

NON-JUMPS OF HYPERGRAPHS

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ABSTRACT. A density $\alpha \in [0, 1)$ is a jump for r if there is some $c > 0$ such that there does not exist a family of r -uniform hypergraphs $\mathcal{F}$ with Turán density $\pi(\mathcal{F})$ in $(\alpha, \alpha + c)$. Erdős conjectured [3] that all $\alpha \in [0, 1)$ are jumps for any r . This was disproven by Frankl and Rödl when they provided examples of non-jumps [4]. In this paper, we provide a method for finding non-jumps for $r = 3$ using patterns. As a direct consequence, we find a few more examples of non-jumps for $r = 3$.

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1. INTRODUCTION

An r -uniform hypergraph is a pair (V, E) where V is a set of vertices and E is a set of r -tuples of vertices called edges. For convenience, we use r -graph in place of r -uniform hypergraph here. For an r -graph G , let $V(G)$ and $E(G)$ denote the set of vertices and edges of G , respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. The density of an r -graph is defined as follows.

Definition 1.1. The density of an r -graph G is

$$d(G) = \frac{e(G)}{\binom{n}{r}}.$$

For some r -graph G and some size n , the extremal number $\text{ex}(n, G)$ denotes the maximum number of edges in an r -graph on n vertices that does not contain G as an induced subgraph. The extremal number for a family of r -graphs is defined the same, except we forbid induced subgraphs isomorphic to any member of the family. The Turán density of a family describes the limiting behavior of the extremal number for a family of r -graphs.

Definition 1.2. The Turán density of a family of r -graphs $\mathcal{F}$ is defined as

$$\pi(\mathcal{F}) = \lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

Such a limit exists by an averaging argument. Jumps describe gaps in the set of Turán densities of all families of r -graphs.

Definition 1.3. We say that $\alpha \in [0, 1)$ is a jump for r if there exists some $c > 0$, depending on α and r , such that there does not exist any family of r -graphs $\mathcal{F}$ with $\pi(\mathcal{F}) \in (\alpha, \alpha + c)$.

Oftentimes a different equivalent definition of hypergraph jumps is used in place of this one. See [7] for a proof of the equivalence.

Proposition 1.4. *The following are equivalent:*

- (1) α is a jump for r .
- (2) there exists some $d > 0$, depending on α and r , such that given any $\epsilon > 0$ and any integer $m \geq r$ there is some integer $N > 0$, depending on α , r , m , and ϵ , such that any r -graph on $n \geq N$ vertices with at least $(\alpha + \epsilon)\binom{n}{r}$ edges contains some subgraph on m vertices with at least $(\alpha + d)\binom{m}{r}$ edges.

By the Erdős-Stone Theorem and another result from Erdős [2], all $\alpha \in [0, 1)$ are jumps for $r = 2$. In addition, Erdős' work in [2] shows that all $\alpha \in [0, \frac{r!}{r^r})$ are jumps for r . In general, little is known about behavior in the interval $[\frac{r!}{r^r}, 1)$. A conjecture of Erdős [3] says that $\frac{r!}{r^r}$ is a jump for r . This problem remains open, but progress has been made in its direction. Baber and Talbot [1] showed that $\alpha \in [0.2299, 0.2316) \cup [0.2871, 8/27)$ are jumps for $r = 3$. Aside from the interval $[\frac{r!}{r^r}, 1)$, these are the only known jumps for $r \geq 3$. However, slightly more can be said about non-jumps.

Frankl and Rödl were the first to find examples of non-jumps [4]. They showed that for $r \geq 3$ and $\ell > 2r$ the density $1 - \frac{1}{\ell^{r-1}}$ is not a jump for r . Since then, a number of non-jumps and sequences of non-jumps have been found [9] [10] [11] [8]. Peng [9] showed that if $\alpha \frac{r!}{r^r}$ is a non-jump for $r \geq 3$, then $\alpha \frac{p!}{p^p}$ is a non-jump for any $p \geq r$. Recently, Shaw [10] showed that the smallest non-jumps that can be found using the Frankl-Rödl method are $\frac{6}{121}(5\sqrt{5} - 2)$ for $r = 3$ and $2 \cdot \frac{r!}{r^r}$ for $r \geq 4$.

