

ON THE h -POLYNOMIALS OF CYCLOTOMIC STANDARD GRADED COMMUTATIVE ALGEBRAS

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ABSTRACT. We call a standard graded commutative $\mathbb{k}$ -algebra *cyclotomic* if its h -polynomial has all its roots on the unit circle in the complex plane. Complete intersections provide typical examples of cyclotomic algebras, since the h -polynomial of any standard graded complete intersection is a product of polynomials of the form $1 + t + \cdots + t^{m-1}$. We refer to such polynomials as being of *type CI*. A natural question is whether there exists a cyclotomic standard graded $\mathbb{k}$ -algebra whose h -polynomial is not of type CI. In this paper, we give a partial answer to this question. We show that the h -polynomial $h_R(t)$ of a cyclotomic standard graded $\mathbb{k}$ -algebra R is of type CI whenever $h_R(1) \in \{1, 4, 6\}$ or $h_R(1)$ is prime. On the other hand, if $n \geq 8$ and n is not prime, then there exists a cyclotomic standard graded $\mathbb{k}$ -algebra R whose h -polynomial $h_R(t)$ is not of type CI and satisfies $h_R(1) = n$.

1. INTRODUCTION

Throughout this paper, let $\mathbb{k}$ denote a field. We say that R is a *standard graded* $\mathbb{k}$ -algebra if $R = \bigoplus_{n \geq 0} R_n$ is a commutative graded $\mathbb{k}$ -algebra with $R_0 = \mathbb{k}$ and finitely generated in degree one. (A standard graded $\mathbb{k}$ -algebra is also called a *homogeneous* $\mathbb{k}$ -algebra in other literature.)

1.1. Background. The Hilbert series of a (not necessarily commutative) graded $\mathbb{k}$ -algebra is one of the most fundamental invariants in the study of graded $\mathbb{k}$ -algebras. It provides a concise analytic expression encoding the dimensions of the homogeneous components and thus captures the growth and structural features of the $\mathbb{k}$ -algebra as a $\mathbb{k}$ -vector space. It is a classical and well-known fact that if a standard graded $\mathbb{k}$ -algebra is Cohen-Macaulay, then the numerator polynomial of its Hilbert series, called the *h -polynomial*, has all positive coefficients. This fact is one of the cornerstones in graded commutative algebra. Furthermore, several algebraic properties can be effectively characterized in terms of the Hilbert series. A prominent theorem is Stanley's characterization of the Gorenstein property via the symmetry of the Hilbert series ([10]), which has inspired subsequent studies establishing necessary conditions for generalized notions of Gorensteinness, such as almost Gorensteinness ([4]), nearly Gorensteinness ([9]), and Artin-Schelter Gorensteinness ([5]).

It is also known that when a standard graded $\mathbb{k}$ -algebra is a complete intersection, its Hilbert series takes a particularly simple form. In fact, for a positive integer m , let

$$\Psi_m(t) = 1 + t + \cdots + t^{m-1}.$$

Then the h -polynomial of a standard graded complete intersection is of the form of a product of a finite collection of $\Psi_m(t)$'s (see Proposition 2.1). In addition, the Hilbert series also serves as an essential tool in investigating noncommutative graded complete intersections (see [6, 7]).

1.2. Cyclotomic standard graded $\mathbb{k}$ -algebras and polynomials of type CI. In recent years, increasing attention has been directed to the finer analytic behavior of the Hilbert series of standard graded $\mathbb{k}$ -algebras, particularly the roots of their h -polynomials. These roots encode subtle structural information and sometimes admit striking combinatorial interpretations

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(see, e.g., the survey [2] for combinatorial consequences of the real-rootedness of the numerator polynomials of generating series).

Motivated by this perspective, we focus on the following notion for standard graded $\mathbb{k}$ -algebras:

Definition 1.1 (cf. [1, 6]). A standard graded $\mathbb{k}$ -algebra R is called *cyclotomic* if all roots of its h -polynomial $h_R(t)$ lie on the unit circle in the complex plane.

Monic polynomials with integer coefficients whose roots all lie on the unit circle are sometimes called *Kronecker polynomials*. It is known that every such polynomial can be expressed as a product of cyclotomic polynomials. Note that the h -polynomials of cyclotomic $\mathbb{k}$ -algebras are *not necessarily cyclotomic polynomials themselves*, but rather products of cyclotomic polynomials. We say that a polynomial is of *type CI* if it can be written as a product of $\Psi_m(t)$'s. In particular, the h -polynomials of standard graded complete intersections are of type CI. However, the converse does not hold in general; see, for example, [10, Example 3.7]. Borzi and D'Ali showed that the converse does hold when R is Koszul [1, Theorem 3.9]. Moreover, it was proved that a cyclotomic polynomial $\Phi_m(t)$ coincides with $h_R(t)$ of some standard graded $\mathbb{k}$ -algebra R if and only if m is prime and R is a hypersurface [1, Theorem 4.3].

Despite these results, in general, one cannot expect the ring-theoretic structure of R to be determined by the cyclotomicity of its h -polynomial. This naturally leads to the following question:

Question 1.2. Which polynomials can occur as the h -polynomials of cyclotomic standard graded $\mathbb{k}$ -algebras, other than those of type CI?

For example, [1, Theorem 4.3] indicates that a single cyclotomic polynomial $\Phi_m(t)$ is rarely equal to the h -polynomial of a standard graded $\mathbb{k}$ -algebra.

1.3. Main Results. The aim of this paper is to make a contribution to this question. Our main result is stated as follows.

Theorem 1.3 (Main Result). (1) *The h -polynomial $h_R(t)$ of any cyclotomic standard graded $\mathbb{k}$ -algebra R is of type CI if $h_R(1) \in \{1, 4, 6\}$ or $h_R(1)$ is prime.*
 (2) *There exists a cyclotomic standard graded $\mathbb{k}$ -algebra R whose h -polynomial $h_R(t)$ is not of type CI and satisfies $h_R(1) = n$ if n is a non-prime number at least 8.*

Note that $h_R(1)$ is nothing but the multiplicity of R (see [3, Proposition 4.1.9]), which is one of the important invariants of graded modules.

1.4. Structure of the paper. We briefly describe the structure of this paper. In Section 2, we fix our notation (Hilbert series, h -polynomials, simplicial complexes, etc.), review fundamental notions (cyclotomic polynomials, Stanley-Reisner theory, and related topics), and recall classical results, namely Macaulay's theorem and the g -theorem, which will be used in the proof of Theorem 1.3. Sections 3 and 4 are devoted to the proofs of parts (1) and (2) of Theorem 1.3, respectively.

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2. PRELIMINARIES

In this section, we prepare several preliminary notions and results used throughout this paper.

