

Stabilization of Time-Varying Perturbed Quantum Systems via Reduced Filters

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Abstract

In practical applications, quantum systems are inevitably subject to significant uncertainties, including unknown initial states, imprecise physical parameters, and unmodeled environmental noise, all of which pose major challenges to robust quantum feedback control. This paper proposes a feedback stabilization strategy based on a reduced quantum filter that achieves robustness against time-varying Hamiltonian perturbations and additional dissipative effects, without requiring prior knowledge of the initial state or exact system parameters. The proposed filter estimates only $\mathcal{O}(N)$ real variables corresponding to the diagonal elements of the system density matrix in a quantum non-demolition basis in contrast to the $\mathcal{O}(N^2)$ variables required by a full stochastic master equation, where N is the Hilbert space dimension. This dimensionality reduction substantially simplifies real-time computation and feedback implementation while preserving both convergence and robustness guarantees. Rigorous analysis further establishes global exponential stability of the target subspace. The results provide a scalable framework for robust and efficient measurement-based feedback control applicable to high-dimensional perturbed open quantum systems.

1 INTRODUCTION

The theory of open quantum systems [11], describing systems interacting with an external environment, has profoundly impacted quantum information science [31]. Such interactions inevitably induce quantum dissipation and decoherence, leading to information loss, one of the most critical challenges in quantum control and quantum computation. Developing feedback control strategies to mitigate decoherence is crucial for the scalability and reliability of quantum technologies, including quantum computing, quantum chemistry, and quantum information processing [13, 1].

A widely adopted framework for analyzing quantum control in the presence of continuous measurement is based on Stochastic Master Equations (SMEs) [6, 10, 3]. In this framework, an open quantum system interacts with its environment and a probe system, where the probe system is continuously monitored. SME-based feedback control has been a key enabler for various quantum technologies [46], leading to significant advances in various domains [46]: *Quantum state protection*, extending coherence lifetimes [45] and protecting macroscopic Schrödinger cat

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states [32]; *Quantum error suppression and correction*, mitigating decoherence and enabling real-time error suppression [27]; *Quantum simulation*, stabilizing Bose-Einstein condensates to improve quantum simulations of condensed matter systems [38]; *Quantum transport in nano-structures*: offering new opportunities in quantum device engineering; *Quantum sensing and metrology*, increasing sensor sensitivity [14]; *Entanglement distribution*, enabling robust remote entanglement generation and stabilization, critical for quantum communication and distributed computing [35]. Recent research has also explored its role in connecting gravity and quantum matter in the so-called Newtonian limit [33].

Despite these successes, two major challenges remain for the practical implementation of SME-based feedback control:

- *Computational complexity of quantum filtering*: Traditional full-state quantum filters track $\mathcal{O}(N^2)$ parameters for an N -level quantum system, making real-time feedback control infeasible for large quantum systems [43]. Several model reduction methods have been proposed to address this challenge [44, 30, 15, 16, 2, 17], but their applicability remains restricted.
- *Robustness to uncertainties*: Unknown initial states, detector inefficiencies, model inaccuracies, and time-varying parameters (e.g., coupling strengths, measurement efficiencies) reduce the effectiveness of a control strategy. Existing robust methods often assume structured uncertainties or full-state estimation [26, 22, 23, 18], limiting practical applicability. These issues are also closely related to non-Markovian effects [11, 5, 4].

Earlier stabilization approaches have largely relied on engineering Lindbladian dynamics [42, 29, 39, 9], focusing on deterministic dissipator design while paying less attention to measurement back-action, i.e., the diffusion terms in the SME. Although valuable structural conditions were derived, such methods often assume precise knowledge of the system state and parameters. In contrast, our earlier work [20, 23, 22, 26] has shown that directly leveraging measurement back-action makes it possible to design feedback laws that guarantee exponential stabilization, while remaining robust against both unknown initial states and parameter uncertainties within admissible ranges. This highlights a clear advantage of measurement-based feedback strategies over purely Lindbladian designs. Specifically, in [21], we first introduced a reduced-filter-based approach for stabilizing perturbed systems with time-invariant coefficients.

This work extends our prior results by developing a rigorous framework for robust feedback stabilization of time-varying perturbed quantum systems using a reduced-order filter. This approach addresses both the computational challenges and robustness issues that arise when dealing with large quantum systems subject to time-varying perturbations. Our key idea is to construct a reduced filter that tracks only $\mathcal{O}(N)$ real parameters, corresponding to the diagonal entries in a Quantum Non-Demolition (QND) basis, rather than the full $\mathcal{O}(N^2)$ -dimensional state. This substantially reduces computational costs and enables scalable real-time feedback control for large quantum systems. Our main contributions are summarized as follows:

1. **Reduced filter construction**: We rigorously establish the existence and well-posedness of the reduced filter and derive sufficient conditions for its systematic construction in an intuitive manner (Theorem 4.1).
2. **Robust exponential stabilization**: We prove that the reduced-filter-based feedback globally exponentially stabilizes the quantum system to the desired target subspace,

under time-varying model perturbations. Unlike prior methods, our approach eliminates the need for full-state estimation or precise system parameters. Additionally, we relax constraints on the feedback controller compared to our previous work [23, 26] (Theorem 4.10).

Beyond rigorous proofs, we provide intuitive insights into the reduced filter's construction and the rationale behind the proposed conditions, ensuring robust stabilization.

This paper is organized as follows: In Section 2, we introduce the stochastic dynamical model of open quantum systems under continuous-time measurements and present the control problem. Section 3 discusses the large-time behavior of uncontrolled systems, focusing on exponential quantum state reduction. In Section 4, we present the reduced-filter-based feedback design and explore its robustness for stabilizing the open quantum system. Section 5 provides a numerical example of a three-level system. Finally, Section 6 concludes the paper and discusses future directions for research.

Notation. The imaginary unit is denoted by i . For $X \in \mathcal{B}(\mathcal{H})$, the adjoint is X^* , where $\mathcal{B}(\mathcal{H})$ denotes the space of all linear operators on $\mathcal{H}$, and $\mathbf{I}$ denotes the identity operator. The set of Hermitian operators is $\mathcal{B}_*(\mathcal{H}) := \{X \in \mathcal{B}(\mathcal{H}) \mid X = X^*\}$. The trace of $X \in \mathcal{B}(\mathcal{H})$ is $\text{Tr}(X)$, and the Hilbert–Schmidt norm is $\|X\| := \text{Tr}(XX^*)^{1/2}$. For $v \in \mathcal{H}$, $\|v\|$ denotes the standard vector norm. Define $L^\infty([0, \infty), \mathcal{B}(\mathcal{H})) := \{X : [0, \infty) \rightarrow \mathcal{B}(\mathcal{H}) \mid \text{each entry is measurable and } \sup_{t \geq 0} \|X(t)\| < \infty\}$. The commutator of $X, Y \in \mathcal{B}(\mathcal{H})$ is $[X, Y] := XY - YX$. For $x \in \mathbb{C}$, $\Re\{x\}$ denotes the real part. The Hölder space of functions from A to B is denoted $\mathcal{C}^{1,\alpha}(A, B)$ for $\alpha \in (0, 1]$. For a positive integer m , we write $[m] := \{1, \dots, m\}$. If $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$ and $X \in \mathcal{B}(\mathcal{H})$, then in a basis adapted to this decomposition X has block form

$$X = \begin{bmatrix} X_S & X_P \\ X_Q & X_R \end{bmatrix},$$

where X_S, X_R, X_P and X_Q are matrices representing operators from $\mathcal{H}_S$ to $\mathcal{H}_S$, from $\mathcal{H}_R$ to $\mathcal{H}_R$, from $\mathcal{H}_R$ to $\mathcal{H}_S$, from $\mathcal{H}_S$ to $\mathcal{H}_R$, respectively.

2 Stochastic dynamical model

We consider an open quantum system undergoing m continuous-time homodyne or heterodyne measurements on a N -dimensional Hilbert space $\mathcal{H}$. The state of the system is associated to a density matrix on $\mathcal{H}$,

$$\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{B}(\mathcal{H}) \mid \rho = \rho^* \geq 0, \text{Tr}(\rho) = 1\}.$$

Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ supporting an m -dimensional Wiener process $W(t)$. On this space, the conditional evolution of the state given the measurement outcomes is described by the Stochastic Master Equation (SME):

$$d\rho(t) = \mathcal{L}_\gamma^u(t, \rho(t))dt + \mathcal{P}(t, \rho(t))dt + \sum_{k=1}^m \sqrt{\theta_k(t)} \mathcal{G}_{L_k}(\rho(t))dW_k(t), \quad (1)$$

$$dY_k(t) = \sqrt{\theta_k(t)} \text{Tr}((L_k^* + L_k)\rho(t))dt + dW_k(t), \quad (2)$$

with $\rho(0) \in \mathcal{S}(\mathcal{H})$. Here,

- $\theta_k(t) = \eta_k(t)\gamma_k(t) \in [\underline{\theta}_k, \bar{\theta}_k]$ is a deterministic measurable function combining measurement efficiency $\eta_k(t) \in [\underline{\eta}_k, \bar{\eta}_k] \subset (0, 1]$ and coupling strength $\gamma_k(t) \in [\underline{\gamma}_k, \bar{\gamma}_k]$ with $\underline{\gamma}_k > 0$, associated with the k -th probe, where $\underline{\theta}_k = \underline{\eta}_k \underline{\gamma}_k$ and $\bar{\theta}_k = \bar{\eta}_k \bar{\gamma}_k$.
- Lindblad generator is

$$\mathcal{L}_\gamma^u(t, \rho) := -i[H_0(t) + u_t H_1, \rho] + \sum_{k=1}^m \gamma_k(t) \mathcal{D}_{L_k}(\rho),$$

where $\mathcal{D}_{L_k}(\rho) := L_k \rho L_k^* - \frac{1}{2} L_k^* L_k \rho - \frac{1}{2} \rho L_k^* L_k$. Here $H_0(t), H_1 \in \mathcal{B}_*(\mathcal{H})$ represent the free and control Hamiltonians, and the measurement operator $\sqrt{\gamma_k(t)} L_k \in \mathcal{B}(\mathcal{H})$ describes interaction with the k -th probe.

- The measurement back-action associated with k -th probe is described by

$$\mathcal{G}_{L_k}(\rho) := L_k \rho + \rho L_k^* - \text{Tr}((L_k + L_k^*)\rho)\rho.$$

- Perturbations are modeled by

$$\mathcal{P}(t, \rho) := -i[\tilde{H}_0(t), \rho] + \sum_{k=1}^{\bar{m}} \mathcal{D}_{C_k(t)}(\rho).$$

The terms $H_0(t), \tilde{H}_0(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}))$ and $C_k(t) \in L^\infty([0, \infty), \mathcal{B}(\mathcal{H}))$ are deterministic.

The continuous semi-martingale $Y_k(t)$ represents measurement record of k -th probe with quadratic variation $\langle Y_k(t), Y_k(t) \rangle = t$. Its natural filtration $\mathcal{F}_t^Y := \sigma\{Y(s), 0 \leq s \leq t\}$, i.e., the smallest σ -algebra containing all measurement outcomes up to time t , coincides with $\sigma\{W(s), 0 \leq s \leq t\}$ by [41, Proposition 5.2.14]. The control input u_t is a bounded real-valued $\mathcal{F}_t^Y$ -adapted process. Existence, uniqueness, and invariance of solutions to (1) within $\mathcal{S}(\mathcal{H})$ follow arguments similar to those in [29, Section 3] and [3, Chapter 5].

We now consider a decomposition: $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_d$, with orthogonal projections $\Pi_0, \dots, \Pi_d$ onto each subspace. Impose the following assumption on the system which is called Quantum Non-Demolition (QND) measurement :

$$\begin{aligned} \mathbf{A1}: \quad & H_0(t) = \text{diag}[H_0(t), \dots, H_d(t)] \text{ with } H_j(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}_j)); \\ & L_k = \sum_{j=0}^d l_{k,j} \Pi_j \text{ with } l_{k,j} \in \mathbb{C}. \end{aligned}$$

System operators $H_0(t)$ and L_k are simultaneously block-diagonal with respect to the above decomposition.

