the unique normal extension of $\sigma|_{\mathfrak{L}(\Lambda_1,\Lambda_2)}$ to $\mathcal{N}(\Lambda_1, \Lambda_2)$ given by the previous lemma.

We will only show σ can be extended to a locally normal family of representations with respect to $\mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$, since this can always be extended to a family with respect to $\mathcal{C}^{\Lambda_a}$, and thus determines a locally normal representation of $\mathfrak{B}^{\Lambda_a}$. Given $\Lambda \in \mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$, choose $\Lambda_1 \in \mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ for which $\Lambda \Subset \Lambda_1$, and set

$$\widetilde{\sigma}_\Lambda = \widetilde{\sigma}_{\Lambda,\Lambda_1}|_{\mathfrak{B}(\Lambda)''}.$$

We claim that the mapping $\Lambda \in \mathcal{C}^{\Lambda_a} \mapsto \widetilde{\sigma}_\Lambda$ is a well defined locally normal representation. By construction each $\widetilde{\sigma}_\Lambda$ is normal, so we need only show that $\widetilde{\sigma}$ is well defined.

To see why this is, let $\Lambda_2 \in \mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ be such that $\Lambda \Subset \Lambda_2$. We need to show that

$$\widetilde{\sigma}_{\Lambda,\Lambda_1}|_{\mathfrak{B}(\Lambda)''} = \widetilde{\sigma}_{\Lambda,\Lambda_2}|_{\mathfrak{B}(\Lambda)''}.$$

Choose $\Lambda_3, \Lambda_4 \in \mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ for which

$$\Lambda_j \subset \Lambda_3 \Subset \Lambda_4$$

for $j = 1, 2$. Then

$$\mathcal{N}(\Lambda, \Lambda_j) \subset \mathfrak{B}(\Lambda_j)'' \subset \mathfrak{B}(\Lambda_3)'' \subset \mathcal{N}(\Lambda_3, \Lambda_4),$$

and intersecting with $\mathfrak{K}^{\Lambda_a}$ gives us $\mathfrak{L}(\Lambda, \Lambda_j) \subset \mathfrak{L}(\Lambda_3, \Lambda_4)$. Thus, $\widetilde{\sigma}_{\Lambda_3,\Lambda_4}|_{\mathcal{N}(\Lambda,\Lambda_j)}$ is a normal extension of $\sigma|_{\mathfrak{L}(\Lambda,\Lambda_j)}$ to $\mathcal{N}(\Lambda, \Lambda_j)$. But by uniqueness, this is just $\widetilde{\sigma}_{\Lambda,\Lambda_j}$. Thus we have that

$$\widetilde{\sigma}_{\Lambda,\Lambda_1}|_{\mathfrak{B}(\Lambda)''} = \widetilde{\sigma}_{\Lambda_3,\Lambda_4}|_{\mathfrak{B}(\Lambda)''} = \widetilde{\sigma}_{\Lambda,\Lambda_2}|_{\mathfrak{B}(\Lambda)''}$$

as desired. □

Lemma 3.23. *Let $\rho : \mathfrak{B}^{\Lambda_a} \to B(\mathcal{H})$ be a locally normal representation. Then*

$$\rho\left(\mathfrak{B}^{\Lambda_a}\right)' = \rho\left(\mathfrak{K}^{\Lambda_a}\right)'.$$

Proof. Since $\mathfrak{K}^{\Lambda_a} \subset \mathfrak{B}^{\Lambda_a}$, we have $\rho\left(\mathfrak{B}^{\Lambda_a}\right)' \subset \rho\left(\mathfrak{K}^{\Lambda_a}\right)'$. To see the other direction, let $\Lambda \in \mathcal{C}^{\Lambda_a}$, and choose $\Lambda_1, \Lambda_2 \in \mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ with

$$\Lambda \subset \Lambda_1 \Subset \Lambda_2.$$

Let $\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle$ denote the C*-subalgebra of $\mathfrak{K}^{\Lambda_a}$ generated by the unit and $K(\Lambda_1, \Lambda_2)$. Then $\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle$ is unital, unlike $K(\Lambda_1, \Lambda_2)$, which implies that $\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle'' = \mathcal{N}(\Lambda_1, \Lambda_2)$, and therefore contains $\mathfrak{B}(\Lambda)''$. To see why, note that $\mathcal{N}(\Lambda_1, \Lambda_2)$ is Type I, and so by Proposition 3.12 there exists a unitary $U : \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{K}_2$ for Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ for which

$$\mathcal{N}(\Lambda_1, \Lambda_2) = U^*(B(\mathcal{K}_1) \otimes \mathbb{C})U \text{ and } K(\Lambda_1, \Lambda_2) = U^*(K_1 \otimes \mathbb{C})U,$$

where K_1 denotes the compact operators on $\mathcal{K}_1$. Since $K_1'' = B(\mathcal{K}_1)$, we also have $\langle \mathbb{I}_{\mathcal{K}_1}, K_1\rangle'' = B(\mathcal{K}_1)$. Moreover, since $\langle \mathbb{I}_{\mathcal{K}_1}, K_1\rangle$ is unital,

$$(\langle \mathbb{I}_{\mathcal{K}_1}, K_1\rangle \otimes \mathbb{C}\mathbb{I}_{\mathcal{K}_2})'' = B(\mathcal{K}_1) \otimes \mathbb{C}\mathbb{I}_{\mathcal{K}_2}.$$

Conjugating with U gives $\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle'' = \mathcal{N}(\Lambda_1, \Lambda_2)$.

Since $\mathcal{N}(\Lambda_1, \Lambda_2) \subset \mathfrak{B}(\Lambda_2)''$ and $\rho|_{\mathfrak{B}(\Lambda_2)''}$ is unitarily implemented, it follows directly that $\rho(\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle'') = \rho(\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle)''$. Thus,

$$\rho\left(\mathfrak{B}(\Lambda)''\right) \subset \rho(\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle'') = \rho(\langle \mathbb{I}, K(\Lambda_1, \Lambda_2)\rangle)'' \subset \rho(\mathfrak{K}^{\Lambda_a})''.$$

Since $\rho\left(\mathfrak{B}(\Lambda)''\right) \subset \rho\left(\mathfrak{K}^{\Lambda_a}\right)''$ for each $\Lambda \in \mathcal{C}^{\Lambda_a}$, we have that $\rho\left(\mathfrak{B}^{\Lambda_a}\right) \subset \rho\left(\mathfrak{K}^{\Lambda_a}\right)''$, and so $\rho\left(\mathfrak{K}^{\Lambda_a}\right)' \subset \rho\left(\mathfrak{B}^{\Lambda_a}\right)'$. $\square$

Remark 3.24. Note that in the case, $\rho : \mathfrak{B}^{\Lambda_a} \to B(\mathcal{H})$ is *localizable*, meaning that $\rho|_{\mathfrak{B}}$ satisfies the SSC, we also have that $\rho(\mathfrak{B})' = \rho(\mathfrak{B}^{\Lambda_a})'$. Indeed, $\rho(\mathfrak{B}) \subset \rho(\mathfrak{B}^{\Lambda_a})$, so $\rho(\mathfrak{B}^{\Lambda_a})' \subset \rho(\mathfrak{B})'$. Conversely, for every $\Lambda \in \mathcal{C}^{\Lambda_a}$,

$$\rho(\mathfrak{B}(\Lambda)'') \subset \rho(\mathfrak{B}(\Lambda))'' \subset \rho(\mathfrak{B})'',$$

so $\rho(\mathfrak{B}^{\Lambda_a})'' \subset \rho(\mathfrak{B})''$, and thus $\rho(\mathfrak{B})' = \rho(\mathfrak{B}_a^{\Lambda})'$. Hence, in this case we have

$$\rho(\mathfrak{B})' = \rho(\mathfrak{B}^{\Lambda_a})' = \rho(\mathfrak{K}^{\Lambda_a})'.$$

Recall that Theorem 3.15 states that every representation ρ satisfying the SSC can be written as a direct integral of irreducible representations ρ_x which themselves satisfy the SSC for almost every x. We now give a proof of this result.

Proof of Theorem 3.15. Let Λ_a be any cone in $\mathbb{R}^2$. Denote the extension of ρ to $\mathfrak{B}^{\Lambda_a}$ by ρ^{Λ_a}, and the restriction of ρ^{Λ_a} to $\mathfrak{K}^{\Lambda_a}$ by $\rho^{\mathfrak{K}}$. As pointed out in Remark 3.18, that any ρ_x is unitarily equivalent to φ when both are restricted to any cone in $\mathcal{C}^{\Lambda_a}$ is equivalent to the claim that ρ_x can be extended to locally normal representation $\rho_x^{\Lambda_a}$ of $\mathfrak{B}^{\Lambda_a}$ on $\mathcal{H}_x$. We will show that for any choice of Λ_a this can be done for a full measure set of $x \in X$.

Recall that the direct integral decomposition of ρ into irreducibles given by

$$\mathcal{H} \cong \int_X^{\oplus} \mathcal{H}_x d\mu(x) \text{ and } \rho \cong \int_X^{\oplus} \rho_x d\mu(x)$$

corresponds to a choice of maximally abelian sub-von Neumann algebra of $\rho(\mathfrak{B})'$ by [Tak02, Thm. IV.8.32]. But as shown in Remark 3.24, we have that

$$\rho(\mathfrak{B})' = \rho^{\Lambda_a}(\mathfrak{B}^{\Lambda_a})' = \rho^{\mathfrak{K}}(\mathfrak{K}^{\Lambda_a})' \tag{3.3}$$

meaning that all three commutants have the same maximally abelian subalgebras. We can therefore decompose ρ^{Λ_a} and $\rho^{\mathfrak{K}}$ into irreducibles over the same decomposition of $\mathcal{H}$, giving

$$\rho^{\Lambda_a} \cong \int_X^{\oplus} \rho_x^{\Lambda_a} d\mu(x) \text{ and } \rho^{\mathfrak{K}} \cong \int_X^{\oplus} \rho_x^{\mathfrak{K}} d\mu(x).$$

Since $\rho^{\Lambda_a}|_{\mathfrak{B}} = \rho$ and $\rho^{\Lambda_a}|_{\mathfrak{K}^{\Lambda_a}} = \rho^{\mathfrak{K}}$, we must have $\rho_x^{\Lambda_a}|_{\mathfrak{B}} = \rho_x$ and $\rho_x^{\Lambda_a}|_{\mathfrak{K}^{\Lambda_a}} = \rho_x^{\mathfrak{K}}$ for all x outside a null set.

Since ρ^{Λ_a} is a locally normal representation, $\rho^{\mathfrak{K}}$ is non-degenerate on each $K(\Lambda_1, \Lambda_2)$ for $\Lambda_1 \Subset \Lambda_2$ and $\Lambda_1, \Lambda_2 \in \mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ by Lemma 3.22. This means that $\rho_x^{\mathfrak{K}}$ must be non-degenerate on $K(\Lambda_1, \Lambda_2)$ for all x outside of a null set. Since $\mathcal{C}_{\mathbb{Q}}^{\Lambda_a}$ is countable, $\rho_x^{\mathfrak{K}}$ is non-degenerate on every such $K(\Lambda_1, \Lambda_2)$ for almost every x.