2.1. Hilbert series of standard graded $\mathbb{k}$ -algebras. Let $R = \bigoplus_{n \geq 0} R_n$ be a standard graded (commutative) $\mathbb{k}$ -algebra. For $n \geq 0$, define $H(R, n) = \dim_{\mathbb{k}} R_n$. We call the function $H(R, -) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ the *Hilbert function* of R . Let $\text{Hilb}(R, t)$ denote the *Hilbert series* of R , that is, $\text{Hilb}(R, t) = \sum_{n \geq 0} H(R, n) t^n$. Then $\text{Hilb}(R, t)$ can be written in the form

$$\text{Hilb}(R, t) = \frac{h_R(t)}{(1-t)^d},$$

where d is the Krull dimension of R and $h_R(t)$ is a polynomial with integer coefficients satisfying $h_R(1) > 0$ (see, e.g., [3, Corollary 4.1.8]). We call the polynomial $h_R(t)$ appearing in the numerator the *h -polynomial* of R . It is known that $h_R(1)$ coincides with the multiplicity of R ([3, Proposition 4.1.9]).

Note that the Hilbert function of R can be recovered from its h -polynomial and Krull dimension. Indeed, if $h_R(t) = \sum_{i \geq 0} h_i t^i$ and d is the Krull dimension of R , then

$$H(R, n) = \sum_{i \geq 0} h_i \binom{d-1+n-i}{n-i}. \quad (2.1)$$

For more details on basic facts about Hilbert functions and Hilbert series, see [3, Section 4].

Regarding the h -polynomials of standard graded complete intersections, we have the following result.

Proposition 2.1 (cf. [10, Corollary 3.4]). *Let R be a standard graded $\mathbb{k}$ -algebra. If R is a complete intersection, then the h -polynomial of R is of type CI, that is, $h_R(t) = \prod_{i=1}^r \Psi_{m_i}(t)$ for some non-negative integer r and positive integers $m_1, \dots, m_r$.*

2.2. Cyclotomic polynomials. Next, we recall some fundamental facts about cyclotomic polynomials. For an introduction to cyclotomic polynomials, see, e.g., [8].

For a positive integer m , let $\Phi_m(t)$ denote the m -th cyclotomic polynomial, i.e.,

$$\Phi_m(t) = \prod_{\substack{1 \leq k \leq m \\ \gcd(k, m) = 1}} (t - e^{2\pi k \sqrt{-1}/m}).$$

We collect some well-known properties of cyclotomic polynomials:

(1) $\Phi_m(t)$ is an irreducible polynomial in $\mathbb{Z}[t]$ of degree $\varphi(m)$, where

$$\varphi(m) = |\{k \in \{1, \dots, m\} : \gcd(k, m) = 1\}|.$$

(2) $\Phi_m(t)$ is palindromic, i.e., $t^{\varphi(m)} \Phi_m(t^{-1}) = \Phi_m(t)$.

(3) For $m \geq 2$,

$$\Phi_m(1) = \begin{cases} p & \text{if } m \text{ is a power of a prime } p, \\ 1 & \text{otherwise.} \end{cases} \quad (2.2)$$

(4) For a prime power p^c ,

$$\Phi_{p^c}(t) = \Phi_p(t^{p^{c-1}}). \quad (2.3)$$

(5) For any m ,

$$\Psi_m(t) = \prod_{\substack{1 \leq d \leq m \\ d|m}} \Phi_d(t).$$

In particular, $\Psi_m(t) = \Phi_m(t)$ if m is prime.

(6) Any monic polynomial with integer coefficients whose roots all lie on the unit circle in the complex plane can be expressed as a product of cyclotomic polynomials.

Example 2.2. We list the cyclotomic polynomials of small degrees:

$$\begin{aligned} \deg 1 : \Phi_1(t) &= t - 1, \quad \Phi_2(t) = 1 + t; \\ \deg 2 : \Phi_3(t) &= 1 + t + t^2, \quad \Phi_4(t) = \Phi_2(t^2) = 1 + t^2, \quad \Phi_6(t) = 1 - t + t^2; \\ \deg 4 : \Phi_5(t) &= 1 + t + t^2 + t^3 + t^4, \quad \Phi_8(t) = \Phi_2(t^4) = 1 + t^4, \\ \Phi_{10}(t) &= 1 - t + t^2 - t^3 + t^4, \quad \Phi_{12}(t) = \Phi_6(t^2) = 1 - t^2 + t^4. \end{aligned}$$

2.3. Simplicial complexes and their h -polynomials. Here, we review simplicial complexes and their h -polynomials.

For a given non-empty finite set V , let 2^V denote the collection of all subsets of V (including the empty set). A subset $\Delta \subset 2^V$ is called a *simplicial complex* if it satisfies the following conditions:

- $\{v\} \in \Delta$ for every $v \in V$;
- if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$.

Note that $\emptyset \in \Delta$ for every simplicial complex Δ .

An element $F \in \Delta$ is called an i -face if $|F| = i + 1$. In particular, $\emptyset$ is the unique (-1) -face. We define the *dimension* of Δ by

$$\dim \Delta = \max\{|F| - 1 : F \in \Delta\}.$$

We then set

$$f_i(\Delta) = |\{F \in \Delta : F \text{ is an } i\text{-face}\}| \quad \text{and} \quad f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{d-1}(\Delta)),$$

where $d - 1 = \dim \Delta$. The vector $f(\Delta)$ is called the f -vector of Δ . The h -polynomial of Δ is defined by

$$\sum_{i=0}^d h_i(\Delta) t^{d-i} = \sum_{i=0}^d f_{i-1}(\Delta) (t-1)^{d-i}. \quad (2.4)$$

(Note that the h -polynomial carries the same information as the f -vector via (2.4).)

For a finite set V , let $\Delta_V = 2^V$. Clearly, Δ_V is a simplicial complex of dimension $|V| - 1$, which is usually called the *simplex* on V . We define the *boundary complex* of a simplex by $\partial\Delta_V = \Delta_V \setminus \{V\}$. We write Δ_d (resp. $\partial\Delta_d$) for Δ_V (resp. $\partial\Delta_V$) when $|V| = d$. Then, as an easy exercise, one obtains

$$h_{\Delta_d}(t) = 1 \quad \text{and} \quad h_{\partial\Delta_d}(t) = \Psi_d(t). \quad (2.5)$$

2.4. Stanley-Reisner rings of simplicial complexes. We now provide a brief introduction to the theory of Stanley-Reisner rings (cf. [3, Section 5]). This theory plays a crucial role in analyzing the Hilbert series of standard graded $\mathbb{k}$ -algebras.

For a simplicial complex Δ on $V = \{1, \dots, d\}$, define

$$\mathbb{k}[\Delta] := \mathbb{k}[x_1, \dots, x_d] / I_\Delta, \quad \text{where} \quad I_\Delta = \left(\prod_{i \in F} x_i : F \subset V, F \notin \Delta \right).$$

The standard graded $\mathbb{k}$ -algebra $\mathbb{k}[\Delta]$ (resp. the ideal I_Δ) is called the *Stanley-Reisner ring* (resp. the *Stanley-Reisner ideal*) of Δ . The following facts are well known:

- The Krull dimension of $\mathbb{k}[\Delta]$ is equal to $\dim \Delta + 1$.
- The h -polynomial of $\mathbb{k}[\Delta]$ coincides with the h -polynomial of Δ .
- The Cohen-Macaulayness and Gorensteinness of $\mathbb{k}[\Delta]$ can be characterized in terms of the reduced homology of Δ .

