

Distributed Adaptive Estimation over Sensor Networks with Partially Unknown Source Dynamics

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Abstract: This paper studies distributed adaptive estimation over sensor networks with partially known source dynamics. We present parallel continuous-time and discrete-time designs in which each node runs a local adaptive observer and exchanges information over a directed graph. For both time scales, we establish stability of the network coupling operators, prove boundedness of all internal signals, and show convergence of each node's estimate to the source despite model uncertainty and disturbances. We further derive input-to-state stability (ISS) bounds that quantify robustness to bounded process noise. A key distinction is that the discrete-time design uses constant adaptive gains and per-step regressor normalization to handle sampling effects, whereas the continuous-time design does not. A unified Lyapunov framework links local observer dynamics with graph topology. Simulations on star, cyclic, and path networks corroborate the analysis, demonstrating accurate tracking, robustness, and scalability with the number of sensing nodes.

Keywords: Adaptive Estimation, Distributed Systems, Input-to-State Stability (ISS), Multi-Agent Systems, Networked Observers

1. INTRODUCTION

Distributed state estimation over sensor networks has emerged as an underlying problem in multi-agent systems, with applications spanning environmental monitoring, surveillance systems, distributed robotics, and cooperative control [Hu et al. (2020); Yang et al. (2022)]. In these scenarios, a network of sensing agents must collectively estimate the state of a target (source) whose dynamics may be known or only partially known. The distributed nature of the problem necessitates algorithms that coordinate local observations and computations over networks.

Existing approaches to distributed state estimation broadly fall into consensus-based observers and distributed Kalman filtering. Consensus-based methods [Wang et al. (2022); Vivas et al. (2015); Rong et al. (2020)] achieve agreement among agents through iterative averaging but typically assume the source dynamics are perfectly known. Distributed Kalman filters [Olfati-Saber (2007); Li et al. (2019); Talebi and Werner (2018)] handle stochastic uncertainty but require knowledge of noise statistics and system matrices. Recent extensions [Zhu et al. (2023)] address time-varying topologies and communication constraints, yet a fundamental assumption persists: the source dynamics matrices are known a priori.

In many practical scenarios, however, the source dynamics contain parametric uncertainty—for instance, an environmental process with unknown reaction rates, or a vehicle with uncertain mass and damping coefficients. Recent

works have explored distributed adaptive estimation to address such uncertainties, combining local learning with networked coordination [Raj et al. (2020); Wang and Meng (2021); Stegagno and Yuan (2019)]. While these studies achieve asymptotic convergence under ideal or noise-free conditions, they typically neglect bounded process disturbances and do not establish input-to-state stability (ISS).

This motivates us to study adaptive distributed estimation in directed sensor networks where each node must simultaneously estimate the source state and identify unknown dynamics while coordinating only with in-neighbors. This paper develops continuous- and discrete-time frameworks for such settings, providing rigorous input-to-state stability (ISS) guarantees via Lyapunov analysis and linear matrix inequality (LMI) conditions, and explicitly revealing how network coupling, adaptation dynamics, and disturbance robustness interact.

Our main contributions are:

- (1) A unified network-level formulation (via Kronecker structures) that decouples local observer design from graph topology, enabling modular stability analysis;
- (2) For continuous-time systems: a smooth gradient-based adaptive laws whose closed-loop error converges under a spectral/LMI condition on the coupling operator $\mathcal{H}$, together with an ISS bound in the presence of bounded source disturbances;
- (3) For discrete-time systems: a gradient-based adaptive law analyzed under analogous (Point 2) spectral/LMI conditions on the coupling operator $\mathcal{S}$, but augmented

with regressor normalization and a constrained adaptation gain (within a stability range) to address sampling and non-strongly convex regressors. The design guarantees bounded parameter iterates, error convergence, as well as an ISS bound with respect to bounded source disturbances.

- (4) For both time scales, explicit trade-offs are established among network connectivity, local damping, and source growth, including robustness to norm-bounded uncertainties; providing transparent design guidelines;
- (5) Simulation studies on star, cyclic, and path graphs (and larger networks) that validate tracking performance, ISS robustness, and scalability with increasing numbers of sensing nodes.

Notations. I_p is the $p \times p$ identity, and $\text{diag}\{A_i\} \in \mathbb{R}^{pn \times pm}$ for $i = 1, \dots, p$ forms a block-diagonal matrix with blocks $A_i \in \mathbb{R}^{n \times m}$. Vectors of ones and zeros in $\mathbb{R}^p$ are $\mathbf{1}_p$ and $\mathbf{0}_p$. The Kronecker product is $A \otimes B$, $\text{tr}(A)$ is the trace, $\lambda(A)$ denotes eigenvalues, and $\rho(A) = \max_i |\lambda_i(A)|$ is the spectral radius.

2. COMMUNICATION NETWORK

We consider a network of $m+1$ agents indexed by $\mathcal{V} = \{0, 1, \dots, m\}$, where agent 0 is the source (system being estimated) and agents $1, \dots, m$ are sensing nodes that run local adaptive observers and exchange information.

Communication is modeled by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and nonnegative weights $\mathcal{W} = \{w_{ij}\}$. An edge $(i, j) \in \mathcal{E}$ means agent i receives information from agent j with weight $w_{ij} > 0$; otherwise $w_{ij} = 0$. The in-neighborhood of agent i is $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. The weights $\{w_{ij}\}$ are algorithmic design coefficients (not physical signal strengths); how much weight agent i places on information from agent j .

For analysis, we partition $\mathcal{G}$ into two induced subgraphs: (i) the sensing-only subgraph $\mathcal{G}_m = (\mathcal{V}_m, \mathcal{E}_m, \mathcal{W}_m)$ with $\mathcal{V}_m = \{1, \dots, m\}$; (ii) the source-to-sensing subgraph $\mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0, \mathcal{W}_0)$ with $\mathcal{V}_0 = \{0\} \cup \{i \in \mathcal{V}_m : (i, 0) \in \mathcal{E}\}$. Thus, $\mathcal{E}_m$ contains sensing-to-sensing links and $\mathcal{E}_0$ contains source-to-sensing links (see Fig. 1).

For $\mathcal{G}_m$, define the in-degree $d_i := \sum_{j: (i,j) \in \mathcal{E}_m} w_{ij}$, $i \in \mathcal{V}_m$ and $\mathbb{D}_m = \text{diag}\{d_1, \dots, d_m\}$. The adjacency and Laplacian of the sensing subgraph are $[\mathbb{A}_m]_{ij} = w_{ij}$ if $(i, j) \in \mathcal{E}_m$ (with 0 otherwise) and $\mathbb{L}_m := \mathbb{D}_m - \mathbb{A}_m$. For $\mathcal{G}_0$, source-to-sensing connections are encoded by a diagonal matrix $\mathbb{A}_0 := \text{diag}\{w_{10}, \dots, w_{m0}\}$ where $[\mathbb{A}_0]_{ii} = w_{i0}$ if $(i, 0) \in \mathcal{E}_0$ and 0 otherwise.

Let $\mathbb{W} := \text{diag}\{w_1, \dots, w_m\}$ with $w_i := d_i + w_{i0}$ (total incoming weight). The augmented Laplacian is

$$\mathbb{L} := \mathbb{L}_m + \mathbb{A}_0 = \mathbb{W} - \mathbb{A}_m.$$

We propose the *balanced* condition $\mathbb{W} = I_m$, or $d_i + w_{i0} = 1$ ($\sum_{j \in \mathcal{N}_i} w_{ij} = 1$), $\forall i \in \mathcal{V}_m$. Under this condition¹,

$$(\mathbb{L} - \mathbb{A}_0)\mathbf{1}_m = \mathbf{0}_m \quad \Leftrightarrow \quad (\mathbb{A}_m + \mathbb{A}_0)\mathbf{1}_m = \mathbf{1}_m. \quad (1)$$

Remark 1. (Source Reachability). We assume: (i) at least one sensing node directly receives information from the

¹ If $\mathbb{W} \neq I_m$, normalization via $\tilde{w}_{ij} := w_{ij}/w_i$ for $(i, j) \in \mathcal{E}_m$ and $\tilde{w}_{i0} := w_{i0}/w_i$ yields $\tilde{\mathbb{W}} = I_m$ and preserves (1); this is not required for implementation but simplifies stability analysis.