In this paper, we provide a way to find non-jumps using patterns. This comes from Shaw's work [10] about Frankl and Rödl's method of finding non-jumps. As corollaries, we find a few examples of non-jumps. Specifically, we show the following.

Theorem 1.5. *The density $64/81$ is not a jump for $r = 3$.*

Theorem 1.6. *Let $n \in \mathbb{Z}^+$ and let $k = \sqrt{3n - 2}$. Then, $1 - \frac{3n^2 - 2n + k^3}{(n+k)^3}$ is not a jump for $r = 3$.*

2. BLOWUPS AND LAGRANGIANS

Our proofs rely heavily on the idea of hypergraph blowups and Lagrangians. We refer to the notation used in Keevash's survey on the topic [6].

Definition 2.1. Let G be an r -graph on n vertices $v_1, \dots, v_n$ and $t = (t_1, \dots, t_n)$. A t -blowup of G is an r -graph $G(t)$ such that each vertex v_i of G is replaced by t_i

copies and an r -tuple of vertices in $G(t)$ is an edge if and only if the corresponding r -tuple of vertices in G is an edge.

We use (v, i) to refer to the i th copy of vertex v in $G(t)$. So, each vertex v of G corresponds to an independent set of t_i vertices $(v, 1), \dots, (v, t_i)$ in $G(t)$, and each edge of G corresponds to an r -partite r -graph in $G(t)$. For example, a $(2, 2)$ -blowup of a single edge on two vertices v and w is a square with vertices $(v, 1), (v, 2), (w, 1), (w, 2)$.

Let $p_G(t)$ denote the number of edges in $G(t)$. Then, $p_G(t) = \sum_{e \in E(G)} \prod_{i \in e} t_i$. Suppose we want to find the largest possible density for $G(t)$ as $|t| \rightarrow \infty$, where $|t| = \sum_{i \in n} t_i$. For any t ,

$$\lim_{m \rightarrow \infty} d(G(mt)) = \lim_{m \rightarrow \infty} \frac{1}{\binom{m|t|}{r}} p_G(mt) = r! p_G(t_1/|t|, \dots, t_n/|t|).$$

So, it suffices to maximize $r! p_G(x)$ over all x in the standard simplex $S = \{x \mid x_1 + \dots + x_n = 1, x_i \geq 0 \ \forall i \in [n]\}$. Since this sort of optimization appears frequently, a name is given to the essential component to be maximized.

Definition 2.2. The Lagrangian of an r -graph G is defined as

$$\lambda(G) = \max_{x \in S} p_G(x).$$

There is also a name for the maximum density.

Definition 2.3. The blowup density of an r -graph G is defined as

$$b(G) = r! \lambda(G).$$

We can think of the x_i s of x as optimal weights in $[0, 1]$ that give the whole graph G a weight $w(G) = \lambda(G)$.

3. PATTERNS

To guide our search for non-jumps, we will need patterns as defined by Hou, Li, Yang, and Zhang in [5]. A pattern follows all of the same rules as a hypergraph except edges are multisets of vertices, where a multiset is defined as a set with repetitions allowed.

Definition 3.1. An r -pattern P is a pair $(n, \mathcal{E})$, where $n \geq r$ is an integer representing the number of vertices of P and $\mathcal{E}$ is a collection of r -multisets on $[n]$ representing the edges of P .

For example, a 3-pattern on 3 vertices may have the edge set $\mathcal{E} = \{112, 123, 223\}$. A blowup of a pattern is defined in the same way it is for hypergraphs. Let P be an r -pattern. If $t = (t_1, \dots, t_n)$, then $P(t)$ is an r -pattern where each vertex i is replaced by t_i copies and an r -multiset in $P(t)$ is an edge if and only if the corresponding r -multiset in P is an edge. A simple blowup $P[t]$ of P is defined as the blowup $P(t)$ with all edges containing repeat vertices removed. For example, if $P = (3, \mathcal{E})$ and $t = (2, 2, 1)$ then $P[t]$ has edge set

$$\mathcal{E}' = \{(1, 1)(2, 1)3, (1, 2)(2, 1)3, (1, 1)(2, 2)3, (1, 2)(2, 2)3, (1, 1)(1, 2)3, (2, 1)(2, 2)3\}.$$