For $j \in \{0, \dots, d\}$, define

$$\begin{aligned} \mathbf{d}_j(\rho) &:= \|\rho - \Pi_j \rho \Pi_j\|, \\ \mathbf{B}_r(\mathcal{H}_j) &:= \{\rho \in \mathcal{S}(\mathcal{H}) \mid \mathbf{d}_j(\rho) < r\}, \\ \mathcal{I}(\mathcal{H}_j) &:= \{\rho \in \mathcal{S}(\mathcal{H}) \mid \text{Tr}(\Pi_j \rho) = 1\}. \end{aligned}$$

Definition 2.1. The subspace $\mathcal{H}_j$ is called invariant almost surely if, for all $\rho(0) \in \mathcal{I}(\mathcal{H}_j)$, $\rho(t) \in \mathcal{I}(\mathcal{H}_j)$ for all $t > 0$ almost surely.

Definition 2.2. An invariant subspace $\mathcal{H}_j$ is almost surely Global Exponential Stable (GES) if $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbf{d}_j(\rho(t)) < 0$ almost surely, for all $\rho(0) \in \mathcal{S}(\mathcal{H})$. The left-hand side is the *sample Lyapunov exponent*.

Problem setting: We study feedback stabilization of (1) toward the target subspace $\mathcal{H}_0$ using a reduced filter instead of full state estimation. For consistency with earlier notation, we identify $\mathcal{H}_S$ with $\mathcal{H}_0$.

Throughout, we focus on perturbations that preserve the invariance of the target subspace $\mathcal{H}_0$. Intuitively, it means that the perturbations do not induce transitions that drive the state of system out of $\mathcal{H}_0$, and then stabilization towards this subspace remains feasible. Following [40], we formalize this assumption as:

A2: For all $t \geq 0$, $\tilde{H}_{0,P}(t) = 0$, and $\forall k \in [m]$, $C_{k,Q}(t) = 0$ and $\sum_{k=1}^m C_{k,S}^*(t)C_{k,P}(t) = 0$.

Here, $\tilde{H}_{0,P}(t)$ denotes the coupling between the subspace $\mathcal{H}_0$ and its orthogonal complement $\bigoplus_{i=1}^d \mathcal{H}_i$ in the perturbed Hamiltonian, while $C_{k,Q}(t)$ and $C_{k,P}(t)$ represent, respectively, the off-diagonal blocks of the noise operators that connect $\mathcal{H}_0$ with the external subspaces. Assumption **A2** therefore guarantees that neither the Hamiltonian nor the dissipative channels induce leakage out of $\mathcal{H}_0$, ensuring that the subspace is invariant under the perturbed dynamics.

3 Stochastic Evolution of States in QND basis

We first analyze the uncontrolled dynamics ($u_t \equiv 0$) to identify the state components that govern large-time behavior. In the QND setting, the diagonal weights $\text{Tr}(\rho(t)\Pi_i)$ carry the relevant asymptotic information; see also [8].

Under **A2**, invariant sets of the perturbed SME (1) are determined by the block structure of $\tilde{H}_0(t)$ and $\{C_k(t)\}_{k \in [m]}$:

Case 1. There exists a non-empty subset $E \subseteq \{0, \dots, d\}$ such that, for all $j \in E$, $\mathcal{I}(\mathcal{H}_j)$ is invariant.

Case 2. For generic $\tilde{H}_0(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}))$ and $C_k(t) \in L^\infty([0, \infty), \mathcal{B}(\mathcal{H}))$, the only invariant sets are those in **Case 1**. Extra invariant subsets arise only on a measure-zero set of nongeneric parameters (additional algebraic constraints on block entries).

If the stronger structural assumption below holds, then **Case 2** occurs with $E = \{0, \dots, d\}$.

$$\begin{aligned} \textbf{A-qsr:} \quad & \tilde{H}_0(t) = \text{diag}[\tilde{H}_0(t), \dots, \tilde{H}_d(t)] \text{ with } \tilde{H}_j(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}_j)); \\ & C_k(t) = \text{diag}[C_{k,0}(t), \dots, C_{k,d}(t)] \text{ with } C_{k,j}(t) \in L^\infty([0, \infty), \mathcal{B}(\mathcal{H}_j)). \end{aligned}$$

We impose the following identifiability assumption on the measurement operators:

A3: For all $i \neq j$, there exists at least one $k \in [m]$ such that $\Re\{l_{k,i}\} \neq \Re\{l_{k,j}\}$.

Define

$$\mathfrak{E}_l := \min_{i \neq j} \left[\sum_{k=1}^m \underline{\theta}_k (\Re\{l_{k,i}\} - \Re\{l_{k,j}\})^2 \right].$$

Assumption **A3** ensures that different eigenstates of the measurement operators can be distinguished through their measurement records. The following theorem establishes that, in the absence of control input, the stochastic dynamics induced by continuous measurements drive the quantum state exponentially toward one of the invariant subspaces.

Theorem 3.1 (Exponential quantum state reduction). *Assume that $u_t \equiv 0$, and **A1**, **A3** and **A-qsr** hold. For any $\rho(0) \in \mathcal{S}(\mathcal{H})$, the system (1) converges towards $\mathcal{I}(\mathcal{H}) := \bigcup_{j=0}^d \mathcal{I}(\mathcal{H}_j)$ in mean and almost surely with Lyapunov exponent less than or equal to $-\mathfrak{E}_l/2$. Moreover, the probability of convergence to $\mathcal{I}(\mathcal{H}_j)$ is $\text{Tr}(\Pi_j \rho(0))$ for $j \in \{0, \dots, d\}$.*

Proof. Define $I := \{k \mid \text{Tr}(\Pi_j \rho_0) = 0\}$ and $\mathcal{S}_I := \{\rho \in \mathcal{S}(\mathcal{H}) \mid \text{Tr}(\Pi_j \rho) = 0 \text{ iff } j \in I\}$. It is straightforward to verify that $\mathcal{S}_I$ is almost surely invariant under the dynamics of system (1). Consider the following candidate Lyapunov function, related to the Bhattacharyya distance, also known as classical fidelity [7, Chapter 2.5],

$$V(\rho) = \sum_{i \neq j} \sqrt{\text{Tr}(\Pi_i \rho) \text{Tr}(\Pi_j \rho)} \geq 0, \quad (3)$$

with $V(\rho) = 0$ if and only if $\rho \in \mathcal{I}(\mathcal{H})$. Due to the invariance of $\mathcal{S}_I$, V is twice continuously differentiable when restricted to $\mathcal{S}_I$. For all $\rho \in \mathcal{S}_I$, we have

$$\begin{aligned} \mathcal{L}V(\rho) &= -\frac{1}{2} \sum_{i \neq j} \sqrt{\text{Tr}(\rho \Pi_i) \text{Tr}(\rho \Pi_j)} \left[\sum_k \theta_k(t) (\Re\{l_{k,i}\} - \Re\{l_{k,j}\})^2 \right] \\ &\leq -\mathfrak{E}_l V(\rho)/2, \end{aligned} \quad (4)$$

where the strict positivity of $\mathfrak{E}_l$ follows from **A3**. Detailed derivations are provided in Appendix B.

Applying similar stochastic Lyapunov arguments as in [20, Theorem 5] and [22, Theorem 2.5], we conclude that $\rho(t)$ converges exponentially to $\mathcal{I}(\mathcal{H})$, both in expectation and almost surely, with Lyapunov exponent at most $-\mathfrak{E}_l/2$. Moreover, the probability of convergence to $\mathcal{I}(\mathcal{H}_j)$ is $\text{Tr}(\Pi_j \rho(0))$. $\square$

In the QND regime, the measurement operators L_k commute with the system Hamiltonian $H_0(t)$, and also perturbations $\tilde{H}_0(t)$ and $C_k(t)$ under **A-qsr**. Hence, the off-diagonal elements of $\rho(t)$ in the QND basis vanish asymptotically (Theorem 3.1). Thus, the diagonal vector $(\text{Tr}(\rho(t)\Pi_0), \dots, \text{Tr}(\rho(t)\Pi_d))$ contains the statistical information for control design. Compared to utilizing full state information, i.e., estimating the actual state via the estimator $\hat{\rho}(t) \in \mathcal{S}(\mathcal{H})$ [26], relying solely on the diagonal entries dramatically reduces complexity: the number of estimated parameters decreases from $N^2 - 1$ with $N = \dim\{\mathcal{H}\}$ to $d + 1$ degrees of freedom subject to the normalization constraint $\sum_{j=0}^d \text{Tr}(\rho(t)\Pi_j) = 1$, i.e., only d independent parameters, where $d + 1 \leq N$.

4 Stabilization by the reduced filter

In this section, we explore the robustness of the state feedback stabilization strategy proposed in [22] for system (1), subject to perturbations $\mathcal{P}(\rho)$ and uncertainties in the initial state $\rho(0)$, the free Hamiltonian $H_0(t)$, and the model parameters $\gamma_k(t), \eta_k(t)$. Throughout, we assume that the measurement operators L_k are known, while the parameters $\gamma_k(t) > 0$ may not be precisely identified. This assumption can be interpreted as allowing proportional time-varying perturbations in L_k . Importantly, this technical assumption plays a crucial role in deriving the main result, as it emphasizes that accurate knowledge of the measurement operators is essential for ensuring robust stability.

Due to relation (2), we rewrite system (1) as follows:

$$\begin{aligned} d\rho(t) = & \mathcal{L}_\gamma^u(t, \rho(t))dt + \mathcal{P}(t, \rho(t))dt + \sum_{k=1}^m \sqrt{\theta_k(t)} \mathcal{G}_{L_k}(\rho(t)) \\ & \times \left(dY_k(t) - \sqrt{\theta_k(t)} \text{Tr}((L_k^* + L_k)\rho(t))dt \right), \end{aligned} \quad (5)$$

with $\rho(0) \in \mathcal{S}(\mathcal{H})$. From a practical perspective, the initial state, the free Hamiltonian, the model parameters and the perturbation cannot be precisely known. Following the approach in [26], we construct the following SME to estimate the state $\rho(t)$ replicating the structure of the actual system (1),

$$d\hat{\rho}(t) = \mathcal{L}_{\hat{\gamma}}^u(t, \hat{\rho}(t))dt + \sum_{k=1}^m \sqrt{\hat{\theta}_k} \mathcal{G}_{L_k}(\hat{\rho}(t)) \left(dY_k(t) - \sqrt{\hat{\theta}_k} \text{Tr}((L_k^* + L_k)\hat{\rho}(t))dt \right), \quad (6)$$

where $\hat{\rho}(0) \in \mathcal{S}(\mathcal{H})$, $\hat{H}_0 = \text{diag}[\hat{H}_0, \dots, \hat{H}_d]$, and $\hat{\theta}_k = \hat{\eta}_k \hat{\gamma}_k$ with $\hat{\gamma}_k \in [\underline{\gamma}_k, \bar{\gamma}_k]$, $\hat{\eta}_k \in [\underline{\eta}_k, \bar{\eta}_k]$. Unlike the true system, which may involve time-varying $H_0(t)$ and $\theta_k(t)$, the estimator uses fixed values selected from prior information, reducing online computational complexity. The feedback law is then implemented as $u_t = u(\hat{\rho}(t))$, requiring real-time numerical integration of (6).

4.1 Motivation for a reduced filter

As the system dimension increases, integrating the full filter (6) in real time becomes computationally demanding. This motivates the design of controllers that exploit only partial information rather than full state estimation.

Before introducing our reduced filter for feedback control, we recall two key mechanisms established in [25, 22, 26] to interpret the full filter:

1. The term $u_t[H_1, \hat{\rho}]$ ensures that $\mathcal{I}(\mathcal{H}_0)$ is the only invariant subset for the estimator. Moreover, under **A2**, if $\hat{\rho}(t)$ approaches an undesired invariant subset such as $\mathcal{I}(\mathcal{H}_j)$ with $j \in E$, this term forces $\hat{\rho}(t)$ away from such undesired invariant subsets.
2. The diffusive term $\mathcal{G}_{L_k}(\hat{\rho})$ accounts for the measurement back-action on the system. It ensures that $\hat{\rho}(t)$, starting from any initial state outside the invariant subsets, is driven toward the target invariant subset.