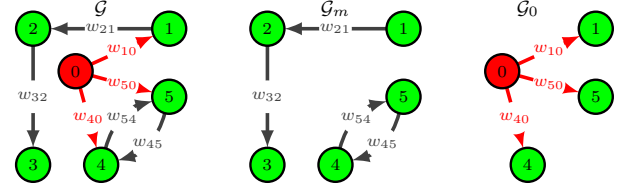


Fig. 1. Example of a graph $\mathcal{G}$ with $m = 5$, sensing-to-sensing subgraph $\mathcal{G}_m$, and source-to-sensing subgraph $\mathcal{G}_0$, showing the decoupling and assignment of w_{ij} .

source ($w_{i0} > 0$ for some i), and (ii) every sensing node is reachable from the source via a directed path in $\mathcal{G}$. Under these assumptions and the balanced condition, the augmented Laplacian $\mathbb{L}$ is *positive stable*: all eigenvalues have strictly positive real parts, $\Re\{\lambda_i(\mathbb{L})\} > 0$ for all i .

This property ensures that $\mathbb{L}$ admits a Lyapunov characterization essential for the input-to-state (ISS) analysis.

3. PROBLEM FORMULATION

We study distributed estimation where:

- Source (agent 0) has uncertain linear dynamics—system matrices contain unknown but bounded perturbations,
- Each sensing node runs local adaptive observer, exchanges estimates with in-neighbors,
- Not all nodes sense source directly; information propagates through $\mathcal{G}$.

We consider both continuous-time and discrete-time cases, as there are fundamental differences in constructing adaptive laws, including constant-gain adaptation and regressor normalization (due to not strongly convex cost function). Moreover, we assume *full-state output* in both time scales (continuous and discrete) for clarity. This is realistic for applications like GPS tracking where measured position is the state.

3.1 Problem 1: Unknown Continuous-time Dynamics

The unknown continuous-time source evolves as

$$\dot{x}_0(t) = F_0 x_0(t) + G_0 v_0(t) + d_0(t), \quad (2a)$$

$$y_0(t) = x_0(t), \quad (2b)$$

where $x_0(t) \in \mathbb{R}^n$ is the state, $v_0(t) \in \mathbb{R}^p$ is the reference input and $y_0(t) \in \mathbb{R}^n$ is the output. The process noise $d_0(t) \in \mathbb{R}^n$ is bounded as $\|d_0(t)\| \leq d^*$ for some constant $d^* > 0$. The matrices F_0 and G_0 are *unknown*, defined as

$$F_0 := F_* + \Delta F \in \mathbb{R}^{n \times n}, \quad G_0 := G_* + \Delta G \in \mathbb{R}^{n \times p}.$$

The nominal parts F_* and G_* are known; the perturbations ΔF and ΔG are unknown but bounded, $\|\Delta F\|_2 \leq f$ and $\|\Delta G\|_2 \leq g$, for some constants $f, g > 0$.

Sensing node $i \in \{1, \dots, m\}$ implements

$$\dot{x}_i(t) = H_i x_i(t) + (\hat{F}_i(t) - H_i) z_i(t) + \hat{G}_i(t) v_0(t), \quad (3a)$$

$$z_i(t) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) + w_{i0} y_0(t), \quad \mathcal{N}_i \subseteq \mathcal{V}, \quad (3b)$$

where $x_i(t) \in \mathbb{R}^n$ is the i -th sensing node estimate. Both $\hat{F}_i(t) \in \mathbb{R}^{n \times n}$ and $\hat{G}_i(t) \in \mathbb{R}^{n \times p}$ are adaptive estimates of F_0 and G_0 , respectively, and $H_i \in \mathbb{R}^{n \times n}$ is user-designed Hurwitz matrix. Here, $z_i(t)$ is weighted aggregate of in-neighbor estimates plus (if applicable) source output.

Main objective: Design a distributed adaptive law such that, under the balanced condition (1) and the source-reachability assumption in Remark 1, all sensing node estimates are ISS stable with respect to the bounded disturbance on the source, i.e.,

$$\limsup_{t \rightarrow \infty} \|x_i(t) - z_i(t)\| \leq c_c d^*, \quad \forall i, \quad (4a)$$

for $c_c > 0$ and, in the disturbance-free case $d^* = 0$

$$\lim_{t \rightarrow \infty} e_i(t) := x_i(t) - z_i(t) = 0. \quad (4b)$$

3.2 Problem 2: Uncertain Discrete-time Dynamics

To avoid confusion with the continuous-time setting, we adopt separate notation in the discrete-time case. The unknown discrete-time source evolves as

$$x_{0,k+1} = A_0 x_{0,k} + B_0 u_{0,k} + \delta_{0,k}, \quad (5a)$$

$$y_{0,k} = x_{0,k}, \quad (5b)$$

where $x_{0,k} \in \mathbb{R}^n$ defines the source state while $u_{0,k} \in \mathbb{R}^p$ and $y_{0,k} \in \mathbb{R}^n$ denote the reference input and the source output. The term $\delta_{0,k} \in \mathbb{R}^n$ represents process noise, bounded as $\|\delta_{0,k}\| \leq \delta^*$ for some constant $\delta^* > 0$. The matrices A_0 and B_0 are *unknown*, denoted as

$$A_0 := A_* + \Delta A \in \mathbb{R}^{n \times n}, \quad B_0 := B_* + \Delta B \in \mathbb{R}^{n \times p},$$

yet the pair (A_*, B_*) is known with bounds $\|\Delta A\|_2 \leq a$, $\|\Delta B\|_2 \leq b$, for some constants $a, b > 0$.

Next, sensing node $i \in \{1, \dots, m\}$ evolves as

$$x_{i,k+1} = S_i x_{i,k} + (\hat{A}_{i,k} - S_i) z_{i,k} + \hat{B}_{i,k} u_{0,k}, \quad (6a)$$

$$z_{i,k} = \sum_{j \in \mathcal{N}_i} w_{ij} x_{j,k} + w_{i0} y_{0,k}, \quad \mathcal{N}_i \subseteq \mathcal{V}, \quad (6b)$$

where $x_{i,k} \in \mathbb{R}^n$ is the i -th sensing node estimate. The terms $\hat{A}_{i,k} \in \mathbb{R}^{n \times n}$ and $\hat{B}_{i,k} \in \mathbb{R}^{n \times p}$ are the estimates of the pair (A_0, B_0) with user-designed Schur-stable matrix $S_i \in \mathbb{R}^{n \times n}$. Finally, $z_{i,k}$ is weighted aggregate of in-neighbor estimates (including the source if $0 \in \mathcal{N}_i$).

Main objective: Design a distributed adaptive law such that, under (1) and Remark 1, all sensing node estimates are ISS stable with respect to the bounded disturbance on the source, i.e.,

$$\limsup_{k \rightarrow \infty} \|x_{i,k} - z_{i,k}\| \leq c_d \delta^* \quad \forall i, \quad (7a)$$

for $c_d > 0$ and, in the disturbance-free case $\delta^* = 0$

$$\lim_{k \rightarrow \infty} e_{i,k} := x_{i,k} - z_{i,k} = 0. \quad (7b)$$

3.3 Unified Notation

To avoid repetition, we define the shorthand notation for the stacked variables (cf. (2a)–(7b)) used in the following sections. For any source signal $\beta_0(t), \beta_{0,k} \in \mathbb{R}^q$,

$$\bar{\beta}_0(t) = \mathbf{1}_m \otimes \beta_0(t) \in \mathbb{R}^{mq}, \quad (8a)$$

$$\bar{\beta}_{0,k} = \mathbf{1}_m \otimes \beta_{0,k} \in \mathbb{R}^{mq}. \quad (8b)$$

For any sensing-node signal $\beta_i(t), \beta_{i,k} \in \mathbb{R}^q$,

$$\bar{\beta}(t) = [\beta_1^\top(t), \dots, \beta_m^\top(t)]^\top \in \mathbb{R}^{mq}, \quad (9a)$$

$$\bar{\beta}_k = [\beta_{1,k}^\top, \dots, \beta_{m,k}^\top]^\top \in \mathbb{R}^{mq}. \quad (9b)$$

For matrices $E_0, E_i, \hat{E}_i(t), \hat{E}_{i,k} \in \mathbb{R}^{q_1 \times q_2}$,

$$\mathbf{E}_0 = I_m \otimes E_0 \in \mathbb{R}^{mq_1 \times mq_2}, \quad (10a)$$

$$\mathbf{E} = \text{diag}\{E_1, \dots, E_m\} \in \mathbb{R}^{mq_1 \times mq_2}, \quad (10b)$$

$$\hat{\mathbf{E}}(t) = \text{diag}\{\hat{E}_1(t), \dots, \hat{E}_m(t)\} \in \mathbb{R}^{mq_1 \times mq_2}, \quad (10c)$$

$$\hat{\mathbf{E}}_k = \text{diag}\{\hat{E}_{1,k}, \dots, \hat{E}_{m,k}\} \in \mathbb{R}^{mq_1 \times mq_2}. \quad (10d)$$

With these definitions, we are ready to study the adaptive laws and their stability in both time scales.

