Macdonald Index From Refined Kontsevich-Soibelman Operator

George Andrews*
Department of Mathematics, Penn State University, University Park, PA 16802

Anindya Banerjee†
Department of Physics, University of Cincinnati, Cincinnati, OH 45211

Ranveer Kumar Singh‡
NHETC and Department of Physics and Astronomy,
Rutgers University, 126 Frelinghuysen Rd., Piscataway NJ 08855, USA

Runkai Tao§
NHETC and Department of Physics and Astronomy,
Rutgers University, 126 Frelinghuysen Rd., Piscataway NJ 08855, USA
(Dated: November 18, 2025)

We propose a refinement of the Kontsevich-Soibelman operator for a class of "special" 4d $\mathcal{N}=2$ superconformal field theories characterized by the following conditions: (1) their Coulomb branch admits a source/sink chamber, i.e., a chamber in which the BPS quiver consists of only source and sink nodes, (2) The nodes with valency greater than 2 of the BPS quiver in a source/sink chamber are either all sources or all sinks. We present strong evidence that the trace of this refined operator is related to the Macdonald index of the theory. In particular, we conjecture new closed form expressions for the Macdonald indices of the $(A_1, \mathfrak{g})$ Argyres-Douglas theories for any simply-laced Lie algebra $\mathfrak{g}$.

I. INTRODUCTION

All known examples of interacting 4d $\mathcal{N}=2$ quantum field theories have a continuous moduli space of vacua known as the Coulomb branch. This is a singular manifold, and powerful techniques for analyzing the (protected or BPS) spectrum and dynamics of these theories may be derived from the operation of moving on this moduli space [1, 2]. In particular, many observables are related to representations of the monodromy group associated with the singularities.

In this work, we will make use of such an object, the q-deformed version [3] of the Kontsevich-Soibelman monodromy operator $M(q)$ [4, 5], to understand the spectrum of BPS states in these theories [6]. In particular, we will focus on superconformal field theories, where the full $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetry is non-anomalous and by virtue of the state-operator correspondence, one may study BPS local operators. Furthermore, in such SCFTs, the monodromy operator satisfies $M^r = I$ for some finite positive integer r [7, 8]. We will propose a prescription to generalize this operator for a "special" class of 4d $\mathcal{N}=2$ SCFTs. We will use this generalized operator to study the Schur subsector of the local operator spectrum of these theories [9], and compute a refined counting function associated with this sector known as the Macdonald index [10]. See [11–15] for other existing ways of computing the Macdonald index of 4d $\mathcal{N}=2$ theories. We expect a similar generalization to work for a much larger class of SCFTs, and present our results here as preparation for future work.

Note Added. After we submitted our paper to the arxiv, the paper [16] appeared which also conjectures expressions for the Macdonald indices of $(A_1, \mathfrak{g})$ theories using the KS monodromy operator.

II. KONTSEVICH-SOIBELMAN OPERATOR

Any 4d $\mathcal{N}=2$ theory at a generic point on the Coulomb branch is IR free and is characterized by a $\mathrm{U}(1)^r$ gauge theory. The massive spectrum of the theory consists of particles carrying electric and magnetic charges valued in the charge lattice Γ equipped with a Dirac pairing $\langle\cdot,\cdot\rangle$. The spectrum of BPS states is particularly interesting because they behave nicely under certain supersymmetry-preserving RG flows. The spectrum of BPS states with a given charge $\gamma\in\Gamma$ is captured by the protected spin character (PSC) defined as follows [17]: let $\mathcal{H}^{\gamma}_{\mathrm{int}}$ be the internal Hilbert space of the BPS particle with charge γ. Then the PSC of γ is defined by

$$\Omega(\gamma,y) := \mathrm{Tr}_{\mathcal{H}^{\gamma}_{\mathrm{int}}}(-1)^R y^{J+R} = \sum_{n\in\mathbb{Z}}\Omega_n(\gamma)y^n \ , \qquad (1)$$

where J is the generator of the Cartan subalgebra of the little group SU(2) of the Lorentz group SO(1,3) and R is the Cartan generator of the SU(2) R-symmetry group. As one moves in the moduli space, the BPS spectrum is

* gea1@psu.edu
† ab1702@scarletmail.rutgers.edu
‡ rks158@scarletmail.rutgers.edu
§ runkai.tao@physics.rutgers.edu

piecewise constant but may jump discontinuously across certain codimension one subloci, called walls of marginal stability. To understand this jump, one needs to construct wall-crossing invariants. Define the quantum torus algebra to be the formal span of generators

$$\mathcal{A} := \mathrm{Span}_{\mathbb{C}}\{X_\gamma : \gamma \in \Gamma\} \ , \tag{2}$$

with the multiplication on $\mathcal{A}$ defined as

$$X_\gamma X_{\gamma'} = q^{\frac{\langle \gamma,\gamma' \rangle}{2}} X_{\gamma+\gamma'} = q^{\langle \gamma,\gamma' \rangle} X_{\gamma'} X_\gamma \ , \tag{3}$$

where q is a formal variable. For each $\gamma \in \Gamma$, we construct the following element of $\mathcal{A}$:

$$U_\gamma = \prod_{n\in\mathbb{Z}} E_q\left((-1)^n q^{n/2} X_\gamma\right)^{(-1)^n \Omega_n(\gamma)} \ , \tag{4}$$

where $E_q(X)$ is the quantum dilogarithm

$$E_q(X) := \prod_{i=0}^{\infty}\left(1+q^{i+\frac{1}{2}}z\right)^{-1} = \sum_{n=0}^{\infty}\frac{\left(-q^{\frac{1}{2}}z\right)^n}{(q)_n} \ . \tag{5}$$

