ON TWO NOTIONS OF CURVATURE ON SINGULAR SURFACES

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Abstract. In this paper, we investigate the equivalence of two distinct notions of curvature bounds on singular surfaces. The first notion involves inequalities of the form $\omega \geq \kappa\mu$ (resp. $\omega \leq \kappa\mu$) where ω is the curvature measure and μ the Hausdorff measure. The second notion is the classical Alexandrov curvature bound CBB (resp. CAT). We demonstrate that these two definitions are, in fact, equivalent. Specifically, we fill an important gap in the theory by showing that the inequalities imply the corresponding Alexandrov CBB (resp. CAT) bound. One striking application of our result is that, in combination with a result of Petrunin, the lower bound $\omega \geq \kappa\mu$ implies $\mathrm{RCD}(\kappa, 2)$.

1. Introduction

In the second half of the twentieth century, the Leningrad school and A. D. Alexandrov developed synthetic notions of curvature bounds on metric spaces. One of these is the well-known curvature bounded below (CBB) formalized in the seminal paper [6] by Yu. Burago, M. Gromov, and G. Perel'man. At the same time, Alexandrov and Zalgaller introduced another theory of surfaces of bounded integral curvature (BIC) in the book [2]. On a metric surface S of this type the curvature takes the form of a signed Radon measure ω with locally bounded variation. One says that the curvature is bounded below by $\kappa \in \mathbb{R}$ if the inequality $\omega \geq \kappa\mu$ holds, where μ denotes the area measure (2-dimensional Hausdorff measure, cf. Section 2.2). The aim of this paper is to show the equivalence between these two approaches to curvature bound on BIC surfaces. Our first main result is the following:

Theorem A. *Assume S is a BIC surface whose curvature measure satisfies $\omega \geq \kappa\mu$ for some $\kappa \in \mathbb{R}$ and S has no cusp, meaning that there are no points $p \in S$ with $\omega(\{p\}) \geq 2\pi$. Then S is* $\mathrm{CBB}(\kappa)$.

One can actually drop the assumption of absence of cusp when $\kappa \geq 0$. The converse of this theorem is already well-known in the literature: if S is a $\mathrm{CBB}(\kappa)$ surface of Hausdorff dimension 2 then it is a BIC surface without any cusp. see [12] for a geometric proof and [5] for an analytic proof.

From Petrunin's work [9], $\mathrm{CBB}(\kappa)$ spaces together with the 2-dimensional Hausdorff measure satisfy the $\mathrm{RCD}(\kappa, 2)$ condition. Hence, an immediate and very important corollary is:

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Corollary. *Assume S is a BIC surface whose curvature measure satisfies $\omega \geq \kappa\mu$ for some $\kappa \in \mathbb{R}$. Moreover, when $\kappa < 0$, S is supposed to be whithout cusp. Then S together with its 2-dimensional Hausdorff measure is* $\mathrm{RCD}(\kappa, 2)$.

We also establish an analog of Theorem A for curvature bounded above by κ, i.e. when $\omega \leq \kappa\mu$. The second main result is the following:

Theorem B. *Assume S is a BIC surface whose curvature measure satisfies $\omega \leq \kappa\mu$ for some $\kappa \leq 0$. Then S is locally* $\mathrm{CAT}(\kappa)$.

The converse of this theorem has been shown very recently in the paper [7]. The conclusion of locally CAT is sharp. For example, if one take a flat cylinder and a triangle on a circular section with all three sides with same length then the angles are π. Whereas the angles of a plane triangle with same side length has angles equal to $\pi/3$. So the flat cylinder cannot be globally CAT(0) even though it has $\omega = 0$.

Theorem 1 and Theorem 2 have already been noted by Reshetnyak in [10, p.140], however, the results were only proved for $\kappa = 0$ (and in a rather sketchy manner). We remark that the proof for arbitrary κ becomes much more complicated, roughly speaking due to the fact that in the computation of the relative angle excess a new term depending on the area of the model triangle appears. Our new insight is to use an estimate that relates the distortion between area of triangles and area of plane developments to curvature measure. The present paper thus fills an important gap in the theory.

