ON THE COMBINATORICS OF TABLEAUX — CLASSIFICATION OF LATTICES UNDERLYING SCHENSTED CORRESPONDENCES

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Abstract. The celebrated Robinson–Schensted algorithm and each of its variants that have attracted substantial attention can be constructed using Fomin's "growth diagram" construction from a modular lattice that is also a weighted-differential poset. We classify all such lattices that meet certain criteria; the main criterion is that the lattice is distributive. Intuitively, these criteria seem excessively strict, but all known Fomin lattices satisfy all of these criteria, with the sole exception of one family that is not even distributive, the Young–Fibonacci lattices. Disappointingly, we discover no new Fomin lattices.

Contents

Date: Nov 10, 2025.

2020 *Mathematics Subject Classification.* Primary 06A11; Secondary 05A17, 06B99.

Key words and phrases. differential poset, Fomin lattice, graded graph, growth diagram, Robinson–Schensted algorithm, Young diagram, Young tableaux.

1. Introduction

The celebrated Robinson–Schensted algorithm[Rob1938][Schen1961][Fom1994][Fom1995] and each of its variants that have attracted substantial attention can be constructed[Wor2023, sec. 2] using Fomin's "growth diagram" construction from a modular lattice that is also a (weighted-)differential poset[Stan1988]. We define these requirements as a "Fomin lattice" that has a minimum element $\hat{0}$. We classify all Fomin lattices that meet certain criteria. Our main criterion is that the lattice is distributive, with secondary criteria on the set of points (the join-irreducible elements of the lattice). Intuitively, these criteria seem excessively strict, but all known Fomin lattices satisfy all of these criteria, with the sole exception of one family that is not even distributive, the Young–Fibonacci lattices. This work adds to the classes of lattices within which all of the Fomin lattices have been classified. Disappointingly, we discover no new Fomin lattices.

The history of this problem in the literature is discussed in para. 2.1. Para. 2.2 enumerates the relevant properties of structures. All known Fomin lattices are described in para. 2.3. They consist of Young's lattice of partitions, the shifted Young's lattice of strict partitions, the lattices of partitions into a limited number of parts, the Young-Fibonacci lattices, the cylindric partition lattices, the lattice of ideals of a "skew strip", some trivial cases, and certain derivatives of other Fomin lattices. The previous classification theorems for Fomin lattices are discussed in para. 2.4.

Our new classification theorem is in sec. 7. The class of lattices classified by our theorem is incomparable with the classes of previous classification theorems; it does not supersede them. However, it is an incremental contribution to the classification of Fomin lattices (and thus Robinson–Schensted algorithms) and is the only one that applies to "weighted-differential" lattices, that is, where the weight on coverings is not identically 1.

Our proof techniques are largely elementary. They consist of a careful accounting of the consequences of the differential condition (7) in lattices that satisfy our criteria. This "local characterization" (sec. 5) is a succession of restatements of the differential condition (lem. 5.7, 5.13, 5.14, 5.18, 5.25, and 5.26). These results culminate in lem. 5.27, that no point covers three distinct points or is covered by three distinct points. Knowing this, the points must be placeable on a two-dimensional grid (lem. 6.4). We then proceed to the "global characterization" (sec. 6) of how the points can be placed on the grid. This culminates in the classification theorem 7.1.

Finally, we discuss possible future directions for research (sec. 8), which assesses the prospect of various relaxations of the criteria of the classification theorem.

Thus this article consists of:

- background information including the previous literature, a catalog of properties, known Fomin lattices, and a recapitulation of properties of the Cartesian product (sec. 2),
- a statement of the definitions essential for the classification (sec. 3),
- a construction supporting the discussion of future directions (sec. 4),
- the characterization of Fomin lattices that meet our criteria (sec. 5 and 6),
- the classification theorem for Fomin lattices that meet our criteria (sec. 7), and
- a discussion of future directions for research (sec. 8).

The development essential to prove our classification theorem 7.1 is contained in sec. 3, 5, 6, and 7.

2. Background

2.1. Previous literature. The literature on differential structures can be classified based on the generality of the structures being considered. The generality can be described along two dimensions:

(1) the generality of the underlying structures (lattice, poset, etc.) being considered and
(2) whether the covering pairs of elements of the structure are all given the weight 1 (the *unweighted* case) or are allowed varying weights (the *weighted* case).

In regard to the underlying structures, from most general to least general, the major categories are

(1) Dual graded graph (two graphs or multi-graphs on the same set of vertices with the same grading with separate weights or edge multiplicities in the two graphs between pairs of points),
(2) Graded poset,
(3) Lattice (which by the differential property must be modular),
(4) Distributive lattice,
(5) Distributive lattice with further conditions (as in th. 7.1).

Significant works in the literature, and distinctive parts of them, can be classified in the following table. In some cases a work's definitions and general analysis are considerably more general than all of the examples described in the work; in that case the classification of the definitions are marked "D" and classification of the examples are marked "E". Some works that are largely a description of examples (often novel examples) are only marked with "E". The works with the best previous classification results (detailed at the end of para. 2.3) are listed with "C" in the appropriate category.

Underlying structure	Weighted	Unweighted
Dual graded graph	Fomin [Fom1994, Def. 1.3.4]	Qing [Qing2008, sec. 2.2] E
Graded poset		Stanley [Stan1988, sec. 2.2] DC Lewis [Lew2007, sec. 3] E
(Modular) lattice	Stanley [Stan1990, sec. 3] this article D	Stanley [Stan1988] E Byrnes [Byrn2012, Ch. 6] C
Distributive lattice	Fomin [Fom1994, sec. 2.2]	Elizalde [El2025] E
Distributive lattice with further restrictions	this article E	

2.2. Properties considered in this article. A large number of properties of structures are considered in this area of combinatorics. This section catalogs the properties that are relevant to this article and their definitions. This article classifies in sec. 7 structures which have all of these properties. The properties are here generally ordered from weakest to strongest. Some theorems relating the properties are included. For a casual reading of this article, this section can be skipped and referred back to as necessary. Section 3 gives terse definitions that suffice for reading characterizations in this article.

(A) *The structure is a dual graded graph (DGG):* [Fom1994] There is a graded set of vertices. In the original formulation each grade contains a finite number of vertices, but we do not include that criterion here. (But see property (D).) Vertices in a grade have edges to some vertices in adjacent ranks, forming a graph. Each edge has an assigned weight, with the weights being drawn from a set of *values* V. In the original formulation, each edge is an otherwise unweighted multi-edge, so the weights are positive integers ($V = \mathbb{Z}$), but we do not include that criterion here.

A dual graded graph is two such graphs on the same graded set of vertices, and thus there are up to two weights between any pair of vertices in adjacent grades. The edges of the first graph are conceptualized as being directed upward (toward higher grades) and the edges of the second graph are conceptualized as being directed downward.

There is some ambiguity in the presentation of a DGG; if a pair of vertices has a non-zero weight between them in only one graph, we may add an edge between them in the other graph that has a weight of 0 without changing any of the essential properties of the DGG. This allows the two graphs to have the same set of edges, with the two graphs differing only in the weights of the edges.

Critically, the DGG has the *differential property*: It has a constant r called the *differential degree* and the DGG is called r-*differential.* Assume that weight 0 edges have been added so that both graphs have the same edges. Let $x \lessdot y$ mean that y is in the rank above x and (x, y) is an edge. When $x \lessdot y$, let $w_1(x \lessdot y)$ be the weight of the edge (x, y) in the first graph and $w_2(x \lessdot y)$ be the weight of the edge (x, y) in the second graph. Then the differential property is that for every pair of vertices x and y in the same grade,

$$\sum_{z \,|\, x \lessdot z, y \lessdot z} w_1(x \lessdot z) w_2(x \lessdot z) - \sum_{w \,|\, w \lessdot x, w \lessdot y} w_2(w \lessdot x) w_1(w \lessdot y) = \begin{cases} 0 & \text{if } x \neq y \\ r & \text{if } x = y \end{cases} \tag{1}$$

(B) *The structure is a differential poset:* [Stan1988] This is a narrower type of DGG. Both graphs have the same set of edges and the weights on the edges of the second graph are all 1. In the original formulation, the weights on the edges of the first graph are also 1, but we do not include that criterion here. Differential posets with weights that are not 1 are also called *weighted-differential* posets.

The graphs are the Hasse diagram of a poset. In the original formulation, the poset is assumed to have a minimum element, $\hat{0}$, but we do not include that criterion here. A differential poset is required to be *unique-cover-modular*.

Definition 2.1. *A poset J is* cover-modular[1] *if when $x, y \in J$ and x and y both cover an element of J, then they are both covered by an element of J (which may not be unique). Dually, we require that when $x, y \in J$ and x and y both are covered by an element of J, then they both cover an element of J (which may not be unique).*

Definition 2.2. *A poset J is* unique-cover-modular *if when $x, y \in J$ and either x and y both cover an element of J or x and y are covered by an element of J, then they both cover a unique element of J and they are both covered by a unique element of J.*

Theorem 2.3. [Birk1967, §II.8 Th. 14] *If a locally-finite poset has $\hat{0}$ and is cover-modular, then it is graded.*

(C) *The structure is locally-finite:* In both the DGG and differential poset formulations, the structure (seen as a poset) is assumed to be locally-finite.

(D) *The structure has finite covers:* In both the DGG and differential poset formulations, every vertex has a finite number of upward and downward edges, or a finite number of upward and downward covers.

[Fom1994] requires each rank of vertices to be finite. If the structure has a $\hat{0}$, this is equivalent to finite covers, but without a $\hat{0}$, it is a stronger property. (See (H).)

(E) *The structure has unique-modular-covers and the weights are projective-constant:* In this case, if $x \neq y$, (1) is satisfied automatically. If we define $w(x \lessdot y) = w_1(x \lessdot y) w_2(x \lessdot y)$, then the $x = y$ case of (1) is equivalent to: For every vertex x,

$$\sum_{z \mid x \lessdot z} w(x \lessdot z) - \sum_{w \mid w \lessdot x} w(w \lessdot y) = r \tag{2}$$

Thus if we require unique-modular-covering and projective-constancy, the DGG and weighted-differential poset structures effectively coincide with only one weight function. Differential posets that are not weighted-differential are automatically projective-constant.

Definition 2.4. *The weights on a DGG or differential poset are* projective-constant *iff (for a DGG) the weight functions w_1 and w_2 or (for a differential poset) the weight function w have the following property: the function f is constant on coverings which are* cover-projective*: If $x \wedge y \lessdot x$ and $y \lessdot x \vee y$, then $f(x \wedge y \lessdot x) = f(y \lessdot x \vee y)$ and $f(x \wedge y \lessdot y) = f(x \lessdot x \vee y)$.*

(F) *The structure is a differential poset that is a lattice.*

Lemma 2.5. [Birk1967, §II.8 Th. 16] *If a lattice is locally-finite and cover-modular, then it is modular.*

(G) *The structure is a differential poset that is a modular lattice:* If the structure is a locally-finite modular lattice, then if the weight function is projective-constant, it is constant on projectivity classes[Birk1967, §V.5] of coverings.

We define "Fomin lattices" to be structures with the above set of properties.

(H) *The structure has a $\hat{0}$:* We consider the existence of a minimum vertex as a separate property. [Stan1988] defines all differential posets to have a $\hat{0}$.

(I) *The weighting and differential degree are admissible:* However see lem. 3.2 for how we abuse the term "admissible".

Definition 2.6. *A weighting w and differential degree r are* admissible *if the set of values V (the target set of w) is $\mathbb{Z}$, all of the values of w are positive, and r is positive.*

Depending on the degree of abstraction of the discussion, the requirements for the set of values V varies. The minimum requirement is that V be a module over a unitial ring of *scalars* S.[2] Usually we will be unconcerned with the details of S other than it necessarily contains a homomorphic image of $\mathbb{Z}$ and thus we can multiply elements of V by integer coefficients. If $V = \mathbb{Z}$, then $\mathbb{Z}$ is *isomorphically* embedded in S.

[1] We choose this name because of its resemblance to the property of coverings in modular lattices. But in general, J is not necessarily modular, indeed, it need not even be a lattice. A poset with this property and finite length is called a "modular poset" in [Birk1967, § II.8].

[2] Remember, V is a module over a ring, not a module over an algebra, so it is not assumed to have an internal product.

(J) *The structure is a distributive lattice:* In this case, we define J to be the poset of *points* (join-irreducibles) of the lattice, so the lattice is isomorphic to the set of finite order ideals of J.
(K) *The poset of points J is unique-cover-modular:* Note that this requirement is on the poset of points, not a requirement on the structure itself (which all Fomin lattices satisfy).
(L) *The structure is a lattice that cannot be factored into two nontrivial lattices:* If the lattice is distributive this is equivalent to J cannot be partitioned into two disjoint non-empty posets $J = J_1 \sqcup J_2$.[3]

2.3. Known Fomin lattices. All known Fomin lattices can be constructed as follows:

(1) Any lattice satisfying (A) can be made into a *degenerate* Fomin lattice by setting $r = 0$ and $w = 0$.
(2) [Fom1994, Exam. 2.1.2 and 2.2.7] Young's lattice, $\mathbb{Y}$, the lattice of partitions, with $r = 1$ and $w = 1$ is an admissible Fomin lattice with $\hat{0}$.
(3) [Sag1979][Wor1984][Fom1994, Exam. 2.2.8] The shifted Young's Lattice, $\mathbb{SY}$, the lattice of partitions into unequal parts, with $r = 1$ and $w = 1$ or 2 is an admissible Fomin lattice with $\hat{0}$.
(4) [Fom1994, Exam. 2.2.5] For any integer $k \geq 1$, the k-row Young's lattice, $\mathbb{Y}_k$, the lattice of partitions into at most k parts (or dually, into an arbitrary number of parts that are all $\leq k$), with $r = k$ and w having a particular range of values is an admissible Fomin lattice with $\hat{0}$.
(5) [Fom1994, Exam. 2.2.1] The upward semi-infinite chain $\mathbb{N}$ with $r = k$ and w having a range of values (which is $\mathbb{Y}_1$, a special case of (4)) is an admissible Fomin lattice with $\hat{0}$.
(6) [Stan1988, sec. 5][Roby1991, Ch. 5] For any integer $k \geq 1$, the Young–Fibonacci lattice of degree k, $\mathbb{YF}_k$, with $r = k$ and $w = 1$ is an admissible Fomin lattice with $\hat{0}$.
(7) [El2025, sec. 2 Fig. 2] For any integers $d, L \geq 1$, the *cylindric partition* lattice $\mathbb{CP}_{dL}$ with $r = 0$ and $w = 1$ is a Fomin lattice but is not admissible and has no $\hat{0}$.[4]
(8) [Fom1994, Exam. 2.2.10] defines *t-SkewStrip*, which is a special case of (7): *t-SkewStrip* $= \mathbb{CP}_{2t}$.
(9) [Fom1994, Exam. 2.2.11] $\mathbb{SS}_t = \mathrm{Ideal_f}(t\text{-}\mathit{SkewStrip}) = \mathrm{Ideal_f}(\mathbb{CP}_{2t})$ with $r = 0$ and $w = 1$ or 2 is a Fomin lattice but is not admissible and has no $\hat{0}$.[5]
(10) [Fom1994, Lem. 2.2.3] Any cartesian product of a finite number of Fomin lattices that all have the same value set V, with the weight of a covering in the product lattice equal to the weight within the factor from which the covering is derived and the differential degree equal to the sum of the differential degrees of the factors, is a Fomin lattice. The product lattice has $\hat{0}$ if all factors have $\hat{0}$ and is admissible if all factors are admissible.
(11) Given a Fomin lattice, the same lattice but multiplying the differential degree and weighting by the same constant gives a Fomin lattice. The resulting lattice is admissible if the original lattice is admissible and the constant is a positive integer.