The Kontsevich-Soibelman (KS) operator is defined as

$$\mathcal{O}(q) := \prod_{\gamma\in\Gamma}^{\curvearrowright} U_\gamma \ , \tag{6}$$

where the product is taken in the order of increasing $\mathrm{Arg}(Z(\gamma))$ with respect to an arbitrarily chosen phase θ, where $Z(\gamma)$ is the central charge corresponding to the charge γ. The celebrated result of Kontsevich and Soibelman [18] is that the operator $\mathcal{O}(q)$ is a wall crossing invariant. Thus, if one can relate physically relevant quantities to this operator, then one can compute it at any point in the moduli space and glean information about the UV theory at the origin of the moduli space. This was the motivation behind the conjecture of Cordova-Shao [19] relating the Schur index of a 4d $\mathcal{N}=2$ SCFT to the trace of the KS operator.

A. Trace Of KS Operator And Schur Index

Let us begin by quoting the conjecture from [19]: it states that the Schur index of a 4d $\mathcal{N}=2$ SCFT is given by an appropriately defined trace:

$$\mathcal{I}_{\mathrm{S}}(q, z_1, \ldots, z_{N_f}) := (q)_\infty^{2r}\,\mathrm{Tr}\,\mathcal{O}(q) \ , \tag{7}$$

where r is the rank of the theory, $z_1, \ldots, z_{N_f}$ are fugacities for flavor symmetries and

$$(a)_\infty := (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i) \ . \tag{8}$$

The factor $(q)_\infty^{2r}$ is simply the Schur index of r free U(1) vector multiplets which is the contribution of the free $\mathrm{U}(1)^r$ vector multiplets at a generic point on the Coulomb branch. Recall that a charge $\gamma \in \Gamma$ is called a flavor charge if $\langle\gamma,\Gamma\rangle = 0$. Let $\{\gamma_{f_i}\}_{i=1}^{N_f}$ be an integral basis of the sublattice of flavor charges. For any charge γ, let

$$\gamma_f = \sum_{i=1}^{N_f} f_i(\gamma)\gamma_{f_i} \ , \tag{9}$$

be the projection on to the subspace of flavor charges. We define

$$\mathrm{Tr}[X_\gamma] = \begin{cases} \prod_i \mathrm{Tr}\left[X_{\gamma_{f_i}}\right]^{f_i(\gamma)} & \langle\gamma,\Gamma\rangle = 0 \ , \\ 0 & \text{else} \ , \end{cases} \ . \tag{10}$$

Imposing the cyclicity of the trace, the trace of any element of $\mathcal{A}$ can then be written in terms of q and N_f variables $\mathrm{Tr}[X_{\gamma_{f_i}}]$. These variables are functions of the flavor fugacities $z_1, \ldots, z_n$ and the explicit functional relation can be determined as explained in [19]. In [19], the trace of the KS operator was computed for a large class of examples and the conjecture (7) was verified.

III. THE REFINED KS OPERATOR

In this section, we define a refined KS operator for a "special" class of 4d $\mathcal{N}=2$ SCFTs.

A. BPS Quiver And Source/Sink Chamber

The BPS quiver of a 4d $\mathcal{N}=2$ theory [20] captures the spectrum of BPS states of the theory at a given point the Coulomb branch. We will not go into relevance of quivers in 4d $\mathcal{N}=2$ theories in this work, and instead refer readers to [20–23] for more details. The rules to draw the BPS quiver are as follows [7, 24]:

1. Draw a node for each basis element of the charge lattice.
2. If for two nodes γ_i, γ_j, $\langle\gamma_i,\gamma_j\rangle > 0$, draw $\langle\gamma_i,\gamma_j\rangle$ arrows from node i to node j and if $\langle\gamma_i,\gamma_j\rangle < 0$ then draw $|\langle\gamma_i,\gamma_j\rangle|$ arrows from j to i.

Note that the quiver depends on a choice of basis and choosing a different basis gives a different quiver. Some 4d $\mathcal{N}=2$ theories admit a source/sink chamber, meaning that the charge lattice at a given point in this chamber admits a basis in which the BPS quiver has only source and sink nodes [25]. The (conjectured) crucial property [7, 24] of the source/sink chamber is that the central charges corresponding to the nodes satisfy

$$\mathrm{Arg}(Z(\gamma)) > \mathrm{Arg}(Z(\gamma')) \ , \quad \text{for all } \gamma \in \text{sink}, \ \gamma' \in \text{source} \ . \tag{11}$$

Moreover, it is clear that

$$[X_\gamma, X_\gamma] = 0 \ , \quad \gamma,\gamma' \in \text{sources, or,} \quad \gamma,\gamma' \in \text{sinks} \ . \tag{12}$$

Thus, one can compute the KS operator in the source/sink chamber:

$$\mathcal{O}(q) = \prod_{\gamma\in\text{source}} U_{-\gamma} \prod_{\gamma\in\text{sink}} U_{-\gamma} \prod_{\gamma\in\text{source}} U_{\gamma} \prod_{\gamma\in\text{sink}} U_{\gamma} \ . \tag{13}$$

B. Refined KS Operator

We consider 4d $\mathcal{N}=2$ SCFTs which satisfy the following two properties:

1. The theory admits a source/sink chamber.

2. The nodes in the BPS quiver at a point in the source/sink chamber with valency greater than 2 are either all sources or all sinks.

