# On Chamber-regular $\tilde{C}_2$-Lattices

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**Abstract**

We construct the first examples of chamber-regular lattices on $\tilde{C}_2$-buildings. Assuming a conjecture of Kantor our list of examples becomes a classification for chamber-regular $\tilde{C}_2$-lattices on locally-finite $\tilde{C}_2$-buildings. The links of special vertices in the buildings we construct, are all isomorphic to (the incidence graph of) the unique generalized quadrangle $Q$ of order (3,5). In particular our constructions involve chamber-regular actions on $Q$. These actions on $Q$ are the first (and if Kantor's conjecture holds the only) chamber-regular actions on a finite generalized quadrangle and therefore interesting in their own right. Moreover $Q$ is not Moufang and therefore none of our examples is a Bruhat-Tits building and all our lattices are exotic building lattices.

## 1 Introduction

Let $Q$ be the unique generalized quadrangle of order (3,5) (see [DZ76] or [PT09, 6.2.4] for uniqueness). We observed that $Q$ admits 11 chamber-regular actions and we use these actions to construct chamber-regular lattices on $\tilde{C}_2$-buildings. A theorem of Seitz implies that a finite quadrangle admitting a chamber-regular action is necessarily non-Moufang (see [Sei73] or [Mal98, Theorem 4.8.7]). On the other hand non-Moufang quadrangles with rich symmetry are rare. It has been conjectured by Kantor [Kan91] that the only finite, non-Moufang quadrangles with chamber-transitive automorphism group are $Q$ and the quadrangle arising from the Lunelli-Sce hyperoval $Q_{\mathrm{LS}}$. One can easily check with a computer, that $Q_{\mathrm{LS}}$ does not admit a chamber-regular action. So assuming Kantor's conjecture the examples we provide are the only chamber-regular actions on finite generalized quadrangles. And if Kantor's conjecture holds, the constructed lattices are the only chamber-regular lattices on locally-finite $\tilde{C}_2$-buildings.

**Theorem 1** (Main Theorem)**.** *There are exactly 3044 chamber-regular lattices (up to isomorphism) on $\tilde{C}_2$-buildings whose links of special vertices are the generalized quadrangle of order (3,5).*

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Note that being uniform lattices on Euclidean buildings the constructed lattices enjoy Kazhdan's property (T) [Opp24].

This article is structured as follows. In the Sections 2 and 3 we introduce triangles of groups and Euclidean buildings and discuss some elementary properties. In Section 4 we define the unique generalized quadrangle $Q$ of order (3,5) and construct two chamber-regular actions explicitly. The data that encodes the 11 chamber-regular action on $Q$ can be found in Appendix A. These actions have been computed with the computer algebra system GAP [25]. Using these actions we construct chamber-regular $\tilde{C}_2$-lattices and classify the results up to group isomorphism in Section 5. Appendix B contains some more combinatorial data that is needed for the classification.

# 2 Triangles of groups

A triangle of groups is a commutative diagram consisting of seven groups. It is a compact way to encode group actions on simply-connected, 2-dimensional simplicial complexes that are transitive on 2-simplices. Triangles of groups have been studied by Stallings–Gersten (see [SG91], [Sta91]). In Section 5 we construct chamber-regular $\tilde{C}_2$-lattices through triangles of groups.

## 2.1 Triangles of groups and developements

**Definition 2.** Let $A, E_1, E_2, E_3, V_1, V_2, V_3$ be groups. Let $\alpha_i : A \to E_i$ and $\epsilon_{ij} : E_i \to V_j$ be monomorphisms for $1 \leq i, j \leq 3, i \neq j$. We call this collection of groups and monorphisms a non-degenerate triangle of groups if the following hold:

1. The diagram in Figure 1 commutes,
2. $\varepsilon_{ik}(E_i) \cap \varepsilon_{jk}(E_j) = \varepsilon_{ik} \circ \alpha_i(A) = \varepsilon_{jk} \circ \alpha_j(A)$ for $\{i, j, k\} = \{1, 2, 3\}$,
3. $\langle \varepsilon_{ik}(E_i), \varepsilon_{jk}(E_j) \rangle = V_k$ and $\varepsilon_{ik}(E_i) \neq \varepsilon_{ik}(E_i) \cap \varepsilon_{jk}(E_j) \neq \varepsilon_{jk}(E_j)$ for $\{i, j, k\} = \{1, 2, 3\}$.

The groups in a triangle of groups $T$ are called local groups. The group $A$ is called face group, the groups $E_1, E_2, E_3$ are called edge groups and the group $V_1, V_2, V_3$ are called vertex groups.

*Remark* 3. A possibly degenerate triangle of groups is a commutative diagram as above not necessarily satisfying properties 2 and 3. In this article we only consider non-degenerate triangles of groups and just call them triangles of groups.

In Figure 1, the edge groups and vertex groups are indexed by $\{1, 2, 3\}$. This enumeration is arbitrary but it will be convenient to have such an enumeration. This motivates the definition of a type-function.

**Definition 4.** Let $T$ be triangle of groups. Then any bijection from the set of vertex groups to $\{1, 2, 3\}$ is called a type function on $T$. If we have a type

$$
\begin{tikzcd}
V_2 & & E_3 \arrow[ll, "\varepsilon_{32}"'] \arrow[rr, "\varepsilon_{31}"] & & V_1 \\
& & A \arrow[u, "\alpha_3"'] \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & \\
& E_1 \arrow[uul, "\varepsilon_{12}"] \arrow[dr, "\varepsilon_{13}"'] & & E_2 \arrow[uur, "\varepsilon_{21}"'] \arrow[dl, "\varepsilon_{23}"] & \\
& & V_3 & &
\end{tikzcd}
$$

Figure 1: A triangle of groups.

function, we can extend it to the set of edge groups by requiring that the edge group with type $i$ embeds in the vertex groups with types $j$ and $k$ for $\{i, j, k\} = \{1, 2, 3\}$.

For the rest of this section we fix $T$ as the triangle of groups from Figure 1. We equip it with the type function $f$ indicated by the notation $f : V_i \mapsto i,\ E_i \mapsto i$.

**Definition 5.** Let $T'$ be another triangle of groups with local groups $A'$, $(E'_i)_{1\leq i\leq 3}$ and $(V'_i)_{1\leq i\leq 3}$ and let $f'$ be the type function indicated by the notation.

1. A family of isomorphisms $\phi_A : A \to A', \phi_{E_i} : E_i \to E'_i, \phi_{V_i} : V_i \to V'_i$ is called type-preserving isomorphism of between $(T, f)$ and $(T', f')$ if $\phi_{E_i} \circ \alpha_i = \alpha'_i \circ \phi_A$ and $\phi_{V_j} \circ \epsilon_{ij} = \epsilon'_{ij} \circ \phi_{E_i}$ for $1 \leq i, j \leq 3, i \neq j$. If it is clear which type-functions we are referring to, we might omit them in the notation.

2. We call $T$ and $T'$ isomorphic, if they are type-preservingly isomorphic for some type-functions on $T$ and $T'$.

*Remark* 6. Note that an isomorphism between triangles of groups is always determined by the isomorphisms between the edge groups.

We now describe the action associated to *developable* triangles of groups.

**Definition 7.**

1. We define the fundamental group $\pi_1(T)$ of $T$ as the colimit of $T$ (considered as commutative diagram).

2. We call $T$ developable if the canonical morphisms from the vertex groups to the fundamental group $\pi_1(T)$ are injective.

If $T$ is developable, then we call the images of the local groups in $\pi_1(T)$ also just local groups and we denote them by $A, E_i$ and $V_i$ as well.

**Definition 8.** Assume that $T$ is developable.

1. We define the abstract development $D^{\mathrm{Ab}}(T)$ as the following abstract simplicial complex of dimension 2.
   (a) The vertices of $D^{\mathrm{Ab}}(T)$ are the left cosets of the images of the vertex groups in $\pi_1(T)$.
   (b) Now a set of vertices is a simplex if and only if the corresponding cosets have non-empty intersection.
2. We define the development $D(T)$ of $T$ as the geometric realization of $D^{\mathrm{Ab}}(T)$.

Note that the type-function $f$ induces type-functions on both developments. That is a function defined on the vertices onto $\{1, 2, 3\}$ that maps adjacent vertices to different types.

The following observation for the abstract development is crucial.

*Observation* 9.

1. The maximal simplices are of dimension two. If three vertices form a 2-simplex than the intersection of the corresponding cosets is a coset $xA$ of the face group $A$. In that case the three vertices correspond to the cosets $xV_1, xV_2, xV_3$.
2. If two vertices form a 1-simplex then the intersection of the corresponding cosets is a coset of an edge group $xE_i$. In that case the vertices correspond to the cosets $xV_j, xV_k$, whereas $\{i, j, k\} = \{1, 2, 3\}$.
3. The fundamental group $\pi_1(T)$ acts canonically on $D(T)$ and this action is type-preserving and transitive on 2-simplices.

The following result a consequence of Theorem II.12.21 in [BH99].

**Proposition 10.** *The development (of a developable triangle of groups) is connected and simply-connected.*

It is important to mention that we can also associate triangles of groups to suitable actions.

**Definition 11.** Let $X$ be connected, simply connected, pure two dimensional simplicial complex $X$ such that every vertex link is a connected, bipartite graph without leaves and multiple edges. Assume that $X$ is equipped with a type function and let $\Gamma$ be a group of type-preserving, simplical isometries, that is transitive on 2-simplices. Then we say that the action $(\Gamma, X)$ arises from a triangle groups. In that case for any choice of a 2-simplex $t$ we get a triangle of groups by considering the stabilizers of the faces of $t$. We call $T$ the triangle of groups associated to the action $(\Gamma, X)$ with respect to $t$.

This definition is justified by the following proposition.

**Proposition 12.** *Let $(\Gamma, X)$ be an action arising from a triangle of groups and let $T$ be a triangle of group associated to it. Then the actions $(\Gamma, X)$ and $(\pi_1(T), D(T))$ are type-preservingly isomorphic.*

*Proof.* This is a consequence of Proposition II.12.20 (1) in [BH99]. □

We continue with a sufficient condition for developability, namely non-positive curvature. For this we need to the define the coset graph of a triple $(V, E, F)$ of groups: If $E \neq F$ are subgroups of a group $V$, then the coset graph $\Gamma_{V,E,F}$ of $V$ with respect to $E$ and $F$ is the simplicial graph with vertices $V/E \cup V/F$ and edges $\{\{vE, vF\} \mid v \in V\}$. The graph $\Gamma_{V,E,F}$ is bipartite and $V$ acts on it via left multiplication.

**Definition 13.** 1. The local action of type $i$ of $T$ is the action by the vertex group $V_i$ on its coset graph with respect to $\epsilon_{ji}(E_j)$ and $\epsilon_{ki}(E_i)$. The angle of type $i$ is defined as $\frac{2\pi}{\mathrm{Girth}(\Gamma_i)}$.

2. We call $T$ non-positively curved if the sum of its angles is at most $\pi$.

Note that by non-degenerality of $T$, the graphs in the local actions are connected, have no leaves and have at least two vertices of each type. Furthermore the local actions are transitive on edges.

The following result is a consequence of Theorem II.12.28 in [BH99].

**Proposition 14.** *Assume that $T$ is non-positively curved. Then $T$ is developable and the development can be turned into a complete* $\mathrm{CAT}(0)$*-space by equipping every 2-simplex with the metric of a triangle in $\mathbb{E}^2$ or $\mathbb{H}^2$ with the same angles as $T$. If we do so the fundamental group acts by isometries.*

*Remark* 15. Assume that $T$ is a non-positively curved triangle of groups with local actions on finite graphs. Let $\Gamma$ be the fundamental group and let $X$ be the development equipped with the metric from Proposition 14. Then $X$ is locally-compact and if we equip $\mathrm{Aut}(X)$ with the compact-open topology it becomes a (possibly discrete) locally-compact group. Note that $\Gamma \leq \mathrm{Aut}(X)$ might not be closed. However, if in addition the local groups of $T$ are all finite, then $\Gamma$ is a uniform lattice in $\mathrm{Aut}(X)$.

