ON THE DETECTION OF KNOTTED SPHERES BY THEIR TRACES IN HIGH DIMENSIONS

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Abstract. For every $n \geq 4$, we demonstrate the existence of non-isotopic smooth $(n-2)$-knots in S^n with diffeomorphic traces by generalising the RBG link construction to all dimensions. Conversely, we prove that for every $n \geq 4$, the unknot in S^n is detected by the diffeomorphism type of its surgery and hence by its trace.

1. Introduction

A *knot in* S^n, for $n \geq 3$, is a smooth embedding $K\colon S^{n-2} \hookrightarrow S^n$. The *trace* $X(K)$ of a knot K is the oriented, connected, compact, smooth $(n+1)$-manifold obtained by attaching an $(n-1)$-handle to D^{n+1} along a tubular neighbourhood of K and smoothing corners. In this article, we study knots in ambient dimension $n \geq 4$, where, in contrast to ambient dimension $n = 3$, there is a unique choice of framing (see Section 2.2 for a discussion on framings). (Ambient) isotopic knots have orientation-preservingly diffeomorphic traces. Our main result shows that the converse is not true, i.e. the oriented diffeomorphism type of the trace $X(K)$ does not detect the isotopy class of the knot K.

Theorem A. *For every $n \geq 4$, there exist non-isotopic knots K_1 and K_2 in S^n whose traces $X(K_1)$ and $X(K_2)$ are orientation-preservingly diffeomorphic.*

For classical knots, i.e. those in S^3, the existence of non-isotopic framed knots with orientation-preservingly diffeomorphic traces follows from [Akb91b]. The existence of non-isotopic framed knots in S^3 that share a surgery was already observed in [Lic76].

We provide two independent proofs of Theorem A using different techniques. The first applies to dimensions $n \geq 4$ and gives infinitely many pairs of knots with orientation-preservingly diffeomorphic traces. The second gives infinite families of such knots for dimensions $n \geq 5$ (see Theorem B) and arbitrarily large families for $n = 4$ (see Theorem 6.3).

1.1. **RBG manifold construction.** The first proof is based on our generalisation of the *RBG link construction* from [Pic19, MP23, Tag24] to all dimensions. This simple and elegant method for constructing knots in S^3 with the same trace or surgery was used in a series of papers to settle a host of important questions and conjectures, e.g. in [Pic20, Pic19, BM18, KP25, HPW25]. Our generalisation to *RBG manifolds* yields a direct method for constructing, for $n \geq 4$, framed knotted

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spheres $\Sigma\colon S^k \hookrightarrow S^n$ with orientation-preservingly diffeomorphic traces.[1] The case of most interest to us is that of knots, i.e. when $k = n-2$. In that case, an RBG manifold is, roughly speaking, an $(n+1)$-manifold W with a handle decomposition consisting of a 0-handle D^{n+1}, an $(n-2)$-handle, and two $(n-1)$-handles attached to D^{n+1} in such a way that, after an isotopy, each $(n-1)$-handle cancels the $(n-2)$-handle (see Definition 3.1). The images of the attaching spheres of the $(n-1)$-handles under the diffeomorphisms given by the handle cancellations then form a pair of knots in S^n with the same trace (see Proposition 3.2). Of course, in overly simple situations, these knots will in fact be isotopic (see Section 3.2). However, using a specific instance of our RBG manifold construction, we show the existence of infinitely many mutually distinct pairs of non-isotopic knots as in Theorem A. The precise statement is given in Theorem 4.14. For $n = 4$, our construction can be visualised in Kirby diagrams (see Section 3.5).

1.2. **Distinguishing knots with the same trace.** In Section 4, we discuss methods by which we can or cannot distinguish knots with orientation-preservingly diffeomorphic traces. In dimension $n = 3$, there are many computable knot invariants that can be used to distinguish such knots in S^3, e.g. the knot group, hyperbolic volume and knot polynomials. In higher dimensions, this is a more difficult task. For $n \geq 4$, if two knots in S^n have diffeomorphic traces, then many of their invariants are the same, such as the knot groups and abelian invariants that can be derived from it (see Proposition 4.1).