4. ADAPTIVE ESTIMATION FOR PROBLEM 1

4.1 Error Dynamics in Networked Systems

In network-level form, the source evolves as

$$\dot{\bar{x}}_0(t) = \mathbf{F}_0 \bar{x}_0(t) + \mathbf{G}_0 \bar{v}_0(t) + \bar{d}_0(t), \quad (11a)$$

$$\bar{y}_0(t) = \bar{x}_0(t), \quad (11b)$$

while the sensing nodes obey

$$\dot{\bar{x}}(t) = \mathbf{H} \bar{x}(t) + (\hat{\mathbf{F}}(t) - \mathbf{H}) \bar{z}(t) + \hat{\mathbf{G}}(t) \bar{v}_0(t), \quad (12a)$$

$$\bar{z}(t) = (\mathbb{A}_m \otimes I_n) \bar{x}(t) + (\mathbb{A}_0 \otimes I_n) \bar{y}_0(t). \quad (12b)$$

Here, the signals $\bar{x}_0(t)$, $\bar{v}_0(t)$, $\bar{d}_0(t)$, and $\bar{y}_0(t)$ follow (8a) whereas $\bar{x}(t)$ and $\bar{z}(t)$ follow (9a). The matrices $\mathbf{F}_0$ and $\mathbf{G}_0$ are expressed as in (10a) while the matrix $\mathbf{H}$ is denoted as in (10b). Also, both $\hat{\mathbf{F}}(t)$ and $\hat{\mathbf{G}}(t)$ are defined as in (10c).

Before proceeding, we denote the following assumption.

Assumption 1. There exist optimal parameter values of the source dynamics in (11a), denoted as $\mathbf{F}^*$ and $\mathbf{G}^*$, such that $\mathbf{F}^* := \mathbf{F}_0$ and $\mathbf{G}^* := \mathbf{G}_0$.

Then, we introduce the parameter errors

$$\Phi_c(t) \triangleq \hat{\mathbf{F}}(t) - \mathbf{F}_0, \quad \Psi_c(t) \triangleq \hat{\mathbf{G}}(t) - \mathbf{G}_0. \quad (13)$$

From (4b), the estimation error defined as in (9a) is

$$\bar{e}(t) = (\mathbb{L} \otimes I_n) \bar{x}(t) - (\mathbb{A}_0 \otimes I_n) \bar{y}_0(t) := \bar{x}(t) - \bar{z}(t). \quad (14)$$

Thus, the estimation error dynamics inspired by [Wafi et al. (2024); Wafi and Siami (2024)] become

$$\begin{aligned} \dot{\bar{e}}(t) &= (\mathbb{L} \otimes I_n) \dot{\bar{x}}(t) - (\mathbb{A}_0 \otimes I_n) \dot{\bar{x}}_0(t) \\ &= (\mathbb{L} \otimes I_n) [\mathbf{H} \bar{x}(t) + (\hat{\mathbf{F}}(t) - \mathbf{H}) \bar{z}(t) + \hat{\mathbf{G}}(t) \bar{v}_0(t)] \\ &\quad - (\mathbb{A}_0 \otimes I_n) [\mathbf{F}_0 \bar{x}_0(t) + \mathbf{G}_0 \bar{v}_0(t) + \bar{d}_0(t)] \\ &= (\mathbb{L} \otimes I_n) [\mathbf{H} \bar{x}(t) + \hat{\mathbf{F}}(t) \bar{z}(t) - \mathbf{H} \bar{z}(t) + \hat{\mathbf{G}}(t) \bar{v}_0(t)] \\ &\quad + \mathbf{F}_0 \bar{e}(t) - (\mathbb{L} \otimes I_n) [\mathbf{F}_0 \bar{x}(t) + \mathbf{G}_0 \bar{v}_0(t) + \bar{d}_0(t)] \\ &= (\mathbb{L} \otimes I_n) [\hat{\mathbf{F}} \bar{z}(t) + \mathbf{H} \bar{e}(t) + \hat{\mathbf{G}}(t) \bar{v}_0(t)] \\ &\quad + (\mathbb{A}_m \otimes I_n) \mathbf{F}_0 \bar{e}(t) \\ &\quad - (\mathbb{L} \otimes I_n) [\mathbf{F}_0 \bar{z}(t) + \mathbf{G}_0 \bar{v}_0(t) + \bar{d}_0(t)] \\ &= \mathcal{H} \bar{e}(t) + (\mathbb{L} \otimes I_n) \Xi_c(t) \bar{\eta}_c(t) - \Omega \bar{d}_0(t), \end{aligned} \quad (15)$$

where

$$\mathcal{H} := [(\mathbb{L} \otimes I_n) \mathbf{H} + (\mathbb{A}_m \otimes I_n) \mathbf{F}_0], \quad (16)$$

adaptive parameter $\Xi_c(t) := [\Phi_c(t), \Psi_c(t)]$, time-varying regressor $\bar{\eta}_c(t) := [\bar{z}^\top(t), \bar{v}_0^\top(t)]^\top$, and $\Omega := (\mathbb{L} \otimes I_n)$.

4.2 Stability of the Coupling Operator $\mathcal{H}$

The stability of (15) depends on whether or not $\mathcal{H}$ in (16) is Hurwitz. We now establish sufficient conditions.

Proposition 1. Let $\mathcal{H} := (\mathbb{L} \otimes I_n) \mathbf{H} + (\mathbb{A}_m \otimes I_n) \mathbf{F}_0$ where $\mathbb{L} \succ 0$ is the (positive-stable) augmented Laplacian under the balanced and source-reachability assumptions (cf. (1) and Remark 1), $\mathbf{H} = \text{diag}\{H_1, \dots, H_m\}$ with each $H_i \in \mathbb{R}^{n \times n}$ Hurwitz, and $\mathbf{F}_0 = I_m \otimes F_0$ for some $F_0 \in \mathbb{R}^{n \times n}$. Define

$$\alpha_H := \min_{i \in \{1, \dots, m\}} |\Re\{\lambda_{\max}(H_i)\}| > 0,$$

$$\alpha_F := \max \Re\{\lambda(F_0)\}.$$

If

$$\lambda_{\min}(\mathbb{L}) \alpha_H > \|\mathbb{A}_m\|_2 \alpha_F, \quad (17)$$

then $\mathcal{H}$ is Hurwitz.

To complement this spectral characterization in (17), we further reformulate the stability requirement as a linear matrix inequality (LMI) that explicitly couples network topology and local sensing node dynamics.

Lemma 1. Let $\mathcal{H} := (\mathbb{L} \otimes I_n)\mathbf{H} + (\mathbb{A}_m \otimes I_n)\mathbf{F}_0$ with $\mathbb{L} \succ 0$, $\mathbf{H} = \text{diag}\{H_1, \dots, H_m\}$ (each $H_i \in \mathbb{R}^{n \times n}$ Hurwitz), and $\mathbf{F}_0 = I_m \otimes F_0$. If there exists a matrix $P_{\mathbb{L}} \succ 0$ such that the following *separable* LMI holds:

$$(\mathbb{L}^\top P_{\mathbb{L}} + P_{\mathbb{L}} \mathbb{L}) \otimes \frac{1}{2}(\mathbf{H} + \mathbf{H}^\top) + 2(P_{\mathbb{L}} \mathbb{A}_m) \otimes \frac{1}{2}(\mathbf{F}_0 + \mathbf{F}_0^\top) \prec 0, \quad (18)$$

then $\mathcal{H}$ is Hurwitz. Moreover, $V(\bar{e}(t)) = \bar{e}^\top(t)(P_{\mathbb{L}} \otimes I_n)\bar{e}(t)$ satisfies $\dot{V}(t) < 0$ for all $\bar{e}(t) \neq 0$.

The following corollary provides a more practical version of the LMI in Lemma 1, expressed in terms of scalar damping and growth bounds on the local and source dynamics.

Corollary 1. Suppose there exist two scalars $\alpha_H > 0$ and $\alpha_F \in \mathbb{R}$ such that $\frac{1}{2}(H_i + H_i^\top) \preceq -\alpha_H I_n, \forall i$ as well as $\frac{1}{2}(F_0 + F_0^\top) \preceq \alpha_F I_n$. If there exists $P_{\mathbb{L}} \succ 0$ with

$$-\alpha_H(\mathbb{L}^\top P_{\mathbb{L}} + P_{\mathbb{L}} \mathbb{L}) + 2\alpha_F P_{\mathbb{L}} \mathbb{A}_m \prec 0, \quad (19)$$

then (18) holds and thus $\mathcal{H}$ is Hurwitz.

Remark 2. Inequality (19) reveals a tradeoff: larger network connectivity (larger $\mathbb{L}$ in the Loewner sense) and more damping in H_i (larger α_H) counteract destabilizing source growth (larger α_F) and strong sensing nodes coupling (large $\mathbb{A}_m$).

