Finite groups, commuting probability, and coprime automorphisms

Eloisa Detomi, Robert M. Guralnick, Marta Morigi, and Pavel Shumyatsky

Abstract. Given two subgroups H, K of a finite group G, the probability that a pair of random elements from H and K commutes is denoted by $\Pr(H, K)$. Suppose that a finite group G admits a group of coprime automorphisms A and let $\epsilon > 0$. We show that, if for any distinct primes $p, q \in \pi(G)$ there is an A-invariant Sylow p-subgroup P and an A-invariant Sylow q-subgroup Q of G for which $\Pr([P, A], [Q, A]) \geq \epsilon$, then $F_2([G, A])$ has ϵ-bounded index in $[G, A]$ (Theorem 1.2). Here $F_2(K)$ stands for the second term of the upper Fitting seris of a group K. We also show that, if $G = [G, A]$ and for any prime p dividing the order of G there is an A-invariant Sylow p-subgroup P such that $\Pr([P, A], [P, A]^x) \geq \epsilon$ for all $x \in G$, then G is bounded-by-abelian-by-bounded (Theorem 1.4).

1. Introduction

Given two subsets X, Y of a finite group G, we write $\Pr(X, Y)$ for the probability that two random elements $x \in X$ and $y \in Y$ commute. The number $\Pr(G, G)$ is called the commuting probability of G. It is well-known that $\Pr(G, G) \leq 5/8$ for any nonabelian group G. Another important result is the theorem of P. M. Neumann [**26**] which states that if G is a finite group and ϵ is a positive number such that $\Pr(G, G) \geq \epsilon$, then G has a normal subgroup R such that both

2020 *Mathematics Subject Classification.* 20D20; 20D45; 20P05.

Key words and phrases. Commuting probability, Sylow subgroups, coprime automorphisms.

The first and the third authors are members of GNSAGA (INDAM) and were funded by Project 2022PSTWLB (subject area: PE - Physical Sciences and Engineering) "Group Theory and Applications". The second author was partially supported by NSF grant DMS 1901595 and Simons Foundation Fellowship 609771. The fourth author was supported by FAPDF and CNPq.

the index $|G : R|$ and the order of the commutator subgroup $[R, R]$ are ϵ-bounded (see also [**14**]). When a group G has a structure as in P. M. Neumann's theorem, we say that G is bounded-by-abelian-by-bounded. More generally, throughout the article we use the expression "$(a, b, \dots)$-bounded" to mean that a quantity is bounded from above by a number depending only on the parameters $a, b, \dots$. A number of further results on commuting probability in finite groups can be found in [**18, 8, 13, 12**].

It is well-known that a finite group is nilpotent if and only if any two Sylow subgroups of coprime orders commute. The following theorem, established in [**12**], provides a probabilistic variation of this fact.

Theorem 1.1. *If $\epsilon > 0$ and G is a finite group such that for any distinct primes $p, q \in \pi(G)$ there is a Sylow p-subgroup P and a Sylow q-subgroup Q of G for which $\mathrm{Pr}(P, Q) \geq \epsilon$, then $F_2(G)$ has ϵ-bounded index in G.*

As usual, here $F_i(G)$ stands for the ith term of the upper Fitting series of the group G.

In this paper we handle similar issues for finite groups admitting coprime automorphisms. An automorphism α of a finite group G is said to be coprime if $(|G|, |\alpha|) = 1$. If a group A acts on a group G, we write $[G, A]$ to denote the subgroup generated by all $g^{-1}g^{\alpha}$, where $\alpha \in A$ and $g \in G$. Note that $[G, A]$ is an A-invariant normal subgroup of G. If A is a group of coprime automorphisms of G, then $[G, A]$ is nilpotent if and only if $[P, A]$ and $[Q, A]$ commute whenever P is an A-invariant Sylow p-subgroup and Q is an A-invariant Sylow q-subgroup with $p \neq q$ (see for example [**3**, Theorem 1.4]). Here we will prove

Theorem 1.2. *Let $\epsilon > 0$, and let G be a finite group admitting a group of coprime automorphisms A such that for any distinct primes $p, q \in \pi(G)$ there is an A-invariant Sylow p-subgroup P and an A-invariant Sylow q-subgroup Q of G for which $\mathrm{Pr}([P, A], [Q, A]) \geq \epsilon$. Then $F_2([G, A])$ has ϵ-bounded index in $[G, A]$.*

It is noteworthy that under the hypothesis of Theorem 1.2 the index of $F([G, A])$ in $[G, A]$ can be arbitrarily large (see the example in Section 4).

Our next result is related to the theorem established in [**11**] that if P is a Sylow subgroup of a finite group G such that $\mathrm{Pr}(P, P^x) \geq \epsilon$ for all $x \in G$, then the index $[P : O_p(G)]$ is ϵ-bounded while $O_p(G)$ is bounded-by-abelian-by-bounded. We will prove the following theorem.

Theorem 1.3. *Let G be a finite group admitting a group of coprime automorphisms A. Let P be an A-invariant Sylow p-subgroup of G and* *assume that*

$$\Pr([P, A], [P, A]^x) \geq \epsilon$$

for all $x \in G$. Then the order of $[P, A]$ modulo $O_p(G)$ is ϵ-bounded.

Next, we deal with groups G in which the above condition holds for every prime divisor of the order of G. It turns out that in this case the structure of G is as restricted as in P. M. Neumann's theorem.

THEOREM 1.4. *Let G be a finite group admitting a group of coprime automorphisms A. Assume that $G = [G, A]$ and for any prime p dividing the order of G there is an A-invariant Sylow p-subgroups P such that*

$$\Pr([P, A], [P, A]^x) \geq \epsilon$$

for all $x \in G$. Then G is bounded-by-abelian-by-bounded.

Unsurprisingly, the proofs of all main results in this paper depend on the classification of finite simple groups.

2. Coprime action

We say that a group A acts coprimely on a group G if the automorphisms of G induced by the elements of A are coprime. We denote by $C_G(\alpha)$ the fixed-point subgroup $\{x \in G \mid x^\alpha = x\}$ of an automorphism α and by $I_G(\alpha)$ the set of all elements of the form $g^{-1}g^\alpha$, where $g \in G$. Thus, $[G, \alpha]$ is generated by $I_G(\alpha)$. Observe that $|I_G(\alpha)| = |G : C_G(\alpha)|$.

In what follows we use the following well-known facts (see for example [**7**]), often without mention.

LEMMA 2.1. *Let a group A act coprimely on a finite group G. The following holds:*

(i) *$G = [G, A]C_G(A)$ and $[G, A] = [G, A, A]$;*
(ii) *if G is abelian, then $G = [G, A] \times C_G(A)$;*
(iii) *if N is any A-invariant normal subgroup of G, we have $C_{G/N}(A) = C_G(A)N/N$;*
(iv) *the group G possesses an A-invariant Sylow p-subgroup for each prime $p \in \pi(G)$, any two A-invariant Sylow p-subgroups are conjugate by an element of $C_G(A)$, and any A-invariant p-subgroup of G is contained in an A-invariant Sylow p-subgroup;*
(v) *if N is any normal subgroup of G such that $[N, A] = 1$, then $[G, A]$ centralizes N.*

Throughout, by a simple group we mean a finite nonabelian simple group. We will often use without special references the well-known corollary of the classification that if a simple group G admits a group of coprime automorphisms A of order $e \neq 1$, then $G = L(q)$ is a group of Lie type, $A = \langle \alpha \rangle$ is cyclic, and α is a field automorphism. Furthermore, $C_G(\alpha) = L(q_0)$ is a group of the same Lie type defined over a smaller field such that $q = q_0^e$ (see [**16**]).

Lemma 2.2. *Let C be a positive integer and G a finite simple group of Lie type in characteristic p admitting a nontrivial coprime automorphism α. Suppose that order of $[P, \alpha]$ is at most C whenever P is an α-invariant Sylow p-subgroup of G. Then the order of G is C-bounded.*

Proof. The proof follows the same lines as the proof of [**2**, Lemma 2.4]. We will repeat it here for the reader's convenience. Let $G = L(q)$ and let $C_G(\alpha) = L(q_0)$, where $q = q_0^e$ and $e = |\alpha|$. Note that $e \geq 3$, because G is a finite simple group and α is coprime. We have that $|P| = q_0^{|\alpha|t}$ for some integer t and $|C_P(\alpha)| = q_0^t$, therefore

$$|[P, \alpha]| \geq q_0^{t(|\alpha|-1)} > |P|^{1/2}.$$

Comparing the orders of G and P (see e.g. [**15**, Table I]), we see that $|G| \leq |P|^3$ and the order of G is C-bounded. □

We will need the following easy remark (see e.g. [**2**]).

Lemma 2.3. *Assume that G is a finite group admitting a coprime automorphism α such that $G = [G, \alpha]$. If $|I_G(\alpha)| \leq m$, then the order of G is m-bounded.*

In the sequel, we denote by $\pi(G)$ the set of prime divisors of the order of a group G and, somewhat abusing the terminology, we assume that a Sylow p-subgroup is trivial if $p \notin \pi(G)$. If π is a set of primes, then $O_\pi(G)$ denotes the largest normal π-subgroup of G and $O_{\pi'}(G)$ denotes the largest normal subgroup of G whose order is not divisible by primes in π. Moreover, $F(G)$ denotes the Fitting subgroup of G, which is the largest normal nilpotent subgroup of G. Recall that, if G is a finite soluble group, then $F(G)$ contains its centralizer.

