The Sobolev space $W_2^{1/2}$: Simultaneous improvement of functions by a homeomorphism of the circle

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Abstract. It is known that for every continuous real-valued function f on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ there exists a change of variable, i.e., a self-homeomorphism h of $\mathbb{T}$, such that the superposition $f \circ h$ is in the Sobolev space $W_2^{1/2}(\mathbb{T})$. We obtain new results on simultaneous improvement of functions by a single change of variable in relation to the space $W_2^{1/2}(\mathbb{T})$. The main result is as follows: there does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in \mathrm{Lip}_{1/2}(\mathbb{T})$. Here $\mathrm{Lip}_{1/2}(\mathbb{T})$ is the class of all functions on $\mathbb{T}$ satisfying the Lipschitz condition of order $1/2$.

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1. Introduction. Given an integrable function f on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, consider its Fourier expansion:

$$f(t) \sim \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikt}, \qquad t \in \mathbb{T}.$$

Recall that the Sobolev space $W_2^{1/2}(\mathbb{T})$ is the space of all (integrable) functions f satisfying

$$\|f\|_{W_2^{1/2}(\mathbb{T})} = \left(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2|k|\right)^{1/2} < \infty. \tag{1}$$

In what follows by $C(\mathbb{T})$ we denote the space of all continuous complex-valued functions on $\mathbb{T}$ (with the usual sup-norm). Given a modulus of continuity ω, i.e., a nondecreasing continuous function on $[0, +\infty)$ with $\omega(0) = 0$, by $\mathrm{Lip}_\omega(\mathbb{T})$ we denote the class of all functions $f \in C(\mathbb{T})$ satisfying $\omega(f, \delta) = O(\omega(\delta))$, $\delta \to +0$, where

$$\omega(f, \delta) = \sup_{|t_1-t_2|\le\delta} |f(t_1) - f(t_2)|, \qquad \delta \ge 0.$$

For $0 < \alpha \le 1$ we just write Lip_α instead of $\mathrm{Lip}_{\delta^\alpha}$. By $V(\mathbb{T})$ we denote the space of all functions of bounded variation on $\mathbb{T}$.

It is well known that one can improve certain properties of a continuous function related to its Fourier series by an appropriate change of variable, i.e., by a self-homeomorphism of $\mathbb{T}$. The first result on this subject is the Bohr–Pál theorem stating that for every real-valued f in $C(\mathbb{T})$ there exists a self-homeomorphism h of $\mathbb{T}$ such that the superposition $f \circ h$ belongs to the space $U(\mathbb{T})$ of functions with uniformly convergent Fourier series. In fact, the original proof of Bohr and Pál yields

$$f \circ h = g + \psi, \tag{2}$$

where $g \in W_2^{1/2} \cap C(\mathbb{T})$ and $\psi \in V \cap C(\mathbb{T})$. This implies $f \circ h \in U(\mathbb{T})$, since $W_2^{1/2} \cap C(\mathbb{T})$ and $V \cap C(\mathbb{T})$ are subsets of $U(\mathbb{T})$.

Subsequently, for certain function spaces the question of whether every continuous function can be transformed by a suitable homeomorphic change of variable into a function that belongs to a given space was studied by various authors. Some of these studies concern the possibility of simultaneous improvement of several functions by means of a single change of variable. For a survey on the subject see [3], [11]; later results were obtained in [2], [5–9].[1] It is worth noting that the proof by Bohr and Pál is based on the Riemann's theorem on conformal mappings, while investigations that followed mostly involve real analysis methods.

We note, in particular, that an immediate consequence of [12, Corollary 1] (see also [5, Sec. 3], [2], [8]) is that the term ψ in (2) can be omitted, i.e., the following refined version of the Bohr–Pál theorem holds: for every real-valued $f \in C(\mathbb{T})$ there exists a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$.

The first result on simultaneous improvement was obtained in [4] in relation to the space $U(\mathbb{T})$: if K is a compact set in $C(\mathbb{T})$, then there exists a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in U(\mathbb{T})$ for every $f \in K$. In other words, given a modulus of continuity ω, there is a change of variable h such that $f \circ h \in U(\mathbb{T})$ for every $f \in \mathrm{Lip}_\omega(\mathbb{T})$. This result naturally led to a question if it is possible to attain the condition $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in K$. A negative answer was obtained in [5, Theorem 4]; as it turned out, given a real-valued $u \in C(\mathbb{T})$, the property that for every real-valued $v \in C(\mathbb{T})$ there is a homeomorphism h such that both $u \circ h$ and $v \circ h$ are in $W_2^{1/2}(\mathbb{T})$ is equivalent to the boundness of variation of u.