## 2.2 Global actions with given local actions

Assume we have an action $(\Gamma, X)$ arising from a triangle of groups. For every vertex $x \in X$ of type $i$ we can consider the action of its stabilizer on its link. We call this action the local action of $(\Gamma, X)$ in type $i$. Since $\Gamma$ is transitive on vertices of the same type, we get the same action for every vertex of type $i$. In this section we want to study the number of actions (up to isomorphism) arising from triangles of groups with given local actions. We are mostly interested in actions that are regular on 2-simplices and on a non-positively curved space $X$, so we fix the following notation. For $1 \leq i \leq 3$ let $(V_i, \Lambda_i)_i$ be an edge-regular action on a connected bipartite graph without leaves, that preserves the bipartition. Furthermore assume

$$\sum_{i=1}^{3} \frac{2}{\mathrm{Girth}(\Gamma_i)} \leq 1.$$

We also fix type functions $f_i$ on $\Lambda_i$ onto $\{j,k\}$ for $\{i,j,k\} = \{1,2,3\}$. Let $e_i = \{x_j^i, x_k^i\}$ be an edge of $\Lambda_i$ with $f_i(x_j^i) = j$ and $f_i(x_k^i) = k$. Denote the stabilizer of $x_j^i$ in $V_i$ with $E_j^i$ and assume that $E_j^i \cong E_j^k$ for any choice $\{i,j,k\} = \{1,2,3\}$. Then we have the following correspondence.

*Observation* 16. The isomorphism classes of actions arising from triangles of groups in class $\mathcal{A}$ (defined below) are in 1-to-1 correspondence with the number of isomorphism classes of the triangles of groups in class $\mathcal{B}$ (defined below). If we have a triangle of groups in $\mathcal{B}$, its (type-preserving) isomorphism class corresponds to the (type-preserving) isomorphism class of the action $(\pi_1(T), D(T))$.

(a) The class $\mathcal{A}$ contains the actions $(\Gamma, X)$ arising from triangles of groups such that the local action in type $i$ is type-preservingly isomorphic to $(V_i, \Lambda_i)$.

(b) The class $\mathcal{B}$ contains the triangles of groups with trivial face groups, vertex groups $(V_i)_i$, edge groups $(E_i)_i$ and non-trivial monomorphisms $\epsilon_{ij} : E_i \to V_j$ such that $\epsilon_{ij}(E_i) = E_i^j$ for $1 \leq i, j \leq 3, i \neq j$. We equip the triangles of groups with the type-function indicated by the notation.

*Proof.* Proposition 12 implies that every action in $\mathcal{A}$ arises from a triangle of groups in $\mathcal{B}$. It is clear that two isomorphic triangles of groups yield isomorphic actions. To see the other direction, first notice that if we start with a triangle of groups $T$, then any triangle of groups associated to $(\pi_1(T), D(T))$ is isomorphic to $T$. On the other hand for two isomorphic actions one can easily construct an isomorphisms between any triangles of groups associated to them. □

In what follows we describe a method to compute the number of type-preserving isomorphism classes of triangles of groups with fixed local actions in a convenient way. We fix abstract groups $E_i$ isomorphic to $E_i^j$ and $E_i^k$ and fix monomorphisms $\kappa_{ij} : E_i \to V_j$, such that $\kappa_{ij}(E_i) = E_i^j \leq V_j$. Then for any triangle $T$ in family $\mathcal{B}$ from Observation 16 there exists a choice of $\gamma_{ij} \in \mathrm{Aut}(E_i)$ with $1 \leq i \neq j \leq 3$ such that $T$ is type-preservingly isomorphic to the triangle $T((\gamma_{ij})_{ij})$ defined in Figure 2, where as usual we equip $T((\gamma_{ij})_{ij})$ with the type-function indicated by the notation. We denote the family of these triangles with $\mathcal{T} := \{T((\gamma_{ij})_{ij}) \mid \gamma_{ij} \in \mathrm{Aut}(E_i)\}$.

**Definition 17.** If we have two subgroups $E_j \neq E_k$ of a group $V_i$. We define the local automorphism group $\Sigma(V_i, E_j, E_k)$ of this triple as follows

$$\Sigma(V_i, E_j, E_k) := \{\sigma \in \mathrm{Aut}(V_i) \mid \sigma(E_j) = E_j,\ \sigma(E_k) = E_k\}.$$

We denote by $\Sigma^i$ the local automorphism group of $(V_i, E_j^i, E_k^i)$. For $\sigma^j \in \Sigma^j$ let $\sigma_i^j \in \mathrm{Aut}(E_i)$ be the unique automorphism satisfying $\epsilon_{ij} \circ \sigma_i^j = \sigma^j \circ \epsilon_{ij}$.

$$
\begin{tikzcd}
V_2 & & E_3 \arrow[ll, "\kappa_{32} \circ \gamma_{32}"'] \arrow[rr, "\kappa_{31} \circ \gamma_{31}"] & & V_1 \\
& & 1 \arrow[u] \arrow[dl] \arrow[dr] & & \\
& E_1 \arrow[uul, "\kappa_{12} \circ \gamma_{12}"] \arrow[dr, "\kappa_{13} \circ \gamma_{13}"] & & E_2 \arrow[uur, "\kappa_{21} \circ \gamma_{21}"'] \arrow[dl, "\kappa_{23} \circ \gamma_{23}"'] & \\
& & V_3 & &
\end{tikzcd}
$$

Figure 2: The triangle of groups $T((\gamma_{ij})_{ij})$.

We denote this map by $\Theta_i^j : \sigma^j \mapsto \sigma_i^j$. Now define $\Theta^j : \Sigma^j \to \mathrm{Aut}(E_i) \times \mathrm{Aut}(E_k)$, $\sigma_j \mapsto (\sigma_i^j, \sigma_k^j)$ and note that $\Theta^j$ is injective. Finally we define the group $\mathcal{I} := \prod_i \mathrm{Aut}(E_i) \times \prod_i \Sigma^i$ and a left action of $\mathcal{I}$ on the family $\mathcal{T}$.

**Definition 18.** Let $\phi = (\phi_{E_i})_i \times (\phi_{V_i})_i \in \mathcal{I}$ and $T := T((\gamma_{ij})_{ij}) \in \mathcal{T}$. Then we define $\phi.T \in \mathcal{T}$ as follows.

$$\phi.T := T\left( (\underbrace{\Theta_i^j(\phi_{V_j})}_{\in \mathrm{Aut}(E_i)} \circ \underbrace{\gamma_{ij}}_{\in \mathrm{Aut}(E_i)} \circ \underbrace{\phi_{E_i}^{-1}}_{\in \mathrm{Aut}(E_i)})_{ij} \right)$$

**Lemma 19.** *We have that $T, T' \in \mathcal{T}$ are type-preservinlgy isomorphic if and only if they lie in the same orbit under $\mathcal{I}$.*

*Proof.* Direct calculations show that if $\phi \in \mathcal{I}$, then it is a type-preserving isomorphism from $T$ to $\phi.T$. On the other hand if $T, T' \in \mathcal{T}$ are type-preservingly isomorphic via $\phi$, then in fact $T' = \phi.T$. $\square$

*Observation* 20. We can identify the family $\mathcal{T}$ with $C = \prod_i \mathrm{Aut}(E_i)^2$ by identifying $(T(\gamma_{ij})_{ij})$ with the tuple $(\gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{23}, \gamma_{31}, \gamma_{32})$. Then the maps $\iota_E$ and $\iota_V$ defined below are group monomorphisms from $\prod_i E_i$ and $\prod_i \Sigma^i$ to $C$.

$$\iota_E : \quad \mathrm{Aut}(E_1) \times \mathrm{Aut}(E_2) \times \mathrm{Aut}(E_3) \to C,$$
$$(\phi_{E_1}, \phi_{E_2}, \phi_{E_3}) \mapsto (\phi_{E_1}, \phi_{E_1}, \phi_{E_2}, \phi_{E_2}, \phi_{E_3}, \phi_{E_3})$$

$$\iota_V : \quad \Sigma^1 \times \Sigma^2 \times \Sigma^3 \to C,$$
$$(\phi_{V_1}, \phi_{V_2}, \phi_{V_3}) \mapsto (\Theta_1^2(\phi_{V_2}), \Theta_1^3(\phi_{V_3}), \Theta_2^1(\phi_{V_1}), \Theta_2^3(\phi_{V_3}), \Theta_3^1(\phi_{V_1}), \Theta_3^2(\phi_{V_2}))$$

Furthermore $(\sigma_E, \sigma_V) \in \mathcal{I}$ acts on $T \in C$ via $\iota_E(\sigma_E) T \iota_V(\sigma_V^{-1})$. In particular the type-preserving isomorphism classes in $C$ correspond to double cosets in $C$ with respect to its subgroups $\iota_E\left(\prod_i E_i\right)$ and $\iota_V\left(\prod_i \Sigma^i\right)$.

## 3 Euclidean buildings of dimension 2

Euclidean buildings are non-positively curved cell complexes that are covered by copies of the Euclidean space (flats). The maximal flats are called apartments. The gluing of the apartments respects a cell structure on the Euclidean space, that is induced by a discrete reflection group. Together with symmetric spaces, Euclidean buildings provide the main examples of higher rank CAT(0)-spaces. In this article we will focus on Euclidean buildings of dimension two, meaning that the apartments are copies of the Euclidean plane.

### 3.1 Definitions

If $W$ is a discrete reflection group on the Euclidean space with compact fundamental domain, the reflection axes induce a cellular structure on the Euclidean space. If we consider the group generated by reflections in the Euclidean plane along the sides of a triangle with angles $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ or $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$, the induced cellar complexes are called model apartments of type $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$ respectively.

**Definition 21.** Let $X$ be a simplicial complex and call every simplicial subcomplex that is isomorphic to one of the model apartments an apartment. Then $X$ is an (irreducible) Euclidean building of dimension two if the following hold.

1. For every two simplices, there is an apartment containing both.
2. For every two apartments, there is a simplicial isomorphism between both apartments that is the identity on the intersection.

Note that the two conditions in the definitions assure that we can define the distance of two points in a building by measuring the distance in an enveloping apartment. This metric turns a Euclidean building into a CAT(0)-space. In particular Euclidean buildings are contractible. If the maximal number of triangles that share an edge in $X$ is finite then $X$ is called locally finite. In that case the building is a locally compact space.

We will be interested in groups that act geometrically on Euclidean buildings.

**Definition 22.** Let $X$ be a locally-finite, two-dimensional Euclidean building and let $\Gamma \leq \mathrm{Aut}(X)$. Then $\Gamma$ is called lattice on $X$ if the action is cocompact and the stabilizers of simplices are finite.

Note that a uniform lattice on $X$ is necessary a uniform lattice in the locally compact group $\mathrm{Aut}(X)$.

### 3.2 The local approach

In the seventies Bruhat and Tits developed a method to associate a Euclidean building to a semi-simple algebraic group over a field with non-Archimedean valuation [BT72], [BT84]. The resulting buildings are called Bruhat-Tits building and are the classical examples of Euclidean buildings. Tits and Weiss showed that if a Euclidean building is of dimension at least three (and irreducible) then it is Bruhat-Tits [Wei09]. However in dimension two a Euclidean building might not be Bruhat-Tits and we call these buildings exotic. Constructing exotic buildings (preferably with a cocompact automorphism group) is already a non-trivial task and there is no algebraic recipe to do so. Several constructions of exotic buildings, including the ones in this article, are based on the fact that buildings can be recognized locally [Tit81]. We state the two-dimensional version below (Proposition 23) using the terminology of a generalized $m$-gon, which is just a bipartite graph with diameter $m$ and girth $2m$.