Building upon Corollary 1, we extend the LMI framework to accommodate norm-bounded uncertainties in the source dynamics, ensuring robustness of the stability condition.

Corollary 2. Let $F_0 = F_* + \Delta F$ with $\frac{1}{2}(F_* + F_*^\top) \preceq \alpha_* I_n$ and $\|\Delta F\|_2 \leq f$. If there exist $P_{\mathbb{L}} \succ 0$ and a scalar $\tau > 0$ such that

$$-\alpha_H(\mathbb{L}^\top P_{\mathbb{L}} + P_{\mathbb{L}} \mathbb{L}) + 2P_{\mathbb{L}} \mathbb{A}_m \otimes \frac{1}{2}(F_* + F_*^\top) + \tau I_m \otimes I_n \prec 0, \quad (20)$$

and simultaneously

$$2\|P_{\mathbb{L}} \mathbb{A}_m\|_2 f < \tau, \quad (21)$$

then $\mathcal{H}$ is Hurwitz for all admissible ΔF .

With the stability of the coupling operator $\mathcal{H}$ ensured by Proposition 1 and its LMI conditions, we proceed to design the adaptive laws for the parameter estimates.

4.3 Adaptive Laws and Stability

We first retain the disturbance-free result (asymptotic convergence), then extend it to input-to-state stability (ISS) in the presence of bounded source disturbance in.

Theorem 1. Consider the networked system in (11a)–(12a), with error dynamics (15) and $\bar{d}_0(t) = \mathbf{0}_{mn}$. Let the communication graph $(\mathcal{V}, \mathcal{E})$ satisfies the balanced and source-reachability conditions (cf. (1) and Remark 1), and that the matrix $\mathcal{H}$ in (16) is Hurwitz. Let $P_c = P_c^\top \succ 0$ and $Q_c = Q_c^\top \succ 0$ satisfy the Lyapunov equation

$$\mathcal{H}^\top P_c + P_c \mathcal{H} = -Q_c. \quad (22)$$

Using $\Phi_c(t)$ and $\Psi_c(t)$ as in (13), then the adaptive laws

$$\dot{\Phi}_c(t) = -\Gamma_\phi(\mathbb{L} \otimes I_n)^\top P_c \bar{e}(t) \bar{z}^\top(t), \quad (23a)$$

$$\dot{\Psi}_c(t) = -\Gamma_\psi(\mathbb{L} \otimes I_n)^\top P_c \bar{e}(t) \bar{v}_0^\top(t), \quad (23b)$$

with gain matrices $\Gamma_\phi, \Gamma_\psi \succ 0$, guarantee that all closed-loop signals remain bounded and that $\bar{e}(t) \rightarrow 0$ as $t \rightarrow \infty$.

Remark 3. From LaSalle's invariance principle: the solutions of the closed-loop system approach the largest invariant set in $\mathbb{S} = \{\bar{e}(t) \in \mathbb{R}^{mn} : \dot{V}(\bar{e}(t), \Phi_c(t), \Psi_c(t)) = 0\}$, implying $\bar{e}(t) \rightarrow 0$ and bounded $\Phi_c(t), \Psi_c(t)$.

We now establish an ISS property when the source is subject to a bounded disturbance $\|d_0(t)\| \leq d^*$ with the same adaptive laws (23).

Theorem 2. Consider the networked system (11a)–(12a) with error dynamics (15), $\Omega := (\mathbb{L} \otimes I_n)$ and $\|\bar{d}_0(t)\| \leq \bar{d}^*$ for all $t \geq 0$. Assume the graph satisfies (1) and Remark 1, and $\mathcal{H}$ in (16) is Hurwitz. Let $P_c = P_c^\top \succ 0$ and $Q_c = Q_c^\top \succ 0$ solve

$$\mathcal{H}^\top P_c + P_c \mathcal{H} = -Q_c. \quad (24)$$

Apply the adaptive laws (23) with $\Gamma_\phi, \Gamma_\psi \succ 0$. Then there exists a constant $c_c(P_c, Q_c, \Omega) > 0$, such that

$$\limsup_{t \rightarrow \infty} \|\bar{e}(t)\| \leq c_c \bar{d}^*. \quad (25)$$

In particular, if $\bar{d}^* = 0$, then $\bar{e}(t) \rightarrow 0$ as $t \rightarrow \infty$.

This completes the continuous-time analysis: Theorem 1 provides asymptotic convergence in the disturbance-free case, while Theorem 2 ensures ISS under bounded source disturbances. In the discrete-time setting, the behavior changes significantly—the fixed adaptive gains and normalization of regressor signals become essential to preserve stability and guarantee bounded adaptation.

5. ADAPTIVE ESTIMATION FOR PROBLEM 2

5.1 Error Dynamics in Networked Systems

In network-level form, the discrete-time source evolves as

$$\bar{x}_{0,k+1} = \mathbf{A}_0 \bar{x}_{0,k} + \mathbf{B}_0 \bar{u}_{0,k} + \bar{\delta}_{0,k}, \quad (26a)$$

$$\bar{y}_{0,k} = \bar{x}_{0,k}, \quad (26b)$$

and the sensing node dynamics implement

$$\bar{x}_{k+1} = \mathbf{S} \bar{x}_k + (\hat{\mathbf{A}}_k - \mathbf{S}) \bar{z}_k + \hat{\mathbf{B}}_k \bar{u}_{0,k}, \quad (27a)$$

$$\bar{z}_k = (\mathbb{A}_m \otimes I_n) \bar{x}_k + (\mathbb{A}_0 \otimes I_n) \bar{y}_{0,k}. \quad (27b)$$

The quantities $\bar{x}_{0,k}$, $\bar{u}_{0,k}$, $\bar{\delta}_{0,k}$, and $\bar{y}_{0,k}$ are defined via (8b), while $\bar{x}_k$ and $\bar{z}_k$ use the stacking in (9b). The matrices $\mathbf{A}_0$ and $\mathbf{B}_0$ follow (10a) while the matrix $\mathbf{S}$ follows (10b). The estimate matrices $\hat{\mathbf{A}}_k$ and $\hat{\mathbf{B}}_k$ are denoted as in (10d).

We first assume the existence of the optimal estimates.

Assumption 2. There exist optimal parameter values of the source dynamics in (26a), denoted as $\mathbf{A}^*$ and $\mathbf{B}^*$, such that $\mathbf{A}^* := \mathbf{A}_0$ and $\mathbf{B}^* := \mathbf{B}_0$.

Using

$$\bar{e}_k = (\mathbb{L} \otimes I_n) \bar{x}_k - (\mathbb{A}_0 \otimes I_n) \bar{y}_{0,k} = \bar{x}_k - \bar{z}_k, \quad (28)$$

and introducing the parameter errors

$$\Phi_{d,k} \triangleq \hat{\mathbf{A}}_k - \mathbf{A}_0, \quad \Psi_{d,k} \triangleq \hat{\mathbf{B}}_k - \mathbf{B}_0, \quad (29)$$

the estimation error dynamics, with structure analogous to (15), can be expressed as

$$\bar{e}_{k+1} = \mathcal{S} \bar{e}_k + (\mathbb{L} \otimes I_n) \Xi_{d,k} \bar{\eta}_{d,k} - \Omega \bar{\delta}_{0,k}. \quad (30)$$

Recall that $\Omega := (\mathbb{L} \otimes I_n)$, then

$$\mathcal{S} := [(\mathbb{L} \otimes I_n) \mathbf{S} + (\mathbb{A}_m \otimes I_n) \mathbf{A}_0], \quad (31)$$

adaptive parameter $\Xi_{d,k} := [\Phi_{d,k}, \Psi_{d,k}]$ and time-varying regressor $\bar{\eta}_{d,k} = [\eta_{d_1,k}^\top, \dots, \eta_{d_m,k}^\top]^\top := [\bar{z}_k^\top, \bar{u}_{0,k}^\top]^\top$.

5.2 Stability of the Coupling Operator $\mathcal{S}$

To guarantee exponential convergence in the discrete-time setting, we next establish a sufficient spectral condition ensuring that the coupled operator $\mathcal{S}$ governing the error dynamics is Schur-stable.

Proposition 2. Let $\mathcal{S} := (\mathbb{L} \otimes I_n)\mathbf{S} + (\mathbb{A}_m \otimes I_n)\mathbf{A}_0$, where $\mathbb{L} \succ 0$ is the balanced Laplacian, $\mathbf{S} = \text{diag}\{S_1, \dots, S_m\}$ with each S_i Schur-stable, and $\mathbf{A}_0 = I_m \otimes A_0$ for some $A_0 \in \mathbb{R}^{n \times n}$. Define

$$\rho_S := \max_i \rho(S_i) < 1, \quad \rho_A := \rho(A_0),$$

where $\rho(\cdot)$ denotes the spectral radius. If

$$\lambda_{\min}(\mathbb{L})(1 - \rho_S) > \|\mathbb{A}_m\|_2 \rho_A, \quad (32)$$

then $\mathcal{S}$ is Schur-stable.