[1] In [5] $C(\mathbb{T})$ stands for the space of all *real-valued* continuous functions on $\mathbb{T}$.

Thus, in general, there is no single change of variable which will bring two real-valued functions in $C(\mathbb{T})$ into $W_2^{1/2}(\mathbb{T})$. This amounts to the existence of a complex-valued $f \in C(\mathbb{T})$ such that $f \circ h \notin W_2^{1/2}(\mathbb{T})$ whenever h is a self-homeomorphism of $\mathbb{T}$.

In the present paper we obtain new results on simultaneous improvement of functions by a single change of variable in relation to the space $W_2^{1/2}(\mathbb{T})$.

For an arbitrary function $F \in C(\mathbb{T})$ consider the family $S_F = \{F(\cdot + \theta) : \theta \in \mathbb{T}\}$ of translations of F and the family $K_F = \{F * \lambda : \lambda \in P(\mathbb{T})\}$ of convolutions of F with probability measures ($P(\mathbb{T})$ is the set of all probability measures on $\mathbb{T}$). Both S_F and K_F are compact sets in $C(\mathbb{T})$ and $S_F \subseteq K_F$. Clearly, if $\lambda \in P(\mathbb{T})$, then $|\widehat{F * \lambda}(k)| \leq |\widehat{F}(k)|$, $k \in \mathbb{Z}$. Thus, if F is in $W_2^{1/2}(\mathbb{T})$, then S_F and K_F are just subsets of $W_2^{1/2}(\mathbb{T})$. In Section 2 we show that if F is not in $W_2^{1/2}(\mathbb{T})$, then, under certain additional assumption on F, there does not exist a homeomorphism which will bring all functions in S_F into $W_2^{1/2}(\mathbb{T})$. In Section 3, without any additional assumptions on F, we obtain a similar result for K_F.

In Section 4 we obtain the main result of the paper: there does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in \mathrm{Lip}_{1/2}(\mathbb{T})$. This is a direct consequence of the result on translations (as well as of the result on convolutions) and the well-known fact that $\mathrm{Lip}_{1/2}(\mathbb{T}) \nsubseteq W_2^{1/2}(\mathbb{T})$. We note that in [7] it was shown that if $\alpha < 1/2$, then there exist two real-valued functions in $\mathrm{Lip}_\alpha(\mathbb{T})$ such that there is no single change of variable which will bring them into $W_2^{1/2}(\mathbb{T})$. The author does not know if such a pair of functions can be found in $\mathrm{Lip}_{1/2}(\mathbb{T})$. It is also worth noting that for $\alpha > 1/2$ functions in $\mathrm{Lip}_\alpha(\mathbb{T})$ do not require an improvement, since in this case $\mathrm{Lip}_\alpha(\mathbb{T}) \subseteq W_2^{1/2}(\mathbb{T})$. The imbedding follows from the well-known equivalence of the seminorm $\|\cdot\|_{W_2^{1/2}(\mathbb{T})}$ (see (1)) and the seminorm $|||\cdot|||_{W_2^{1/2}(\mathbb{T})}$ defined by

$$|||f|||_{W_2^{1/2}(\mathbb{T})} = \left(\int\int_{[0,2\pi]^2} \frac{|f(x) - f(y)|^2}{|x-y|^2} dxdy \right)^{1/2}. \tag{3}$$

The concluding Section 5 contains certain remarks, open problems and the shortest, known to the author, proof of the refined version of the Bohr–Pál theorem.

2. Translations of a continuous function of analytic type. Let $F \in C(\mathbb{T})$. For each $\theta \in \mathbb{T}$ define the function F_θ by $F_\theta(t) = F(t+\theta)$. Consider the family S_F of translations of F:

$$S_F = \{F_\theta : \theta \in \mathbb{T}\}.$$

(Clearly, S_F is a compact set in $C(\mathbb{T})$.)