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2. Background

2.1. **Angles in metric spaces.**

We denote by $\mathbb{M}^2(\kappa)$, called *model spaces*, the connected two-dimensional Riemannian manifold of constant sectional curvature κ and write ϖ^κ its diameter. Let X be a metric space with distance function d. For any $\kappa \in \mathbb{R}$ and any points $p, q, r \in X$, we define the *model triangle* $\tilde{\Delta}^\kappa(pqr)$ to be any triangle $[\tilde{p}\tilde{q}\tilde{r}]$ in the model space $\mathbb{M}^2(\kappa)$ such that

$$d_{\mathbb{M}^2(\kappa)}(\tilde{p}, \tilde{q}) = d(p, q), \quad d_{\mathbb{M}^2(\kappa)}(\tilde{q}, \tilde{r}) = d(q, r), \quad d_{\mathbb{M}^2(\kappa)}(\tilde{r}, \tilde{p}) = d(r, p).$$

Note that when $\kappa \leq 0$, the model triangle always exists. Otherwise, when κ is positive, we require that the following condition holds,

$$d(p, q) + d(q, r) + d(r, p) < 2\varpi^\kappa.$$

This construction allows us to define the *model angle*, written $\tilde{\measuredangle}^\kappa(p^q_r)$, by the angle at the vertex $\tilde{p}$ in the model triangle $[\tilde{p}\tilde{q}\tilde{r}]$. Let $p, x, y \in X$ be a triple of points such that p is different from x and y. A pair of minimizing geodesics (γ, γ') such that γ (resp. γ') is from p to x (resp. y) is called a *hinge* and is denoted $[p^x_y]$. Given a hinge $[p^x_y]$, we define its *(upper) angle* to be

$$\measuredangle[p^x_y] := \limsup_{\substack{q,r\to p\\ q,r\neq p}} \tilde{\measuredangle}^\kappa(p^q_r).$$

This definition doesn't depend on the κ chosen. We say that this angle is *defined* if the lim sup is actually a limit. Two hinges $[p^x_y]$ and $[p^y_z]$ are said to be *adjacent* if the union of the geodesics from p to x and from p to z is actually a geodesic from x to z.

Another notion of angle is defined as follow. Consider a hinge $[p^x_y]$ and all possible sequence of x_n, y_n such that

- $x_n \in [px], y_n \in [py], x_n \neq p, y_n \neq p$, $x_n \to p$ or $y_n \to p$,
- each pair x_n, y_n are joined by a minimizing geodesic such that in the case $x_n \to p$ the curve $[x_n y_n]$ converges to a segment of $[py]$ or if $y_n \to p$ the curve $[x_n y_n]$ converges to a segment of $[px]$.

The lower limit taken on all sequences satisfying these conditions

$$\measuredangle^{\mathrm{ls}}[p^x_y] := \liminf \tilde{\measuredangle}^0(p^{x_n}_{y_n})$$

is called the *lower strong angle*. Lower strong angles mainly serve to obtain symmetric estimates to those derived from the upper angle.

A *triangle* T in X is the data of three points and three minimizing geodesics joining them. The notation T^κ stands for a model triangle of T in the model surface of constant curvature κ and we write $|T^\kappa|$ its area measure. Let $\delta_\kappa(T)$ denotes the *relative excess* of T with respect to T^κ, i.e.

$$\delta_\kappa(T) := (\alpha + \beta + \gamma) - (\alpha_\kappa + \beta_\kappa + \gamma_\kappa)$$

where α, β, γ are the angles of T and $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ are the angles of T^κ. It will be also useful to express it as

$$\delta_\kappa(T) = \delta_0(T) - \delta_0(T^\kappa). \tag{1}$$

When the relative excess is computed with $\kappa = 0$, we write simply it $\delta(T)$ and call it *excess*. When $\mathcal{T}$ is a finite family of non-overlapping triangles, then

$$\delta(\mathcal{T}) := \sum_{T\in\mathcal{T}} \delta(T).$$

With indeed $\delta(\mathcal{T}) = 0$ when $\mathcal{T} = \varnothing$. A triangle T is *convex relative to the boundary* if no couple of points on the boundary of T can be joined by a curve lying outside T and shorter than the part of the boundary joining the points. A triangle is *simple* if it is homeomorphic to a disc and convex relative to the boundary.

The next two propositions are general estimates for the distortion between angles in a metric space and model angle. They first appeared in the paper [4] by Alexandrov. They play a central role in the proofs of the main theorems.