In view of the control objective: stabilizing system (1) toward $\mathcal{I}(\mathcal{H}_0)$ equivalently $\text{Tr}(\rho(t)\Pi_0) \rightarrow 1$, and the large-time behavior in Theorem 3.1, which shows that the off-diagonal blocks $\Pi_i \rho(t) \Pi_j$ vanish for $i \neq j$. It is natural to focus only on the diagonal entries of $\rho(t)$ in the QND basis. We therefore construct an $\mathbb{R}^{d+1}$ -dimensional estimator that tracks these diagonal components. By capturing precisely the information relevant for stabilization, this reduced estimator provides an effective input to the feedback controller while avoiding the complexity of full state estimation via $\hat{\rho}(t)$.

Since the measurement operators $(L_k)_{k \in [m]}$ are diagonal in the QND basis, their contribution to the measurement back-action can be simplified to:

$$\text{Tr}(\mathcal{G}_{L_k}(\rho)\Pi_n) = 2\text{Tr}(\rho\Pi_n) \left(\Re\{l_{k,n}\} - \sum_{j=0}^d \Re\{l_{k,j}\} \text{Tr}(\rho\Pi_j) \right). \quad (7)$$

Motivated by this structure, and by the analysis in Section 3, we introduce a reduced-order estimator whose state $\hat{q}(t) \in \mathbb{R}^{d+1}$ tracks the diagonal components $\text{Tr}(\rho(t)\Pi_n)$ based on the measurement output $\mathcal{F}_t^Y$. Inspired by [12, 22], we propose the following autonomous Stochastic Differential Equation (SDE):

$$d\hat{q}_n(t) = u_t \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j(t) dt + 2\hat{q}_n(t) \sum_{k=1}^m \sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}(t)) \left(dY_k(t) - 2\sqrt{\hat{\theta}_k} \Lambda_k(\hat{q}(t)) dt \right), \quad (8)$$

where

$$\Phi_n^k(\hat{q}) := \Re\{l_{k,n}\} - \Lambda_k(\hat{q}), \text{ with } \Lambda_k(\hat{q}) := \sum_{j=0}^d \Re\{l_{k,j}\} \hat{q}_j,$$

and $\hat{q}_n = e_n^* \hat{q}$, and $\{e_0, \dots, e_d\}$ forms an orthonormal basis of $\mathbb{R}^{d+1}$. $u_t := u(\hat{q}(t))$ represents the feedback controller.

Moreover, we assume that the matrix $\Gamma \in \mathbb{R}^{(d+1) \times (d+1)}$ satisfies the following condition:

C1: $\sum_{i=0}^d \Gamma_{i,j} = 0$ for all $j \in \{0, \dots, d\}$, $\Gamma_{j,j} < 0$ for all $j \in \{0, \dots, d\}$, and $\Gamma_{i,j} \geq 0$ for all $i \neq j \in \{0, \dots, d\}$.

We observe that the second and third terms on the right-hand side of (8) effectively reproduce the measurement back-action, as they structurally resemble the second summand in (5) when considering the diagonal elements in (7). The first term, $u_t \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j$, under Condition **C1**, serves to replicate the effect of $u_t[H_1, \hat{\rho}]$. Regarding **C1**, the constraint $\sum_{i=0}^d \Gamma_{i,j} = 0$ ensures that $\sum_n \hat{q}_n = 1$, while the remaining conditions guarantee the positivity of $\hat{q}_n \geq 0$, ensuring

$$\hat{q} \in \bar{\mathcal{O}}_{d+1} := \left\{ \hat{q} \in [0, 1]^{d+1} \mid \sum_{n=0}^d \hat{q}_n = 1 \right\}.$$

Thus, $\hat{q}$ is a valid candidate probability vector approximating the diagonal of $\hat{\rho}$ in the QND basis.

4.2 Stochastic model for the reduced filter

Based on (2), we rewrite (8) as the following Wiener process-driven SDE for $n \in \{0, \dots, d\}$,

$$d\hat{q}_n(t) = u_t \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j(t) dt + 2\hat{q}_n(t) \sum_{k=1}^m \sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}(t)) (dW_k(t) + \mathfrak{T}_k(t, \rho(t), \hat{q}(t)) dt), \quad (9)$$

where

$$\mathfrak{T}_k(t, \rho, \hat{q}) := \sqrt{\hat{\theta}_k(t)} \text{Tr}((L_k + L_k^*)\rho) - 2\sqrt{\hat{\theta}_k} \Lambda_k(\hat{q}),$$

and $\rho(t)$ is the solution to SME (1) with $u_t := u(\hat{q}(t))$ as the feedback controller, which satisfies the following assumption:

A4: $u \in \mathcal{C}^{1,\alpha}(\bar{\mathcal{O}}_{d+1}, \mathbb{R}_+)$ with $\alpha \in (0, 1]$, $u(e_0) = 0$ and $u(e_n) > 0$ for all $n > 0$, where $\{e_n\}_{n=0}^d$ is the orthonormal basis of $\mathbb{R}^{d+1}$.

Define

$$\mathcal{O}_{d+1} := \{\hat{q} \in (0, 1)^{d+1} \mid \sum_{n=0}^d \hat{q}_n = 1\}.$$

Before analyzing the stability properties of the system, it is essential to ensure that the coupled system (1)–(9) is well-posed and that the reduced state remains within the probability simplex $\mathcal{O}_{d+1}$ for all time. The next theorem provides these well-posedness and invariance properties.

Theorem 4.1. *Suppose **A4** and **C1** hold. Then, for all $(\rho(0), \hat{q}(0)) \in \mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$, the coupled system (1)–(9) has a unique global solution $(\rho(t), \hat{q}(t)) \in \mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$ for all $t \geq 0$ almost surely.*

Proof. First, suppose that u_t is a real, bounded positive process adapted to $\mathcal{F}_t^Y$. Existence, uniqueness, and invariance of the solution $\rho(t)$ in $\mathcal{S}(\mathcal{H})$ for (1) follow directly from [29, Section 3] or [3, Chapter 5].

Denote the explosion time from the initial state $\hat{q}(0)$ by $\tau_e(\hat{q}(0), \omega) : \mathbb{R}^{d+1} \times \Omega \rightarrow [0, \infty]$. Since the coefficients of (9) are locally Lipschitz, by [28, Theorem 5.2.8], the SDE (9) admits a unique local strong solution on $[0, \tau_e)$ almost surely.

Summing (9) over $n \in \{0, \dots, d\}$ yields

$$\begin{aligned} d \sum_{n=0}^d \hat{q}_n(t) = & u_t \sum_{n=0}^d \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j(t) dt \\ & + 2 \sum_{k=1}^m \sqrt{\hat{\theta}_k} \Lambda_k(\hat{q}(t)) \left(1 - \sum_{n=0}^d \hat{q}_n(t) \right) (dW_k(t) + \mathfrak{T}_k(t, \rho(t), \hat{q}(t)) dt). \end{aligned}$$

Since the coefficients of the above equation are locally Lipschitz continuous, for any given initial state $\hat{q}(0) \in \mathcal{O}_{d+1}$, there is a unique local solution on $t \in [0, \tau_e)$. According to the condition **C1**, $\sum_{n=0}^d \Gamma_{n,j} = 0$. It follows $\sum_{n=0}^d \hat{q}_n(t) = 1$ when $\hat{q}(0) \in \mathcal{O}_{d+1}$ for all $t \in [0, \tau_e)$ almost surely. It implies that, for all $\hat{q}(0) \in \mathcal{O}_{d+1}$, $\sum_{n=0}^d \hat{q}_n(t) = 1$ till the explosion time.

For all $\hat{q}(0) \in \mathcal{O}_{d+1}$, choose $k_0 \geq 0$ sufficiently large so that $\hat{q}_n(0) > 1/k_0$ for all $n \in \{0, \dots, d\}$. For each integer $k \geq k_0$, define two stopping times

$$\begin{aligned} \tau_k &:= \inf\{t \in [0, \tau_e) \mid \hat{q}_n(t) \leq 1/k \text{ for some } n\}, \\ \sigma_k &:= \inf\{t \in [0, \tau_e) \mid \hat{q}_n(t) \notin (1/k, k) \text{ for some } n\}. \end{aligned}$$

Clearly, $\sigma_k \leq \tau_k$ and $\sigma_k \rightarrow \tau_e$ as $k \rightarrow \infty$. Because $\sum_{n=0}^d \hat{q}_n(t) = 1$ holds almost surely for all $t < \tau_e$, it follows that $\tau_k = \sigma_k$ almost surely.

Consider the function

$$V(\hat{q}) = \sum_{n=0}^d \log \hat{q}_n^{-1} \in \mathcal{C}^2(\mathcal{O}_{d+1}, \mathbb{R}_+),$$

whose infinitesimal generator is given by

$$\mathcal{L}V(\hat{\rho}) = -u_t \mathcal{A}(\hat{q}) + \mathcal{B}(t, \rho, \hat{q}),$$

where

$$\begin{aligned} \mathcal{A}(\hat{q}) &:= \sum_{n=0}^d \frac{1}{\hat{q}_n} \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j, \\ \mathcal{B}(t, \rho, \hat{q}) &:= 2 \sum_{n,k} \sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}) \left[\sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}) - \mathfrak{T}_k(t, \rho, \hat{q}) \right]. \end{aligned}$$

Due to compactness of $\mathcal{S}(\mathcal{H}) \times \bar{\mathcal{O}}_{d+1}$, there exists a finite $c_1 > 0$ such that $\mathcal{B}(t, \rho, \hat{q}) \leq c_1$ for all $(t, \rho, \hat{q}) \in \mathbb{R}_+ \times \mathcal{S}(\mathcal{H}) \times \bar{\mathcal{O}}_{d+1}$. Define

$$\bar{\mathcal{O}}_{d+1}^k := \{\hat{q} \in \mathbb{R}^{d+1} \mid \hat{q}_n \geq 1/k \text{ for } n \in \{0, \dots, d\}, \text{ s.t. } \sum_{n=0}^d \hat{q}_n = 1\} \subset \mathcal{O}_{d+1}.$$

For all $\hat{q} \in \bar{\mathcal{O}}_{d+1}^k$, due to the compactness and continuity, there exists $c_2 > 0$ such that $-\mathcal{A}(\hat{q}) \leq c_2 V(\hat{q})$ and $c_1 \leq c_2 V(\hat{q})$. Since u_t is positive and bounded, it follows that

$$\mathcal{L}V(\hat{q}) \leq c_3 V(\hat{q}), \quad \forall \hat{q} \in \bar{\mathcal{O}}_{d+1}^k,$$

for some constant $c_3 > 0$.

For $\hat{q} \in \mathcal{O}_{d+1} \setminus \bar{\mathcal{O}}_{d+1}^k$, at least one $\hat{q}_i$ approaches zero. We rewrite

$$\mathcal{A}(\hat{q}) = \sum_{n=0}^d \Gamma_{n,n} + \sum_{n=0}^d \frac{1}{\hat{q}_n} \sum_{n \neq j} \Gamma_{n,j} \hat{q}_j,$$

with $\Gamma_{n,n} < 0$ and $\Gamma_{n,j} \geq 0$ for $n \neq j$.

Suppose $\hat{q}_n$ approaches zero. There are three possible cases:

1. If $\sum_{n \neq j} \Gamma_{n,j} \hat{q}_j$ converges to a finite positive constant, then $\frac{1}{\hat{q}_n} \sum_{n \neq j} \Gamma_{n,j} \hat{q}_j$ diverges to infinity.
2. If $\sum_{n \neq j} \Gamma_{n,j} \hat{q}_j$ converges to zero slower than $\hat{q}_n$, then $\frac{1}{\hat{q}_n} \sum_{n \neq j} \Gamma_{n,j} \hat{q}_j$ diverges to infinity.
3. If $\sum_{n \neq j} \Gamma_{n,j} \hat{q}_j$ converges to zero at the same or faster rate than $\hat{q}_n$, then $\frac{1}{\hat{q}_n} \sum_{n \neq j} \Gamma_{n,j} \hat{q}_j$ converges to a finite constant.