By $C^+(\mathbb{T})$ we denote the class of all continuous functions of analytic type on $\mathbb{T}$, i.e., of those $F \in C(\mathbb{T})$ which satisfy $\widehat{F}(k) = 0$ for all $k < 0$.

It is obvious that if $F \in W_2^{1/2}(\mathbb{T})$, then $S_F \subseteq W_2^{1/2}(\mathbb{T})$. On the other hand, the following theorem holds.

Theorem 1. *Let $F \in C^+(\mathbb{T})$. Suppose that $F \notin W_2^{1/2}(\mathbb{T})$. Then there does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in S_F$.*

To prove the theorem we will need Lemma 1 below. It has a technical character and will also be used in the next section. Before we proceed to the lemma, note that the bilinear form

$$B(x,y) = \frac{1}{2\pi}\int_{\mathbb{T}} x(t)dy(t),$$

defined for $x \in C(\mathbb{T})$, $y \in V(\mathbb{T})$, is invariant with respect to self-homeomorphisms h of $\mathbb{T}$, that is, $B(x\circ h, y\circ h) = B(x,y)$. Note also that if x and y are in $W_2^{1/2}(\mathbb{T})$, then $\sum_{k\in\mathbb{Z}} |\widehat{x}(-k)\, ik\, \widehat{y}(k)| < \infty$.

Lemma 1. *Let $x \in C(\mathbb{T})$, $y \in V(\mathbb{T})$ and $x, y \in W_2^{1/2}(\mathbb{T})$. Then*

$$\text{(i)} \qquad \frac{1}{2\pi}\int_{\mathbb{T}} x(t)dy(t) = \sum_{k\in\mathbb{Z}} \widehat{x}(-k)\, ik\, \widehat{y}(k);$$

$$\text{(ii)} \qquad \left|\frac{1}{2\pi}\int_{\mathbb{T}} x(t)dy(t)\right| \le \|x\|_{W_2^{1/2}(\mathbb{T})}\|y\|_{W_2^{1/2}(\mathbb{T})}.$$

Proof. Clearly, (ii) follows from (i). To verify (i), observe that

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikt}dy(t) = -\frac{1}{2\pi}\int_{\mathbb{T}} y(t)de^{-ikt} = ik\widehat{y}(k), \qquad k \in \mathbb{Z},$$

so (i) holds in the case when x is a trigonometric polynomial. In the general case it suffices to approximate x by the Fejér sums $\sigma_N(x)$:

$$\sigma_N(x)(t) = \sum_{|k|\le N} \widehat{x}(k)\left(1 - \frac{|k|}{N}\right) e^{ikt}, \quad N = 1, 2, \dots. \tag{4}$$

Indeed, we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} \sigma_N(x)(t) dy(t) = \sum_{|k|\le N} \widehat{x}(-k)\left(1 - \frac{|k|}{N}\right) ik\, \widehat{y}(k). \tag{5}$$

Let $N \to \infty$. The sequence $\sigma_N(x)$, $N = 1, 2, \dots$, converges uniformly to x, so

$$\frac{1}{2\pi}\int_{\mathbb{T}} \sigma_N(x)(t) dy(t) \to \frac{1}{2\pi}\int_{\mathbb{T}} x(t) dy(t)$$

and it remains to notice that the right-hand side in (5) tends to the right-hand side in (i). The lemma is proved.

Proof of Theorem 1. We have $F \in C(\mathbb{T})$ and

$$F(t) \sim \sum_{n\ge 0} c_n e^{int},$$

where, by the assumption,

$$\sum_{n\ge 0} |c_n|^2 n = \infty. \tag{6}$$

Suppose that, contrary to the assertion of the theorem, there exists a self-homeomorphism h of $\mathbb{T}$ such that $F_\theta \circ h \in W_2^{1/2}(\mathbb{T})$ for every $\theta \in \mathbb{T}$. Consider the sets $T_m \subseteq \mathbb{T}$, $m = 1, 2, \dots$, defined by

$$T_m = \{\theta \in \mathbb{T} : \|F_\theta \circ h\|_{W_2^{1/2}(\mathbb{T})} \le m\}.$$