By another application of Lemma 3.22, we therefore have that $\rho_x^{\Lambda_a}$ is locally normal for almost every x, and hence $\rho_x|_{\mathfrak{B}(\Lambda)}$ is unitarily implemented for almost every x and every $\Lambda \in \mathcal{C}^{\Lambda_a}$. In other words, we have shown that the set

$$Z(\Lambda_a) = \{x \in X \mid \rho_x|_{\mathfrak{B}(\Lambda)} \text{ is unitarily implemented for every } \Lambda \in \mathcal{C}^{\Lambda_a}\}$$

is a full measure set. Since our choice of Λ_a was arbitrary and $\mathcal{C}_{\mathbb{Q}}$ is countable, we see that $\bigcap_{\Lambda_a \in \mathcal{C}_{\mathbb{Q}}} Z(\Lambda_a)$ is a full measure set as well. Since every cone Λ is in $\mathcal{C}^{\Lambda_a}$ for some $\Lambda_a \in \mathcal{C}_{\mathbb{Q}}$, we have that $\rho_x|_{\mathfrak{B}(\Lambda)}$ is unitarily implemented for almost every x and every $\Lambda \in \mathcal{C}$, and therefore ρ_x satisfies SSC(φ) for a full measure set of x in X. $\square$

Recall that we are assuming throughout that π_1 has properly infinite cone algebras, which follows from the assumption that ω_1 is a gapped ground state, and therefore satisfies the conditions of Theorem 3.15. With this fact and Theorem 3.15 in hand we are ready to prove the following.

Proposition 3.25 (Type I property). *Let π_1 satisfy the approximate split property, Definition 3.14, and let σ be a representation satisfying* SSC(π_1). *If in addition π_1, has at most countably many irreducible superselection sectors then σ can be decomposed as a countable direct sum of irreducible representations satisfying* SSC(π_1). *Moreover, if σ is a factor representation, then $\sigma(\mathfrak{A}_1)''$ is Type I.*

Proof. Let $\sigma : \mathfrak{A}_1 \to B(\mathcal{H})$ be a representation that satisfies SSC(π_1). Let us decompose σ again into irreducible representations

$$\mathcal{H} \cong \int_X^{\oplus} \mathcal{H}_x d\mu(x) \text{ and } \sigma \cong \int_X^{\oplus} \sigma_x d\mu(x)$$

for a measure space (X, Σ, μ). By Theorem 3.15, σ_x satisfies SSC(π_1) for all x in a full measure subset of X. Thus, by possibly removing a set of zero measure, we may without loss of generality take σ_x to satisfy SSC(π_1) for every $x \in X$. Moreover, since $\mathfrak{A}_1$ is UHF, each σ_x is a faithful representation on a separable Hilbert space, and thus we make the identification $\mathcal{H}_x \equiv \mathcal{K}$ for all $x \in X$. Thus, σ_x is a map into $B(\mathcal{K})$ for each $x \in X$, and for each $A \in \mathfrak{A}_1$, $X \ni x \mapsto \sigma_x(A) \in B(\mathcal{K})$ is a measurable function. Furthermore, we identify σ_x and σ_y whenever $\sigma_x \cong \sigma_y$. Since there are only countably many unitary equivalence classes satisfying SSC(π_1), we index representatives from each of these classes by a countable set J, so that for each $x \in X$ there exists $j \in J$ for which $\sigma_x = \sigma_j$. Note this means we can partition X as

$$X = \bigcup_{j \in J} \{x \in X \mid \sigma_x = \sigma_j\}.$$

Let us denote $S_j = \{x \in X \mid \sigma_x = \sigma_j\}$. We first need to show that each S_j is measurable. To see why, notice that since $\sigma_j \neq \sigma_k$ whenever $j \neq k$, we can find $A_{jk} \in \mathfrak{A}_1$ such that $\sigma_j(A_{jk}) \neq \sigma_k(A_{jk})$. Since the map $f_{jk}(x) = \sigma_x(A_{jk})$ is measurable, the set

$$S_j(k) := f_{jk}^{-1}\{\sigma_j(A_{jk})\} = \{x \in X \mid \sigma_x(A_{jk}) = \sigma_j(A_{jk})\}$$

is measurable, since the singleton $\{\sigma_j(A_{jk})\}$ belongs to the Borel sigma algebra on $B(\mathcal{K})$. Hence, we have that $\bigcap_{k \neq j} S_j(k)$ is measurable as well, since $J \backslash \{j\}$ is a countable set. But for each $k \neq j$, $\sigma_j(A_{jk}) \neq \sigma_k(A_{jk})$, so $S_j = \bigcap_{k \neq j} S_j(k)$, so S_j is measurable as required.

Now let $I = \{j \in J \mid \mu(S_j) > 0\}$. Then

$$\sigma \cong \int_X^{\oplus} \sigma_x d\mu(x) \cong \bigoplus_{i \in I} \int_{S_i}^{\oplus} \sigma_x d\mu(x).$$

But for each $i \in I$, $\sigma_x = \sigma_i : \mathfrak{A}_1 \to B(\mathcal{K})$, so $\int_{S_i}^{\oplus} \sigma_x d\mu(x)$ is the representation on the Hilbert space $L^2(S_i, \mu|_{S_i}; \mathcal{K})$ that acts on the fibre $\mathcal{K}$ by σ_i. Thus, it is unitarily equivalent to

$$A \mapsto \sigma_i(A) \otimes \mathbb{I} \in B(\mathcal{K}) \otimes \mathbb{I} \subset B(\mathcal{K} \otimes L^2(S_i, \mu|_{S_i}))$$

and is therefore a Type I factor representation by Proposition 3.12. Any Type I factor representation on a separable Hilbert space can be written as a direct sum of countably many copies of some irreducible representation, in this case σ_i, and so we can write

$$\int_{S_i}^{\oplus} \sigma_x d\mu(x) \cong \bigoplus_{k=1}^{d_i} \sigma_i,$$

where d_i is the (possibly infinite but countable) dimension of $L^2(S_i, \mu|_{S_i})$. Ergo

$$\sigma \cong \bigoplus_{i\in I} \bigoplus_{k=1}^{d_i} \sigma_i,$$

which establishes our first claim that σ is a countable direct sum. Moreover, since each of the σ_i's are inequivalent, we see that σ will be a direct sum of inequivalent Type I factors whenever I contains more than one element, in which case σ could not be a factor. Thus, if σ is a factor, I must be a singleton, meaning

$$\sigma \cong \bigoplus_{k=1}^{d_i} \sigma_i,$$

and is therefore Type I. □

Remark 3.26. In addition to justifying the Type I condition, Proposition 3.25 establishes sufficient conditions under which we can decompose an arbitrary reducible representation that satisfies the SSC into a direct sum of irreducible superselection sectors. Indeed, any representation ρ can be decomposed as a direct integral of irreducibles over some measure space,

$$\rho \cong \int_X^{\oplus} \rho_x d\mu(x).$$

However, even if ρ satisfies the SSC, there is no a priori reason to believe that

(i) ρ_x satisfies the SSC for all x in some full measure set in X, or
(ii) the direct integral can be written as a direct sum.

Firstly, (i) may not hold if the SSC fails for each ρ_x individually and only becomes satisfied after the representations are averaged over an integral. Furthermore, (ii) is not guaranteed to hold for general representations unless the Hilbert space is finite dimensional.

We stress here that these issues arise not only in the context of stacking, but more generally in the context of superselection sector theory. Note that Proposition 3.25 implies that it is sufficient to consider only irreducible sectors. However, a general sector may still be a countable direct sums of irreducibles. In applications, it is often to restrict to sectors which are *dualizable*. Physically, these means there is a 'conjugate sector'. By restricting to dualizable sectors, it follows that it is enough to consider only *finite* direct sums [LR97], i.e., the category of dualizable superselection sectors becomes semi-simple. This is particularly useful in the context of *fusion rules*. Briefly, one of the main results of Doplicher–Haag–Roberts theory is that we can endow the superselection sectors with a monoidal product, which we will write as $\widehat{\otimes}$ (this is *not* the tensor product of representations!). The general construction can be found in [Oga22] for the models we are considering. A natural question is then how the tensor product $\pi_i \widehat{\otimes} \pi_j$ of two irreducible representations satisfying the SSC behaves. In particular, we get

$$\pi_i \widehat{\otimes} \pi_j \cong \bigoplus_{k \in I} N_{ij}^k \pi_k.$$

The set $\{N_{ij}^k\}$ are called the *fusion rules* of the theory. If we restrict to dualizable sectors only, this is a finite direct sum, and each N_{ij}^k is a non-negative integer.

3.5. **Topological entanglement entropy and the approximate split property.** Here we discuss the validity of the approximate split property, which is essential to ensure that almost all irreducible factors in the decomposition of a representation satisfying the SSC do satisfy the SSC themselves, see Theorem 3.15.

We first recall some basic definitions. Let ω be a state on $\mathcal{A}$. If $\Lambda \subset \Gamma$ is a finite set, $\omega|_{\mathcal{A}(\Lambda)}$ is a state on some full matrix algebra and hence can be represented by a density operator $\rho_\Lambda \in \mathcal{A}(\Lambda)$ such that $\omega(A) = \mathrm{Tr}(\rho_\Lambda A)$ for all $A \in \mathcal{A}(\Lambda)$. For disjoint, finite regions $A, C \subset \Gamma$, we denote $AC = A \cup C$ and define the *quantum mutual information* as

$$I_\omega(A : C) := S(\rho_A) + S(\rho_C) - S(\rho_{AC}) = S(\rho_{AC}, \rho_A \otimes \rho_C).$$

Here $S(\rho)$ is the von Neumann entropy of the density operator ρ, and $S(\rho, \sigma)$ is the quantum relative entropy.

We will show that the vanishing of the mutual information

$$I_\omega(A : C) = 0 \tag{3.4}$$

whenever A and C are sufficiently far removed implies the approximate split property. Note that $I_\omega(A : C) = 0$ if and only if $\rho_{AC} = \rho_A \otimes \rho_C$, namely ω is a product state for observables separated far enough. Here, we mean that there is some $\xi > 0$ such that equation (3.4) is satisfied for any pair of finite sets A and C with $d(A, C) > \xi$.

Before we get to the split property, let us mention that there are several interesting models with topological order for which condition (3.4) is satisfied. This is true in, for example, the toric code, if ω is its frustration free ground state. This can be seen from the explicit description of ω given in [AFH07]. In fact, (3.4) follows from the following strict form of the area law which is expected to hold for the renormalizaton fixpoints of topologically ordered phases [KP06; LW06]: There are α and γ such that for (large enough) compact and simply connected regions A,

$$S(\rho_A) = \alpha |\partial A| - n_A \gamma. \tag{3.5}$$

Here $|\partial A|$ is the length of the boundary A and n_A is the number of disconnected components of ∂A. While α depends on the microscopic details of the Hamiltonian, the topological entanglement entropy γ is related to the quantum dimension of the anyons in the model.

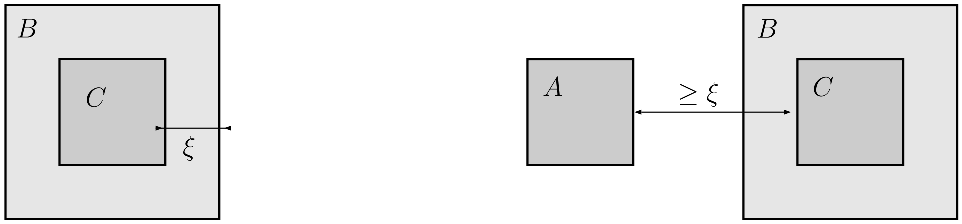
More generally, it follows from one of the axioms of [SKK20] that are postulated to hold for general gapped ground states. One considers an annulus B and the disk C in its interior. Their Axiom A0 says that

$$S(\rho_{BC}) + S(\rho_C) - S(\rho_B) = 0 \tag{3.6}$$

for regions B, C as in Figure 1a.

To go from equation (3.6) to (3.4), one can use strong subadditivity of the entropy [LR73] in the form

$$S(\rho_A) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC}).$$



(1a) The setting of equation 3.6. (1b) The vanishing of $I_\omega(A:C)$ whenever $\mathrm{dist}(A,C) \geq \xi$.

Figure 1. The entanglement entropy and the mutual information.