**Proposition 23.** *Let $X$ be a two-dimensional, simply connected simplicial complex. Assume there is a type function on $X$ that associates to each vertex one of three types. Assume that the links of vertices of type $i$ are generalized $m_i$-gons for the same $m_i$. Furthermore assume that $\sum_i \frac{1}{m_i} = 1$. Then $X$ is a Euclidean building.*

We apply the proposition to triangles of groups.

**Corollary 24.** *Let $T$ be a triangle of groups such that its local actions only involve generalized polygons and such that its angle sum is $\pi$. Then the development of $T$ is a Euclidean building of dimension 2.*

### 3.3 Recovering the action of a building lattice

In this subsection we show the following. Let $(\Gamma, X)$ be an action on a Euclidean building with finite vertex stabilzers, that arises from a triangle of groups. Then the action is already determined by the group structure of $\Gamma$. We will use this result in Section 5.

**Lemma 25.** *Let $T$ be a triangle of groups with finite local groups and whose universal cover is a Euclidean building. Let $\Gamma$ be the fundamental group and let $X$ be the development. Identify simplices of $X$ with cosets in $\Gamma$. Then the following hold:*

1. *The map $xL \mapsto \mathrm{Ad}(x)(L)$ is a well-defined bijection from the set of vertices in $X$ to the set of maximal finite subgroups in $\Gamma$.*

2. *Let $\mathcal{V}$ be the set of maximal finite subgroups in $\Gamma$. The map $xL \mapsto \mathrm{Ad}(x)(L)$ is a well-defined surjective map from the set of edges in $X$ to the set $\mathcal{E}$ which is the set of maximal elements in the following family.*

$$\bar{\mathcal{E}} := \{E \leq \Gamma \mid \textit{there exists different } V_1, V_2 \in \mathcal{V} \textit{ both containing } E\}.$$

*Proof.* By Theorem 14 the development $X$ can be equipped with a complete CAT(0)-metric. Then by Theorem II.2.8 in [BH99] any finite group in $\Gamma$ has a non-empty convex fix point set. But the stabilizers of points are conjugates of the local groups. In particular vertex stabilizers are precisely the maximal finite subgroups. We have shown that the first map is well-defined and surjective. To see that it is injective assume that two vertices have the same stabilizer. Then this stabilizer also fixes the geodesic between the vertices but this geodesic contains points which are not vertices. Stabilizers of such points are non-maximal finite subgroups, so this yields a contradiction. Now consider a subgroup $E$ that embeds in different vertex stabilizers. Then again $E$ fixes a geodesic between the corresponding vertices, so it is contained in an edge stabilizer. This shows that the maximal groups in $\bar{\mathcal{E}}$ are all edge stabilizers. We continue with an observation about the fixed point set of an edge stabilizer. Since interior points of chambers have a strictly smaller stabilizer than edges, the fixed point set of any edge stabilizer must be a (possibly finite) tree supported on vertices and edges in the building $X$. Furthermore in this tree the angle between two outgoing edges at a vertex must be $\pi$, since the tree is convex in $X$. Now assume there exist edge stabilizers $E \subsetneq E'$. Then the fixed tree of $E'$ is a proper subtree of the fixed tree of $E$. In particular there exist a vertex $x$ and two edges $e, e'$ both incident with $x$ such that $E$ and $E'$ are the stabilizers of $e$ and $e'$. In the building $X$ the edges $e$ and $e'$ must be of different type, otherwise their stabilizers would have the same order. If the universal cover is of type $\tilde{C}_2$ this cannot happen, because edges of different types cannot form an angle of size $\pi$. If the building is of type $\tilde{A}_2$ every edge contains the same number of chambers, so every edge stabilizer must be of the same size. In type $\tilde{G}_2$ edges of different type only can form an angle of size $\pi$ if they intersect at a vertex whose link is an $A_2$-building. In that case the stabilizer of these edges must be of the same size. Therefore the set of edge stabilizers are precisely the elements in $\mathcal{E}$. This shows well-definedness and surjectivity. □

**Proposition 26.** *Let $T, T'$ be triangles of groups with finite local groups and whose universal covers are Euclidean buildings. Let $\Gamma, \Gamma'$ be their fundamental groups. Then $T$ and $T'$ are isomorphic if and only if $\Gamma$ and $\Gamma'$ are isomorphic.*

*Proof.* We will reconstruct the action of $\Gamma$ on the developement $X$ out of group theoretic properties of $\Gamma$. Let $\mathcal{V}$ and $\mathcal{E}$ be as in Lemma 25. We claim that up to conjugation in $\Gamma$ there exists a 3-set $\{V_1, V_2, V_3\}$ in $\mathcal{V}$ satisfying the following:

(i) The groups $V_1, V_2, V_3$ are pairwise not conjugate.

(ii) The groups $V_i, V_j$ intersect in an element $E_k \in \mathcal{E}$ with $\{i, j, k\} = \{1, 2, 3\}$.

(iii) The groups $E_i, E_j, E_k$ are pairwise distinct.

(iv) The groups $V_1, V_2, V_3$ generate $\Gamma$.

Let $(V_1, V_2, V_3)$ be a triple satisfying the properties (i) to (iii). If $\Gamma$ acts as the fundamental group of a triangle of groups on the corresponding development $X$ then there are three vertices $x_1, x_2, x_3 \in X$ stabilized by $V_1, V_2, V_3$ respectively. Since the groups $V_1, V_2, V_3$ are pairwise not conjugate, the vertices $x_1, x_2, x_3$ are necessarily of three different types in the building $X$. Let $\{i, j, k\} = \{1, 2, 3\}$ and let $\gamma_k$ be the geodesic between $x_i$ and $x_j$. These geodesics $\gamma_1, \gamma_2, \gamma_3$ must be edge

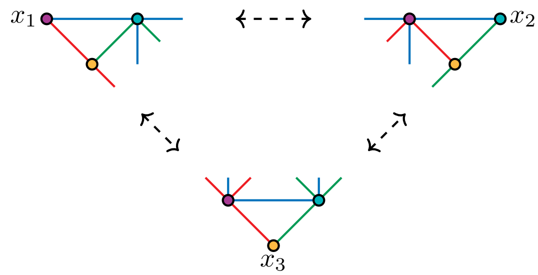


Figure 3: The configuration of $x_1, x_2, x_3$ in the proof of Proposition 26 in type $\tilde{C}_2$.

paths stabilized by the groups $E_1, E_2, E_3$. The geodesics $\gamma_i, \gamma_j$ only intersect in the vertex stabilized by $V_k$, because otherwise there would be at least two vertices fixed by $\langle E_i, E_j \rangle$ and since $E_i \neq E_j$ neither $E_i$ nor $E_j$ would be maximal among the groups that are contained in two vertex stabilizers. In particular none of the angles in the geodesic triangle $\Delta(x_1, x_2, x_3)$ is 0. On the other hand $\Delta(x_1, x_2, x_3)$ is a geodesic triangle in a Euclidean building whose sides are edge paths. In particular if none of the angles is 0, their sum must be $\pi$. Now we can apply the Proposition II.2.9 in [BH99] (Flat Triangle Lemma) and deduce that $\Delta(x_1, x_2, x_3)$ is isometric to its Euclidean comparison triangle. By Theorem VI.7.2 in [Bro88] the triangle $\Delta(x_1, x_2, x_3)$ is contained in an apartment. If $\Delta(x_1, x_2, x_3)$ is a chamber in the development, then clearly condition (iv) is satisfied. If $\Delta(x_1, x_2, x_3)$ is not chamber, we claim that $V_1, V_2, V_3$ do not generate $\Gamma$. In particular if our claim holds, then the by group theoretic properties of $\Gamma$ we can determine a fundamental chamber in the development it acts on and more we can determine the monomorphisms between the stabilizers of simplices in that fundamental chamber. In particular $\Gamma$ determines the triangle of groups it arises from. So assume $\Delta(x_1, x_2, x_3)$ is not a chamber. Let $A := \cap V_i$. Then the groups $((V_i)_i, (E_i)_i, A)$ form a triangle of groups $T'$ with the same local actions as the triangle of groups $\Gamma$ arises from. In particular $T'$ is developable and the development $X'$ of $T'$ is a Euclidean building with the same vertex links as $X$. Therefore the fundamental group $\Gamma' := \langle V_1, V_2, V_3 \rangle$ is a subgroup of $\Gamma$ and the development $X'$ is a subcomplex of $X$. Now $\Gamma'$ acts regular on the orbit $\Gamma'.\Delta(x_1, x_2, x_3)$ of its fundamental chamber, so it is not as transitive as $\Gamma$. In particular $\Gamma'$ is a proper subgroup of $\Gamma$, which is what we claimed. $\square$

# 4 The generalized quadrangle of order $(3, 5)$

In this section we define the generalized quadrangle $Q$ of order (3,5) and describe two chamber-regular action on $Q$. It is a member of the family of slanted symplectic quadrangles, which are Payne-derivations of symplectic quadrangles. The slanted symplectic quadrangle $W^\diamond(\mathbb{F})$ over the field $\mathbb{F}$ can be constructed as follows. Let $M(x,y,z) \in \mathrm{SL}_4(\mathbb{F})$ be the matrix defined below and let $S$ be the group consisting of elements of this form. Let $X_0 = \{M(0,0,\mu) \mid \mu \in \mathbb{F}\}$ and for each $p = \langle (a,b) \rangle_{\mathbb{F}}$ in the projective line define the the subgroup $X_{[a:b]} = \{M(\mu a, \mu b, 0) \mid \mu \in \mathbb{F}\}$. Now the slanted symplectic quadrangle $W^{\diamond}(\mathbb{F})$ is the incidence geometry with points $S$ and lines being the left cosets of the subgroups $X_0$ and $X_{[a:b]}$. Note that $S$ acts point-regularly on $W^{\diamond}(\mathbb{F})$ by left multiplication. If $\mathbb{F}$ is finite of order $q$ the quadrangle $W^{\diamond}(\mathbb{F})$ is of order $(q-1, q+1)$.

$$M(x,y,z) := \begin{pmatrix} 1 & -x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If $\mathbb{F}$ is of order $q = 2^k$, then $S$ is elementary abelian of order $2^{3k}$. In particular this is the case for $q = 4$ which yields the unique quadrangle $Q$ of order (3,5). We fix $q = 4$. Let $\alpha \in \mathbb{F}_4$ be the element satisfying $\alpha^2 + \alpha + 1 = 1$. The assignment $\Phi$ indicated below extends to an isomorphism from $S$ (defined over $\mathbb{F}_4$) to the vector space $V := \mathbb{F}_2^6$. We denote the standard basis with $(e_i)_i$.

$$\begin{aligned} \Phi: \ & M(1,0,0) \mapsto e_1, && M(\alpha,0,0) \mapsto e_2, && M(0,1,0) \mapsto e_3, \\ & M(0,\alpha,0) \mapsto e_4, && M(0,0,1) \mapsto e_5, && M(0,0,\alpha) \mapsto e_6. \end{aligned}$$

The images of the subgroups $X_0$ and $X_{[a:b]}$ are the following subspaces.

$$\begin{aligned} U_1 &:= \langle e_5, e_6 \rangle_{\mathbb{F}_2} &&= \Phi\left(X_0\right), \\ U_2 &:= \langle e_1, e_2 \rangle_{\mathbb{F}_2} &&= \Phi\left(X_{[1:0]}\right), \\ U_3 &:= \langle e_3, e_4 \rangle_{\mathbb{F}_2} &&= \Phi\left(X_{[0:1]}\right), \\ U_4 &:= \langle e_1 + e_3 + e_5, e_2 + e_4 + e_5 + e_6 \rangle_{\mathbb{F}_2} &&= \Phi\left(X_{[1:1]}\right), \\ U_5 &:= \langle e_1 + e_4 + e_6, e_2 + e_3 + e_4 + e_5 \rangle_{\mathbb{F}_2} &&= \Phi\left(X_{[1:\alpha]}\right), \\ U_6 &:= \langle e_2 + e_3 + e_6, e_1 + e_2 + e_4 + e_5 \rangle_{\mathbb{F}_2} &&= \Phi\left(X_{[\alpha:1]}\right). \end{aligned}$$

Now consider the matrices $A, B, C$ from Table 1. We make the following observation

*Observation* 27. 1. The matrix $A$ is of order six and $\langle A \rangle$ acts regularly (by left multiplication) on the subspaces $U_i$. In particular the affine group $V \rtimes \langle A \rangle$ acts regularly on the chambers of $Q$.