Lemma 2.4. *Let G be a finite group admitting a coprime automorphism α such that $G = [G, \alpha]$. If G is soluble, then $p \in \pi(G)$ if and only if $I_P(\alpha) \neq 1$ for some α-invariant Sylow p-subgroup P of G.*

Proof. Let G be a counterexample of minimal order. Then G possesses a nontrivial α-invariant Sylow p-subgroup P such that $I_P(\alpha) = 1$. Because of minimality, $O_{p'}(G) = 1$ and $C_G(O_p(G)) \leq O_p(G)$. As $P \leq C_G(\alpha)$, it follows from Lemma 2.1 that $O_p(G) \leq Z(G)$. So $G \leq C_G(O_p(G)) \leq O_p(G)$ and therefore $G = P$. Since $G = [G, \alpha]$, we get that $G = 1$, a contradiction. □

We remark that the above lemma fails for non-soluble groups. Indeed, let q be an odd prime power such that the group $G = \mathrm{PSL}_2(q)$ admits a coprime automorphism $\alpha \neq 1$. Then $C_G(\alpha) = \mathrm{PSL}_2(q_0)$, where $q = q_0^e$ and $e = |\alpha|$. Note that e is odd, because α is coprime, therefore $|G|/|C_G(\alpha)| = (q_0^{2e}-1)/(q_0-1)$ is odd as well. It follows that any α-invariant Sylow 2-subgroup of G is contained in $C_G(\alpha)$. On the other hand, it is clear that $G = [G, \alpha]$.

We will need Theorem B of [**20**]. We record it in the next lemma for the reader's convenience.

If A is a finite group and k is a field of characteristic not dividing $|A|$, then any kA-module V is a direct sum of simple components. Let S be a given simple kA-module and let $m_{A,S}(V)$ denote the number of simple components of V isomorphic to S.

Lemma 2.5. *Let A be an arbitrary finite group. Then there exists a number $\gamma = \gamma_A > 0$, depending only on A, with the following property: Let A act coprimely on a finite soluble group G, and let k be any field with characteristic not dividing $|A|$. Let V be any simple kAG-module and let S be any kA-module which appears as a component of the restriction V_A. Then $m_{A,S}(V) \geq \gamma \dim V$.*

Lemma 2.6. *Let G be a finite soluble group admitting a coprime automorphism α such that $G = [G, \alpha]$. Let M be a minimal α-invariant normal subgroup of G and assume that $|I_M(\alpha)| \leq n$. Then either M is central or the order of M is $(|\alpha|, n)$-bounded.*

Proof. If $I_M(\alpha) = 1$, then $M \leq C_G(\alpha)$ and so M is contained in the centre of G by Lemma 2.1.

Assume that $|I_M(\alpha)| \geq 2$. Because of the minimality of M it follows that M is an elementary abelian p-group for some prime p. Let $|M| = p^t$. We regard M as an irreducible $G\langle\alpha\rangle$-module. As $|I_M(\alpha)| \geq 2$, there is a nontrivial simple $\langle\alpha\rangle$-submodule S of M. Let γ be the constant as in Lemma 2.5, which only depends on $|\alpha|$. Note that $M = [M, \alpha] \times C_M(\alpha)$, by Lemma 2.1, and all $\langle\alpha\rangle$-submodules of M isomorphic to S are contained in $[M, \alpha]$. If S appears in M with multiplicity m_S then $\dim[M, \alpha] \geq m_S \geq \gamma \dim M = \gamma t$. Note that $|[M, \alpha]| = |M : C_M(\alpha)| = |I_M(\alpha)| \leq n$. Therefore $n \geq |[M, \alpha]| \geq p^{\gamma t}$, which implies that $|M| = p^t \leq n^{\frac{1}{\gamma}}$ is $(|\alpha|, n)$-bounded. $\square$

A group is said metanilpotent if it possesses a normal nilpotent subgroup N such that G/N is nilpotent. In the sequel $\gamma_\infty(G)$ stands for the intersection of the terms of the lower central series of a group G.

Lemma 2.7. [**4**, Lemma 2.4] *If G is a finite metanilpotent group, then $\gamma_\infty(G) = \prod_p [K_p, H_{p'}]$, where K_p is a Sylow p-subgroup of $\gamma_\infty(G)$ and $H_{p'}$ is a Hall p'-subgroup of G.*

Lemma 2.8. *Assume that G is a finite soluble group admitting a coprime automorphism α such that $G = [G, \alpha]$. Let $D = \gamma_\infty(G)$ and suppose that $|I_D(\alpha)| \leq n$. Then the order of D is $(|\alpha|, n)$-bounded.*

Proof. We assume that $D \neq 1$. As $D = [D, G]$, it follows that D is not central in G and therefore $|I_D(\alpha)| \geq 2$ by Lemma 2.1.

We will argue by induction on n. Let N be the maximal normal subgroup of G contained in $C_G(\alpha)$. Note that $N \leq Z(G)$ by Lemma 2.1. We pass to the quotient $\bar{G} = G/N$ and let $\bar{D} = DN/N$. Let $\bar{M}$ be a minimal α-invariant normal subgroup of $\bar{G}$ contained in $\bar{D}$. Observe that $[\bar{M}, \alpha] \neq 1$, whence $I_{\bar{D}/\bar{M}}(\alpha) < I_D(\alpha)$ and, by induction, the index of $\bar{M}$ in $\bar{D}$ is $(|\alpha|, n)$-bounded. In view of Lemma 2.6 the order of $\bar{M}$ is also $(|\alpha|, n)$-bounded. Hence $\bar{D}$ has $(|\alpha|, n)$-bounded order.

Going back to G we get that the centre of D has $(|\alpha|, n)$-bounded index in D. By Schur's theorem [**27**, 4.12] the derived subgroup D' has $(|\alpha|, n)$-bounded order. Passing to the quotient group G/D', we may assume that D is abelian and hence G is metanilpotent. If P is a Sylow p-subgroup of D, by Lemma 2.7 we have $P = [P, H]$, where H is a Hall p'-subgroup of G. As P is abelian, $P = [P, H] \times C_P(H)$, whence $C_P(H) = 1$ and so $P \cap Z(G) = 1$. This holds for every prime divisor p of the order of D. Therefore $D \cap Z(G) = 1$. As $N \leq Z(G)$ and the order of D is $(|\alpha|, n)$-bounded modulo N, the lemma follows. □

If G is a finite soluble group, the Fitting height $h(G)$ of G is the length of a shortest normal series all of whose quotients are nilpotent. In their seminal paper [**21**] Hall and Higman showed that a finite group of exponent e possesses a normal series of e-bounded length all of whose quotients are either nilpotent or isomorphic to a direct product of non-abelian simple groups. Therefore a finite soluble group of exponent e has e-bounded Fitting height.

Remark 2.9. *Let N be a normal subgroup of G and let $K/N = F(G/N)$. If $|N| \leq m$, then the index of $F(G)$ in K is m-bounded.*

Indeed, $C = C_K(N)$ is nilpotent and the index of C in K is at most $(m-1)!$. It is clear that $C \leq F(G)$ so the claim follows.

Lemma 2.10. *Let G be a finite soluble group admitting a coprime automorphism α such that $G = [G, \alpha]$. Assume that $|I_{F(G)}(\alpha)| \leq n$. Then $|G|$ is n-bounded.*

Proof. Set $F = F(G)$. If $n = 1$, then $F \leq C_G(\alpha)$ and so $F \leq Z(G)$, whence $G = F \leq C_G(\alpha)$. Since $G = [G, \alpha]$, it follows that $G = 1$ and the lemma holds. Therefore we assume that $n \geq 2$.

As $|I_F(\alpha)| \leq n$ and $\langle \alpha \rangle$ acts on $I_F(\alpha)$ by permuting its elements, the kernel of this action has index at most $n!$. Therefore there exists a positive integer $j \leq n!$, such that α^j centralizes $I_F(\alpha)$. Then α^j centralizes the whole subgroup $[F, \alpha] = \langle I_F(\alpha) \rangle$. As $F = [F, \alpha] C_F(\alpha)$, it follows that α^j centralizes F. Therefore, by Lemma 2.1, $[G, \alpha^j]$ centralizes F. Since G is soluble, we deduce that $[G, \alpha^j] \leq F$. Thus,

$$[G, \alpha^j] = [G, \alpha^j, \alpha^j] \leq [F, \alpha^j] = 1.$$

It follows that the automorphism α has order dividing j, which is n-bounded.

Lemma 2.6 implies that there is a normal series

$$1 = M_1 \leq \cdots \leq M_s = F,$$

all of whose factors are either central in G or of $(|\alpha|, n)$-bounded order. Actually, the non-central factors have n-bounded order because the order of α is n-bounded. As $G/C_G(M_{i+1}/M_i)$ acts on each factor M_{i+1}/M_i by automorphisms, there is an n-bounded number e such that G^e centralizes all factors M_{i+1}/M_i. It follows from Kaluzhnin's theorem [**23**, Theorem 16.3.1] that $G^e/C_{G^e}(F)$ is nilpotent. Therefore G^e is metanilpotent, as $C_{G^e}(F) \leq F$. By the Hall-Higman theory [**21**] G has n-bounded Fitting height $h = h(G)$.

We now argue by induction on h. If $h = 1$ then $G = F$ and the result follows from Lemma 2.3. So assume $h > 1$. Set $D = \gamma_\infty(G)$ and observe that $h(D) = h - 1$. Moreover $F([D, \alpha])$ is subnormal in G, hence contained in F. Thus, by induction, the order of $[D, \alpha]$ is n-bounded. We deduce from Lemma 2.8 that the order of D is n-bounded.

As G/D is nilpotent, it follows from Remark 2.9 that the index of F in G is n-bounded. We have

$$|I_G(\alpha)| = |G : C_G(\alpha)| \leq |G : F|\,|F : C_F(\alpha)| = |G : F|\,|I_F(\alpha)|.$$

So $|I_G(\alpha)|$ is n-bounded and Lemma 2.3 yields the desired result. □

3. Commuting Probability

If X, Y are subsets of a finite group G, we have

$$\Pr(X, Y) = \frac{|\{(x, y) \in X \times Y \mid xy = yx\}|}{|X|\,|Y|}.$$

Note that $\Pr(X,Y) = \Pr(Y,X)$ and

$$\Pr(X,Y) = \frac{1}{|Y|}\sum_{y\in Y}\frac{|C_X(y)|}{|X|} = \frac{1}{|X|}\sum_{x\in X}\frac{|C_Y(x)|}{|Y|},$$

where, as usual, $C_Y(x)$ denotes the set of all elements of Y commuting with x.