Hence, $-\mathcal{A}(\hat{q})$ either diverges negatively or remains bounded by a finite positive constant. Note that, in these cases, $V(\hat{\rho})$ approaches infinity. Thus, there exists a finite constant $c_4 > 0$ such that $\mathcal{L}V(\hat{q}) \leq c_3 V(\hat{q})$ for all $\hat{q} \in \mathcal{O}_{d+1} \setminus \bar{\mathcal{O}}_{d+1}^k$. Therefore, there exists $c > 0$ such that

$$\mathcal{L}V(\hat{q}) \leq cV(\hat{q}), \quad \forall \hat{q} \in \mathcal{O}_{d+1}.$$

Next, we show $\tau_e = \infty$ almost surely by contradiction inspired by analogous result established in [28, Lemma 4.3.2]. Assume $\mathbb{P}(\tau_e < \infty) > 0$, and then there is $T > 0$ sufficient large such that $\mathbb{P}(\tau_e \leq T) > 0$. Given that $\mathcal{L}V(\hat{q}) \leq cV(\hat{q})$ for all $\hat{q} \in \mathcal{O}_{d+1}$, define $f(\hat{q}, t) = e^{-ct} V(\hat{q})$, whose infinitesimal generator is given by

$$\mathcal{L}f(\hat{q}, t) = e^{-ct} (-cV(\hat{q}) + \mathcal{L}V(\hat{q})) \leq 0,$$

for all $\hat{q} \in \mathcal{O}_{d+1}$. Since $\hat{q}(0) \in \bar{\mathcal{O}}_{d+1}^{k_0} \subset \bar{\mathcal{O}}_{d+1}^k$, we have $\hat{q}(T \wedge \tau_k) \in \bar{\mathcal{O}}_{d+1}^k \subset \mathcal{O}_{d+1}$. By Itô formula, we obtain

$$\mathbb{E}(f(\hat{q}(T \wedge \tau_k), T \wedge \tau_k)) = V(\hat{q}(0)) + \mathbb{E}\left(\int_0^{T \wedge \tau_k} \mathcal{L}f(\hat{q}(s), s) ds\right) \leq V(\hat{q}(0)).$$

Conditioned on the event $\{\tau_e \leq T\}$, we deduce

$$f(\hat{q}(T \wedge \tau_k), T \wedge \tau_k) = f(\hat{q}(\tau_k), \tau_k) = e^{-c(T \wedge \tau_k)} V(\hat{q}(\tau_k)) \geq e^{-cT} V(\hat{q}(\tau_k)) \geq e^{-cT} \log k.$$

It implies

$$\mathbb{E}(e^{-cT} \log k \mathbb{1}_{\{\tau_e \leq T\}}) \leq \mathbb{E}(f(\hat{q}(T \wedge \tau_k), T \wedge \tau_k) \mathbb{1}_{\{\tau_e \leq T\}}) \leq \mathbb{E}(f(\hat{q}(T \wedge \tau_k), T \wedge \tau_k)) \leq V(\hat{q}(0)).$$

Consequently,

$$\mathbb{P}(\tau_e \leq T) \leq e^{cT} \frac{V(\hat{q}(0))}{\log k}.$$

Letting $k \rightarrow \infty$, we have $\mathbb{P}(\tau_e \leq T) = 0$, which contradicts the assumption. Thus, we conclude that $\tau_e = \infty$ almost surely in the case where u_t is a real, bounded, and positive process adapted to $\mathcal{F}_t^Y$.

Now, consider the case where $u_t = u(\hat{q})$ with $u \in \mathcal{C}^{1,\alpha}(\bar{\mathcal{O}}_{d+1}, \mathbb{R}_+)$. Due to the compactness of $\mathcal{S}(\mathcal{H}) \times \bar{\mathcal{O}}_{d+1}$, we can find an open set $\mathcal{E} \in \mathcal{B}(\mathcal{H}) \times \mathbb{R}^{d+1}$ such that $\mathcal{S}(\mathcal{H}) \times \bar{\mathcal{O}}_{d+1} \subset \mathcal{E}$. Let $\mathcal{X}(\rho, \hat{q}) : \mathcal{B}(\mathcal{H}) \times \mathbb{R}^{d+1} \rightarrow [0, 1]$ be a smooth function with compact support such that $\mathcal{X}(\rho, \hat{q}) = 1$ for $(\rho, \hat{q}) \in \mathcal{E}$. Additionally, define $U \in \mathcal{C}^{1,\alpha}(\mathbb{R}^{d+1}, \mathbb{R}_+)$ such that $U(\hat{q}) = u(\hat{q})$ for all $\hat{q} \in \bar{\mathcal{O}}_{d+1}$. Then, the following coupled equations

$$\begin{aligned} d\varrho(t) &= \mathcal{X}(\varrho(t), \hat{q}(t)) \left[\mathcal{L}_\gamma^U(\varrho(t))dt + \mathcal{P}(\rho(t))dt \sum_{k=1}^m \sqrt{\theta_k(t)} \mathcal{G}_{L_k}(\rho(t))dW_k(t) \right], \\ d\hat{q}_n(t) &= \mathcal{X}(\varrho(t), \hat{q}(t)) \left[U(\hat{q}(t)) \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j(t)dt + 2\hat{q}_n(t) \sum_{k=1}^m \sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}_n(t)) \right. \\ &\quad \left. \times (dW_k(t) + \mathfrak{T}_k(t, \varrho(t), \hat{q}(t))dt) \right], \end{aligned}$$

have global Lipschitz coefficients, ensuring a unique strong solution with almost surely continuous adapted paths [34]. Since $\mathcal{X}$ has compact support, $(\varrho(t), \hat{q}(t))$ is bounded, making $U(\hat{q}(t))$ an almost surely continuous, real bounded adapted process. Now, consider the coupled system (1)–(9) with $u_t = U(\hat{q}(t))$ and $(\rho(0), \hat{q}(0)) = (\varrho(0), \hat{q}(0)) \in \mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$. As both solutions $(\rho(t), \hat{q}(t))$ and $(\varrho(t), \hat{q}(t))$ have a unique solution, the solutions must coincide up to the first exit time from $\mathcal{E}$. Moreover, $(\rho(t), \hat{q}(t))$ remains in $\mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$ for all $t \geq 0$ almost surely, and thus $(\varrho(t), \hat{q}(t))$ will never exit from $\mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$. Hence, $(\rho(t), \hat{q}(t)) = (\varrho(t), \hat{q}(t))$ for all $t \geq 0$ almost surely. The proof is complete. $\square$

4.3 Feedback stabilization to target subspaces

We adapt methods from [22, 26] to ensure robust GES of $\mathcal{I}(\mathcal{H}_0)$ for the system (1). This stabilization is achieved without requiring knowledge of the initial state or precise values of the free Hamiltonian $H_0(t)$ and the model parameters $\{\gamma_k(t)\}_{k=1}^m$ and $\{\eta_k(t)\}_{k=1}^m$, and remains effective in the presence of perturbations $\mathcal{P}(\rho)$ satisfying **A2**.

We systematically analyze the behavior of the coupled system (1)–(9). The key aspects of our analysis include:

1. *Instability of undesired sets.* Based on the structure of perturbations discussed in Section 3, we identify $\text{card}(E)$ undesired invariant subsets $\mathcal{I}(\mathcal{H}_n) \times e_0$. Under assumptions **A2** and **A4**, these subsets persist despite the feedback controller. Inspired by Khas'minskii's recurrence conditions [19, Theorem 3.9], we establish instability of these non-desired subsets (Lemma 4.5).
2. *Recurrence.* Utilizing the support theorem [37], we demonstrate that the trajectory $(\rho(t), \hat{q}(t))$ of the coupled system almost surely enters any neighborhood of the target subset $\mathcal{I}(\mathcal{H}_0) \times e_0$ in finite time (Proposition 4.8).
3. *Exponential stability.* Inspired by Theorem 3.1, we use local Lyapunov methods to prove stability in probability. Combined with the recurrence property and the strong Markov

property of $(\rho(t), \hat{q}(t))$, local stability in probability implies almost sure asymptotic stability of the subset $\mathcal{I}(\mathcal{H}_0) \times e_0$ ([20, Theorem 6.3]). Finally, we employ Lyapunov techniques to estimate the sample Lyapunov exponent explicitly (Theorem 4.10).

Define, for all $k \in [m]$,

$$\begin{aligned}\bar{\mathfrak{c}}_k &:= \Re\{l_{k,0}\} - \min_{n \in [d]} \Re\{l_{k,n}\}, \\ \underline{\mathfrak{c}}_k &:= \Re\{l_{k,0}\} - \max_{n \in [d]} \Re\{l_{k,n}\}.\end{aligned}$$

Impose the following assumption on measurement operators:

A5: For each $k \in [m]$, $\bar{\mathfrak{c}}_k \leq 0$ or $\underline{\mathfrak{c}}_k \geq 0$.

This assumption is essential for establishing the instability of undesired invariant subsets, the recurrence property, and for estimating the Lyapunov exponent. Note that under **A3**, the case where $\bar{\mathfrak{c}}_k = 0$ or $\underline{\mathfrak{c}}_k = 0$ for all $k \in [m]$ is excluded.

4.3.1 Instability of undesired invariant subsets

Here, we provide Lyapunov-type sufficient conditions ensuring the instability of an invariant subspace of the coupled system (1)–(9), assuming the perturbation structure satisfies **Case 2**. Additionally, we estimate the average escape time of trajectories from a neighborhood of such invariant subspaces.

The control Hamiltonian H_1 determines how the feedback input u_t can induce transitions between invariant subspaces. If H_1 fails to couple certain eigenspaces, the associated subspace $\mathcal{I}(\mathcal{H}_n)$ may remain invariant regardless of the applied control. To formalize this requirement, we adopt a Hautus-type controllability condition [36, Chapter 3.3] ensuring that H_1 connects all relevant eigenspaces. Let $\mathcal{E}_\lambda(X)$ be the eigenspace of X corresponding to the eigenvalue λ .

A6: For each $j \in [d]$, and for all eigenvalue λ of $\Pi_j H_1 \Pi_j$,

$$\bigcap_{i \neq j} \text{Ker}\{\Pi_i H_1 \Pi_j\} \cap \mathcal{E}_\lambda(\Pi_j H_1 \Pi_j) = \{0\}.$$

The following lemma provides a necessary and sufficient condition ensuring that H_1 acts non-trivially on each $\mathcal{I}(\mathcal{H}_n)$.

Lemma 4.2. *Under Assumption **A1**, the necessary and sufficient condition of $[H_1, \rho] \neq 0$ for all $\rho \in \mathcal{I}(\mathcal{H}_n)$ and all $n \in [d]$ is Assumption **A6**.*

Proof. Fix $n \in [d]$ and let $\rho \in \mathcal{I}(\mathcal{H}_n)$, then $\rho = \Pi_n \rho = \rho \Pi_n = \Pi_n \rho \Pi_n$. $[H_1, \rho] = 0$ is equivalent to $\Pi_k [H_1, \rho] \Pi_k = \Pi_i [H_1, \rho] \Pi_j = 0$ for all k and $i \neq j$. We first focus on the diagonal part,

$$\Pi_k [H_1, \rho] \Pi_k = [\Pi_n H_1 \Pi_n, \rho] = 0, \quad \forall k \in \{0, \dots, d\}.$$

Then, ρ admits the decomposition $\rho = \sum_\lambda P_\lambda \rho P_\lambda$, where P_λ is the eigenprojection of $\Pi_n H_1 \Pi_n$ corresponds to the eigenvalue λ . Next, we have

$$\Pi_i [H_1, \rho] \Pi_j = 0, \quad \forall i \neq j \Leftrightarrow \Pi_i H_1 \Pi_n \rho = 0, \quad \forall i \neq n.$$

Thus, for all $\rho \in \mathcal{I}(\mathcal{H}_n)$, $[H_1, \rho] \neq 0$ if and only if $\bigcap_{i \neq n} \text{Ker}\{\Pi_i H_1 \Pi_n\} \cap \mathcal{E}_\lambda(\Pi_n H_1 \Pi_n) = \{0\}$, for all λ . The lemma then follows for arbitrary $n \in [d]$. $\square$

The following corollary provides a simple algebraic condition ensuring that **A6** holds.