The following corollary extends this result to the case of norm-bounded uncertainty in the source dynamics, ensuring robustness of the Schur stability condition.

Corollary 3. (Robust Schur Stability). Suppose $A_0 = A_* + \Delta A$ with $\rho(A_*) = \rho_*$ and $\|\Delta A\|_2 \leq a$. If (32) holds with ρ_A replaced by $\rho_* + a$, then $\mathcal{S}$ remains Schur-stable for all admissible uncertainties ΔA .

Next, we analyze how the choice of the adaptive gains and the normalization mechanism affect performance and robustness, two aspects absent in the continuous-time case.

5.3 Normalized Adaptive Laws and Stability

We start with a fundamental challenge. In (30), notice that the error dynamics relate the estimation and parameter errors, motivating a regression-based update for $\Xi_{d,k}$.

To simplify the adaptive law design, we first consider the disturbance-free case $\bar{\delta}_{0,k} = \mathbf{0}_{mn}$, corresponding to the nominal dynamics. In this setting, the prediction mismatch [Fisher and Annaswamy (2023)] can be expressed as

$$\bar{\epsilon}_{k+1} := \Xi_{d,k} \bar{\eta}_{d,k} = [(\mathbb{L}^\top \mathbb{L})^{-1} \mathbb{L}^\top \otimes I_n](\bar{e}_{k+1} - \mathcal{S} \bar{e}_k), \quad (33)$$

where $\bar{\epsilon}_{k+1} = [\epsilon_{1,k+1}^\top, \dots, \epsilon_{m,k+1}^\top]^\top$. Then, the corresponding quadratic loss is

$$\mathcal{J}_k = \frac{1}{2} \|\bar{\epsilon}_{k+1}\|^2 = \frac{1}{2} \|\Xi_{d,k} \bar{\eta}_{d,k}\|^2, \quad (34)$$

whose gradient follows directly as

$$\nabla \mathcal{J}_k = \bar{\epsilon}_{k+1} \bar{\eta}_{d,k}^\top. \quad (35)$$

However, the loss $\mathcal{J}_k$ is in general not strongly convex due to the time-varying regressor norm $\|\bar{\eta}_{d,k}\|^2$. Without strong convexity, (35) can exhibit unbounded parameter updates. To address this and ensure a uniformly smooth optimization, we normalize the loss by

$$N_k = \max\{\varphi, \|\bar{\eta}_{d,k}\|^2\}, \quad \varphi > 0, \quad (36)$$

(e.g., $\varphi = 0.01$) yielding the normalized cost

$$\mathbf{J}_k = \frac{\|\bar{\epsilon}_{k+1}\|^2}{2N_k}, \quad \nabla \mathbf{J}_k = \frac{\bar{\epsilon}_{k+1} \bar{\eta}_{d,k}^\top}{N_k}. \quad (37)$$

The normalization (37) guarantees a 1-smooth loss surface, allowing the construction of a gradient-based adaptive law. Since N_k requires global information (not feasible for networked systems as agent i does not have access to global measurements to compute $\|\bar{\eta}_{d,k}\|^2$), we adopt local greedy estimates $\hat{N}_{i,k} = \max\{\varphi, \|\eta_{d,i,k}\|^2\}$, leading to

$$\nabla \hat{\mathbf{J}}_k = \text{diag} \left\{ \frac{\epsilon_{1,k+1} \eta_{d1,k}^\top}{\hat{N}_{1,k}}, \dots, \frac{\epsilon_{m,k+1} \eta_{dm,k}^\top}{\hat{N}_{m,k}} \right\}. \quad (38)$$

Finally, the first-order adaptive law is

$$\hat{\Xi}_{d,k+1}^\top = \hat{\Xi}_{d,k}^\top - \gamma \nabla \hat{\mathbf{J}}_k, \quad \gamma > 0, \quad (39)$$

where γ is a constant adaptive gain, common to all agents for simplicity of analysis. This formulation naturally connects the distributed adaptation process with the normalized gradient flow, ensuring robust convergence even under non-strongly convex regressors.

Before establishing the closed-loop stability of the discrete-time network, we first analyze the parameter adaptation mechanism in (39). The next lemma confirms that the resulting parameter iterates remain bounded under a mild gain condition, which is fundamental for guaranteeing overall stability of the coupled error dynamics.

Lemma 2. Under the adaptive update (39), let the parameter error be defined as $\Xi_{d,k} := \hat{\Xi}_{d,k} - \Xi_{d,k}^*$ where $\Xi_{d,k}^*$ denotes the ideal (stationary) parameter matrix. Then $\Xi_{d,k}$ remains bounded for all k if the adaptive gain satisfies $0 < \gamma < 2$, with $V_k = \|\Xi_{d,k}\|_F^2$ serving as a Lyapunov function candidate.

The range $0 < \gamma < 2$ guarantees a monotonic decrease of the Lyapunov energy and thus bounded parameter evolution. Having ensured that the adaptation process itself is stable, we now study the joint dynamics of the estimation error and the adaptive update in the disturbance-free case.

Theorem 3. Consider the closed-loop network composed of (30) with $\bar{\delta}_{0,k} = \mathbf{0}_{mn}$, (33), and the estimator update (39). Suppose $\varphi > 0$ and $0 < \gamma < 2$. If $\mathcal{S}$ in (31) is Schur-stable, then $\lim_{k \rightarrow \infty} \|\bar{e}_k\| = 0$.

Theorem 3 shows asymptotic stability in the disturbance-free case. We now reintroduce bounded process noise in the source and show that the same normalized adaptive law renders the estimation error ISS with respect to $\bar{\delta}_{0,k}$ —the discrete-time counterpart of Theorem 2.

Theorem 4. Consider the networked system (26a)–(27a) with error dynamics (30), $\Omega := (\mathbb{L} \otimes I_n)$ and $\|\bar{\delta}_{0,k}\| \leq \bar{\delta}^*$ for all k . Assume $\mathcal{S}$ in (31) is Schur-stable, and let $Q_d = Q_d^\top \succ 0$ and $P_d = P_d^\top \succ 0$ satisfy

$$\mathcal{S}^\top P_d \mathcal{S} - P_d = -Q_d, \quad (40)$$

Apply the normalized adaptive law (39) with $\gamma \in (0, 2)$. Then there exists a constant $c_d > 0$, depending only on (P_d, Q_d, Ω) , such that

$$\limsup_{k \rightarrow \infty} \|\bar{e}_k\| \leq c_d \bar{\delta}^*. \quad (41)$$

In particular, if $\bar{\delta}^* = 0$, then $\bar{e}_k \rightarrow 0$ as $k \rightarrow \infty$.

In summary, the discrete-time scheme achieves (i) asymptotic convergence without disturbances and (ii) ISS under bounded process noise, with performance shaped by the Schur stability of $\mathcal{S}$, the constant gain $\gamma \in (0, 2)$, and the per-agent normalization. This completes the discrete-time analysis and aligns it with the continuous-time guarantees.

We next validate these theoretical results through numerical simulations, demonstrating convergence of the estimation errors across network topologies and confirming scalability through computational complexity analysis as the number of sensing nodes increases.

6. NUMERICAL RESULTS

We validate the theoretical results through simulations on three network topologies—*star*, *cyclic*, and *path* (see Fig. 2 for details), each satisfying (1) and Remark 1. For both time scales, we consider $m = 4$ sensing nodes with $n = 2$ dimensional space. For scalability tests, we extend to $m \in \{100, 150, 200, 300\}$ to measure computational complexity².

Source dynamics: For continuous time,

$$F_* = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}, \quad G_* = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with input $v_0(t) = 0.7 \sin(0.5t) + 1.5 \cos(1.0t + \pi/6)$, with bounded disturbance $d_0(t) = 5.5[\sin(0.1t), 0.5 \cos(0.3t)]^\top$. For discrete time,

$$A_* = \begin{bmatrix} 0.9 & 0.1 \\ -0.1 & 0.95 \end{bmatrix}, \quad B_* = \begin{bmatrix} 0.05 \\ 0.10 \end{bmatrix},$$

with input $u_{0,k} = 0.9 \sin(0.05k) + 0.6 \cos(0.1k + \pi/5)$ and disturbance $\delta_{0,k} = 0.05[0.7 \sin(0.05k), 0.5 \cos(0.09k)]^\top$. Unknowns are introduced as $\|\Delta F\|_2 = 0.55$, $\|\Delta G\|_2 = 0.50$, $\|\Delta A\|_2 = 0.1$, and $\|\Delta B\|_2 = 0.07$ to obtain F_0 , G_0 , A_0 , and B_0 respectively.