The (G, G') Argyres-Douglas theories introduced in [7] are prime examples of this special class of theories. Indeed, their moduli space admits a source/sink chamber. Each node in the BPS quiver in the source/sink chamber represents a single hypermultiplet with trivial internal degrees of freedom. Moreover, from the general rules laid out in [7, 24] for the BPS quiver of (G, G') theories, one can prove that the second condition is also satisfied. Define the functions

$$\begin{aligned} E_{q,T}(X_\gamma) &= \sum_{n=0}^{\infty} \frac{(-(qT)^{\frac{1}{2}}X_\gamma)^n}{(q)_n}, \\ \widetilde{E}_{q,T}(X_\gamma) &= \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}}X_\gamma)^n}{(qT)_n} \ . \end{aligned} \tag{14}$$

Define the refined KS operator in the source/sink chamber by

$$\begin{aligned} \mathcal{O}(q,T) := &\prod_{\gamma\in\text{source}} \prod_{n\in\mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n\Omega_n(-\gamma)} \\ &\times \prod_{\gamma\in\text{sink}} \prod_{n\in\mathbb{Z}} \widetilde{E}_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n\Omega_n(-\gamma)} \\ &\times \prod_{\gamma\in\text{source}} \prod_{n\in\mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_{\gamma})^{(-1)^n\Omega_n(\gamma)} \\ &\times \prod_{\gamma\in\text{sink}} \prod_{n\in\mathbb{Z}} E_{q}((-1)^n q^{n/2} X_{\gamma})^{(-1)^n\Omega_n(\gamma)} \ . \end{aligned} \tag{15}$$

If the BPS quiver has a source node with valency greater than 2, then we define the refined KS operator as:

$$\begin{aligned} \mathcal{O}(q,T) := &\prod_{\gamma\in\text{source}} \prod_{n\in\mathbb{Z}} \widetilde{E}_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n\Omega_n(-\gamma)} \\ &\times \prod_{\gamma\in\text{sink}} \prod_{n\in\mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_{-\gamma})^{(-1)^n\Omega_n(-\gamma)} \\ &\times \prod_{\gamma\in\text{source}} \prod_{n\in\mathbb{Z}} E_{q}((-1)^n q^{n/2} X_{\gamma})^{(-1)^n\Omega_n(\gamma)} \\ &\times \prod_{\gamma\in\text{sink}} \prod_{n\in\mathbb{Z}} E_{q,T}((-1)^n q^{n/2} X_{\gamma})^{(-1)^n\Omega_n(\gamma)} \ . \end{aligned} \tag{16}$$

Note that, at the moment, we can define the refined KS operator only in the source/sink chamber.

IV. MACDONALD INDEX FROM MODIFIED KS OPERATOR: A CONJECTURE

Recall that the Macdonald index of the free vectormultiplet is given by

$$\mathcal{I}_{\mathrm{M}}^{\mathrm{vec}}(q,T) = (q)_\infty (qT)_\infty \ . \tag{17}$$

We conjecture that the Macdonald index of the class of 4d $\mathcal{N}=2$ SCFTs for which the modified KS operator is defined is given by

$$\mathcal{I}_{\mathrm{M}}(q,T,z_1,\ldots,z_{N_f}) = (q)_\infty^r (qT)_\infty^r \,\mathrm{Tr}\,\mathcal{O}(q,T) \ , \tag{18}$$

where r is the rank of the theory. Assuming the conjecture in [19], this formula trivially satisfies the correct Schur limit $T\to 1$.

A. Evidence For the Conjecture

In this section, we use our conjecture to calculate the Macdonald index of $(A_1, \mathfrak{g})$ Argyres-Douglas theories.

1. (A_1, A_k) Theory

Let us start with $k=2n$ case. The BPS quiver of the (A_1, A_{2n}) theory in the source/sink chamber is shown in Figure 1. Each node in the quiver represents a single

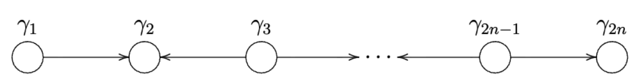

FIG. 1. BPS quiver of the (A_1, A_{2n}) theory in the source/sink chamber. Figure adapted from [19].

hypermultiplet. The refined KS operator is thus given by

$$\begin{aligned}\mathcal{O}(q,T) = &\prod_{i=1}^{n} E_{q,T}(X_{-\gamma_{2i-1}}) \prod_{i=1}^{n} \widetilde{E}_{q,T}(X_{-\gamma_{2i}}) \\ &\times \prod_{i=1}^{n} E_{q,T}(X_{\gamma_{2i-1}}) \prod_{i=1}^{n} E_{q}(X_{\gamma_{2i}}) \ .\end{aligned} \tag{19}$$

Using the quantum torus algebra and following the calculation in [19], we obtain

$$\begin{aligned}\mathcal{I}_{\mathrm{M}}^{A_{2n}}(q,T) = &(q)_\infty^n (Tq)_\infty^n \\ \times &\sum_{k_1,\ldots,k_{2n}\geq 0} \frac{T^{\sum_{i=1}^{n} k_{2i-1}} q^{\sum_{i=1}^{n} k_{2i} + \sum_{i=1}^{2n-1} k_i k_{i+1}}}{\prod_{i=1}^{n} (q)_{k_{2i-1}}^2 \prod_{i=1}^{n} (q)_{k_{2i}} (Tq)_{k_{2i}}}.\end{aligned} \tag{20}$$