Proposition 1. *Let $T = [pqr]$ be a triangle in a metric space (X, d). Define*

$$\mu_\kappa := \sup_{\substack{x\in[pq],\\ y\in[pr]}} \delta_\kappa[pxy].$$

Then

$$\alpha - \alpha_\kappa \leq \mu_\kappa,$$

where α is the upper angle at p and α_κ the model angle.

Proposition 2. *Let $T = [pqr]$ be a triangle in a metric space (X, d) such that, for all $x \in [pq]$ and $y \in [pr]$, there exists a unique minimizing geodesic $[xy]$. Assume strong angles exist between $[xy]$ and both $[xp], [yp]$. For any $\kappa \in \mathbb{R}$, define*

$$\nu_\kappa^{ls} := \inf_{\substack{x\in[pq],\\ y\in[pr]}} \delta_\kappa^{ls}[pxy].$$

where $\delta_\kappa^{ls}[pxy]$ is the relative excess computed with lower strong angles. Then

$$\alpha^{ls} - \alpha_\kappa \geq \nu_\kappa^{ls}, \tag{2}$$

where α^{ls} is the lower strong angle at p and α_κ the model angle.

2.2. **Bounded integral curvature.**

Definition 1. A metric space S with distance function d is called a *surface with locally bounded integral curvature*, in short a *BIC surface*, if

- S is a connected, oriented, 2-dimensional smooth manifold without boundary;
- $d : S \times S \to [0, +\infty)$ is a complete intrinsic distance inducing the manifold topology of S;
- for every compact K there exists a constant $C = C(K) > 0$ such that $\delta(\mathcal{T}) \leq C$ for any finite collection $\mathcal{T}$ of non-overlapping simple triangles contained in K.

We fix a BIC surface S for the rest of this section. Let μ denote the 2-dimensional Hausdorff measure induced by d. A signed measure of interest on these surfaces is the *curvature measure* denoted ω. The positive and negative parts of this latter can be defined over open sets $U \subset S$ by

$$\omega^+(U) := \sup_{\mathcal{T}} \sum_{T\in\mathcal{T}} \delta(T), \quad \omega^-(U) := -\inf_{\mathcal{T}} \sum_{T\in\mathcal{T}} \delta(T),$$

with both supremum taken on finite families $\mathcal{T}$ of non-overlapping simple triangles contained in U. We extend $\omega^\pm$ to any other Borel subset $E \subset S$ by

$$\omega^\pm(E) := \inf_{U\supset E \text{ open}} \omega^\pm(U),$$

and so $\omega(E) = \omega^+(E) - \omega^-(E)$. The measure ω^+ is always locally finite, while local finiteness of ω^- is a deep fact of the theory (see [2, Theorem 15 p.134]). By [11, Theorem 2.18], ω is thus a signed Radon measure. A point $p \in S$ is a *cusp* if $\omega(\{p\}) \geq 2\pi$. This name comes the fact that, if $\omega(\{p\}) > 2\pi$ then p is a point at infinity and if $\omega(\{p\}) = 2\pi$ then p may be at finite or infinite distance from other points.

Definition 2. Let $\kappa \in \mathbb{R}$. We say that S has *curvature bounded below by* κ, shortly S is BICB(κ), if the inequality $\omega \geq \kappa\mu$ holds in the sense of set functions. Analogously, we say that S has *curvature bounded above by* κ, shortly S is BICA(κ), if the inequality $\omega \leq \kappa\mu$ holds in the sense of set functions.

For a simple closed curve $L \subset S$ bounding a domain D homeomorphic to a disc, the *Gauss-Bonnet formula* holds:

$$\kappa_l(L) + \omega(D) = 2\pi,$$

where $\kappa_l(L)$ is the left turn of L (see [8, Chapter 13]).

The following estimate on the distortion of area of the triangle by the curvature measure holds.

Proposition 3. *Let T be a triangle in S with* $\mathrm{diam}(T) < \ell$. *Then*

$$-\frac{1}{2}\omega^-(T)\ell^2 \le \mu(T) - |T^0| \le \frac{1}{2}\omega^+(T)\ell^2. \tag{3}$$

Proof. This theorem is a specialization of [2, Theorem 3 p.262]. □

These surfaces admit a local isothermal coordinates representation.