Corollary 4.3. *For all $i \neq n \in \{0, \dots, d\}$, the block matrix of $\Pi_i H_1 \Pi_n$ from $\mathcal{H}_n$ to $\mathcal{H}_i$ is injective, then **A6** is satisfied.*

Next, to quantify the effect of time-varying measurement perturbations, we introduce the normalized parameter

$$\chi_k(t) := (\theta_k(t)/\hat{\theta}_k)^{1/2} \in [\underline{\chi}_k, \bar{\chi}_k]$$

where $\underline{\chi}_k = (\underline{\theta}_k/\hat{\theta}_k)^{1/2}$ and $\bar{\chi}_k = (\bar{\theta}_k/\hat{\theta}_k)^{1/2}$ for all $k \in [m]$, and impose the following condition:

C2: For all $k \in [m]$,

$$2\bar{\chi}_k - 1 < \min_{n \in [d]} \left\{ \frac{\Re\{l_{k,0}\}}{\Re\{l_{k,n}\}} \middle| \Re\{l_{k,n}\} \neq 0 \right\} \text{ if } \bar{\mathbf{c}}_k \leq 0,$$

while

$$2\underline{\chi}_k - 1 > \max_{n \in [d]} \left\{ \frac{\Re\{l_{k,0}\}}{\Re\{l_{k,n}\}} \middle| \Re\{l_{k,n}\} \neq 0 \right\} \text{ if } \underline{\mathbf{c}}_k \geq 0.$$

Define

$$\mathbf{R}_{k,n}(t) := (\Re\{l_{k,n}\} - \Re\{l_{k,0}\})((2\chi_k(t) - 1)\Re\{l_{k,n}\} - \Re\{l_{k,0}\}).$$

The next lemma ensures that, under the above conditions, the weighted sum of these terms remains strictly positive, which is crucial for establishing instability.

Lemma 4.4. *If **A3**, **A5** and **C2** are satisfied, then for all $\chi_k(t) \in [\underline{\chi}_k, \bar{\chi}_k]$, $\sum_{k=1}^m \hat{\theta}_k \mathbf{R}_{k,n}(t) > 0$.*

Proof. Fix $n \in [d]$.

- If $\bar{\mathbf{c}}_k \leq 0$ and $\Re\{l_{k,n}\} \neq 0$, then by **C2**,

$$2\chi_k(t) - 1 \leq 2\bar{\chi}_k - 1 < \Re\{l_{k,0}\}/\Re\{l_{k,n}\},$$

which together with $\Re\{l_{k,n}\} \geq \Re\{l_{k,0}\}$ implies $\mathbf{R}_{k,n}(t) \geq 0$, with equality if and only if $\Re\{l_{k,n}\} = \Re\{l_{k,0}\}$.

- If $\underline{\mathbf{c}}_k \geq 0$ and $\Re\{l_{k,n}\} \neq 0$, a symmetric argument shows $\mathbf{R}_{k,n}(t) \geq 0$, again with equality if and only if $\Re\{l_{k,n}\} = \Re\{l_{k,0}\}$.
- If $\Re\{l_{k,n}\} = 0$, then $\mathbf{R}_{k,n}(t) = \Re\{l_{k,0}\}^2 \geq 0$, with equality if and only if $\Re\{l_{k,0}\} = 0$.

Therefore, in all cases $\mathbf{R}_{k,n}(t) \geq 0$, and under assumptions **A3** and **A5**, strict positivity of the weighted sum follows: $\sum_{k=1}^m \hat{\theta}_k \mathbf{R}_{k,n}(t) > 0$. $\square$

Define

$$d_0(\hat{q}) := \|\hat{q} - e_0\|, \\ B_r(e_0) := \{\hat{q} \in \bar{\mathcal{O}}_{d+1} \mid d_0(\hat{q}) < r\}.$$

The following lemma establishes that each non-target invariant subspace is locally unstable, i.e., trajectories initialized sufficiently close to such subspaces will almost surely leave their neighborhood in finite time.

Lemma 4.5. *Suppose **A3**, **A4**, **A5** and **C1** and **C2** hold. In addition, assume there exists a non-empty subset $E \subset [d]$ such that $\mathcal{I}(\mathcal{H}_n)$ for all $n \in E$ are invariant for the perturbed system (1) when $u_t \equiv 0$. Then, almost all $\tilde{H}(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}))$ and $C_k(t) \in L^\infty([0, \infty), \mathcal{B}(\mathcal{H}))$, there exists $\lambda > 0$ such that for all $(\rho(0), \hat{q}(0)) \in \mathbf{B}_\lambda(\mathcal{H}_n) \times B_\lambda(e_0) \cap \mathcal{O}_{d+1}$ with $n \in E$, the trajectory $(\rho(t), \hat{q}(t))$ of the coupled system (1)–(9) exits from $\mathbf{B}_\lambda(\mathcal{H}_n) \times B_\lambda(e_0)$ in finite time almost surely.*

Proof. Consider the function $V_n(\hat{q}) = \log \hat{q}_n^{-1} \in \mathcal{C}^2(\mathcal{O}_{d+1}, \mathbb{R}_+)$. Its infinitesimal generator is given by

$$\mathcal{L}V_n(\hat{q}) = -u(\hat{q})A_n(\hat{q}) + B_n(\rho, \hat{q}),$$

where

$$A_n(\hat{q}) := \frac{1}{\hat{q}_n} \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j,$$

$$B_n(t, \rho, \hat{q}) := 2 \sum_{k=1}^m \sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}) \left[\sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}) - \mathfrak{T}_k(t, \rho, \hat{q}) \right].$$

Under **C1** and **A4**, we deduce that, for all $\hat{q} \in \mathcal{O}_{d+1}$,

$$u(\hat{q})A_n(\hat{q}) = u(\hat{q})(\Gamma_{n,n} + \sum_{n \neq j} \Gamma_{n,j} \frac{\hat{q}_j}{\hat{q}_n}) \geq u(\hat{q})\Gamma_{n,n} \geq 0.$$

Moreover, according to Lemma 4.4, we have

$$\lim_{(\rho, \hat{q}) \rightarrow \mathcal{I}(\mathcal{H}_n) \times e_0} B_n(t, \rho, \hat{q}) = -2 \sum_{k=1}^m \hat{\theta}_k \mathbf{R}_{k,n}(t) < 0.$$

It implies

$$\limsup_{(\rho, \hat{q}) \rightarrow \mathcal{I}(\mathcal{H}_n) \times e_0} \mathcal{L}V_n(\hat{q}) \leq \lim_{(\rho, \hat{q}) \rightarrow \mathcal{I}(\mathcal{H}_n) \times e_0} B_n(\rho, \hat{q}) < 0.$$

Therefore, there exist $\delta, \lambda > 0$ such that

$$\mathcal{L}V(\hat{q}) \leq -\delta, \quad \forall (\rho, \hat{q}) \in \mathbf{B}_\lambda(\mathcal{H}_n) \times B_\lambda(e_0) \cap \mathcal{O}_{d+1}.$$

Define τ_λ as the first exiting time from $\mathbf{B}_\lambda(\mathcal{H}_n) \times B_\lambda(e_0)$. According to Theorem 4.1, $\hat{q}(t) \in \mathcal{O}_{d+1}$ for all $t \geq 0$ almost surely. Applying the Itô formula on $V(\hat{q}(t))$, we obtain $\mathbb{E}(\tau_\lambda) \leq V(\hat{q}(0))/\delta < \infty$. Then, we conclude the proof by applying the Markov inequality. $\square$

4.3.2 Recurrence property

We now provide sufficient conditions ensuring that trajectories of the coupled system (1)–(9) are recurrent in neighborhoods of the desired invariant subset. The recurrence analysis relies on the support theorem [37], which characterizes the reachable sets of SDEs through associated deterministic control systems. Consider the deterministic control system associated with the Stratonovich form [34] of (1)–(9), for $n \in \{0, \dots, d\}$,

$$\dot{\rho}_v(t) = \tilde{\mathcal{L}}_{\gamma, \eta}^u(t, \rho_v(t)) + \mathcal{P}(t, \rho_v(t)) \sum_{k=1}^m \sqrt{\theta_k(t)} \mathcal{G}_{L_k}(\rho_v(t)) V_k(t), \quad (10)$$

$$\dot{\hat{q}}_{v,n}(t) = \mathfrak{f}_n^u(\hat{q}_v(t)) + \Delta_n(\hat{q}_v(t)) + 2\hat{q}_{v,n}(t) \sum_{k=1}^m \sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}_v(t)) V_k(t), \quad (11)$$

where $\rho_v(0) = \rho(0)$, $\hat{q}_v(0) = \hat{q}(0)$, and

$$V_k(t) := v_k(t) + \sqrt{\theta_k(t)} \text{Tr}((L_k + L_k^*)\rho_v(t)),$$

where $v_k(t) \in \mathcal{V}$ is the bounded control input. Here

$$\begin{aligned} \tilde{\mathcal{L}}_{\gamma, \eta}^u(t, \rho) &:= -i[H_0(t) + uH_1, \rho] + \sum_{k=1}^m \frac{\gamma_k(t)}{2} (2(1 - \eta_k(t))L_k \rho L_k^* - (L_k^* L_k + \eta_k(t)L_k^2)\rho \\ &\quad - \rho(L_k^* L_k + \eta_k(t)L_k^{*2}) + \eta_k(t)\text{Tr}((L_k + L_k^*)^2 \rho)\rho), \\ \mathfrak{f}_n^u(\hat{q}) &= u \sum_{j=0}^d \Gamma_{n,j} \hat{q}_j - 4\hat{q}_n \sum_{k=1}^m \sqrt{\hat{\theta}_k} \Phi_n^k(\hat{q}) \Lambda_k(\hat{q}), \\ \Delta_n(\hat{q}) &= 2\hat{q}_n \sum_k \hat{\theta}_k [\sum_j \hat{q}_j \Re\{l_{k,j}\} \Phi_j^k(\hat{q}) - (\Phi_n^k(\hat{q}))^2]. \end{aligned}$$

The invariance of $\mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$ for the deterministic system (10)–(11) follows directly from the support theorem.

We now introduce a Kalman-type controllability assumption on the control Hamiltonian H_1 to guarantee sufficient coupling between the target subspace $\mathcal{H}_0$ and its orthogonal complement $\bigoplus_{i=1}^d \mathcal{H}_i$. This ensures that the set

$$\mathcal{I}_0 := \{\mathcal{S}(\mathcal{H}) \mid \text{Tr}(\rho \Pi_0) = 0\}$$

which contains all potential invariant subsets of the dynamics in **Case 2**, i.e., $\mathcal{I}(H_n)$ with $n \in E$, is non-invariant under the evolution. Establishing the non-invariance of $\mathcal{I}_0$ is essential for stabilization. Let $\Pi_0^\perp := \mathbf{I} - \Pi_0$ denote the orthogonal projector onto $\bigoplus_{i=1}^d \mathcal{H}_i$. Consider the block operators of the control Hamiltonian $H_{1,R}$ from $\bigoplus_{i=1}^d \mathcal{H}_i$ to $\bigoplus_{i=1}^d \mathcal{H}_i$ and $H_{1,Q}$ from $\mathcal{H}_0$ to $\bigoplus_{i=1}^d \mathcal{H}_i$. We impose the following assumption:

A7: There exists a $n \in \mathbb{Z}_+$ such that

$$\text{rank}\{[\mathbf{I}, H_{1,R}, \dots, H_{1,R}^n] H_{1,Q}\} \geq N - 1 - \dim\{\mathcal{H}_0\}.$$

The next lemma shows that, under the required assumptions, the control Hamiltonian H_1 guarantees the immediate exit of trajectories from $\mathcal{I}_0$.