Sensing nodes: Each node implements a local observer with following details. *Continuous*: $H_i = \text{diag}([-2, -2])$ (Hurwitz), with $\Gamma_\phi = \Gamma_\psi = 10I_m$; *Discrete*: $S_i = \text{diag}([0.5, 0.5])$ (Schur-stable), with $\gamma = 1.3$ and $\varphi = 0.01$.

Initial conditions: The source state is initialized as $x_0(0) = [1.0, -0.5]^\top$ (continuous) or $x_{0,0} = [1.0, -0.6]^\top$ (discrete). Each node state satisfies $x_i(0) \in \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$, $i = 1, \dots, m$, and in simulations we draw $x_i(0)$ with components independently from $\text{Unif}[-1, 1]$. We design $\hat{F}_i(0) = \hat{G}_i(0) = 0$ and $\hat{A}_{i,0} = \hat{B}_{i,0} = 0$ as initial parameter estimates.

For both time scales and all three network topologies (star, cyclic, path), the trajectories of all sensing nodes closely track the source state, and the aggregated error norms decay, as illustrated in Figs. 3a–3c for continuous time and Figs. 3e–3g for discrete time, with the corresponding bounded error norms shown in Figs. 3d and 3h. The star topology performs best, which is expected since every sensing node directly receives source information. The cyclic topology uses only partial source access (40% source weighting) yet still achieves fast agreement through network coupling, demonstrating that distributed coordination can achieve near-centralized accuracy. The path topology exhibits slower convergence due to weaker connectivity and longer information flow, but still reaches a bounded error consistent with input-to-state stability (ISS). These results are consistent with the theoretical analysis: the continuous-time analysis (Proposition 1, Corollary 2, Theorem 2) guarantees stability of the network operator $\mathcal{H}$ and ISS with respect to bounded disturbances, and the discrete-time analysis (Proposition 2, Corollary 3, Theorem 4) provides the analogous guarantees for $\mathcal{S}$ in sampled time.

² All simulations use MATLAB R2023a on a standard desktop. Continuous time: `ode45`, tolerances $10^{-6}/10^{-8}$, horizon $T = 30$ s. Discrete time: $K = 300$ samples. For complexity, we record total CPU time via `tic-toc` per topology and network size.

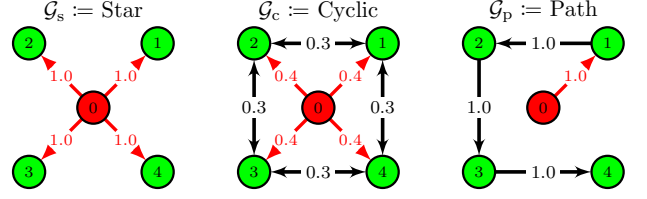


Fig. 2. Three network topologies ($\mathcal{G}_s, \mathcal{G}_c, \mathcal{G}_p$) with weights used in the simulations.

Table 1. Total runtime t_{tot} (s) vs. number of sensing nodes m for each topology.

m	Continuous			m	Discrete		
	$\mathcal{G}_s$	$\mathcal{G}_c$	$\mathcal{G}_p$		$\mathcal{G}_s$	$\mathcal{G}_c$	$\mathcal{G}_p$
100	0.66	0.65	1.65	100	0.45	0.43	0.38
150	0.76	0.86	1.95	150	0.68	0.69	0.71
200	0.98	1.23	2.63	200	0.85	0.84	0.84
300	1.88	2.68	5.38	300	1.17	1.18	1.16

Table 1 reports the total runtime t_{tot} as the number of sensing nodes m increases for each topology. The discrete-time algorithm performs explicit updates with per-step complexity $\mathcal{O}(m)$, as each iteration requires only local state propagation, neighbor aggregation, and a low-rank adaptive update per node. Consequently, total runtime scales linearly with m . The continuous-time implementation, based on the `ode45` solver, exhibits the same theoretical $\mathcal{O}(m)$ complexity but with larger constants due to adaptive step-size control and numerical stiffness. Among the topologies, the path graph consistently takes longer because weaker coupling slows information propagation, leading the solver to take smaller steps. The star and cyclic graphs achieve faster convergence and shorter runtimes, remaining below a few seconds even at $m=300$. Both continuous and discrete formulations thus demonstrate linear computational scaling and practical efficiency for large sensor networks.

Both continuous and discrete distributed adaptive estimators achieve accurate tracking, ISS robustness, and near-linear computational growth, confirming the theoretical predictions and demonstrating the framework’s scalability to large directed networks.

7. CONCLUSION AND FUTURE WORK

We studied distributed adaptive estimation for both continuous and discrete time scales with a partially unknown source. In both cases we established stability of the network operators— $\mathcal{H}$ (continuous) and $\mathcal{S}$ (discrete)—and proved convergence of the estimation error; moreover, with bounded source disturbances we derived ISS bounds for both time scales. The main distinction is architectural: continuous-time uses smooth adaptation, whereas discrete-time requires per-step normalization with a constant gain to handle sampling and non-strongly convex regressors. Simulations across multiple topologies corroborate the theory, showing that all estimates accurately track the source despite uncertainties and disturbances. The framework also scales efficiently with the number of sensing nodes, demonstrating both robustness and computational feasibility for larger networks.

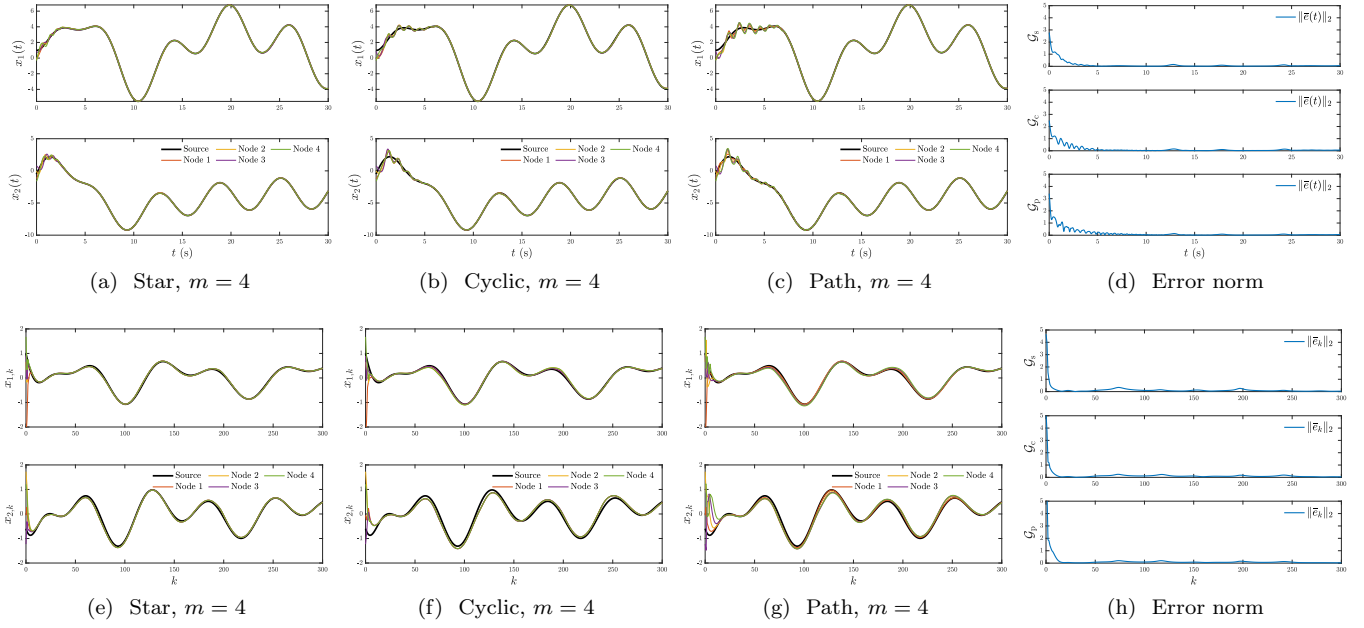


Fig. 3. Trajectories of sensing nodes for continuous-time cases (a)–(c) and discrete-time cases (e)–(g), along with corresponding error norms in (d) and (h).