For first few values of n, one can compute the series and find perfect match with the known form of the Macdonald index [12–14, 26]. In fact, we can go further and prove that the above series is identical to the known form of the Macdonald index [27]:

$$\mathcal{I}_{\mathrm{M}}^{A_{2n}}(q,T) = \sum_{\ell_1,\ldots,\ell_n \geq 0} \frac{T^{\ell_1+\ldots+\ell_n} q^{\ell_1^2+\ldots+\ell_n^2+\ell_1+\ldots+\ell_n}}{(q)_{\ell_1-\ell_2}(q)_{\ell_2-\ell_3}\ldots(q)_{\ell_{n-1}\ell_n}(q)_{\ell_n}} \ . \tag{21}$$

This nontrivial identity is a strong evidence of our conjecture. The BPS quiver of the (A_1, A_{2n}) theory in the source/sink chamber is shown in Figure 2. The computation of the trace for (A_1, A_{2n+1}) is similar to the (A_1, A_{2n}) case albeit with the complication of a flavor generator. Following the calculation in [19], we obtain

FIG. 2. BPS quiver of the (A_1, A_{2n+1}) theory in the source/sink chamber. Figure adapted from [19].

$$\begin{aligned}\mathcal{I}_{\mathrm{M}}^{A_{2n+1}}(q,T,z) = (q)_\infty^n (Tq)_\infty^n \sum_{\substack{\ell_1,\cdots,\ell_{2n+1}\geq 0\\ k_1,\cdots,k_{2n+1}\geq 0}} &\frac{(-1)^{\sum_{i=1}^{2n+1}(k_i+\ell_i)} T^{\frac{1}{2}\sum_{i=1}^{n+1}(k_{2i-1}+\ell_{2i-1})} q^{\frac{1}{2}\sum_{i=1}^{2n+1}(k_i+\ell_i)+\sum_{j=1}^{n}\ell_{2j}(\ell_{2j-1}+\ell_{2j+1})}}{\prod_{i=1}^{n+1}(q)_{k_{2i-1}}(q)_{\ell_{2i-1}} \prod_{i=1}^{n}(Tq)_{k_{2i}}(q)_{\ell_{2i}}} \\ &\times \mathrm{Tr}[X_{\gamma_f}]^{\ell_1-k_1} \prod_{i=1}^{n} \delta_{k_{2i},\ell_{2i}} \prod_{j=1}^{n} \delta_{(-1)^{j+1}k_1+k_{2j+1},(-1)^{j+1}\ell_1+\ell_{2j+1}},\end{aligned} \tag{22}$$

where we take $\mathrm{Tr}[X_{\gamma_f}] = z^2$ for $n=1$ and $\mathrm{Tr}[X_{\gamma_f}] = (-1)^{(n+1)}z$ for $n>1$ [19].

Let $\chi_{\boldsymbol{n}}(z)$ denote the character of the n-dimensional representation of SU(2). The first few terms in the expansion are shown below.

$$\begin{aligned}&\mathcal{I}_{\mathrm{M}}^{A_3}(q,T,z) = 1 + qT\chi_{\boldsymbol{3}} + q^2T\left(\chi_{\boldsymbol{1}}+\chi_{\boldsymbol{3}}\right) + q^2T^2\chi_{\boldsymbol{5}} + q^3T\left(\chi_{\boldsymbol{1}}+\chi_{\boldsymbol{3}}\right) + q^3T^2(\chi_{\boldsymbol{3}}+\chi_{\boldsymbol{5}}) + q^3T^3\chi_{\boldsymbol{7}} + q^4T(\chi_{\boldsymbol{1}}+\chi_{\boldsymbol{3}}) \\ &\quad + q^4T^2(\chi_{\boldsymbol{1}}+2\chi_{\boldsymbol{3}}+2\chi_{\boldsymbol{5}}) + q^4T^3(\chi_{\boldsymbol{5}}+\chi_{\boldsymbol{7}}) + q^4T^4\chi_{\boldsymbol{9}} + \ldots \\ &\mathcal{I}_{\mathrm{M}}^{A_5}(q,T,z) = 1 + qT + q^{\frac{3}{2}}T^{\frac{3}{2}}(z+z^{-1}) + q^2(2T+T^2) + q^{\frac{5}{2}}(T^{\frac{3}{2}}+T^{\frac{5}{2}})(z+z^{-1}) + 2q^3(T+T^2) \\ &\quad + q^3T^3(1+z^{-2}+z^2) + q^{\frac{7}{2}}(T^{\frac{3}{2}}+2T^{\frac{5}{2}}+T^{\frac{7}{2}})(z+z^{-1}) + q^4(2T+5T^2+T^3+(1+z^2+z^{-2})(T^3+T^4)) + \ldots\end{aligned} \tag{23}$$

This matches with the results of [11, 13].

2. (A_1, D_k) *Theory*

The BPS quiver of the (A_1, D_k) theory in the source/sink chamber is shown in Figure 3,4. Let us start with the $k=2n+1$ case. Note that the only node with valency greater than 2 is a source, so the refined KS

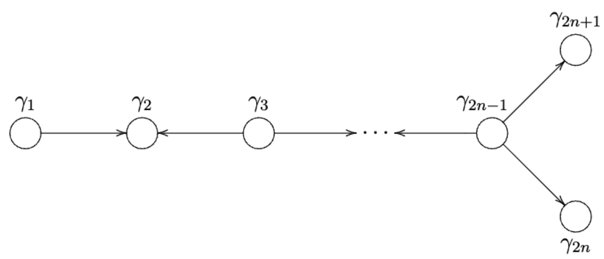

FIG. 3. BPS quiver of the (A_1, D_{2n+1}) theory in the source/sink chamber. Figure adapted from [19].