2.4. Previous classifications. The previous classifications of Fomin lattices with $\hat{0}$ are:

(1) [Stan1988, Prop. 5.5] If a Fomin lattice with $\hat{0}$ is distributive and has $w = 1$, then it is isomorphic to $\mathbb{Y}^{\times r}$.[6]
(2) [Byrn2012, Ch. 6] If a Fomin lattice with $\hat{0}$ has $r = 1$ and $w = 1$, then it is either $\mathbb{Y}$ or $\mathbb{YF}_1$.

In particular, there are no previous classifications for the weighted case, when w is not identically 1.

2.5. Cartesian product. Para. 2.3 item (10) above shows that cartesian product preserves Fomin lattices. Surprisingly, the converse is also true.

Theorem 2.7. *If L is a Fomin lattice with differential degree r and weighting w, and L is the cartesian product of two lattices $L = L_1 \times L_2$, then*

(1) L_1 is a Fomin lattice with some differential degree r_1 and weighting w_1.
(2) L_2 is a Fomin lattice with some differential degree r_2 and weighting w_2.
(3) $r = r_1 + r_2$.
(4) For any $x \lessdot y$ in L_1 and z in L_2, $w_1(x \lessdot y) = w((x, z) \lessdot (y, z))$.

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[3] We use $\sqcup$ for disjoint unions of sets and posets because "disjoint union" is the coproduct in the categories of sets and posets.

[4] [El2025] uses the notation $\mathcal{L}_{d,L}$ for $\mathbb{CP}_{dL}$. The cylindric partitions have been known since at least [GessKratt1997] but [El2025] seems to be the first report that all of the $\mathbb{CP}_{dL}$ are 0-differential. (Though [Fom1994] reports that the $\mathbb{CP}_{2t}$ are 0-differential.)

[5] We define $\mathrm{Ideal_f}(P)$, where P is a poset, to be the lattice of finite lower order ideals of P.

[6] We use $\bullet^{\times n}$ to denote the n-fold cartesian exponential to avoid ambiguity with other uses of superscripts.

(5) *For any w in L_1 and $x \lessdot y$ in L_2, $w_2(x \lessdot y) = w((w,x) \lessdot (w,y))$.*

Proof. First we show that for any $x \lessdot y$ in L_1 and any $z_1, z_2 \in L_2$,

$$w((x,z_1) \lessdot (y,z_1)) = w((x,z_2) \lessdot (y,z_2)). \tag{3}$$

Because of property (A)(4), w is constant on cover-projective pairs of elements of L, so equation (3) holds if $z_1 \lessdot z_2$. Since any two elements $z_1, z_2 \in L_2$ can be connected by a finite chain of upward and downward covers, equation (3) holds for any z_1 and z_2. Thus the w_1 in conclusion (4) exists.

Similarly, for any $w_1, w_2 \in L_1$ and any $x \lessdot y$ in L_2

$$w((w_1,x) \lessdot (w_1,y)) = w((w_2,x) \lessdot (w_2,y)). \tag{4}$$

Thus the w_2 in conclusion (5) exists.

We now show that r_1 and r_2 exist, that is, that L_1 is a Fomin lattice with differential degree r_1 and weighting w_1 and similarly for L_2. For any $x \in L_1$ and any $z_1, z_2 \in L_2$, examining (x, z_1),

$$\sum_{(p,q)\,|\,(p,q)\lessdot(x,z_1)} w((p,q) \lessdot (x,z_1)) + r = \sum_{(t,u)\,|\,(x,z_1)\lessdot(t,u)} w((x,z_1) \lessdot (t,u))$$

$$\sum_{p\,|\,p\lessdot x} w((p,z_1) \lessdot (x,z_1)) + \sum_{q\,|\,q\lessdot z_1} w((x,q) \lessdot (x,z_1)) + r = \sum_{t\,|\,x\lessdot t} w((x,z_1) \lessdot (t,z_1)) + \sum_{u\,|\,z_1\lessdot u} w((x,z_1) \lessdot (x,u))$$

$$\sum_{p\,|\,p\lessdot x} w_1(p \lessdot x) + \sum_{q\,|\,q\lessdot z_1} w_2(q \lessdot z_1) + r = \sum_{t\,|\,x\lessdot t} w_1(x \lessdot t) + \sum_{u\,|\,z_1\lessdot u} w_2(z_1 \lessdot u) \tag{5}$$

Similarly, examining (x, z_2) we derive

$$\sum_{p\,|\,p\lessdot x} w_1(p \lessdot x) + \sum_{q\,|\,q\lessdot z_2} w_2(q \lessdot z_2) + r = \sum_{t\,|\,x\lessdot t} w_1(x \lessdot t) + \sum_{u\,|\,z_2\lessdot u} w_2(z_2 \lessdot u) \tag{6}$$

Subtracting (6) from (5) gives

$$\sum_{q\,|\,q\lessdot z_1} w_2(q \lessdot z_1) - \sum_{q\,|\,q\lessdot z_2} w_2(q \lessdot z_2) = \sum_{u\,|\,z_1\lessdot u} w_2(z_1 \lessdot u) - \sum_{u\,|\,z_2\lessdot u} w_2(z_2 \lessdot u)$$

$$\sum_{u\,|\,z_1\lessdot u} w_2(z_1 \lessdot u) - \sum_{q\,|\,q\lessdot z_1} w_2(q \lessdot z_1) = \sum_{u\,|\,z_2\lessdot u} w_2(z_2 \lessdot u) - \sum_{q\,|\,q\lessdot z_2} w_2(q \lessdot z_2)$$

This shows that $\sum_{u\,|\,z\lessdot u} w_2(z \lessdot u) - \sum_{q\,|\,q\lessdot z} w_2(q \lessdot z)$ is the same for $z = z_1$ and $z = z_2$ for any $z_1, z_2 \in L_2$. We define r_2 as this value.

Similarly, we show that $\sum_{t\,|\,z\lessdot t} w_1(z \lessdot t) - \sum_{q\,|\,q\lessdot z} w_1(q \lessdot z)$ is independent of $z \in L_1$, and we define r_1 as this value.

From these facts it is straightforward to prove the conclusions of the theorem. ☐

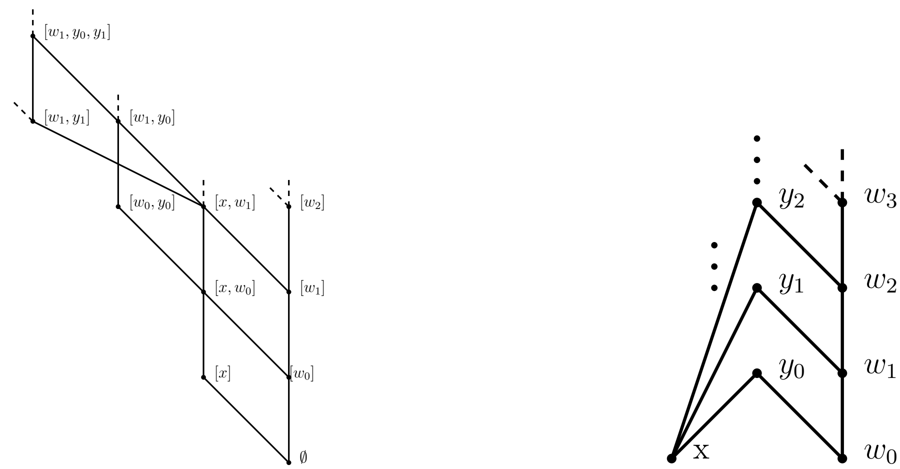
Lemma 2.8. *If L is a Fomin lattice with differential degree r and weighting w, and if L is the cartesian product of two Fomin lattices $L = L_1 \times L_2$, with L_1 having differential degree r_1 and weighting w_1 and L_2 having differential degree r_2 and weighting w_2, then*

(1) *If L has a minimum element $\hat{0}_L$, then L_1 has a minimum element $\hat{0}_{L_1}$ and L_2 has a minimum element $\hat{0}_{L_2}$.*
(2) *If L has a minimum element $\hat{0}_L$ and is admissible, and L_1 and L_2 are not the trivial lattice, then L_1 and L_2 are admissible.*
(3) *If L is distributive, then L_1 and L_2 are distributive.*

Proof.

Regarding (1) and (3): These are straightforward lattice reasoning.

Regarding (2): By the construction in th. 2.7, the set of values V_{L_1} for L_1 and the set of values V_{L_2} for L_2 are the same as the set of values V_L of L, and thus are $\mathbb{Z}$. Every value of w_1 and every value of w_2 are values of w for suitable arguments, so all values of w_1 and w_2 are positive. Since L has a $\hat{0}_L$, L_1 has a $\hat{0}_{L_1}$ and L_2 has a $\hat{0}_{L_2}$. By lem. 3.2, both L_1 and L_2 are admissible. ☐



(A) Lattice L has finite upward covers (B) Poset $J = J(L)$ does not have finite upward covers

FIGURE 1. Counterexample in rem. 2.12

Remark 2.9.

(1) A McLarnan map on L may not factor into McLarnan maps on L_1 and L_2 in any nice way.

(2) The trivial lattice is a Fomin lattice: Its w must be the empty function and $r = 0$. It is the identity of the cartesian product of Fomin lattices. By our definition, this is not an admissible weighting. Thus lem. 2.8(2) requires the hypothesis that L_1 and L_2 are not the trivial lattice.[7]

(3) Lem. 2.8(2) requires the hypothesis that $\hat{0}_L$ exists: Consider $\mathbb{Y}$, the lattice of partitions, which is an admissible Fomin lattice with $w = 1$ and $r = 1$. Consider $\mathbb{Y}^*$, the lattice dual of $\mathbb{Y}$, which is a Fomin lattice with $w = 1$ and $r = -1$ and so is not admissible. Define $2\mathbb{Y}$ to be the Fomin lattice $\mathbb{Y}$ with $w = 2$ and $r = 2$, which is admissible. Then $L = 2\mathbb{Y} \times \mathbb{Y}^*$ is a Fomin lattice with $w = 1$ or 2 and $r = 1$, which is admissible. Thus $L = 2\mathbb{Y} \times \mathbb{Y}^*$ is an admissible Fomin lattice that is a product of Fomin lattices, one of which is not admissible.

2.6. Cartesian product of distributive Fomin lattices. The results in this paragraph are only relevant under the assumption that L is distributive and that its poset of join-irreducibles is unique-cover-modular. Hence, they are not significant until para. 5.5. However, since these results are not new, we place them in this section.

Lemma 2.10. *Let L be a finitary*[8] *distributive lattice with poset of join-irreducible elements $J_L = J(L)$. Then L is the cartesian product of two non-trivial lattices $L = L_1 \times L_2$ iff J_L is the disjoint union of two non-empty posets $J_L = J_1 \sqcup J_2$, in which case $L_1 = \mathrm{Ideal_f}(J_1)$ and $L_2 = \mathrm{Ideal_f}(J_2)$.*

Lemma 2.11. *Let J be a poset (typically, the poset $J(L)$ of join-irreducibles of a distributive lattice L) and $J = J_1 \sqcup J_2$. Then*

(1) If J is cover-modular, then J_1 and J_2 are cover-modular. (def. 2.1.)

(2) If J is unique-cover-modular, then J_1 and J_2 are unique-cover-modular. (def. 2.2.)

Remark 2.12. *Suppose L is a distributive lattice, L is finitary, and L has finite upward covers. It is not necessary that J has finite upward covers, as is shown by the counterexample shown in fig. 1.*

Define J to be the poset with elements $x, y_0, y_1, y_2, \ldots, w_0, w_1, w_2, \ldots$ and the coverings:

- $w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots$,
- $x \lessdot y_i$ for all i, *and*
- $w_i \lessdot y_i$ for all i.

Define L to be the lattice $L = \mathrm{Ideal_f}(J)$ of all finite ideals of J. By Birkhoff's Representation Theorem[Stan2012, Prop. 3.4.3][Birk1967, §III.3 Th. 3] *L is distributive, L is finitary, and J is poset-isomorphic to the set of join-irreducible elements of L. The elements of L are*

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[7] Though our presentation of the theory might be made more uniform if we defined the trivial Fomin lattice to be admissible.

[8] "Finitary" means that all principal ideals are finite. For lattices, it is equivalent to $\hat{0}$ exists and the lattice is locally finite.

- $[w_i] = \{w_0, w_1, w_2, \ldots, w_i\}$ for all $i \geq -1$ *(generating $\emptyset$ when $i = -1$),*
- $[x, w_i] = \{x, w_0, w_1, w_2, \ldots, w_i\}$ for all $i \geq -1$ *(generating $\{x\}$ when $i = -1$), and*
- $[w_i, y_{n_1}, y_{n_2}, y_{n_3}, \ldots y_{n_j}] = \{x, w_0, w_1, w_2, \ldots, w_i, y_{n_1}, y_{n_2}, y_{n_3}, \ldots y_{n_j}\}$ for all $j \geq 1$, all $n_1 < n_2 < n_3 < \cdots < n_j$, and $n_j \leq i$ *(some non-empty finite set of $y_\bullet$ added to a compatible ideal of the preceding type),*

Where "$[a, b, c, \ldots]$" is the ideal of J generated by the elements $a, b, c, \ldots$.

We see that each of the above types of elements of L (ideals of J) has only a finite number of ideals that cover it in L, that is, contain one more element than it does. So L has finite upward covers. We see that x has all of the $y_\bullet$ as upward covers in J. So J does not have finite upward covers.

3. Fomin lattice essentials

The celebrated Robinson–Schensted algorithm[Rob1938][Schen1961][Fom1994][Fom1995] and its variants that have attracted substantial attention can be constructed[Wor2023, sec. 2] from the following machinery.[9] The core of the machinery is a "Fomin lattice" which satisfies the properties discussed in para. 2.2 and tersely defined below.

Definition 3.1.

(A) A Fomin lattice,[10] *which comprises:*
  (1) a lattice L that
    (a) is modular,
    (b) is locally finite, and
    (c) has finite upward and downward covers for each element;
  (2) a differential degree *r which is a member of a set of* values *V;*
  (3) a weighting *or* weight *$w(\bullet \lessdot \bullet)$, which is a function from* coverings *(or* prime quotients*), pairs $x \lessdot y$ of elements of L, to V, for which L is a* weighted-differential lattice[Stan1988][Fom1994]*:*

$$\sum_{y \,|\, y \lessdot x} w(y \lessdot x) + r = \sum_{z \,|\, x \lessdot z} w(x \lessdot z) \qquad \text{for all } x \in L \tag{7}$$

  *(which is often described as r-*differential*); and*
  (4) for which w is constant on projectivity classes[Birk1967, Ch. V sec. 5] *of coverings or equivalently, on coverings which are* cover-projective*: if $x \wedge y \lessdot x$ and $y \lessdot x \vee y$, then $w(x \wedge y \lessdot x) = w(y \lessdot x \vee y)$ and $w(x \wedge y \lessdot y) = w(x \lessdot x \vee y)$.*

(B) The lattice L has a minimum element, $\hat{0}$.

(C) A McLarnan map[11] *or* R-correspondence *M, which is a realization of the weighted-differential condition:*

  for all $x \in L$, M_x is a bijection between $\{y \in L \,|\, y \lessdot x\} \sqcup \{1, 2, 3, \ldots r\}$ and $\{z \in L \,|\, x \lessdot z\}$.

(D) Depending on the degree of abstraction of the discussion, the requirements for the set of values V varies. The minimum requirement is that V be a module over a unitial ring of scalars *S. Usually we will be unconcerned with the details of S other than it necessarily contains a homomorphic image of $\mathbb{Z}$ and thus we can multiply elements of V by integer coefficients. If $V = \mathbb{Z}$ (which implies $\mathbb{Z}$ is* isomorphically *embedded in S), $r > 0$, and all of the values of w are positive, then the weighting is defined to be* admissible, *which is required to construct a Robinson–Schensted algorithm.*

Lemma 3.2. *If L is a Fomin lattice with $\hat{0}$, L is not the trivial (one-element) lattice, $V = \mathbb{Z}$, and all values of w are positive, then r is positive.*

Proof.
For $x = \hat{0}$, (7) becomes $r = \sum_{z \,|\, x \lessdot z} w(\hat{0} \lessdot z)$. But by hypothesis, there is at least one z in the range of the sum and $w(\hat{0} \lessdot z) > 0$, which ensures the sum is positive. That proves r is positive. $\square$

[9] Though since a Robinson–Schensted variant can be constructed from any dual graded graph, it is not clear why the rest of the broader class has not been as productive. See sec. 8(9).