Future research will focus on extending the proposed adaptive estimation framework toward control-oriented architectures. Coupling the estimator with distributed or non-fragile controllers [Javanfar et al. (2025)] can enable real-time regulation of uncertain systems using the identified source dynamics. Further directions include learning-based and geometric control formulations—such as neural-network feedback [Hedesh et al. (2025)], reinforcement-learning adaptation under packet loss [Wafi and Siami (2023)], and convex data-driven control with Riemannian metrics [Oliveira et al. (2025)]. Integrating safety principles and density-based coordination [Bozdog et al. (2025)], as well as robust active control in network systems [Zolfaghari and Moradi (2025)], strengthen the connection between adaptive estimation, control, and safe multi-agent interaction.

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Appendix A. PROOFS

Proof of Proposition 1: Since $\mathbb{L} \succ 0$, there exists $P_{\mathbb{L}} \succ 0$ such that

$$\mathbb{L}^{\top} P_{\mathbb{L}} + P_{\mathbb{L}} \mathbb{L} \succeq 2\lambda_{\min}(\mathbb{L})P_{\mathbb{L}}.$$

Let $P := P_{\mathbb{L}} \otimes I_n$ and consider $V(\bar{e}(t)) = \bar{e}^{\top}(t)P\bar{e}(t)$. Along $\dot{\bar{e}}(t) = \mathcal{H}\bar{e}(t)$,

$$\begin{aligned} \dot{V}(t) &= \bar{e}^{\top}(t)(\mathcal{H}^{\top}P + P\mathcal{H})\bar{e}(t) \\ &= \bar{e}^{\top}(t)[(\mathbb{L}^{\top}P_{\mathbb{L}} + P_{\mathbb{L}}\mathbb{L}) \otimes \tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\top}) \\ &\quad + 2(P_{\mathbb{L}}\mathbb{A}_m) \otimes \tfrac{1}{2}(\mathbf{F}_0 + \mathbf{F}_0^{\top})]\bar{e}(t). \end{aligned}$$

Because each H_i is Hurwitz, there exists $\alpha_H > 0$ with $\tfrac{1}{2}(H_i + H_i^{\top}) \preceq -\alpha_H I_n$; hence $\tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\top}) \preceq -\alpha_H I_{mn}$. Also, letting $\alpha_F = \max \Re\{\lambda(F_0)\}$ yields $\tfrac{1}{2}(\mathbf{F}_0 + \mathbf{F}_0^{\top}) \preceq \alpha_F I_{mn}$. Therefore,

$$\dot{V}(t) \leq -2\lambda_{\min}(\mathbb{L})\alpha_H\|\bar{e}(t)\|^2 + 2\|P_{\mathbb{L}}\mathbb{A}_m\|_2\alpha_F\|\bar{e}(t)\|^2.$$

Since $\|P_{\mathbb{L}}\mathbb{A}_m\|_2 \leq \|P_{\mathbb{L}}\|_2\|\mathbb{A}_m\|_2$ and $P_{\mathbb{L}}$ can be scaled without loss of generality, the sufficient condition (17) guarantees $\dot{V}(t) < 0$ for all $\bar{e}(t) \neq 0$, hence $\mathcal{H}$ is Hurwitz.

Proof of Lemma 1: Compute $\dot{V}(t)$ along $\dot{\bar{e}}(t) = \mathcal{H}\bar{e}(t)$:

$$\dot{V}(t) = \bar{e}^{\top}(t)[\mathcal{H}^{\top}(P_{\mathbb{L}} \otimes I_n) + (P_{\mathbb{L}} \otimes I_n)\mathcal{H}]\bar{e}(t).$$

Using Kronecker identities and diagonality of $\mathbf{H}$ and $\mathbf{F}_0$ yields the left-hand side of (18), implying $\dot{V}(t) < 0$.

Proof of Corollary 1: The result follows directly from Lemma 1 by substituting the matrix inequalities for both $\tfrac{1}{2}(\mathbf{H} + \mathbf{H}^{\top}) \preceq -\alpha_H I_{mn}$ and $\tfrac{1}{2}(\mathbf{F}_0 + \mathbf{F}_0^{\top}) \preceq \alpha_F I_{mn}$ into (18), yielding (19).

Proof of Corollary 2: Decompose $\tfrac{1}{2}(\mathbf{F}_0 + \mathbf{F}_0^{\top})$ as

$$\tfrac{1}{2}(\mathbf{F}_* + \mathbf{F}_*^{\top}) + \tfrac{1}{2}(\Delta\mathbf{F} + \Delta\mathbf{F}^{\top})$$

and upper bound

$$\bar{e}^{\top}(t)[2P_{\mathbb{L}}\mathbb{A}_m \otimes \tfrac{1}{2}(\Delta\mathbf{F} + \Delta\mathbf{F}^{\top})]\bar{e}(t) \leq 2\|P_{\mathbb{L}}\mathbb{A}_m\|_2 f\|\bar{e}(t)\|^2$$

Apply the standard $\mathcal{S}$ -procedure (or quadratic bound) with slack τ to dominate the uncertainty contribution.

Proof of Theorem 1: Define the Lyapunov function

$$\begin{aligned} V(t) &= \bar{e}^{\top}(t)\mathbf{P}_c\bar{e}(t) + \text{tr}[\Phi_c^{\top}(t)\Gamma_{\phi}^{-1}\Phi_c(t)] \\ &\quad + \text{tr}[\Psi_c^{\top}(t)\Gamma_{\psi}^{-1}\Psi_c(t)]. \end{aligned} \quad (\text{A.1})$$

Differentiating (A.1) along (15) and using (22) gives

$$\begin{aligned} \dot{V}(t) &= \bar{e}^{\top}(t)(\mathcal{H}^{\top}P_c + P_c\mathcal{H})\bar{e}(t) \\ &\quad + 2\bar{e}^{\top}(t)P_c(\mathbb{L} \otimes I_n)[\Phi_c(t)\bar{z}(t) + \Psi_c(t)\bar{v}_0(t)] \\ &\quad + 2\text{tr}[\dot{\Phi}_c^{\top}(t)\Gamma_{\phi}^{-1}\Phi_c(t)] + 2\text{tr}[\dot{\Psi}_c^{\top}(t)\Gamma_{\psi}^{-1}\Psi_c(t)]. \end{aligned}$$

Substituting (23) yields perfect cancellation of the cross terms, leading to

$$\dot{V}(t) = -\bar{e}(t)^{\top}Q_c\bar{e}(t) \leq 0.$$

Therefore, the Lyapunov function $V(t)$ is nonincreasing and all trajectories are bounded. Because $\dot{V}(t) = 0$ if and only if $\bar{e}(t) = 0$, the largest invariant set contained in $\{\dot{V}(t) = 0\}$ corresponds to $\bar{e}(t) \equiv 0$, $\dot{\Phi}_c = \dot{\Psi}_c = 0$. By LaSalle's invariance principle, $\bar{e}(t) \rightarrow 0$ and the adaptive parameters converge to constant steady-state values.

Proof of Theorem 2: Consider the Lyapunov function $V(t)$ given in (A.1) with $\Phi_c(t), \Psi_c(t)$ as in (13). Along the error dynamics and using the adaptive laws (23), all mixed terms involving $\Xi_c(t)\bar{\eta}_c(t)$ cancel by construction. Using (24) we obtain

$$\dot{V}(t) = -\bar{e}(t)^{\top}Q_c\bar{e}(t) + 2\bar{e}(t)^{\top}P_c\Omega\bar{d}_0(t).$$

For any $\varepsilon \in (0, 1)$, by Cauchy–Schwarz and completing the square

$$\begin{aligned} 2\bar{e}(t)^{\top}P_c\Omega\bar{d}_0(t) &\leq \varepsilon\bar{e}(t)^{\top}Q_c\bar{e}(t) \\ &\quad + \tfrac{1}{\varepsilon}\lambda_{\max}(P_c\Omega\Omega^{\top}P_c)\|\bar{d}_0(t)\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \dot{V}(t) &\leq -(1 - \varepsilon)\bar{e}(t)^{\top}Q_c\bar{e}(t) \\ &\quad + \tfrac{1}{\varepsilon}\lambda_{\max}(P_c\Omega\Omega^{\top}P_c)\|\bar{d}_0(t)\|^2. \end{aligned}$$

Using $\bar{e}(t)^{\top}Q_c\bar{e}(t) \geq \lambda_{\min}(Q_c)\|\bar{e}(t)\|^2$ and $\bar{e}(t)^{\top}P_c\bar{e}(t) \geq \lambda_{\min}(P_c)\|\bar{e}(t)\|^2$, we get

$$\dot{V}(t) \leq -\alpha_c\|\bar{e}(t)\|^2 + \beta_c\|\bar{d}_0(t)\|^2,$$

where $\alpha_c := (1 - \varepsilon)\lambda_{\min}(Q_c)$ and $\beta_c := \tfrac{1}{\varepsilon}\lambda_{\max}(P_c\Omega\Omega^{\top}P_c)$. By standard ISS comparison arguments, it follows that

$$\limsup_{t \rightarrow \infty} \|\bar{e}(t)\| \leq \sqrt{\frac{\beta_c}{\alpha_c\lambda_{\min}(P_c)}}\bar{d}^* =: c_c\bar{d}^*,$$

which establishes (25). If $\bar{d}^* = 0$, then $\dot{V}(t) \leq -\alpha\|\bar{e}(t)\|^2$ and thus $\bar{e}(t) \rightarrow 0$.