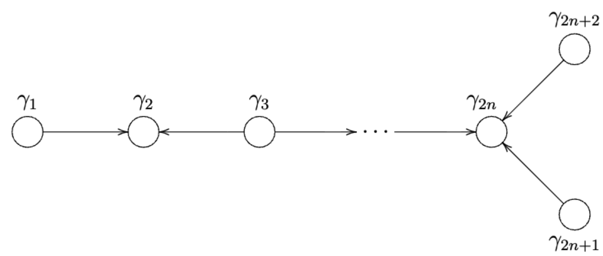

FIG. 4. BPS quiver of the (A_1, D_{2n+2}) theory in the source/sink chamber. Figure adapted from [19].

operator is given by

$$\begin{aligned}\mathcal{O}(q,T) = &\prod_{i=1}^{n} \widetilde{E}_{q,T}\left(X_{-\gamma_{2i-1}}\right) \prod_{i=1}^{n} E_{q,T}\left(X_{-\gamma_{2i}}\right) E_{q,T}\left(X_{-\gamma_{2n+1}}\right) \\ &\times \prod_{i=1}^{n} E_{q}\left(X_{\gamma_{2i-1}}\right) \prod_{i=1}^{n} E_{q,T}\left(X_{\gamma_{2i}}\right) E_{q,T}(X_{\gamma_{2n+1}}) \ .\end{aligned} \tag{24}$$

Following the calculation of trace in [19], we obtain

$$\mathcal{I}_{\mathrm{M}}^{D_{2n+1}}(q,T,z) = (q)_\infty^n (Tq)_\infty^n \sum_{\substack{\ell_1,\cdots,\ell_{2n+1}\geq 0 \\ k_{2n},k_{2n+1}\geq 0}} \frac{q^{\sum_{i=1}^{2n+1}\ell_i + \frac{1}{2}\sum_{i,j=1}^{2n+1} b_{ij}^{D_{2n+1}}\ell_i\ell_j} T^{\ell_{2n}+\ell_{2n+1}+\sum_{i=1}^{n-1}\ell_{2i}} z^{2(\ell_{2n+1}-k_{2n+1})}}{(q)_{k_{2n}}(q)_{k_{2n+1}}(q)_{\ell_{2n}}(q)_{\ell_{2n+1}}\prod_{i=1}^{n-1}(q)_{\ell_{2i}}^2 \prod_{i=1}^{n}(q)_{\ell_{2i-1}}(Tq)_{\ell_{2i-1}}} \delta_{k_{2n}+k_{2n+1},\ell_{2n}+\ell_{2n+1}} \ , \tag{25}$$

where z is the flavor fugacity, $b_{ij}^{D_{2n+1}} = -C_{ij}^{D_{2n+1}} + 2\delta_{ij}$ and $C_{ij}^{D_{2n+1}}$ is the Cartan matrix of the D_{2n+1} Lie algebra. The first few terms for (A_1, D_5) is shown below:

$$\begin{aligned}\mathcal{I}_{\mathrm{M}}^{D_5}(q,T,z) = &1 + qT\chi_{\mathbf{3}} + q^2T(\chi_{\mathbf{1}}+\chi_{\mathbf{3}}) + q^2T^2\chi_{\mathbf{5}} \\ &+ q^3T(\chi_{\mathbf{1}}+\chi_{\mathbf{3}}) + q^3T^2(2\chi_{\mathbf{3}}+\chi_{\mathbf{5}}) + q^3T^3\chi_{\mathbf{7}} \\ &+ q^4T(\chi_{\mathbf{1}}+\chi_{\mathbf{3}}) + q^4T^2(5\chi_{\mathbf{3}}+2\chi_{\mathbf{5}}) \\ &+ q^4T^3(\chi_{\mathbf{7}}+2\chi_{\mathbf{5}}) + q^4T^4\chi_{\mathbf{9}} + \ldots\end{aligned} \tag{26}$$

The computation for (A_1, D_{2n+2}) is similar. We obtain

$$\begin{aligned}\mathcal{I}_{\mathrm{M}}^{D_{2n+2}}(q,T,x,y) = &(q)_\infty^n (Tq)_\infty^n \sum_{\substack{\ell_1,\cdots,\ell_{2n+2}\geq 0 \\ k_1,\cdots,k_{2n+2}\geq 0}} \frac{(-1)^{\sum_{i=1}^{2n+2}(k_i+\ell_i)} q^{\frac{1}{2}\sum_{i=1}^{2n+2}(k_i+\ell_i)+\frac{1}{2}\sum_{i,j=1}^{2n+2} b_{ij}^{D_{2n+2}}\ell_i\ell_j}}{(q)_{k_{2n+1}}(q)_{\ell_{2n+1}}(q)_{k_{2n+2}}(q)_{\ell_{2n+2}}\prod_{i=1}^{n}(Tq)_{k_{2i}}(q)_{\ell_{2i}}\prod_{i=1}^{n}(q)_{k_{2i-1}}(q)_{\ell_{2i-1}}} \\ &\times T^{\frac{1}{2}(k_{2n+1}+k_{2n+2}+\ell_{2n+1}+\ell_{2n+2})+\frac{1}{2}\sum_{i=1}^{n}(k_{2i-1}+\ell_{2i-1})} \left(\mathrm{Tr}\left[X_{\gamma_{f_1}}\right]\right)^{\ell_{2n+2}-k_{2n+2}} \left(\mathrm{Tr}\left[X_{\gamma_{f_2}}\right]\right)^{\ell_1-k_1} \\ &\times \left(\prod_{i=1}^{n}\delta_{k_{2i},\ell_{2i}}\right)\left(\prod_{i=1}^{n-1}\delta_{(-1)^{i+1}k_1+k_{2i+1},(-1)^{i+1}\ell_1+\ell_{2i+1}}\right)\delta_{(-1)^{n+1}k_1+k_{2n+1}+k_{2n+2},(-1)^{n+1}\ell_1+\ell_{2n+1}+\ell_{2n+2}},\end{aligned} \tag{27}$$