[10] We call it a Fomin lattice because all of the required properties are explicitly or implicitly given in [Fom1994] and [Fom1995].

[11] We name the correspondence for McLarnan, who seems to have been the first to emphasize[McLar1986, p. 22] the arbitrariness of the choice of the correspondence and thus that there is no "natural" correspondence. The set of possible correspondences for a particular Fomin lattice is usually uncountable. We choose "map" because of its consonance with "McLarnan".

In light of the above lemma, we will abuse language by using “w is admissible” or “the lattice is admissible” to mean that the weighting is admissible, that is, V is $\mathbb{Z}$, $r > 0$, and all of the values of w are positive.

Lemma 3.3. *If a lattice L forms a Fomin lattice with two pairs of differential degrees and weightings (r, w) and (r', w'), then L forms a Fomin lattice with any integer linear combination of the two: the differential degree $r'' = \alpha r + \beta r'$ and weighting $w''(x \lessdot y) = \alpha w(x \lessdot y) + \beta w'(x \lessdot y)$ for any $\alpha, \beta \in \mathbb{Z}$. Thus, the set of pairs of degrees and weightings for L forms a subspace of an infinite-dimensional module over S, the subspace of solutions of the set of linear equations (7).*

If (r, w) and (r', w') are admissible, $\alpha \geq 0$, $\beta \geq 0$, and not both $\alpha = \beta = 0$, then (r'', w'') is admissible.

Lemma 3.4. *Because L is modular and w satisfies the projectivity condition (A)(4), the Kurosh–Ore theorem*[Dil1961, Th. 3.3] *shows that L is graded and that the multiset of the weights of the coverings composing a path from some x to some $y \geq x$ in L depends only on x and y and not on the particular path.*

Definition 3.5. *We define the* number of colorings *of $x \in L$, denoted by $c(x)$, to be the product of the weights of the coverings composing an arbitrary path from $\hat{0}$ to x in L.*

Theorem 3.6. *Given this machinery, the general Robinson–Schensted insertion algorithm can be constructed using Fomin's growth diagram construction provided the weighting is admissible. The set of permutations of $\{1, \ldots, n\}$ whose elements are r-colored can be bijectively mapped to the set*

$$\begin{aligned}\{(P, Q, c) \,|\, &\text{there exists } x \in L \text{ of rank } n \text{ for which} \\ &P \text{ is a path in } L \text{ from } \hat{0} \text{ to } x \text{ and} \\ &Q \text{ is a path in } L \text{ from } \hat{0} \text{ to } x \text{ and} \\ &c \text{ maps each covering } y \lessdot z \text{ in } P \text{ into an element of } \{1, 2, 3, \ldots, w(y \lessdot z)\}\}\end{aligned}$$

where the particular bijection depends on the McLarnan map that is chosen. (We use “path” rather than the more usual “saturated chain” to emphasize the importance of the coverings composed of adjacent elements of the chain, which bear the weights.)

Definition 3.7. *Each element x of L with rank n is called a* diagram *of* size *n. Each path from $\hat{0}$ to x is called a* tableau *of* shape *x and thus size n. We define $f(x)$ as the number of paths from $\hat{0}$ to x, that is, the number of tableaux of shape x.*

Theorem 3.8. *The most celebrated consequence of the general insertion algorithm is the immediate consequence of the above bijection:*

$$n!\, r^n = \sum_{\text{diagrams } x \text{ with size } n} f(x)^2 c(x). \tag{8}$$

4. Auxiliary result

The result here supports the discussion of future directions (sec. 8). It is placed here to avoid interacting with the assumptions used in sec. 5 and 6.

4.1. Partitioning a lattice. Let L be a locally finite modular lattice and let S be a set of coverings in L that is closed under cover-projectivity, that is, if $x \lessdot y$ is in S and it is cover-projective to $x' \lessdot y'$, then $x' \lessdot y'$ is in S. Since L is locally finite and modular, this is equivalent to S is the union of a set of projectivity classes of coverings in L.

We will define a relationship $\sim$ between elements of L. Intuitively, $x \sim y$ if there is a (finite) path between x and y composed of (upward and downward) covering steps, none of which are in S. But it's easier to analyze $\sim$ by defining it in a different way.

Lemma 4.1. *Given $x \leq y$ in L and any two paths between them, one path contains a covering in S iff the other path contains a covering in S.*

Proof.

Since L is modular and locally finite, the interval $[x, y]$ is modular and finite. Choose any projectivity class C of coverings in L. The Kurosh–Ore theorem can be applied $[x, y]$ to show that one path contains a covering in C iff the other path contains a covering in C. The lemma then follows. ☐

Thus the following definition does not depend on the particular path chosen between $x \wedge y$ and $x \vee y$.

Definition 4.2. *We define $x \sim y$ iff no covering in a path from $x \wedge y$ to $x \vee y$ is in S.*

Lemma 4.3. *If $x \leq y$ in L, then $x \sim y$ iff no covering in a path from x to y is in S.*

Lemma 4.4. *If $x \leq y \leq z$ in L and $x \sim z$, then $x \sim y$ and $y \sim z$.*

Proof.
By lem. 4.3, no covering in any path from x to z is in S. We can choose such a path from x to y and then from y to z. Since no covering in that path is in S, no covering from x to y or from y to z is in S. Then by lem. 4.3, $x \sim y$ and $y \sim z$. ☐

Lemma 4.5. *Conversely, if $x \leq y \leq z$ in L, $x \sim y$, and $y \sim z$, then $x \sim y$.*

Lemma 4.6. *If $x, y \in L$, then $x \sim x \vee y$ iff $x \wedge y \sim y$.*

Proof.
This follows directly from the diamond isomorphism theorem and lem. 4.3. ☐

Lemma 4.7. *If $x, y \in L$, $x \sim y$ implies $x \wedge y \sim x \vee y$, $x \sim x \vee y \sim y$, and $x \sim x \wedge y \sim y$.*

Lemma 4.8. *If $x, y \in L$, $x \sim x \vee y \sim y$ implies $x \sim y$. Dually, $x \sim x \wedge y \sim y$ implies $x \sim y$.*

Proof.
Given $x \sim x \vee y$, by lem. 4.6, $x \wedge y \sim y$. Since $x \wedge y \leq y \leq x \vee y$, $x \wedge y \sim y$, and $y \sim x \vee y$, By lem. 4.5, $x \wedge y \sim x \vee y$ and so $x \sim y$. ☐

Lemma 4.9. *Given $x, y, z \in L$, if $x \sim y$ and $y \sim z$, then $x \sim z$.*

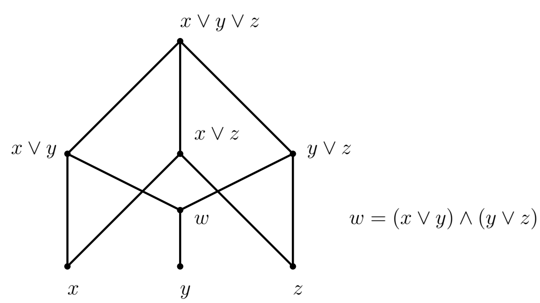


Figure 2. Construction for lem. 4.9.

Proof.
We know $x \sim y$ and $y \sim z$. By lem. 4.7, $x \sim x \vee y$, $x \vee y \sim y$, $y \sim y \vee z$, and $y \vee z \sim z$. Define $w = (x \vee y) \wedge (y \vee z)$. Of necessity, $y \leq w \leq x \vee y$ and $y \leq w \leq y \vee z$. By lem. 4.4, $w \sim x \vee y$ and $w \sim y \vee z$. By lem. 4.8, $x \vee y \sim y \vee z$ and by lem. 4.7, $x \vee y \sim (x \vee y) \vee (y \vee z) = x \vee y \vee z$. Since $x \leq x \vee y \leq x \vee y \vee z$, by lem. 4.5, $x \sim x \vee y \vee z$. Since $x \leq x \vee z \leq x \vee y \vee z$, by lem. 4.4, $x \sim x \vee z$. Similarly, $z \sim x \vee z$. Then by lem.4.8, $x \sim z$. ☐

Lemma 4.10. *$\sim$ is an equivalence relation on the elements of L.*

Proof.
That $\sim$ is reflexive and symmetric are trivial. Lem. 4.9 shows that $\sim$ is transitive. ☐

Lemma 4.11. *Given $x, y, z \in L$, if $x \sim y$ and $x \sim z$, then $x \sim y \wedge z$ and $x \sim y \vee z$.*

Proof.
By lem. 4.9, $y \sim z$, and by lem. 4.7, $y \sim y \wedge z$. Then by lem. 4.5, $x \sim y \wedge z$. Dually, $x \sim y \vee z$. ☐

Lemma 4.12. *Given $x, y, z, w \in L$ with $y \leq z \leq w$, if $x \sim y$ and $x \sim w$, then $x \sim z$.*

Proof.
By lem. 4.9, $y \sim w$, and by lem. 4.4, $y \sim z$. Then by lem. 4.9, $x \sim z$. □

Now we fix an element $z \in L$. z will usually be $\hat{0}$.

Definition 4.13. *We define $L_{/Sz}$ to be $\{x \in L \,|\, x \sim z\}$ (where $\sim$ is determined by S).*

Intuitively, we construct $L_{/Sz}$ by constructing the Hasse diagram of L, removing from it the edges which are coverings in S, selecting from the resulting graph the connected component that contains z, and considering that component as the Hasse diagram of a lattice, which is $L_{/Sz}$.

Lemma 4.14. *$L_{/Sz}$ is a convex sublattice of L.*

Proof.
That $L_{/Sz}$ is convex is shown by lem. 4.12. That $L_{/Sz}$ is a sublattice is shown by lem. 4.11. □

We apply this construction to a Fomin lattice that has a non-admissible weighting.

Definition 4.15. *Given a Fomin lattice L with weighting w and $\hat{0}$, we define its* truncation *$L_{/}$ to be $L_{/S\hat{0}}$, where $S = \{x \lessdot y \,|\, w(x \lessdot y) = 0\}$, the set of coverings with weight 0. We define $w_{/}(x \lessdot y)$ for any $x, y \in L/$ to be equal to $w(x \lessdot y)$ (for L).*

Theorem 4.16. *Given a Fomin lattice L with $\hat{0}$, its truncation $L_{/}$ is a Fomin lattice.*

Proof.
All of the requirements for $L_{/}$ to be a Fomin lattice (def. 3.1) are trivially inherited from L except for (7). But by construction, (7) for a particular $x \in L_{/}$ contains a subset of the terms of (7) for $x \in L$, and the terms of the latter that are not in the former are ones for which w has the value 0. Thus (7) is true for all $x \in L_{/}$. □

Theorem 4.17. *Given a distributive Fomin lattice L with $\hat{0}$ with points $J(L)$, its truncation $L_{/}$ is isomorphic to* $\mathrm{Ideal_f}(J_{/})$ *where $J_{/}$ is the subposet $\{j \in J \,|\, \text{not } (\exists k)(k \leq j \text{ and } w(j) = 0)\}$.*

Theorem 4.18. *Given a Fomin lattice L with $\hat{0}$, if for every covering $x \lessdot y$ for which $w(x \lessdot y) < 0$, there exists a z, w for which $z \leq w \leq x$ and $w(z \leq w) = 0$, then $L_{/}$ is a Fomin lattice with an admissible weighting.*

Theorem 4.19. *Given a distributive Fomin lattice L with $\hat{0}$ with points $J(L)$, if for every point j for which $w(j) < 0$ then there exists a k for which $k \leq j$ and $w(k) = 0$, then $L_{/}$ is a Fomin lattice with an admissible weighting.*

Of course, truncation could be done using a different element $z \in L$ than $\hat{0}$, but if L has *hatzero*, then any sublattice of L has a minimum element.

The only known application of truncation is when L is $\mathbb{Y}$, J is the quadrant, and the weighting on J is $w((i, j) = \alpha + \beta i + \gamma j$.[Fom1994, Exam. 2.2.7] If $\alpha = r$, $\beta = r/k$, and $\gamma = -r/k$ for some k (positive or negative) that divides r, then $L_{/}$ is the k-strip (if k is positive) or the dual of the k-strip (if k is negative).

5. Local characterization

Henceforth we will characterize admissible Fomin lattices under a certain set of assumptions. As we proceed through the analysis, additional, stricter assumptions will be made. This may seem an excessively narrow approach, but all known Fomin lattices with $\hat{0}$ other than $\mathbb{YF}_r$ satisfy all of these assumptions.

5.1. Distributive lattice.

Assumption 5.1. *Henceforth, we will restrict our attention to distributive Fomin lattices with $\hat{0}$. Specifically, we assume L is a Fomin lattice with $\hat{0}$ with differential degree r, values V, and weighting w, and that L is distributive. (We do not, at this point, assume that w is admissible.)*

Theorem 5.2. [Stan2012, Prop. 3.4.3][Birk1967, §III.3, Th. 3] *Since we have assumed that $\hat{0}$ exists, L is finitary, that is, all if its principal ideals are finite. L is isomorphic to the set of finite order ideals of the poset $J = J(L)$ of its join-irreducible elements, which we denote $L = \mathrm{Ideal_f}(J)$. Given a covering $x \lessdot y$ in L, $x \setminus y$ (considered as the set difference of two ideals of J) contains exactly one element of J, and the projectivity requirement (condition (A)(4)) on w reduces to: There is a weighting w from J to V, and for every $x \lessdot y$ in L, $w(x \lessdot y) = w(y \setminus x)$.*

The weighted-differential condition (7) is equivalent to

$$\sum_{a \,|\, a \text{ maximal in } x} w(a) + r = \sum_{b \,|\, b \text{ minimal in } J \setminus x} w(b) \qquad \text{for all } x \in \mathrm{Ideal_f}(J). \tag{9}$$

Because the image of w in L is the same as the image of w on J, we abuse language by conflating the two meanings of w when we are dealing them as entire functions. In particular, each of them has all of its values positive iff the other does, and so "w is admissible" is used to mean both.

Definition 5.3. *The elements of J are called* points *or* boxes*.*

Lemma 5.4.

(1) J is locally finite.
(2) J is finitary.

Proof.
Regarding (1): If there was some $a \leq b$ in J for which $[a, b]$ is infinite, then the set of all principal ideals of the elements in $[a, b]$, $\{[x] \,|\, a \leq x \leq b\}$ would be an infinite subset of L with all of its elements $\geq [a]$ and $\leq [b]$. That contradicts that L is locally finite.

Regarding (2): Similarly, if there was an $a \in J$ whose principal ideal $[a]$ in J was infinite, then $\{[x] \,|\, x \leq b\}$ would be an infinite subset of L with all of its elements $\leq [a]$. That contradicts that, given our assumptions, th. 5.2 shows that L is finitary. □

Lemma 5.5. *The lattice of diagrams is isomorphic to the lattice of finite ideals of points under union and intersection, the size of a diagram is the cardinality of its corresponding ideal (the number of points it contains), and the number of tableaux with a particular shape is the number of linear orderings of its points (considered as a subposet of J). The number of colorings $c(x)$ of a diagram x is the product of the weights w of its points: $c(x) = \prod_{a \in x} w(x)$.*

Definition 5.6. *Given any diagram x, its* insertion points *are the points that are the minimal elements of $J \setminus x$, those which can be added to x to make a diagram (whose size is 1 larger than the size of x). The* deletion points *of x are the points that are the maximal elements of x, those which can be deleted from x to make a diagram (whose size is 1 smaller than the size of x). For any diagram x, we define D_x to be the set of deletion points of x, and I_x to be the set of insertion points of x.*

The summation on the left of (9) is over the deletion points of x and the summation on the right of (9) is over the insertion points of x. Thus, (9) is equivalent to

$$\sum_{a \in I_x} w(a) + r = \sum_{b \in D_x} w(b) \qquad \text{for all } x \in \mathrm{Ideal_f}(J). \tag{10}$$

Applying (9) to the ideal $\emptyset$ gives

$$r = \sum_{z \,|\, z \text{ minimal in } J} w(z) \tag{11}$$

In many cases J has a unique minimal element, which we denote $\hat{0}_J$, and in that case, (11) reduces $w(\hat{0}_J) = r$.