The next lemma is essentially Lemma 2.2 of [**12**] and it is useful when considering quotients, subgroups, or direct products of groups. In the sequel it is often used without explicit mention.

Lemma 3.1. *Let G be a finite group and let H, K be subgroups of G. Then*

1. *If N is a normal subgroup of G, then* $\Pr(HN/N, KN/N) \geq \Pr(H,K)$.
2. *If $H_0 \leq H$, then* $\Pr(H_0,K) \geq \Pr(H,K) \geq \frac{1}{|H:H_0|}\Pr(H_0,K)$.
3. *If $G = G_1 \times G_2$, $H_i \leq G_i$ and $K_i \leq G_i$, then*

$$\Pr(H_1 \times H_2, K_1 \times K_2) = \Pr(H_1,K_1)\Pr(H_2,K_2).$$

The next lemma is Lemma 2.4 of [**11**].

Lemma 3.2. *Let P be a p-subgroup and Q be a q-subgroup of a finite group G. If $[P,Q] \neq 1$, then* $\Pr(P,Q) \leq 3/4$.

We will also need some technical results, which are analogous to some results in [**12, 13**] for A-invariant subgroups, where A is a group of automorphisms of G.

Lemma 3.3. *Let $\epsilon > 0$. There exists an ϵ-bounded integer m with the following property: If G is a finite group with a group A acting on G by automorphisms and H, K are A-invariant subgroups of G with $\Pr(H,K) \geq \epsilon > 0$, then there exists an A-invariant normal subgroup H_0 of H such that:*

1. *$|H : H_0| \leq m$;*
2. *$|K : C_K(x)| \leq m$ for every $x \in H_0$.*

Proof. Note that the set

$$X = \{x \in H \mid |x^K| \leq 2/\epsilon\}$$

is A-invariant. Following line by line the proof of Lemma 2.8 of [**12**], we find an A-invariant subgroup $T \leq H$ such that the indices $|H : T|$ and $|K : C_K(x)|$ are ϵ-bounded for every $x \in T$.

Let U be the maximal normal subgroup of H contained in T. Clearly, the index $|H : U|$ is ϵ-bounded. Since each U^a is normal in H, for every $a \in A$, there exist ϵ-boundedly many elements $a_i \in A$

such that $U^A = \prod_i U^{a_i}$. Set $H_0 = U^A$ and notice that $|K : C_K(x^{a_i})|$ is ϵ-bounded for every $x \in U$ and for every a_i. As the number of a_i is ϵ-bounded, we deduce that $|K : C_K(x)|$ is ϵ-bounded for every $x \in H_0$. The result follows. $\square$

PROPOSITION 3.4. *Let a group A act on a finite group G, and let H be an A-invariant subgroup of G such that $\Pr(H,G) \geq \epsilon > 0$. Then there is an A-invariant normal subgroup $U \leq G$ and an A-invariant normal subgroup B of H such that the indices $[G : U]$, $[H : B]$, and the order of the commutator subgroup $[B,U]^G$ are ϵ-bounded.*

PROOF. The proof uses the same arguments as those in the proof of Proposition 1.2 of [**13**]. We outline it here for the reader's convenience, pointing out which relevant subgroups are A-invariant, and refer to the paper [**13**] for further details.

By Lemma 3.3 there exists an A-invariant normal subgroup B of H of ϵ-bounded index in H such that $|G : C_G(x)|$ is ϵ-bounded for every $x \in B$. Set $L = \langle B^G \rangle$ and note that L is A-invariant. It follows from [**5**, Theorem 1.1] that the commutator subgroup $[L,L]$ has ϵ-bounded order. By Lemma 3.1 (i) we can replace G with the quotient group $G/[L,L]$ and still assume that $\Pr(H,G) \geq \epsilon$ and the indices $|H : B|$ and $|G : C_G(x)|$ are ϵ-bounded, for every $x \in B$. Therefore now L is abelian.

Again by Lemma 3.3, as $\Pr(G,H) = \Pr(H,G) \geq \epsilon$, there exists an A-invariant normal subgroup U of G such that $|G : U| \leq m$ and $|H : C_H(x)| \leq m$ for every $x \in E$, where the integer m is ϵ-bounded. Now $|b^G|$ is ϵ-bounded, for every $b \in B$ and $|y^B| \leq m$ for every $y \in U$. As L is abelian, it follows from Lemma 2.2 of [**13**] that $[B,U]$ has ϵ-bounded order.

Moreover $[B,U]$ is normalized by U, therefore it has ϵ-boundedly many conjugates in G, all of them normalizing each other. Hence, $[B,U]^G$ has ϵ-bounded order. This concludes the proof. $\square$

4. The soluble case

In the sequel, we will work with groups satisfying the following hypothesis.

HYPOTHESIS 4.1. *Let $\epsilon > 0$ and let G be a finite group admitting a coprime automorphism α such that for any distinct primes $p, q \in \pi(G)$ there exists a Sylow p-subgroup P and a Sylow q-subgroup Q in G, both α-invariant, for which $\Pr([P,\alpha],[Q,\alpha]) \geq \epsilon$.*

We note that under Hypothesis 4.1 the index $|[G,\alpha] : F([G,\alpha])|$ can be arbitrarily large. Indeed, let C be the cyclic group of order

3 and let α be the involutory automorphism of C. Let $p_1, p_2, \dots, p_s$ be distinct primes greater than 3. For $p \in \{p_1, p_2, \dots, p_s\}$ let C_p be the cyclic group of order p and B_p the base of the wreath product $C_p \wr C\langle\alpha\rangle$ of C_p by $C\langle\alpha\rangle$. Let $H_p = [B_pC, \alpha]$. Observe that $C < H_p$ and α induces an involutory automorphism of H_p such that $H_p = [H_p, \alpha]$. Moreover, $|H_p : F(H_p)| = 3$. Now let G be the direct product of H_{p_i} for $i = 1, \dots, s$. In a natural way α induces an involutory automorphism of G such that $G = [G, \alpha]$. For any primes $p, q \in \pi(G)$ other than 3 the Sylow p-subgroup and Sylow q-subgroup of G commute. If P is a Sylow p-subgroup for $p \geq 5$ and S an α-invariant Sylow 3-subgroup, then $\Pr(P, S) > 1/3$. Note that $|G : F(G)| = 3^s$, which can be arbitrarily large.

Observe that if a group G satisfies Hypothesis 4.1 and H is an α-invariant normal subgroup of G, then H satisfies Hypothesis 4.1 as well.

Remark 4.2. *If G satisfies Hypothesis 4.1 and P is an α-invariant Sylow p-subgroup of G, then for every $q \neq p$ there exists an α-invariant Sylow q-subgroup Q of G for which* $\Pr([P, \alpha], [Q, \alpha]) \geq \epsilon$.

Indeed, any two α-invariant Sylow p-subgroups of G are conjugate by an element of $C_G(\alpha)$. So, if P^x and Q are respectively an α-invariant Sylow p-subgroup and an α-invariant Sylow q-subgroup of G such that $\Pr([P^x, \alpha], [Q, \alpha]) \geq \epsilon$ with $x \in C_G(\alpha)$, then

$$\Pr([P, \alpha], [Q^{x^{-1}}, \alpha]) = \Pr([P^x, \alpha], [Q, \alpha]) \geq \epsilon.$$

Lemma 4.3. *Assume Hypothesis 4.1 with $G = PQ$, where P is a normal Sylow p-subgroup and Q an α-invariant Sylow q-subgroup such that $Q = [Q, \alpha]$. Then there exists an ϵ-bounded integer m such that $|G : F(G)| \leq m$.*

Proof. As $F(G/P'O_q(G)) = F(G)/P'O_q(G)$, we may assume that $F(G) = P$ and P is abelian.

Observe that $|Q : C_Q(y)| = |G : C_G(y)|$ for every $y \in [P, \alpha]$. So, taking into account that $Q = [Q, \alpha]$ and that $\Pr([P, \alpha], Q) \geq \epsilon$ by Remark 4.2, we have

$$\begin{aligned} \Pr([P, \alpha], G) &= \frac{1}{|[P, \alpha]|} \sum_{y \in [P, \alpha]} \frac{|C_G(y)|}{|G|} \\ &= \frac{1}{|[P, \alpha]|} \sum_{y \in [P, \alpha]} \frac{|C_Q(y)|}{|Q|} = \Pr([P, \alpha], Q) \geq \epsilon. \end{aligned}$$

By Proposition 3.4 there is an A-invariant normal subgroup $U \leq G$ and an A-invariant subgroup $P_0 \leq [P, \alpha]$ such that the indices $|G : U|$ and $[[P, \alpha] : P_0]$, and the order of $[P_0, U]^G$ are ϵ-bounded.

Set $N = [P_0, U]^G$ and $K/N = F(G/N)$.

By Remark 2.9, the index of $F(G)$ in K is ϵ-bounded and, in order to bound the index of $F(G)$ in G, it is enough to bound the index of K in G. So we now assume that $N = [P_0, U]^G = 1$.

As $G = PQ$ and P is abelian, U acts coprimely on $[P, U]$. In particular, $C_{[P,U]}(U) = 1$ and so $[P, U] \cap P_0 = 1$. Since P_0 has ϵ-bounded index in $[P, \alpha]$, it follows that $[[P, U], \alpha]$ has ϵ-bounded order. In particular $|I_{[P,U]}(\alpha)|$ is ϵ-bounded.

Let $H = [P, U]Q$ and let $V = C_Q([P, U])$; then $V = O_q(H)$ and $F(H) = [P, U]V$. Let $\bar{Q} = Q/V$, $\bar{H} = H/V$ and note that $\bar{H}$ is isomorphic to the semidirect product $[P, U]\bar{Q}$. As $C_{\bar{Q}}([P, U]) = 1$, it follows that $F(\bar{H}) = [P, U]$.