We consider two finite sets A, C separated by a distance of at least ξ. Around C, but disjoint from A, we can choose a region B such that BC is a region for which Axiom (3.6) holds, see Figure 1b. We then have

$$\begin{aligned} I_\omega(A:C) &= S(\rho_A) + S(\rho_C) - S(\rho_{AC}) \\ &= S(\rho_A) + S(\rho_B) - S(\rho_{BC}) - S(\rho_{AC}) \leq 0. \end{aligned}$$

We used (3.6) in the second equality and the strong subadditivity to conclude the inequality. Since $I_\omega(A:C) \geq 0$, this implies that $I_\omega(A:C) = 0$, and hence equation (3.4).

Just like the strict area law 3.5, the strict equalities (3.4) and 3.6 are only expected to hold for the renormalization fixpoints. However, its consequence, namely the approximate split property, is stable under the action of locally generated automorphisms and therefore holds throughout the phase, see [NO22].

Remark 3.27. For infinite dimensional operator algebras, the von Neumann entropy is not a good quantity as it is often infinite. Following Araki [Ara75], the *relative* entropy can be defined in a general von Neumann algebraic setting, so that one could hope to make sense of $I_\omega(A:C)$ for infinite regions A and C. Here however one has to be careful what exactly is a bipartite system. More precisely, if ω_{AB} is a state on some von Neumann algebra $\mathcal{M}$, there need not be a natural decomposition $\mathcal{M} = \mathcal{M}_A \overline{\otimes} \mathcal{M}_B$, so that there is no natural meaning of $\omega_A \otimes \omega_B$ as a state. In fact, the split property is equivalent to saying that there is a $*$-isomorphism between $\mathcal{M}_A \vee \mathcal{M}_B$ and $\mathcal{M}_A \overline{\otimes} \mathcal{M}_B$.

We now prove that the approximate split property, Definition 3.14, is a consequence of the strict equality (3.4). The following theorem is a generalisation of a similar proof for the toric code [Naa11]. To state the result, we need a slightly stronger assumption than approximate Haag duality, which is a slight variant of what is sometimes called *bounded spread Haag duality* or *weak algebraic duality* (see e.g. [Jon24]).

Definition 3.28. Consider the notation of Definition 2.4. Then $(\mathcal{H}, \pi_0)$ satisfies *bounded spread Haag duality* if it satisfies approximate Haag duality with $f_\delta(t) = 0$ and $U = \mathbb{I}$.

Note that this implies that for any cone Λ, we can find some slightly larger and translated cone Λ' (with $\Lambda \subset \Lambda'$) such that $\mathcal{R}'_{\Lambda^c} \subset \mathcal{R}_{\Lambda'}$.

Theorem 3.29. *Suppose that ω is a pure state satisfying condition* (3.4). *Assume furthermore that π_ω satisfies bounded spread Haag duality. Let $\Lambda_1 \subset \Lambda_2$ be an inclusion of cones whose boundaries are sufficiently far removed from each other, in the sense that $I_\omega(A:C) = 0$ whenever $A \subset \Lambda_1$ and $C \subset \Lambda_2^c$. Then π_ω satisfies the approximate split property, in the sense* *that there is a Type I factor $\mathcal{N}$ such that $\pi_\omega(\mathfrak{A}(\Lambda_1))'' \subset \mathcal{N} \subset \pi_\omega(\mathfrak{A}(\widehat{\Lambda}_2))''$, where $\widehat{\Lambda}_2$ is some cone containing Λ_2.*

Proof. The condition that $I_\omega(A : C) = 0$ implies that $\rho_{AC} = \rho_A \otimes \rho_C$. Hence $\omega(ac) = \omega(a)\omega(c)$ for all $a \in \mathcal{A}(A)$ and $c \in \mathcal{A}(C)$ for all finite sets $A \subset \Lambda_1$ and $C \subset \Lambda_2^c$. Hence it follows that this is true for all $a \in \mathcal{A}(\Lambda_1)_{\mathrm{loc}}$ and $c \in \mathcal{A}(\Lambda_2^c)_{\mathrm{loc}}$.

Write $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ for the GNS representation of ω. Let $\mathcal{R}_{\Lambda_1} := \pi_\omega(\mathcal{A}(\Lambda_1))''$ and $\mathcal{R}_{\Lambda_2^c} := \pi_\omega(\mathcal{A}(\Lambda_2^c))''$. The map $a \mapsto \langle \Omega, a\Omega \rangle$ is a σ-weakly continuous extension of ω from $\pi_\omega(\mathcal{A})$ to $\mathcal{B}(\mathcal{H}_\omega)$, which we also denote by ω. Consider $a \in \mathcal{R}_{\Lambda_1}$ and $c \in \mathcal{R}_{\Lambda_2^c}$ with norm bounded by one. From Kaplansky's density theorem (and Theorem II.2.6 of [Tak02]), there are nets $a_\lambda \in \mathcal{A}(\Lambda_1)_{\mathrm{loc}}$ and $c_\mu \in \mathcal{A}(\Lambda_2^c)_{\mathrm{loc}}$ such that $a_\lambda \to a$ and $c_\mu \to c$ in the σ-weak topology. We then have

$$\omega(ac_\mu) = \omega(\lim_\lambda a_\lambda c_\mu) = \lim_\lambda \omega(a_\lambda c_\mu) = \lim_\lambda \omega(a_\lambda)\omega(c_\mu) = \omega(a)\omega(c_\mu).$$

Here we used that for fixed b, $a \mapsto ab$ is continuous in the σ-weakly topology, and that ω is a normal state, and hence σ-weak continuous. Taking the limit over μ in a similar way shows that $\omega(ac) = \omega(a)\omega(c)$. But this implies that the product state $\omega \otimes \omega$ on the *algebraic* tensor product $\mathcal{R}_{\Lambda_1} \odot \mathcal{R}_{\Lambda_2^c}$ can be extended to a σ-weakly continuous state on $\mathcal{R} := \mathcal{R}_{\Lambda_1} \vee \mathcal{R}_{\Lambda_2^c}$. Note that by locality, we have $\mathcal{R}_{\Lambda_1} \subset \mathcal{R}'_{\Lambda_2^c}$ and that from a similar argument as in Lemma 3.7 it follows that $\mathcal{R}$ is a factor. It then follows by a result of Takesaki [Tak58] that $\mathcal{R}$ is $*$-isomorphic to the *von Neumann* tensor product $\mathcal{R}_{\Lambda_1} \overline{\otimes} \mathcal{R}_{\Lambda_2^c}$. Since the $\mathcal{R}_\Lambda$ are factors (cf. the proof of Lemma 3.7), if follows from Theorem 1 and Corollary 1(iv) of [DL83] that there exists a Type I factor $\mathcal{N}$ such that $\mathcal{R}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{R}'_{\Lambda_2^c}$. By bounded spread Haag duality, it follows that there is some cone $\widehat{\Lambda}_2$ containing Λ_2 such that $\mathcal{R}'_{\Lambda_2^c} \subset \mathcal{R}_{\widehat{\Lambda}_2}$, which completes the proof. $\square$

Remark 3.30. The assumptions necessary for Theorem 3.29 hold for a large class of models. Kim and Ranard [KR24] show that every model satisfying the 'bootstrap axioms' (which as we have noted imply our entropy condition) can be mapped to a Levin–Wen string-net model using a finite depth quantum circuit. These Levin–Wen string-net models satisfy bounded spread Haag duality (see the proof of [OPR25, Cor. 4.1.5]). Since they are connected to the model we started with by a FDQC, they have the same property.

It follows by Theorem 3.29 and the stability result of [NO22] that the approximate split property holds or all states in the same phase as a Levin–Wen model. These are conjectured to exhaust all commuting projector topologically ordered models in 2D where the local algebras are full matrix algebras, which are all believed to be non-chiral. (The latter condition is necessary, see for example [Haa23]).

3.6. **Invertible states.** The classification of representations satisfying the superselection criterion of a stacked system yields the triviality of invertible states, in the sense of Definition 2.9, which was the original motivation of introducing this class: If a state is invertible, its GNS representation has trivial superselection sectors. We begin with recalling a useful characterization of pure states with equivalent GNS representations (see e.g. [Naa17, Prop. 3.2.8] for a proof).

Proposition 3.31. *Let $\mathfrak{B}$ be a quasi-local algebra on Γ, and let ϕ and ψ be two pure states on $\mathfrak{B}$. The following are equivalent:*

(i) The GNS representations of ϕ and ψ are equivalent.
(ii) For every $\varepsilon > 0$, there is some finite set $X \subset \Gamma$ such that for any finite $Y \subset \Gamma \backslash X$ and $A \in \mathfrak{B}(Y)$, we have $|\phi(A) - \psi(A)| < \varepsilon \|A\|$.

Definition 3.32. We say two states on the same C*-algebra are equivalent if their corresponding GNS representations are unitary equivalent.

Proposition 3.33. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be quasi-local algebras on the lattice Γ. Let ϕ and ψ be inequivalent pure states on $\mathfrak{B}_1$ and let χ be a pure state on $\mathfrak{B}_2$. Then $\phi \otimes_{\mathrm{s}} \chi$ and $\psi \otimes_{\mathrm{s}} \chi$ are inequivalent pure states on $\mathfrak{B}_1 \otimes_{\mathrm{s}} \mathfrak{B}_2$.*

Proof. By Proposition 3.31, there exists some $\varepsilon > 0$ such that for every finite $X \subset \Gamma$, there exists a finite set $Y \subset \Gamma \backslash X$ and $A \in \mathfrak{B}_1(Y)$ for which

$$|\phi(A) - \psi(A)| \geq \varepsilon \|A\|.$$

However, this means we have

$$\begin{aligned}|\phi \otimes_{\mathrm{s}} \chi(A \otimes_{\mathrm{s}} \mathbb{I}) - \psi \otimes_{\mathrm{s}} \chi(A \otimes_{\mathrm{s}} \mathbb{I})| &= |(\phi(A) - \psi(A))\chi(\mathbb{I})| \\ &= |\phi(A) - \psi(A)| \geq \varepsilon \|A\| = \varepsilon \|A \otimes_{\mathrm{s}} \mathbb{I}\|.\end{aligned}$$

since $A \otimes_{\mathrm{s}} \mathbb{I} \in (\mathfrak{B}_1 \otimes_{\mathrm{s}} \mathfrak{B}_2)(Y)$, we have that $\phi \otimes_{\mathrm{s}} \chi$ and $\psi \otimes_{\mathrm{s}} \chi$ must be inequivalent as well. □

Corollary 3.34 (Corollary D)**.** *Let ω be an invertible state of a quantum spin system $\mathfrak{A}$ and let π be its GNS representation. For all irreducible ρ satisfying* $\mathrm{SSC}(\pi)$*, we have $\rho \cong \pi$.*

Proof. Let $\overline{\mathfrak{A}}$ be a quasi-local algebra on Γ and $\overline{\omega}$ be a state on $\overline{\mathfrak{A}}$ for which $\omega \otimes_{\mathrm{s}} \overline{\omega}$ is related to a product state under the action of an LGA. We write $\overline{\pi}$ for the GNS representation of $\overline{\omega}$.

Suppose ρ is an irreducible representation of $\mathfrak{A}$ satisfying $\mathrm{SSC}(\pi)$, and let ϕ be a state in the sector defined by ρ (so that ρ is the GNS representation associated to ϕ). Then by Theorem A, $\rho \otimes_{\mathrm{s}} \overline{\pi}$ satisfies $\mathrm{SSC}(\pi \otimes_{\mathrm{s}} \overline{\pi})$. By [NO22], the only irreducible superselection sector for $\pi \otimes_{\mathrm{s}} \overline{\pi}$ is $\pi \otimes_{\mathrm{s}} \overline{\pi}$ itself, so we must have $\rho \otimes_{\mathrm{s}} \overline{\pi} \cong \pi \otimes_{\mathrm{s}} \overline{\pi}$. Thus, $\phi \otimes_{\mathrm{s}} \overline{\omega}$ is equivalent to $\omega \otimes_{\mathrm{s}} \overline{\omega}$, but by Proposition 3.33, ϕ must be equivalent to ω, and therefore $\rho \cong \pi$. □

4. Stacking as a categorical product

So far, we have mostly considered the superselection sectors only. The main result of the DHR approach is that these sectors have a rich structure, namely they are braided C* (or in our case, in fact W*-) categories. We refer the interested reader to [GLR85; Yam07; HNP24] for the notion of W*-categories, but we aim to provide a concrete definition of the categories we are interested in below. In this section, we prove that whole structure is compatible with stacking in the sense of Theorem C.