Lemma 4.6. *Suppose that **A1**, **A2**, **A4**, **A6** and **A7** holds, then for all $\rho(0) \in \mathcal{I}_0$, the trajectory $\rho_v(t)$ to the system (10) exits $\mathcal{I}_0$ immediately.*

Proof. Under **A4**, we suppose that $u_t = u(\hat{q}_v(t)) > 0$ for all $t \geq t_0$. If the lemma false, then there exists a $\delta > 0$ arbitrarily small such that $\rho_v(t) \in \mathcal{I}_0$ for all $t \in [t_0, t_0 + \delta]$. Hence,

$$\Pi_0 \rho_v(t) \Pi_0 = \rho_v(t) \Pi_0 = \Pi_0 \rho_v(t) = 0.$$

By **A1** and **A2**, for all $k \in [m]$, we have,

$$[L_k, \Pi_0] = [H_0(t), \Pi_0] = [\tilde{H}_0(t), \Pi_0] = 0, \quad \forall t \geq 0.$$

It implies that, for all $t \in [t_0, t_0 + \delta]$,

$$\Pi_0 \dot{\rho}_v(t) \Pi_0 = \sum_{k=1}^{\bar{m}} \Pi_0 C_k(t) \rho_v(t) C_k(t)^* \Pi_0 \geq 0.$$

However, since we cannot access the full information of the perturbations, and $C_k(t)$ act only positively in breaking the invariance of $\mathcal{I}_0$, it suffices to consider the “worst-case” scenario in the first-order derivative analysis, i.e., $C_k(t) = 0$ for all $k \in [\bar{m}]$ and $t \in [t_0, t_0 + \delta]$. Define

$$A := \Pi_0^\perp H_1 \Pi_0^\perp, \quad B := \Pi_0^\perp H_1 \Pi_0.$$

Then, we have

$$\dot{\rho}_v(t) \Pi_0 = \mathbf{i} u_t \rho_v(t) H_1 \Pi_0 = \mathbf{i} u_t \rho_v(t) B = 0, \quad \forall t \in [t_0, t_0 + \delta],$$

which implies $\rho_v(t) B = 0$ and consequently $B^* \rho_v(t) B = 0$. Moreover, under $C_k(t) = 0$, we get

$$B^* \dot{\rho}_v(t) B = \sum_{k=1}^m \gamma_k(t) (1 - \eta_k(t)) B^* L_k \rho_v(t) L_k^* B = 0.$$

Thus, for all $i \in [m]$, $\rho_v(t) L_i^* B = 0$. Since $H_0(t)$, $\tilde{H}_0(t)$ and L_k has zero cross-block $\Pi_0^\perp(\cdot) \Pi_0$, they cannot contribute to exiting from $\mathcal{I}_0$. Only H_1 does, through the non-zero block B . Next, we calculate the derivative of the following equality,

$$B^* A \rho_v(t) B - B^* \rho_v(t) A B = 0, \quad \forall t \in [t_0, t_0 + \delta],$$

which yields,

$$B^* A \dot{\rho}_v(t) B - B^* \dot{\rho}_v(t) A B = \mathbf{i} u_t B^* A \rho_v(t) A B = 0.$$

It implies that $B^* A \rho_v(t) A B = 0$, thus $\rho_v(t) A B = 0$. Repeating this argument recursively gives, for all $t \in [t_0, t_0 + \delta]$,

$$\rho_v(t) B = \rho_v(t) A B = \cdots = \rho_v(t) A^n B = 0,$$

for some $n \in \mathbb{Z}_+$. Due to **A7**, we deduce $\text{rank}(\rho_v(t)) \leq 1$ for all $t \in [t_0, t_0 + \delta]$. This leads to a contradiction, since by Lemma A.2, Lemma A.1 and the support theorem [37], $\text{rank}(\rho_v(t)) > 1$ for all $t > 0$. The proof is complete. $\square$

Remark 4.7. Assumption **A6** guarantees the non-invariance of $\bigcup_{n \in [d]} \mathcal{I}(\mathcal{H}_n) \subset \mathcal{I}_0$, and **A7** ensures the non-invariance of $\mathcal{I}_0$ itself. Both properties can be simultaneously guaranteed by the following stronger Kalman-type rank condition:

A-ctrl: There exists a $n \in \mathbb{Z}_+$ such that

$$\text{rank}\{\mathbf{I}, H_{1,R}, \dots, H_{1,R}^n H_{1,Q}\} \geq N - \dim\{\mathcal{H}_0\}.$$

This stronger condition does not rely on the non-invariance of the set of pure states (Lemma A.2) and clearly implies both **A6** and **A7**.

For all $k \in [m]$ and $n \in \{0, \dots, d\}$, define

$$\Psi_n^k(\rho) := \Re\{l_{k,n}\} - \sum_j \Re\{l_{k,j}\} \text{Tr}(\rho \Pi_j).$$

The following proposition shows that, under suitable regularity and coupling conditions, the trajectories almost surely enter a small neighborhood of the desired subspace in finite time.

Proposition 4.8. *Suppose that $\bar{\eta}_k < 1$ for all $k \in [m]$, and **C1-C2** and **A1-A7** hold. For all $(\rho(0), \hat{q}(0)) \in \mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$ and any $\zeta > 0$, the trajectory of the coupled system (1)–(9) enters $\mathbf{B}_\zeta(\mathcal{H}_0) \times B_\zeta(e_0)$ in finite time almost surely, for almost all $\tilde{H}(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}))$ and $C_k(t) \in L^\infty([0, \infty), \mathcal{B}(\mathcal{H}))$.*

Proof. By Lemma 4.6, if $\text{Tr}(\Pi_0 \rho_v(0)) = 0$, there exists a finite $t_1 > 0$ such that $\text{Tr}(\Pi_0 \rho_v(t_1)) > 0$. From (10), we have, for all $t \geq t_1$,

$$\begin{aligned} \text{Tr}(\Pi_0 \dot{\rho}_v(t)) &= \text{Tr}[\Pi_0(\tilde{\mathcal{L}}_{\gamma, \eta}^u(t, \rho_v(t)) + \mathcal{P}(t, \rho_v(t)))] \\ &\quad + 2 \sum_{k=1}^n \sqrt{\theta_k(t)} \Psi_0^k(\rho_v(t)) \text{Tr}(\Pi_0 \rho_v(t)) V_k(t), \end{aligned}$$

where $\text{Tr}(\Pi_0 \rho_v(t_1)) > 0$. Since $\mathcal{S}(\mathcal{H}) \times \bar{\mathcal{O}}_{d+1}$ is compact, the first two terms on the right-hand side and those in (11) are uniformly bounded from above in this domain. Moreover, a direct calculation yields

$$\bar{\mathbf{c}}_k(1 - \text{Tr}(\rho \Pi_0)) \geq \Psi_0^k(\rho) \geq \underline{\mathbf{c}}_k(1 - \text{Tr}(\rho \Pi_0)), \quad (12)$$

$$\bar{\mathbf{c}}_k(1 - \hat{q}_0) \geq \Phi_0^k(\hat{q}) \geq \underline{\mathbf{c}}_k(1 - \hat{q}_0). \quad (13)$$

Under **A5**, we deduce

$$\begin{aligned} \bigcap_{k=1}^m \{\rho \in \mathcal{S}(\mathcal{H}) \mid \Psi_0^k(\rho) = 0\} &= \mathcal{I}(\mathcal{H}_0), \\ \bigcap_{k=1}^m \{\hat{q} \in \bar{\mathcal{O}}_{d+1} \mid \Phi_0^k(\hat{q}) = 0\} &= e_0. \end{aligned}$$

Thus, for any $\zeta > 0$, there exist at least one $k \in [n]$ and $\delta > 0$ such that $|\Psi_0^k(\rho)| \geq \delta$ for all $\rho \in \mathcal{S}(\mathcal{H}) \setminus \mathbf{B}_\zeta(\mathcal{H}_0)$, and $|\Phi_0^k(\hat{q})| \geq \delta$ for all $\hat{q} \in \bar{\mathcal{O}}_{d+1} \setminus B_\zeta(e_0)$. Then by choosing $V_k(t) = K / \min\{\Psi_0^k(\rho_v(t)), \Phi_0^k(\hat{q}_v(t))\}$, with $K > 0$ sufficiently large, we can guarantee that $(\rho_v(t), \hat{q}_v(t))$ enter $\mathbf{B}_\zeta(\mathcal{H}_0) \times B_\zeta(e_0)$ in finite time.

We apply the support theorem [37] to extend the finite-time entrance property from the deterministic system to the stochastic trajectories of (1)–(9). Since the deterministic flow reaches $\mathbf{B}_\zeta(\mathcal{H}_0) \times B_\zeta(e_0)$ in finite time, there exists a constant $\bar{p} > 0$ such that the stochastic trajectory enters the same neighborhood within finite time with probability at least $\bar{p}$, uniformly over all initial conditions.

Finally, due to the compactness of $\mathcal{S}(\mathcal{H}) \times \bar{\mathcal{O}}_{d+1}$ and the Feller continuity of the trajectories $(\rho(t), \hat{q}(t))$, together with the strong Markov property and Lemma 4.5, one can apply the same arguments as in [25, Lemma 4.10] to conclude that the process enters $\mathbf{B}_\zeta(\mathcal{H}_0) \times B_\zeta(e_0)$ in finite time almost surely. In particular, compactness ensures uniform tightness of the trajectories, while Feller continuity guarantees continuous dependence on initial data, ensuring the recurrence property of the deterministic system to extend to the stochastic one. Combined with the strong Markov property, these features imply that the hitting probability of the target neighborhood accumulates to one as time evolves, thereby yielding almost-sure finite-time entrance. This completes the proof. $\square$

4.3.3 Exponential stability

For each $k \in [m]$, set

$$\underline{\mathbf{l}}_k := \min\{|\underline{\mathbf{c}}_k|, |\bar{\mathbf{c}}_k|\}, \quad \bar{\mathbf{l}}_k := \max\{|\underline{\mathbf{c}}_k|, |\bar{\mathbf{c}}_k|\},$$

and define

$$\mathfrak{A}_k := \mathfrak{l}_k^2 \min \left\{ \chi_k^2, 1 \right\},$$

$$\mathfrak{B}_k := 2\bar{\mathfrak{l}}_k |\Re\{l_{k,0}\}| \max \left\{ 1 - \underline{\chi}_k, \bar{\chi}_k - 1 \right\}.$$

We impose the following condition to guarantee local stability in probability of the target subspace:

C3: If $|\Re\{l_{k,0}\}| = 0$, then $\mathfrak{l}_k > 0$. Otherwise, $\mathfrak{A}_k > \mathfrak{B}_k$ for all $k \in [m]$.

By a straightforward calculation, we have the following lemma.

Lemma 4.9. *Under condition C3, for all $\chi_k(t) \in [\underline{\chi}_k, \bar{\chi}_k]$, we have*

$$\mathfrak{l}_k^2 \min\{\chi_k(t)^2, 1\} - 2\bar{\mathfrak{l}}_k |(\chi_k(t) - 1)\Re\{l_{k,0}\}| > \mathfrak{A}_k - \mathfrak{B}_k > 0.$$

Define the coefficient

$$\mathfrak{C}_{\underline{\chi}, \bar{\chi}} := \sum_k \hat{\theta}_k (2\mathfrak{A}_k - \mathfrak{B}_k)^2 / 2\mathfrak{A}_k,$$

which is used to estimate the Lyapunov exponent. Building on the previous stability and recurrence results, the following theorem presents the main result of this paper, under mild regularity and coupling conditions, the target subspace is globally exponentially stable almost surely.

Theorem 4.10. *Suppose $\bar{\eta}_k < 1$ for all $k \in [n]$, conditions C1-C3 and assumptions A1-A7 hold. Then, for all $\rho(0) \in \mathcal{S}(\mathcal{H})$, the target subspace $\mathcal{H}_0$ is almost sure GES for the perturbed system (1), for almost all $\tilde{H}(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}))$ and $C_k(t) \in L^\infty([0, \infty), \mathcal{B}(\mathcal{H}))$, with the sample Lyapunov exponent less than or equal to $-\mathfrak{C}_{\underline{\chi}, \bar{\chi}}/2$.*

Proof. Define the auxiliary function

$$\mathbf{V}_0(\rho, \hat{q}) = 1 - \text{Tr}(\rho \Pi_0) + 1 - \hat{q}_0,$$

and consider the candidate Lyapunov function

$$V(\rho, \hat{q}) = \mathbf{V}_0(\rho, \hat{q})^x \geq 0, \quad x \in (0, 1),$$

where the equality holds if and only if $(\rho, \hat{q}) \in \mathcal{I}(\mathcal{H}_0) \times e_0$. By Theorem 4.1, if the initial condition satisfies $\hat{q}_0(0) < 1$, then $\hat{q}_0(t) < 1$ for all $t \geq 0$ almost surely. Consequently, $V(\rho(t), \hat{q}(t)) > 0$ almost surely.