Proof of Proposition 2: Using the discrete-time Lyapunov criterion, $\mathcal{S}$ is Schur-stable if $\exists P_{\mathcal{S}} \succ 0$ such that $\mathcal{S}^{\top}P_{\mathcal{S}}\mathcal{S} - P_{\mathcal{S}} \prec 0$. By separability of the Kronecker structure and the diagonal form of $\mathbf{S}$, the sufficient condition (32) ensures contraction of the term $(\mathbb{L} \otimes I_n)\mathbf{S}$ dominates the coupling effect of $(\mathbb{A}_m \otimes I_n)\mathbf{A}_0$, yielding $\rho(\mathcal{S}) < 1$.

Proof of Corollary 3: The uncertain source matrix can be written as $\mathbf{A}_0 = I_m \otimes (A_* + \Delta A) = \mathbf{A}_* + I_m \otimes \Delta A$. From the submultiplicative property of the induced norm, $\|(\mathbb{A}_m \otimes I_n)\mathbf{A}_0\|_2 \leq \|\mathbb{A}_m\|_2(\rho_* + \|\Delta A\|_2) \leq \|\mathbb{A}_m\|_2(\rho_* + a)$. Hence, if the modified condition in (32) holds with ρ_A replaced by $\rho_* + a$, the perturbation cannot shift any eigenvalue of $\mathcal{S}$ outside the unit circle, and $\mathcal{S}$ remains Schur-stable for all admissible ΔA .

Proof of Lemma 2: Define the Lyapunov function $V_k = \|\Xi_{d,k}\|_F^2 = \text{tr}(\Xi_{d,k}^\top \Xi_{d,k})$. Using the adaptive update (39), the parameter recursion can be written as

$$\Xi_{d,k+1} = \Xi_{d,k} - \gamma \nabla \hat{\mathbf{J}}_k,$$

where the local gradient is

$$\nabla \hat{\mathbf{J}}_k = \text{diag} \left\{ \frac{\epsilon_{1,k+1} \eta_{d_1,k}^\top}{\hat{N}_{1,k}}, \dots, \frac{\epsilon_{m,k+1} \eta_{d_m,k}^\top}{\hat{N}_{m,k}} \right\}.$$

Then, the one-step difference of V_k is

$$\begin{aligned} \Delta V_k &:= V_{k+1} - V_k = \|\Xi_{d,k+1}\|_F^2 - \|\Xi_{d,k}\|_F^2 \\ &= \|\Xi_{d,k} - \gamma \nabla \hat{\mathbf{J}}_k\|_F^2 - \|\Xi_{d,k}\|_F^2. \end{aligned} \quad (\text{A.2})$$

Expanding (A.2) and rearranging gives

$$\Delta V_k = -2\gamma \text{tr}(\Xi_{d,k}^\top \nabla \hat{\mathbf{J}}_k) + \gamma^2 \|\nabla \hat{\mathbf{J}}_k\|_F^2. \quad (\text{A.3})$$

From the definition of $\nabla \hat{\mathbf{J}}_k$, we have

$$\text{tr}(\Xi_{d,k}^\top \nabla \hat{\mathbf{J}}_k) = \sum_{i=1}^m \frac{1}{\hat{N}_{i,k}} \epsilon_{i,k+1}^\top \Xi_{d_i,k} \eta_{d_i,k}.$$

Using the regression form $\epsilon_{i,k+1} = \Xi_{d_i,k} \eta_{d_i,k}$ and the fact that $\hat{N}_{i,k} \geq \|\eta_{d_i,k}\|^2$, it follows that

$$\Delta V_{i,k} = -\frac{\gamma}{\hat{N}_{i,k}} \left(2 - \frac{\gamma \|\eta_{d_i,k}\|^2}{\hat{N}_{i,k}} \right) \|\epsilon_{i,k+1}\|^2 \leq 0,$$

for $0 < \gamma < 2$ and thus $\Delta V_k = \sum_i \Delta V_{i,k} \leq 0$ globally. Hence V_k is nonincreasing and bounded below by zero, implying bounded $\Xi_{d,k}$ for all k .

Proof of Theorem 3: From Lemma 2, $\Xi_{d,k}$ is bounded and V_k nonincreasing. Summing (A.3) over k gives

$$\sum_{k=0}^{\infty} \frac{\|\bar{\epsilon}_{k+1}\|^2}{N_k} \leq \frac{V_0}{\gamma(2-\gamma)} < \infty,$$

which implies $\|\bar{\epsilon}_{k+1}\|/N_k \rightarrow 0$ as $k \rightarrow \infty$. With bounded regressors $\|\eta_{d,k}\|^2$, it follows that $\bar{\epsilon}_k \rightarrow 0$.

Substituting $\bar{\epsilon}_k$ into the error recursion (30) yields

$$\bar{\epsilon}_{k+1} = \mathcal{S} \bar{\epsilon}_k + (\mathbb{L} \otimes I_n) \bar{\epsilon}_{k+1}.$$

Since $\mathcal{S}$ is Schur-stable, there exist constants $c > 0$ and $\alpha \in (0, 1)$ such that $\|\mathcal{S}^\ell\| \leq c\alpha^\ell$ for all $\ell \geq 0$. Unfolding the recursion gives

$$\bar{\epsilon}_k = \mathcal{S}^{k-k_0} \bar{\epsilon}_{k_0} + \sum_{t=k_0}^{k-1} \mathcal{S}^{k-1-t} (\mathbb{L} \otimes I_n) \bar{\epsilon}_{t+1}.$$

The first term decays exponentially, and the second term is a convolution of an exponentially decaying kernel with a vanishing sequence $\bar{\epsilon}_{t+1} \rightarrow 0$, hence also tends to zero. Then, $\lim_{k \rightarrow \infty} \|\bar{\epsilon}_k\| = 0$, showing asymptotic convergence.

Proof of Theorem 3: Define the Lyapunov function

$$V_k = \bar{\epsilon}_k^\top P_d \bar{\epsilon}_k + \text{tr}(\Phi_{d,k}^\top \Gamma_\phi^{-1} \Phi_{d,k}) + \text{tr}(\Psi_{d,k}^\top \Gamma_\psi^{-1} \Psi_{d,k}).$$

Under the update (39), the cross-terms in $\Xi_{d,k} \bar{\eta}_{d,k}$ cancel, and from (40),

$$V_{k+1} - V_k \leq -\bar{\epsilon}_k^\top Q_d \bar{\epsilon}_k + 2\bar{\epsilon}_k^\top P_d \Omega \bar{\delta}_{0,k}.$$

Applying the discrete-time inequality

$$2x_k^\top Y z_k \leq \varepsilon x_k^\top Q_d x_k + \frac{1}{\varepsilon} \lambda_{\max}(Y Q_d^{-1} Y^\top) \|z_k\|^2,$$

for any $\varepsilon \in (0, 1)$ gives

$$\begin{aligned} V_{k+1} - V_k &\leq -(1 - \varepsilon) \bar{\epsilon}_k^\top Q_d \bar{\epsilon}_k \\ &\quad + \frac{1}{\varepsilon} \lambda_{\max}(P_d \Omega Q_d^{-1} \Omega^\top P_d) \|\bar{\delta}_{0,k}\|^2. \end{aligned}$$

Using $\bar{\epsilon}_k^\top Q_d \bar{\epsilon}_k \geq \lambda_{\min}(Q_d) \|\bar{\epsilon}_k\|^2$ and $V_k \geq \lambda_{\min}(P_d) \|\bar{\epsilon}_k\|^2$, we obtain a standard ISS recursion

$$V_{k+1} \leq (1 - \alpha_d) V_k + \beta_d \|\bar{\delta}_{0,k}\|^2,$$

for some $\alpha_d, \beta_d > 0$. By the discrete-time comparison lemma, this implies

$$\limsup_{k \rightarrow \infty} \|\bar{\epsilon}_k\| \leq \sqrt{\frac{\beta_d}{\alpha_d \lambda_{\min}(P_d)}} \bar{\delta}^* =: c_d \bar{\delta}^*,$$

proving (41). When $\bar{\delta}^* = 0$, the negative-definite term dominates and $\bar{\epsilon}_k \rightarrow 0$.