Proposition 4. *Around every point $p \in S$, there is a chart (U, φ) such that the following hold*

(1) U is geodesically convex and its closure is homeomorphic to a disc;

(2) There is a function u, defined on $\varphi(U) \subset \mathbb{C}$, of the form

$$u(z) = -\frac{1}{\pi}\int_{\varphi(U)} \ln|z-\zeta|\, d(\varphi_\#\omega)(\zeta) + h(z)$$

where h is harmonic over $\varphi(U)$;

(3) φ is an isometry between U equipped with d and $\varphi(U)$ equipped with d_u defined by

$$d_u(x,y) = \inf_\gamma \int_\gamma e^u |dz|.$$

Proof. See [8, Chapter 7] and [2, Theorem 1 p.58]. □

2.3. **Alexandrov spaces.** A full exposition of Alexandrov spaces can be found in [3]. Here we just recall the definitions we need.

Definition 3. Let X be a complete geodesic space. The space X is said to have

- *curvature bounded below by κ*, shortly X *is* CBB(κ), if the following conditions hold: for any hinge $[p^x_y]$, the angle $\measuredangle[p^x_y]$ is defined and

  $$\measuredangle[p^x_y] \ge \tilde{\measuredangle}^\kappa(p^x_y).$$

  Moreover, for any two adjacent hinges $[p^x_y]$ and $[p^y_z]$,

  $$\measuredangle[p^x_y] + \measuredangle[p^y_z] \le \pi.$$

- *curvature locally bounded below by κ*, shortly *locally* CBB(κ), if for every point of X there exists a ball B centered at this point such that the following conditions hold: when $p \in B$, the angle $\measuredangle[p^x_y]$ is defined and for any two adjacent hinges $[p^x_y]$ and $[p^y_z]$,

  $$\measuredangle[p^x_y] + \measuredangle[p^y_z] \le \pi.$$

  Moreover, for any hinge $[p^x_y]$ contained in B,

  $$\measuredangle[p^x_y] \ge \tilde{\measuredangle}^\kappa(p^x_y).$$

We have that if X is CBB(κ') for every $\kappa' < \kappa$ then X is CBB(κ). A local to global property ensures that locally $CBB(\kappa)$ implies $CBB(\kappa)$.

We are also interested in spaces with curvature bounded above.

Definition 4. Let X a metric space. The space X is said to have

- *curvature bounded above by* κ, in short X *is* CAT(κ), if for every points $p, q, x, y \in X$, one of the two following inequalities holds:
$$\tilde{\measuredangle}^{\kappa}(p^{y}_{x}) \leq \tilde{\measuredangle}^{\kappa}(p^{q}_{x}) + \tilde{\measuredangle}^{\kappa}(p^{y}_{q}) \quad \text{or} \quad \tilde{\measuredangle}^{\kappa}(q^{y}_{x}) \leq \tilde{\measuredangle}^{\kappa}(q^{p}_{x}) + \tilde{\measuredangle}^{\kappa}(q^{y}_{p}),$$
or one of the corresponding model angle is not defined.
- *curvature locally bounded above by* κ, in short X *is locally* CAT(κ), if around every point there is a neighborhood U such that the four point comparison holds for every quadruplet in U.

A useful equivalent definition of CAT(κ) is the following one. See [3, Theorem 9.14] for a proof.

Proposition 5. *In a geodesic space X, the* CAT(κ) *condition is equivalent to the condition: for any points $p, x, y \in X$ of perimeter $< \varpi^{\kappa}$, the angle $\measuredangle[p^{x}_{y}]$ is defined and*
$$\measuredangle[p^{x}_{y}] \leq \tilde{\measuredangle}^{\kappa}(p^{x}_{y}).$$

3. Proofs

3.1. **Curvature bounded below.** The next proposition rules out cusps from surfaces with non-negative curvature. This result was already known by Reshetnyak (see [10, p.130]). We recall it for completeness.

Lemma 1. *On a* BICB(0) *surface S, $\omega(\{p\}) < 2\pi$ for every $p \in S$.*

Proof. Recall that $p \in S$ is said to be *at infinity* if for every $q \in S, d(q, p) = +\infty$. From [8, Theorem 4.110], we know that a point with $\omega(\{p\}) > 2\pi$ is at infinity which is not possible by definition. Local finiteness of ω implies that the set of cusps is discrete. Being a point at infinity is a local property. We can restrict our analysis to a closed disc S of the plane with $\operatorname{diam}(S) < 1$ for the Euclidean distance and containing only the origin as cusps.