Nevertheless, we successfully use the conjugacy class of the meridians in the knot groups $\pi_1(K_i)$, $i \in \{1,2\}$, to distinguish between knots K_1 and K_2 with the same trace. The idea is to count the number of homomorphisms of $\pi_1(K_i)$ into a specific finite group A that map an element representing the conjugacy class of the meridian of K_i to a fixed, well-chosen element of A. This number is an ambient isotopy invariant of the knot (see Section 4.2), which suffices to distinguish our pairs of knots K_1 and K_2 with the same trace (see Section 4.3).

Remark 1.1. As an application of Theorem A, we disprove a generalisation of the *light bulb theorem* to links. Recall that the classical light bulb trick shows that any 1-knot in $S^1 \times S^2$ intersecting $\{*\} \times S^2$ in a single point is isotopic to $S^1 \times \{*\}$. Gabai showed that a corresponding result holds for 2-spheres in $S^2 \times S^2$ as well [Gab20]. In Corollary 4.15, we show that the analogue of these results for 2-component links in $S^2 \times S^{n-2}$ does not hold. That is, for each $n \geq 4$, we use our construction to produce a 2-component link L in $S^2 \times S^{n-2}$ such that each component is isotopic to the standard $\{z\} \times S^{n-2}$ for $z \in S^2$, but L is not isotopic to $\{z_1, z_2\} \times S^{n-2}$ for $z_1 \neq z_2 \in S^2$. For $n \in \{3,4\}$, such links are easy to construct (see Remark 4.16), using the light bulb trick. For $n \geq 5$, we do not have a light bulb theorem, so we need our construction.

1.3. **Infinite families for $n \geq 5$.** Our second proof of Theorem A is based on a result by Plotnick [Plo83b]; see the discussion in Section 1.4 below. This gives the following generalisation of Theorem A for $n \geq 5$.

[1] The trace is defined analogously for knotted spheres Σ; see Section 2.1.

Theorem B. *Let $n \geq 5$. There is an infinite collection $\{X_i\}_{i\geq 1}$ of pairwise non-diffeomorphic smooth $(n+1)$-manifolds such that, for each integer $i \geq 1$, there exists an infinite collection $\{K_j^i\}_{j\geq 1}$ of pairwise non-isotopic knots in S^n whose trace is orientation-preservingly diffeomorphic to X_i, i.e.*

$$X(K_j^i) \cong X_i \qquad \text{for all integers} \quad i, j \geq 1.$$

In particular, for each $i \geq 1$, this results in an infinite family of pairwise non-isotopic knots in S^n with orientation-preservingly diffeomorphic traces.

Using the same method, we can construct arbitrarily large families of non-isotopic knots in S^4 with orientation-preservingly diffeomorphic traces (see Theorem 6.3). Moreover, for $n = 4$, we also obtain the analogous result to Theorem B in the topological category (see Theorem 6.2).

1.4. **Surgeries, traces and Gluck twists.** Our RBG manifold construction only produces *pairs* of knots with orientation-preservingly diffeomorphic traces. To prove the existence of infinitely many knots sharing the same trace as in Theorem B, we need to employ different methods. Our main insight is summarised in the following result. For $n \geq 4$, let $S^n(K)$ denote the manifold obtained by performing surgery on S^n along the knot K. See Section 2.2 for details.