2. The matrices $B$ and $C$ are of order two and three respectively and $\langle B, C \rangle \cong \mathrm{Sym}(3)$ acts regularly (by left multiplication) on the subspaces $U_i$. In particular the affine group $V \rtimes \langle B, C \rangle$ acts regularly on the chambers of $Q$.

*Remark* 28. The groups $V \rtimes \langle A \rangle$ and $V \rtimes \langle B, C \rangle$ are isomorphic to the groups $L_7$ and $L_{11}$ from Table 2 in Appendix A.

The following proposition was checked with a computer.

$$A := \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

$$C := \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad D := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Table 1: Matrices over $\mathbb{F}_2$.

**Proposition 29.** *1. The affine group $V \rtimes \langle A, C\rangle$ is the full automorphism group of $Q$. In particular the automorphism group of $Q$ contains a normal subgroup acting regularly on the points.*

2. *Up to conjugacy there are two chamber-regular subgroups of $\mathrm{Aut}(Q)$ that contain the normal subgroup $V$. These are exactly the groups from the previuos observation.*

3. *Since the $\langle A, C\rangle$ permutes the subspaces $U_1, \dots, U_6$ we get an homomophism $\Psi$ to $\mathrm{Sym}(6)$. The kernel is cyclic of order three and generated by the matrix $D$ indicated in Table 1. The image of $A$ and $C$ under $\Psi$ are $(1, 5, 2, 3, 4, 6)$ and $(1, 3, 5)(2, 4, 6)$ respectively. In particular $\Psi$ is surjective and we have the following (non-split) exact sequence.*

$$1 \to C_3 \cong \langle D\rangle \to \langle A, C\rangle \to \mathrm{Sym}(6) \to 1.$$

*Remark* 30. We point out that the quadrangle $Q$ admits way more regular actions than the ones we discuss in the previous discussion. In fact (up to conjugacy in the automorphism group) there are 58 groups acting regular on the points, six groups acting regular on the lines and 11 groups acting regular on the chambers of $Q$. Several of the point- and line-regular actions arise as subactions of the chamber-regular ones. In Appendix A we provide explicit presentations for the 11 chamber-regular groups.

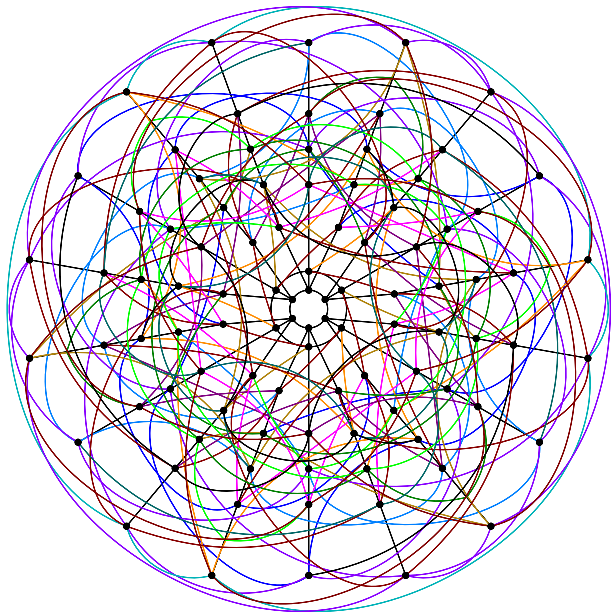

Figure 4: The generalized quadrangle of order (5,3), the dual of $Q$.

## 5 Chamber-regular actions on $\tilde{C}_2$-buildings

In this section we classify lattices acting chamber-regularly on $\tilde{C}_2$-buildings whose links of special vertices are the unique generalized quadrangle of order (3,5). Actually we classify all triangles of groups that give rise to such an action and this boils down to listing all possible local actions and working out which ones are compatible with each other. Then we count how many different triangles of groups can be constructed out of each matching. The following three lemmas capture all actions on generalized polygons that can arise as local actions in one of these triangles of groups. The tables these lemmas are referring to have been calculated with a computer.

**Lemma 31.** *Let $Q$ be the unique generalized quadrangle of order (3,5). Let $G$ be a group of automorphisms of $Q$ that is regular on the edges. Then the action $(G, Q)$ is isomorphic to the action of one of the groups $L_1, \ldots, L_{11}$, indicated in Table 2 in Appendix A, on its coset graph with respect to the subgroups $\langle a_i \rangle$ and $\langle b_i \rangle$. Furthermore these actions are pairwise not isomorphic.*

**Lemma 32.** *Let $K_{4,4}$ be the complete bipartite graph with partition size* $(4,4)$*. Let $G$ be a group of automorphisms of $K_{4,4}$ that is regular on the edges. Then the action $(G, K_{4,4})$ is isomorphic to the action of one of the groups $L_{12}, \dots, L_{21}$, indicated in Table 3 in Appendix A, on its coset graph with respect to the subgroups $\langle a_i \rangle$ and $\langle b_i \rangle$. Furthermore these actions are pairwise not isomorphic.*

**Lemma 33.** *Let $K_{6,6}$ be the complete bipartite graph with partition size* $(6,6)$*. Let $G$ be a group of automorphisms of $K_{6,6}$ that is regular on the edges. Then the action $(G, K_{6,6})$ is isomorphic to the action of one of the groups $L_{22}, \dots, L_{35}$, indicated in Table 4 in Appendix A, on its coset graph with respect to the subgroups $\langle a_i \rangle$ and $\langle b_i \rangle$. Furthermore these actions are pairwise not isomorphic.*

*Remark* 34. While the groups $L_1, \dots, L_{11}$ acting on $Q$ are pairwise non-isomorphic, several of the groups $L_{12}, \dots, L_{35}$ acting on $K_{44}$ and $K_{66}$ are indeed isomorphic. In fact we have that $L_{13} \cong L_{18} \cong L_{19}$, $L_{15} \cong L_{16} \cong L_{21}$, $L_{22} \cong L_{26} \cong L_{27} \cong L_{28} \cong L_{30} \cong L_{31}$, $L_{23} \cong L_{24} \cong L_{32} \cong L_{33} \cong L_{34}$ and $L_{29} \cong L_{35}$. For the groups $L_{12}, \dots, L_{35}$ we provide a short description of the isomorphism type in table 8 in Appendix B.

From now on we call the groups $L_i$ local actions and we call the subgroups $\langle a_i \rangle$ and $\langle b_i \rangle$ edge groups of $L_i$. As in Subsection 2.2 we fix abstract groups isomorphic to the edge groups. We just denote them by their isomorphism type and will refer to them as model edge groups.

$$
\begin{aligned}
C_4 &:= \langle e_1 \mid e_1^4 \rangle, & C_2 \times C_2 &:= \langle e_1, e_2 \mid e_1^2, e_2^2, (e_1 e_2)^2 \rangle, \\
C_6 &:= \langle e_1 \mid e_1^6 \rangle, & \mathrm{Sym}(3) &:= \langle e_1, e_2 \mid e_1^2, e_2^3, (e_1 e_2)^2 \rangle.
\end{aligned}
$$

The assignments $e_i \mapsto a_i$ and $e_i \mapsto b_i$ define monomorphisms from the model edge groups to $L_i$. We denote these monomorphism by $\kappa_A^i$ and $\kappa_B^i$ and refer to them as standard embeddings. We make a quick observation that will be needed later.

*Observation* 35. The local action $L_i$ admits a group automorphism that swaps its edge groups if and only if $i \in \{12, 13, 16, 17, 22, 25, 26, 29\}$. In fact if $i$ is in this list the assignment $a_i \leftrightarrow b_i$ extends to an involutory automorphism of $L_i$.

We will list the possible triangles of groups using the following notation.

*Notation* 36.

1. Let $T$ be the triangle of groups indicated in Figure 5 equipped with the type function indicated by notation. Then we denote this triangle with $\mathbf{T}(V_1, V_2, V_3, E_1, E_2, E_3, (\varepsilon_{ij})_{ij})$.

2. Let $(r, s, t)$ be a triple with $1 \leq r, s \leq 11$, $12 \leq t \leq 21$ and such that $E_t \cong \mathrm{im}(\kappa_B^r) \cong \mathrm{im}(\kappa_B^s)$, $E_s \cong \mathrm{im}(\kappa_A^r) \cong \mathrm{im}(\kappa_B^t)$ and $E_r \cong \mathrm{im}(\kappa_A^s) \cong \mathrm{im}(\kappa_A^t)$, where $E_r, E_s, E_t$ are model edge groups. We denote the family of of triangles $\mathbf{T}(L_r, L_s, L_t, E_r, E_s, E_t, (\varepsilon_{ij})_{ij})$ satisfying the conditions below with $\mathcal{T}^{(1)}_{(r,s,t)}$.

$$
\begin{aligned}
\mathrm{im}(\varepsilon_{12}) &= \mathrm{im}(\kappa_A^s), & \mathrm{im}(\varepsilon_{13}) &= \mathrm{im}(\kappa_A^t), & \mathrm{im}(\varepsilon_{21}) &= \mathrm{im}(\kappa_A^r), \\
\mathrm{im}(\varepsilon_{23}) &= \mathrm{im}(\kappa_B^t), & \mathrm{im}(\varepsilon_{31}) &= \mathrm{im}(\kappa_B^r), & \mathrm{im}(\varepsilon_{32}) &= \mathrm{im}(\kappa_B^s).
\end{aligned}
$$

$$\begin{tikzcd}
V_2 & & E_3 \arrow[ll, "\varepsilon_{32}"'] \arrow[rr, "\varepsilon_{31}"] & & V_1 \\
& E_1 \arrow[ul, "\varepsilon_{12}"] \arrow[dr, "\varepsilon_{13}"] & 1 \arrow[l] \arrow[r] \arrow[u] & E_2 \arrow[ur, "\varepsilon_{21}"'] \arrow[dl, "\varepsilon_{23}"] & \\
& & V_3 & &
\end{tikzcd}$$

Figure 5: The triangle of groups $\mathbf{T}(V_1, V_2, V_3, E_1, E_2, E_3, (\varepsilon_{ij})_{ij})$.