We first establish the local stability in probability of the target subspace $\mathcal{H}_0$. That is, for every $\varepsilon \in (0, 1)$ and for every $r > 0$, there exists $\delta = \delta(\varepsilon, r) > 0$ such that,

$$\mathbb{P}(V(\rho(t), \hat{q}(t)) < r \text{ for } t \geq 0) \geq 1 - \varepsilon,$$

whenever $V(\rho(0), \hat{q}(0)) < \delta$.

The infinitesimal generator of $V(\rho, \hat{q})$ is given by

$$\mathcal{L}V(\rho, \hat{q}) = xV(\rho, \hat{q}) \left[\frac{\mathfrak{F}(t, \rho, \hat{q})}{\mathbf{V}_0(\rho, \hat{q})} - \frac{2(1-x) \sum_k \mathfrak{G}_k(t, \rho, \hat{q})}{\mathbf{V}_0(\rho, \hat{q})^2} \right],$$

where

$$\begin{aligned}\mathfrak{F}(t, \rho, \hat{q}) &= u(\hat{q}) \left[\sum_j \Gamma_{0,j} \hat{q}_j - i \text{Tr}([H_1, \rho] \Pi_0) \right] \\ &\quad + 2\hat{q}_0 \sum_k \Phi_0^k(\hat{q}) \mathfrak{T}_k(t, \rho, \hat{q}), \\ \mathfrak{G}_k(t, \rho, \hat{q}) &= \hat{\theta}_k \left[\chi_k(t) \Psi_0^k(\rho) \text{Tr}(\rho \Pi_0) + \Phi_0^k(\hat{q}) \hat{q}_0 \right]^2.\end{aligned}$$

Since $u(\hat{q}) \in \mathcal{C}^{1,\alpha}$ with $\alpha \in (0, 1)$,

$$|u(\hat{q})| \leq c|1 - \hat{q}_0|^{1+\alpha} \leq cV_0^{1+\alpha}$$

for some $c > 0$. Moreover,

$$\begin{aligned}|\Phi_0^k(\hat{q})| &= \left| \Re\{l_{k,0}\} \sum_{j=0}^d \hat{q}_j - \sum_{n=0}^d \Re\{l_{k,n}\} \hat{q}_n \right| \\ &= \left| \sum_{n=0}^d (\Re\{l_{k,0}\} - \Re\{l_{k,n}\}) \hat{q}_n \right| \\ &\leq \bar{l}_k(1 - \hat{q}_0) \leq \bar{l}_k V_0(\rho, \hat{q}).\end{aligned}$$

Therefore, due to the compactness of $\mathcal{S}(\mathcal{H})$ and $\bar{\mathcal{O}}_{d+1}$,

$$\begin{aligned}\frac{\mathfrak{F}(t, \rho, \hat{q})}{V_0(\rho, \hat{q})} &\leq \frac{|u(\hat{q})|}{V_0(\rho, \hat{q})} \left| \sum_j \Gamma_{0,j} \hat{q}_j - i \text{Tr}([H_1, \rho] \Pi_0) \right| + \frac{2\hat{q}_0 \sum_k |\Phi_0^k(\hat{q})| |\mathfrak{T}_k(t, \rho, \hat{q})|}{V_0(\rho, \hat{q})} \\ &\leq cV_0(\rho, \hat{q})^\alpha + 2\hat{q}_0 \sum_k \bar{l}_k |\mathfrak{T}_k(t, \rho, \hat{q})|,\end{aligned}$$

for some $c > 0$. Together with the relation (12)–(13), we have

$$\frac{\sum_{k=1}^m \mathfrak{G}_k(t, \rho, \hat{q})}{V_0(\rho, \hat{q})^2} \geq \sum_{k=1}^m \hat{\theta}_k l_k^2 \min \{ \chi_k(t)^2 \text{Tr}(\rho \Pi_0)^2, \hat{q}_0^2 \}.$$

Putting the estimates together,

$$\mathcal{L}V(\rho, \hat{q}) \leq -V(\rho, \hat{q}) \mathbb{C}_x(t, \rho, \hat{q}),$$

where

$$\mathbb{C}_x(t, \rho, \hat{q}) = 2x(1-x) \sum_k \hat{\theta}_k l_k^2 \min \{ \chi_k(t)^2 \text{Tr}(\rho \Pi_0)^2, \hat{q}_0^2 \} - cxV_0(\rho, \hat{q})^\alpha + 2x\hat{q}_0 \sum_k \bar{l}_k |\mathfrak{T}_k(t, \rho, \hat{q})|.$$

By continuity arguments and Lemma 4.9, there exists $x \in (0, 1)$ such that

$$\begin{aligned}\lim_{(\rho, \hat{q}) \rightarrow \mathcal{I}(\mathcal{H}_0) \times e_0} \mathbb{C}_x(t, \rho, \hat{q}) &= 2x \sum_k \hat{\theta}_k \left[(1-x) l_k^2 \min \{ \chi_k(t)^2, 1 \} - 2\bar{l}_k |(\chi_k(t) - 1) \Re\{l_{k,0}\}| \right] \\ &\geq \mathbb{C}_x > 0,\end{aligned}$$

where

$$\mathbb{C}_x = 2x \sum_k \hat{\theta}_k [(1-x) \mathfrak{A}_k - \mathfrak{B}_k].$$

Consequently,

$$\limsup_{(\rho, \hat{q}) \rightarrow \mathcal{I}(\mathcal{H}_0) \times e_0} \frac{\mathcal{L}V(\rho, \hat{q})}{V(\rho, \hat{q})} \leq \lim_{(\rho, \hat{q}) \rightarrow \mathcal{I}(\mathcal{H}_0) \times e_0} -\mathbb{C}_x(t, \rho, \hat{q}) = -\mathbb{C}_x < 0.$$

Due to the continuity of $\mathbb{C}_x(t, \rho, \hat{q})$, there exists a $\lambda > 0$ such that

$$\mathcal{L}V(\rho, \hat{q}) \leq 0, \quad \forall (\rho, \hat{q}) \in \mathbf{B}_\lambda(\mathcal{H}_0) \times B_\lambda(e_0),$$

and by applying the similar arguments as in [20, Theorem 6.3], then local stability in probability is ensured.

Next, according to Proposition 4.8, for almost all $\tilde{H}(t) \in L^\infty([0, \infty), \mathcal{B}_*(\mathcal{H}))$ and $C_k(t) \in L^\infty([0, \infty), \mathcal{B}(\mathcal{H}))$, the coupled system is almost surely recurrent. That is, for any initial condition $(\rho(0), \hat{q}(0)) \in \mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$, the trajectory $(\rho(t), \hat{q}(t))$ almost surely enters any neighborhood of $\mathcal{I}(\mathcal{H}_0) \times e_0$ in finite time.

Combining the above two properties, *local stability in probability* and *almost sure recurrence*, one can invoke the Strong Markov property to establish the almost sure convergence of $V(\rho(t), \hat{q}(t))$ to zero (see [20, Theorem 6.3] for a detailed argument; see also [19, Theorem 5.5.7]). Indeed, since the process $(\rho(t), \hat{q}(t))$ almost surely returns infinitely often to any neighborhood of $\mathcal{I}(\mathcal{H}_0) \times e_0$, and due to local stability in probability, it remains in such neighborhoods with non-zero probability, the probability that the trajectory leaves these neighborhoods infinitely many times is zero. Consequently, $V(\rho(t), \hat{q}(t))$ converges to zero almost surely. Therefore, the target subspace $\mathcal{H}_0$ is almost surely asymptotically stable for all $(\rho(0), \hat{q}(0)) \in \mathcal{S}(\mathcal{H}) \times \mathcal{O}_{d+1}$.

Finally, we derive the almost sure global exponential stability and provide an estimation of the sample Lyapunov exponent. Observe that

$$\begin{aligned} & \liminf_{(\rho, \hat{q}) \rightarrow \mathcal{I}(\mathcal{H}_0) \times e_0} \sum_{k=1}^n \hat{\theta}_k \left[\text{Tr} \left(\nabla_\rho V(\rho, \hat{q}) \frac{\chi_k(t) \mathcal{G}_{L_k}(\rho)}{V(\rho, \hat{q})} \right) + \frac{\nabla_{\hat{q}} V(\rho, \hat{q})^\top \Phi^k(\hat{q})}{V(\sigma, \hat{\sigma})} \right]^2 \\ & \geq 4x \sum_k \hat{\theta}_k \mathbb{I}_k^2 \min \{ \chi_k(t)^2, 1 \} \geq 2\mathbb{K}_x, \end{aligned}$$

where $\Phi^k(\hat{q}) = 2[\hat{q}_0 \Phi_0^k(\hat{q}) \cdots \hat{q}_d \Phi_d^k(\hat{q})]^\top$ and $\nabla_\rho V(\rho, \hat{q})$ is the Fréchet gradient, and

$$\mathbb{K}_x = 2x \sum_k \hat{\theta}_k \mathbb{I}_k^2 \min \{ \underline{\chi}_k^2, 1 \}.$$

Therefore, by using arguments as in the proof of [20, Theorem 6.3] again, we obtain

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log V(\rho(t), \hat{q}(t)) \leq - \max_{x \in (0, 1)} \{ \mathbb{C}_x + \mathbb{K}_x \} = -\mathfrak{C}_{\underline{\chi}, \bar{\chi}} < 0, \quad a.s.$$

Moreover, since $(\mathbf{d}_0(\rho))^2 \leq cV(\rho, \hat{q})$ for some positive constant c , it follows that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbf{d}_0(\rho(t)) \leq -\mathfrak{C}_{\underline{\chi}, \bar{\chi}}/2 < 0, \quad a.s.$$

This completes the proof. $\square$

Remark 4.11. In Theorem 4.10 we assumed that the feedback controller satisfies $u \in \mathcal{C}^{1, \alpha}$. However, the proof shows that global exponential stability also holds under the condition that $u \in \mathcal{C}^1$ and $u(\hat{q}) = 0$ for all $\hat{q} \in B_\delta(e_0)$, where $\delta > 0$ is arbitrarily small.

As an example of application of the previous results, we consider the following feedback law. Define

$$u(\hat{q}) = a(1 - \hat{q})^b, \tag{14}$$

with $a > 0$ and $b > 1$. Therefore, **A4** holds true.

5 Numerical example: Three-level systems

We consider a three-level system undergoing one-channel homodyne detection along the z -axis, with target state $\text{diag}(1, 0, 0)$. The system operators are specified as follows:

$$H_0(t) = \omega(t)J_z, \quad L = J_z = \text{diag}(1, 0, -1),$$

and

$$H_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix},$$

with perturbations, for all $k \in [\bar{m}]$,

$$\tilde{H}_0(t) = \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, C_k(t) \in \left\{ \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\},$$

where partial information about the perturbation is available, i.e., $\tilde{H}_0(t)$ and $\{C_k(t)\}_{k \in [\bar{m}]}$ satisfy the above assumption, which can be verified using operator identification techniques.

The free Hamiltonian frequency is chosen to be sinusoidally modulated,

$$\omega(t) = 1.5 \left(1 + 0.2 \sin \frac{2\pi t}{T} \right).$$

Its time dependence reflects slow oscillations of the external magnetic field caused by line-frequency interference or deliberately applied AC modulation.

The key physical parameters are assumed to be time-varying. The effective coupling strength $\gamma(t)$ and the measurement efficiency $\eta(t)$ are chosen as

$$\gamma(t) = \gamma_0 \left(1 + 0.2 \sin \frac{2\pi t}{T} \right), \quad \eta(t) = \eta_0 \left(1 + 0.1 \text{tri} \frac{t}{T} \right),$$

where $\text{tri}(\cdot)$ is a symmetric triangle wave with range $[-1, 1]$. Thus, $\theta(t) = \eta(t)\gamma(t) \in [\underline{\theta}, \bar{\theta}]$ where $\underline{\theta} = 0.72\eta_0\gamma_0$ and $\bar{\theta} = 1.32\eta_0\gamma_0$, reflecting atomic position-field coupling changes (sine) and slow efficiency drift (triangle).