Given an ideal x and a point p that is an insertion point of x, that is, minimal in $J \setminus x$, we can construct the ideal $x' = x \sqcup \{p\}$. Thus $J \setminus x = (J \setminus x') \sqcup \{p\}$ and p is a deletion point of x'.

The difference between (10) applied to x' and (10) applied to x is

$$\sum_{a \in D_{x'} \setminus D_x} w(a) - \sum_{b \in D_x \setminus D_{x'}} w(b) = \sum_{a \in I_{x'} \setminus I_x} w(a) - \sum_{b \in I_x \setminus I_{x'}} w(b) \qquad \text{for all } x \in \mathrm{Ideal_f}(J),\ p \in I_x. \tag{12}$$

The domains of the four summations of (12) are:

(1) $D_{x'} \setminus D_x$: The set of maximal elements of x' that are not maximal elements of x, which is $\{p\}$,
(2) $D_x \setminus D_{x'}$: The set of maximal elements of x that are not maximal elements of x',
(3) $I_{x'} \setminus I_x$: The set of minimal elements of $J \setminus x'$ that are not minimal elements of $J \setminus x$,
(4) $I_x \setminus I_{x'}$: The set of minimal elements of $J \setminus x$ that are not minimal elements of $J \setminus x'$, which is $\{p\}$.

Lemma 5.7. *Condition (9) (for every $x \in \mathrm{Ideal_f}(J)$) is equivalent to the combination of (11) and (12) (for every $x \in \mathrm{Ideal_f}(J)$ and $p \in I_x$).*

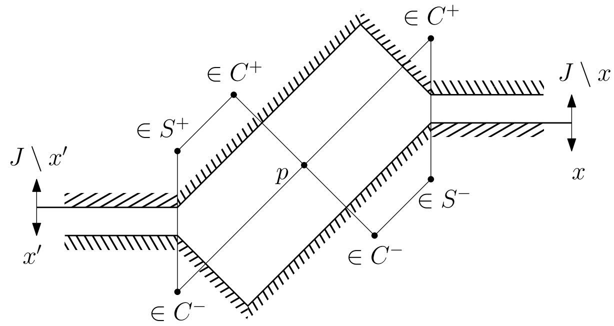


FIGURE 3. Adding a point p to an ideal x making an ideal x'

Proof.

Regarding $\Rightarrow$: This has been proved by the above derivation of (11) and (12) from (9).

Regarding $\Leftarrow$: Since every $y \in L$ is the top of a finite chain with $\hat{0}$ as bottom, we can can prove this by induction on increasing length of the chain. The base case is a chain of length zero, which implies $y = \hat{0}$. In that case, (9) with $x = y = \hat{0}$ is equivalent to (11).

If the chain has length > 0, let y' be the element below y in the chain. By induction, (9) is true for $x = y'$. Now consider (12) with $x = y'$ and p the unique element of $y \setminus y'$. We can add these two and by the above derivation obtain (9) for $x = y$. □

Definition 5.8. *Given a point $p \in J$, define C^-_p, or simply C^- when p is understood, (C^- for "downward covers") as the set of all elements of J which are covered by p. Similarly, define C^+_p, or simply C^+, (C^+ for "upward covers") as the set of all elements of J which cover p. Define N_p, or simply N, (N for "neighborhood") to be $\{x \in J \mid (\exists a \in C^-, b \in C^+)\, a \leq x \leq b\}$, the convex subset of J generated by C^- and C^+. Clearly $C^+ \subset N$ and $C^- \subset N$. Define S_p, or simply S, (S for "siblings") to be $N_p \setminus C^- \setminus C^+ \setminus \{p\}$, N_p with C^-, C^+, and p deleted.*

Lemma 5.9. *By construction, every element of C^+ is $>$ every element of C^-, but no element of C^+ covers any element of C^-. Every element of S is incomparable to p, $>$ some element of C^-, and $<$ some element of C^+. If J is graded, every element of S has the same rank as p and S is an antichain. However, in general, S need not be an antichain.*

Lemma 5.10. *Given an ideal x and a point p that is an insertion point of x, $C^- \subset x$ and $C^+ \subset J \setminus x'$.*

Definition 5.11. *Given an ideal x and a point p that is an insertion point of x, define $S^-_{xp} = S^- = S \cap x$ and $S^+_{xp} = S^+ = S \cap (J \setminus x')$.*

Lemma 5.12. *Since $p \notin S$, by construction, S is the disjoint union of S^- and S^+, and if $s^- \in S^-$ and $s^+ \in S^+$ are comparable, then $s^- \leq s^+$. So S^- is an ideal of S and S^+ is a filter of S.*

The generic relationships of elements of these sets are shown in fig. 3.

Lemma 5.13. *The four summation domain sets of (12) are:*

(1) $D_{x'} \setminus D_x = \{p\}$

(2) $D_x \setminus D_{x'}$ is the set of maximal elements of x which are covered by p. This is the same as the elements of C^- which are not $\leq$ (or equivalently, not covered by) any element of S^-. We denote this property by $\not\leq S^-$.

(3) $I_{x'} \setminus I_x$ is the set of minimal elements of $J \setminus x'$ which cover p. This is the same as the elements of C^+ which are not $\geq$ (or equivalently, do not cover) any element of S^+. We denote this property by $\not\geq S^+$.

(4) $I_x \setminus I_{x'} = \{p\}$

This shows (12) is equivalent to

$$w(p) - \sum_{a \in C^-,\ a \not\leq S^-} w(a) = \sum_{b \in C^+,\ b \not\geq S^+} w(b) - w(p)$$

$$\sum_{a \in C^-_p,\ x \not\leq S^-_{xp}} w(a) + \sum_{b \in C^+_p,\ y \not\geq S^+_{xp}} w(b) = 2w(p) \qquad \text{for all } x \in \mathrm{Ideal_f}(J),\ p \in I_x. \tag{13}$$

5.2. Reorganizing the linear equations. Looking at (13) as a set of linear equations in the values of w, we see that each equation is determined by choosing first an $x \in \mathrm{Ideal_f}(J)$ and then a $p \in I_x$. We will now reorganize the statement of the set of linear equations (13), but indexed first by an arbitrary choice of $p \in J$ (which determines C^+ and C^-), and then by the values of S^+ and S^- consistent with p.

Lemma 5.14. *The set of equations (13) is the same set as the equations*

$$\sum_{a \in C_p^-,\ a \not\lessgtr T^-} w(a) + \sum_{b \in C_p^+,\ b \ngtr T^+} w(b) = 2w(p) \qquad \text{for all } p \in J,\ S_p = T^- \sqcup T^+, \text{ with } T^- \text{ an ideal and } T^+ \text{ a filter.} \tag{14}$$

Proof.
A particular equation in the set (13) is indexed first by an $x \in \mathrm{Ideal_f}(J)$ and then a $p \in I_x$. p determines C_p^+ and C_p^-, and p and x together determine S_{xp}^- and S_{xp}^+. Necessarily, $S_p = S_{xp}^+ \sqcup S_{xp}^+$, so setting $T^- = S_{xp}^-$ and $T^+ = S_{xp}^+$ shows that the equation is in the set (14).

Conversely, a particular equation in the set (14) is indexed first by a $p \in J$ and then a partition of S_p into an ideal T^- and a filter T^+. Define $x = \{z \in J \mid z < p\} \sqcup T^- = (p) \sqcup T^-$, the principal ideal of J generated by p, less p itself, plus T^-. By construction, x is an ideal of J, p is in insertion point of x, $S_{xp}^- = T^-$ and $S_{xp}^+ = T^+$. This shows that the equation is in the set (13). ☐

5.3. Orphans.

Definition 5.15. *Define $x \in C^+$ as an* upward orphan *(relative to p) if x does not cover any element of S, which is equivalent to x covering only one element n in N (namely p). Similarly, we define $x \in C^-$ as a* downward orphan *(relative to p) if x is not covered by any element of S, which is equivalent to x being covered by only one element n in N (namely p).*

In (14), if we choose $T^- = \emptyset$ and $T^+ = S_p = S$, it becomes

$$\sum_{a \in C^-} w(a) + \sum_{b \in C^+,\ b \ngtr S} w(b) = 2w(p) \qquad \text{for all } p \in J$$

By construction, S contains every element covered by any element of C^+ except for p, so the second sum is over all upward orphans. Thus

Lemma 5.16.

$$\sum_{a \mid a \lessdot p} w(a) + \sum_{b \mid b \gtrdot p,\ b \text{ upward orphan}} w(b) = 2w(p) \qquad \text{for all } p \in J \tag{15}$$

Equation (15) allows the weight of a point to be computed as half of the sum of the weights of the elements it covers, if the point is not covered by any upward orphans.

If we choose $T^- = S_p$ and $T^+ = \emptyset$, then (14) becomes

Lemma 5.17.

$$\sum_{a \mid a \lessdot p,\ a \text{ downward orphan}} w(a) + \sum_{b \mid b \gtrdot p} w(b) = 2w(p) \qquad \text{for all } p \in J \tag{16}$$

5.4. Partitioning S into three sets. Given p, let us partition S_p into three disjoint parts $S_p = L \sqcup \{m\} \sqcup U$ where L is an ideal of S, U is a filter of S, and m is an element between L and U. Setting $E^- = L$ and $E^+ = \{m\} \sqcup U$ in (14) and then setting $F^- = L \sqcup \{m\}$ and $F^+ = U$ in (14) gives

$$\sum_{a \mid a \in C^-,\ a \nless E^-} w(a) + \sum_{b \mid b \in C^+,\ b \ngtr E^+} w(b) = 2w(p) \qquad \text{and}$$
$$\sum_{a \mid a \in C^-,\ a \nless F^-} w(a) + \sum_{b \mid b \in C^+,\ b \ngtr F^+} w(b) = 2w(p) \qquad \text{for all } p \in J \text{ and } S_p = L \sqcup \{m\} \sqcup U$$

Since $E^- \subset F^-$, $x \not\leq F^-$ implies $x \not\leq E^-$. And since $F^+ \subset E^+$, $x \not\geq E^+$ implies $x \not\geq F^+$. Thus subtracting these two equations gives

$$\sum_{a \,|\, a \in C^-,\ a \not\leq E^-,\ \text{not } a \not\leq F^-} w(a) = \sum_{b \,|\, b \in C^+,\ b \not\geq F^+,\ \text{not } b \not\geq E^+} w(b) \qquad \text{for all } p \in J \text{ and } S_p = L \sqcup \{m\} \sqcup U$$

This is equivalent to

$$\sum_{a \,|\, a \in C^-,\ a \not\leq L,\ a \leq m} w(a) = \sum_{b \,|\, b \in C^+,\ b \not\geq U,\ b \geq m} w(b) \qquad \text{for all } p \in J \text{ and } S_p = L \sqcup \{m\} \sqcup U \tag{17}$$

Lemma 5.18. *Given a $p \in J$, (14) holds for every partition of S_p into an ideal T^- and a filter T^+ iff the following conditions are true for p:*

(1) (15) holds and
(2) (17) holds for every partition of S_p into three disjoint sets $S_p = L \sqcup \{m\} \sqcup U$, where L is an ideal of S_p and U is a filter of S_p.

Proof.
Regarding $\Rightarrow$: This has been proved by the above derivations of (15) and (17).

Regarding $\Leftarrow$: For a given p, we prove (14) for any partition $S_p = T^- \sqcup T^+$ where T^- is an ideal and T^+ is a filter by induction upward on the size of T^-. If $T^- = \emptyset$, then by hypothesis (15) holds for p, which is equivalent to (14) for $T^- = \emptyset$.

If $T^- \neq \emptyset$, then choose m as a maximal element of T^- and define $L = T^- \setminus \{m\}$ and $U = T^+$, so that $S_p = L \sqcup \{m\} \sqcup U$ with L being an ideal and U being a filter. Since $|L| = |T^-| - 1$, by induction (14) holds for the partition $S_p = L \sqcup (\{m\} \sqcup U)$, which is

$$\sum_{a \in C_p^-,\ a \not\leq L} w(a) + \sum_{b \in C_p^+,\ b \not\geq \{m\} \sqcup U} w(b) = 2w(p) \tag{18}$$

By hypothesis, (17) holds for the partition $S = L \sqcup \{m\} \sqcup U$, which is

$$\sum_{a \,|\, a \in C_p^-,\ a \not\leq L,\ a \leq m} w(a) = \sum_{b \,|\, b \in C_p^+,\ b \not\geq U,\ b \geq m} w(b) \tag{19}$$

Subtracting the left side of (19) from the first term of (18) and adding the right side of (19) from the second term of (18) gives

$$\sum_{a \in C_p^-,\ a \not\leq L,\ a \not\leq m} w(a) + \sum_{b \in C_p^+,\ b \not\geq U} w(b) = 2w(p)$$

This equation is the same as (14) for the partition $S_p = T^- \sqcup T^+$:

$$\sum_{a \in C_p^-,\ a \not\leq T^-} w(a) + \sum_{b \in C_p^+,\ b \not\geq T^+} w(b) = 2w(p)$$

Thus by induction (14) holds for every p and partition of S_p into an ideal T^- and a filter T^+. $\square$

5.5. Unique-cover-modular.

Assumption 5.19. *Henceforth, we assume the poset of points is unique-cover-modular (def. 2.1).*

The following example disproves a conjecture that was in preliminary versions of this article.

Remark 5.20. *The poset t-Plait in* [Fom1994, Exam. 2.2.10 and Fig. 7] *is locally finite and cover-modular but is not graded.*[12]

Definition 5.21. *For any $p \in J$ and any $x \in S_p$, we define x^+ (relative to p) to be the unique element of C_p^+ which is $> x$ and we define x^- (relative to p) to be the unique element of C_p^- which is $< x$.*

[12] "Graded" means that every saturated chain from a to b has the same length, but there need not be a $\hat{0}$. In general, the rank function of a graded poset ranges over the integers, not just the nonnegative integers.

5.6. Admissible weight.

Assumption 5.22. *Henceforth, we assume that w is an admissible weighting, that is, the set of values V, the target of w, is $\mathbb{Z}$, and all of the values of w are positive.*

This assumption is not a loss of generality for our larger purpose, because if a non-admissible weighting could be used to construct an admissible weighting using th. 4.16 et al., our analysis would reveal the resulting admissible weighting directly.

Lemma 5.23. *For $x, y \in S$, if $x^+ = y^+$ and $x^- = y^-$, then $x = y$.*

Proof.
Assume there is $x, y \in S$, with $x \neq y$, $x^+ = y^+$, and $x^- = y^-$. Since $x \neq y$, $x \not\leq y$ or $y \not\leq x$. Without loss of generality, assume $y \not\leq x$. Define $X = \{s \in S \,|\, s^- = x^- \text{ and } s^+ = x^+ \text{ and } s \leq x\} = [x] \cap [x^-, x^+]$, the principal ideal generated by x within the interval $[x^-, x^+]$. By construction, $x \in X$ and $y \notin X$.

Because of unique-cover-modularity, given $a \in C^-$, $a \leq z$ for some $z \in X$ iff $a = x^-$ iff $a \leq x$. So $a \not\leq X$ iff $a \not\leq x$ Because $y \in S \setminus X$ and $y^+ = x^+$, given $b \in C^+$, $b \geq z$ for some $z \in S$ iff $b \geq z$ for some $z \in S \setminus X$. So $b \not\geq S$ iff $b \not\geq S \setminus X$.