3. Let $(r, s, t)$ be a triple with $1 \le r, s \le 11$, $22 \le t \le 35$ and such that $E_t \cong \mathrm{im}(\kappa_A^r) \cong \mathrm{im}(\kappa_A^s)$, $E_s \cong \mathrm{im}(\kappa_B^r) \cong \mathrm{im}(\kappa_B^t)$ and $E_r \cong \mathrm{im}(\kappa_B^s) \cong \mathrm{im}(\kappa_A^t)$, where $E_r, E_s, E_t$ are model edge groups. We denote the family of of triangles $\mathbf{T}(L_r, L_s, L_t, \ldots, (\varepsilon_{ij})_{ij})$ satisfying the conditions below with $\mathcal{T}^{(2)}_{(r,s,t)}$.

$$\begin{aligned}
&\mathrm{im}(\varepsilon_{12}) = \mathrm{im}(\kappa_B^s), \qquad && \mathrm{im}(\varepsilon_{13}) = \mathrm{im}(\kappa_A^t), \qquad && \mathrm{im}(\varepsilon_{21}) = \mathrm{im}(\kappa_B^r),\\
&\mathrm{im}(\varepsilon_{23}) = \mathrm{im}(\kappa_B^t), \qquad && \mathrm{im}(\varepsilon_{31}) = \mathrm{im}(\kappa_A^r), \qquad && \mathrm{im}(\varepsilon_{32}) = \mathrm{im}(\kappa_A^s).
\end{aligned}$$

**Lemma 37.** *Let $\Gamma$ be a lattice acting regularly on the chambers of a building of type $\tilde{C}_2$ with vertex links $W^\diamond(4)$ and $K_{4,4}$. Then an isomorphic copy of the corresponding triangle of groups lies in exactly one of the following 163 families.*

1. $\mathcal{T}^{(1)}_{(1,1,t)}$ *for* $t \in \{12, 13, 14\}$ *(three families),*

2. $\mathcal{T}^{(1)}_{(r,s,t)}$ *for* $r, s \in \{6, 9\}, r \le s,$ $t \in \{12, 13, 14\}$ *(nine families),*

3. $\mathcal{T}^{(1)}_{(9,6,14)}$ *(one family),*

4. $\mathcal{T}^{(1)}_{(r,s,t)}$ *for* $r, s \in \{3, 5, 7, 8\}, r \le s,$ $t \in \{15, 16, 17\}$ *(30 families),*

5. $\mathcal{T}^{(1)}_{(r,s,15)}$ *for* $r, s \in \{3, 5, 7, 8\}, s < r$ *(six families).*

6. $\mathcal{T}^{(1)}_{(r,s,t)}$ *for* $r, s \in \{2, 4, 10, 11\}, r \le s,$ $t \in \{15, 16, 17\}$ *(30 families),*

7. $\mathcal{T}^{(1)}_{(r,s,15)}$ *for* $r, s \in \{2, 4, 10, 11\}, s < r$ *(six families).*

8. $\mathcal{T}^{(1)}_{(1,s,t)}$ *for* $s \in \{3, 5, 7, 8\},$ $t \in \{18, 19, 20, 21\}$ *(16 families),*

9. $\mathcal{T}^{(1)}_{(r,s,t)}$ *for* $r \in \{2, 4, 10, 11\},$ $s \in \{6, 9\},$ $t \in \{18, 19, 20, 21\}$ *(32 families).*

**Lemma 38.** *Let $\Gamma$ be a lattice acting regulary on the chambers of a building of type $\tilde{C}_2$ with vertex links $W^{\diamond}(4)$ and $K_{6,6}$. Then an isomorphic copy of the corresponding triangle of groups lies in exactly one of the following 232 families.*

1. $\mathcal{T}^{(2)}_{(1,1,t)}$ *for* $t \in \{22, 23, 24, 25\}$ *(four families),*
2. $\mathcal{T}^{(2)}_{(r,s,t)}$ *for* $r, s \in \{3, 5, 7, 8\}, r \le s,$ $t \in \{22, 23, 24, 25\}$ *(40 families),*
3. $\mathcal{T}^{(2)}_{(r,s,t)}$ *for* $r, s \in \{3, 5, 7, 8\}, s < r,$ $t \in \{23, 24\}$ *(12 families),*
4. $\mathcal{T}^{(2)}_{(r,s,t)}$ *for* $r, s \in \{6, 9\}, r \le s,$ $t \in \{26, 27, 28, 29\}$ *(12 families),*
5. $\mathcal{T}^{(2)}_{(9,6,t)}$ *for* $t \in \{27, 28\}$ *(two families),*
6. $\mathcal{T}^{(2)}_{(r,s,t)}$ *for* $r, s \in \{2, 4, 10, 11\}, r \le s,$ $t \in \{26, 27, 28, 29\}$ *(40 families),*
7. $\mathcal{T}^{(2)}_{(r,s,t)}$ *for* $r, s \in \{2, 4, 10, 11\}, s < r,$ $t \in \{27, 28\}$ *(12 families),*
8. $\mathcal{T}^{(1)}_{(1,s,t)}$ *for* $s \in \{6, 9\},$ $t \in \{30, 31, 32, 33, 34, 35\}$ *(12 families).*
9. $\mathcal{T}^{(1)}_{(r,s,t)}$ *for* $r \in \{3, 5, 7, 8\},$ $s \in \{2, 4, 10, 11\},$ $t \in \{30, 31, 32, 33, 34, 35\}$ *(96 families).*

*Proof of Lemmas 37, 38.* Consider all compatible combinations of local actions $L_i$ with actions on the quadrangle $Q$ at types 1 and 2. Then mirroring the triangle will yield an isomorphic triangle and moreover if the vertex group of type 3 appears in the list of Observation 35 then swapping the images of the edge groups of type 1 and 2 in the vertex group of type 3 will yield an isomorphic triangle. In the Lemma we just list a representative system for all compatible combinations with the just described properties up to the just described operations. It is clear that an isomorphic copy of a triangle can lie in at most one of these families. $\square$

We will now determine the number of type-preserving isomorphism classes of triangles of groups in each family from the Lemmas 37 and 38. To do so we will use Observation 20 and introduce some more necessary notation. If we have a local action $L_i$ and its local automorphism group $\Sigma(L, \langle a_j \rangle, \langle b_k \rangle)$, then by identifying the edge groups with the corresponding model edge groups $E_A, E_B$ via the standard embeddings we obtain a group $\Sigma^i \le \mathrm{Aut}(E_A) \times \mathrm{Aut}(E_B)$ equipped with projections $\pi^i_A : \Sigma^i \to \mathrm{Aut}(E_A)$ and $\pi^i_B : \Sigma^i \to \mathrm{Aut}(E_B)$. We call $\Sigma^i$ also local automorphism group of $L_i$. Often we have that $\Sigma^i = \mathrm{im}(\pi^i_A) \times \mathrm{im}(\pi^i_B)$ and in that case we call $\Sigma^i$ decomposable and denote $\mathrm{im}(\pi^i_A)$ and $\mathrm{im}(\pi^i_B)$ with $\Sigma^i_A$ and $\Sigma^i_B$ respectively. We computed the groups $\Sigma^i$ and listed them in the Tables 5, 6, 7 in Appendix B. We point out the following.

*Observation* 39. For $1 \le i \le 35$ the local automorphism group $\Sigma^i$ is indecomposable if and only if $i \in \{24, 28, 31, 33\}$.

In particular all edge-regular actions on $K_{4,4}$ and $Q$ have decomposable local automorphism groups. Therefore all families from Lemma 37 and most families from Lemma 38 only contain triangles of groups with decomposable local actions. In that case there is a convenient minimal representative system for the double cosets corresponding to the isomorphism classes in the family.

**Lemma 40.** *1. Let $\mathcal{T} = \mathcal{T}^{(1)}_{(r,s,t)}$ be a family from Lemma 37 and let $R_r, R_s, R_t$ be minimal representative systems for the following double cosets spaces*

$$\Sigma^t_A \backslash \operatorname{Aut}(E_r)/\Sigma^s_A, \qquad \Sigma^r_A \backslash \operatorname{Aut}(E_s)/\Sigma^t_B, \qquad \Sigma^s_B \backslash \operatorname{Aut}(E_t)/\Sigma^r_B.$$

*Then the following family is a minimal representative system for isomorphism classes of triangles of groups in $\mathcal{T}$.*

$$\begin{aligned}\{\boldsymbol{T}(L_r, L_s, L_t, E_r, E_s, E_t, (\varepsilon_{ij})_{ij}) \mid \varepsilon_{13} &= \kappa^t_A,\ \varepsilon_{12} \in \kappa^s_A \circ R_r,\\ \varepsilon_{21} &= \kappa^r_A,\ \varepsilon_{23} = \kappa^t_B \circ R_s,\\ \varepsilon_{32} &= \kappa^s_B,\ \varepsilon_{31} = \kappa^r_B \circ R_t\}.\end{aligned}$$

*2. Let $\mathcal{T} = \mathcal{T}^{(2)}_{(r,s,t)}$ be a family from Lemma 38 with only decomposable local actions and let $R_r, R_s, R_t$ be minimal representative systems for the following double cosets spaces*

$$\Sigma^t_A \backslash \operatorname{Aut}(E_r)/\Sigma^s_B, \qquad \Sigma^r_B \backslash \operatorname{Aut}(E_s)/\Sigma^t_B, \qquad \Sigma^s_A \backslash \operatorname{Aut}(E_t)/\Sigma^r_A.$$

*Then the following family is a minimal representative system for isomorphism classes of triangles of groups in $\mathcal{T}$.*

$$\begin{aligned}\{\boldsymbol{T}(L_r, L_s, L_t, E_r, E_s, E_t, (\varepsilon_{ij})_{ij}) \mid \varepsilon_{13} &= \kappa^t_A,\ \varepsilon_{12} \in \kappa^s_B \circ R_r,\\ \varepsilon_{21} &= \kappa^r_B,\ \varepsilon_{23} = \kappa^t_B \circ R_s,\\ \varepsilon_{32} &= \kappa^s_A,\ \varepsilon_{31} = \kappa^r_A \circ R_t\}.\end{aligned}$$

*Proof.* Let $\mathcal{T}^{(1)}_{(r,s,t)}$ be one of the considered families. Then $\mathcal{T}^{(1)}_{(r,s,t)}$ can be identified with the product $C = \operatorname{Aut}(E_r)^2 \times \operatorname{Aut}(E_s)^2 \times \operatorname{Aut}(E_t)^2$ by identifying the triangles of groups with monomorphisms $(\epsilon_{ij})_{ij}$ with the tuple $(\gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{23}, \gamma_{31}, \gamma_{32})$, where $\gamma_{ij}$ is the unique automorphism of the corresponding model edge group with $\kappa_{ij} \circ \gamma_{ij} = \epsilon_{ij}$. Let $D_r \cong \operatorname{Aut}(E_r), D_s \cong \operatorname{Aut}(E_s), D_t \cong \operatorname{Aut}(E_t)$ be the diagonal subgroups of the first two, the middle two and the last two components respectively. Now let $R$ be a minimal representative system for the following double coset space.

$$D_r \times D_s \times D_t \backslash C/\Sigma^s_A \times \Sigma^t_A \times \Sigma^r_A \times \Sigma^t_B \times \Sigma^r_B \times \Sigma^s_B.$$

Now by Observation 20 such system $R$ corresponds to a minimal representative system of triangles of groups in $\mathcal{T}^{(1)}_{(r,s,t)}$ up to type-preserving isomorphism. Clearly $R_r \times 1 \times 1 \times R_s \times R_t \times 1$ is such a system, which proves the first statement. The same proof with adjusted indices proves the second statement. $\square$

Note that for our model edge group any double coset space is either of size one, two, three or six. We now go through the cases with three decomposable local actions and list the number of type-preserving isomorphism classes.