Throughout this section we consider quantum systems $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with reference representations π_1 and π_2 satisfying Assumption 2.3 (properly infiniteness of cone algebras), Assumption 2.5 (approximate Haag duality), and the assumptions of Proposition 3.25 (approximate split property and countably many sectors). The derivation of the monoidal structure, including the braiding, in this setting is worked out in [Oga22]. We recall the main points here.

Any representation π satisfying $\mathrm{SSC}(\pi_1)$ has a unitarily equivalent representation represented on the Hilbert space of the reference representation π_1. In fact, fix a cone Λ, and let

$V_\Lambda : \mathcal{H}_\pi \to \mathcal{H}_{\pi_1}$ be the unitary setting up the equivalence in (2.1), i.e. $V_\Lambda \pi(A) V_\Lambda^* = \pi_1(A)$ for all $A \in \mathfrak{A}_1(\Lambda^c)$. Define a representation $\pi_\Lambda(A) := V_\Lambda \pi(A) V_\Lambda^*$. Then π_Λ is a representation on the same Hilbert space as the reference representation π_1, and $\pi_\Lambda(A) = \pi_1(A)$ for all $A \in \mathfrak{A}_1(\Lambda)$. We also say that π_Λ is *localized* in Λ. This motivates the following definition.

Definition 4.1. The category $\Delta_{\mathfrak{A}_1}$ has as objects

$$\{\pi : \mathfrak{A}_1 \to B(\mathcal{H}_{\pi_1}) : \pi \text{ satisfies } \mathrm{SSC}(\pi_1) \text{ and } \pi(A) = \pi_0(A) \text{ for all } A \in \mathfrak{A}_1(\Lambda^c)\}$$

We call such a representation *localized* in Λ. The morphisms are the intertwiners between such representations.

Remark 4.2. This category turns out to be independent of the choice of cone Λ (up to unitary equivalence), and is also equivalent to the category of representations satisfying $\mathrm{SSC}(\pi_1)$. This has certainly been well-known for a long time, but is often used only implicitly. For a discussion of this point, see e.g. [BCNS25] or [Bha+25]. Since the choice of Λ is not important, we will not use it in our notation $\Delta_{\mathfrak{A}_1}$.

The essential idea to define the monoidal structure (and braiding) is to pass from *representations* to *endomorphisms*. Fix a cone Λ_a such that the Λ cone is admissible with respect to Λ_a, see equation (3.2). We write $\mathfrak{A}_i^{\Lambda_a}$ for the auxiliary algebras (3.1) for $\mathfrak{A}_i$, and $\mathfrak{A}_{12}^{\Lambda_a}$ for the auxiliary algebra[3] obtained from the stacked system. Recall that we can identify $\pi_1(A)$ with A for all $A \in \mathfrak{A}_1$. Then it can be shown (using approximate Haag duality in an essential way) that any $\pi \in \Delta_{\mathfrak{A}_1}$ extends uniquely to a $*$-endomorphism ρ_Λ of $\mathfrak{A}_1^{\Lambda_a}$, that is, $\rho_\Lambda(\pi_1(A)) = \pi(A)$ for all $A \in \mathfrak{A}_1$. For every admissible cone Λ', this ρ_Λ is weakly continuous on $\pi_1(\mathfrak{A}_1(\Lambda'))''$. This allows us to identify objects in $\Delta_{\mathfrak{A}_1}$ with certain endomorphisms of $\Delta_{\mathfrak{A}_1}$, and we will do so below when convenient. We refer the reader to [Oga22] for details.

The simple objects in $\Delta_{\mathfrak{A}_i}$ correspond to irreducible representations that satisfy $\mathrm{SSC}(\pi_i)$. Theorem A and Proposition 3.33 give an injection of objects in $\Delta_{\mathfrak{A}_1}$ to objects in the stacked theory $\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$. A fortiori, there is an injection mapping pairs of objects in $\Delta_{\mathfrak{A}_1}$ and $\Delta_{\mathfrak{A}_2}$ to $\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$. Theorem B in turn shows that this map is a surjection. Not surprisingly, this map can be lifted to non-irreducible sectors, and this can be done in a way that is compatible with the monoidal structure and braiding. We now construct a new W*-braided tensor category $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ which is equivalent to the category of the decoupled stacked system sector theory $\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$ via a 'stacking' functor ι. The category $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ is a variant of the Deligne tensor product of Abelian categories (see e.g. [EGNO15]) suitable for categories with infinite direct sums. A similar construction has been considered in [HNP24] in an abstract setting.

Definition 4.3. We define the category $\Delta_{\mathfrak{A}_1} \boxtimes_0 \Delta_{\mathfrak{A}_2}$ as follows: the objects are pairs $(\rho_{\mathfrak{A}_1}, \rho_{\mathfrak{A}_2})$, where the $\rho_{\mathfrak{A}_i} \in \Delta_{\mathfrak{A}_i}$, $i = 1, 2$, are irreducible. The morphisms are defined as $\mathrm{Hom}((\rho_{\mathfrak{A}_1}, \sigma_{\mathfrak{A}_1}), (\rho_{\mathfrak{A}_2}, \sigma_{\mathfrak{A}_2})) := \mathrm{Hom}_{\Delta_{\mathfrak{A}_1}}(\rho_{\mathfrak{A}_1}, \sigma_{\mathfrak{A}_1}) \otimes \mathrm{Hom}_{\Delta_{\mathfrak{A}_2}}(\rho_{\mathfrak{A}_2}, \sigma_{\mathfrak{A}_2})$, where the tensor product is that of vector spaces.

Lemma 4.4. *There is a full and faithful $*$-functor $F_0 : \Delta_{\mathfrak{A}_1} \boxtimes_0 \Delta_{\mathfrak{A}_2} \to \Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$*

Proof. Let $(\rho_{\mathfrak{A}_1}, \rho_{\mathfrak{A}_2}) \in \boxtimes_0$. By definition of $\otimes_{\mathrm{s}}$, it follows that

$$(\rho_{\mathfrak{A}_1} \otimes_{\mathrm{s}} \rho_{\mathfrak{A}_2})(A_1 \otimes A_2) = \pi_1(A_1) \otimes_{\mathrm{s}} \pi_2(A_2)$$

[3] There are approaches to sector theory that do not require the choice of an auxiliary algebra, see e.g. [Bha+25; BCNS25], but so far details have only been worked out under the assumption of *strict* Haag duality.

for all $A_i \in \mathfrak{A}_i$. From Theorem 3.11 it follows that $\rho_{\mathfrak{A}_1} \otimes_{\mathrm{s}} \rho_{\mathfrak{A}_2}$ satisfies the superselection criterion. It follows that $\rho_{\mathfrak{A}_1} \otimes_{\mathrm{s}} \rho_{\mathfrak{A}_2} \in \Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$. Hence we can define the functor F_0 on objects by $F_0((\rho_{\mathfrak{A}_1}, \rho_{\mathfrak{A}_2})) := \rho_{\mathfrak{A}_1} \otimes_{\mathrm{s}} \rho_{\mathfrak{A}_2}$.

If $S \in \mathrm{Hom}(\rho_{\mathfrak{A}_1}, \sigma_{\mathfrak{A}_1})$ and $T \in \mathrm{Hom}(\rho_{\mathfrak{A}_2}, \sigma_{\mathfrak{A}_2})$, we set

$$F_0((S,T)) := S \otimes_{\mathrm{s}} T \in \mathrm{Hom}(\rho_{\mathfrak{A}_1} \otimes_{\mathrm{s}} \rho_{\mathfrak{A}_2}, \sigma_{\mathfrak{A}_1} \otimes_{\mathrm{s}} \sigma_{\mathfrak{A}_2}).$$

This functor F_0 clearly is linear, preserves the $*$-operation, and is faithful. It is also full, since $\rho_{\mathfrak{A}_1} \otimes \rho_{\mathfrak{A}_2}$ is irreducible, and hence $\mathrm{Hom}_{\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}}(F_0((\rho_{\mathfrak{A}_1}, \rho_{\mathfrak{A}_2})), F_0((\sigma_{\mathfrak{A}_1}, \sigma_{\mathfrak{A}_2})))$ is either one-dimensional or the zero vector space. But from the proof of Proposition 3.33 it follows that it is one-dimensional if and only if $\rho_{\mathfrak{A}_i} \cong \sigma_{\mathfrak{A}_i}$ The claim then follows. $\square$

This result means that we can regard $\Delta_{\mathfrak{A}_1} \boxtimes_0 \Delta_{\mathfrak{A}_2}$ as a full subcategory of $\Delta_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$, and we will do so from now on.

Remark 4.5. Alternatively, the functor can be defined using the endomorphism picture. Since it is unclear if $\mathfrak{A}_1^{\Lambda_a} \otimes \mathfrak{A}_2^{\Lambda_a} = \mathfrak{A}_{12}^{\Lambda_a}$ (where the tensor product is that of C*-algebras represented on Hilbert spaces), this is less straightforward. Note that if $\rho_{\mathfrak{A}_1}$ is an endomorphism obtained from some representation satisfying $\mathrm{SSC}(\pi_1)$, it follows that $A \mapsto \rho_{\mathfrak{A}_1}(\pi_1(A))$ satisfies the superselection criterion, and $\rho_{\mathfrak{A}_1}$ is uniquely determined by $\rho_{\mathfrak{A}_1}(\pi_1(A))$ for all $A \in \mathfrak{A}_1$. The same is of course true for the other stacked layer, and we can take the stacked representation. As shown in the proof above, this is in $\Delta_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$, and we can *uniquely* assign a $*$-endomorphism $\rho_{\mathfrak{A}_1 \otimes_{\mathrm{s}} \mathfrak{A}_2}$ to it. It follows that we can define the functor on endomorphisms instead.

From Proposition 3.25, we know that a representation satisfying $\mathrm{SSC}(\pi_1)$ decomposes as a countable (in particular, possibly infinite) direct sum of irreducible representations satisfying $\mathrm{SSC}(\pi_1)$. It remains to be seen that infinite direct sums are well-defined in the category $\Delta_{\mathfrak{A}_1}$. Let us first recall the definition of a subobject and a direct sum in W*-categories.

Definition 4.6 ([GLR85]). The category $\Delta_{\mathfrak{A}}$ has *subobjects* if for every $\rho \in \Delta_{\mathfrak{A}}$ and $P = P^2 = P^* \in \mathrm{Hom}(\rho, \rho)$, there is some object $\sigma \in \Delta_{\mathfrak{A}}$ and $V \in \mathrm{Hom}(\sigma, \rho)$ such that $VV^* = P$ and $V^*V = \mathbb{I}$.

If I is a set, and $\rho_i \in \Delta_{\mathfrak{A}}$ for $i \in I$ a collection of objects. Then ρ is the *direct sum* of $\{\rho_i\}_{i \in I}$ if there are $V_i \in \mathrm{Hom}(\rho_i, \rho)$ such that

$$V_i^* V_j = \delta_{i,j} \mathbb{I} \qquad \sum_{i \in I} V_i V_i^* = \mathbb{I},$$

where convergence of the sum is in the strong operator topology. We write $\rho = \bigoplus_{i \in I} \rho_i$ for the direct sum.

Direct sums, if they exist, are unique up to unitary isomorphism. Alternatively, one can define direct sums in W*-categories in terms of an universal property, see e.g. [FW19].