The Lagrangian for an r -pattern is defined in a similar way to how we defined it for hypergraphs. Let $m_e(i)$ denote the multiplicity of i in an edge e . Then for any

pattern P and n -tuple t ,

$$|\mathcal{E}(P[t])| = \sum_{e \in \mathcal{E}(P)} \prod_{i \in e} \binom{t_i}{m_e(i)}.$$

As $t_i \rightarrow \infty$ for all $i \in [n]$,

$$\sum_{e \in \mathcal{E}(P)} \prod_{i \in e} \binom{t_i}{m_e(i)} \rightarrow \sum_{e \in \mathcal{E}(P)} \prod_{i \in e} \frac{t_i^{m_e(i)}}{m_e(i)!}.$$

Let $p_P(t)$ denote the expression on the right. Then, for any t ,

$$\lim_{m \rightarrow \infty} d(P[mt]) = \lim_{m \rightarrow \infty} \frac{1}{\binom{m|t|}{r}} p_P(mt) = r! p_P(t_1/|t|, \dots, t_n/|t|).$$

Just as before, if we wish to maximize the asymptotic density of $P[t]$, then it suffices to maximize $r! p_P(x)$ over the standard simplex S . As before, the Lagrangian of P is defined as $\lambda(P) = \max_{x \in S} p_P(x)$, and its blowup density as $b(P) = r! \lambda(P)$.

4. FINDING NON-JUMPS

Now, we outline the strategy for finding non-jumps, building on the work of Shaw [10]. First, we introduce the Frank-Rödl construction $\text{FR}_v(P)$ from a pattern P .

Definition 4.1. Suppose we are given an r -pattern $P = (n, \mathcal{E})$ and $v \in [n]$. Let t be defined such that $t_v = r$ and $t_i = 1$ for all other vertices $i \in [n]$. Let P' be an r -pattern on the vertices of $P(t)$ with

$$\mathcal{E}(P') = \{e \in \mathcal{E}(P(t)) \mid m_e((v, i)) \leq 1 \quad \forall i \in \{2, \dots, r\}\},$$

Then, $\text{FR}_v(P)$ is defined as

$$\text{FR}_v(P) = P' \cup \{(v, 1) \cdots (v, n)\}.$$

This construction is useful for providing a sufficient condition for α to be a non-jump for r . The proof of this can be found in [10].

Theorem 4.2. *Let P be an r -pattern, and v be a vertex of P such that a maximal weighting of P assigns v positive weight. Suppose*

$$\lambda(\text{FR}_v(P)) = \lambda(P) < 1.$$

Then, $r! \lambda(P)$ is not a jump for r -graphs.

As the name suggests, this method was first used by Frankl and Rödl [4] to show the existence of non-jumps. Using this technique, we prove the densities given in Theorem 1.5 and Theorem 1.6 are non-jumps. First, we need a lemma for patterns with equivalent vertices. Two vertices in a pattern are deemed equivalent if their labelings can be swapped to produce a pattern isomorphic to the original.

Lemma 4.3. *Let $P = (n, \mathcal{E})$ be a 3-pattern, and let i and j be vertices in P . If i and j are equivalent, then either $w_i = w_j$ or one of w_i and w_j is 0 for an optimal weighting of P , where w_v represents the weighting of vertex v .*

Proof. Suppose we know the optimal weights of all of the vertices in $[n] \setminus \{i, j\}$. Then, let $w(P)$ be the optimal weight of P and C_1, C_2, C_3 be values depending only on the weights of vertices in $[n] \setminus \{i, j\}$ such that

$$w(P) = (w_i + w_j)C_1 + (w_i^2 + w_j^2)C_2 + (w_i w_j)C_3.$$

By the Lagrange multiplier method, either at least one of w_i, w_j is 0 or $\frac{\partial}{\partial w_i} w(P) = \frac{\partial}{\partial w_j} w(P)$. In the second case,

$$\begin{aligned} 2w_i C_2 + w_j C_3 &= 2w_j C_2 + w_i C_3 \\ (2C_2 - C_3)(w_i - w_j) &= 0. \end{aligned}$$

If $C_3 = 2C_2$, then

$$w(P) = (w_i + w_j)C_1 + (w_i + w_j)^2 C_2.$$

So, the relative weights of w_i and w_j do not affect the weight of the pattern. Therefore, $w_i = w_j$ gives us an optimal weighting. $\square$

We also need the blowup density of a specific small 3-pattern.