**Lemma 41.** *1. Each of the three families $\mathcal{T}^{(1)}_{(1,1,t)}$ for $t \in \{12, 13, 14\}$ contains two type-preserving isomorphism classes (six classes).*

2. *Each of the nine families $\mathcal{T}^{(1)}_{(r,s,t)}$ for $r, s \in \{6, 9\}$, $r \leq s$, $t \in \{12, 13, 14\}$ and also the family $\mathcal{T}^{(1)}_{(9,6,14)}$ contain six type-preserving isomorphism classes (60 classes).*

3. *Each of the twenty families $\mathcal{T}^{(1)}_{(r,s,t)}$ for $r, s \in \{3, 5, 7, 8\}$, $r \leq s$, $t \in \{15, 16\}$ and also each of the six families $\mathcal{T}^{(1)}_{(r,s,15)}$ for $s < r \in \{3, 5, 7, 8\}$ contains nine type-preserving isomorphism classes (234 classes).*

4. *Each of the 10 families $\mathcal{T}^{(1)}_{(r,s,17)}$ for $r, s \in \{3, 5, 7, 8\}$, $r \leq s$ contains one type-preserving isomorphism classes (ten classes).*

5. *Each of the twenty families $\mathcal{T}^{(1)}_{(r,s,t)}$ for $r \leq s \in \{2, 4, 10, 11\}$, $t \in \{15, 16\}$ and also each of the six families $\mathcal{T}^{(1)}_{(r,s,15)}$ for $r, s \in \{2, 4, 10, 11\}$, $s < r$ contains 18 type-preserving isomorphism classes (468 classes).*

6. *Each of the 10 families $\mathcal{T}^{(1)}_{(r,s,17)}$ for $r, s \in \{2, 4, 10, 11\}, r \leq s$ contains two type-preserving isomorphism classes (20 classes).*

7. *Each of the 12 families $\mathcal{T}^{(1)}_{(1,s,t)}$ for $s \in \{3, 5, 7, 8\}$, $t \in \{18, 19, 21\}$ contains three type-preserving isomorphism classes (36 classes).*

8. *Each of the four families $\mathcal{T}^{(1)}_{(1,s,20)}$ for $s \in \{3, 5, 7, 8\}$ contains one type-preserving isomorphism class (four classes).*

9. *Each of the 24 families $\mathcal{T}^{(1)}_{(r,s,t)}$ for $r \in \{2, 4, 10, 11\}$, $s \in \{6, 9\}$, $t \in \{18, 19, 21\}$ contains nine type-preserving isomorphism classes (216 classes).*

10. *Each of the eight families $\mathcal{T}^{(1)}_{(r,s,20)}$ for $r \in \{2, 4, 10, 11\}$, $s \in \{6, 9\}$ contains three type-preserving isomorphism classes (24 classes).*

**Lemma 42.** *1. Each of the three families $\mathcal{T}^{(2)}_{(1,1,t)}$ for $t \in \{22, 23, 25\}$ contains two type-preserving isomorphism classes (six classes).*

2. *Each of the 30 families $\mathcal{T}^{(2)}_{(r,s,t)}$ for $r, s \in \{3, 5, 7, 8\}$, $r \leq s$, $t \in \{22, 23, 25\}$ and also each of the six families $\mathcal{T}^{(2)}_{(r,s,23)}$ for $s, r \in \{3, 5, 7, 8\}$, $s < r$ contains six type-preserving isomorphism classes (216 classes).*

3. *Each of the three families $\mathcal{T}^{(2)}_{(r,s,26)}$ for $r, s \in \{6, 9\}$, $r \leq s$ contains two type-preserving isomorphism classes (six classes).*

4. *Each of the four families* $\mathcal{T}^{(2)}_{(r,s,27)}$ *for* $r, s \in \{6, 9\}$ *contains six type-preserving isomorphism classes (24 classes).*

5. *Each of the three families* $\mathcal{T}^{(2)}_{(r,s,29)}$ *for* $r, s \in \{6, 9\}$*,* $r \leq s$ *contains 18 type-preserving isomorphism classes (54 classes).*

6. *Each of the 10 families* $\mathcal{T}^{(2)}_{(r,s,26)}$ *for* $r, s \in \{2, 4, 10, 11\}$*,* $r \leq s$ *contains six type-preserving isomorphism classes (60 classes).*

7. *Each of the 16 families* $\mathcal{T}^{(2)}_{(r,s,27)}$ *for* $r, s \in \{2, 4, 10, 11\}$ *contains 12 type-preserving isomorphism classes (192 classes).*

8. *Each of the 10 families* $\mathcal{T}^{(2)}_{(r,s,29)}$ *for* $r, s \in \{2, 4, 10, 11\}$*,* $r \leq s$ *contains 24 type-preserving isomorphism classes (240 classes).*

9. *Each of the four families* $\mathcal{T}^{(2)}_{(1,s,t)}$ *for* $s \in \{6, 9\}$*,* $t \in \{30, 34\}$ *contains six type-preserving isomorphism classes (24 classes).*

10. *Each of the four families* $\mathcal{T}^{(2)}_{(1,s,t)}$ *for* $s \in \{6, 9\}$*,* $t \in \{32, 35\}$ *contains two type-preserving isomorphism classes (eight classes).*

11. *Each of the 32 families* $\mathcal{T}^{(2)}_{(r,s,t)}$ *for* $r \in \{3, 5, 7, 8\}$*,* $s \in \{2, 4, 10, 11\}$*,* $t \in \{30, 34\}$ *contains 12 type-preserving isomorphism classes (384 classes).*

12. *Each of the 32 families* $\mathcal{T}^{(2)}_{(r,s,t)}$ *for* $r \in \{3, 5, 7, 8\}$*,* $s \in \{2, 4, 10, 11\}$*,* $t \in \{32, 35\}$ *contains six type-preserving isomorphism classes (192 classes).*

*Proof of Lemmas 41, 42.* We just applied Lemma 40 to each case. □

The cases with one non-decomposable local action remain.

**Lemma 43.**

1. *The family* $\mathcal{T}^{(2)}_{(1,1,24)}$ *contains four type-preserving isomorphism classes.*

2. *Each of the 16 families* $\mathcal{T}^{(2)}_{(r,s,24)}$ *for* $r, s \in \{3, 5, 7, 8\}$ *contains six type-preserving isomorphism classes (96 classes).*

3. *Each of the four families* $\mathcal{T}^{(2)}_{(r,s,28)}$ *for* $r, s \in \{6, 9\}$ *contains 12 type-preserving isomorphism classes (48 classes).*

4. *Each of the 16 families* $\mathcal{T}^{(2)}_{(r,s,28)}$ *for* $r, s \in \{2, 4, 10, 11\}$ *contains 12 type-preserving isomorphism classes (192 classes).*

5. *Each of the two families* $\mathcal{T}^{(2)}_{(1,s,31)}$ *for* $s \in \{6, 9\}$ *contains 12 type-preserving isomorphism classes (24 classes).*

6. *Each of the two families* $\mathcal{T}^{(2)}_{(1,s,33)}$ *for* $s \in \{6, 9\}$ *contains four type-preserving isomorphism classes (eight classes).*

7. *Each of the 16 families $\mathcal{T}^{(2)}_{(r,s,31)}$ for $r \in \{3,5,7,8\}$, $s \in \{2,4,10,11\}$ contains 12 type-preserving isomorphism classes (192 classes).*

8. *Each of the 16 families $\mathcal{T}^{(2)}_{(r,s,33)}$ for $r \in \{3,5,7,8\}$, $s \in \{2,4,10,11\}$ contains six type-preserving isomorphism classes (96 classes).*

*Proof.* Let $\mathcal{T}^{(2)}_{(r,s,t)}$ be one of the families in the lemma. Then $\mathcal{T}^{(2)}_{(r,s,t)}$ can be identified with the product $\mathrm{Aut}(E_r)^2 \times \mathrm{Aut}(E_s)^2 \times \mathrm{Aut}(E_t)^2$ as in the proof of Lemma 40. Note that $\Sigma^i_A = 1$ for $1 \le i \le 11$. In particular if $M'$ is a minimal representative system for the double cosets in $(*)$ indicated below, then the system $M$ in $(**)$ indicated below corresponds to a minimal representative set of type-preserving isomorphism classes in $\mathcal{T}^{(2)}_{(r,s,t)}$. In $(*)$ the groups $D_r \cong \mathrm{Aut}(E_r)$ and $D_s \cong \mathrm{Aut}(E_s)$ are the diagonal subgroups in the first two components and the latter two components respectively.

$$(D_r \times D_s)\backslash(E_r \times E_r \times E_s \times E_s)/(\Sigma^a_B \times \Sigma^t \times \Sigma^r_B), \tag{$*$}$$

$$M = M' \times \mathrm{Aut}(E_t) \times 1. \tag{$**$}$$

The cases 1, 2, 3, 5, 6, 7, 8 in the Lemma are now straightforward. In case 4 we have $E_r = E_s = \mathrm{Sym}(3)$ and $\Sigma^s_B = \Sigma^r_B = \langle \mathrm{Ad}(e_1)\rangle$. The group $\Sigma^t$ is the the diagonal subgroup of $\mathrm{Aut}(\mathrm{Sym}(3))^2$. In particular there are at least two double cosets in $M$ and in fact there are exactly two. Let $t = \mathrm{Ad}(e_1e_2)$. We claim that $M := \{(1,1,1,1),(t,1,1,1)\}$ is a representative system. Let $U$ be the union of the double cosets of the elements in $M$. Let $x, y \in \mathrm{Aut}(\mathrm{Sym}(3))$ be arbitrary. Then there is a $z \in \langle \mathrm{Ad}(e_1)\rangle$ such that the element $(yx, z, 1, 1)$ lies in $U$. In particular $(yx,1,1,z) \in U$ but then $(yx,1,1,1) \in U$. Hence $(x,1,y,1) \in U$ and this implies $U = \mathrm{Aut}(\mathrm{Sym}(3))^4$. □

**Proposition 44.** *Up to type-preserving isomorphism there are 3144 triangles of groups that give rise to a chamber-regular action on a $\tilde{C}_2$-building whose links of special vertices are the unique generalized quadrangle of order $(3,5)$.*

Now we compute the number of triangles of groups up to (possibly non-type-preserving) isomorphism. From the previous discussion one can easily extract a representative system up to type-preserving isomorphism. Furthermore this system is separated into families and members of different families are not isomorphic. Note that in most of the families two members can only be isomorphic via a type-preserving isomorphism. This holds exactly for the families $\mathcal{T}^{(i)}_{(r,s,t)}$ with $r \neq s$ or with $t$ not in the list in Observation 35. Therefore for these families the number of type-preserving isomorphism classes is the number of isomorphism classes. In what follows we study the remaining families.

**Lemma 45.** 1. *Each of the two families $\mathcal{T}^{(1)}_{(1,1,t)}$ for $t \in \{12,13\}$ contains two isomorphism classes (four classes).*

2. *Each of the four families $\mathcal{T}^{(1)}_{(r,r,t)}$ for $r \in \{6,9\}$, $t \in \{12,13\}$ contains five isomorphism classes (20 classes).*

3. *Each of the four families* $\mathcal{T}^{(1)}_{(r,r,16)}$ *for* $r \in \{3,5,7,8\}$ *contains six isomorphism classes (24 classes).*

4. *Each of the four families* $\mathcal{T}^{(1)}_{(r,r,16)}$ *for* $r \in \{2,4,10,11\}$ *contains 12 isomorphism classes (48 classes).*

5. *Each of the four families* $\mathcal{T}^{(1)}_{(r,r,17)}$ *for* $r \in \{2,4,10,11\}$ *contains two isomorphism classes (eight classes).*

6. *Each of the two families* $\mathcal{T}^{(2)}_{(1,1,t)}$ *for* $t \in \{22,25\}$ *contains two isomorphism classes (four classes).*

7. *Each of the eight families* $\mathcal{T}^{(2)}_{(r,r,t)}$ *for* $r \in \{3,5,7,8\}$*,* $t \in \{22,25\}$ *contains five isomorphism classes (40 classes).*

8. *Each of the two families* $\mathcal{T}^{(2)}_{(r,r,26)}$ *for* $r \in \{6,9\}$ *contains two isomorphism classes (four classes).*

9. *Each of the two families* $\mathcal{T}^{(2)}_{(r,r,29)}$ *for* $r \in \{6,9\}$ *contains 12 isomorphism classes (24 classes).*

10. *Each of the four families* $\mathcal{T}^{(2)}_{(r,r,26)}$ *for* $r \in \{2,4,10,11\}$ *contains five isomorphism classes (20 classes).*

11. *Each of the four families* $\mathcal{T}^{(2)}_{(r,r,29)}$ *for* $r \in \{2,4,10,11\}$ *contains 15 isomorphism classes (60 classes).*