Applying (14) to $S = \emptyset \sqcup S$ gives

$$\sum_{a \in C^-,\ a \not\leq \emptyset} w(a) + \sum_{b \in C^+,\ b \not\geq S} w(b) = 2w(p)$$
$$\sum_{a \in C^-} w(a) + \sum_{b \in C^+,\ b \not\geq S \setminus X} w(b) = 2w(p)$$

Applying (14) to $S = X \sqcup (S \setminus X)$ gives

$$\sum_{a \in C^-,\ a \not\leq X} w(a) + \sum_{b \in C^+,\ b \not\geq S \setminus X} w(b) = 2w(p)$$
$$\sum_{a \in C^-,\ a \not\leq x} w(a) + \sum_{b \in C^+,\ b \not\geq S \setminus X} w(b) = 2w(p)$$

Taking the difference of these gives $w(x^-) = 0$. By the assumption that w is admissible, this is a contradiction. $\square$

Lemma 5.24. *For any $p \in J$, S_p is an antichain.*

Proof.
Assume there exists $x < y$ in S_p. By unique-cover-modularity, $x^+ = y^+$ and $x^- = y^-$. By lem. 5.23, $x = y$, which is a contradiction. Thus S_p is an antichain. $\square$

Thus, for any p, any subset of S_p is both an ideal and a filter, which allows (14) and (17) to be used quite broadly.

Consider how unique-cover-modularity simplifies (17): For any given $m \in S$, the a on the left-hand side of (17) can only be m^-. Similarly, the b on the right-hand side can only be m^+. Thus (17) is equivalent to:

$$\left.\begin{matrix} \text{if } m^- \not\leq L & w(m^-) \\ \text{otherwise} & 0 \end{matrix}\right\} = \begin{cases} w(m^+) & \text{if } m^+ \not\geq U \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

If we set $L = \emptyset$ and $U = S \setminus \{m\}$,

$$w(m^-) = \begin{cases} w(m^+) & \text{if } m^+ \not\geq U \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Given the assumption of unique-cover-modularity, $m^+ \geq z$ for some $z \in U$ iff $m^+ = z^+$ for some $z \in U$. So $m^+ \not\geq U$ iff there is no $z \in U$ with $m^+ = z+$. Given the assumption of admissibility, $w(m^-) > 0$, which is contradicted by (21) unless there is no $z \in S$ with $z^+ = m^+$ and $z \neq m$.

Dually, if we set $U = \emptyset$ and $L = S \setminus \{m\}$, we get a contradiction unless there is no $z \in S$ with $z^- = m^-$ and $z \neq m$.

Thus for every $m \neq n$ in S, $m^- \neq n^-$ and $m^+ \neq n^+$; the set of pairs $\{(m^-, m^+) \,|\, m \in S\}$ is a bijection between subsets of C^- and C^+. In addition, (20) reduces to

$$w(m^-) = w(m^+) \qquad \text{for all } m \in S_p,\ p \in J \tag{22}$$

Note that x^+ and x^- in (22) implicitly depend on p.

Lemma 5.25. *Given a $p \in J$, (17) holds for every partition of S_p into three disjoint sets $S_p = L \sqcup \{m\} \sqcup U$ iff the following conditions are true for p:*

(1) the set of pairs $\{(m^-, m^+) \,|\, m \in S_p\}$ is a bijection between subsets of C^- and C^+ and
(2) for every $m \in S_p$, $w(m^-) = w(m^+)$.

Proof.
Regarding $\Rightarrow$: This has been shown by the above derivations.
Regarding $\Leftarrow$: By the above derivation, for any $S_p = L \sqcup \{m\} \sqcup U$, (17) is the same as (20). By hypothesis (1), $m^- \leq z$ for some $z \in L$ iff $m^- = z^-$ iff $m = z$. But by construction, $m \notin L$. Thus $m^- \not\leq L$ is always true. Dually, $m^+ \not\leq U$ is always true. Thus (20) is the same as

$$w(m^-) = w(m^+) \tag{23}$$

which is assumed as hypothesis (2). ☐

At this point, let us summarize our progress:

Lemma 5.26. *Given*

(assump. 5.1) L is a Fomin lattice,
(assump. 5.1) L has a $\hat{0}$,
(assump. 5.1) L is distributive, and
(assump. 5.19) J is unique-cover-modular,

a weighting $w(\bullet \lessdot \bullet)$ of coverings in L into $V = \mathbb{Z}$ satisfies the differential condition (7) and is admissible (assump. 5.22) iff the corresponding $w(\bullet)$ from J into V satisfies

(assump. 5.22) $w(p) > 0$ for every $p \in J$,
(eq. 11) $r = \sum_{z \,|\, z \text{ minimal in } J} w(z)$,
(eq. 15) $\sum_{a \,|\, a<p} w(a) + \sum_{b \,|\, b>p,\ b \text{ upward orphan}} w(b) = 2w(p)$ for all $p \in J$,
(lem. 5.25(1)) the set of pairs $\{(m^-, m^+) \,|\, m \in S_p\}$ is a bijection between subsets of C_p^- and C_p^+ for all $p \in J$,
(lem. 5.25(2)) $w(m^-) = w(m^+)$ for every $m \in S_p$, $p \in J$.

Proof.
This is proved by chaining together th. 5.2, lem. 5.7, lem. 5.13, lem. 5.14, lem. 5.18, and lem. 5.25. ☐

5.7. No upward or downward triple covers.

Lemma 5.27. *No element of J is covered by three or more distinct elements.*

Proof.
The construction for this poof is shown in fig. 4.

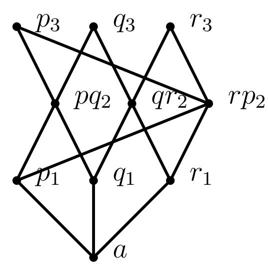


FIGURE 4. Construction for the proof of lem. 5.27

Assume that we have $a \in J$ covered by distinct p_1, q_1 and r_1. By unique-cover-modularity, there is a pq_2 covering p_1 and q_1, a qr_2 covering q_1 and r_1, and an rp_2 covering r_1 and p_1. By lem. 5.23 (applied to p_1, q_1, and r_1), pq_2, qr_2, and rp_2 are distinct. By (22) (applied to p_1, q_1, and r_1), $w(a) = w(pq_2) = w(qr_2) = w(rp_2)$.

By unique-cover-modularity, there is a p_3 covering rp_2 and pq_2, a q_3 covering pq_2 and qr_2, and an r_3 covering qr_3 and rp_3. Suppose two of p_3, q_3, and r_3 are the same, say p_3 is the same as q_3. Define $b = p_3 = q_3$, so b covers pq_2, qr_2, and rp_2. Then by (22) (applied to pq_2, qr_2, and rp_2), $w(b) = w(p_1) = w(q_1) = w(r_1)$. Since w is admissible, (15) implies $2w(pq_2) \geq w(p_1) + w(q_1) = 2w(b)$. So $w(pq_2) \geq w(b)$ and similarly $w(qr_2) \geq w(b)$ and $w(rp_2) \geq w(b)$. Applying (15) to b and the above gives $2w(b) \geq w(pq_2) + w(qr_2) + w(rp_2) \geq 3w(b)$. Since $w(b) > 0$, this is a contradiction. Thus p_3, q_3, and r_3 are distinct.

By (22), $w(p_3) = w(p_1)$, $w(q_3) = w(q_1)$, and $w(r_3) = w(r_1)$. Applying (15) and admissibility,

$$
\begin{aligned}
2w(p_1) + 2w(q_1) + 2w(r_1) &= 2w(p_3) + 2w(q_3) + 2w(r_3) \\
2w(p_3) + 2w(q_3) + 2w(r_3) &\geq 2w(pq_2) + 2w(qr_2) + 2w(rp_2) \qquad (24) \\
2w(pq_2) + 2w(qr_2) + 2w(rp_2) &\geq 2w(p_1) + 2w(q_1) + 2w(r_1) \qquad (25)
\end{aligned}
$$

So

$$
w(p_1) + w(q_1) + w(r_1) = w(pq_2) + w(qr_2) + w(rp_2) = w(p_3) + w(q_3) + w(r_3)
$$

Since the inequalities (24) and (25) are tight, (15) and the assumption that w is admissible requires that pq_2, qr_2, rp_2, p_3, q_3, and r_3 cover no elements other than the ones we mentioned above (and shown in fig. 4) and have no upward orphans. Applying (15) to pq_2, qr_2, and rp_2 thus gives

$$
\begin{aligned}
w(a) = w(pq_2) &= w(p_1)/2 + w(q_1)/2 \qquad (26) \\
w(a) = w(qr_2) &= w(q_1)/2 + w(r_1)/2 \qquad (27) \\
w(a) = w(rp_2) &= w(r_1)/2 + w(p_1)/2. \qquad (28)
\end{aligned}
$$

Together these prove $w(p_1) = w(q_1) = w(r_1) = w(a)$ and $w(p_3) = w(q_3) = w(r_3) = w(a)$.

By (16) and that w is admissible, $2w(a) \geq w(p_1) + w(q_1) + w(r_1) = 3w(a)$. Since $w(a) > 0$, this is a contradiction.

Thus, there is no $a \in J$ that is covered by three distinct elements. □

Lemma 5.28. *No element of J covers three or more distinct elements.*

Proof.

This is proved dually to lem. 5.27, by interchanging the use of (15) and (16) and replacing "upward orphan" with "downward orphan". □

Remark 5.29. Lem. 5.27 and 5.28 are the deeper explanation of the long-known fact[Fom1994, sec. 2.2] that $\mathrm{Ideal_f}(\mathbb{N}^k)$ for $k > 2$ has no admissible weighting as a differential lattice. Similarly, they show why $\mathbb{SS}_t = \mathrm{Ideal_f}(\mathbb{CP}_{2t}) = \mathrm{Ideal_f}(t\text{-}\mathit{SkewStrip})$ is a Fomin lattice (para. 2.3(9) but $\mathrm{Ideal_f}(\mathbb{CP}_{kt})$ with $k > 2$ has no admissible weightings. (Compare $\mathbb{CP}_{32} = \mathcal{L}_{3,2}$ in [El2025, Fig. 2] with $\mathbb{CP}_{33} = \mathcal{L}_{3,3}$ in [El2025, Fig. 3].)

The equations implied by lem. 5.26 for any particular $p \in J$ are characterized by the structure of its neighborhood N_p as a poset.

Lemma 5.30. *N_p is characterized up to isomorphism as a poset by the numbers $|C^-|$, $|C^+|$ and $|S|$: All elements of C^- are covered by p, all elements of C^+ cover p, and S forms a matching between subsets of C^- and C^+, with each element of S covering a distinct element of C^- and being covered by a distinct element of C^+. In addition, $|C^-| \leq 2$, $|C^+| \leq 2$, and $|N| = |C^-| + |C^+| + |S| + 1$.*

Proof.

This follows from lem. 5.24, 5.26, 5.27, and 5.28. □

For each possible set of values of $|C^-|$, $|C^+|$, $|S|$, and $|N|$, fig. 5.7 shows the resulting structure of N, and the consequent constraints per lem. 5.26.

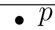
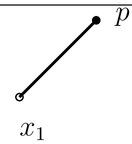
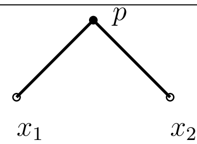
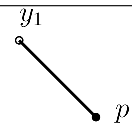
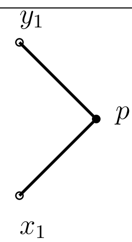
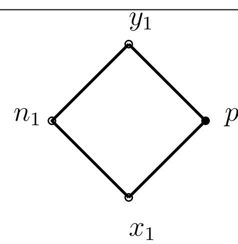
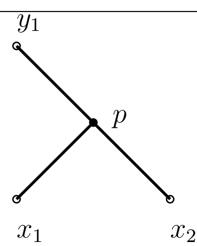
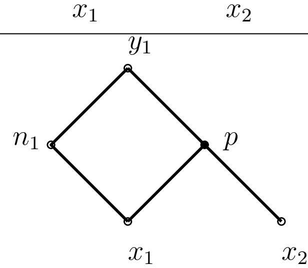
$\lvert C^+\rvert$	$\lvert C^-\rvert$	$\lvert S\rvert$	$\lvert N\rvert$	Neighborhood	Constraints
0	0	0	1		$2w(p) = 0$
0	1	0	2		$2w(p) = w(x_1)$
0	2	0	3		$2w(p) = w(x_1) + w(x_2)$
1	0	0	2		$2w(p) = w(y_1)$
1	1	0	3		$2w(p) = w(x_1) + w(y_1)$
1	1	1	4		$2w(p) = w(x_1)$ $w(x_1) = w(y_1)$
1	2	0	4		$2w(p) = w(x_1) + w(x_2) + w(y_1)$
1	2	1	5		$2w(p) = w(x_1) + w(x_2)$ $w(x_1) = w(y_1)$

FIGURE 5.1. The possible neighborhoods of p and their constraints, part 1

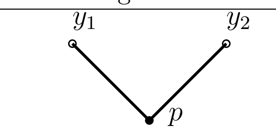
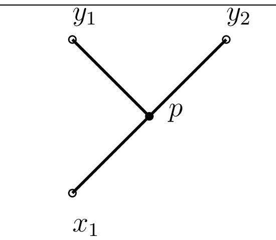
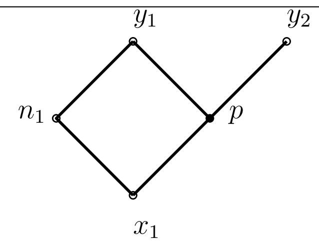
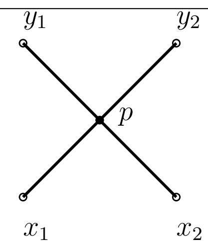
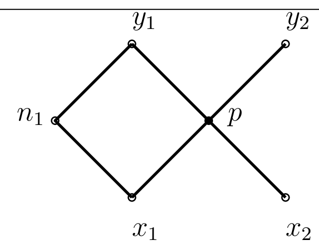
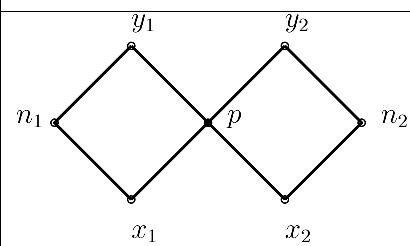
$\lvert C^+\rvert$	$\lvert C^-\rvert$	$\lvert S\rvert$	$\lvert N\rvert$	Neighborhood	Constraints
2	0	0	3		$2w(p) = w(y_1) + w(y_2)$
2	1	0	4		$2w(p) = w(x_1) + w(y_1) + w(y_2)$
2	1	1	5		$2w(p) = w(x_1) + w(y_2)$ $w(x_1) = w(y_1)$
2	2	0	5		$2w(p) = w(x_1) + w(x_2) + w(y_1) + w(y_2)$
2	2	1	6		$2w(p) = w(x_1) + w(x_2) + w(y_2)$ $w(x_1) = w(y_1)$
2	2	2	7		$2w(p) = w(x_1) + w(x_2)$ $w(x_1) = w(y_1)$ $w(x_2) = w(y_2)$

FIGURE 5.2. The possible neighborhoods of p and their constraints, part 2

6. Global characterization

In this section, we continue the assumptions we have made in section 5. Thus, all points $p \in J$ have a neighborhood as described by lem. 5.30, and w must satisfy the consequent constraints of lem. 5.26.

6.1. Cartesian products. If a lattice is the cartesian product product of Fomin lattices, para. 2.3 item (10 shows that it is is a Fomin lattice. It is straightforward that the product lattice preserves any of the properties that we have been discussing (5.1, 5.19, and 5.22) that are true of the factors. Conversely, if a Fomin lattice is the cartesian product of nontrivial lattices, th. 2.7, lem. 2.8, 2.10, and 2.11 show that its factors are Fomin lattices that preserve any of the properties that are true of the product lattice. Thus, we can restrict our attention to lattices that cannot be cartesian-factored without excluding any Fomin lattices from our classification.