Since $[P, U] = [P, U, U] \leq [P, U, Q] = [P, U, \bar{Q}]$, we have that $\bar{H} = \bar{Q}^{\bar{H}}$. Observe that $\bar{Q} = [\bar{Q}, \alpha] \leq [\bar{H}, \alpha]$. The latter subgroup is normal in $\bar{H}$ so it follows that $\bar{H} = [\bar{H}, \alpha]$. Moreover, $I_{F(\bar{H})}(\alpha) = I_{[P,U]}(\alpha)$ and so $|I_{F(\bar{H})}(\alpha)|$ is ϵ-bounded. We are in a position to apply Lemma 2.10 and deduce that the order of $\bar{H}$ is ϵ-bounded. So the order of $\bar{Q}$ is ϵ-bounded as well. Therefore V has ϵ-bounded index in Q and so PV has ϵ-bounded index in G. Moreover, PV is a normal subgroup of G.

We claim that $R = PU \cap PV$ is nilpotent. Indeed, as V acts coprimely on P,

$$[P, PU \cap PV] = [P, PU \cap PV, PU \cap PV] \leq [[P, PU], V] = 1.$$

Therefore $F(G)$ contains R and thus has ϵ-bounded index in G, as required. □

In what follows we need the next observation.

Remark 4.4. *Let G be a finite metanilpotent group and Q a Sylow q-subgroup of G. If $M = O_{q'}(F(G))$, then $C_Q(M) \leq F(G)$.*

Lemma 4.5. *Under Hypothesis 4.1 assume that G is soluble and let m be as in Lemma 4.3. If $q > m$ is a prime and Q is an α-invariant Sylow q-subgroup, then $[Q, \alpha] \leq F(G)$.*

Proof. Assume that the lemma is false and let G be a counterexample of minimal order. Let $q > m$ be a prime and let Q be an α-invariant Sylow q-subgroup of G such that $[Q, \alpha] \not\leq F(G)$. By considering the quotient group $G/F(G)$ and taking into account minimality, we get that $[Q, \alpha] \leq F_2(G)$. So, again by minimality, $G = F_2(G)$, that is, G is metanilpotent. By Remark 4.2, for every prime $p \neq q$ that divides $|F(G)|$, there exists an α-invariant Sylow p-subgroup P of G such that $\Pr([P,\alpha],[Q,\alpha]) \geq \epsilon$. The Sylow p-subgroup P_1 of $F(G)$ is contained in P and so $\Pr([P_1,\alpha],[Q,\alpha]) \geq \epsilon$. Now we apply Lemma 4.3 to the subgroup $H = P_1[Q,\alpha]$ and deduce that $[Q,\alpha] \leq F(H)$, as $q > m$. Since $P_1 \leq F(H)$, it follows that $[Q,\alpha]$ commutes with P_1. As this happens for every prime $p \neq q$, we conclude that $[Q,\alpha] \leq F(G)$ (see Remark 4.4). □

We can now prove Theorem 1.2 in the particular case where G is soluble and A is cyclic.

LEMMA 4.6. *Under Hypothesis 4.1 assume that G is soluble and $G = [G,\alpha]$. Then the index $|G : F_2(G)|$ is ϵ-bounded.*

PROOF. Since G is soluble and $G = [G,\alpha]$, it follows from Lemma 2.4 that if p divides the order of G, then there exists an α-invariant Sylow p-subgroup P of G such that $[P,\alpha] \neq 1$.

Let Q be an α-invariant Sylow q-subgroup of G for a prime $q > m$, where m is as in Lemma 4.3. By Lemma 4.5, $[Q,\alpha] \leq F(G)$. Therefore, by Lemma 2.4, q does not divide the order of $G/F(G)$.

Passing to the quotient over $F(G)$, we are reduced to the case where the prime divisors of $|G|$ do not exceed m. Therefore $\pi(G)$ contains only ϵ-boundedly many primes. We will show that under this assumption $F(G)$ has ϵ-bounded index in G and this will imply the desired result.

Let $N = F_2(G)$ and note that N satisfies Hypothesis 4.1.

Let Q be an α-invariant Sylow q-subgroup of N and set

$$M = O_{q'}(F(G)) = P_1 \times \cdots \times P_r,$$

where P_i is a Sylow p_i-subgroup of $F(G)$. We know that r is ϵ-bounded. Note that each P_i is contained in every α-invariant Sylow p_i-subgroup of G, so by virtue of Lemma 3.1 and Remark 4.2, the group $P_i[Q,\alpha]$ satisfies the hypotheses of Lemma 4.3. We deduce that the index of the centralizer C_i of P_i in $[Q,\alpha]$ is ϵ-bounded. Thus the intersection $C = \cap C_i$ of all centralizers has ϵ-bounded index in $[Q,\alpha]$ and it centralizes M. As N is metanilpotent, $C \leq F(N) = F(G)$ (see Remark 4.4). Therefore the order of $[Q,\alpha]F(G)/F(G)$ is ϵ-bounded.

This holds for every prime q that divides the order of $N/F(G)$, and there are ϵ-boundedly many such primes. As $N/F(G)$ is nilpotent, $[N,\alpha]$ is the direct product of the subgroups $[Q,\alpha]F(G)/F(G)$, hence $[N,\alpha]$ has ϵ-bounded order modulo $F(G)$. This means that $[F(G/F(G)),\alpha]$ has ϵ-bounded order. Now Lemma 2.10 tells us that $G/F(G)$ has ϵ-bounded order. This completes the proof. □

5. The case when G is a simple group

This section is devoted to the proof of Theorem 1.2 when G is a simple group and $A = \langle\alpha\rangle \neq 1$ is cyclic. Recall that in this case G is a group of Lie type defined over the field $\mathbb{F}_q$ and α is a field automorphism. Furthermore, $C_G(\alpha)$ is a group of the same Lie type defined over a smaller field $\mathbb{F}_{q_0}$ such that $q = q_0^e$, where $e = |\alpha|$.

We will use notation and terminology introduced in [**9**], which we briefly recall hereafter. Let $L(q)$ be a finite simple Chevalley group, with set of roots Φ and set of fundamental roots $\Pi = \{r_1, \dots, r_\ell\}$. For every root $r \in \Phi$ we denote by

$$X_r = \{\, x_r(t) \mid t \in \mathbb{F}_q \,\}$$

the corresponding root subgroup of $L(q)$.

Any automorphism φ of the field $\mathbb{F}_q$ induces a field automorphism (also denoted by φ) of $L(q)$ defined by

$$\big(x_r(t)\big)^\varphi \;=\; x_r\big(t^\varphi\big).$$

We fix an ordering $\{r_1 < r_2 < \cdots < r_\ell < \dots\}$ of the set Φ^+ of positive roots. Then the subgroup

$$U \;=\; \prod_{r\in\Phi^+} X_r$$

is an α-invariant Sylow p-subgroup of $L(q)$ and every element $x \in U$ can be written in a unique way as a product

$$x \;=\; \prod_{r\in\Phi^+} x_r(t_r),$$

with $t_r \in \mathbb{F}_q$.

We will make use of the following observation.

Remark 5.1. *Let $x, y \in U$ and write*

$$x \;=\; x_{r_1}(t_1)\cdots x_{r_\ell}(t_\ell)\, z, \quad y \;=\; x_{r_1}(u_1)\cdots x_{r_\ell}(u_\ell)\, w,$$

where $z, w \in \prod_{r\in(\Phi^+\setminus\Pi)} X_r$ and $t_i, u_i \in \mathbb{F}_q$.

Then, using the Chevalley commutator formulas [**9**, Theorem 5.2.2] *we can write*

$$xy \;=\; x_{r_1}(t_1+u_1)\cdots x_{r_\ell}(t_\ell+u_\ell)\, u, \qquad \text{with } u \in \prod_{r\in(\Phi^+\setminus\Pi)} X_r.$$

We will now recall some basic facts about twisted groups of Lie type.

Let $L(q^s)$ be a group of Lie type whose Dynkin diagram has a non-trivial symmetry ρ of order s. If τ denotes the corresponding graph automorphism, suppose that $L(q^s)$ admits a nontrivial field automorphism φ such that the automorphism

$$\sigma = \varphi\tau$$

satisfies $\sigma^s = 1$. Then, the twisted group

$${}^sL(q)$$

is defined as the subgroup of $L(q^s)$ consisting of the elements fixed element-wise by σ.

The structure of ${}^sL(q)$ is very similar to that of a Chevalley group. If Φ is the root system of $L(q^s)$, the automorphism σ determines a partition

$$\Phi = \bigcup_i S_i.$$

If S is one of the equivalence classes in this partition, we define

$$X_S = \langle\, X_r \mid r \in S \,\rangle \ \subseteq L(q^s),$$

and

$$X_S^1 = \{\, x \in X_S \ \mid \ x^\sigma = x \,\} \ \subseteq {}^sL(q).$$

The group ${}^sL(q)$ is generated by the subgroups X_S^1. In fact, the subgroups X_S^1 play the role of the root subgroups. In particular, the subgroup

$$U^1 \ = \prod_{S_i \subseteq \Phi^+} X_{S_i}^1$$

is an α-invariant Sylow p-subgroup of ${}^sL(q)$.

Lemma 5.2. *Let G be a group of Lie type in characteristic p admitting a nontrivial coprime automorphism α. Let P be an α-invariant Sylow p-subgroup of G. Then $[P,\alpha]$ contains a regular unipotent element x with $C_G(x) \leq P$.*

Proof. First assume that $G = L(q)$ is an untwisted finite simple group of Lie type defined over the field $\mathbb{F}_q$. We may assume that the subgroup P is the subgroup $U = \prod_{r\in\Phi^+} X_r$, defined above. Let $t \in \mathbb{F}_q$ be an element which is not fixed by α, that is, $t^\alpha - t \neq 0$. Consider the element

$$x = \prod_{r\in\Pi} x_r(t^\alpha - t).$$

Then x is a regular unipotent element by Proposition 5.1.3 of [**10**]. Note that $x \in [P,\alpha]$ because $x_r(t^\alpha - t) = x_r(t)^{-1}x_r(t)^\alpha \in [P,\alpha]$ for each r.