Proposition 4.7. *The category $\Delta_{\mathfrak{A}_1}$ has subobjects and countable direct sums.*

Proof. It has already been shown that the category has subobjects and finite direct sums (see [Oga22, Lemma 5.7]). It remains to show that we also have countable direct sums, which is proven in a similar way. Since $\pi_1(\mathfrak{A}_1(\Lambda))''$ is properly infinite by assumption, there exist (via the Halving Lemma, [KR97]) countably many isometries $V_i \in \pi_1(\mathfrak{A}(\Lambda))''$ with orthogonal ranges such that

$$V_i^* V_j = \delta_{i,j}\mathbb{I} \text{ and } \sum_i V_i V_i^* \xrightarrow{\text{SOT}} \mathbb{I},$$

see for example the proof of Theorem 6.3.4 of [KR97]. Then we can define the infinite direct sum $\sigma(A) := \sum_i V_i \sigma_i(A) V_i^*$, with the sum converging in the strong operator topology. This is again a representation of $\mathfrak{A}_1$, and by locality it follows that $\sigma(A) = \pi_1(A)$ for all $A \in \mathfrak{A}_1(\Lambda^c)$. Since the σ_i satisfy the superselection criterion, a short computation gives that the same is true for σ, and hence $\sigma \in \Delta_{\mathfrak{A}_1}$. The verification that this is indeed a direct sum of the σ_i in the category is analogous to the finite case. □

A category which has subobjects and direct sums is called *Cauchy complete* in [HNP24]. Hence $\Delta_{\mathfrak{A}}$ is Cauchy complete in that sense, with the caveat that we only require *countable* (and not arbitrary) direct sums.

Since $\Delta_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ has countable direct sums, we can add them to our category $\Delta_{\mathfrak{A}_1} \boxtimes_0 \Delta_{\mathfrak{A}_2}$.

Definition 4.8. We write $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ for the full subcategory of $\Delta_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ obtained by adding all (finite or countable) direct sums of objects to $\Delta_{\mathfrak{A}_1} \boxtimes_0 \Delta_{\mathfrak{A}_2}$.

Since $\Delta_{\mathfrak{A}_1} \boxtimes_0 \Delta_{\mathfrak{A}_2}$ has subobjects (the only subobjects are trivial), by construction $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ also has subobjects.

Remark 4.9. More generally, one can give an abstract construction of this category, see for example [HNP24]. Briefly, we can define a tensor product of two Cauchy complete W*-categories $\mathcal{C}, \mathcal{D}$. This category will in general not include all direct sums, but one can take a 'completion' to add them. Formally, if $\mathcal{C}$ is a W*-category, we can define a new category with as objects formal direct sums $\bigoplus_{i\in I} \rho_i$ with morphisms $\bigoplus_{i \in I} T_i$, such that the set $\{T_i\}_{i\in I}$ is bounded. Similarly, if $\mathcal{C}$ does not have subobjects, we can add them as well. The resulting category is denoted $\mathcal{C}^{\overline{\oplus}}$ and is called the *Cauchy completion* of $\mathcal{C}$ (it does not matter if we first add direct sums, or first add subobjects). Applying this to the tensor product of two Cauchy-complete W*-categories as mentioned earlier gives [HNP24, Definition 3.33]. When applied to $\Delta_{\mathfrak{A}_1}$ and $\Delta_{\mathfrak{A}_2}$, this precisely gives a category that is equivalent (as W*-categories) to $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$, when we restrict to adding at most countable direct sums only. In the notation of [Yam07], this means we are working in the category $\mathscr{S}\mathscr{R}ep$ of normal representations on *separable* Hilbert spaces only.

We now make some observations that will be useful later. Suppose that $\rho \in \Delta_{\mathfrak{A}_1}$ and $\sigma \in \Delta_{\mathfrak{A}_2}$. Then it follows from Proposition 3.25 that $\rho \cong \bigoplus_{i\in I} \rho_i$ and $\sigma \cong \bigoplus_{j \in J} \sigma_j$, with the ρ_i and σ_j irreducible. Write $\{V_i\}$ and $\{W_j\}$ for the corresponding isometries as in Definition 4.6. Note that it follows that $Z_{ij} := V_i \otimes W_j \in \mathfrak{A}_{12}^{\Lambda_a}$ forms a (at most) countable set of isometries such that $Z_{ij}^* Z_{k\ell} = \delta_{i,k}\delta_{j,\ell} = \mathbb{I} \otimes \mathbb{I}$ and $\sum_{i,j} Z_{i,j} Z_{i,j}^*$ converges to $\mathbb{I} \otimes \mathbb{I}$ in the strong operator topology.

It follows that we can identify the pair (ρ, σ) with an object in $\boxtimes$, and we will denote it by $\rho \boxtimes \sigma$. From the construction it follows that $\boxtimes$ distributes over direct sums in both variables. That is, the construction of $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ is compatible with the direct sums in the original categories in the expected way. More concretely, we can define a functor $\overline{F} : \Delta_{\mathfrak{A}_1} \times \Delta_{\mathfrak{A}_2} \to \Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ by setting $\overline{F}((\rho,\sigma)) = \rho \boxtimes \sigma$ (and similarly for the morphisms in the category).

This functor is bilinear, and satisfies

$$\overline{F}\left(\bigoplus_{i\in I}\rho_i,\bigoplus_{j\in J}\sigma_j\right)=\left(\bigoplus_{i\in I}\rho_i\right)\boxtimes\left(\bigoplus_{j\in J}\sigma_j\right)\cong\bigoplus_{i\in I,j\in J}\rho_i\boxtimes\sigma_j=\bigoplus_{i\in I,j\in J}\overline{F}((\rho_i,\sigma_j))$$

for all (at most countable) direct sums.

Note that by definition $\Delta_{\mathfrak{A}_1}\boxtimes\Delta_{\mathfrak{A}_2}$ is a full subcategory of $\Delta_{\mathfrak{A}_1\otimes\mathfrak{A}_2}$. Hence a natural question is if they are equivalent, that is, if the inclusion functor is essentially surjective. But by Proposition 3.25, it follows that every object in $\Delta_{\mathfrak{A}_1\otimes\mathfrak{A}_2}$ can be written as a countable direct sum of irreducibles. By Theorem B, each of this irreducibles is equivalent to $\rho\otimes_{\mathrm{s}}\sigma$, with ρ and σ irreducible. But this is an object of $\Delta_{\mathfrak{A}_1}\boxtimes_0\Delta_{\mathfrak{A}_2}$, and hence the following lemma follows.

Lemma 4.10. *The inclusion functor* $\iota:\Delta_{\mathfrak{A}_1}\boxtimes\Delta_{\mathfrak{A}_2}\hookrightarrow\Delta_{\mathfrak{A}_1\otimes_{\mathrm{s}}\mathfrak{A}_2}$ *is an equivalence of* W*-*categories.*

Remark 4.11. Again, this can be proven more abstractly, and it is enough to check that $\Delta_{\mathfrak{A}_1}\boxtimes_0\Delta_{\mathfrak{A}_2}$ is a subcategory of generators for $\Delta_{\mathfrak{A}_1\otimes\mathfrak{A}_2}$, see [HNP24, Lemma 3.16].

Note that the categories $\Delta_{\mathfrak{A}_i}$ come with a distinguished object, namely the reference representation π_i. Hence we can define full and faithful functors $\Delta_{\mathfrak{A}_i}\to\Delta_{\mathfrak{A}_1}\times\Delta_{\mathfrak{A}_2}$ by setting $\rho\mapsto(\rho,\pi_2)$ (resp. $\sigma\mapsto(\pi_1,\sigma)$). Similarly, we can define inclusions $F_i:\Delta_{\mathfrak{A}_i}\to\Delta_{\mathfrak{A}_1}\boxtimes\Delta_{\mathfrak{A}_2}$ by setting $F_1(\rho):=\rho\boxtimes\pi_2$ (resp. $F_2(\sigma):=\pi_1\boxtimes\sigma$). Finally, stacking with the reference representation of the other layer leads to full and faithful functors $\iota_i:\Delta_{\mathfrak{A}_i}\hookrightarrow\Delta_{\mathfrak{A}_1\otimes_{\mathrm{s}}\mathfrak{A}_2}$. This gives the following commutative diagram:

$$\begin{tikzcd}
 & \Delta_{\mathfrak{A}_1}\times\Delta_{\mathfrak{A}_2} \arrow[d, dashed, "\bar{F}"] & \\
\Delta_{\mathfrak{A}_1} \arrow[ur, hook] \arrow[r, "F_1"] \arrow[dr, "\iota_1"'] & \Delta_{\mathfrak{A}_1}\boxtimes\Delta_{\mathfrak{A}_2} \arrow[d, dashed, "\iota"', "\cong"] & \Delta_{\mathfrak{A}_2} \arrow[ul, hook'] \arrow[l, "F_2"'] \arrow[dl, "\iota_2"] \\
 & \Delta_{\mathfrak{A}_1\otimes_{\mathrm{s}}\mathfrak{A}_2} &
\end{tikzcd} \tag{4.1}$$

Here, the functors in the bottom half of the diagram (including the horizontal ones) are all W*-functors.

To complete the picture, it remains to consider the monoidal structure ('fusion') and the braiding on $\Delta_{\mathfrak{A}_1}\boxtimes\Delta_{\mathfrak{A}_2}$. To define this, if $\pi\in\Delta_{\mathfrak{A}_1}$, write π^a for its extension to an endomorphism of $\mathfrak{A}_1^{\Lambda_a}$. Then for $\rho,\sigma\in\Delta_{\mathfrak{A}_1}$, we can define $\rho\widehat{\otimes}\sigma:=\rho^a\circ\sigma$. Here we use $\widehat{\otimes}$ for the tensor product in the category, to distinguish it from the (many) other tensor products we have considered so far. It can be shown that $\rho\widehat{\otimes}\sigma$ an object in $\Delta_{\mathfrak{A}_1}$. To define the monoidal product on the morphisms, consider $S\in\mathrm{Hom}_{\Delta_{\mathfrak{A}_1}}(\rho,\rho')$ and $T\in\mathrm{Hom}_{\Delta_{\mathfrak{A}_1}}(\sigma,\sigma')$. Using approximate Haag duality, one can show that $S,T\in\mathfrak{A}_1^{\Lambda_a}$. This is in fact true for any intertwiner between representations ρ and σ (strictly) localized in two cones Λ_1 and Λ_2 in the sense of Definition 4.1 as long are both cones are admissible (this is important when defining the braiding). Using this observation, we can define $S\widehat{\otimes}T=S\rho^a(T)$. This gives a strict monoidal category $(\Delta_{\mathfrak{A}_1},\widehat{\otimes},\pi_1)$. Finally, we can define a braiding $\epsilon_{\rho,\sigma}\in\mathrm{Hom}_{\Delta_{\mathfrak{A}_1}}(\rho\widehat{\otimes}\sigma,\sigma\widehat{\otimes}\rho)$. The idea behind the definition of the braiding is that we move one of the representations (say, the one in the second variable) far away, show that we can then interchange both in the monoidal product, and move the representation back. Again we refer the reader to [Oga22] for the full details.