Assumption 6.1. *Henceforth, we assume that L cannot be factored into two nontrivial lattices, which is equivalent to assuming that J cannot be partitioned into two disjoint non-empty posets $J = J_1 \sqcup J_2$.*

6.2. J has a minimum point.

Lemma 6.2. *Given any two elements $a, b \in J$, we can connect them by a sequence of elements*

$$a = x_0, x_1, x_2, \dots, x_i, \dots, x_{n-1}, x_n = b$$

with

$$a = x_0 \gtrdot x_1 \gtrdot x_2 \gtrdot \cdots \gtrdot x_i \lessdot \cdots \lessdot x_{n-1} \lessdot x_n = b,$$

for some $0 \leq i \leq n$. (That is, the sequence first descends, then ascends.)

Proof.
Because J is locally finite and cannot be separated into the disjoint union of two non-empty posets, a and b can be connected with a sequence of elements $a = x_0, x_1, x_2, \dots, x_n = b$ where each adjacent pair x_i, x_{i+1} has either $x_i \lessdot x_{i+1}$ or $x_i \gtrdot x_{i+1}$. Given such a sequence, if there is an i such that $0 < i < n$ and $x_{i-1} \lessdot x_i \gtrdot x_{i+1}$, that is, x_i is a local maximum, by the assumption of unique-cover-modularity, there is a y such that $x_{i-1} \gtrdot y \lessdot x_{i+1}$. Thus $a = x_0, x_1, x_2, \dots, x_{i-1}, y, x_{i+1}, \dots x_n = b$ is another such sequence.

We can iterate this transformation on the sequence, and clearly any series of such transformations must terminate in a sequence for which there is no $i < i < n$ for which x_i which is a local maximum. Thus, any final sequence must have some $0 \leq i \leq n$ for which for all $0 \leq j < i$, $x_j \gtrdot x_{j+1}$ and for all $i \leq j < n$, $x_j \lessdot x_{j+1}$. □

Lemma 6.3. *J has a minimum element, which we denote $\hat{0}_J$.*

Proof.
Since J is non-empty and finitary by lem. 5.4(2), J must contain at least one minimal element m. Assume there is an element $a \not\geq m$ in J. Then by lem. 6.2, there is a sequence of elements $m = x_0, x_1, x_2, \dots, x_n = a$ where for some $0 \leq i \leq n$, for any $0 \leq j < i$, $x_j \gtrdot x_{j+1}$ and for any $i \leq j < n$, $x_j \lessdot x_{j+1}$. But since m is minimal, there is no y with $m \gtrdot y$, and so $i = 0$. If $n > 0$, then $m = x_0 \lessdot x_1 \leq x_n = a$. That implies $m < a$, which contradicts that $a \not\geq m$. So $n = 0$ and $m = x_0 = a$, which is also a contradiction. Thus there is no $a \not\geq m$ in J. □

Lemma 6.4. *J is graded. We define ρ as the rank function on J.*

Proof.
Proving this is straightforward give that J is locally finite, is unique-cover-modular, and has a minimum element $\hat{0}_J$. □

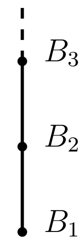
Remark 6.5. *In lem. 6.4, the hypothesis that $\hat{0}_J$ exists is necessary. A counterexample is the poset "3-Plait" in* [Fom1994, Fig. 7 and Exam. 2.2.10]. *Further counterexamples are the posets of points of the lattices of cylindrical partitions.*[El2025]

6.3. The bottom elements. First we define the sequence, finite or infinite, of elements at the bottom of J that form a chain, with each element (above $\hat{0}_J$) being the unique cover of the one below.

Definition 6.6. *We define $B_1 = \hat{0}_J$. Then iteratively for each $i \geq 0$, if B_i has a unique upward cover, we define B_{i+1} to be that cover. If there is a $B_\bullet$ that does not have a unique cover (and thus by lem. reflem:no-triples has either zero or two covers), we define N to be the index of that $B_\bullet$. Otherwise, we define N to be ∞.*

The mnemonic for "$B_\bullet$" is that these are the *bottom* elements of J.

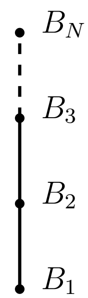
In the case $N = \infty$, the $B_\bullet$ form an upward semi-infinite path, and they are the entirety of J.



The constraints on w are $w(B_1) = r$, $2w(B_1) = w(B_2)$, and, for all $i \geq 2$, $2w(B_i) = w(B_{i-1}) + w(B_{i+1})$, which have the unique solution $w(B_i) = ir$. Thus we have:

Case 6.7. *(para. 2.3(5)) J is the upward semi-infinite path B_i for $i \geq 1$ with $r > 0$ and $w(B_i) = ir$. L is isomorphic to $\mathbb{N}$.*

If N is not ∞, then there are two cases: B_N has zero upward covers and B_N has two upward covers. If B_N has zero upward covers, then J consists of the finite path $B_0 \lessdot B_1 \lessdot B_2 \lessdot \cdots \lessdot B_N$.



By lem. 5.26, the constraints on w are

- $w(B_1) = r$;
- if $N = 2$, $2w(B_1) = 0$;
- if $N \geq 2$, $2w(B_1) = w(B_2)$;
- for all $2 \leq i < N$, $2w(B_i) = w(B_{i-1}) + w(B_{i+1})$; and
- if $N \geq 2$, $2w(B_N) = w(B_{N-1})$.

For all finite N, these require $r = 0$, and so L is not admissible.

6.4. J is two-dimensional. The remaining case is that B_N has exactly two upward covers.

Definition 6.8. *We define J^+ as the elements of J that are not a B_i for which $i < N$ (of which there are at least three, one of which is B_N itself and two of which are the covers of B_N).*

Lemma 6.9. *All $p \in J^+$ are $\geq B_N$, and each has a finite rank $\rho(p) \geq \rho(B_N)$ by lem. 6.4.*

We will "draw the Hasse diagram of J in the plane" by defining a set of locations T_{xy} for all integers $x \geq 0$ and $y \geq 0$ and assigning every $p \in J^+$ to a distinct location T_{xy} with $x + y = \rho(p) - \rho(B_N)$. The mnemonic for "$T_{\bullet\bullet}$" is that they are the *top* elements of J. That is, B_N is assigned to T_{00} and elements of J^+ are assigned "upward" from T_{00}. If $p \lessdot q$ in J^+ and $p = T_{xy}$ then either $q = T_{x+1,y}$ or $q = T_{x,y+1}$. Intuitively, the $T_{\bullet\bullet}$ form a quarter-plane extending upward, with T_{00} at the bottom center, the x coordinate increasing to the upper-left and the y coordinate increasing to the upper-right. We define $v = x + y$; v is how far T_{xy} is vertically above T_{00}. We define $h = y - x$; h is how far T_{xy} is left of T_{00} (if negative) or right of T_{00} (if positive). Note that while v ranges over all nonnegative integers, h ranges over integers $-v \leq h \leq v$ which have the same parity as v. The assignment of elements of J^+ realizes its Hasse diagram, with each cover relation being either one step to the upper left (incrementing x) or one step to the upper right (incrementing y). An example assignment is fig. 6.

We construct the assignments by first assigning $T_{00} = B_N$ and successively assigning each rank of J^+ above $\rho(B_N)$ to locations so as to satisfy these rules.

Lemma 6.10. *We assign the elements of J^+ to locations in $\{T_{xy} \,|\, x \geq 0 \text{ and } y \geq 0\}$ by this process:*

(1) Assign B_N to T_{00}. Thus all elements of rank $\rho(B_N)$, which are only B_N, are assigned to locations with $v = 0$.

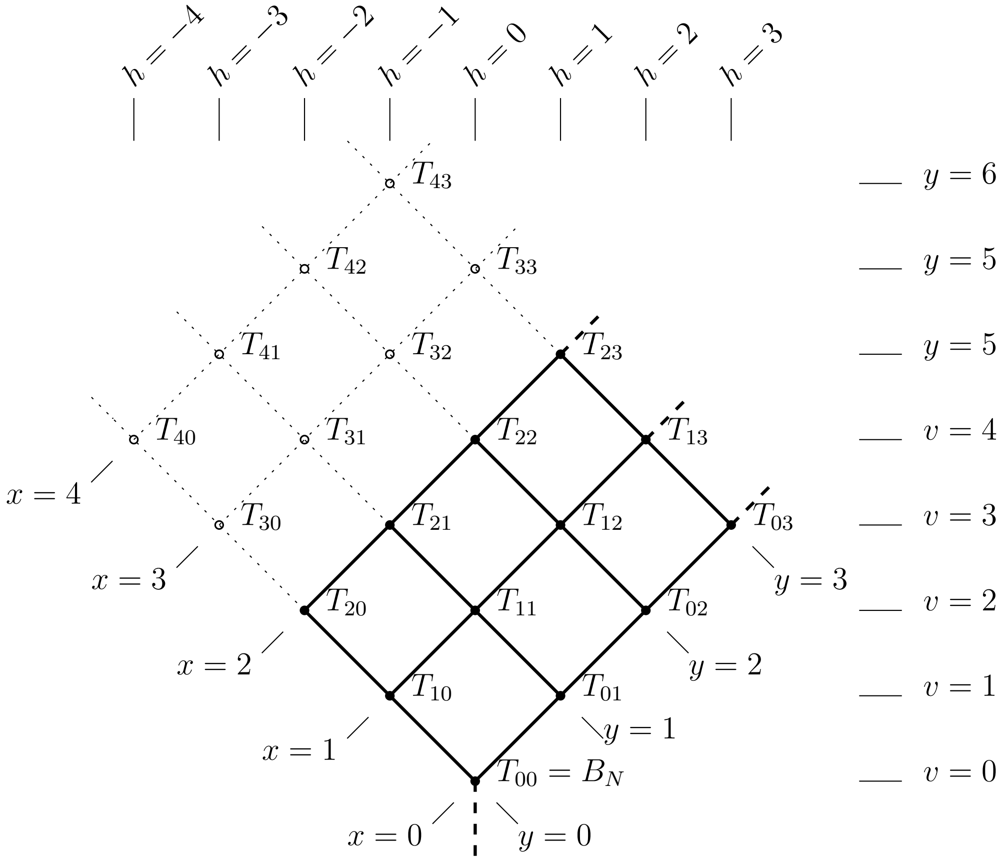


FIGURE 6. An example of assigning elements of J^+ to locations $T_{\bullet\bullet}$. Solid circles are assigned locations, empty circles are unassigned locations. Solid lines are coverings in J.

(2) *Iteratively assign the elements of J^+ with each rank R from $R = \rho(B_N) + 1$ upward. Terminate if we reach a rank and all elements of J^+ have been assigned. (Intuitively, elements are assigned to $T_{\bullet\bullet}$ going upward.)*
   (a) *Iteratively consider each element p of rank $R-1$ (which has already been assigned a location T_{xy}) in order of increasing h (which means decreasing x and increasing y). (Intuitively, elements are examined left-to-right.)*
      (i) *If p is the unique element of rank R, then assign one cover of p (if it exists) to $T_{x+1,y}$ and the other cover of p (if p has two covers) to $T_{x,y+1}$.*
      (ii) *Otherwise, if p is the element of rank R with minimum h (maximum x, minimum y, leftmost), $T_{x-1,y+1}$ must be assigned and its cover $T_{xy} \vee T_{x-1,y+1}$ (which must exist) is assigned to $T_{x,y+1}$, and its other cover (if it exists) is assigned to $T_{x+1,y}$.*
      (iii) *Otherwise, if p is the element of rank R with maximum h (minimum x, maximum y, rightmost), $T_{x+1,y-1}$ must be assigned and its cover $T_{x+1,y-1} \vee T_{xy}$ (which must exist) is assigned to $T_{x+1,y}$, and its other cover (if it exists) is assigned to $T_{x,y+1}$.*
      (iv) *Otherwise, $T_{x-1,y+1}$ and $T_{x+1,y-1}$ must both be assigned, its cover $T_{xy} \vee T_{x-1,y+1}$ is assigned to $T_{x,y+1}$, and its other cover $T_{x+1,y-1} \vee T_{xy}$ is assigned to $T_{x+1,y}$.*

This algorithm has these properties:

(A) Elements p of J^+ are assigned to locations T_{xy} where $v = x+y = \rho(p) - \rho(B_N)$. Thus, there are elements with values of v either ranging from 0 to some $v_{\max}$ or from 0 upward indefinitely (in which case we define $v_{\max} = \infty$).

(B) The elements assigned to locations with a particular value of v have values h that are a contiguous subset of the integers $-v \leq h \leq v$ which have the same parity as v. The minimum x of an assigned location with a particular

value of v is defined to be $h_v^{\min}$ and the maximum x of an assigned location is with a particular value of v is defined to be $h_v^{\max}$.

(C) For any $0 \leq v < v_{\max}$, $h_{v+1}^{\min} = h_v^{\min} \pm 1$ and $h_{v+1}^{\max} = h_v^{\max} \pm 1$.

(D) Any adjacent pair of assigned locations T_{xy} and $T_{x-1,y+1}$ are the covers of $T_{x-1,y}$ and are covered by $T_{x,y+1}$.

(E) If p is assigned to T_{xy}, then $q \gtrdot p$ iff q is assigned to either $T_{x+1,y}$ or $T_{x,y+1}$. That is, q's v value is one more than p's and its h value is either one more or one less than p's.

The algorithm is nondeterministic in case (2)(a)(1) if p has two covers; either cover may be assigned to either of the two locations. In particular, this happens when assigning the two elements that cover B_N. In all other steps, it is deterministic.

Proof.

The properties stated in the lemma and the correctness of the algorithm are proved by mutual induction. A lot of the details of the proof are straightforward and will be omitted. The critical facts are

(1) J^+ is graded. (lem. 6.4)
(2) Each element of J^+ has at most two downward covers and two upward covers. (lem. 5.27)
(3) If a location T_{xy} is in the middle of its rank, that is, has $h_v^{\min} < h < h_v^{\max}$, then it has exactly two downward covers (one shared with $T_{x+1,y-1}$ and one shared with $T_{x-1,y+1}$), which implies by unique-cover-modularity that it has exactly two upward covers shared with the same elements.
(4) If a location has an upward cover that is not shared with the adjacent element on its rank, it has h either $h_v^{\min}$ or $h_v^{\max}$ and by unique-cover-modularity that cover is not an upward cover of any other element.

□

Because the assigned locations on each rank are contiguous and very similar to the assigned locations on the ranks above and below, the possible J^+ are determined up to isomorphism by the outline of the set of assigned locations, the values $h_\bullet^{\min}$ and $h_\bullet^{\max}$.

6.5. The bottom of J. We can now examine the constraints implied by the neighborhoods of J.

Definition 6.11. *We define $m = w(B_N) = w(T_{00})$. We define $\delta^- = w(T_{10}) - m$ and $\delta^+ = w(T_{01}) - m$.*[13]

Both δ^- and δ^+ are well-defined since $T_{00} = B_N$ must have two upward covers, which are T_{10} and T_{01}. The mnemonic for "m" is that it is the weight of the "middle" element of J, namely B_N. The mnemonic for "δ^-" is that it is a difference in the *negative* direction of h, that is, the "left" side of J. Similarly, the mnemonic for "δ^+" is that it is a difference in the *positive* direction of h, that is, the "right" side of J.

Lemma 6.12. *The constraints on w due to applying lem. 5.26 to B_1 and all the B_i for $i < N$ are*

$$w(B_i) = ir \qquad \text{for all } 1 \leq i \leq N \tag{29}$$

In particular, $m = Nr$.

Lemma 6.13. *The constraints on w due to applying lem. 5.26 to B_N are*

$$\delta^- + \delta^+ = -(N-1)r. \tag{30}$$

In consequence, $\delta^- + \delta^+$ is not positive and not both δ^+ and δ^- are positive.