From (2.5), we obtain the following equalities, which will be used several times in the sequel:

$$\text{Hilb}(\mathbb{k}[\Delta_d], t) = \frac{1}{(1-t)^d} \quad \text{and} \quad \text{Hilb}(\mathbb{k}[\partial\Delta_d], t) = \frac{\Psi_d(t)}{(1-t)^{d-1}}.$$

2.5. Macaulay's theorem and g -theorem. Finally, let us review Macaulay's theorem and the g -theorem. To state them, we first recall the notion of a binomial sum expansion.

Given a positive integer a , there exists a unique expression

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(d')}{d'}, \quad (2.6)$$

for any fixed positive integer d , where d' is some positive integer satisfying $d \geq d'$ and $k(d) > k(d-1) > \cdots > k(d') \geq d'$ (see [3, Lemma 4.2.6]). We call the expression (2.6) the *binomial sum expression* of a with respect to d . We then define

$$a^{(d)} = \binom{k(d)+1}{d+1} + \binom{k(d-1)+1}{d} + \cdots + \binom{k(d')+1}{d'+1}.$$

Theorem 2.3 (Macaulay's theorem, cf. [3, Theorem 4.2.10] and [11, Chapter II, Theorem 2.2]). *Let $h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ be a function with $h(0) = 1$. Then there exists a standard graded $\mathbb{k}$ -algebra R satisfying $H(R, n) = h(n)$ for all $n \geq 0$ if and only if $h(n+1) \leq h(n)^{\langle n \rangle}$ for every $n > 0$.*

Example 2.4. We illustrate how to use Macaulay's theorem. Let $h(t) = \sum_{i=0}^s h_i t^i$ be a polynomial such that $h_0 = h_1 = \cdots = h_{a-1} = 1$ and $h_a > 1$ for some $1 < a \leq s$. Then $h(t)$ cannot be the h -polynomial of any standard graded $\mathbb{k}$ -algebra. Indeed, suppose there exists such an algebra R , and let d denote its Krull dimension. Then

$$\begin{aligned} H(R, a-1) &= \sum_{i \geq 0} h_i \binom{d-1+a-1-i}{a-1-i} = \binom{d+a-2}{a-1} + \binom{d+a-3}{a-2} + \cdots + \binom{d-1}{0} \\ &= \binom{d+a-1}{a-1}, \\ \text{while } H(R, a) &= \sum_{i \geq 0} h_i \binom{d-1+a-i}{a-i} = \binom{d+a-1}{a} + \cdots + \binom{d}{1} + h_a \binom{d-1}{0} \\ &= \binom{d+a}{a} + h_a - 1. \end{aligned}$$

Since $H(R, a-1)^{\langle a-1 \rangle} = \binom{d+a}{a}$, we obtain

$$H(R, a-1)^{\langle a-1 \rangle} - H(R, a) = 1 - h_a < 0,$$

a contradiction to Macaulay's theorem.

Example 2.5. We next describe another polynomial that cannot occur as the h -polynomial of any standard graded $\mathbb{k}$ -algebra. Consider $h(t) = 1 + 2t + t^2 + 3t^5$, and suppose that there exists a standard graded $\mathbb{k}$ -algebra R of Krull dimension d with this h -polynomial. Then, from (2.1), we have

$$\begin{aligned} H(R, 4) &= \binom{d+3}{4} + 2\binom{d+2}{3} + \binom{d+1}{2} \quad \text{and} \\ H(R, 5) &= \binom{d+4}{5} + 2\binom{d+3}{4} + \binom{d+2}{3} + 3. \end{aligned}$$

Without loss of generality, we may assume that $d \geq 2$. By a direct computation, the binomial sum expansion of $H(R, 4)$ is

$$\binom{d+3}{4} + 2\binom{d+2}{3} + \binom{d+1}{2} = \binom{d+4}{4} + \binom{d+1}{3} + \binom{d-1}{2} + \binom{d-2}{1}.$$

Hence,

$$H(R, 4)^{\langle 4 \rangle} = \binom{d+5}{5} + \binom{d+2}{4} + \binom{d}{3} + \binom{d-1}{2}.$$

In this case, one can show that

$$H(R, 4)^{\langle 4 \rangle} - H(R, 5) = -1,$$

which contradicts Macaulay's theorem. However, establishing this fact is far from trivial.

As illustrated by this example, the computation of binomial expansions is generally rather involved, and hence it is not easy to apply Macaulay's theorem to determine whether a given polynomial can be realized as an h -polynomial.

For a simplicial polytope P with vertex set V , the collection

$$\{F \subset V : \text{conv}(F) \text{ is a proper face of } P\}$$

forms a simplicial complex on V , called the *boundary complex* of P . Note that the boundary complex of a simplex of dimension $d-1$ is precisely $\partial\Delta_d$.

A characterization of the h -polynomials of boundary complexes of simplicial polytopes is known as the *g-theorem*, one of the most celebrated results in Stanley-Reisner theory.

Theorem 2.6 (*g-theorem*, cf. [11, Chapter III, Theorem 1.1]). *Let $(h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$. Then there exists a simplicial polytope P of dimension d whose boundary complex has the h -polynomial $\sum_{i=0}^d h_i t^i$ if and only if the following two conditions are satisfied:*

- (1) $h_i = h_{d-i}$ for $0 \leq i \leq \lfloor d/2 \rfloor$, and
- (2) $h_0 = 1$ and $0 \leq h_{i+1} - h_i \leq (h_i - h_{i-1})^{\langle i \rangle}$ for $1 \leq i \leq \lfloor d/2 \rfloor - 1$.

3. PROOF OF THEOREM 1.3 (1)

This section is devoted to proving Theorem 1.3 (1).

Lemma 3.1. *Let Δ be a simplicial complex of dimension $D-1$ and let $f_{D-1}(\Delta) = a$. Then*

$$f_{D-2}(\Delta) \geq \begin{cases} aD - \binom{a}{2} & \text{if } a \leq D+1, \\ \binom{D+1}{2} & \text{if } a > D+1. \end{cases}$$

Proof. If $a = 1$, since there should be at least D faces of dimension $(D-2)$, we get $f_{D-2}(\Delta) \geq D$.