Lemma 4.4. *Let $P = \{112, 122\}$. Then,*

$$\lambda(\text{FR}_1(P)) = \lambda(P) = \frac{1}{8}.$$

Proof. Let w be an optimal weighting of P and w_1 and w_2 be the weights of the vertices. Since both 1 and 2 are equivalent, by Lemma 4.3 either $w_1 = w_2$ or one of w_1 or w_2 is 0. Since we cannot have the second case, $w_1 = w_2 = \frac{1}{2}$. So,

$$\lambda(P) = \frac{w_1^2 w_2}{2} + \frac{w_1 w_2^2}{2} = \frac{1}{8}.$$

Now, let w' be an optimal weighting of $\text{FR}_1(P)$ and $w_{(1,1)}, w_{(1,2)}, w_{(1,3)}$, and w_2 be the weights of the vertices. By Lemma 4.3, either $w_{(1,2)} = w_{(1,3)}$ or one of $w_{(1,2)}$ or $w_{(1,3)}$ is 0. The second case does not give us an optimal weighting, so we only consider the first. Let $a = w_{(1,1)}, b = 2w_{(1,2)} = w_{(1,2)} + w_{(1,3)}$, and $c = w_2$. Then,

$$w'(\text{FR}_1(P)) = \frac{ab^2}{4} + \frac{a^2 c}{2} + \frac{b^2 c}{4} + abc + \frac{(a+b)c^2}{2}.$$

We are subject to the constraint $a + b + c = 1$. Assume a, b , and c are all positive. Then, by the Lagrange multiplier method,

$$\frac{\partial}{\partial a} w'(\text{FR}_1(P)) = \frac{\partial}{\partial b} w'(\text{FR}_1(P)) = \frac{\partial}{\partial c} w'(\text{FR}_1(P)).$$

We have

$$(4.5) \quad \frac{\partial}{\partial a} w'(\text{FR}_1(P)) = \frac{b^2}{4} + ac + bc + \frac{c^2}{2}$$

$$(4.6) \quad \frac{\partial}{\partial b} w'(\text{FR}_1(P)) = \frac{ab}{2} + \frac{bc}{2} + ac + \frac{c^2}{2}$$

$$(4.7) \quad \frac{\partial}{\partial c} w'(\text{FR}_1(P)) = \frac{a^2}{2} + \frac{b^2}{4} + ab + (a+b)c.$$

Subtracting (4.5) from (4.7) gives us $b = \frac{c^2 - a^2}{2a}$, and subtracting (4.5) from (4.6) gives us $b = 2(a - c)$. So, if $c < a$, we get $b < 0$ from the first equality, and if $c > a$ we get $b < 0$ from the last equality. So, we must have $a = c$. However, this implies that $b = 0$, a contradiction. So, one of a, b , or c must be 0. We cannot have $a = 0$

for an optimal weighting. If $c = 0$, then the optimal weight is $\frac{a^2b}{4} \leq \frac{1}{27}$. If $b = 0$, then we are left with the expression for $w(P)$. So, $\lambda(\text{FR}_1(P)) = \lambda(P) = \frac{1}{8}$. This implies that $\frac{3}{4}$ is a non-jump for $r = 3$. $\square$

Now, we can prove what we want.