Lemma 6.14. *If T_{xy} and $T_{x+1,y+1}$ are both assigned, then either $T_{x+1,y}$ or $T_{x,y+1}$ or both are assigned and $w(T_{xy}) = w(T_{x+1,y+1})$. As a consequence, if there is a sequence of assigned locations $T_{x+i,y+i}$ for $0 \leq i \leq n$ then all of these locations have the same value of w. (This set of locations has the same value of h and v values $x+y+2i$.)*

[13] The following derivations would be a bit simpler if δ^- and δ^+ were defined with the opposite signs. We use the current definitions because when the poset $T_{\bullet\bullet}$ is reoriented into the standard English format for diagrams, one of them is the increment of the weights of points when stepping from left to right and one of them is the increment of the weights of points when stepping from top down.

6.6. The left and right sides of J. We start by analyzing the right side of J, the elements assigned to T_{xy} where $y \geq x$.

Definition 6.15. *If there is a maximum value of y for which T_{0y} is assigned, we define that value to be $y_{\max}$. In that case, $y_{\max} \geq 1$. If T_{0y} is assigned for all $y \geq 0$, we define $y_{\max} = \infty$.*

Note that we have *not* proven that $y_{\max}$ is the largest value of y for which some T_{xy} is defined. We use the convention that $i < \infty$ for any (finite) integer i.

Lemma 6.16.

$$w(T_{0y}) = m + y\delta^+ \qquad \text{for all } 1 \leq y \leq y_{\max} \tag{31}$$

$y_{\max}$ is finite iff $\delta^+ < 0$. If $y_{\max}$ is finite, then

$$y_{\max} = -m/\delta^+ - 1. \tag{32}$$

which is equivalent to $w(T_{0,y_{\max}}) = -\delta^+$ and

$$m = -(y_{\max} + 1)\delta^+. \tag{33}$$

In particular, $-\delta^+$ must divide m

Proof.
Because T_{10} is assigned, unique-cover-modularity implies that T_{1y} is assigned for all $y \leq y_{\max}$. This implies that for any $1 \leq y \leq y_{\max}$, none of the T_{1y} are upward orphans of the the corresponding T_{0y}, as $T_{1,y-1}$ is assigned. Thus the constraints on w due to applying lem. 5.26 to T_{0y} for $1 \leq y < y_{\max}$ are $2w(T_{0y}) = w(T_{0,y-1}) + w(T_{0,y+1})$, which collectively imply (31).

If $y_{\max}$ is finite, then by definition $T_{0,y_{\max}+1}$ is not assigned and the constraint on w for $T_{0,y_{\max}}$ is $2w(T_{0,y_{\max}}) = w(T_{0,y_{\max}-1})$. Applying (31) and lem. 6.14 gives $2(m + y_{\max}\delta^+) = m + (y_{\max} - 1)\delta^+$, which implies (32).

Conversely, assume $\delta^+ \geq 0$. Then (32) cannot be satisfied by any finite $y_{\max} \geq 1$. Hence by the preceding paragraph, there can be no finite $y_{\max}$. ☐

Lemma 6.17. *For all $x < y$, applying (31) and lem. 6.14 gives*

$$w(T_{xy}) = w(T_{0,y-x}) = m + (y - x)\delta^+ = m + h\delta^+ \tag{34}$$

where $h = y - x$ as defined above.

Definition 6.18. *Define $c_{\max}$ as the maximum integer such that for all $0 \leq i \leq c_{\max}$, T_{ii} is assigned, or if all T_{ii} are assigned, then $c_{\max} = \infty$.*

Note that we have *not* proved that there is no T_{jj} assigned with $j > c_{\max}$. The mnemonic for "$c_{\max}$" is that it is the index of the top element of the *centerline* extending upward from T_{00}.

Lemma 6.19. *By unique-cover-modularity, all T_{xy} are assigned for all x, y with $x < y$, $x \leq c_{\max}$, and $y \leq y_{\max}$.*

Lemma 6.20. *If $y \geq x + 2$, $T_{x+1,y-1}$ is not assigned, and $T_{x,y-1}$, T_{xy}, and $T_{x-1,y}$ are assigned, then $T_{x+1,y}$ is not assigned. (See fig. 7a for a diagram of the configuration.)*

Proof.
The constraint for T_{xy} is

$$2w(T_{xy}) = w(T_{x,y-1}) + w(T_{x-1,y}) + w(T_{x+1,y}) + w(T_{x,y+1})$$

where the third term on the right is present iff $T_{x+1,y}$ is an upward orphan of T_{xy} and the fourth term on the right is present iff $T_{x,y+1}$ is an upward orphan of T_{xy}. If we define $\alpha = w(T_{xy})$, then this simplifies to

$$2\alpha = (\alpha - \delta^+) + (\alpha - \delta^+) + w(T_{x+1,y}) + w(T_{x,y+1}).$$

Since both $w(T_{x+1,y})$ and $w(T_{x,y+1})$ must be positive, both $T_{x+1,y}$ and $T_{x,y+1}$ must be either unassigned or not upward orphans of T_{xy}. Since by hypothesis $T_{x+1,y-1}$ is unassigned, $T_{x+1,y}$ would be an upward orphan if it was assigned. Thus $T_{x+1,y}$ is not assigned. ☐

Lemma 6.21. *If $y \geq x + 1$, $T_{x-1,y+1}$ is not assigned, and $T_{x-1,y}$, T_{xy}, and $T_{x,y-1}$ are assigned, then $T_{x,y+1}$ is not assigned. (See fig. 7b for a diagram of the configuration.)*

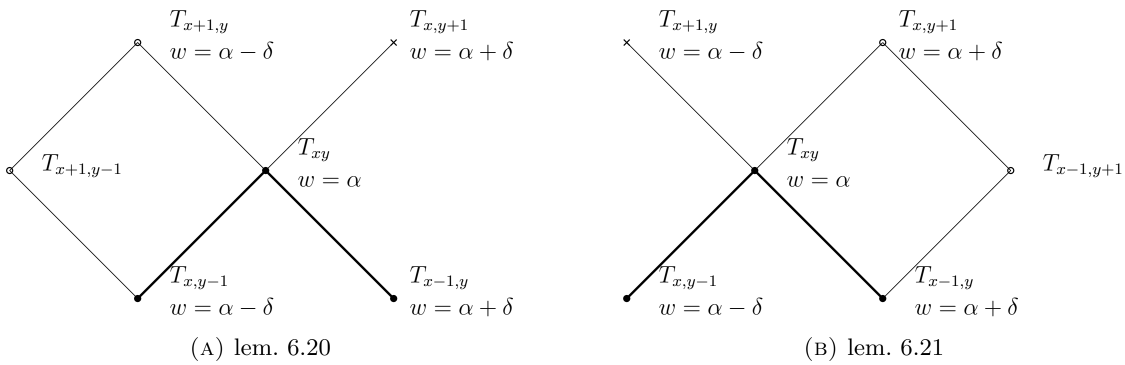


FIGURE 7. Configurations for lem. 6.20 and 6.21, showing the weights of the elements relative to $\alpha = w(T_{xy})$. Solid dots indicate assigned elements, empty dots indicate unassigned elements, and crosses can be either.

Proof.
This is proved by a left-right reversal of the proof of lem. 6.20. □

Lemma 6.22. *Given the values of $y_{\max}$ and $c_{\max}$, the set of assigned T_{xy} of J^+ with $x \leq y$ is as shown in the corresponding diagram in fig. 8.*

Proof.
In each diagram of fig. 8, the $T_{\bullet\bullet}$ on the centerline that are shows as assigned and unassigned is determined by the corresponding hypothesis regarding $c_{\max}$. Similarly, the $T_{\bullet\bullet}$ on the y axis that are shows as assigned and unassigned is determined by the corresponding hypothesis regarding $y_{\max}$.

The assigned $T_{\bullet\bullet}$ represented by the shaded areas are determined by lem 6.19.

The series of unassigned $T_{\bullet\bullet}$ extending upward and to the right (increasing y) from $T_{c_{\max}+1,c_{\max}+1}$ are determined by lem. 6.20. The series of unassigned $T_{\bullet\bullet}$ extending upward and to the left (increasing x) from $T_{0,y_{\max}+1}$ are determined by lem. 6.21.

In case (D), the non-allocation $T_{c_{\max}+1,y_{\max}+1}$ is determined by lem. 6.14, since neither $T_{c_{\max}+1,y_{\max}}$ $T_{c_{\max},y_{\max}+1}$ is assigned. □

The analysis of the left side of J, the elements assigned to T_{xy} where $x \geq y$, is the left-right reversal to the analysis of the right side.

Definition 6.23. *If there is a maximum value of x for which T_{x0} is assigned, we define that value to be $x_{\max}$. In that case, $x_{\max} \geq 1$. If T_{x0} is assigned for all $x \geq 0$, we define $x_{\max} = \infty$.*

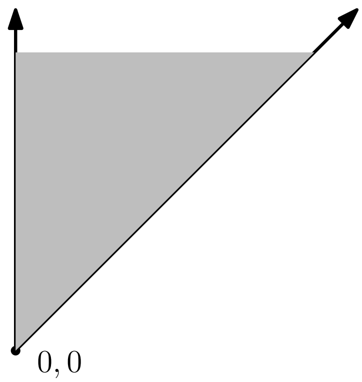
Lemma 6.24. *Given the values of $x_{\max}$ and $c_{\max}$, the set of assigned T_{xy} of J^+ with $x \geq y$ is the left-right reversal of diagram in fig. 8 for $y_{\max} = -x_{\max}$ and $c_{\max}$.*

6.7. The global structure of J. Thus, the overall structure of the allocation of the lower ranks of J^+ can be described by two letters, the first being the case of fig. 8 that describes the lower part of the left-hand side (where $h \leq 0$) and the second letter being the case that describes the lower part of the right-hand side (where $h \geq 0$). Since swapping the two letters corresponds to left-right reversing the allocations, leaving the poset J isomorphic, without loss of generality, we list only letter pairs where the first is alphabetically before or the same as the second.

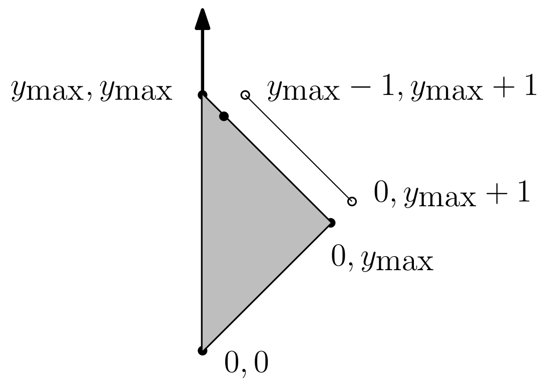
Most combinations of left and right cases determine the entire assignment of J^+ as the allocations and non-allocations in the two cases determine the allocation of the entirety of J^+. E.g. (AA) specifies that all $T_{\bullet\bullet}$ are assigned. But also (CE) specifies that all $T_{\bullet,c_{\max}+1}$ are unassigned, which requires that no $T_{\bullet y}$ is assigned for $y > c_{\max}$. On the other hand, (AB) does not directly specify the allocation of T_{xy} when $y_{\min} \leq x < y$.

Lemma 6.25. *The possible structures of the allocation of the lower ranks of J^+ are (AA), (AB), and (CE).*

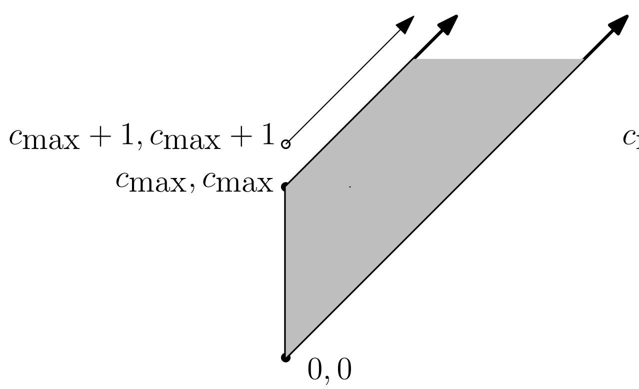
Proof.



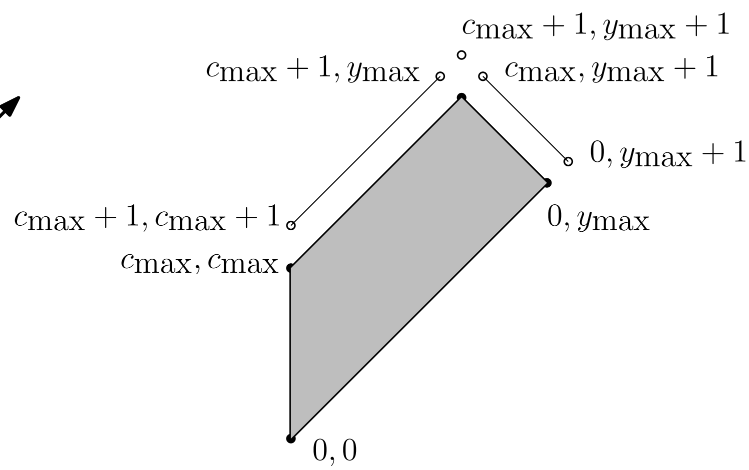
(A) $c_{\max} = \infty$ and $y_{\max} = \infty$



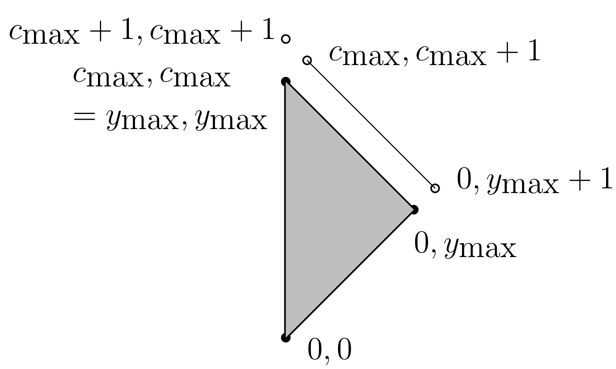
(B) $c_{\max} = \infty$ and $y_{\max}$ finite



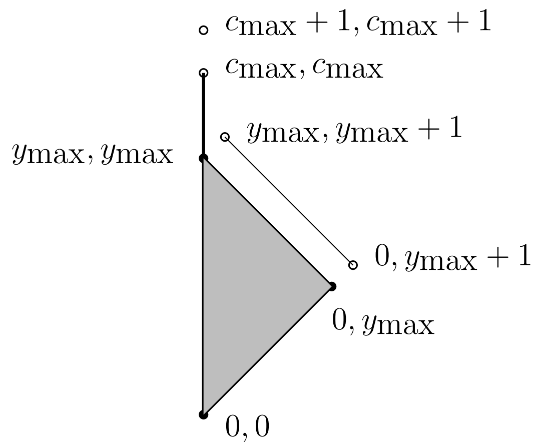
(C) $c_{\max}$ finite and $y_{\max} = \infty$



(D) $c_{\max}$ finite, $y_{\max}$ finite, and $y_{\max} > c_{\max}$



(E) $c_{\max}$ finite, $y_{\max}$ finite, and $y_{\max} = c_{\max}$



(F) $c_{\max}$ finite, $y_{\max}$ finite, and $y_{\max} < c_{\max}$

FIGURE 8. The possible sets of assignments on the left side of J^+. Solid dots, heavy lines, and shading indicate assigned $T_{\bullet\bullet}$'s. Empty dots and light lines indicate unassigned $T_{\bullet\bullet}$'s.

Of necessity, the left side and the right side have the same value of $c_{\max}$, and in particular it is either finite or infinite. Thus the possible structures are composed of letters A and B (if $c_{\max} = \infty$), or of letters C, D, E, and F (if $c_{\max}$ is finite).

The following cases lead to contradictions.

(BB): The allocation of $T_{y_{\max},y_{\max}}$ and $T_{y_{\max}+1,y_{\max}+1}$ violates lem. 6.14, since neither $T_{y_{\max}+1,y_{\max}}$ nor $T_{y_{\max},y_{\max}+1}$ is assigned.