Now assume that $G = {}^sL(q)$ is of twisted type. We may assume that the subgroup P is the subgroup $U^1 = \prod_{S_i\subseteq\Phi^+} X_{S_i}^1$, defined above.

Fix an equivalence class $S \subseteq \Phi^+$. By Proposition 13.6.3 of [**9**], we have that $S = \{a_1, \dots, a_j, a_{j+1}, \dots, a_k\}$, with $j \leq 3$, $\{a_1, \dots, a_j\} \subseteq \Pi$ and $\{a_{j+1}, \dots, a_k\} \in (\Phi^+ \setminus \Pi)$. Moreover there exist automorphisms φ_i of $\mathbb{F}_{q^s}$ such that, for every $t \in \mathbb{F}_{q^s}$, in X_S^1 there exists an element of the form

$$x_S(t) = x_{a_1}(t) \cdots x_{a_j}(t^{\varphi_j})\, z, \qquad \text{with } z \in \prod_{r \in (\Phi^+ \setminus \Pi)} X_r.$$

Let $t \in \mathbb{F}_q$ be an element which is not fixed by α, that is, $t^\alpha - t \neq 0$. For each equivalence class $S_i \subseteq \Phi^+$ consider the element

$$x_i(t) = x_{a_1}(t) \cdots x_{a_j}(t^{\varphi_j})\, z_i \in X_{S_i}^1, \qquad \text{with } z_i \in \prod_{r \in (\Phi^+ \setminus \Pi)} X_r$$

as above. By Remark 5.1 we can write

$$[x_i(t), \alpha] = x_{a_1}(t^\alpha - t) \cdots x_{a_j}(t^{\varphi_j \alpha} - t^{\varphi_j})\, u_i$$

where $t^{\varphi_i \alpha} - t^{\varphi_i} \neq 0$ and $u_i \in \prod_{r \in (\Phi^+ \setminus \Pi)} X_r$. Let

$$x = \prod_{S_i \subseteq \Phi^+} [x_i(t), \alpha].$$

Again by Remark 5.1, we can write x in a unique way as

$$x = \prod_{r \in \Pi} x_r(t_r^\alpha - t_r)\, u,$$

with $t_r^\alpha - t_r \neq 0$ and $u \in \prod_{r \in (\Phi^+ \setminus \Pi)} X_r$. Then x is a regular unipotent element by Proposition 5.1.3 of [**10**]. Note that $x \in [P, \alpha]$ because $[x_i(t), \alpha] \in [P, \alpha]$ for each i.

The fact that, in both the twisted and untwisted cases, $C_G(x) \leq P$ follows from Corollary 4.6 of [**22**]. $\square$

The following corollary is a straightforward consequence of the previous lemma.

Corollary 5.3. *Let G be a group of Lie type in characteristic p admitting a nontrivial coprime automorphism α. Let P be an α-invariant Sylow p-subgroup of G. Then $C_G([P, \alpha]) \leq P$.*

If n is a natural number and p is a prime, the p-part n_p (respectively, the p'-part $n_{p'}$) of n is the largest p-power dividing n (respectively, the largest divisor of n coprime to p).

Lemma 5.4. *Under Hypothesis 4.1 with $\alpha \neq 1$, assume that G is simple. Then G is a group of Lie type and the characteristic p of G is ϵ-bounded.*

Proof. Recall that $G = L(q)$ is a finite simple group of Lie type and $q = q_0^e = p^{te}$, where $e = |\alpha| \geq 3$, because the order of a coprime automorphism of a simple group must be odd.

Let m be as in Lemma 3.3 and assume that $p > m$. Let U be an α-invariant Sylow p-subgroup of G. Let $r \neq p$ be a prime dividing $|G : C_G(\alpha)|$, and let R be an α-invariant Sylow r-subgroup of G such that $\Pr([U,\alpha],[R,\alpha]) \geq \epsilon$. Since $p > m$, $[U,\alpha]$ centralizes a normal subgroup R_0 of index at most m in R. It follows from Corollary 5.3 that $R_0 \leq U$, whence $R_0 = 1$ and $|[R,\alpha]| \leq m$. Since $R = [R,\alpha]C_R(\alpha)$, it follows that for every prime divisor $r \neq p$ of $|G : C_G(\alpha)|$ we have that the r-part of $|G : C_G(\alpha)|$ is at most m. This implies that $r \leq m$. Thus the p'-part of $|G : C_G(\alpha)|$ has order at most m^m. Checking the values of $|L(q_0^e) : L(q_0)|$ (see [**15**, TABLE I, p. 8]), we see that the p'-part $|G : C_G(\alpha)|_{p'}$ of $|G : C_G(\alpha)|$ is at least q_0. Therefore $q_0 \leq |G : C_G(\alpha)|_{p'} \leq m^m$, which proves that p is ϵ-bounded. □

Proposition 5.5. *Under Hypothesis 4.1 with $\alpha \neq 1$, assume that G is simple. Then $|G|$ is ϵ-bounded.*

Proof. As G is simple, it is a group of Lie type defined over a field of size q and $C_G(\alpha)$ is a finite simple group of the same Lie type and same rank, defined over a field of size q_0 such that $q = q_0^{|\alpha|} = p^{f|\alpha|}$, for some prime p and some integer f. By Lemma 5.4 we know that p is ϵ-bounded.

According to [**19**] for every finite simple group G in the list of [**19**, Table 2], there is an exponent e such that a Zsigmondy prime r for (q,e), that is a primitive prime divisor of $q^e - 1$, divides the order of G but does not divide the order of any parabolic subgroup of G.

Assume that G is one of the groups in [**19**, Table 2], and let e be the corresponding exponent. We remark that, as $q^e = p^{f|\alpha|e}$, a Zsigmondy prime for $(p, f|\alpha|e)$ is also a Zsigmondy prime for (q,e) and therefore divides the order of G.

Consider a Zsigmondy prime r for $(p, f|\alpha|e)$ and note that $r \geq f|\alpha|e$, as the order of p modulo r is precisely $f|\alpha|e$. As r does not divide $p^t - 1$ for $t < f|\alpha|e$, in particular r does not divide $q_0^s - 1$ for $s < |\alpha|e$, and therefore r does not divide the order of $C_G(\alpha)$ (see e.g. [**15**, Table I]). It follows that, if R is an α-invariant r-subgroup of G, then $[R,\alpha] = R$.

Moreover, since the centralizer of a unipotent element is always contained in a parabolic subgroup (see e.g. [**24**, Theorem 26.5, (Borel-Tits)]) we deduce that there is no r-element that centralizes a p-element of G. Therefore, if P is an α-invariant Sylow p-subgroup of G such that

$\Pr([P,\alpha],R) = \Pr([P,\alpha],[R,\alpha]) \geq \epsilon$, then

$\{(x,y) \in [P,\alpha] \times R \mid [x,y]=1\} = \{(x,1) \mid x \in [P,\alpha]\} \cup \{(1,y) \mid y \in R\}$

and

$$\Pr([P,\alpha],R) = \frac{1}{|R|} + \frac{1}{|[P,\alpha]|} - \frac{1}{|[P,\alpha]||R|} \geq \epsilon.$$

Thus one of $|R|$ and $|[P,\alpha]|$ is ϵ-bounded.

If $|R|$, and hence r, is ϵ-bounded, then $f|\alpha|e$ is ϵ-bounded, as $r \geq f|\alpha|e$. So the rank of G is ϵ-bounded, since e is defined as in [**19**, Table 2]. Moreover, as $q = p^{f|\alpha|}$, also q is ϵ-bounded. Hence, we conclude that the order of G is ϵ-bounded.

If $|[P,\alpha]|$ is ϵ-bounded, then the order of G is ϵ-bounded by Lemma 2.2.

We are left with the cases where G is not in the list of [**19**, Table 2]. As discussed in [**19**, Section 4.1], we have that $G = L_2(p)$ for some Mersenne prime p or the type of G is one of

$$G_2(2), A_5^+(2), A_2^-(2), A_3^-(2), B_3(2), C_3(2), D_4^+(2).$$

In all these cases, $q = q_0 = p$, against the fact that α is a nontrivial field automorphism of G. The proof is now complete. □

6. The case when A is cyclic

This section is devoted to the proof of Theorem 1.2 when A is cyclic.

Lemma 6.1. *Assume that G is group admitting a coprime automorphism α such that $G = [G,\alpha]$ and for any distinct primes $p,q \in \pi(G)$ there exists a Sylow p-subgroup P and a Sylow q-subgroup Q in G, both α-invariant, for which $[\,[P,\alpha],[Q,\alpha]\,] = 1$. Then G is nilpotent.*

Proof. Assume by contradiction that that the statement is false and let G be a counterexample of minimal order. Let N_1, N_2 be two minimal normal minimal α-invariant subgroups of G, then G/N_1 and G/N_2 are both nilpotent. If $N_1 \neq N_2$, then G embeds into the direct product $G/N_1 \times G/N_2$, which is nilpotent, a contradiction. So G has a unique minimal normal α-invariant subgroup N, which is either elementary abelian or a direct product of isomorphic (nonabelian) simple groups.

Suppose that N is an elementary abelian q-group. Let Q be a Sylow q-subgroup of G; as G/N is nilpotent and $[G/N,\alpha] = G/N$, it follows that Q is normal in G. Moreover $N \leq Z(Q)$, by minimality.

Let P be a Sylow p-subgroup of G with $p \neq q$. As $[G/N,\alpha] = G/N$ and G/N is nilpotent, it follows that $[P,\alpha]N = PN$. Therefore $[P,\alpha] = P$ for every Sylow p-subgroup P with $p \neq q$.