The function h is harmonic on S, so there is a constant $C > 0$ such that,
$$\begin{aligned}
\exp(u(z)) &= \exp\left(-\frac{1}{2\pi}\int_{S} \ln|z-\zeta|\, d\omega(\zeta) + h(z)\right) \\
&= |z|^{-1} \exp\left(-\frac{1}{2\pi}\int_{S\setminus\{0\}} \ln|z-\zeta|\, d\omega(\zeta) + h(z)\right) \\
&= C|z|^{-1}.
\end{aligned}$$
Hence every curve starting at the origin will have an infinite length. This is contradictory because the distance of a BIC surface is supposed to be finite. $\square$

Theorem 1. *Suppose that S is a* BICB(κ) *surface for some $\kappa \in \mathbb{R}$. Then, for each $\kappa' < \kappa$, there exists a $\ell > 0$ such that for any triangle $T = [pqr]$*

of S homeomorphic to a disc and whose diameter is less than ℓ, the upper angle is not less than the model angle for κ', i.e.

$$\measuredangle[p_r^q] \geq \tilde{\measuredangle}^{\kappa'}(p_r^q).$$

The following proof is inspired by the proof of [1, Lemma 1 p.379].

Proof. Let ω denote the curvature measure of S. By a procedure of excision and pasting described in the proof of [2, Theorem 10 p.216], we can admit that every triangle $[pxy]$ are triangles homeomorphic to a disc and absolutely convex.

Let κ and κ' be as prescribed in the statement of the theorem. One can observe in the inequality

$$\alpha^{\mathrm{ls}} - \alpha_\kappa \geq \nu_\kappa^{\mathrm{ls}}$$

of theorem 2 that the proof is done if $\nu_{\kappa'}^{\mathrm{ls}}$ is shown to be non-negative. In other words, if $\delta_{\kappa'}[pxy] \geq 0$ for every $x \in [pq], y \in [pr]$. Having (1) and Gauss-Bonnet theorem in mind, it boils down to showing that, for every $x \in [pq], y \in [pr]$,

$$\delta_0[pxy] \geq \kappa' \left| [pxy]^{\kappa'} \right|.$$

Starting with the curvature hypothesis $\omega \geq \kappa\mu$ applied to any triangle T,

$$\delta_0(T) = \omega(T) \geq \kappa\mu(T) \tag{4}$$

we will estimate from below the right-hand term $\kappa\mu(T)$. The rest of the proof splits in two parts, one for the case κ non-negative and one for κ negative.

($\boldsymbol{\kappa \geq 0}$) Directly from the area distortion estimate (3) applied once on T and once on T^κ, we have

$$\begin{aligned} \mu(T) &\geq |T^0| - \frac{1}{2}\omega^-(T)\ell^2 \\ &\geq |T^\kappa| - \frac{1}{2}\omega^+(T^\kappa)\ell^2 \\ &= \left(1 - \frac{1}{2}\kappa\ell^2\right)|T^\kappa|. \end{aligned}$$

When we plug it back into (4), it gives us

$$\delta_0(T) \geq \kappa\left(1 - \frac{1}{2}\kappa\ell^2\right)|T^\kappa|.$$

So by choosing d such that

$$\kappa' = \kappa\left(1 - \frac{1}{2}\kappa\ell^2\right)$$

and by the fact that $|T^\kappa| \geq |T^{\kappa'}|$, we get

$$\delta_0(T) \geq \kappa'|T^{\kappa'}|$$

which is the desired inequality.