There is a natural monoidal product and braiding on $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$, coming from the corresponding operations of $\Delta_{\mathfrak{A}_1}$ (resp. $\Delta_{\mathfrak{A}_2}$). For pairs $\rho_{\mathfrak{A}_1} \boxtimes \rho_{\mathfrak{A}_2}$ and $\sigma_{\mathfrak{A}_1} \boxtimes \sigma_{\mathfrak{A}_2}$, we define

$$(\rho_{\mathfrak{A}_1} \boxtimes \rho_{\mathfrak{A}_2}) \widehat{\boxtimes} (\sigma_{\mathfrak{A}_1} \boxtimes \sigma_{\mathfrak{A}_2}) := \left(\rho_{\mathfrak{A}_1} \widehat{\otimes} \sigma_{\mathfrak{A}_1}\right) \boxtimes \left(\rho_{\mathfrak{A}_2} \widehat{\otimes} \sigma_{\mathfrak{A}_2}\right).$$

The monoidal product on morphisms is also given component-wise. These constructions can be extended to the full category $\boxtimes$ by defining $\widehat{\boxtimes}$ on countable direct sums. For example,

$$\left(\bigoplus_{i\in I} \rho_i \boxtimes \sigma_i\right) \widehat{\boxtimes} \left(\bigoplus_{j\in J} \tau_j \boxtimes \zeta_j\right) := \bigotimes_{i\in I, j\in J} (\rho_i \widehat{\otimes} \tau_j) \boxtimes (\sigma_i \widehat{\otimes} \zeta_j).$$

The monoidal unit is given by $\pi_1 \boxtimes \pi_2$. Using the construction above, similarly the braiding $\varepsilon^i_{\rho_{\mathfrak{A}_i}, \sigma_{\mathfrak{A}_i}}$ of $\Delta_{\mathfrak{A}_i}$ extends to $\boxtimes$, by setting

$$\varepsilon^{\boxtimes}_{\rho_{\mathfrak{A}_1} \boxtimes \rho_{\mathfrak{A}_2}} := \varepsilon^1_{\rho_{\mathfrak{A}_1}, \sigma_{\mathfrak{A}_1}} \boxtimes \varepsilon^2_{\rho_{\mathfrak{A}_2}, \sigma_{\mathfrak{A}_2}},$$

and extended as earlier to all of $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$. This makes $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ into a braided W*-category $(\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}, \widehat{\boxtimes}, \pi_1 \boxtimes \pi_2)$.[4]

Remark 4.12. We emphasize that the category $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ represents a pairing of *decoupled* quantum systems. That is, there are no 'string excitations' which connect the two layers $\Delta_{\mathfrak{A}_1}$ and $\Delta_{\mathfrak{A}_2}$. With this in mind, the constructed monoidal product and braiding morphism respect this property and are given as formal tensor products of the product or braiding in individual layers.

In view of Remark 4.5, note that $\mathfrak{A}_1^{\Lambda_a} \otimes_{\mathrm{s}} \mathfrak{A}_2^{\Lambda_a} \subset \mathfrak{A}_{12}^{\Lambda_a}$. We then have the following lemma.

Lemma 4.13. *Let $\rho_i \in \Delta_{\mathfrak{A}_i}$. Then the following equation holds for all $A \in \mathfrak{A}_1^{\Lambda_a} \otimes_{\mathrm{s}} \mathfrak{A}_2^{\Lambda_a}$:*

$$(\rho_1^a \otimes_{\mathrm{s}} \rho_2^a)(A) = (\rho_1 \otimes_{\mathrm{s}} \rho_2)^a(A).$$

Here on the left-hand side, we first extend to the auxiliary algebra and then stack, while on the right-hand side we do it the other way round (note that the extension there is to $\mathfrak{A}_{12}^{\Lambda_a}$).

Proof. Recall from the definition of ρ_i^a (see [Oga22, Lem. 2.13]) that it is enough to define ρ_i^a on cone algebras $\pi_i(\mathfrak{A}(\Lambda'))''$ for all admissible cones Λ'. Given such a cone, it follows from the selection criterion that there is a unitary $V_i \in B(\mathcal{H}_{\pi_i})$, where V_i is a charge transporter that transports ρ_i to a representation that is contained in the cone Λ_a (and sufficiently far away from Λ'. We then have $\rho_i^a(A) := V_i A V_i^*$ for $A \in \mathfrak{A}_1(\Lambda')''$. But the unitary $V_2 \otimes_{\mathrm{s}} V_2$ is such a charge transporter for $\rho_1 \otimes_{\mathrm{s}} \rho_2$. Hence for $A_i \in \mathfrak{A}_i(\Lambda')''$, we have

$$(\rho_1 \otimes_{\mathrm{s}} \rho_2)^a(A_1 \otimes_{\mathrm{s}} A_2) = (V_1 \otimes_{\mathrm{s}} V_2)(A_1 \otimes_{\mathrm{s}} A_2)(V_1 \otimes_{\mathrm{s}} V_2) = V_1 A V_1^* \otimes_{\mathrm{s}} V_2 \otimes_{\mathrm{s}} V_2 = (\rho_1^a \otimes_{\mathrm{s}} \rho_2^a)(A_1 \otimes_{\mathrm{s}} A_2).$$

The result then follows for general $A \in \mathfrak{A}_1^{\Lambda_a} \otimes_{\mathrm{s}} \mathfrak{A}_2^{\Lambda_a}$ by using that all maps are $*$-endomorphisms. $\square$

[4]Unlike $\Delta_{\mathfrak{A}_1}$, the category $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ is no longer a *strict* tensor category because direct sums are only unique up to unitary isomoprhisms.

Recall that $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ is a full subcategory of $\Delta_{\mathfrak{A}_1 \otimes_s \mathfrak{A}_2}$, which has its own monoidal product and braiding defined through the sector theory. It turns out that these agree with those on $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$.

Proposition 4.14. *The embedding functor $\iota : \Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2} \to \Delta_{\mathfrak{A}_1 \otimes_s \mathfrak{A}_2}$ is a braided monoidal functor.*

Proof. To see that ι is a monoidal functor with respect to the tensor products defined, it is sufficient to show this on the simple objects, as ι is a normal $*$-functor (in the language of W*-categories), meaning it sends direct sums to direct sums. Because of Remark 4.5, it is enough to show that

$$(\rho_1 \otimes_s \rho_2)^a \circ (\sigma_1 \otimes_s \sigma_2)(A) = (\rho_1^a \circ \sigma_1 \otimes_s \rho_2^a \circ \sigma_2)(A)$$

for all $A \in \mathfrak{A}_1 \otimes_s \mathfrak{A}_2$. But this follows from Lemma 4.13, since $\sigma_i(A_i) \in \mathfrak{A}_1^{\Lambda_a}$. It is then readily checked that the tensor products on the morphisms are the same as well.

Finally, we would like to see that this functor is braided with respect to the braiding in $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$. Again it is enough to verify it for 'simple tensors' of the form $\rho \boxtimes \sigma$, using naturality of the braiding and that ι preserves direct sums. Hence the problem reduces to verifying that

$$\varepsilon^1_{\rho_{\mathfrak{A}_1}, \sigma_{\mathfrak{A}_1}} \boxtimes \varepsilon^2_{\rho_{\mathfrak{A}_2}, \sigma_{\mathfrak{A}_2}} = \varepsilon_{\rho_{\mathfrak{A}_1} \otimes_s \sigma_{\mathfrak{A}_1}, \rho_{\mathfrak{A}_2} \otimes_s \sigma_{\mathfrak{A}_2}}.$$

To see this, recall that the braiding is defined by choosing a sequence of unitaries that moves one of the (i.e., a sequence of charge transporters). But by the same observation as in the proof of Lemma 4.13, choosing these sequences for each of the stacked layers gives a similar sequence for the stacked representations, by stacking the unitaries. These can then be used to define the braiding ε of $\Delta_{\mathfrak{A}_1 \otimes_s \mathfrak{A}_2}$, and the equality above follows. □

Recall that just for ordinary functors, if we have a braided monoidal full and faithful functors between braided categories which is also essentially surjective, this implies that the two categories are *braided* equivalent. Hence the results of this section can be summarised with the following theorem.

Theorem 4.15 (Theorem C)**.** *The categories $\Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ and $\Delta_{\mathfrak{A}_1 \otimes_s \mathfrak{A}_2}$ are equivalent as braided* W**-categories.*

And so we have constructed a braided tensor category which is equivalent to the stacked system sector theory. The following remarks demonstrate the advantages of having done so.

Remark 4.16. The work in [Oga25] establishes a braided faithful functor between the individual layer category $\Delta_{\mathfrak{A}_1}$ and the stacked system category $\Delta_{\mathfrak{A}_1 \otimes_s \mathfrak{A}_2}$ via a *-isomorphism $\Theta : \pi_\varphi(\mathfrak{A}_1)'' \to \pi(\mathfrak{A}_1)''$ with φ the restriction of a state on $\mathfrak{A}_1 \otimes_s \mathfrak{A}_2$ to one algebra $\mathfrak{A}_1$. This construction can be extended to the stacking functor ι via the inclusion $F_1 : \Delta_{\mathfrak{A}_1} \hookrightarrow \Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$ given by pairing with the trivial (reference) representation, as defined above.

We need to see that $\iota \circ F_1 : \Delta_{\mathfrak{A}_1} \to \Delta_{\mathfrak{A}_1 \otimes_s \mathfrak{A}_2}$ preserves the monoidal structure of the categories. Since a composition of braided monoidal functors gives a braided monoidal functor, it only remains to see that the inclusion F_1 is indeed a functor of this nature. To see it is monoidal, consider $\rho_1 \hat{\otimes} \rho_2 \in \Delta_{\mathfrak{A}_1}$. Then we have

$$F_1(\rho_1 \hat{\otimes} \rho_2) = (\rho_1 \hat{\otimes} \rho_2) \boxtimes \pi_2 = (\rho_1 \boxtimes \pi_2) \hat{\boxtimes} (\rho_2 \boxtimes \pi_2)$$

by using the monoidal product in this category. Hence, $F_1(\rho_1 \hat{\otimes} \rho_2) = F_1(\rho_1) \hat{\boxtimes} F_1(\rho_2)$.

The proof that F_1 preserves the braiding is similar. Therefore, also the composition $\iota \circ F_1$ carries over the monoidal structure of the categories. A similar argument can be made for $F_2 : \Delta_{\mathfrak{A}_2} \to \Delta_{\mathfrak{A}_1} \boxtimes \Delta_{\mathfrak{A}_2}$. That is to say, the functors in the lower part of diagram 4.1 are all braided monoidal functors.

Remark 4.17. The above results show that the sectors of the stacked system are related to the sectors of the two layers. In applications, it is often desirable to restrict to the full subcategory $\Delta^f_{\mathfrak{A}_1}$ of sectors which are dualisable, namely, they have conjugates. Having dual (conjugate) objects is usually assumed in algebraic descriptions of the theory of anyons [Kit06, Appendix E], and indeed is part of the requirements of a UMTC. Physically, this means that we only consider anyons for which there is a conjugate charge, namely an anti-particle. It follows that all Hom-spaces in the category are finite dimensional, which implies that every sector in the category can be written as a finite direct sum of irreducible sectors [LR97]. In this case, we have that $\Delta^f_{\mathfrak{A}_1} \boxtimes \Delta^f_{\mathfrak{A}_2} \cong \Delta^f_{\mathfrak{A}_1 \otimes_s \mathfrak{A}_2}$, where in the construction of $\boxtimes$ we now only add *finite* direct sums. In this case, $\boxtimes$ is essentially the Deligne product of monoidal categories (apart from the technical point that our categories strictly speaking are not Abelian as we exclude the zero representation for convenience).

5. Examples

5.1. **Stacking quantum double models.** Quantum double models [Kit03] form an important class of examples of anyonic systems, and therefore it is useful to analyze the stacking operation specifically in this context. Indeed, the quantum double model of any finite group has only finitely many superselection sectors [BV25], satisfies bounded spread Haag duality [OPR25], and satisfies the approximate split property, so our main results apply directly to this situation. As an example, we show that the quantum double model for direct product groups $G \times H$ can be naturally interpreted in terms of the stacking operation. Namely, the quantum double model for $G \times H$ is equivalent to the stacking of the model for $D(G)$ and that for $D(H)$, where $D(G)$ is the quantum double of the group algebra $\mathbb{C}[G]$. If we write Δ^f_G for the braided category of (dualizable) sectors of the quantum double model for G, this analysis implies that $\Delta^f_{G\times H} \cong \Delta^f_G \boxtimes \Delta^f_H$ as unitary braided monoidal categories.