where $b_{ij}^{D_{2n+2}} = -C_{ij}^{D_{2n+2}} + 2\delta_{ij}$ and $C_{ij}^{D_{2n+2}}$ is the Cartan matrix of the D_{2n+2} Lie algebra and we follow the normalization for $\mathrm{SU}(2)\times\mathrm{U}(1)$ flavor fugacities given in [19]:

$$\mathrm{Tr}\left[X_{\gamma_{f_1}}\right] = y^2, \quad \mathrm{Tr}\left[X_{\gamma_{f_2}}\right] = \begin{cases} xy & \text{if } n \text{ is odd },\\ -\frac{x}{y} & \text{if } n \text{ is even }.\end{cases} \tag{28}$$

For (A_1, D_4), the flavor symmetry is enhanced to SU(3) and the fugacities x, y are related to the SU(3) flavor fugacities as $x = z_2^{-3/2}, y = z_1 z_2^{-1/2}$ [19]. Let $\widetilde{\chi}_{\boldsymbol{n}}$ denote the character of the n-dimensional representation of SU(3). Then the first terms in the expansion of the Macdonald index is given by

$$\begin{aligned}
&\mathcal{I}_{\mathrm{M}}^{D_4}(q,T,z_1,z_2) = 1 + qT\widetilde{\chi}_{\mathbf{8}} + q^2T(\widetilde{\chi}_{\mathbf{1}} + \widetilde{\chi}_{\mathbf{8}}) + q^2T^2\widetilde{\chi}_{\mathbf{27}} + q^3T(\widetilde{\chi}_{\mathbf{1}} + \widetilde{\chi}_{\mathbf{8}}) + q^3T^2(\widetilde{\chi}_{\mathbf{8}} + \widetilde{\chi}_{\mathbf{10}} + \widetilde{\chi}_{\overline{\mathbf{10}}} + \widetilde{\chi}_{\mathbf{27}}) + q^3T^3\widetilde{\chi}_{\mathbf{64}} + \dots\\
&\mathcal{I}_{\mathrm{M}}^{D_6}(q,T,x,y) = 1 + qT(\chi_{\mathbf{1}} + \chi_{\mathbf{3}}) + q^{\frac{3}{2}}T^{\frac{3}{2}}(x + x^{-1})\chi_{\mathbf{2}} + q^2T(2\chi_{\mathbf{1}} + \chi_{\mathbf{3}}) + q^2T^2(\chi_{\mathbf{1}} + \chi_{\mathbf{3}} + \chi_{\mathbf{5}}) + q^{\frac{5}{2}}T^{\frac{3}{2}}(x + x^{-1})\chi_{\mathbf{2}}\\
&\quad + q^{\frac{5}{2}}T^{\frac{5}{2}}(x + x^{-1})(\chi_{\mathbf{2}} + \chi_{\mathbf{4}}) + q^3T(2\chi_{\mathbf{1}} + \chi_{\mathbf{3}}) + q^3T^2(2\chi_{\mathbf{1}} + 4\chi_{\mathbf{3}} + \chi_{\mathbf{5}}) + q^3T^3\left((\chi_{\mathbf{1}} + (x^{-2} + x^2 + 1)\chi_{\mathbf{3}} + \chi_{\mathbf{5}} + \chi_{\mathbf{7}})\right) + \dots
\end{aligned} \tag{29}$$

The Macdonald index for (A_1, D_4) matches with [11, Table 2] and (A_1, D_6) matches with the general formula of [11].

3. (A_1, E_n) Theory

The BPS quiver for the (A_1, E_6) theory is shown in Figure 5. The refined KS operator is given by

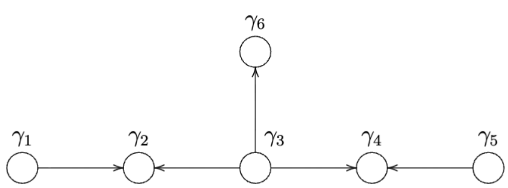

FIG. 5. BPS quiver of the (A_1, E_6) theory in the source/sink chamber. Figure adapted from [19].

$$\begin{aligned}
\mathcal{O}(q,T) = &\prod_{i=1}^{3} \widetilde{E}_{q,T}(X_{-\gamma_{2i-1}}) \prod_{i=1}^{3} E_{q,T}(X_{-\gamma_{2i}})\\
&\times \prod_{i=1}^{3} E_q(X_{\gamma_{2i-1}}) \prod_{i=1}^{3} E_{q,T}(X_{\gamma_{2i}}) \ .
\end{aligned} \tag{30}$$

Computing the trace, the Macdonald index is given by

$$\begin{aligned}
&\mathcal{I}_{\mathrm{M}}^{E_6}(q,T)\\
&= (q)_\infty^3 (qT)_\infty^3 \sum_{\ell_1,\cdots,\ell_6=0}^{\infty} \frac{T^{\ell_2+\ell_4+\ell_6} q^{\sum_{i=1}^{6}\ell_i + \frac{1}{2}\sum_{i,j=1}^{6} b_{ij}^{E_6}\ell_i\ell_j}}{\prod_{i=1}^{3}(qT)_{\ell_{2i-1}}(q)_{\ell_{2i-1}} \prod_{i=1}^{3}(q)_{\ell_{2i}}^2}\\
&= 1 + q^2T + q^3(T + T^2) + q^4(T + 2T^2) + q^5(T + 2T^2)\\
&+ q^6\left(T + 3T^2 + 2T^3\right) + \dots
\end{aligned} \tag{31}$$

where $b_{ij}^{E_6} = -C_{ij}^{E_6} + 2\delta_{ij}$ and $C_{ij}^{E_6}$ is the Cartan matrix of the E_6 Lie algebra. The BPS quiver of (A_1, E_8) theory is shown in Figure 6. The computation of the Macdonald index of (A_1, E_8) theory identical to that of (A_1, E_6)