($\kappa < 0$) The reasoning mirrors that of the non-negative case. Once again by area distortion estimate (3), we have

$$\begin{aligned}\mu(T) &\leq \frac{1}{2}\omega^+(T)\ell^2 + |T^0| \\ &\leq \frac{1}{2}(\delta_0(T) + \omega^-(T))\ell^2 + \frac{1}{2}\omega^-(T^{\kappa'})\ell^2 + |T^{\kappa'}| \\ &\leq \frac{1}{2}(\delta_0(T) + |\kappa|\mu(T))\ell^2 + |T^{\kappa'}|\left(1 + \frac{1}{2}|\kappa'|\ell^2\right).\end{aligned}$$

Thus

$$\left(1 - \frac{1}{2}|\kappa|\ell^2\right)\mu(T) \leq \frac{1}{2}\delta_0(T)\ell^2 + |T^{\kappa'}|\left(1 + \frac{1}{2}|\kappa'|\ell^2\right)$$

And so, for ℓ small enough, we plug it back into (4),

$$\delta_0(T) \geq \frac{1}{1 - \frac{1}{2}|\kappa|\ell^2}\left(\frac{1}{2}\kappa\delta_0(T)\ell^2 + \kappa|T^{\kappa'}|\left(1 + \frac{1}{2}|\kappa'|\ell^2\right)\right).$$

Simplifying the last inequality,

$$\delta_0(T) \geq \kappa\left(1 + \frac{1}{2}|\kappa'|\ell^2\right)|T^{\kappa'}|.$$

By shrinking ℓ even more if necessary, we let

$$\kappa'' := \kappa\left(1 + \frac{1}{2}|\kappa'|\ell^2\right) > \kappa'$$

to obtain

$$\delta_0(T) \geq \kappa''|T^{\kappa'}| \geq \kappa'|T^{\kappa'}|$$

which is the desired inequality. This achieves the proof. □

Lemma 2. *Assume S is a* BICB(κ) *surfaces. Then S is non-branching.*

Proof. Let ω denote the curvature measure of S. Let γ_0, γ_1 be two branching geodesics starting at a point p and ending at the points x, y respectively. We define

$$t_{\max} := \sup\{t | \gamma_0(s) = \gamma_1(s) \text{ for all } s \in [0, t_{\max})\}.$$

By continuity, $\gamma_0(t_{\max}) = \gamma_1(t_{\max})$. Let's call this splitting point z. We have, by definition of the turns,

$$\kappa_l(\gamma_i) + \kappa_r(\gamma_i) = \omega(\gamma_i^\circ)$$

where γ_i° stands for the image of γ_i without the endpoints. However, by no cusp assumption or lemma 1 and [8, Theorem 4.121], the turns are negatives so

$$\kappa_l(\gamma_i) = \kappa_r(\gamma_i) = \omega(\gamma_i^\circ) = \omega(\{z\}) = 0.$$

This shows that the complete angle at z is 2π. The formula given by [8, Theorem 4.156] applied on both γ_i gives us that the angles at z between the two branches of γ_i cut at z is π. Therefore the angle of the hinge $[z_y^x]$ must be 0. The sector induced by the hinge $[z_y^x]$ is geodesically convex. Indeed, if a geodesic between two points in the sector crosses one of the γ_i then it is possible to cut it with a segment of γ_i to shorten it which is contradictory. Now we consider two points q, r respectively on each of the branches of the hinge. Two cases can happen. First, the geodesic between q

and r is fully included in $[zx]\cup[zy]$. This is impossible otherwise, again using [8, Theorem 4.156], the angle of the sector will be π which is impossible. On the other hand, if we take q, r close enough to z it is possible to get a triangle homeomorphic to a disc. But by hinge comparison we have,

$$\measuredangle[z_y^x] = 0 \Rightarrow \tilde{\measuredangle}^\kappa\left(z_{\mathrm{geod}_{[zy]}(t)}^{\mathrm{geod}_{[zx]}(t)}\right) = 0$$

for t small enough. This means that γ_0 and γ_1 coincides after z, which contradicts the hypothesis. □

Proof of theorem A. Let $\kappa' < \kappa$. We first show that S is locally CBB. For this, we pick a point $o \in S$ and consider a ball B centered in o with a radius small enough such that any triangle contained in B has diameter smaller than the constant given by the theorem 1. Let $T = [qr]$ be a triangle contained in B. By lemma 2, (S, d) is non-branching so the sides of T form a Jordan curve and then T is homeomorphic to a disc. Thus, we have the hinge comparison inequality for κ', namely