(CC), (CD), and (DD): The allocation of $T_{c_{\max},c_{\max}}$, $T_{c_{\max}+1,c_{\max}}$, and $T_{c_{\max},c_{\max}+1}$ but not $T_{c_{\max}+1,c_{\max}+1}$ violates unique-cover-modularity.

(CF): The constraint for $T_{y_{\max}+1,y_{\max}}$ (which is assigned on the left side) is

$$2w(T_{y_{\max}+1,y_{\max}}) = w(T_{y_{\max}+1,y_{\max}-1}) + w(T_{y_{\max},y_{\max}}) + w(T_{y_{\max}+1,y_{\max}+1})$$

because $T_{y_{\max}+1,y_{\max}+1}$ is assigned and $T_{y_{\max},y_{\max}+1}$ is not, making $T_{y_{\max}+1,y_{\max}+1}$ an upward orphan of $T_{y_{\max}+1,y_{\max}}$. That equation can be simplified to

$$2(m + \delta^-) = (m + 2\delta^+) + m + w(T_{y_{\max}+1,y_{\max}+1})$$

which is impossible since $w(T_{y_{\max}+1,y_{\max}+1})$ is positive.

(DE), (DF), (EE), (EF), and (FF): The constraint for T_{11} (which must be assigned) is

$$2w(T_{11}) = w(T_{10}) + w(T_{01})$$

That simplifies to $2m = (m + \delta^-) + (m + \delta^+)$, so $\delta^- + \delta^+ = 0$. But in this case both $x_{\max}$ and $y_{\max}$ are finite, so both δ^+ and δ^- must be negative, which is a contradiction.

Thus only (AA), (AB), and (CE) are possibilities. □

Lemma 6.26. *If the structure of the allocation J^+ is (AA), then the structure of J is:*

(1) All $T_{\bullet\bullet}$ are assigned.
(2) $N = 1$ and $m = r$.
(3) $w(T_{xy}) = r$ for all x, y.

Thus,

Case 6.27. *(para. 2.3(2)) J is isomorphic to the quadrant of the plane, $\{(x, y) \mid x \text{ and } y \text{ are integers} \geq 0\}$, and L is $\mathbb{Y}$, Young's lattice, the lattice of partitions, with weighting r times the conventional weighting of 1.*

Proof.

By the hypothesis (AA), all $T_{\bullet\bullet}$ are assigned.

The constraint for T_{11} is

$$2w(T_{11}) = w(T_{10}) + w(T_{01})$$

That simplifies to $2m = (m + \delta^-) + (m + \delta^+)$, so $\delta^- + \delta^+ = 0$. In this case $x_{\max} = y_{\max} = \infty$, so both δ^+ and δ^- must be nonnegative, and thus $\delta^- = \delta^+ = 0$.

By (30), $N = 1$. Thus $T_{00} = B_1$.

By (29), $m = w(B_1) = r$. Since $\delta^- = \delta^+ = 0$, by (31) and its left-right reverse, $w(T_{\bullet\bullet}) = r$. □

Lemma 6.28. *If the structure of the allocation J^+ is (AB), then the structure of J is:*

(1) T_{xy} is assigned iff $y \leq x + 1$.
(2) $N = 2$.
(3) $w(T_{xy}) = 2r$ if $y \leq x$.
(4) $w(T_{xy}) = r$ if $y = x$.

Thus,

Case 6.29. *(para. 2.3(3)) J is isomorphic to the octant of the plane, $\{(x, y) \mid x \text{ and } y \text{ are integers with } x \geq 0 \text{ and } y \leq x\}$, and L is $\mathbb{SY}$, the shifted Young's lattice, the lattice of partitions into distinct parts, with weighting r times the conventional weighting of 1 (for "diagonal" elements) and 2 (for "non-diagonal" elements).*

Proof.

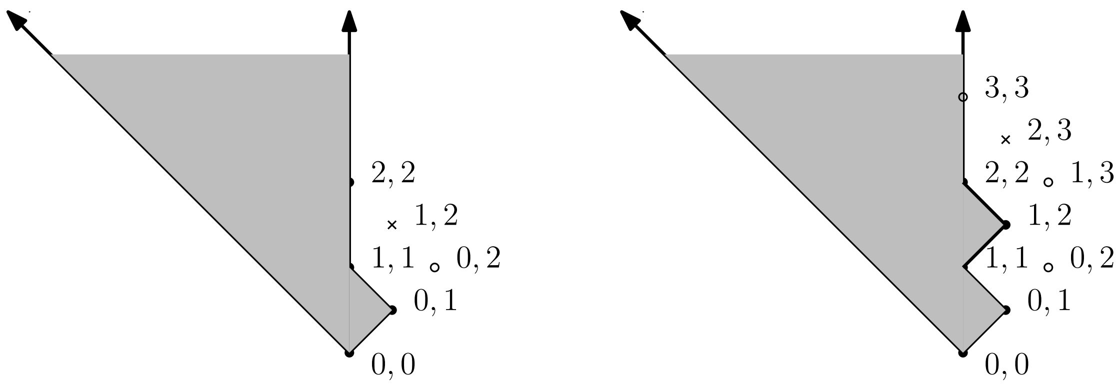
By the hypothesis (AB), all T_{xy} with $x \geq y$ are assigned, all T_{xy} with $x \leq y \leq y_{\max}$ are assigned, and all $T_{x,y_{\max}+1}$ with $x \leq y_{\max} - 1$ are unassigned.

If $y_{\max} \geq 2$, the constraint for T_{11} (which must be assigned) is

$$2w(T_{11}) = w(T_{10}) + w(T_{01})$$

because neither T_{21} or T_{12} are upward orphans of T_{11}. That simplifies to $2m = (m+\delta^-)+(m+\delta^+)$, so $\delta^- + \delta^+ = 0$. In this case $x_{\max} = \infty$ and $y_{\max}$ is finite, so δ^+ must be negative and δ^- must be nonnegative, which is a contradiction. Thus $y_{\max} = 1$, which implies $m = -2\delta^+$.

The allocations and non-allocations that are known so far are as shown in fig. 9a.



(A) Lem. 6.28, first step. (B) Lem. 6.28, second step.

FIGURE 9. Solid dots indicate assigned elements, empty dots indicate unassigned elements, and crosses can be either.

If T_{02} is not assigned, then the constraint for T_{11} is

$$2w(T_{11}) = w(T_{10}) + w(T_{01})$$

That simplifies to $2m = (m + \delta^-) + (m + \delta^+)$, so $\delta^- + \delta^+ = 0$. But by the same argument as above, this is a contradiction. Thus T_{12} is assigned.

The constraint for T_{11} is

$$2w(T_{11}) = w(T_{10}) + w(T_{01}) + w(T_{12})$$

That simplifies to $2m = (m + \delta^-) + (m + \delta^+) + (m + \delta^+)$ and since $m = -2\delta^+$, $\delta^- = 0$.

The allocations and non-allocations that are known so far are as shown in fig. 9b.

This analysis can be iterated, examining the constraints for T_{ii} for increasing $i \geq 1$, giving the complete allocation of J^+ shown in fig. 10.

The constraint for T_{00} is

$$2w(T_{00}) = w(B_N) = w(B_{N-1}) + w(T_{10}) + w(T_{01})$$

which simplifies to $2Nr = (N - 1)r + (m + \delta^-) + (m + \delta^+)$. Since $m = Nr = -2\delta^+$ and $\delta^- = 0$, this reduces to $r = -\delta^+$, and thus $N = 2$ and $m = 2r$. □

Lemma 6.30. *If the structure of the allocation J^+ is (CE), then the structure of J is:*

(1) $r = \alpha k$ for some positive integers α, k.
(2) T_{xy} is assigned iff $y < k$.
(3) $N = 1$ and $m = r$.
(4) $w(T_{xy}) = \alpha(k + x - y)$ for all $0 \leq x$ and $0 \leq y < k$.

Thus,

Case 6.31. *(para. 2.3(4)) J is isomorphic to the k-row strip, $\{(x, y) \,|\, x \text{ and } y \text{ are integers}, 0 \leq x \text{ and } 0 \leq y < k\}$, and L is $\mathbb{Y}_k$, the k-row Young's lattice, the lattice of partitions with at most k parts, with weighting $\alpha > 0$ times the conventional weighting $w(T_{xy}) = k + x - y$.*

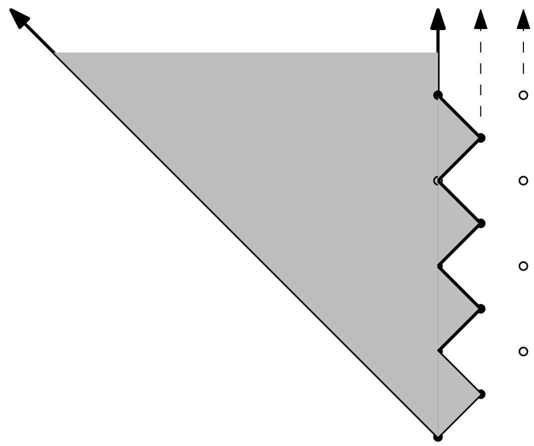


FIGURE 10. Solid dots indicate assigned elements and empty dots indicate unassigned elements.

Proof.

By the hypothesis (CE), $c_{\max} = y_{\max}$. All T_{xy} are assigned for $0 \leq y < y_{\max}$. Also, all $T_{x,y_{\max}+1}$ are unassigned which constrains all T_{xy} with $y \geq y_{\max}$ to be unassigned.

The constraint for T_{11} (which must be assigned) is

$$2w(T_{11}) = w(T_{10}) + w(T_{01})$$

That simplifies to $2w = (w + \delta^-) + (w + \delta^+)$, so $\delta^- + \delta^+ = 0$. In this case $x_{\max} = \infty$ and $y_{\max}$ is finite, δ^+ is a negative integer and $\delta^- = -\delta^+$ is a positive integer. By (30), $N = 1$. Thus $T_{00} = B_1$. By (29), $m = w(B_1) = r$.

Define $k = y_{\max} + 1$ and $\alpha = -\delta^+ = \delta^-$. Thus by (33), $r = m = k\alpha$. By (31) and its left-right reverse, $w(T_{xy}) = \alpha(k + x - y)$. ☐

7. Classification theorem

Theorem 7.1. *The only Fomin lattices L that satisfy the assumptions*

(assump. 5.1) The lattice L is distributive. We define the poset of points (join-irreducibles) to be J, so $L = \mathrm{Ideal_f}(J)$.
(assump. 5.19) The J is unique-cover-modular.
(assump. 5.22) The weighting w is admissible, that is, the set of values V, the target of w, is $\mathbb{Z}$, and all of the values of w are positive. The differential degree r is positive.
(assump. 6.1) The lattice cannot be factored into two nontrivial lattices, which is equivalent to assuming that J cannot be partitioned into two disjoint non-empty posets $J = J_1 \sqcup J_2$.

are the cases

6.7: (para. 2.3(5)) J is the upward semi-infinite path B_i for $i \geq 1$ with $r > 0$ and $w(B_i) = ir$. L is isomorphic to $\mathbb{N}$.
6.27: (para. 2.3(2)) J is isomorphic to the quadrant of the plane, $\{(x, y) \,|\, x \text{ and } y \text{ are integers} \geq 0\}$, and L is $\mathbb{Y}$, Young's lattice, the lattice of partitions, with weighting r times the canonical weighting of 1.
6.29: (para. 2.3(3)) J is isomorphic to the octant of the plane, $\{(x, y) \,|\, x \text{ and } y \text{ are integers with } x \geq 0 \text{ and } y \leq x\}$, and L is $\mathbb{SY}$, the shifted Young's lattice, the lattice of partitions into distinct parts, with weighting r times the canonical weighting of 1 and 2.
6.31: (para. 2.3(4)) J is isomorphic to the k-row strip, $\{(x, y) \,|\, x \text{ and } y \text{ are integers, } 0 \leq x \text{ and } 0 \leq y < k\}$, and L is $\mathbb{Y}_k$, the k-row Young's lattice, the lattice of partitions with at most k parts, with weighting $\alpha > 0$ times the canonical weighting $w(T_{xy}) = k + x - y$.

Proof.

This is the summation of 6.25, 6.26, 6.28, and 6.30. ☐

8. Future directions

The strength of the assumptions of our classification theorem (th. 7.1 is disappointing, but they are satisfied by all known Fomin lattices except the one family that is not even distributive, viz., the Young–Fibonacci lattices and cartesian products that include them. In future work, we hope to expand the classification to larger classes. The following considerations come to mind.

There are a number of ways of relaxing the assumption of admissibility.

(1) Considering sets of values V that are not integers $\mathbb{Z}$ may be of abstract interest but doesn't seem to be directly relevant to Robinson–Schensted algorithms. However, investigations of "generic" weightings of a lattice can exploit this generality. For instance, looking at weightings of the quadrant as J, we could start with the generic formula $w((i,j)) = \alpha+\beta r+\gamma i+\delta j$ and see what values of α, β, γ, and δ allow the differential criterion to be satisfied. This can be modeled by setting $V = \mathbb{R}^4$ and translating $\alpha + \beta r + \gamma i + \delta j$ into $(\alpha, \beta, \gamma, \delta)$ and r into $(0, 1, 0, 0)$. Once the possible weightings for this V are determined, then a homomorphism from V to $\mathbb{Z}$ can be applied and the result weightings that are admissible can be identified. From these admissible weightings Robinson–Schensted algorithms can be instantiated.

    When L, w, and V are jointly constructed from some complex algebraic structure (such as [Fom1994, sec. 2.3]) this sort of flexibility may be particularly useful.

(2) Allowing values of w to be negative (when $V = \mathbb{Z}$) does not seem to be relevant to construction Robinson–Schensted algorithms.

(3) Allowing values of w to be 0 may be useful as an intermediate step in constructing admissible weightings, but it will reveal no new Fomin lattices since th. 4.16 et al. shows that a Fomin lattice with some edges weighted 0 can be truncated into connected components which are each Fomin lattices with all edges having nonzero weights.

(4) Allowing $r \leq 0$ (with $V = \mathbb{Z}$) seems to have limited potential as then either L is nontrivial or does not have $\hat{0}$. However, see the discussion of Fomin lattices without $\hat{0}$ in (5).

There are also a number of ways of relaxing our structural assumptions about L.

(5) Relaxing the assumption that L has a $\hat{0}$ seems like it would be tractable. But lem. 6.4 and rem. 6.5 show that the existence of $\hat{0}$ has non-obvious consequences in that it causes unique-cover-modularity to imply that L is graded. Thus removing the requirement of $\hat{0}$ may introduce unexpected complexity.

    Currently, it is impossible to use a Fomin lattice without $\hat{0}$ to build a Fomin lattice with $\hat{0}$, and thus a Robinson–Schensted algorithm. But the truncation theorems (th. 4.16 et al.) suggest it's worth looking for processes by which they might be exploited. The existence of Fomin lattices without $\hat{0}$ that do not resemble any Fomin lattices with $\hat{0}$ (para. 2.3(7)(8)) suggests this might be productive.

(6) Relaxing the assumption that poset of points is unique-cover-modular to just cover-modular seems like it may be tractable. The structure of neighborhoods would become more complicated but perhaps a careful analysis of the graphs of possible coverings would show that a tractable set of constraints characterize the weights of the points.

(7) Relaxing the assumption that the poset of points is cover-modular seems to be more difficult as it is used in the proof that no element of L has more than two upward or downward covers (lem. 5.27) and a number of the lemmas for global characterization.

(8) Classifying modular but non-distributive Fomin lattices seems to be the most difficult problem. There seems to be no tractable representation of general modular lattices that parallels Birkhoff's Representation Theorem for distributive lattices, and our entire analysis is based on characterizing L in the manner of Birkhoff's representation.

On the largest scale,

(9) All of the Robinson–Schensted algorithms that have attracted attention in the past has been derivable as growth diagrams based on what we call Fomin lattices, viz., modular lattices whose weightings are constant on projectivity classes. However the growth diagram construction can be based on any pair of dual graded graphs which is a much broader and more complicated class of structures. There needs to be an investigation why only the Fomin lattices have appeared in previous works.

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