Note that the sets T_m, $m = 1, 2, \dots$, are closed (we leave the proof to the reader) and at the same time

$$\mathbb{T} = \bigcup_{m=1}^{\infty} T_m,$$

whence, using the Baire category theorem, we see that at least one of the sets T_m, say T_{m_0}, contains an interval $I \subseteq \mathbb{T}$. So $\|F_\theta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq m_0$ for all $\theta \in I$. Replacing, if necessarily, h with $h + \gamma_I$, where γ_I is the center of I, one can assume that $I = (-\delta_0, \delta_0)$, where $0 < \delta_0 \leq \pi$. Thus,

$$\|F_\theta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq m_0 \quad \text{for all} \quad \theta \in (-\delta_0, \delta_0). \tag{7}$$

For $0 < \delta < \delta_0$ we let

$$F^\delta(t) = \frac{1}{2\delta} \int_{-\delta}^{\delta} F(t+\theta) d\theta.$$

It follows from (7) that $F^\delta \circ h \in W_2^{1/2}(\mathbb{T})$ and

$$\|F^\delta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq c \quad \text{for all} \quad \delta \in (0, \delta_0), \tag{8}$$

where $c > 0$ does not depend on δ. To verify this, it suffices to use the equivalence of the seminorms $\|\cdot\|_{W_2^{1/2}(\mathbb{T})}$ and $|||\cdot|||_{W_2^{1/2}(\mathbb{T})}$ (we leave details to the reader).

It is clear that F^δ is continuous and of bounded variation. So $F^\delta \circ h$ is also continuous and of bounded variation. Using Lemma 1 (see (ii)) and (8), we have that for all $\delta \in (0, \delta_0)$

$$\left| \frac{1}{2\pi} \int_{\mathbb{T}} \overline{F^\delta(t)} dF^\delta(t) \right| = \left| \frac{1}{2\pi} \int_{\mathbb{T}} \overline{F^\delta \circ h(t)} d(F^\delta \circ h)(t) \right| \leq$$

$$\leq \|\overline{F^\delta \circ h}\|_{W_2^{1/2}(\mathbb{T})} \|F^\delta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq c^2, \tag{9}$$

where the bar stands for the complex conjugation.

Let

$$\chi_\delta = \frac{1}{2\delta} 1_{(-\delta,\,\delta)},$$

where $1_{(-\delta,\delta)}$ is the indicator function of the interval $(-\delta, \delta)$. Obviously, $F^\delta = F * \chi_\delta$ and

$$\widehat{\chi_\delta}(k) = \frac{1}{2\pi} \frac{\sin k\delta}{k\delta}, \qquad k \neq 0.$$

Applying Lemma 1 (see (i)), we obtain

$$\frac{1}{2\pi} \int_{\mathbb{T}} \overline{F^\delta(t)} dF^\delta(t) = \sum_{k \in \mathbb{Z}} \overline{\widehat{F^\delta}(k)} ik \widehat{F^\delta}(k) = i \sum_{k \in \mathbb{Z}} |\widehat{F^\delta}(k)|^2 k =$$

$$= i \sum_{k\in\mathbb{Z}} |\widehat{F}(k)|^2 |\widehat{\chi_\delta}(k)|^2 k = i \sum_{n\geq 1} |c_n|^2 \left(\frac{1}{2\pi}\frac{\sin n\delta}{n\delta}\right)^2 n.$$

Thus (see (9)), for all $\delta \in (0, \delta_0)$

$$\sum_{n\geq 1} |c_n|^2 \left(\frac{1}{2\pi}\frac{\sin n\delta}{n\delta}\right)^2 n \leq c^2.$$

Choose a positive integer N. We see that for all $\delta \in (0, \delta_0)$

$$\sum_{n=1}^{N} |c_n|^2 \left(\frac{\sin n\delta}{n\delta}\right)^2 n \leq (2\pi c)^2.$$

Let $\delta \to +0$. We obtain

$$\sum_{n=1}^{N} |c_n|^2 n \leq (2\pi c)^2,$$

which contradicts (6), since N was chosen arbitrarily. The theorem is proved.