In addition, environmental noise is modeled by Lindblad operators of the form

$$C_1(t) = \sqrt{\gamma_\phi(t)} J_z, \quad \gamma_\phi(t) = 1.5 \left(1 + 0.2 \cos \frac{2\pi t}{T} \right),$$

describing dephasing with periodic fluctuation due to background magnetic field or laser phase noise, and

$$C_2(t) = \sqrt{\gamma_\downarrow(t)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \gamma_\downarrow(t) = 1.5 \left(1 + 0.1 \frac{t}{T} \right),$$

which models selective pumping from $\text{diag}(0, 0, 1)$ to $\text{diag}(0, 1, 0)$ caused by polarization-dependent spontaneous emission or auxiliary fields. The time-varying perturbation matrices can be specified as

$$\tilde{H}_0(t) = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0.5 + 0.2 \sin(2\pi t/T) \\ 0 & 0.5 + 0.2 \sin(2\pi t/T) & 0.4 \end{bmatrix}.$$

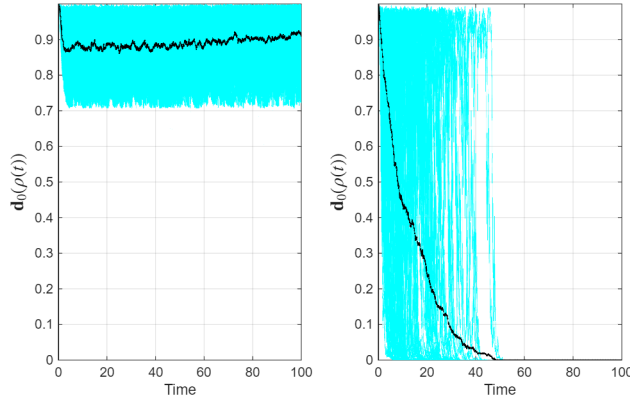


Figure 1: Three-level system under slow modulation ($T = 100\tau$). Left: no feedback ($u \equiv 0$). Right: reduced-filter feedback $u(\hat{q})$. Initial conditions: $\rho(0) = \text{diag}(0, 0, 1)$, $\hat{q}(0) = [1, 1, 1]^\top/3$. Black curve: mean over 100 realizations.

The diagonal terms represent static level shifts, while the constant off-diagonal coupling 0.5 models a systematic coherent perturbation between $\text{diag}(0, 0, 1)$ and $\text{diag}(0, 1, 0)$. The additional sinusoidal modulation $0.2 \sin(2\pi t/T)$ accounts for bounded time-varying effects such as line-frequency magnetic-field noise or parametric drift in the driving field.

The Hilbert space decomposes into three one-dimensional subspaces, with the target state chosen as $\text{diag}(1, 0, 0)$. Under this setting, assumptions **A1–A3**, **A5–A7** and condition **C2** are satisfied. We compute $\underline{\mathfrak{c}} = 1$, $\bar{\mathfrak{l}} = 1$ and $\bar{\mathfrak{l}} = 2$. The physical parameters are set as $\eta_0 = 0.4$, $\gamma_0 = 1.6$ yielding $\underline{\theta} = 0.4608$ and $\bar{\theta} = 0.8448$, choosing $\hat{\eta} = 0.5$ and $\hat{\gamma} = 1.2$ making **C3** hold.

Let $\tau := 1/(\eta_0\gamma_0)$ denote the characteristic measurement time scale. To evaluate robustness, we consider both slow and fast deterministic modulations: $T = 100\tau$ to emulate quasi-static drifts, and $T = 2\tau$ to impose rapidly varying perturbations on the reduced filter.

We implement (9) with

$$\Gamma = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

which satisfies **C1**. and feedback of the form (14) with $a = 4$, $b = 2$ satisfying **A4**. Figures 1–2 show $\mathbf{d}_0(\rho(t))$ over 100 trajectories under slow/fast modulation, with and without feedback. Exponential convergence is observed despite time-varying parameters, in agreement with the predicted sample Lyapunov exponent $-\mathfrak{C}_{\underline{\chi}, \bar{\chi}}/2 \approx -0.1218$, supporting Theorem 4.10.

6 CONCLUSIONS AND FUTURE WORKS

We studied robust feedback stabilization of perturbed quantum systems under QND measurements using a reduced quantum filter. The proposed method ensures stabilization toward a target subspace when the free Hamiltonian, coupling strengths, measurement efficiencies, and perturbations are time-varying, by estimating only the diagonal elements of the system state in the non-demolition basis. Our analysis highlights invariance-preserving perturbations and the critical role of accurately known measurement operators in ensuring effectiveness. Future work will extend these results to broader classes of perturbations and more general measurement settings, further advancing scalable and robust quantum feedback control.

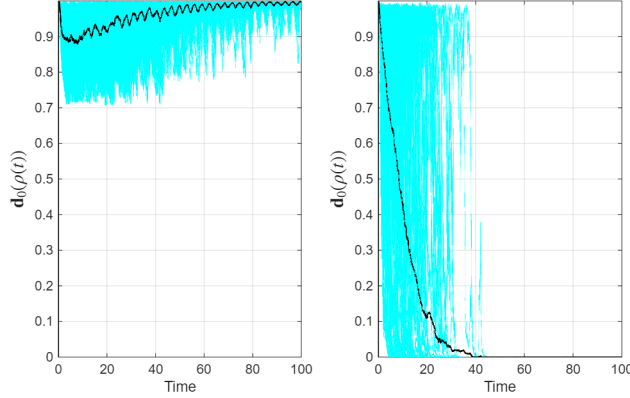


Figure 2: Three-level system under fast modulation ($T = 2\tau$). Left: no feedback ($u \equiv 0$). Right: reduced-filter feedback $u(\hat{q})$. Initial conditions: $\rho(0) = \text{diag}(0, 0, 1)$, $\hat{q}(0) = [1, 1, 1]^\top / 3$. Black curve: mean over 100 realizations.

Appendices

A Invariant properties of quantum trajectories

This appendix states several auxiliary results used in the proofs of instability and recurrence. These lemmas concern the invariance properties of the stochastic dynamics and are analogous to those established in [20, Section 4] and [24, Lemma 7] for SME (1).

The first lemma is a direct analogue of [20, Section 4], its proof is therefore omitted.

Lemma A.1. *Assume that **A4** holds. The rank of $\rho(t)$ is almost surely non-decreasing.*

Let $\mathcal{I}_P := \{\rho \in \mathcal{S}(\mathcal{H}) \mid \text{Tr}(\rho^2) = 1\}$ denote the set of pure states.

Lemma A.2. *Assume that **A4** and **A6** are satisfied. In addition, suppose that there exists a $k \in [m]$ such that $\bar{\eta}_k < 1$. Then, for all initial state $\rho(0) \in \mathcal{I}_P \setminus \mathcal{I}(\mathcal{H}_0)$, $\rho(t)$ is mixed (i.e., $\text{Tr}(\rho(t)^2) < 1$) for all $t > 0$ almost surely.*

Proof. For all $\rho \in \mathcal{I}_P$, we have $\rho^2 = \rho$ and $\text{Tr}(\rho A \rho B) = \text{Tr}(\rho A) \text{Tr}(\rho B)$ with $A, B \in \mathcal{B}(\mathcal{H})$. By Itô formula, we get $d\text{Tr}(\rho(t)^2) = \mathfrak{A}(t, \rho(t))dt + \sum_k \mathfrak{B}_k(t, \rho(t))dW_k(t)$. By a straightforward calculation, it follows that, for all $\rho \in \mathcal{I}_P$, $\mathfrak{B}_k(t, \rho) = 0$ for all k , and

$$\begin{aligned} \mathfrak{A}(t, \rho) = & -2 \sum_{k=1}^m \gamma_k(t) (1 - \eta_k(t)) [\text{Tr}(L_k^* L_k \rho) - \text{Tr}(L_k^* \rho) \text{Tr}(L_k \rho)] \\ & - 2 \sum_{k=1}^{\tilde{m}} [\text{Tr}(C_k^*(t) C_k(t) \rho) - \text{Tr}(C_k^*(t) \rho) \text{Tr}(C_k(t) \rho)] \end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \text{Tr}(L_k^* L_k \rho) & \geq \text{Tr}(L_k^* \rho) \text{Tr}(L_k \rho), \\ \text{Tr}(C_k^*(t) C_k(t) \rho) & \geq \text{Tr}(C_k^*(t) \rho) \text{Tr}(C_k(t) \rho), \end{aligned}$$

where the equalities hold if and only if $\rho \in \bigcup_{i=0}^d \mathcal{I}(\mathcal{H}_i)$. By Lemma 4.2, the trajectory exits $\bigcup_{i=1}^d \mathcal{I}(\mathcal{H}_i)$ immediately. Then, the state becomes mixed immediately and remains mixed afterwards almost surely due to Lemma A.1. The proof is complete. $\square$

B Proof of Inequality (4)

Computing the infinitesimal generator $\mathcal{L}V(\rho)$ via Itô formula provides a more direct and compact approach than differentiating the Lyapunov function (3) explicitly, especially since the quantum state ρ is a complex matrix.

Under assumptions **A1** and **A-qsr**, and for $u = 0$, the cyclic property of the trace implies that, for any $i \in \{0, \dots, d\}$,

$$\begin{aligned} d\text{Tr}(\rho(t)\Pi_i) &= \sum_{k=1}^m \sqrt{\theta_k(t)} (\text{Tr}(\rho(t)L_k\Pi_i) + \text{Tr}(\rho(t)L_k^*\Pi_i) - \text{Tr}(L_k\rho(t) + \rho(t)L_k^*)\text{Tr}(\rho(t)\Pi_i))dW_k(t) \\ &= \sum_{k=1}^m \sqrt{\theta_k(t)} \mathcal{Z}_{k,i}(\rho(t))\text{Tr}(\rho(t)\Pi_i)dW_k(t), \end{aligned}$$

where $\mathcal{Z}_{k,i}(\rho) := 2\Re\{l_{k,i}\} - \text{Tr}(L_k\rho + \rho L_k^*)$. By Itô product rule, we have

$$\begin{aligned} d\text{Tr}(\rho(t)\Pi_i)\text{Tr}(\rho(t)\Pi_j) &= \text{Tr}(\rho(t)\Pi_i)\text{Tr}(\rho(t)\Pi_j) \sum_{k=1}^m \left[\theta_k(t) \mathcal{Z}_{k,i}(\rho(t)) \mathcal{Z}_{k,j}(\rho(t)) dt \right. \\ &\quad \left. + \sqrt{\theta_k(t)} (\mathcal{Z}_{k,i}(\rho(t)) + \mathcal{Z}_{k,j}(\rho(t))) dW_k(t) \right]. \end{aligned}$$

Due to the invariance of $\mathcal{S}_I$, V is twice continuously differentiable when restricted to $\mathcal{S}_I$. Applying Itô formula to $\sqrt{\text{Tr}(\rho(t)\Pi_i)\text{Tr}(\rho(t)\Pi_j)}$ and collecting the drift terms yields the following infinitesimal generator,

$$\begin{aligned} &\mathcal{L} \sqrt{\text{Tr}(\rho(t)\Pi_i)\text{Tr}(\rho(t)\Pi_j)} \\ &= -\frac{1}{8} \sqrt{\text{Tr}(\rho\Pi_i)\text{Tr}(\rho\Pi_j)} \sum_{k=1}^m \theta_k(t) (\mathcal{Z}_{k,i}(\rho) - \mathcal{Z}_{k,j}(\rho))^2 \\ &= -\frac{1}{2} \sqrt{\text{Tr}(\rho\Pi_i)\text{Tr}(\rho\Pi_j)} \sum_{k=1}^m \theta_k(t) (\Re\{l_{k,i}\} - \Re\{l_{k,j}\})^2 \\ &\leq -\frac{\mathfrak{E}_l}{2} \sqrt{\text{Tr}(\rho\Pi_i)\text{Tr}(\rho\Pi_j)}. \end{aligned}$$

Therefore, we conclude

$$\mathcal{L}V(\rho) \leq -\frac{\mathfrak{E}_l}{2} V(\rho).$$

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