*Proof.* As in the proofs of the Lemma 40 and 43 we identify the triangles in a family with $C = \mathrm{Aut}(E_r)^2 \times \mathrm{Aut}(E_s)^2 \times \mathrm{Aut}(E_t)^2$. Now let $\mathcal{T}$ be family considered in the lemma and let $T \in \mathcal{T}$. We obtain another (isomorphic) triangle $T' \in \mathcal{T}$ by replacing the type-function $f$ with $(1,2) \circ f$ and we call $T'$ the mirror of $T$. If $T$ corresponds to the tuple $(\gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{23}, \gamma_{31}, \gamma_{32})$ then by Observation 35 its mirror is type-preservingly isomorphic to the triangle in $\mathcal{T}$ corresponding to $(\gamma_{21}, \gamma_{23}, \gamma_{12}, \gamma_{13}, \gamma_{32}, \gamma_{31})$. In the cases 1, 2, 6, 7, 8 and 10 in the lemma a minimal representative system up to type-preserving automorphism corresponds to the following set in $C$: $\{(\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \gamma_{31}, \mathrm{id}) \mid \gamma_{31} \in \mathrm{Aut}(E_3)\}$. We deduce that the triangle corresponding to $\{(\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \gamma_{31}, \mathrm{id})$ is (non-type-preservingly) isomorphic to $\{(\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \gamma_{31}^{-1}, \mathrm{id})$ and therefore the number of isomorphism classes in $\mathcal{T}$ equals the cardinality of $\{\{\gamma, \gamma^{-1}\} \mid \gamma \in \mathrm{Aut}(E_3)\}$. Now let $\mathcal{T} = \mathcal{T}^{(1)}_{(r,r,16)}$ for $r \in \{3,5,7,8\}$. Let $R = \{\mathrm{id}, (a_1, a_2), (a_1, a_1 a_2)\} \subseteq \mathrm{Aut}(C_2 \times C_2)$. Then a minimal representative system for type-preserving isomorphism classes in $\mathcal{T}$ corresponds to the set $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \mathrm{id}, \mathrm{id}) \mid \gamma_{13}, \gamma_{23} \in R\}$. Now enumerate $R = \{r_1, r_2, r_3\}$. With the same argumentation we deduce that a minimal representative system for isomorphism classes corresponds to the set $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \mathrm{id}, \mathrm{id}) \mid \gamma_{13} = r_i, \gamma_{23} = r_j, i \leq j\}$. Now let $\mathcal{T} = \mathcal{T}^{(1)}_{(r,r,16)}$ for $r \in \{2,4,10,11\}$. Let $S = \{\mathrm{id}, \mathrm{Ad}(b_2)\} \subseteq \mathrm{Aut}(\mathrm{Sym}(3)$ and let $R = \{r_1, r_2, r_3\}$

be as before. Then a minimal representative system for type-preserving isomorphism classes in $\mathcal{T}$ corresponds to the set $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \gamma_{31}, \mathrm{id}) \mid \gamma_{13}, \gamma_{23} \in R, \gamma_{31} \in S\}$. Since $S$ consists of involutions a representative system for isomorphism classes corresponds to the set $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \gamma_{31}, \mathrm{id}) \mid \gamma_{13} = r_i, \gamma_{23} = r_j, i \leq j, \gamma_{31} \in S\}$. Now let $\mathcal{T} = \mathcal{T}^{(1)}_{(r,r,17)}$ for $r \in \{3,5,7,8\}$. Let $S$ be as before. Then a minimal representative system for type-preserving isomorphism classes corresponds to the set $\{(\mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id}, \gamma_{31}, \mathrm{id}) \mid \gamma_{31} \in S\}$. Since $S$ consists of involutions this is also a minimal representative system for isomorphism classes. Now let $\mathcal{T} = \mathcal{T}^{(2)}_{(r,r,29)}$ for $r \in \{6,9\}$. This time we set $R = \{\mathrm{id}, \mathrm{Ad}(a_2), \mathrm{Ad}(a_1 a_2)\} \subseteq \mathrm{Aut}(\mathrm{Sym}(3))$ and again we enumerate it $R = \{r_1, r_2, r_3\}$. We set $S = \mathrm{Aut}(C_4)$ Then a minimal representative system for the type-preserving isomorphism classes in $\mathcal{T}$ corresponds to the set $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \gamma_{31}, \mathrm{id}) \mid \gamma_{13} = r_i, \gamma_{23} = r_j, i \leq j, \gamma_{31} \in S\}$. By similar arguments as before a minimal representative system for isomorphism classes corresponds to $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \gamma_{31}, \mathrm{id}) \mid \gamma_{13} = r_i, \gamma_{23} = r_j, i \leq j, \gamma_{31} \in S\}$. For the last case let $\mathcal{T} = \mathcal{T}^{(2)}_{(r,r,29)}$ for $r \in \{2,4,10,11\}$. This time let $R = \{r_1, r_2\} = \{\mathrm{id}, \mathrm{Aut}(b_2)\}$ and $S = \mathrm{Aut}(C_2 \times C_2)$. Then a minimal representative system for the type-preserving isomorphism classes in $\mathcal{T}$ corresponds to the set $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \gamma_{31}, \mathrm{id}) \mid \gamma_{13} = r_i, \gamma_{23} = r_j, i \leq j, \gamma_{31} \in S\}$. Note that this time $S$ contains elements that are not involutions. Let $S' \subseteq S$ be a minimal set such that $S' \cup (S')^{-1} = S$. Then a minimal representative system for isomorphism classes corresponds to $\{(\mathrm{id}, \gamma_{13}, \mathrm{id}, \gamma_{23}, \gamma_{31}, \mathrm{id}) \mid \gamma_{13} = r_i, \gamma_{23} = r_j, i \leq j, \gamma_{31} \in S'\}$. □

So in total there are 3044 chamber-regular action. By Proposition 26 the acting groups are pairwise not isomorphic.

**Theorem 46.** *Up to isomorphism there exist 3044 chamber-regular lattices on a $\tilde{C}_2$-building with thickness (4,4,6) or (6,6,4). Each of these groups admits only one such action. Assuming Kantor's conjecture mentioned in the introduction these are the only lattices that act chamber-regularly on a locally-finite $\tilde{C}_2$-building.*

# A Presentations of vertex groups

Table 2: Presentations of chamber-regular groups on $Q$.

| **Grp** | **Presentation** |
|---|---|
| $L_1$ | $\langle a, b \mid a^4,\ b^6,\quad ab^{-1}a^{-1}b^2a^{-1}ba^{-2}b,\quad aba^{-2}b^{-1}a^{-1}ba^{-2}b^{-1},\quad abab^{-1}ab^{-2}a^{-1}b^2 \rangle$ |

| | |
|---|---|
| $L_2$ | $\langle a_1, a_1, b_1, b_2 \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ a_2b_2a_1b_2^{-1}a_2b_2^{-1}a_1b_2,\ (a_1b_2^{-1}a_1b_2)^2,\ a_2b_2a_2b_1a_1b_2^{-1}a_1b_1,\ a_1a_2b_2a_1b_2b_1a_2b_2a_1b_1,\ a_1b_2b_1(a_1b_1b_2^{-1})^3 \rangle$ |
| $L_3$ | $\langle a_1, a_2, b \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ b^6,\ a_1ba_2ba_2b^{-1}a_2b^{-1},\ a_2b^{-1}a_2ba_2ba_1b^{-1},\ (a_1b)^4,\ a_1ba_2b^{-2}a_1b^{-2}a_2b \rangle$ |
| $L_4$ | $\langle a_1, a_2, b_1, b_2 \mid a_1^2, a_2^2, (a_1a_2)^2, b_1^2, b_2^3, (b_1b_2)^2, (a_2b_1a_1b_1)^2, (a_1b_1)^4, a_2b_2a_2b_1a_1b_2^{-1}a_1b_1, (a_1b_2^{-1}a_1b_2)^2, a_2b_2a_1b_2^{-1}a_2a_1b_2a_1b_2^{-1}, a_1b_2^{-1}a_2b_1b_2a_2b_2a_1b_1b_2, a_1b_2a_2b_1b_2^{-1}a_2b_2^{-1}a_1b_2b_1 \rangle$ |
| $L_5$ | $\langle a_1, a_2, b \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ b^6,\ (a_2b)^4,\ (a_2ba_1b)^2,\ (a_1b)^4,\ a_2b^{-1}a_1b^2a_2ba_1b^{-2},\ a_2b^2a_1b^{-2}a_2ba_2b^{-1} \rangle$ |
| $L_6$ | $\langle a, b_1, b_2 \mid a^4,\ b_1^2,\ b_2^3,\ (b_1b_2^{-1})^2,\ (ab_2^{-1}ab_1)^2,\ ab_1a^{-1}b_1ab_1a^{-1}b_1b_2^{-1},\ a^{-1}b_2^{-1}ab_2^{-1}a^{-1}b_1a^{-1}b_1b_2 \rangle$ |
| $L_7$ | $\langle a_1, a_2, b \mid a_1^2, a_2^2, (a_1a_2)^2, b^6,\ (a_2ba_2b^{-1})^2,\ (a_1ba_2b^{-1})^2,\ (a_2ba_1b^{-1})^2,\ (a_1ba_1b^{-1})^2,\ a_1ba_2ba_1b^2a_2b^2,\ (a_1ba_1b^2)^2,\ a_2ba_2ba_1a_2ba_1b^3 \rangle$ |
| $L_8$ | $\langle a_1, a_2, b \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ b^6,\ (ba_1b^{-1}a_2)^2,\ a_2ba_1a_2ba_2b^{-1}a_2b^{-1},\ a_1b^2a_2ba_2b^2a_1b \rangle$ |
| $L_9$ | $\langle a, b_1, b_2 \mid a^4,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ (ab_2)^4,\ ab_2^{-1}ab_1a^{-1}b_2a^{-1}b_1b_2,\ ab_1ab_1b_2ab_2^{-1}a^{-1}b_2^{-1},\ ab_1b_2ab_1ab_2a^{-1}b_2,\ ab_2ab_1ab_1a^2b_2b_1 \rangle$ |
| $L_{10}$ | $\langle a_1, a_2, b_1, b_2 \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ (a_1b_2^{-1}a_1b_2)^2,\ a_2b_2^{-1}a_1b_1a_1b_2a_2b_2b_1,\ a_2a_1b_2^{-1}a_1b_1a_2b_2a_1b_1b_2^{-1},\ a_2b_2^{-1}a_1b_2b_1a_2b_1b_2^{-1}a_1b_2 \rangle$ |

| | | |
|---|---|---|
| $L_{11}$ | $\langle a_1,a_2,b_1,b_2 \mid a_2^2,\ a_1^2,\ (a_1a_2)^2,$ $(a_1b_1)^4,$ $(a_1b_2^{-1}a_1b_1)^2,$ $a_1b_2a_2b_2a_2b_2b_1a_2b_1,$ | $b_1^2,\ b_2^3,\ (b_1b_2)^2,$ $(a_2b_2^{-1}a_1b_2)^2,$ $(a_1b_2^{-1}a_1b_2)^2,$ $a_1b_2a_2b_1a_2b_2^{-1}a_1b_1b_2\rangle$ |

Table 3: Presentations of chamber-regular groups on $K_{44}$.