3. Convolutions of a continuous function with probability measures. Let $F \in C(\mathbb{T})$. Let $P(\mathbb{T})$ be the set of all probability measures on $\mathbb{T}$. Consider the family K_F of convolutions of F with the measures in $P(\mathbb{T})$:

$$K_F = \{F * \lambda : \lambda \in P(\mathbb{T})\}.$$

Clearly, K_F is a compact set in $C(\mathbb{T})$. It is also clear that (as we mentioned in the introduction) if $F \in W_2^{1/2}(\mathbb{T})$, then $K_F \subseteq W_2^{1/2}(\mathbb{T})$. On the other hand, the following theorem holds.

Theorem 2. *Let $F \in C(\mathbb{T})$. Suppose that $F \notin W_2^{1/2}(\mathbb{T})$. Then there does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in K_F$.*

To prove Theorem 2 we will need two lemmas. As in the previous section, for an $F \in C(\mathbb{T})$ and a $\theta \in \mathbb{T}$, we use F_θ to denote the function defined by $F_\theta(t) = F(t + \theta)$.

Lemma 2. *Let $x \in C(\mathbb{T})$. Let φ be a continuous mapping of $\mathbb{T}$ into itself. Then for each $\nu \in \mathbb{Z}$ the function $\theta \to \widehat{x_\theta \circ \varphi}(\nu)$ is continuous on $\mathbb{T}$ and*

$$\frac{1}{2\pi}\int_{\mathbb{T}} |\widehat{x_\theta \circ \varphi}(\nu)|^2 d\theta = \sum_{k\in\mathbb{Z}} |\widehat{x}(k)|^2 |\widehat{e^{ik\varphi}}(\nu)|^2. \tag{10}$$

Proof. The continuity of $\widehat{x_\theta \circ \varphi}(\nu)$ is obvious. To prove (10), assume first that x is a trigonometric polynomial. Then (there is a finite number of nonzero terms in the sum below)

$$x_\theta(t) = x(t+\theta) = \sum_{k\in\mathbb{Z}} \widehat{x}(k) e^{ik(t+\theta)},$$

whence

$$x_\theta \circ \varphi(t) = \sum_{k\in\mathbb{Z}} \widehat{x}(k) e^{ik\varphi(t)} e^{ik\theta}.$$

So

$$\widehat{x_\theta \circ \varphi}(\nu) = \sum_{k\in\mathbb{Z}} \widehat{x}(k) \widehat{e^{ik\varphi}}(\nu) e^{ik\theta},$$

which implies (10). In the general case, consider the Fejér sums $\sigma_N(x), N = 1, 2, \dots,$ (see (4)). We have

$$\frac{1}{2\pi}\int_{\mathbb{T}} |(\sigma_N(x)_\theta \circ \varphi)^\wedge(\nu)|^2 d\theta = \sum_{|k|\le N} |\widehat{x}(k)|^2 \left(1 - \frac{|k|}{N}\right)^2 |\widehat{e^{ik\varphi}}(\nu)|^2.$$

It remains to tend N to ∞. The lemma is proved.

Lemma 3. *Let h be a self-homeomorphism of $\mathbb{T}$. Let $k \in \mathbb{Z}$. Suppose that $e^{ikh} \in W_2^{1/2}(\mathbb{T})$. Then $\|e^{ikh}\|_{W_2^{1/2}(\mathbb{T})} \ge |k|^{1/2}$.*

Proof. Applying Lemma 1 (see (ii)), we obtain

$$|k| = \left|\frac{1}{2\pi}\int_{\mathbb{T}} e^{ikx} de^{-ikx}\right| = \left|\frac{1}{2\pi}\int_{\mathbb{T}} e^{ikh(t)} de^{-ikh(t)}\right| \le \|e^{ikh}\|^2_{W_2^{1/2}(\mathbb{T})}.$$

The lemma is proved.

Note by the way, that for the identity homeomorphism $h_0(t) \equiv t$ it is obvious that $\|e^{ikh_0}\|_{W_2^{1/2}(\mathbb{T})} = |k|^{1/2}$.

Proof of Theorem 2. Assume that h is a self-homeomorphism of $\mathbb{T}$ such that $(F * \lambda) \circ h \in W_2^{1/2}(\mathbb{T})$ for every measure $\lambda \in P(\mathbb{T})$. Let $M(\mathbb{T})$ be the Banach space of all measures μ on $\mathbb{T}$ with the usual norm $\|\mu\|_{M(\mathbb{T})}$ equal to the variation of μ. Each $\mu \in M(\mathbb{T})$ is a linear combination of two probability measures, so we see that $(F * \mu) \circ h \in W_2^{1/2}(\mathbb{T})$ for all $\mu \in M(\mathbb{T})$.