Let $a \leq D+1$ and let $F_1, \dots, F_a$ be the $(D-1)$ -faces of Δ . Let

$$\Delta' = \langle F_1, \dots, F_{a-1} \rangle = \{G \in \Delta : G \subset F_i \text{ for some } i\}.$$

Since $f_{D-2}(\Delta' \cap F_a) \leq a-1$, we see by induction on a that

$$f_{D-2}(\Delta) = f_{D-2}(\Delta') + D - f_{D-2}(\Delta' \cap F_a) \geq (a-1)D - \binom{a-1}{2} + D - (a-1) = aD - \binom{a}{2}.$$

If $a > D+1$, let Δ'' be a subcomplex of Δ consisting of some $(D-1)$ -faces $F_1, \dots, F_{D+1} \in \Delta$. Then $f_{D-2}(\Delta'') \geq (D+1)D - \binom{D+1}{2} = \binom{D+1}{2}$ by the first case. Thus,

$$f_{D-2}(\Delta) \geq f_{D-2}(\Delta'') \geq \binom{D+1}{2},$$

as required. □

The following proposition plays the essential role for the proof of Theorem 1.3 (1).

Proposition 3.2. *Let R be a standard graded $\mathbb{k}$ -algebra whose h -polynomial is palindromic. Then we have $\deg(h_R(t)) \leq h_R(1) - 1$.*

Proof. We employ a similar technique used in the proof of [1, Lemma 4.6].

Let $h_R(t) = \sum_{i=0}^s h_i t^i$, where s is the degree of $h_R(t)$. Since $h_R(t)$ is palindromic, i.e., $h_i = h_{s-i}$ for $i = 0, 1, \dots, s$. We set $a = h_R(1)$.

Let $R \cong S/I$, where S is a polynomial ring and I is an ideal of S . Since the Hilbert series does not change when passing to the initial ideal of I , we can assume without loss of generality that I is a monomial ideal. Moreover, since polarization preserves $h_R(t)$, we can further assume that I is a squarefree monomial ideal. Hence, I is the Stanley-Reisner ideal of some simplicial complex Δ . Let $\dim \Delta = D - 1$. Then $s \leq D$ and $h_\Delta(t) = h_R(t)$, that is, $h_i(\Delta) = h_i$ for every $0 \leq i \leq s$. By (2.4), we see that

$$f_{D-1}(\Delta) = h_R(1) = a, \text{ and}$$

$$f_{D-2}(\Delta) = \sum_{i=0}^s (D-i)h_i(\Delta) = (D-s) \sum_{i=0}^s h_i(\Delta) + \frac{s}{2} \sum_{i=0}^s h_i(\Delta) = \left(D - \frac{s}{2}\right) a.$$

Suppose that $a \leq D + 1$. Then, by Lemma 3.1, we have

$$f_{D-2}(\Delta) = \left(D - \frac{s}{2}\right) a \geq aD - \binom{a}{2}$$

and hence $s \leq a - 1$. This is the desired conclusion.

Suppose that $a > D + 1$. Then, by Lemma 3.1, we have

$$f_{D-2}(\Delta) = \left(D - \frac{s}{2}\right) a \geq \binom{D+1}{2}.$$

This implies $2Da - D^2 - D \geq sa$, and hence, $a(a-s) \geq (D-a)^2 + D > 0$. Therefore, $a > s$, i.e., $s \leq a - 1$. $\square$

By using Proposition 3.2, we can show the following two propositions, which give a proof of Theorem 1.3 (1).

Proposition 3.3. *Let R be cyclotomic and assume that $h_R(1)$ is 1 or a prime p . Then R is regular or a hypersurface defined by a homogeneous polynomial of degree p , respectively. In particular, $h_R(t)$ is of type CI.*

Proof. In the case of $h_R(1) = 1$, Proposition 3.2 directly implies $h_R(t) = 1$, i.e., R is regular.

If $h_R(1) = p$ is prime, then $h_R(t)$ is divisible by $\Phi_p(t^i)$ for some $i \geq 1$. By Proposition 3.2, we see that $\deg(h_R(t)) \leq h_R(1) - 1$. Hence, $i = 1$ and $h_R(t)$ must be $\Phi_p(t)$ itself. Then [1, Theorem 4.3] implies that R is a hypersurface. $\square$

Proposition 3.4. *Let $f(t)$ be a monic polynomial with integer coefficients whose roots all lie on the unit circle. Assume that $\deg(f(t)) \leq f(1) - 1$.*

(1) *If $f(1) = 4$, then $f(t)$ is of type CI.*

(2) *If $f(1) = 6$ and $f(t) = h_R(t)$ for some standard graded $\mathbb{k}$ -algebra R , then $f(t)$ is of type CI.*

Proof. (1) Consider all possible polynomials $f(t)$ with $f(1) = 4$ and $\deg(f(t)) \leq 3$ constructed as a product of cyclotomic polynomials. (See Example 2.2 for cyclotomic polynomials of small degrees.) Then such polynomials are $\Phi_2(t)^2 = (1+t)^2$ and $\Phi_2(t)\Phi_4(t) = 1+t+t^2+t^3$.

(2) Similarly, all possible polynomials $f(t)$ with $f(1) = 6$ and $\deg(f(t)) \leq 5$ are

$$\Phi_2(t)\Phi_3(t) = 1+2t+2t^2+t^3, \quad \Phi_2(t)\Phi_3(t)\Phi_6(t) = \Psi_6(t), \quad \text{and} \quad \Phi_3(t)\Phi_4(t) = 1+t+2t^2+t^3+t^4.$$

The polynomials $\Phi_2(t)\Phi_3(t)$ and $\Psi_6(t)$ are of type CI, while we know that $1+t+2t^2+t^3+t^4$ cannot be the h -polynomial of any standard graded $\mathbb{k}$ -algebra (see Example 2.4). $\square$

4. PROOF OF THEOREM 1.3 (2)

The goal of this section is to give a proof of Theorem 1.3 (2). Specifically, for any non-prime n with $n \geq 8$, we construct a cyclotomic standard graded $\mathbb{k}$ -algebra R such that $h_R(t)$ is not of type CI and $h_R(1) = n$. Such $\mathbb{k}$ -algebras are realized as Stanley-Reisner rings of certain simplicial complexes.

We recall methods for constructing a simplicial complex from two simplicial complexes.

- Consider two simplicial complexes Δ_1 and Δ_2 on V_1 and V_2 , respectively, with $V_1 \cap V_2 \neq \emptyset$. Then $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta' = \Delta_1 \cap \Delta_2$ are simplicial complexes on $V_1 \cup V_2$ and $V_1 \cap V_2$, respectively. Thus, by the inclusion-exclusion principle, we see that

$$\text{Hilb}(\mathbb{k}[\Delta], t) = \text{Hilb}(\mathbb{k}[\Delta_1], t) + \text{Hilb}(\mathbb{k}[\Delta_2], t) - \text{Hilb}(\mathbb{k}[\Delta'], t). \quad (4.1)$$

- Consider two simplicial complexes Δ_1 and Δ_2 on V_1 and V_2 , respectively, and assume that $V_1 \cap V_2 = \emptyset$. Then the *join* of Δ_1 and Δ_2 , denoted by $\Delta_1 * \Delta_2$, is the simplicial complex on $V_1 \cup V_2$ defined by

$$\{F_1 \cup F_2 : F_1 \in \Delta_1, F_2 \in \Delta_2\}.$$

Note that $\dim(\Delta_1 * \cdots * \Delta_\ell) = \dim \Delta_1 + \cdots + \dim \Delta_\ell + \ell - 1$. It is well known that

$$h_{\Delta_1 * \Delta_2}(t) = h_{\Delta_1}(t) \cdot h_{\Delta_2}(t).$$

The following examples correspond to the cases $n = 8, 9, 12$, respectively.