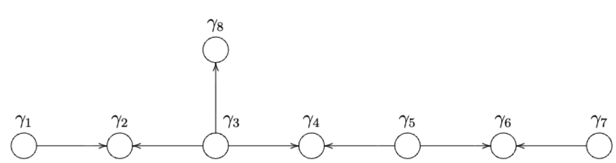

FIG. 6. BPS quiver of the (A_1, E_8) theory in the source/sink chamber. Figure adapted from [19].

theory and is given by

$$
\begin{aligned}
&\mathcal{I}_{\mathrm{M}}^{E_8}(q,T) \\
&= (q)_\infty^4 (qT)_\infty^4 \sum_{\ell_1,\cdots,\ell_8=0}^{\infty} \frac{T^{\sum_{i=1}^{4}\ell_{2i}}\, q^{\sum_{i=1}^{8}\ell_i + \frac{1}{2}\sum_{i,j=1}^{8} b_{ij}^{E_8}\ell_i\ell_j}}{\prod_{i=1}^{4}(qT)_{\ell_{2i-1}}(q)_{\ell_{2i-1}} \prod_{i=1}^{4}(q)_{\ell_{2i}}^2} \\
&= 1 + q^2 T + q^3(T+T^2) + q^4(T+2T^2) \\
&+ q^5(T+2T^2+T^3) + q^6(T+3T^2+3T^3) + \ldots,
\end{aligned}
\tag{32}
$$

where $b_{ij}^{E_8} = -C_{ij}^{E_8} + 2\delta_{ij}$ and $C_{ij}^{E_8}$ is the Cartan matrix of the E_8 Lie algebra. It is expected that $(A_1, E_{6,8}) \cong (A_2, A_{3,4})$ [28]. We confirmed that the Macdonald indices of $(A_1, E_{6,8})$ theory matches with the Macdonald indices of $(A_2, A_{3,4})$ theory computed in [29, 30]. Finally, the computation of the Macdonald index for (A_1, E_7) theory similar to (A_1, E_6) albeit with a flavor generator. We get

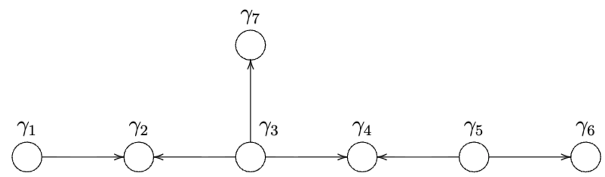

FIG. 7. BPS quiver of the (A_1, E_7) theory in the source/sink chamber. Figure adapted from [19].

$$
\begin{aligned}
\mathcal{I}_{\mathrm{M}}^{E_7}(q,T,z) &= (q)_\infty^3 (qT)_\infty^3 \sum_{\substack{\ell_1,\cdots,\ell_7\geq 0\\ k_1,\cdots,k_7\geq 0}} \frac{(-1)^{\sum_{i=1}^{7}(k_i+\ell_i)} T^{\frac{1}{2}(k_7+\ell_7)+\frac{1}{2}\sum_{i=1}^{3}(k_{2i}+\ell_{2i})} q^{\frac{1}{2}\sum_{i=1}^{7}(k_i+\ell_i)} q^{k_2(\ell_1+\ell_3)+k_4(\ell_3+\ell_5)+k_6\ell_5+k_7\ell_3}}{(q)_{k_7}(q)_{\ell_7}\prod_{i=1}^{3}(qT)_{k_{2i-1}}(q)_{\ell_{2i-1}}\prod_{i=1}^{3}(q)_{k_{2i}}(q)_{\ell_{2i}}} \\
&\qquad \times (-z)^{-\ell_7+k_7}\delta_{k_4+k_7,\ell_4+\ell_7}\delta_{k_6-k_7,\ell_6-\ell_7}\prod_{i\in\{1,2,3,5\}}\delta_{k_i,\ell_i} \\
&= 1 + qT + q^{\frac{3}{2}}T^{\frac{3}{2}}(z+z^{-1}) + q^2(2T+T^2) + q^{\frac{5}{2}}(T^{\frac{3}{2}}+T^{\frac{5}{2}})(z+z^{-1}) \\
&+ q^3(2T+3T^2+T^3(z^{-2}+1+z^2)) + q^{\frac{7}{2}}(T^{\frac{3}{2}}+3T^{\frac{3}{2}}+T^{\frac{7}{2}})(z+z^{-1}) + \ldots,
\end{aligned}
\tag{33}
$$

where z is the flavor fugacity.

V. FUTURE DIRECTIONS

The intriguing relation we have found between Macdonald index of a 4d $\mathcal{N}=2$ SCFT and the trace of the modified KS operator raises a lot of interesting questions. We list here some possible future directions. The first order problem is to perform more checks of our conjecture. Secondly, we would like to understand how the refined KS operator transforms under quiver mutations [7, 24]. More generally, we would like to give a chamber independent prescription for the refined KS operator. Once we have a chamber independent prescription, our formulas for Argyres-Douglas theory gives strong evidence that the modified KS operator is a wall-crossing invariant.