This result can be understood purely algebraically. One can check that $D(G \times H) \cong D(G) \otimes D(H)$ as quasi-triangular Hopf algebras. This implies the result that $\mathbf{Rep}_f D(G \times H) \cong \mathbf{Rep}_f D(G) \boxtimes \mathbf{Rep}_f D(H)$, where $\mathbf{Rep}_f$ is the (unitary braided) category of finite dimensional unitary $D(G)$-modules (cf. [EGNO15, Prop. 1.11.2] or [Müg03c, Sect. 4.2]).

We begin by reviewing some basic concepts, as well describing the setting and conventions we will use for the remainder of this section. Fix two finite groups, G and H, and denote their direct product by $K = G \times H := \{g \times h \mid g \in G, h \in H\}$. For $J = G, H, K$, let $\mathcal{H}_J$ denote the Hilbert space formed by formal linear combinations of group elements,

$$\sum_{j\in J} \lambda_j |j\rangle, \ \lambda_j \in \mathbb{C},$$

where the group elements of J are taken to be an orthonormal basis. Hence, we may write

$$\mathcal{H}_J = \mathrm{span}\{|j\rangle \mid j \in J\}.$$

Since $K = G \times H$, it is clear that we may make the identification $\mathcal{H}_K \equiv \mathcal{H}_G \otimes \mathcal{H}_H$ by identifying $|g \times h\rangle \equiv |g\rangle \otimes |h\rangle$ for any $g \in G$ and $h \in H$.

In addition to forming a basis for $\mathcal{H}_J$, we also identify each group element $j \in J$ with the unitary defined by $j|\tilde{j}\rangle := |j\tilde{j}\rangle$ for any $\tilde{j} \in J$, and then extended to all of $\mathcal{H}_J$ by linearity. This is indeed a unitary since each group element simply permutes the defining basis of $\mathcal{H}_J$ by acting on the underlying group on the left. The group C*-algebra of J is defined as the subalgebra $C^*(J) \subset B(\mathcal{H}_J)$ that is generated by the unitaries corresponding to the elements of J. By our identification $|g \times h\rangle \equiv |g\rangle \otimes |h\rangle$, it is clear we can make the identification $g \times h \equiv g \otimes h$ as unitaries acting on $\mathcal{H}_K$. Thus, we have that $C^*(K) = C^*(G) \otimes C^*(H)$.

Let $\mathfrak{A}_J$ denote the quasi-local algebra on the lattice Γ whose on-site algebras are given by copies of $B(\ell^2(J)) \cong M_{|J|}(\mathbb{C})$. Here we will take our sites to be the edges of Γ rather than the vertices or plaquettes, as is customary for the quantum double model. What is more, the edges will be taken to be directed. Since $|K| = |G| \cdot |H|$, we can identify $M_{|K|}(\mathbb{C}) \cong M_{|G|}(\mathbb{C}) \otimes M_{|H|}(\mathbb{C})$. We do this identification in the obvious way, namely such that the unitary (g, h) corresponds to $g \otimes h$ under the isomorphism. Using this identification, with a slight abuse of notation, we have that $\mathfrak{A}_K = \mathfrak{A}_G \otimes_{\mathrm{s}} \mathfrak{A}_H$.

For each vertex v and plaquette p, denote by S_v and S_p the set of all edges incident with v and the set of edges forming the boundary of p respectively. Let $A_J^v(j)$ denote the vertex operator on the local algebra $\mathfrak{A}_J(S_v)$ associated to $j \in J$, and B_J^p the plaquette operator on the local algebra $\mathfrak{A}_J(S_p)$. We refer the reader to a standard reference, such as [Kit03] for the precise definition of these operators. Also denote

$$A_J^v = \frac{1}{|J|} \sum_{j \in J} A_J^v(j).$$

The Hamiltonian for the quantum double model of J is given by

$$H_J = \sum_v (\mathbb{I} - A_J^v) + \sum_p (\mathbb{I} - B_J^p).$$

Of course, in the infinite volume case this sum does not converge, but $i[H_J, O]$ is well-defined for local operators O and defines a derivation that generates the dynamics of the system, and hence we can talk about ground states. We say a state ω_J on $\mathfrak{A}_J$ is *frustration-free* if $\omega_J(A_J^v) = \omega_J(B_J^p) = 1$ for every vertex v and plaquette p. It can be shown, though we will not do so here, that there is only one frustration-free ground state ω_j on $\mathfrak{A}_J$ [FN15].

Proposition 5.1. *Let ω_G and ω_H be the unique frustration-free ground states on $\mathfrak{A}_G$ and $\mathfrak{A}_H$ respectively. Then the frustration-free ground state on $\mathfrak{A}_K$ is given by $\omega_K = \omega_G \otimes_{\mathrm{s}} \omega_H$.*

Proof. Let v be a vertex and p a plaquette. For every $\phi \in \bigotimes_{e \in S_v} \mathcal{H}_G$ and $\psi \in \bigotimes_{e \in S_v} \mathcal{H}_H$, we have

$$A_K^v(|\phi\rangle \otimes |\psi\rangle) = \frac{1}{|K|} \sum_{g \times h \in K} A_K^v(g \times h)(|\phi\rangle \otimes |\psi\rangle)$$

$$= \frac{1}{|G|} \frac{1}{|H|} \sum_{g \in G} \sum_{h \in H} A_G^v(g)|\phi\rangle \otimes A_H^v(h)|\psi\rangle = A_G^v|\phi\rangle \otimes A_H^v|\psi\rangle.$$

Moreover, for $\phi \in \bigotimes_{e \in S_p} \mathfrak{H}_G$ and $\psi \in \bigotimes_{e \in S_p} \mathfrak{H}_H$, a similar computation yields

$$B_K^p(|\phi\rangle \otimes |\phi\rangle) = B_G^p|\phi\rangle \otimes B_H^p|\psi\rangle.$$

Thus, as quasi-local operators on the whole lattice we have that $A_K^v = A_G^v \otimes_{\mathrm{s}} A_H^v$ and $B_K^p = B_G^p \otimes_{\mathrm{s}} B_H^p$, and so

$$\omega_G \otimes_{\mathrm{s}} \omega_H(A_K^v) = \omega_G(A_G^v)\omega_H(A_H^v) = 1 \text{ and } \omega_G \otimes_{\mathrm{s}} \omega_H(B_K^p) = \omega_G(B_G^p)\omega_H(B_H^p) = 1.$$

By uniqueness of the frustration-free ground state, we therefore have $\omega_K = \omega_G \otimes_{\mathrm{s}} \omega_H$. $\square$

The category of superselection sectors for the quantum double model has been worked out explicitly [BV25; BHNV25]. It can be shown that every irreducible sector is dualizable, and hence it is natural to only consider finite direct sums of irreducible sectors. The corresponding category, which we denote Δ_G^f, is then unitary braided equivalent to $\mathbf{Rep}_f D(G)$. Hence if we apply Theorem C to the result of Proposition 5.1, it follows that $\mathbf{Rep}_f(D(K)) \cong \mathbf{Rep}_f(D(G)) \boxtimes \mathbf{Rep}_f(D(H))$, where each of the categories is equivalent to the corresponding DHR category. This reproduces the result from Hopf algebra theory mentioned earlier.

If we are just interested in the superselection sectors (and not the fusion rules and braiding), this result can be understood more directly. We will now sketch the idea. By the above and Proposition 3.4, we see that if $(\pi_J, \mathcal{H}_J, \Omega_J)$ is the GNS triple corresponding to the frustration-free ground state on the quantum double model $\mathfrak{A}_J$ for $J = G, H, K$, then

$$(\pi_G \otimes_{\mathrm{s}} \pi_H, \mathcal{H}_G \otimes \mathcal{H}_H, \Omega_G \otimes \Omega_H) = (\pi_K, \mathcal{H}_K, \Omega_K).$$

Our goal will therefore be to understand the representations satisfying $\mathrm{SSC}(\pi_K)$ in terms of those satisfying $\mathrm{SSC}(\pi_G)$ and $\mathrm{SSC}(\pi_H)$.

A central role in the analysis of the sector structure is played by so-called ribbon operators $F^{g_1,g_2}(\xi)$, where ξ is a 'ribbon' (we refer to e.g. [Kit03] or [BV25] for the precise definition). These operators create a pair of (conjugate) excitations at the endpoints of the ribbon. Representatives of superselection sectors are then obtained by a limiting procedure, where one end of the ribbon is sent to infinity. It is known that irreducible representations of $D(G)$ are in one-one correspondence with pairs (C, R), where C is a conjugacy class of G, and R an irreducible representation of the centralizer of an element in C [Gou93]. An indeed, the ribbon operators can be grouped into multiplets corresponding to such pairs (C, ρ), and these lead precisely to representatives of each irreducible sector [BV25]. We will now write $\underline{\Delta}_J$ for the set of equivalence classes of irreducible sectors.

Our ultimate goal will be to show that $\underline{\Delta}_K = \underline{\Delta}_G \otimes_{\mathrm{s}} \underline{\Delta}_H := \{[\rho_G \otimes_{\mathrm{s}} \rho_H] \mid [\rho_G] \in \underline{\Delta}_G, [\rho_H] \in \underline{\Delta}_H\}$. Indeed one can accomplish this by unpacking the definition of the ribbon operators and showing that for any $g_1, g_2 \in G$ and $h_1, h_2 \in H$,

$$F^{(g_1 \times h_1, g_2 \times h_2)}(\xi) = F^{(g_1,g_2)}(\xi) \otimes_{\mathrm{s}} F^{(h_1,h_2)}(\xi).$$

However, since the $\underline{\Delta}_J$'s are finite sets, the result can be established by simply counting the possible representations.

Proposition 5.2. *Let G, H be finite groups and $K = G \times H$. Then $|\underline{\Delta}_K| = |\underline{\Delta}_G||\underline{\Delta}_H|$.*

Proof. For each group J, let C_J denote the set of conjugacy classes of J and for each $c \in C_J$ let R_c denote the set of unitary equivalence classes of irreducible representations of N_c. For any $g \times h \in K$,

$$\{(\tilde{g} \times \tilde{h})^{-1}(g \times h)(\tilde{g} \times \tilde{h}) \mid \tilde{g} \times \tilde{h} \in K\} = \{\tilde{g}^{-1}g\tilde{g} \times \tilde{h}^{-1}h\tilde{h} \mid \tilde{g} \in G, \tilde{h} \in H\}$$
$$= \{\tilde{g}^{-1}g\tilde{g} \mid \tilde{g} \in G\} \times \{\tilde{h}^{-1}h\tilde{h} \mid \tilde{h} \in H\},$$

and so $C_K = \{c \times c' \mid c \in C_G, c' \in C_H\}$. This implies immediately $|C_K| = |C_G||C_H|$. Moreover, for any $g \in c \in C_G$ and $h \in c' \in C_H$,

$$\begin{aligned}\{g' \times h' \in K \mid (g' \times h')(g \times h) = (g \times h)(g' \times h')\}\\ = \{g' \in G \mid gg' = g'g\} \times \{h' \in H \mid hh' = h'h\},\end{aligned}$$

so $N_{c\times c'} = N_c \times N_{c'}$. It is a standard fact from the representation theory of finite groups that the irreducible representations of a direct product of groups are just the tensor product of the representations of the individual groups. Thus, $|R_{c\times c'}| = |R_c||R_{c'}|$. Therefore the computation

$$|\underline{\Delta}_K| = \sum_{\tilde{c}\in C_K} |R_{\tilde{c}}| = \sum_{c\in C_G, c'\in C_H} |R_c||R_{c'}| = \Big(\sum_{c\in C_G} |R_c|\Big)\Big(\sum_{c'\in C_H} |R_{c'}|\Big) = |\underline{\Delta}_G||\underline{\Delta}_H|$$

concludes the proof. □

Corollary 5.3. *Let G, H be finite groups and $K = G \times H$. Then $\underline{\Delta}_K = \underline{\Delta}_G \otimes_{\mathrm{s}} \underline{\Delta}_H$.*