Since G/N is nilpotent, N is not contained in $Z(G)$. So, by minimality, $N \cap Z(G) = 1$. By assumptions, for any prime $p \neq q$ there exists an α-invariant Sylow p-subgroup P of G, for which $[P, [N, \alpha]] = 1$. Moreover $[N, \alpha] \leq N \leq Z(Q)$. It follows that $[N, \alpha] \leq Z(G)$. Then $[N, \alpha] \leq N \cap Z(G) = 1$ and, by Lemma 2.1, we conclude that $N \leq Z(G)$, a contradiction.

So N is a direct product of simple groups and $G = N$, by minimality. Again by minimality, G must be a simple group.

Therefore G is a simple group of Lie type, say in characteristic p, and α is a field automorphism. Let P be an α-invariant Sylow p-subgroup of G. Note that $[P, \alpha] \neq 1$. Recall that by Corollary 5.3 the centralizer of $[P, \alpha]$ is contained in P. Therefore for every prime $q \neq p$ there exists an α-invariant Sylow q-subgroup Q of G such that $[Q, \alpha] = 1$ and $Q = C_Q(\alpha) \leq C_G(\alpha)$. Thus the index $|G : C_G(\alpha)|$ is a p-power, a contradiction. □

The following corollary is a straightforward consequence of the above lemma.

COROLLARY 6.2. *Assume that G is a simple group admitting a nontrivial coprime automorphism α. Then there exist two distinct primes $p, q \in \pi(G)$ such that for every α-invariant Sylow p-subgroup P and every α-invariant Sylow q-subgroup Q we have that $[P, \alpha]$ does not commute with $[Q, \alpha]$.*

By a semisimple group we mean the direct product of finite simple groups.

LEMMA 6.3. *Under Hypothesis 4.1, assume that G is semisimple and has no nontrivial proper α-invariant normal subgroups. Then the order of G is ϵ-bounded.*

PROOF. Note that the assumptions imply that $G = [G, \alpha]$. If G is simple, the lemma follows from Proposition 5.5. So we assume that

$$G = S \times S^{\alpha} \cdots \times S^{\alpha^{s-1}},$$

where S is a simple factor and $s \geq 2$.

Note that α^s normalizes S and if P is an α^s-invariant Sylow p-subgroup of S, then $P^{\langle \alpha \rangle}$ is an α-invariant Sylow p-subgroup of G. As any two α-invariant Sylow p-subgroups of G are conjugate by an element of $C_G(\alpha)$, it follows that any α-invariant Sylow p-subgroup of G is of the form $P^{\langle \alpha \rangle}$, where P is an α^s-invariant Sylow p-subgroup of S.

So let $P^{\langle \alpha \rangle}, Q^{\langle \alpha \rangle}$ be as in Hypothesis 4.1. Note that if $x \in S$ and $\bar{\pi} : G \to S$ is the projection on S, then $\bar{\pi}([x, \alpha]) = x^{-1}$. Therefore

$\bar{\pi}([P^{\langle\alpha\rangle}, \alpha]) = P$ and $\bar{\pi}([Q^{\langle\alpha\rangle}, \alpha]) = Q$. It follows from Lemma 3.1 that

$$\Pr(P, Q) \geq \Pr([P^{\langle\alpha\rangle}, \alpha], [Q^{\langle\alpha\rangle}, \alpha]) \geq \epsilon.$$

This proves that for any distinct primes $p, q \in \pi(S)$ there is a Sylow p-subgroup P and a Sylow q-subgroup Q of S such that $\Pr(P, Q) \geq \epsilon$. It follows from Theorem 1.1 of [**12**] that the order of S is ϵ-bounded.

Since $P^{\langle\alpha\rangle} = P \times P^{\alpha} \times \cdots \times P^{\alpha^{s-1}}$, it follows that $C_{P^{\langle\alpha\rangle}}(\alpha)$ consists of elements of the form $(x, x^{\alpha}, \ldots, x^{\alpha^{s-1}})$, where x ranges over $C_P(\alpha^s)$. Since α is coprime, $P^{\langle\alpha\rangle} = [P^{\langle\alpha\rangle}, \alpha]C_{P^{\langle\alpha\rangle}}(\alpha)$, and so $[P^{\langle\alpha\rangle}, \alpha]$ has index at most $|P|$ in $P^{\langle\alpha\rangle}$. Similarly, $|Q^{\langle\alpha\rangle} : [Q^{\langle\alpha\rangle}, \alpha]| \leq |Q|$. Therefore, by Lemma 3.1 (2), we obtain that

$$\Pr(P^{\langle\alpha\rangle}, Q^{\langle\alpha\rangle}) \geq \frac{1}{|P|}\frac{1}{|Q|}\Pr\left([P^{\langle\alpha\rangle}, \alpha], [Q^{\langle\alpha\rangle}, \alpha]\right) \geq \frac{1}{|S|}\epsilon.$$

Since $|S|$ is ϵ-bounded, we deduce that there exists $\tilde{\epsilon}$, depending only on ϵ, such that for any distinct primes $p, q \in \pi(G)$ there is a Sylow p-subgroup $\tilde{P}$ and a Sylow q-subgroup $\tilde{Q}$ of G such that $\Pr(\tilde{P}, \tilde{Q}) \geq \tilde{\epsilon}$. Now the result follows from Theorem 1.1 of [**12**]. This completes the proof. □

Lemma 6.4. *Under Hypothesis 4.1, assume that G is semisimple and $G = [G, \alpha]$. Then $|G|$ is ϵ-bounded.*

Proof. Write

$$G = T_1 \times \cdots \times T_s,$$

where T_i are minimal normal α-invariant subgroups of G. Note that $T_i = [T_i, \alpha]$. As each T_i is a normal subgroup of G, Hypothesis 4.1 holds for every T_i. It follows from Lemma 6.3 that each T_i has ϵ-bounded order. Therefore we only need to prove that s is ϵ-bounded.

Note that only ϵ-boundedly many groups occur as simple factors of T_i and there are only ϵ-boundedly many pairwise non-isomorphic groups among the T_i. So without loss of generality we can assume that all the T_i are isomorphic to a given semisimple group T.

First assume that T is simple. Let p and q be as in Corollary 6.2, so that for any α-invariant Sylow p-subgroup $\tilde{P}$ and any α-invariant Sylow q-subgroup $\tilde{Q}$ of T we have that $[\tilde{P}, \alpha]$ does not commute with $[\tilde{Q}, \alpha]$. It follows from Lemma 3.2 that $\Pr([\tilde{P}, \alpha], [\tilde{Q}, \alpha]) \leq 3/4$. Choose P, Q as in Hypothesis 4.1 and write $P = \prod P_i$, $Q = \prod Q_i$, where the P_i (resp. Q_i) are α-invariant Sylow p-subgroups (resp. q-subgroups) of T_i. Then, by Lemma 3.1 (3),

$$\epsilon \leq \Pr([P, \alpha], [Q, \alpha]) = \prod_{i=1}^{s} \Pr([P_i, \alpha], [Q_i, \alpha]) \leq (3/4)^s.$$

Therefore s is ϵ-bounded.

So assume that each T_i is not simple. Let

$$T = S \times S^{\alpha} \times \cdots \times S^{\alpha^m}$$

where S is a simple group and $m \geq 1$. By Proposition 3.1 of [**12**] there exist two distinct primes p and q such that for every Sylow p-subgroup P_0 and every Sylow q-subgroup Q_0 of S, we have $\Pr(P_0, Q_0) \leq 2/3$. Choose P, Q as in Hypothesis 4.1 with respect to these primes p and q. Let $P_i = P \cap T_i$ and $Q_i = Q \cap T_i$, so that

$$P = \prod_{i=1}^{s} P_i \quad \text{and} \quad Q = \prod_{i=1}^{s} Q_i.$$

For a given index $i \in \{1, \ldots, s\}$, let $\bar{\pi} : T_i \to S$ be the projection on the first component of T_i and and set $P_0 = P \cap S = \bar{\pi}(P_i)$ and $Q_0 = Q \cap S = \bar{\pi}(Q_i)$. If $x \in S$ then $\bar{\pi}([x, \alpha]) = x^{-1}$. Therefore $\bar{\pi}([P_i, \alpha]) = P_0$ and $\bar{\pi}([Q_i, \alpha]) = Q_0$. Thus

$$\Pr([P_i, \alpha], [Q_i, \alpha]) \leq \Pr(P_0, Q_0) \leq 2/3.$$

This holds for every T_i and by Lemma 3.1 (3) we get

$$\epsilon \leq \Pr([P, \alpha], [Q, \alpha]) = \prod_{i=1}^{s} \Pr([P_i, \alpha], [Q_i, \alpha]) \leq (2/3)^s.$$

Therefore s is ϵ-bounded and the proof is complete. $\square$

Recall that the generalized Fitting subgroup $F^*(G)$ of a finite group G is the product of the Fitting subgroup $F(G)$ and all subnormal quasisimple subgroups; here a group is quasisimple if it is perfect and its quotient by the centre is a nonabelian simple group. In every finite group G, the centralizer of $F^*(G)$ is contained in $F^*(G)$. We will denote by $R(G)$ the soluble radical of the finite group G, i.e. the largest normal soluble subgroup of G.

Lemma 6.5. *Under Hypothesis 4.1, assume that the soluble radical of G is trivial and $G = [G, \alpha]$. Then $|G|$ is ϵ-bounded.*

Proof. Taking into account that $R(G) = 1$ write $F^*(G) = S_1 \times \cdots \times S_l$, where the factors S_i are simple. The group $G\langle\alpha\rangle$ acts as a permutation group of the set of simple factors of $F^*(G)$. Let K be the kernel of this action. Note that by Lemma 6.4 the order of $[F^*(G), \alpha]$ is ϵ-bounded. Hence the element α moves only ϵ-boundedly many points. It follows from Lemma 2.5 of [**2**] that G/K has ϵ-bounded order, so there are only ϵ-boundedly many conjugates of $[F^*(G), \alpha]$, all of them normalizing each other, being normal in $F^*(G)$. Therefore $N = [F^*(G), \alpha]^G$ has ϵ-bounded order. Since α centralizes $F^*(G)/N$, by Lemma 2.1 $F^*(G)/N$ is central in G/N. Taking into account that $F^*(G)$ is semisimple we deduce that $F^*(G) = N$ and so $F^*(G)$ has ϵ-bounded order. Since G acts on $F^*(G)$ by conjugation and $F^*(G)$ contains its centralizer, the lemma follows. □

We can now prove Theorem 1.2 in the special case when A is cyclic.