Theorem 4.8. *Let $P = (n, \mathcal{E})$ be a 3-pattern such that a maximal weighting of P assigns vertex 1 positive weight. Suppose $\{122\} \cup \{11i \mid i \in [n] \setminus \{1\}\}$ is a subset of $\mathcal{E}$. Then, P is a 3-pattern satisfying the conditions of Theorem 4.2, assuming $\lambda(P) < 1$.*

Proof. Consider $\text{FR}_1(P)$. Suppose w is an optimal weighting of $\text{FR}_1(P)$ and w_v is the weight of vertex v . Let $a = w_{(1,1)}$, $b = w_{(1,2)} + w_{(1,3)} = 2w_{(1,2)}$, and $k = \sum_{i=2}^n w_i$. Then,

$$\begin{aligned} w(\text{FR}_1(P)) &= \frac{ab^2}{4} + \frac{a^2k}{2} + \frac{b^2k}{4} + abk + \frac{1}{2}(a+b) \left(\sum_{(1,i,i) \in \mathcal{E}} w_i^2 \right) + \\ &\quad (a+b) \left(\sum_{\substack{(1,i,j) \in \mathcal{E} \\ i \neq j}} w_i w_j \right) + \sum_{\substack{(i,j,k) \in \mathcal{E} \\ i \neq j, j \neq k, i \neq k}} w_i w_j w_k + \sum_{(i,i,j) \in \mathcal{E}} \frac{w_i^2 w_j}{2}. \end{aligned}$$

Fix $w_2, \dots, w_n$ to be any value. If $a \neq 0$ and $b \neq 0$, then by the Lagrange multiplier method

$$\frac{\partial}{\partial a} w(\text{FR}_1(P)) = \frac{\partial}{\partial b} w(\text{FR}_1(P)).$$

We have

$$\begin{aligned} \frac{\partial}{\partial a} w(\text{FR}_1(P)) &= \frac{b^2}{4} + ak + bk + \frac{1}{2} \sum_{(1,i,i) \in \mathcal{E}} w_i^2 + \sum_{\substack{(1,i,j) \in \mathcal{E} \\ i \neq j}} w_i w_j \\ \frac{\partial}{\partial b} w(\text{FR}_1(P)) &= \frac{ab}{2} + \frac{bk}{2} + ak + \frac{1}{2} \sum_{(1,i,i) \in \mathcal{E}} w_i^2 + \sum_{\substack{(1,i,j) \in \mathcal{E} \\ i \neq j}} w_i w_j. \end{aligned}$$

So,

$$\begin{aligned} \frac{\partial}{\partial a} w(\text{FR}_1(P)) - \frac{\partial}{\partial b} w(\text{FR}_1(P)) &= 0 \\ \frac{b^2}{4} + \frac{bk}{2} - \frac{ab}{2} &= 0 \\ b + 2(k - a) &= 0. \end{aligned}$$

This can only be true if $a > k$. Now, we assume this is the case and show $b = 0$.

Let w' be an optimal weighting for $\text{FR}_1(P)$ with this assumption and w'_v be the weight of vertex v . Also, let $a' = w'_{(1,1)}$, $b' = w'_{(1,2)} + w'_{(1,3)} = 2w'_{(1,2)}$, and $k' = \sum_{i=2}^n w'_i$. Fix some distribution of the weights $w'_2, \dots, w'_n$ so that $w'_i = c'_i k'$ for constants $c'_2, \dots, c'_n$ in $[0, 1]$. Then, when expressing $w'(\text{FR}_1(P))$ we can collect all of the constants $c'_2, \dots, c'_n$ into two constants c_1 and c_2 to get

$$w'(\text{FR}_1(P)) = \frac{a'b'^2}{4} + \frac{a'^2 k'}{2} + \frac{b'^2 k'}{4} + a'b'k' + c_1(a' + b')k'^2 + c_2 k'^3.$$

By Lemma 4.3, we know that $w'(\text{FR}_1(P)) \geq \frac{1}{8}$. Therefore, in an optimal weighting for $\text{FR}_1(P)$ we have

$$c_2 \geq \frac{1}{k'^3} \left(\frac{1}{8} - \frac{a'b'^2}{4} - \frac{a'^2k'}{2} - \frac{b'^2k'}{4} - a'b'k' - c_1(a' + b')k'^2 \right).$$

Assume a' , b' , and k' are nonzero. Then, the Lagrange multiplier method gives us

$$\frac{\partial}{\partial k'} w'(\text{FR}_1(P)) - \frac{\partial}{\partial a'} w'(\text{FR}_1(P)) = 0.$$