Proof. By Theorem 3.11, if ρ_G, ρ_H are irreducible representations satisfying $\mathrm{SSC}(\pi_G)$, respectively $\mathrm{SSC}(\pi_H)$, then $\rho_G \otimes \rho_H$ satisfies $\mathrm{SSC}(\pi_K)$. Since $U_G \in \mathcal{U}(\mathcal{H}_G)), U_H \in \mathcal{U}(\mathcal{H}_H))$ implies $U_G \otimes U_H \in \mathcal{U}(\mathcal{H}_G \otimes \mathcal{H}_H))$ and $\mathcal{H}_G \otimes \mathcal{H}_H = \mathcal{H}_K$, this implies that for any $[\rho_G] \in \underline{\Delta}_G$ and $[\rho_H] \in \Delta_H$, $[\rho_G \otimes \rho_H] \in \underline{\Delta}_K$. Moreover, by Proposition 3.31 we have that $\rho_G \otimes \rho_H$ is not equivalent to $\sigma_G \otimes \sigma_H$ unless $\rho_G \cong \sigma_G$ and $\rho_G \cong \sigma_H$. We therefore have that $\underline{\Delta}_G \otimes_{\mathrm{s}} \underline{\Delta}_H \subset \underline{\Delta}_K$. Finally,

$$|\underline{\Delta}_G \otimes_{\mathrm{s}} \underline{\Delta}_H| = |\{[\rho_G \otimes \rho_H] \mid [\rho_G] \in \underline{\Delta}_G \text{ and } [\rho_H] \in \underline{\Delta}_H\}| = |\underline{\Delta}_G||\underline{\Delta}_H| = |\underline{\Delta}_K|,$$

and since these are finite sets, we have $\underline{\Delta}_K = \underline{\Delta}_G \otimes_{\mathrm{s}} \underline{\Delta}_H$. □

5.2. **Morita-equivalent Levin–Wen models.** We now turn to Levin–Wen string-net models [LW06; LLB21], which are commuting-projector Hamiltonians associated with a unitary fusion category $\mathcal{D}$. Let ω be the unique frustration-free ground state of the model (the existence of which follows from [JNPW25, Thm. 4.8 and Rem. 2.25] or [QW20]). Then its GNS representation satisfies approximate Haag duality by [OPR25]. Hence we can define the braided C*-category of superselection sectors Δ_ω. This model is believed to have finitely many irreducible superselection sectors (a proof is announced in [BK25]) and satisfies the assumptions in Sect. 3.5. Hence this model has the Type I property.[5]

An important question is how Δ_ω depends on the input category $\mathcal{D}$. While $\mathcal{D}$ determines the microscopic description of the model, in particular the dimension of the local Hilbert spaces at each site and the local interactions, different local descriptions may lead to equivalent emerging 'global' properties, such as the category Δ_ω of superselection sectors. Because the local dimensions for different $\mathcal{D}$'s are in general different, there cannot be an LGA connecting the ground states of such different models. However, a connecting automorphism may be constructed after the addition of degrees of freedom at each site, namely by stacking with a trivial system: One then speaks of *stable equivalence*, see again [Kit13]. Recalling [NO22] that product states have trivial superselection structure, our results confirm that such a stabilization leaves the sector theory invariant indeed. The Levin–Wen

[5]Since we will only need stacking with trivial product states, alternatively one could use the results in [Oga25] instead of the Type I property.

examples illustrate that it is needed, see also [BBR25, Appendix B] for the simplest example of non-interacting fermions.

To state the result, we first recall the notion of Morita equivalence for tensor categories [Müg03a].

Definition 5.4. Two unitary fusion categories $\mathcal{D}_1, \mathcal{D}_2$ are called *Morita equivalent* if they have equivalent centres: $\mathcal{Z}(\mathcal{D}_1) \cong \mathcal{Z}(\mathcal{D}_2)$. Here the equivalence is as unitary braided tensor categories.

Here $\mathcal{Z}(\mathcal{D})$ is the quantum double (or Drinfeld centre) of $\mathcal{D}$, a generalisation of the quantum double construction for Hopf algebras to tensor categories. See e.g. [Müg03b] for the definition in the case of the C*-categories we are considering. The definition of Morita equivalence given above is equivalent to the one given in [Müg03a] by [EGNO15, Thm. 8.12.3].

Proposition 5.5. *The ground states of Levin–Wen string-net models with Morita equivalent input categories have the same superselection structure, namely* $\Delta_{\omega_1} \cong \Delta_{\omega_2}$.

Proof. Let ω_1, ω_2 be the ground states of Levin–Wen models from Morita equivalent categories $\mathcal{D}_1$ and $\mathcal{D}_2$. By [LVSV22], there is a finite-depth quantum circuit mapping between ground states of two Morita equivalent string-net models $\mathcal{D}_1, \mathcal{D}_2$. Since the total dimension of Hilbert spaces for the two models need not match, one first adds ancillary qudits at each site. In the present setting, this amounts to stacking the systems with a product state $\varphi_j, j = 1, 2$ to obtain algebras $\mathfrak{A}_{\omega_1} \otimes_{\mathrm{s}} \mathfrak{A}_{\varphi_1}$ and $\mathfrak{A}_{\omega_2} \otimes_{\mathrm{s}} \mathfrak{A}_{\varphi_2}$ of the same local dimension. On the stacked system one can define an FDQC $\alpha = \alpha_1 \circ \cdots \circ \alpha_n$ such that $(\omega_1 \otimes_{\mathrm{s}} \varphi_1) \circ \alpha = \omega_2 \otimes_{\mathrm{s}} \varphi_2$ Here each 'layer' α_i is an automorphism given by conjugating with unitaries with mutually disjoint support (and the size of the support is uniformly bounded). Such an automorphism α can be seen as a particularly simple LGA. We then have the following commuting diagram, where each equivalence is an equivalence of braided C*-categories:

$$\begin{tikzcd}
\Delta_{\omega_1} \arrow[r, dashed, "\cong"] \arrow[d, "\cong"'] & \Delta_{\omega_2} \\
\Delta_{\omega_1 \otimes_s \varphi_1} \arrow[r, "\alpha"'] & \Delta_{\omega_2 \otimes_s \varphi_2} \arrow[u, "\cong"']
\end{tikzcd}$$

The vertical equivalences follow from Theorem 4.15 and the comment at the beginning of the proof. The horizontal equivalence at the bottom is a consequence of the stability result of [Oga22] applied to the automorphism α of [LVSV22]. This completes the proof. □

Remark 5.6. As pointed out in the introduction, this shows that $\mathcal{Z}(\mathcal{D})$ completely characterizes the sector theory, but it does not require a priori knowledge of it. A proof that the superselection sectors for the Levin–Wen model with as input some unitary fusion category $\mathcal{D}$ is given by $\mathcal{Z}(\mathcal{D})$ has recently been announced [BK25].

Remark 5.7. A natural question is to determine under which additional condition two states ω_1, ω_2 (potentially defined on different systems) having equivalent superselection category are stably equivalent, namely there is an LGA connecting the two states, potentially after stacking with trivial systems. For a special class of models (which roughly correspond to the 'RG flow fix points') this follows from the results in [KR24], which shows that the ground states of these models can be connected to a Levin–Wen string-net state. Although its very definition in a general setting remains unclear, see [Sop24] for recent progress, a non-vanishing chiral central charge is believed to be an obstruction. As far as we are aware, the general case remains conjectural.

5.3. **Symmetry enriched toric code.** As a final example, we consider the *symmetry enriched* toric code model presented in [KVW24]. Consider the square lattice $\mathbb{Z}^2$. As in the usual treatment of the toric code, we put a qubit $\mathbb{C}^2$ on each edge. We give the edges an orientation by always having them point upwards or to the right. If e is an edge, we write $\partial_0 e$ for the vertex it points away from, and $\partial_1 e$ for the vertex it points to. Furthermore, we write $e \ni v$ if the edge e either points away from or towards v. In the symmetry enriched version, put an additional qubit $\mathbb{C}^2$ at each *vertex* in $\mathbb{Z}^2$. To distinguish the degrees of freedom on the edges and vertices, we write σ_e^k for the Pauli matrices on the edges, and τ_v^k for those on the vertices.

Recall that the interactions in the toric code are defined in terms of *star* and *plaquette* operators. For a vertex v, a star is the set of all edges either starting or ending in v (that is, $e \ni v$). Similarly, a plaquette is formed by the edges along a face of the lattice. For a face f, we write $e \in f$ for these edges. The star and plaquette operators are then defined as[6]

$$A_v = \bigotimes_{e \ni v} \sigma_e^x, \qquad B_f = \bigotimes_{e \in f} \sigma_e^z.$$

With this notation, we define the following interaction terms:

$$\widetilde{B}_f := i^{-\sum_{e\in f} \sigma_e^x(\tau^z_{\partial_1 e} - \tau^z_{\partial_0 e})/2} B_f, \qquad \widetilde{Q}_v := \frac{\mathbf{I} + A_v}{2} \tau_v^x i^{-\tau_v^z \sum_{e \ni v} f(e,v)\sigma_e^x/2}.$$

Here, $f(e,v)$ is equal to $+1$ if e points away from v, and -1 if e points toward v. Note that these dynamics couple the vertex with the edge degrees of freedom. It can be shown that this Hamiltonian has a unique (and hence pure) frustration free ground sate, which we denote by ω_{SET}.

We have the following results, which gives an alternative proof of [KVW24, Prop. 7.25], which does not require mimicking the toric code analysis in the symmetry enriched case.

Proposition 5.8. *The category Δ_{SET} is unitarily braided equivalent to Δ_{TC}, the category of superselection sectors of the toric code.*

Proof. We can consider the original toric code model and the sites at the vertices as two layers of a stacked system (cf. Remark 2.6). The first layer is the usual toric code model (see [Kit03; Naa11]) and we write ω_{TC} for its unique frustration free ground state. The toric code satisfies Haag duality and the distal split property, and there are only finitely many irreducible sectors [Naa13; FN15], and hence the Type I property is satisfied and our analysis applies. For the stacked layer of vertices, define the on-site Hamiltonian $H_v := (\mathbf{I} - \tau_v^x)/2$. This has a unique frustration free ground state, which we call ω_{sym}. Note that this is a product state, and hence has trivial superselection structure by [NO22]. From [KVW24, Sect. 7.2.2] there is a finite-depth quantum circuit α such that $\omega_{\mathrm{SET}} \circ \alpha = \omega_{\mathrm{TC}} \otimes_{\mathrm{s}} \omega_{\mathrm{sym}}$. The result then follows from Theorem 4.15 and [Oga22] (note in particular that [Oga22] implies that ω_{SET} satisfies at least approximate Haag duality). □

[6]This is the standard convention for the toric code. These operators are the same as obtained for the quantum double model in Sect. 5.1 for $G = \mathbb{Z}_2$ up to a shift by the identity.

The model has an on-site $\mathbb{Z}_2$-symmetry. Define the unitary $U_v^g := \tau_v^x$ for all vertices v, and $U_e^g := \mathbb{I}$ for all edges, where g is the non-trivial element of $\mathbb{Z}_2$. This defines an action $g \mapsto \beta_g$ of $\mathbb{Z}_2$ on the quasi-local algebra by conjugating on each site. The Hamiltonian above can be shown to be invariant under this action. The on-site symmetry allows us to talk about *symmetry defect sectors*. Briefly, this gives a category of sectors which is graded by the group G. The component with the trivial grading is the category of sectors is the category Δ_{TC}, but the symmetry enriched toric code has a non-trivial graded component that distinguishes it from the toric code model. We refer to [KVW24] for the details.

Declarations

Conflicts of interest. The authors have no conflicts of interest to declare in relation to this manuscript.

Data availability. No new data was created during this study.

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