THEOREM 6.6. *Under Hypothesis 4.1, assume that $G = [G, \alpha]$. Then the index $[G : F_2(G)]$ is ϵ-bounded.*

PROOF. Let $R = R(G)$ be the soluble radical of G. By Lemma 4.6 the index $|[R, \alpha] : F_2([R, \alpha])|$ is ϵ-bounded. As $F_2([R, \alpha])$ is subnormal in G and therefore contained in $F_2(G)$, we deduce that the order of $[R, \alpha]$ modulo $F_2(G)$ is ϵ-bounded. We know from Lemma 6.5 that the index $|G : R|$ is ϵ-bounded. As R normalizes $[R, \alpha]$, there are ϵ-boundedly many G-conjugates of $[R, \alpha]$ and they normalize each other, being all normal in R. Therefore the order of $[R, \alpha]^G$ modulo $F_2(G)$ is ϵ-bounded. Since α centralizes $R/[R, \alpha]^G$, Lemma 2.3 now implies that $R/F_2(G)$ has ϵ-bounded order, and the result follows. □

7. The general case

To extend Theorem 6.6 from the case of one automorphism α to the case where a coprime group of automorphisms A acts on G, we need the following lemma.

LEMMA 7.1. *Let A be a coprime group of automorphisms of a finite group G and assume that $|[G, a]| \le m$ for every $a \in A$. Then $|[G, A]|$ is m-bounded.*

PROOF. Note that, as $[G, \alpha]$ is normal in G for all $\alpha \in A$,

$$[G, A] = \prod_{\alpha \in A} [G, \alpha].$$

So, it is sufficient to show that $|A|$ is m-bounded. Since the order of A is bounded in terms of the orders of its abelian subgroups (see for instance [**6**, Theorem 5.2]), we can assume that A is abelian. Set $K = GA$. Then $|a^K| \le m$ for every $a \in A$.

It follows from Theorem 1.1 of [**5**] that the derived subgroup of $\langle A^G \rangle$ has m-bounded order. Then $[G, A] = [G, A, A] \le [A^G, A]$ has m-bounded order as well. As A acts on $[G, A]$, we deduce that the index of $C_A([G, A])$ in A is m-bounded. Note that if $a \in C_A([G, A])$, then $[G, a, a] = [G, a] = 1$, hence $a = 1$. It follows that the order of A is m-bounded. □

Now we are ready to prove Theorem 1.2:

Proof of Theorem 1.2. Let $\epsilon > 0$ and let G be a finite group admitting a coprime automorphism group A such that $G = [G, A]$ and for any distinct primes $p, q \in \pi(G)$ there are A-invariant Sylow p-subgroup P and Sylow q-subgroup Q in G for which $\Pr([P, A], [Q, A]) \geq \epsilon$. We want to prove that $[G : F_2(G)]$ is ϵ-bounded.

Recall that $G = [G, A] = \prod_{\alpha \in A}[G, \alpha]$. Now we will show that for any $\alpha \in A$, the subgroup $[G, \alpha]$ satisfies the assumptions of Theorem 6.6. Let $p, q \in \pi(G)$ be distinct primes and let P and Q be A-invariant Sylow p and q subgroups such that $\Pr([P, A], [Q, A]) \geq \epsilon$; then $\Pr([P, \alpha], [Q, \alpha]) \geq \epsilon$. Set $\tilde{P} = P \cap [G, \alpha]$ and $\tilde{Q} = Q \cap [G, \alpha]$. Observe that $\tilde{P}$ and $\tilde{Q}$ are α-invariant Sylow subgroups of $[G, \alpha]$. We have

$$\Pr([\tilde{P}, \alpha], [\tilde{Q}, \alpha]) \geq \Pr([P, \alpha], [Q, \alpha]) \geq \epsilon.$$

By Theorem 6.6 we deduce that the index of $F_2([G, \alpha])$ in $[G, \alpha]$ is ϵ-bounded. Note that $F_2([G, \alpha])$ is characteristic in $[G, \alpha]$, which is normal in G, therefore $F_2([G, \alpha]) \leq F_2(G)$.

Now we factor out $F_2(G)$ and we get that $[G, \alpha]$ has ϵ-bounded order for every $\alpha \in A$. It follows from Lemma 7.1 that the order of $[G, A] = G$ is ϵ-bounded. This concludes the proof. $\square$

8. Theorems 1.3 and 1.4

This section is devoted to the proofs of Theorem 1.3 and Theorem 1.4. We start by studying the case when G is a finite simple group.

Proposition 8.1. *Let G be a finite simple group of Lie type in characteristic p admitting a coprime automorphism α. Let P be an α-invariant Sylow p-subgroup of G and assume that*

$$\Pr([P, \alpha], [P, \alpha]^x) \geq \epsilon$$

for all $x \in G$. Then the order of $[P, \alpha]$ is ϵ-bounded.

Proof. Let $G = L(q)$ and $C_G(\alpha) = L(q_0)$ where $q = q_0^e$, with $e = |\alpha|$. We can assume $\alpha \neq 1$. Let m be the ϵ-bounded integer in Lemma 3.3, which we may assume to be bigger than 8.

If the rank r of G is at most m, then by virtue of [**11**, Theorem 1.3] the order of $[P, \alpha]$ is ϵ-bounded.

So assume $r > m$. Then G contains the alternating group $\mathrm{Alt}(m)$ (see the proof of Lemma 6.4 of [**1**]) and therefore the order of G is divisible by all primes less or equal than m. Since $(|\alpha|, |G|) = 1$, we deduce that every prime divisor of the order of α is bigger than m.

We know that α normalizes a Borel subgroup $B = UT$, where T is a torus and U is a Sylow p-subgroup. Then $B^- = U^-T$ is the opposite

Borel subgroup (with $U \cap U^-) = 1$). This is obtained by conjugating B by the longest element n in the Weyl group. We can assume that $P = U$. Let $x \in C_G(\alpha)$ be an element corresponding to n. Note that, in particular, $P \cap P^x = 1$.

By Lemma 3.3, $[P, \alpha]^x$ contains an α-invariant normal subgroup P_0 such that $|[P, \alpha]^x/P_0| \leq m$ and $|[P, \alpha] : C_{[P,\alpha]}(g)| \leq m$ for every $g \in P_0$. Note that $[P, \alpha]^x = [P^x, \alpha]$. Since the order of α is divisible only by primes bigger than m, it acts trivially on the quotient group $[P^x, \alpha]/P_0$. Therefore $[P^x, \alpha] = [P^x, \alpha, \alpha] \leq P_0$, and hence $P_0 = [P^x, \alpha]$.

It follows from Lemma 5.2 that $[P^x, \alpha]$ contains a regular element y with $C_G(y) \leq P^x$, so

$$C_{[P,\alpha]}(y) \leq P \cap P^x = 1.$$

Since $|[P, \alpha] : C_{[P,\alpha]}(y)| \leq m$, we deduce that $[P, \alpha]$ has ϵ-bounded order, as claimed. □

We are now ready to prove Theorem 1.3, which we restate here for convenience

Theorem 1.3. *Let G be a finite group admitting a group of coprime automorphisms A. Let P be an A-invariant Sylow p-subgroup of G and assume that*

$$\Pr([P, A], [P, A]^x) \geq \epsilon$$

for all $x \in G$. Then the order of $[P, A]$ modulo $O_p(G)$ is ϵ-bounded.

Proof. We may assume that $O_p(G) = 1$. Let r be the maximum integer such that G has a composition factor which is isomorphic to a simple group of Lie type of rank r in characteristic p. We will show that r is ϵ-bounded. Then the theorem will be immediate from [**11**, Theorem 1.3]. We can assume $r > 8$.

First, we deal with the case where $A = \langle\alpha\rangle$ is cyclic.

Let M be the characteristic subgroup of G generated by all normal subgroups containing no composition factor which is isomorphic to a simple group of Lie type of rank r in characteristic p. Note that the maximum integer s such that G/M has a composition factor which is isomorphic to a simple group of Lie type of rank s in characteristic p is precisely r. So we can pass to the quotient G/M and without loss of generality assume that $M = 1$.

Then G has at least one minimal α-invariant normal subgroup which is a direct product of simple groups of Lie type of rank r in characteristic p. Let N be any such subgroup. Then N contains an α-invariant normal subgroup $N_0 = S^{\langle\alpha\rangle}$, for some simple group S of Lie type of rank r in characteristic p. Note that $P \cap N_0$ is an α-invariant Sylow p-subgroup of N_0 with

$$\Pr([P \cap N_0, \alpha], [P \cap N_0, \alpha]^x) \geq \epsilon$$

for all $x \in N_0$.

If $N_0 = S$, then the rank r is ϵ-bounded by Proposition 8.1 and Lemma 2.2.

Otherwise

$$N_0 = S \times S^{\alpha} \times \cdots \times S^{\alpha^{c-1}}$$

and an α-invariant Sylow p-subgroup $P \cap N_0$ of N_0 is of the form $\tilde{P} \times \tilde{P}^{\alpha} \times \cdots \times \tilde{P}^{\alpha^{c-1}}$, where $\tilde{P}$ is an α^c-invariant Sylow p-subgroup of S. Note that, if $x \in \tilde{P}$, then $x^{-1}x^{\alpha} \in [P \cap N_0, \alpha]$ has the same order as x. Therefore the exponent of $\tilde{P}$ is the same as the exponent of $[P \cap N_0, \alpha]$, which is ϵ-bounded by Lemma 2.11 of [**11**]. Note that every classical group of Lie rank $r > 8$ contains the alternating group $\mathrm{Alt}(r)$ of degree r (see the proof of Lemma 6.4 of [**1**]); as the exponent of the Sylow p-subgroup of $\mathrm{Alt}(r)$ goes to infinity as r does, it follows that the rank r of S is bounded in terms of the exponent of $\tilde{P}$. This implies that r is ϵ-bounded.