Theorem C. *For $n \geq 5$, let K_1 and K_2 be knots in S^n and let $\varphi\colon S^n(K_1) \to S^n(K_2)$ be an orientation-preserving diffeomorphism between their surgeries. Then either there is an orientation-preserving diffeomorphism $X(K_1) \cong X(K_2)$ or the Gluck twist $T(K_1)$ is a knot in the standard smooth n-sphere with $X(T(K_1)) \cong X(K_2)$.*

We will recall the precise definition of the Gluck twist, analogous to the definition for $n = 4$ by Gluck [Glu62], in Definition 5.1. Roughly speaking, regluing a tubular neighbourhood of a knot K in S^n using a non-trivial diffeomorphism of $S^1 \times S^{n-2}$ gives rise to a knot $T(K)$ in a possibly non-standard S^n (see also Remark 5.2).

We will use Theorem C to deduce Theorem B as follows. For every $n \geq 5$, Plotnick [Plo83b] shows the existence of an infinite family $\mathcal{K}_{p,q,r}$ of pairwise inequivalent knots in S^n with orientation-preservingly diffeomorphic surgeries, starting from a certain Brieskorn sphere $\Sigma(p, q, r)$. By using Theorem C and paying particular attention to the framings of these knots, we can ensure that all the knots in $\mathcal{K}_{p,q,r}$ have orientation-preservingly diffeomorphic traces. We can apply this method to infinitely many distinct Brieskorn spheres, yielding the infinite families in Theorem B.

For $n = 4$, we obtain a weaker result than Theorem C (see Theorem 5.6). In particular, given an orientation-preserving diffeomorphism $\varphi\colon S^4(K_1) \to S^4(K_2)$ for knots K_1, K_2 in S^4, we can only deduce the existence of one of the diffeomorphisms as in Theorem C if an additional condition on the Gluck twist along K_1 is met. However, Section 5 also establishes that any such surgery diffeomorphism $\varphi\colon S^4(K_1) \to S^4(K_2)$ extends to either a trace diffeomorphism $X(K_1) \to X(K_2)$ or to a homeomorphism $X(T(K_1)) \to X(K_2)$.[2] The following theorem provides the complete answer as to which of these two cases occurs, depending on the algebraic topology of the traces.

[2] Note that $T(K_1)$ lies in a possibly non-standard smooth S^n, but the homeomorphism type of its trace $X(T(K_1))$ is still well-defined (see the discussion in Section 5).

This echoes the analogous result of Manolescu–Piccirillo [MP23, Theorem 3.7] based on [Boy86] for extending homeomorphisms in the $n = 3$ case.

Theorem D. *Let K_1 and K_2 be 2-knots in S^4. An orientation-preserving surgery diffeomorphism $\varphi\colon S^4(K_1) \to S^4(K_2)$ extends to an orientation-preserving trace diffeomorphism $\Phi\colon X(K_1) \to X(K_2)$ if and only if the closed 5-manifold $X(K_1) \cup_\varphi -X(K_2)$ is spin.*

1.5. **Detection results.** Neither Theorem A nor its proofs prevent particular knots from being detected by their traces. For example, when $n = 3$, it follows from Property R [Gab87] that if $S^1 \times S^2$ is obtained by surgery along a knot in S^3, then the knot is isotopic to the unknot. We prove a higher-dimensional analogue. Here, an *unknot* U in S^n, for $n \geq 3$, is an embedding $S^{n-2} \hookrightarrow S^n$ that extends to an embedding $D^{n-1} \hookrightarrow S^n$. Up to (ambient) isotopy, this determines the unknot uniquely.

Theorem E. *For $n \geq 4$, let K be a knot in S^n. Suppose that its surgery $S^n(K)$ is (possibly orientation-reversingly) diffeomorphic to the surgery $S^n(U) \cong S^{n-1} \times S^1$ of the unknot U. Then K is isotopic to U. In particular, if K and U have diffeomorphic traces, then K is isotopic to U.*

We prove Theorem E in Section 7 as a corollary of Theorem 7.2, which is a more general detection result for *unknotted* embeddings $S^{n-2} \hookrightarrow S^n$. We call a knot $K\colon S^{n-2} \hookrightarrow S^n$ *unknotted* if its image bounds a ball (see Definition 7.1). The distinction between unknots in S^n and unknotted embeddings $S^{n-2} \hookrightarrow S^n$ is a subtle one, related to whether knots are defined as embeddings or their images (see Remark 2.1).