$$\measuredangle[p_r^q] \geq \tilde{\measuredangle}^{\kappa'}(p_r^q).$$

Now it remains to show the second point of locally CBB. Let $[p_y^x]$ and $[p_z^y]$ be two adjacent hinges with $p \in B$. As p is an interior point of a geodesic, $\omega(\{p\}) = 0$ and so the complete angle at p is 2π. Let $\tilde{\alpha}_1, \tilde{\alpha}_2$ be the two sector angles formed by the two convex sectors induced by $[px]$ and $[pz]$. By [2, Theorem 6 p.120], $\tilde{\alpha}_1, \tilde{\alpha}_2 \geq \pi$ and $\tilde{\alpha}_1 + \tilde{\alpha}_2 = 2\pi$. Thus $\tilde{\alpha}_1 = \tilde{\alpha}_2 = \pi$. So, by the definition of sector angles, we have

$$\measuredangle[p_y^x] + \measuredangle[p_y^z] \leq \pi.$$

As o was arbitrary, this shows that S is locally CBB(κ').

We conclude the proof by the globalization theorem [3, Theorem 8.31] so S is CBB(κ') and then by using that CBB(κ') for every $\kappa' < \kappa$ implies CBB(κ), see [3, Corollary 8.33]. □

Corollary. *The class* BICB(κ) *surfaces without cusp coincides with the class of complete* CBB(κ) *spaces without boundary and of Hausdorff dimension two.*

Proof. One direction follows directly from A and the converse follows from [5, Theorem 3.2]. □

3.2. **Curvature bounded above.**

Theorem 2. *Suppose that S is a* BICA(κ) *surface for some $\kappa \in \mathbb{R}$. Then, for each $\kappa' > \kappa$, there exists a $\ell > 0$ such that for any triangle $T = [pqr]$ of S homeomorphic to a disc and whose diameter is less than ℓ, the upper angle is not more than the model angle for κ', i.e.*

$$\measuredangle[p_r^q] \leq \tilde{\measuredangle}^{\kappa'}(p_r^q).$$

Proof. The proof follows the same lines as for Theorem 1 using opposite inequalities and Proposition 1. □

Lemma 3. *On a* BICA(κ) *surface* S *with* $\kappa \leq 0$*, there is a unique geodesic between two points.*

Proof. Let ω denote the curvature measure of S. Let $p, q \in S$ be two points joined by two different geodesics γ_1, γ_2. As these two geodesics are different, together they form a Jordan curve enclosing a domain D. Let call this enclosing curve γ. Then by Gauss-Bonnet theorem

$$\kappa_l(\gamma) + \omega(D) = 2\pi.$$

However, by [8, Theorem 4.156],

$$\kappa_l(\gamma) = \kappa_l(\gamma_1) + \kappa_l(\gamma_2) + \pi - \theta_{l,\lambda}(p) + \pi - \theta_{l,\lambda}(q)$$

Plugged into Gauss-Bonnet, we obtain

$$\kappa_l(\gamma_1) + \kappa_l(\gamma_2) + \omega(D) = \theta_{l,\lambda}(p) + \theta_{l,\lambda}(q).$$

But the left-hand side is non-positive as the right-hand side is positive. This is a contradiction. □

Proof of theorem B. Let (S, d) be a BICA(κ) surface and let $\kappa' > \kappa$. Let $o \in S$ and let P be a geodesically convex polygon containing o and having a diameter less than the constant ℓ given by theorem 2. We consider a triangle $T = [pqr]$ in P. By the preceding lemma only branching phenomena can appear. If the geodesics $[pq]$ and $[pr]$ branch in p then the hinge comparison holds trivially because

$$\measuredangle[p^q_r] = 0 \leq \tilde{\measuredangle}^{\kappa'}(p^q_r).$$

Otherwise, if the geodesics branches at the vertices q or r we can use the procedure given in the proof of [2, Theorem 10 p.216] to deform, arbitrarily small, T into a triangle homeomorphic to a disc. Then the hinge comparison holds

$$\measuredangle[p^q_r] \leq \tilde{\measuredangle}^{\kappa'}(p^q_r).$$

Thus P is a CAT(κ') space and then by using that CAT(κ') for every $\kappa' > \kappa$ implies CAT(κ), P is CAT(κ). As o was arbitrary, S is locally CAT(κ). □

We close with a few open questions that we have not been able to answer. Even if the no-cusp assumption is very frequent in the theory, this would be interesting to extend lemma 1 to know if BICB(κ) with κ negative implies the absence of cusp. Also for curvature bounded above, is it possible to extend theorem B to positive curvature?

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