Example 4.1. (1) (The case $h_R(1) = 8$) Let R be the Stanley-Reisner ring of the following simplicial complex Δ on $V \cup \{v'\}$, where $|V| = 8$ and $v' \notin V$. We construct it as the union of $\partial\Delta_{|V|}$ (of dimension 6) and $\partial\Delta_2 * \partial\Delta_3 * \underbrace{\Delta_1 * \{v'\}}_{\cong \Delta_2}$ (of dimension 4). Then their intersection is

$\partial\Delta_2 * \partial\Delta_3 * \Delta_1$ (of dimension 3). Note that $\partial\Delta_2 * \partial\Delta_3 * \Delta_1$ can be found inside of $\partial\Delta_V = 2^V \setminus V$. Hence, it follows from (4.1) that

$$\begin{aligned} \text{Hilb}(\mathbb{k}[\Delta], t) &= \frac{h_{\partial\Delta_8}(t)}{(1-t)^7} + \frac{h_{\partial\Delta_2 * \partial\Delta_3 * \Delta_2}(t)}{(1-t)^5} - \frac{h_{\partial\Delta_2 * \partial\Delta_3 * \Delta_1}(t)}{(1-t)^4} \\ &= \frac{\Psi_8(t)}{(1-t)^7} + \frac{\Psi_2(t)\Psi_3(t)}{(1-t)^5} - \frac{\Psi_2(t)\Psi_3(t)}{(1-t)^4} \\ &= \frac{1+t+\cdots+t^7 + (1-(1-t))(1-t)^2(1+t)(1+t+t^2)}{(1-t)^7} \\ &= \frac{1+2t+t^2+t^5+2t^6+t^7}{(1-t)^7} = \frac{\Phi_2(t)^3\Phi_{10}(t)}{(1-t)^7}. \end{aligned}$$

Explicitly,

$$\mathbb{k}[\Delta] \cong \mathbb{k}[x_1, \dots, x_9] / (x_1x_2x_3x_4x_5x_6x_7x_8, x_1x_2x_9, x_3x_4x_5x_9, x_7x_9, x_8x_9).$$

(2) (The case $h_R(1) = 9$) Let R be the Stanley-Reisner ring of Δ which can be constructed as the union of $\partial\Delta_9$ (of dimension 7) and $\partial\Delta_2 * \partial\Delta_2 * \partial\Delta_3 * \Delta_1 * \{v'\}$ (of dimension 5). Then their intersection is $\partial\Delta_2 * \partial\Delta_2 * \partial\Delta_3 * \Delta_1$ (of dimension 4). Thus,

$$\begin{aligned} h_R(t) &= 1+t+\cdots+t^8 + (1-t)^2(1+t)^2(1+t+t^2) - (1-t)^3(1+t)^2(1+t+t^2) \\ &= 1+2t+2t^2-t^4+2t^6+2t^7+t^8 \\ &= \Phi_3(t)^2\Phi_{12}(t). \end{aligned}$$

Explicitly,

$$\mathbb{k}[\Delta] \cong \mathbb{k}[x_1, \dots, x_{10}] / (x_1x_2x_3x_4x_5x_6x_7x_8x_9, x_1x_2x_{10}, x_3x_4x_{10}, x_5x_6x_7x_{10}, x_9x_{10}).$$

(3) (The case $h_R(1) = 12$) Let R be the Stanley-Reisner ring of Δ which can be constructed as the union of $\partial\Delta_{12}$ (of dimension 10) and $\partial\Delta_2 * \partial\Delta_2 * \partial\Delta_3 * \partial\Delta_4 * \Delta_1 * \{v'\}$ (of dimension 8). Then their intersection is $\partial\Delta_2 * \partial\Delta_2 * \partial\Delta_3 * \partial\Delta_4 * \Delta_1$ (of dimension 7). Thus,

$$\begin{aligned} h_R(t) &= 1 + t + \cdots + t^{11} + (1-t)^2 \Psi_2(t)^2 \Psi_3(t) \Psi_4(t) - (1-t)^3 \Psi_2(t)^2 \Psi_3(t) \Psi_4(t) \\ &= 1 + 2t + 3t^2 + 2t^3 - 2t^5 - 2t^6 + 2t^8 + 3t^9 + 2t^{10} + t^{11} \\ &= \Psi_3(t) \Psi_4(t) \Phi_{18}(t). \end{aligned}$$

Explicitly,

$$\mathbb{k}[\Delta] \cong \mathbb{k}[x_1, \dots, x_{13}] / (x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_{10} x_{11} x_{12}, x_1 x_2 x_{13}, x_3 x_4 x_{13}, x_5 x_6 x_7 x_{13}, x_8 x_9 x_{10} x_{11} x_{13}).$$

We prove the following theorem, which implies Theorem 1.3 (2).

Theorem 4.2. (1) *Let $q \geq 5$ be an odd integer. Then there exists a cyclotomic standard graded $\mathbb{k}$ -algebra R such that $h_R(t)$ is not of type CI with $h_R(1) = 2q$.*

(2) *Let $q \geq 3$ be an odd integer. Then there exists a cyclotomic standard graded $\mathbb{k}$ -algebra R such that $h_R(t)$ is not of type CI with $h_R(1) = 3q$.*

(3) *Let $5 \leq p \leq q$ be odd integers. Then there exists a cyclotomic standard graded $\mathbb{k}$ -algebra R such that $h_R(t)$ is not of type CI with $h_R(1) = pq$.*

We postpone the proof of this theorem. So far, assume Theorem 4.2 holds. Then we can prove the desired result. In fact, let R be a cyclotomic standard graded $\mathbb{k}$ -algebra whose h -polynomial is not of type CI. Then, for any positive integer a , the standard graded $\mathbb{k}$ -algebra $R' = R[x]/(x^a)$ has the h -polynomial $h_{R'}(t)$ which is not of type CI and $h_{R'}(1) = a \cdot h_R(1)$. (In fact, we have $h_{R'}(t) = h_R(t) \Psi_a(t)$.)

Let $n \geq 8$ be a non-prime integer.

- If n has a prime divisor at least 5, then n is a multiple of some primes p, q with $pq \geq 10$. Hence, the existence is guaranteed by Theorems 4.2 together with the above discussion.
- Let $n = 2^i 3^j \geq 8$. Then either $(i, j) \geq (3, 0)$, $(i, j) \geq (2, 1)$, or $(i, j) \geq (0, 2)$ holds. Example 4.1 guarantees the existence of the cases $(i, j) = (3, 0)$, $(2, 1)$ and $(0, 2)$.