Clearly, $W_2^{1/2}(\mathbb{T})$ is a Banach space with respect to the norm

$$\| \cdot \|^{\circ}_{W_2^{1/2}(\mathbb{T})} = \left(\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2(|k|+1) \right)^{1/2}.$$

Consider the operator $Q : M(\mathbb{T}) \to W_2^{1/2}(\mathbb{T})$ defined by

$$Q\mu = (F * \mu) \circ h.$$

Using standard argument, namely the closed graph theorem, we see that Q is a bounded operator (we leave to the reader to verify that all assumptions of the closed graph theorem hold). Thus, for every $\mu \in M(\mathbb{T})$ we have

$$\|(F * \mu) \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq \|(F * \mu) \circ h\|^{\circ}_{W_2^{1/2}(\mathbb{T})} \leq c\|\mu\|_{M(\mathbb{T})},$$

where $c > 0$ is independent of μ. Applying this estimate to $\mu = \delta_\theta$, where θ is a point in $\mathbb{T}$ and δ_θ is the unit mass at θ, and taking into account that $F_\theta = F * \delta_\theta$, we obtain

$$\|(F_\theta \circ h\|_{W_2^{1/2}(\mathbb{T})} \leq c \quad \text{for all} \quad \theta \in \mathbb{T}. \tag{11}$$

Note now that for each $k \in \mathbb{Z}$ one can find a positive integer $m(k)$ so that

$$\sum_{|\nu|\leq m(k)} |\widehat{e^{ikh}}(\nu)|^2|\nu| \geq |k|/2.$$

Indeed, if $e^{ikh} \in W_2^{1/2}(\mathbb{T})$, the existence of $m(k)$ follows from Lemma 3, while if $e^{ikh} \notin W_2^{1/2}(\mathbb{T})$, the existence of $m(k)$ is obvious.

Let N be an arbitrary positive integer. We define $M(N)$ by

$$M(N) = \max_{|k|\leq N} m(k).$$

Then, for all $k \in \mathbb{Z}$ with $|k| \le N$ we have

$$\sum_{|\nu| \le M(N)} |\widehat{e^{ikh}}(\nu)|^2 |\nu| \ge |k|/2. \tag{12}$$

At the same time, Lemma 2 implies

$$\frac{1}{2\pi} \int_{\mathbb{T}} |\widehat{F_\theta \circ h}(\nu)|^2 d\theta \ge \sum_{|k| \le N} |\widehat{F}(k)|^2 |\widehat{e^{ikh}}(\nu)|^2,$$

whence, by multiplying by $|\nu|$ and summing over $|\nu| \le M(N)$, we obtain

$$\frac{1}{2\pi} \int_{\mathbb{T}} \Big(\sum_{|\nu| \le M(N)} |\widehat{F_\theta \circ h}(\nu)|^2 |\nu| \Big) d\theta \ge \sum_{|k| \le N} \Big(|\widehat{F}(k)|^2 \sum_{|\nu| \le M(N)} |\widehat{e^{ikh}}(\nu)|^2 |\nu| \Big).$$

Taking (11) and (12) into account, we see that

$$c^2 \ge \sum_{|k| \le N} |\widehat{F}(k)|^2 (|k|/2).$$

Since N was chosen arbitrarily, this yields $F \in W_2^{1/2}(\mathbb{T})$, which contradicts the assumption of the theorem. The theorem is proved.

4. The class $\mathrm{Lip}_{1/2}(\mathbb{T})$. Here we obtain the main result of the paper. Recall that if $\alpha > 1/2$, then $\mathrm{Lip}_\alpha(\mathbb{T}) \subseteq W_2^{1/2}(\mathbb{T})$ (see (3)). On the other hand, the function

$$F(t) = \sum_{n \ge 0} 2^{-n/2} e^{i 2^n t}$$

is in $\mathrm{Lip}_{1/2}(\mathbb{T})$ (see, e.g., [1, Ch. XI, Sec. 6]) but is not in $W_2^{1/2}(\mathbb{T})$. Obviously, for the corresponding family of translations S_F we have $S_F \subseteq \mathrm{Lip}_{1/2}(\mathbb{T})$ (the same is true for K_F). Thus, an immediate consequence of Theorem 1 (as well as that of Theorem 2) is the following Theorem 3.