We want to show $k' > \frac{1}{2}$ in this equation, since this contradicts the assumption $a' > k'$. If c_1 or c_2 increases, then the k' that maximizes $w'(\text{FR}_1(P))$ must not decrease. So it suffices to show $k' > \frac{1}{2}$ for the minimum possible c_2 . If we plug this value for c_2 into the equation, we get

$$\begin{aligned} & \frac{a'^2}{2} + a'b' + (2c_1 - 1)(a' + b')k' + \\ & \frac{3}{k'} \left(\frac{1}{8} - \frac{a'b'^2}{4} - \frac{a'^2k'}{2} - \frac{b'^2k'}{4} - a'b'k' - c_1(a' + b')k'^2 \right) - c_1k'^2 = 0. \end{aligned}$$

Now, if we implicitly differentiate with respect to c_1 holding a' and b' constant we get

$$\begin{aligned} & 2(a' + b')k' + (2c_1 - 1)(a' + b') \frac{dk'}{dc_1} - \frac{3}{k'} \left(\frac{a'^2}{2} + \frac{b'^2}{4} + a'b' + 2c_1(a' + b')k' \right) \frac{dk'}{dc_1} - \\ & 3(a' + b')k' + \frac{3}{k'^2} \left(\frac{a'b'^2}{4} - \frac{1}{8} + \frac{a'^2k'}{2} + \frac{b'^2k'}{4} + a'b'k' + c_1(a' + b')k'^2 \right) \frac{dk'}{dc_1} - \\ & k'^2 - 2c_1k' \frac{dk'}{dc_1} = 0. \end{aligned}$$

Let

$$g(a', b', k', c_1) = (2c_1 - 1)(a' + b') - 2c_1k' + \frac{3}{k'^2} \left(\frac{a'b'^2}{4} - \frac{1}{8} \right) - 3c_1(a' + b').$$

Then,

$$\frac{dk'}{dc_1} = \frac{(a' + b')k' + k'^2}{g(a', b', k', c_1)}.$$

We have $g(a', b', k', c_1) < 0$. Therefore, given that c_2 is a function of c_1 and a' and b' are constant, the k' that maximizes $w'(\text{FR}_1(P))$ increases as c_1 decreases. If we chose any c_2 other than the minimum, then k' would only increase more. However, this implies $k' > \frac{1}{2}$ for $c_1 < \frac{1}{2}$, since $k = \frac{1}{2}$ maximizes $w'(\text{FR}_1(P))$ for $c_1 = \frac{1}{2}$ and $c_2 = 0$ by Lemma 4.4. If $c_1 > \frac{1}{2}$, then we also get $k' > \frac{1}{2}$, since k increases when c_1 increases and c_2 is non-decreasing. This contradicts the assumption that $a' > k'$.

Since $a \leq k'$, one of a' , b' , k' must be zero for an optimal weighting of $\text{FR}_1(P)$. We cannot have $a' = 0$. If $k' = 0$, then $w'(\text{FR}_1(P)) = \frac{a'b'^2}{4} \leq \frac{1}{27}$. Therefore, $b' = 0$ for an optimal weighting. $\square$

5. PROOF OF THEOREM 1.5

Let $P = \{123, 122, 112, 113, 223\}$. By Theorem 4.2 and Theorem 4.8, it suffices to show that vertex 1 receives a positive weight in an optimal weighting for P and

$$\lambda(P) = \frac{32}{243}.$$

This will show us that $3! \cdot \frac{32}{243} = \frac{64}{81}$ is not a jump for $r = 3$. Suppose w is a maximal weighting for P and w_1, w_2, w_3 are the weights of the vertices. Then,

$$w(P) = w_1 w_2 w_3 + \frac{w_1 w_2^2}{2} + \frac{w_1^2 w_2}{2} + \frac{w_1^2 w_3}{2} + \frac{w_2^2 w_3}{2}.$$