| Grp | Presentation |
|---|---|
| $L_{12}$ | $\langle a,b \mid a^4,\ b^4,\ aba^{-1}b^{-1}\rangle$ |
| $L_{13}$ | $\langle a,b \mid a^4, b^4,\ (ab^{-1})^2,\ (ab)^2\rangle$ |
| $L_{14}$ | $\langle a,b \mid a^4, b^4,\ abab^{-1}\rangle$ |
| $L_{15}$ | $\langle a_1,a_2,b_1,b_2 \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ \ b_1^2,\ b_2^2,\ (b_1b_2)^2,\ \ (a_1b_1)^2,\ (a_1b_2)^2,\ (a_2b_1)^2,\ a_2a_1b_2a_2b_2\rangle$ |
| $L_{16}$ | $\langle a_1,a_2,b_1,b_2 \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ \ b_1^2,\ b_2^2,\ (b_1b_2)^2,\ \ (a_1b_1)^2,\ (a_1b_2)^2,\ (a_2b_1)^2,\ a_2a_1b_2a_2b_2b_1\rangle$ |
| $L_{17}$ | $\langle a_1,a_2,b_1,b_2 \mid a_1^2,\ a_2^2,\ (a_1a_2)^2,\ \ b_1^2,\ b_2^2,\ (b_1b_2)^2,\ \ (a_1b_1)^2,\ (a_1b_2)^2,\ (a_2b_1)^2,\ (a_2b_2)^2\rangle$ |
| $L_{18}$ | $\langle a,b_1,b_2 \mid a^4,\ b_1^2,\ b_2^2,\ (b_1b_2)^2,\ ab_1a^{-1}b_1,\ ab_2a^{-1}b_2b_1\rangle$ |
| $L_{19}$ | $\langle a,b_1,b_2 \mid a^4,\ b_1^2,\ b_2^2,\ (b_1b_2)^2,\ ab_1a^{-1}b,\ (ab_2)^2b_1,\ a^2b_2a^{-2}b_2\rangle$ |
| $L_{20}$ | $\langle a,b_1,b_2 \mid a^4,\ b_1^2,\ b_2^2,\ (b_1b_2)^2,\ ab_2a^{-1}b_2,\ ab_1a^{-1}b_1\rangle$ |
| $L_{21}$ | $\langle a,b_1,b_2 \mid a^4,\ b_1^2,\ b_2^2,\ (b_1b_2)^2,\ ab_1a^{-1}b_1,\ (ab_2)^2\rangle$ |

Table 4: Presentations of chamber-regular groups on $K_{66}$.

| Grp | Presentation |
|---|---|
| $L_{22}$ | $\langle a,b \mid a^6,\ b^6,\ (ab)^2,\ (ab^{-1})^2\ \rangle$ |
| $L_{23}$ | $\langle a,b \mid a^6,\ b^6,\ aba^{-1}b\rangle$ |
| $L_{24}$ | $\langle a,b \mid a^6,\ b^6,\ (ab^{-1})^2,\ a^2b^{-1}a^{-2}b\rangle$ |

| | |
|---|---|
| $L_{25}$ | $\langle a, b \mid a^6,\ b^6,\ ab^{-1}a^{-1}b \rangle$ |
| $L_{26}$ | $\langle a_1, a_2, b_1, b_2 \mid a_1^2,\ a_2^3,\ (a_1a_2)^2,\ \ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ \ (a_1b_1)^2,$ <br> $a_1b_2a_1b_2^{-1},\ \ a_2b_1a_2^{-1}b_1,\ \ a_2b_2a_2^{-1}b_2^{-1} \rangle$ |
| $L_{27}$ | $\langle a_1, a_2, b_1, b_2 \mid a_1^2,\ a_2^3,\ (a_1a_2)^2,\ \ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ \ (a_1b_1)^2,$ <br> $a_1b_2a_1b_2^{-1},\ \ a_2b_2a_2^{-1}b_2^{-1},\ \ (a_2b_1^{-1})^2 \rangle$ |
| $L_{28}$ | $\langle a_1, a_2, b_1, b_2 \mid a_1^2,\ a_2^3,\ (a_1a_2)^2,\ \ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ \ (a_1b_1)^2,$ <br> $a_1a_2b_2a_1b_2^{-1},\ \ a_2b_2a_2^{-1}b_2^{-1},\ \ (a_2b_1^{-1})^2 \rangle$ |
| $L_{29}$ | $\langle a_1, a_2, b_1, b_2 \mid a_1^2,\ a_2^3,\ (a_1a_2)^2,\ \ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ \ (a_1b_1)^2,$ <br> $(a_1b_2)^2,\ \ a_2b_2a_2^{-1}b_2^{-1},\ \ (a_2b_1^{-1})^2 \rangle$ |
| $L_{30}$ | $\langle a, b_1, b_2 \mid a^6,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ ab_2a^{-1}b_2,\ (ab_1)^2 \rangle$ |
| $L_{31}$ | $\langle a, b_1, b_2 \mid a^6,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ a^3b_2^{-1}a^{-1}b_2^{-1},\ (ab_1)^2,\ (ab_2^{-1})^2 \rangle$ |
| $L_{32}$ | $\langle a, b_1, b_2 \mid a^6,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ ab_1a^{-1}b_1,\ ab_2^{-1}a^{-1}b_2 \rangle$ |
| $L_{33}$ | $\langle a, b_1, b_2 \mid a^6,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ ab_1a^{-1}b_2^{-1}b_1,\ ab_2^{-1}a^{-1}b_2 \rangle$ |
| $L_{34}$ | $\langle a, b_1, b_2 \mid a^6,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ ab_1a^{-1}b_1,\ ab_2^{-1}a^{-1}b_2^{-1} \rangle$ |
| $L_{35}$ | $\langle a_1, b_1, b_2 \mid a_1^6,\ b_1^2,\ b_2^3,\ (b_1b_2)^2,\ (a_1b_1^{-1})^2,\ a_1b_2a_1^{-1}b_2^{-1} \rangle$ |

# B Actions on the edge groups

As in Section 5 we identify the local automorphism group of $L_i$ with a subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$, where $A, B$ are the model edge groups. In the following tables we list these groups and denote them by $\Sigma^i$. We also introduce some notation for automorphisms of the edge groups. Let $A$ be a model edge group. If $A$ is cyclic, we denote the automorphism, that maps $a \in A$ to $a^{-1}$ by $\rho_A$. If $A = C_2 \times C_2$ we will indicate the action of automorphism on $A$ by cycle notation. If $A = \mathrm{Sym}(3)$ we indicate automorphisms by inner automorphisms, and we denote the conjugation with $a \in A$ by $\mathrm{Ad}(a)$.

Table 5: $\Sigma^i$ for the groups $L_1 \ldots, L_{11}$.

| $i \in$ | $A$ | $B$ | $\Sigma^i$ | $\lvert\Sigma^i\rvert$ |
|---|---|---|---|---|
| $\{1\}$ | $C_4$ | $C_6$ | 1 | 1 |

| | | | | |
|---|---|---|---|---|
| $\{2,4,10,11\}$ | $C_2 \times C_2$ | $\mathrm{Sym}(3)$ | $1 \times \langle \mathrm{Ad}(b_1) \rangle$ | 2 |
| $\{3,5,7,8\}$ | $C_2 \times C_2$ | $C_6$ | $1 \times \mathrm{Aut}(C_6)$ | 2 |
| $\{6,9\}$ | $C_4$ | $\mathrm{Sym}(3)$ | 1 | 1 |

Table 6: $\Sigma^i$ for the groups $L_{12}, \ldots, L_{21}$.

| $\boldsymbol{i \in}$ | $\boldsymbol{A}$ | $\boldsymbol{B}$ | $\boldsymbol{\Sigma^i}$ | $\boldsymbol{\lvert\Sigma^i\rvert}$ |
|---|---|---|---|---|
| $\{12,13,14\}$ | $C_4$ | $C_4$ | $\mathrm{Aut}(C_4) \times \mathrm{Aut}(C_4)$ | 4 |
| $\{15,16\}$ | $C_2 \times C_2$ | $C_2 \times C_2$ | $\langle (a_2, a_1a_2) \rangle \times \langle (b_2, b_1b_2) \rangle$ | 4 |
| $\{17\}$ | $C_2 \times C_2$ | $C_2 \times C_2$ | $\mathrm{Aut}(C_2 \times C_2) \times \mathrm{Aut}(C_2 \times C_2)$ | 36 |
| $\{18,19,21\}$ | $C_4$ | $C_2 \times C_2$ | $\mathrm{Aut}(C_4) \times \langle (b_2, b_1b_2) \rangle$ | 4 |
| $\{20\}$ | $C_4$ | $C_2 \times C_2$ | $\mathrm{Aut}(C_4) \times \mathrm{Aut}(C_2 \times C_2)$ | 12 |

Table 7: $\Sigma^i$ for the groups $L_{22}, \ldots, L_{35}$.

| $\boldsymbol{i \in}$ | $\boldsymbol{A}$ | $\boldsymbol{B}$ | $\boldsymbol{\Sigma^i}$ | $\boldsymbol{\lvert\Sigma^i\rvert}$ |
|---|---|---|---|---|
| $\{22,23,25\}$ | $C_6$ | $C_6$ | $\mathrm{Aut}(C_6) \times \mathrm{Aut}(C_6)$ | 4 |
| $\{24\}$ | $C_6$ | $C_6$ | $\langle (\rho_A, \rho_B) \rangle$ | 2 |
| $\{26\}$ | $\mathrm{Sym}(3)$ | $\mathrm{Sym}(3)$ | $\mathrm{Aut}(\mathrm{Sym}(3)) \times \mathrm{Aut}(\mathrm{Sym}(3))$ | 36 |
| $\{27\}$ | $\mathrm{Sym}(3)$ | $\mathrm{Sym}(3)$ | $\langle \mathrm{Ad}(a_1) \rangle \times \mathrm{Aut}(\mathrm{Sym}(3))$ | 12 |
| $\{28\}$ | $\mathrm{Sym}(3)$ | $\mathrm{Sym}(3)$ | $\langle (\mathrm{Ad}(a_1), \mathrm{Ad}(b_1)), (\mathrm{Ad}(a_2), \mathrm{Ad}(b_2)) \rangle$ | 6 |
| $\{29\}$ | $\mathrm{Sym}(3)$ | $\mathrm{Sym}(3)$ | $\langle \mathrm{Ad}(a_1) \rangle \times \langle \mathrm{Ad}(b_1) \rangle$ | 4 |
| $\{30,34\}$ | $C_6$ | $\mathrm{Sym}(3)$ | $\mathrm{Aut}(C_6) \times \langle \mathrm{Ad}(b_1) \rangle$ | 4 |
| $\{31\}$ | $C_6$ | $\mathrm{Sym}(3)$ | $\langle (\rho_A, \mathrm{Ad}(b_1)) \rangle$ | 2 |
| $\{32,35\}$ | $C_6$ | $\mathrm{Sym}(3)$ | $\mathrm{Aut}(C_6) \times \mathrm{Aut}(\mathrm{Sym}(3))$ | 12 |
| $\{33\}$ | $C_6$ | $\mathrm{Sym}(3)$ | $\langle (\rho_A, \mathrm{Ad}(b_1)), (\mathrm{id}_A, \mathrm{Ad}(b_2)) \rangle$ | 6 |

The following table provides convenient data for vertex groups acting regularly on complete bipartite graphs.

Table 8: Description of the vertex groups $L_{12}, \ldots, L_{35}$.

| Grps | Description | Small Group ID |
|---|---|---|
| $L_{12}$ | $C_4 \times C_4$ | (16,2) |
| $L_{13} \cong L_{18} \cong L_{19}$ | $(C_4 \times C_2) \rtimes C_2$ | (16,3) |
| $L_{14}$ | $C_4 \rtimes C_4$ | (16,4) |
| $L_{15} \cong L_{16} \cong L_{21}$ | $C_2 \times D_8$ | (16,11) |
| $L_{17}$ | $C_2 \times C_2 \times C_2 \times C_2$ | (16,14) |
| $L_{20}$ | $C_4 \times C_2 \times C_2$ | (16,10) |
| $L_{22} \cong L_{26} \cong L_{27} \cong L_{28} \cong L_{30} \cong L_{31}$ | $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$ | (36,10) |
| $L_{23} \cong L_{24} \cong L_{32} \cong L_{33} \cong L_{34}$ | $C_6 \times \mathrm{Sym}(3)$ | (36,12) |
| $L_{25}$ | $C_6 \times C_6$ | (36,14) |
| $L_{29} \cong L_{35}$ | $C_2 \times ((C_3 \times C_3) \rtimes C_2)$ | (36,13) |

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