Theorem 3. *There does not exist a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in W_2^{1/2}(\mathbb{T})$ for every $f \in \mathrm{Lip}_{1/2}(\mathbb{T})$.*

5. Remarks and open problems. 1. Given two real-valued functions u and v in $\mathrm{Lip}_{1/2}(\mathbb{T})$, does there exist a change of variable h such that $u \circ h$ and $v \circ h$ are in $W_2^{1/2}(\mathbb{T})$? (This question was already mentioned in the introduction.)

2. Given a real-valued $F \in C(\mathbb{T})$ and a $\theta \in \mathbb{T}$, $\theta \neq 0$, does there exist an h such that $F \circ h$ and $F_\theta \circ h$ are in $W_2^{1/2}(\mathbb{T})$? What if $F \in \mathrm{Lip}_{1/2}(\mathbb{T})$?

3. It is unclear if one can replace $C^+(\mathbb{T})$ with $C(\mathbb{T})$ in Theorem 1.

4. For $s > 0$ the Sobolev space $W_2^s(\mathbb{T})$ is defined as the space of integrable functions f on $\mathbb{T}$ with $\sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2|k|^{2s} < \infty$. It was shown in [5, Corollary 3] that if K is a compact set in $C(\mathbb{T})$, then there exists a self-homeomorphism h of $\mathbb{T}$ such that $f \circ h \in \bigcap_{s<1/2} W_2^s(\mathbb{T})$ for every $f \in K$.

5. There exists a real-valued $f \in C(\mathbb{T})$ such that $f \circ h \notin \bigcup_{s>1/2} W_2^s(\mathbb{T})$ whenever h is a self-homeomorphism of $\mathbb{T}$. This is a simple consequence of the inclusion $\bigcup_{s>1/2} W_2^s \cap C(\mathbb{T}) \subseteq A(\mathbb{T})$, where $A(\mathbb{T})$ is the Wiener algebra of absolutely convergent Fourier series, and the result obtained in [10] (see also [11, Theorem 3.2]): there exists a real-valued $f \in C(\mathbb{T})$ such that $f \circ h \notin A(\mathbb{T})$ for every self-homeomorphism h.

6. We now provide a very short proof of the refined version of the Bohr–Pál theorem. Implicitly this proof is contained in the proof of Theorem 4 in [5]. The refinement is achieved through a slight modification of the original argument. Suppose that $f \in C(\mathbb{T})$ is real-valued. Without loss of generality we assume that $f(t) > 0$ for all $t \in \mathbb{T}$. Consider the curve γ in the complex plane $\mathbb{C}$ given by $\gamma(t) = f(t)e^{it}$, $t \in [0, 2\pi]$. This is a closed continuous curve without self-intersections. By Ω we denote the interior domain bounded by γ. Consider a conformal mapping G of the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ onto Ω. As is well known, G extends to a homeomorphism of the closure $\overline{D}$ of D onto the closure $\overline{\Omega}$ of Ω and, being thus extended, homeomorphically maps the circle $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ onto the boundary $\partial\Omega$ of Ω. We retain the notation G for the extension and consider the function g defined by $g(t) = G(e^{it})$. Clearly, there is a self-homeomorphism h of the segment $[0, 2\pi]$ such that $g(t) = \gamma(h(t))$, $t \in [0, 2\pi]$. Note that $\pi \sum_{n\geq 0} |\widehat{g}(n)|^2 n$ is the area of Ω. So $\gamma \circ h \in W_2^{1/2}(\mathbb{T})$. At the same time, we have $f \circ h = |\gamma \circ h|$. It remains to observe that for an arbitrary function F the condition $F \in W_2^{1/2}(\mathbb{T})$ implies $|F| \in W_2^{1/2}(\mathbb{T})$, which is clear from the definition of $||| \cdot |||_{W_2^{1/2}(\mathbb{T})}$ (see (3)). Thus, $f \circ h \in W_2^{1/2}(\mathbb{T})$.

A stronger result, based on the Riemann's theorem on conformal mappings, was obtained in [2], see also [8].

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