Acknowledgments. The authors would like to thank Gregory Moore, Leonardo Rastelli and Anirudh Deb for comments on the draft and Ashoke Sen for some useful discussions on related topics. R.K.S. would like to thank LITP, Stanford University for hospitality where this work began. The work of A.B. is supported by the US Department of Energy under grant DE-SC1019775. The work of R.K.S. is supported by the US Department of Energy under grant DE-SC0010008.

[1] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **426** (1994) 19–52, [`hep-th/9407087`].

[2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*, *Nucl. Phys. B* **431** (1994) 484–550, [`hep-th/9408099`].

[3] The original Kontsevich-Soibelman operator was introduced in context of Donaldson-Thomas invariants in [18] and used in context of 4d $\mathcal{N}=2$ theories in [6] to study wall crossing.

[4] T. Dimofte and S. Gukov, *Refined, Motivic, and Quantum*, *Lett. Math. Phys.* **91** (2010) 1, [`arXiv:0904.1420`].

[5] T. Dimofte, S. Gukov, and Y. Soibelman, *Quantum Wall Crossing in N=2 Gauge Theories*, *Lett. Math. Phys.* **95** (2011) 1–25, [`arXiv:0912.1346`].

[6] D. Gaiotto, G. W. Moore, and A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, *Commun. Math. Phys.* **299** (2010) 163–224, [arXiv:0807.4723].

[7] S. Cecotti, A. Neitzke, and C. Vafa, *R-Twisting and 4d/2d Correspondences*, arXiv:1006.3435.

[8] S. Cecotti, J. Song, C. Vafa, and W. Yan, *Superconformal Index, BPS Monodromy and Chiral Algebras*, *JHEP* **11** (2017) 013, [arXiv:1511.01516].

[9] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, *Infinite Chiral Symmetry in Four Dimensions*, *Commun. Math. Phys.* **336** (2015), no. 3 1359–1433, [arXiv:1312.5344].

[10] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, *Gauge Theories and Macdonald Polynomials*, *Commun. Math. Phys.* **319** (2013) 147–193, [arXiv:1110.3740].

[11] M. Buican and T. Nishinaka, *Argyres-Douglas Theories, the Macdonald Index, and an RG Inequality*, *JHEP* **02** (2016) 159, [arXiv:1509.05402].

[12] C. Bhargava, M. Buican, and H. Jiang, *Exact Operator Map from Strong Coupling to Free Fields: Beyond Seiberg-Witten Theory*, *Phys. Rev. Lett.* **132** (2024), no. 3 031602, [arXiv:2306.05507].

[13] J. Song, *Macdonald Index and Chiral Algebra*, *JHEP* **08** (2017) 044, [arXiv:1612.08956].

[14] G. Andrews, A. Banerjee, C. Bhargava, R. K. Singh, and R. Tao, *Argyres-Douglas Theories, Macdonald Indices and Arc Space of Zhu Algebra*, arXiv:2507.06294.

[15] M. J. Kang, C. Lawrie, and J. Song, *Index from a point*, arXiv:2507.12510.

[16] H. Kim, H. Kim, and J. Song, *Macdonald index from 3d TQFT*, arXiv:2511.11186.

[17] D. Gaiotto, G. W. Moore, and A. Neitzke, *Framed BPS States*, *Adv. Theor. Math. Phys.* **17** (2013), no. 2 241–397, [arXiv:1006.0146].

[18] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[19] C. Cordova and S.-H. Shao, *Schur Indices, BPS Particles, and Argyres-Douglas Theories*, *JHEP* **01** (2016) 040, [arXiv:1506.00265].

[20] F. Denef, *Quantum quivers and Hall / hole halos*, *JHEP* **10** (2002) 023, [hep-th/0206072].

[21] F. Denef and G. W. Moore, *Split states, entropy enigmas, holes and halos*, *JHEP* **11** (2011) 129, [hep-th/0702146].

[22] J. Manschot, B. Pioline, and A. Sen, *Wall Crossing from Boltzmann Black Hole Halos*, *JHEP* **07** (2011) 059, [arXiv:1011.1258].

[23] J. Manschot, B. Pioline, and A. Sen, *From Black Holes to Quivers*, *JHEP* **11** (2012) 023, [arXiv:1207.2230].

[24] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, and C. Vafa, *BPS Quivers and Spectra of Complete N=2 Quantum Field Theories*, *Commun. Math. Phys.* **323** (2013) 1185–1227, [arXiv:1109.4941].

[25] A node with all arrows emanating from it is called a source node and a node with all arrows ending on it is called a sink node.

[26] J. Song, *Superconformal indices of generalized Argyres-Douglas theories from 2d TQFT*, *JHEP* **02** (2016) 045, [arXiv:1509.06730].

[27] G. Andrews, A. Banerjee, R. K. Singh, and R. Tao, *BPS Quivers Of 4d $\mathcal{N} = 2$ Theories, Refined Kontsevich-Soibelman Operator And The Superconformal Index*, to-appear.

[28] Y. Wang and D. Xie, *Classification of Argyres-Douglas theories from M5 branes*, *Phys. Rev. D* **94** (2016), no. 6 065012, [arXiv:1509.00847].

[29] A. Watanabe and R.-D. Zhu, *Testing Macdonald Index as a Refined Character of Chiral Algebra*, *JHEP* **02** (2020) 004, [arXiv:1909.04074].

[30] P. Agarwal, S. Lee, and J. Song, *Vanishing OPE Coefficients in 4d $N = 2$ SCFTs*, *JHEP* **06** (2019) 102, [arXiv:1812.04743].