This completes the proof of Theorem 1.3 (2).

In preparation for the proofs of Theorem 4.2 (1) and (2), we state the following lemma.

Lemma 4.3. (1) *Let $q \geq 5$ be an odd integer. Then the following holds:*

$$\Psi_2(t) \Psi_q(t) \Phi_6(t^{(q-1)/2}) = \Psi_{2q} + t(1-t)^2 \Psi_{(q-3)/2}(t) \Psi_{(q-1)/2}(t) \Psi_q(t).$$

(2) *Let $q \geq 3$ be an odd integer. Then the following holds:*

$$\Psi_3(t) \Psi_q(t) \Phi_6(t^{q-1}) = \Psi_{3q} + t(1-t)^2 \Psi_2(t) \Psi_{q-2}(t) \Psi_{q-1}(t) \Psi_q(t).$$

Proof. (1) We can directly check the following:

$$\begin{aligned} (1-t)^2 (\Psi_2(t) \Psi_q(t) \Phi_6(t^{(q-1)/2}) - \Psi_{2q}(t)) &= (1-t^2)(1-t^q)(1-t^{\frac{q-1}{2}} + t^{q-1}) - (1-t)(1-t^{2q}) \\ &= t(1-t)^4 \Psi_{(q-3)/2}(t) \Psi_{(q-1)/2}(t) \Psi_q(t). \end{aligned}$$

This yields the required equation.

(2) We can directly check the following:

$$\begin{aligned} (1-t)^2 (\Psi_3(t) \Psi_q(t) \Phi_6(t^{q-1}) - \Psi_{3q}(t)) &= (1-t^3)(1-t^q)(1-t^{q-1} + t^{2q-2}) - (1-t)(1-t^{3q}) \\ &= t(1-t)^4 \Psi_2(t) \Psi_{q-2}(t) \Psi_{q-1}(t) \Psi_q(t). \end{aligned}$$

This yields the required equation. □

Proofs of Theorem 4.2 (1) and (2). (1) Let Δ be the simplicial complex constructed as the union of $\partial\Delta_{2q}$ (of dimension $2q-2$) and $\partial\Delta_{(q-3)/2} * \partial\Delta_{(q-1)/2} * \partial\Delta_q * \Delta_1 * \{v'\}$ (of dimension $2q-4$), whose intersection is $\partial\Delta_{(q-3)/2} * \partial\Delta_{(q-1)/2} * \partial\Delta_q * \Delta_1$ (of dimension $2q-5$). (We see that this can be found inside of $\partial\Delta_{2q}$ by considering the number of vertices.) It then follows from (4.1) that

$$\begin{aligned} \text{Hilb}(\mathbb{k}[\Delta], t) &= \frac{h_{\partial\Delta_{2q}}(t)}{(1-t)^{2q-1}} + \frac{h_{\partial\Delta_{(q-3)/2} * \partial\Delta_{(q-1)/2} * \partial\Delta_q}(t)}{(1-t)^{2q-3}} - \frac{h_{\partial\Delta_{(q-3)/2} * \partial\Delta_{(q-1)/2} * \partial\Delta_q}(t)}{(1-t)^{2q-4}} \\ &= \frac{\Psi_{2q}(t) + ((1-t)^2 - (1-t)^3)\Psi_{(q-3)/2}(t)\Psi_{(q-1)/2}(t)\Psi_q(t)}{(1-t)^{2q-1}} \\ &= \frac{\Psi_2(t)\Psi_q(t)\Phi_6(t^{(q-1)/2})}{(1-t)^{2q-1}} \quad (\text{by Lemma 4.3 (1)}). \end{aligned}$$

It can be seen that $\mathbb{k}[\Delta]$ is cyclotomic, but $h_{\mathbb{k}[\Delta]}(t) = \Psi_2(t)\Psi_q(t)\Phi_6(t^{(q-1)/2})$ is not of type CI.

(2) Let Δ be the simplicial complex constructed as the union of $\partial\Delta_{3q}$ (of dimension $3q-2$) and $\partial\Delta_2 * \partial\Delta_{q-2} * \partial\Delta_{q-1} * \partial\Delta_q * \Delta_1 * \{v'\}$ (of dimension $3q-4$), whose intersection is $\partial\Delta_2 * \partial\Delta_{q-2} * \partial\Delta_{q-1} * \partial\Delta_q * \Delta_1$ (of dimension $3q-5$). Then

$$\begin{aligned} \text{Hilb}(\mathbb{k}[\Delta], t) &= \frac{h_{\partial\Delta_{3q}}(t)}{(1-t)^{3q-1}} + \frac{h_{\partial\Delta_2 * \partial\Delta_{q-2} * \partial\Delta_{q-1} * \partial\Delta_q}(t)}{(1-t)^{3q-3}} - \frac{h_{\partial\Delta_2 * \partial\Delta_{q-2} * \partial\Delta_{q-1} * \partial\Delta_q}(t)}{(1-t)^{3q-4}} \\ &= \frac{\Psi_{3q}(t) + ((1-t)^2 - (1-t)^3)\Psi_2(t)\Psi_{q-2}(t)\Psi_{q-1}(t)\Psi_q(t)}{(1-t)^{3q-1}} \\ &= \frac{\Psi_3(t)\Psi_q(t)\Phi_6(t^{q-1})}{(1-t)^{2q-1}} \quad (\text{by Lemma 4.3 (2)}). \end{aligned}$$

Hence we get the result. $\square$

For the proof of Theorem 4.2 (3), we introduce several notations and equalities. Given odd integers p, q with $3 \leq p \leq q$, let

$$g_{p,q}(t) = \Psi_p(t)\Psi_q(t)\Phi_6(t^{(p-1)(q-1)/2}) - \Psi_{pq}(t) \quad \text{and} \quad f_{p,q}(t) = \frac{g_{p,q}(t)}{t(1-t)^2\Psi_{q-1}(t)}.$$

Note that $f_{3,q}(t) = \Psi_2(t)\Psi_{q-2}(t)\Psi_q(t)$ by Lemma 4.3 (2).

Lemma 4.4. *For any odd integers p, q with $3 \leq p \leq q$, $f_{p,q}(t)$ is a palindromic polynomial in t of degree $(p-1)q-3$.*

Proof. First, we verify the polynomiality of $f_{p,q}(t)$. Namely, it suffices to show that $g_{p,q}(t)$ (resp. its derivative $g'_{p,q}(t)$) has zeros at 0, 1, and $e^{2\pi j\sqrt{-1}/(q-1)}$ for all $j = 1, \dots, q-2$ (resp. at 1), which can be verified straightforwardly.

Next, we confirm the degree of $f_{p,q}(t)$, but it is also straightforward. In fact, we have $\deg(g_{p,q}(t)) = pq-2$ (note that the subtraction causes cancellation of the leading terms) and $\deg(t(1-t)^2\Psi_{q-1}(t)) = 3+q-2 = q+1$. Hence, $\deg(f_{p,q}(t)) = pq-2-(q+1) = (p-1)q-3$.