Thus, the theorem holds in the case where A is cyclic. The general case is straightforward from Lemma 7.1 applied to the action of A on P. □

Lemma 8.2. *Assume that the finite group G is the product of k normal subgroups $A_1, \ldots, A_k$ such that $\Pr(A_i, A_j) \geq \epsilon$ for all i, j. Then G has a normal subgroup D of (ϵ, k)-bounded index such that the order of D' is (ϵ, k)-bounded.*

Proof. For any pair i, j, by virtue of Lemma 3.3, G contains a normal subgroup $D_{i,j} \leq A_i$ such that the indices $|A_i : D_{i,j}|$ and $|A_j : C_{A_j}(x)|$ are ϵ-bounded for every $x \in D_{i,j}$. Let

$$D_i = \bigcap_{j=1}^{k} D_{i,j}.$$

Then D_i is a normal subgroup of G contained in A_i such that its index in A_i is (ϵ, k)-bounded and every element x of D_i has (ϵ, k)-boundedly many A_j-conjugates, for every $j = 1, \ldots, k$. Moreover, as $G = A_1 \cdots A_k$ and D_i is normal in G, the size of the conjugacy class

$$|x^G| = |(\cdots(x^{A_1})\cdots)^{A_k}|$$

is (ϵ, k)-bounded for every $x \in D_i$.

Note that the normal subgroup $D = D_1 \cdots D_k$ has (ϵ, k)-bounded index in G. Moreover, every element of D is a product of at most k elements from $D_1, \ldots, D_k$, so it has (ϵ, k)-boundedly many G-conjugates.

It follows follows from the proof of Neumann's BFC-theorem [**25**] that the order of D' is (ϵ, k)-bounded (see also [**17, 28**]). □

We can now prove our last main result, Theorem 1.4.

THEOREM 1.4. *Let G be a finite group admitting a group of coprime automorphisms A. Assume that $G = [G, A]$ and for any prime p dividing the order of G there is an A-invariant Sylow p-subgroups P such that*

$$\mathrm{Pr}([P, A], [P, A]^x) \geq \epsilon$$

for all $x \in G$. Then G is bounded-by-abelian-by-bounded.

PROOF. Let $F = F(G)$. It follows from Proposition 1.3 that there exists some ϵ-bounded integer such that

$$|[P, \alpha]F/F| \leq m$$

for every α-invariant Sylow subgroup P of G. Then the index $|G : F|$ is ϵ-bounded by Lemma 2.4 of [**2**].

We will now show that $[F, \alpha]^G$ has ϵ-bounded index in G. Consider the quotient group $\bar{G} = G/[F, \alpha]^G$. As $\bar{G}$ is the union of ϵ-boundedly many cosets $\bar{F}\bar{x}_1, \ldots, \bar{F}\bar{x}_k$, it follows that $I_{\bar{G}}(\alpha) = \{[\bar{x}_1, \alpha], \ldots, [\bar{x}_k, \alpha]\}$ has ϵ-bounded cardinality, therefore $\bar{G}$ has ϵ-bounded order by Lemma 2.3, as required.

So now it is sufficient to prove that $N = [F, \alpha]^G$ is bounded-by-abelian-by–bounded. As $[F, \alpha]$ is normal in F, which has ϵ-bounded index in G, it follows that N is a product of ϵ-boundedly many normal subgroups, all conjugate to $[F, \alpha]$, say

$$N = \prod_{i=1}^{t} [F, \alpha]^{g_i},$$

where $g_i \in G$ and t is ϵ-bounded. If p is a prime and P is the Sylow p-subgroup of F, then $[P, \alpha]$ is the Sylow p-subgroup of $[F, \alpha]$. Actually $[P, \alpha] = O_p([F, \alpha])$ and so $[P, \alpha]$ is normal in F. Therefore the Sylow p-subgroup $[P, \alpha]^G$ of N is the product of $[P, \alpha]^{g_i}$, for $i = 1, \ldots, t$, where each $[P, \alpha]^{g_i}$ is normal in N and t is ϵ-bounded. Since P is contained in a Sylow p-subgroup of G, by the assumptions we have that

$$\mathrm{Pr}\left([P, \alpha]^{g_i}, [P, \alpha]^{g_j}\right) \geq \epsilon$$

for every $i, j = 1, \ldots, t$. It follows from Lemma 8.2 that $[P, \alpha]^G$ is bounded-by-abelian-by-bounded.

Since the bounds do not depend on the prime p, it follows that for p large enough $[P, \alpha]^G$ is abelian. So we deduce that N is a direct product of ϵ-boundedly many bounded-by-abelian-by-bounded Sylow subgroups and an abelian subgroup. We conclude that N is bounded-by-abelian-by-bounded. The proof is complete. $\square$

References

[1] C. Acciarri, R. M. Guralnick, E. Khukhro, P. Shumyatsky, Rank type conditions on commutators in finite groups, arXiv:2404.14599v2.

[2] C. Acciarri, R. M. Guralnick, P. Shumyatsky, Coprime automorphisms of finite groups, Trans. Amer. Math. Soc. 375 (2022), 4549–4565.

[3] C. Acciarri, R. M. Guralnick, P. Shumyatsky, Criteria for solubility and nilpotency of finite groups with automorphisms, Bull. Lond. Math. Soc. 55 (2023), 1340–1346.

[4] C. Acciarri, P. Shumyatsky, A. Thillaisundaram, Conciseness of coprime commutators in finite groups. Bull. Aust. Math. Soc. 89 (2014), 252–258.

[5] C. Acciarri, P. Shumyatsky, A stronger form of Neumann's BFC-theorem, Isr. J. Math. **242** (2021), 269–278.

[6] S. Aivazidis, I. M. Isaacs, Large abelian normal subgroups. Arch. Math. (Basel) 111 (2018), 113–122.

[7] M. Aschbacher, Finite group theory. Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 2000.

[8] T. Burness, R. M. Guralnick, A. Moretó, G. Navarro, On the commuting probability of p-elements in finite groups, Algebra and Number Theory 17 (2023), 1209–1229.

[9] R. W. Carter, Simple groups of Lie type, Pure and Applied Mathematics, vol. 28, John Wiley & Sons, London-New York-Sydney, 1972.

[10] R. W. Carter, Finite groups of Lie type. Conjugacy classes and complex characters. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985.

[11] E. Detomi, R. M. Guralnick, M. Morigi, P. Shumyatsky, Commuting probability for conjugate subgroups of a finite group, Preprint, arXiv:2505.10521.

[12] E. Detomi, A. Lucchini, M. Morigi, P. Shumyatsky, Commuting probability for the Sylow subgroups of a finite group, Israel J. Math. To appear, arXiv:2311.10454.

[13] E. Detomi, P. Shumyatsky, On the commuting probability for subgroups of a finite group, Proc. Roy. Soc. Edinburgh Sect. A 152 (2022), 1551–1564.

[14] S. Eberhard, Commuting probabilities of finite groups, Bull. London Math. Soc. 47 (2015), 796–808.

[15] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups. Mathematical Surveys and Monographs, 40.1. American Mathematical Society, Providence, RI, 1994.

[16] D. Gorenstein, R. Lyons, R. Solomon, The classification of the finite simple groups. Number 3. Part I. Chapter A. Almost simple K-groups. Mathematical Surveys and Monographs, 40.3. American Mathematical Society, Providence, RI, 1998.

[17] R. M. Guralnick, A. Maroti, Average dimension of fixed point spaces with applications, Journal of Algebra 226 (2011), 298–308.
[18] R. M. Guralnick, G. R. Robinson, On the commuting probability in finite groups, J. Algebra 300 (2006), 509–528.
[19] R. M. Guralnick, J. Shareshian, R. Woodroofe, Invariable generation of finite simple groups and rational homology of coset posets. J. Algebra 659 (2024), 686–697.
[20] B. Hartley, I. M. Isaacs, On characters and fixed points of coprime operator groups, J. Algebra 131 (1990), 342–358.
[21] P. Hall, G. Higman, On the p-length of p-soluble groups and reduction theorems for Burnside's problem. Proc. London Math. Soc. 6 (1956), 1–42.
[22] J. E. Humphreys, Conjugacy classes in semisimple algebraic groups. Mathematical Surveys and Monographs, 43. American Mathematical Society, Providence, RI, 1995.
[23] M. I. Kargapolov, Ju. I. Merzljakov, Fundamentals of the theory of groups. Translated from the second Russian edition by Robert G. Burns. Graduate Texts in Mathematics, 62. Springer-Verlag, New York-Berlin, 1979.
[24] G. Malle, D. Testerman, Linear algebraic groups and finite groups of Lie type. Cambridge Studies in Advanced Mathematics, 133. Cambridge University Press, Cambridge, 2011.
[25] B. H. Neumann, Groups covered by permutable subsets, Journal of the London Mathematical Society 29 (1954), 236–248.
[26] P. M. Neumann, Two Combinatorial Problems in Group Theory, Bull. London Math. Soc. 21 (1989), 456–458.
[27] D. J. S. Robinson, Finiteness conditions and generalized soluble groups. Part 1. Springer-Verlag, New York-Berlin, 1972.
[28] J. Wiegold, Groups with boundedly finite classes of conjugate elements, Proceedings of the Royal Society of London. Series A 238 (1957), 389–401.

Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy
Email address: eloisa.detomi@unipd.it

Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA
Email address: guralnic@usc.edu

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy
Email address: marta.morigi@unibo.it

Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil
Email address: pavel@unb.br