Suppose w_1, w_2 , and w_3 are positive. Then, by Lemma 4.3 we have $w_1 = w_2$. Therefore,

$$w(P) = 2w_1^2 w_3 + w_1^3.$$

We are subject to the constraint

$$2w_1 + w_3 = 1.$$

So, using the method of Lagrange multipliers,

$$\begin{aligned} \frac{\partial}{\partial w_1} w(P) &= 2 \frac{\partial}{\partial w_3} w(P) \\ 4w_1 w_3 + 3w_1^2 &= 4w_1^2 \\ w_1 - 4w_3 &= 0. \end{aligned}$$

Then, using the constraint, we get $w_1 = w_2 = 4/9$ and $w_3 = 1/9$, which leaves us $w(P) = 32/243$. If $w_1 = 0$, then $w_2 = \frac{2}{3}$ and $w_3 = \frac{1}{3}$ maximizes $w(P)$. This leaves $w(P) = 1/27 < 32/243$. We get the same result when $w_2 = 0$. If $w_3 = 0$, then by Lemma 4.4 we have $w(P) \leq 1/8 < 32/243$. So, $\lambda(P) = 32/243$.

6. PROOF OF THEOREM 1.6

Let n be fixed. Then, let P be the pattern on $n + 1$ vertices with edges $[n + 1]^{(3)} \cup \{1, 2, 2\} \cup \{1, 3, 3\} \cup \dots \cup \{1, n + 1, n + 1\}$. By Theorem 4.2 and Theorem 4.8, it suffices to show that vertex 1 receives a positive weight in an optimal weighting for P and

$$\lambda(P) = \frac{1}{6} - \frac{3n^2 - 2n + k^3}{6(n + k)^3}.$$

Let w_1 be the weight of the first vertex. By Lemma 4.3, either all of the weights of the other vertices are equal, or some of them are 0 and the rest are equal. Let m be the number of these vertices that are assigned positive weight and w_2 equal m times the weight of one of these vertices. Then,

$$\begin{aligned} w(P) &= \binom{m}{2} \frac{w_1 w_2^2}{m^2} + \binom{m}{3} \frac{w_2^3}{m^3} + \frac{w_1^2 w_2}{2} + \frac{w_1 w_2^2}{2m} \\ &= \frac{w_1 w_2^2}{2} + \binom{m}{3} \frac{w_2^3}{m^3} + \frac{w_1^2 w_2}{2}. \end{aligned}$$

This is increasing in m , so we must have $m = n$. We are subject to the constraint

$$w_1 + w_2 = 1.$$

First, suppose both w_1 and w_2 are positive. Then, by the Lagrange multipliers method,

$$\begin{aligned}\frac{\partial}{\partial w_1} w(P) &= \frac{\partial}{\partial w_2} w(P) \\ \frac{w_2^2}{2} + w_1 w_2 &= w_1 w_2 + \binom{n}{3} \frac{3w_2^2}{n^3} + \frac{w_1^2}{2} \\ \left(1 - \frac{(n-1)(n-2)}{n^2}\right) w_2^2 &= w_1^2 \\ w_1 &= \frac{k}{n} w_2.\end{aligned}$$

We use the constraint equation to get

$$\begin{aligned}w_2 &= \frac{1}{1 + \frac{k}{n}} = \frac{n}{n+k} \\ w_1 &= \frac{k}{n} w_2 = \frac{k}{n+k}.\end{aligned}$$

Finally, plugging these values for w_1 and w_2 into the expression for $w(P)$ we get

$$w(P) = \frac{1}{6} - \frac{3n^2 - 2n + k^3}{6(n+k)^3}.$$

Now, suppose $w_1 = 0$, $w_2 = 1$. Then,

$$w(P) = \binom{n}{3} \frac{1}{n^3} = \frac{(n-1)(n-2)}{6n^2} = \frac{1}{6} - \frac{3n-2}{6n^2}.$$

However,

$$\frac{1}{6} - \frac{3n-2}{6n^2} < \frac{1}{6} - \frac{3(n+k)-2}{6(n+k)^2} < \frac{1}{6} - \frac{3n^2-2n+k^3}{6(n+k)^3}.$$

Clearly, $w_2 = 0$ does not maximize $w(P)$. Therefore, $\lambda(P) = \frac{1}{6} - \frac{3n^2-2n+k^3}{6(n+k)^3}$.

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