Finally, it can be directly checked that $f_{p,q}(t)$ satisfies the relation $t^{(p-1)q-3}f_{p,q}(t^{-1}) = f_{p,q}(t)$. This shows the palindromicity of $f_{p,q}(t)$. $\square$

Example 4.5. For small p, q , let us compute $f_{p,q}(t)$ explicitly:

$$\begin{aligned} f_{5,5}(t) &= 1 + 3t + 6t^2 + 10t^3 + 14t^4 + 18t^5 + 22t^6 + 25t^7 + 26t^8 + 26t^9 + \dots + 3t^{16} + t^{17}, \\ f_{5,7}(t) &= 1 + 3t + 6t^2 + 10t^3 + 14t^4 + 18t^5 + 22t^6 + 26t^7 + 30t^8 + 34t^9 + 37t^{10} + 39t^{11} + 40t^{12} \\ &\quad + 40t^{13} + \dots + 3t^{24} + t^{25}, \\ f_{7,7}(t) &= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 27t^6 + 33t^7 + 39t^8 + 45t^9 + 51t^{10} + 57t^{11} + 62t^{12} \\ &\quad + 67t^{13} + 72t^{14} + 77t^{15} + 82t^{16} + 86t^{17} + 88t^{18} + 89t^{19} + 89t^{20} + \dots + 3t^{38} + t^{39}. \end{aligned}$$

Then we see that

$$(1-t)f_{5,7}(t) = 1 + 2t + 3t^2 + 4t^3 + 4(t^4 + \dots + t^9) + 3t^{10} + 2t^{11} + t^{12} - t^{14} - \dots - t^{26}.$$

Namely, the first parts of the coefficients $c_0, c_1, \dots, c_{12}$ of $(1-t)f_{5,7}(t)$ are

$$c_i = \begin{cases} i+1 & \text{if } i = 0, 1, \dots, 3, \\ 4 & \text{if } i \in \{4, \dots, 9\}, \\ 3 & \text{if } i \in \{10\}, \\ 13-i & \text{if } i = 11, 12. \end{cases}$$

This description matches (4.2) for $(p, q) = (5, 7)$ appearing in the proof of Proposition 4.7.

Moreover, we have

$$(1-t)f_{7,7}(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 6(t^6 + \dots + t^{11}) + 5(t^{12} + \dots + t^{16}) \\ + 4t^{17} + 2t^{18} + t^{19} - t^{21} - \dots - t^{40}.$$

Namely, the first parts of the coefficients $c_0, c_1, \dots, c_{19}$ of $(1-t)f_{7,7}(t)$ are

$$c_i = \begin{cases} i+1 & \text{if } i = 0, 1, \dots, 5, \\ 6 & \text{if } i \in \{6, \dots, 11\}, \\ 5 & \text{if } i \in \{12, \dots, 16\}, \\ 4 & \text{if } i \in \{17\}, \\ 20-i & \text{if } i = 18, 19, \end{cases}$$

This description matches (4.3) for $p = q = 7$ appearing in the proof of Proposition 4.7.

Remark 4.6. In fact, one can verify that the following equality holds:

$$f_{p,q}(t) = \Psi_{p-1}(t) \left(\Psi_{(p-1)(q-1)/2}(t) \right)^2 - t^{p+q-2} \Psi_{(p-3)q/2-p+1}(t) \Psi_{(p-3)q/2-p+2}(t) \\ - t^{2(q-1)+p-1} \Psi_{(p-5)(q-1)/2-(p-1)}(t) \Psi_{(p-5)(q-1)/2-1}(t) \\ - \sum_{k=3}^{(p-3)/2} t^{k(q-1)+p-1} \Psi_{((p-1)/2-k)(q-1)-(p+2-k)}(t) \Psi_{((p-1)/2-k)(q-1)-k+2}(t) \\ - t^{(p-1)(q-1)/2-1} (1+t) \Psi_{(p+3)/2}(t) \Psi_{(p-5)/2}(t)$$

for $7 \leq p \leq q$ and

$$f_{5,q}(t) = \Psi_4(t) \left(\Psi_{2(q-1)}(t) \right)^2 - t^{q+3} \Psi_{q-4}(t) \Psi_{q-3}(t) - t^{2q-3} \Psi_4(t)$$

for $q \geq 5$. We omit to give a proof of this equality since we do not use this.

The following constitutes the crucial part of the proof of Theorem 4.2 (3), and hence of Theorem 1.3 (2).

Proposition 4.7. *Let $5 \leq p \leq q$ be odd integers. Then there exists a simplicial complex (in particular, a boundary complex of a simplicial polytope) of dimension $(p-1)q-5$ with $(p-1)q-1$ vertices whose h -polynomial is equal to $f_{p,q}(t)$.*

Proof. Let $f_{p,q}(t) = \sum_{i=0}^s h_i t^i$, where $s = p(q-1) - 3$, and let $(1-t)f_{p,q}(t) = \sum_{i \geq 0} c_i t^i$. Then $c_i = h_i - h_{i-1}$ for $i = 1, \dots, (p-1)q-3$. Note that $h_1 = 3$. This implies that the number of vertices $f_0(\Delta)$ of a desired simplicial polytope (a simplicial complex) is $(p-1)q-5+1+h_1 = (p-1)q-1$ (see (2.4)).

By a steady computation of the coefficients of the polynomial $(1-t)f_{p,q}(t)$, which coincides with $\frac{g_{p,q}(t)}{t(1-t)\Psi_{q-1}(t)}$, we see that the first half of the coefficients of $(1-t)f_{p,q}(t)$, i.e., $c_0 =$

$1, c_1, \dots, c_{(p-1)q/2-2}$ are of the following form: if $5 \leq p < q$, then

$$c_i = \begin{cases} i+1 & \text{if } i = 0, 1, \dots, p-2, \\ p-1-k & \text{if } i \in I_k \text{ for } k = 0, 1, \dots, (p-3)/2, \\ (p-1)q/2-1-i & \text{if } i = (p-1)(q-1)/2-1, \dots, (p-1)q/2-2, \end{cases} \quad (4.2)$$

where

$$I_k = \{k(q-1) + p-1, \dots, k(q-1) + p-1 + (q-2)\} \text{ for } k = 0, 1, \dots, (p-5)/2, \text{ and} \\ I_{(p-3)/2} = \{(p-3)(q-1)/2 + p-1, \dots, \underbrace{(p-3)(q-1)/2 + p-1 + q-p-2}_{=(p-1)(q-1)/2-2}\};$$

and if $5 \leq p = q$, then

$$c_i = \begin{cases} i+1 & \text{if } i = 0, 1, \dots, p-2, \\ p-1-k & \text{if } i \in I_k \text{ for } k = 0, 1, \dots, (p-3)/2, \\ (p-1)q/2-1-i & \text{if } i = (p-1)(q-1)/2, \dots, (p-1)q/2-2, \end{cases} \quad (4.3)$$

where

$$I_k = \{k(q-1) + p-1, \dots, k(q-1) + p-1 + q-2\} \text{ for } k = 0, 1, \dots, (p-7)/2, \\ I_{(p-5)/2} = \{(p-5)(q-1)/2 + p-1, \dots, \underbrace{(p-5)(q-1)/2 + p-1 + q-3}_{=(p-1)(q-1)/2-2}\}, \text{ and} \\ I_{(p-3)/2} = \{(p-1)(q-1)/2-1\}.$$

See Example 4.5 for illustrations.

Note that $f_{p,q}(t)$ is a palindromic polynomial whose constant term is 1 (see Lemma 4.4). Hence, according to Theorem 2.6, in order for there to exist a simplicial polytope whose h -polynomial is $f_{p,q}(t)$, it suffices to verify the following inequalities:

$$0 \leq c_{i+1} \leq c_i^{(i)} \text{ for each } i = 1, \dots, (p-1)q/2-2.$$

The first inequality $0 \leq c_{i+1}$ holds for each i . Regarding the second one, we can get a binomial sum expression as follows:

$$c_i = \begin{cases} \binom{i+1}{i} & \text{if } i = 0, 1, \dots, p-2, \\ \binom{i}{i} + \binom{i-1}{i-1} + \dots + \binom{i-c_i+1}{i-c_i+1} & \text{if } i = p-1, \dots, (p-1)q/2-2, \end{cases}$$

where we notice that $c_i \leq i$ if $i \geq p-1$. Hence,

$$c_i^{(i)} = \begin{cases} \binom{i+2}{i+1} = i+2 = c_{i+1} & \text{if } i = 0, 1, \dots, p-3, \\ \binom{p}{p-1} = p = c_{p-1} + 1 & \text{if } i = p-2, \\ \binom{i+1}{i+1} + \binom{i}{i} + \dots + \binom{i-c_i+2}{i-c_i+2} = c_i & \text{if } i = p-1, \dots, (p-1)q/2-2. \end{cases}$$

Since $c_i \geq c_{i+1}$ holds for each i with $p-2 \leq i \leq (p-1)q/2-2$ (see (4.2) and (4.3)), we conclude the desired condition. $\square$

Now, we are ready to give a proof of Theorem 4.2 (3).

Proof of Theorem 4.2 (3). By definition of $f_{p,q}(t)$, we see the following equality:

$$\frac{\Psi_p(t)\Psi_q(t)\Phi_6(t^{(p-1)(q-1)/2})}{(1-t)^{pq-1}} = \frac{\Psi_{pq}(t)}{(1-t)^{pq-1}} + \frac{\Psi_{q-1}(t)f_{p,q}(t)}{(1-t)^{pq-3}} - \frac{\Psi_{q-1}(t)f_{p,q}(t)}{(1-t)^{pq-4}}.$$

Since $h_{\partial\Delta_{pq}}(t) = \Psi_{pq}(t)$ and $\partial\Delta_{pq}$ has dimension $pq-2$, by the same argument as before, it is enough to show the existence of a subcomplex of $\partial\Delta_{pq}$ of dimension $pq-5$ whose h -polynomial is equal to $\Psi_{q-1}(t)f_{p,q}(t)$. We can construct such a subcomplex as $\partial\Delta_{q-1} * \Delta' * \Delta_1$ inside $\partial\Delta_{pq}$,

where Δ' is a simplicial complex guaranteed by Proposition 4.7, because the number of vertices is equal to

$$\underbrace{q-1}_{=\partial\Delta_{q-1}} + \underbrace{(p-1)q-1+1}_{=\Delta'} = pq-1,$$

which is less than the number of vertices of $\partial\Delta_{pq}$. Moreover, the dimension is equal to $q-2+(p-1)q-5+0+2=pq-5$.

Therefore, we complete the proof. $\square$

What we have shown in this section is that, if $p=2$ and $q \geq 5$ is odd, or if $p, q \geq 3$ are both odd, then there exists a standard graded $\mathbb{k}$ -algebra (indeed, a simplicial complex) whose h -polynomial is equal to $\Psi_p(t)\Psi_q(t)\Phi_6(t^{(p-1)(q-1)/2})$. It is natural to consider a “more direct” application of Macaulay’s theorem to verify the existence of a standard graded $\mathbb{k}$ -algebra having this h -polynomial. However, such an approach seems to be highly nontrivial. Indeed, one must translate the coefficients of the polynomial $\Psi_p(t)\Psi_q(t)\Phi_6(t^{(p-1)(q-1)/2})$ into the language of Hilbert functions via (2.1), and the resulting expressions quickly become intricate. For instance, let $p=2$ and $q \geq 5$ be odd. Then

$$\begin{aligned} \Psi_2(t)\Psi_q(t)\Phi_6(t^{(q-1)/2}) &= 1 + 2(t + \dots + t^{(q-3)/2}) + t^{(q-1)/2} + t^{q-1} + t^q + t^{(3q-1)/2} \\ &\quad + 2(t^{(3q+1)/2} + \dots + t^{2q-2}) + t^{2q-1}. \end{aligned}$$

Let d be a sufficiently large integer, and define

$$f(n) = \sum_{i \geq 0} h_i \binom{d-1+n-i}{n-i}$$

with (2.1) in mind, where h_i denotes the coefficient of t^i in $\Psi_2(t)\Psi_q(t)\Phi_6(t^{(q-1)/2})$. To apply Macaulay’s theorem, we need to verify that

$$f(n+1) \leq f(n)^{\langle n \rangle} \quad \text{for all } n \geq 1. \quad (4.4)$$

Although these inequalities are guaranteed by our result (the proof of Theorem 4.2), we now attempt a direct verification. For example, we have

$$\begin{aligned} f(q-2) &= \binom{d-1+q-2}{q-2} + 2 \left(\binom{d-1+q-3}{q-3} + \dots + \binom{d-1+(q-5)/2}{(q-5)/2} \right) \\ &\quad + \binom{d-1+(q-3)/2}{(q-3)/2}, \\ f(q-1) &= \binom{d-1+q-1}{q-1} + 2 \left(\binom{d-1+q-2}{q-2} + \dots + \binom{d-1+(q-3)/2}{(q-3)/2} \right) \\ &\quad + \binom{d-1+(q-1)/2}{(q-1)/2} + \binom{d-1}{0}, \end{aligned}$$

and one needs to estimate $f(q-2)^{\langle q-2 \rangle} - f(q-1)$, which requires explicit and complicated computations of the binomial sum expression $f(q-2)^{\langle q-2 \rangle}$. Consequently, a direct verification of (4.4) seems quite difficult. Our approach, which consists of finding a tractable polynomial and applying the g -theorem, is technical but significant, as it allows us to avoid elaborate computations of binomial sum expressions.

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