There are other unknot detection results in high dimensions in the literature. For instance, it was shown in [Sha68, Lev65b, Lev70, Tro73] that if the exterior of a knot in S^n, for $n \geq 5$, is homotopy equivalent to a circle, then it is unknotted. It remains an open question whether a 2-knot in S^4 with infinite cyclic fundamental group of the complement is (smoothly) unknotted. Such knots are known to be topologically unknotted [FQ90, Theorem 11.7A].

The key observation in the proof of Theorem 7.2 and Theorem E is that under their hypotheses, the exterior of the knot K is diffeomorphic to the exterior of U, i.e. $D^{n-1} \times S^1$. This uses the fact that the fundamental group of the surgery along an unknotted embedding is isomorphic to $\mathbb{Z}$ and thus has exactly two possible generators. The proof of Theorem 7.2 is completed for $n \geq 5$ by the results mentioned in the previous paragraph. For $n = 4$ we appeal to [Glu62], where it was shown that if two 2-knots have diffeomorphic exteriors, then either they are isotopic or one is isotopic to the Gluck twist of the other.

1.6. **Structure.** In Section 2, we first present the background material that will be required later on. Section 3 then outlines the construction of RBG manifolds, before Section 4 focuses on knot invariants. Together, these tools allow us to complete the proof of Theorem A. In Sections 5 and 6, we first prove Theorems C and D, then Theorem B. Section 7 contains the proof of Theorem E. Section 8 concludes the paper with a discussion of open questions.

1.7. **Conventions.** Unless stated otherwise, all manifolds are assumed to be oriented, connected, compact and smooth. Maps between such manifolds are assumed to be smooth. Diffeomorphisms are assumed to be orientation-preserving and denoted by $\cong$. In Sections 5 and 6, we will in some places also be concerned with orientation-preserving homeomorphisms, denoted by $\approx$. Homology groups $H_*(\cdot)$ without specified coefficient rings are understood to have $\mathbb{Z}$ coefficients.

The oriented diffeomorphism type of the surgery or trace of a knot K is independent of the orientation of K. Thus, all results in the introduction are meant to be understood as results for unoriented knots. However, for our proofs, the orientations are helpful to precisely describe certain constructions. So in the rest of the text, we will assume all our knots to be oriented.

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2. Background on knotted spheres

The results presented in this section are well known in the field. However, some of them are difficult to find in the literature, so we provide further discussion. Moreover, we will fix some notation and conventions.

2.1. **Knotted spheres.** Let M^n be an (oriented, connected, compact, smooth) n-manifold with $n \geq 3$ and suppose that $1 \leq k \leq n$. A *knotted (k-)sphere in M* is a smooth embedding $\Sigma\colon S^k \hookrightarrow M$, where S^k is endowed with the standard orientation. We will be particularly interested in *knots*, which correspond to the case $k = n-2$, i.e. a *knot in M* is a smooth embedding $K\colon S^{n-2} \hookrightarrow M^n$. We will study knotted spheres (in particular, knots) up to *isotopy* or equivalently *ambient isotopy*. See also Section 2.6.[3]

The image of an embedding $\Sigma\colon S^k \hookrightarrow M$ defines an oriented smooth submanifold $\Sigma(S^k)$ of M of codimension $n-k$. There is a unique closed tubular neighbourhood (up to isotopy of tubular neighbourhoods) of Σ in M [Hir76], denoted by $\nu\Sigma$. We will usually think of $\nu\Sigma$ as (the isotopy class of) a submanifold of M. We will explicitly stress when we need to keep track of a parametrisation of $\nu\Sigma$, and for this reason we will discuss framings in a subsequent section. We denote the *exterior* of Σ by $E_\Sigma := M \setminus \mathring{\nu}\Sigma$, where $\mathring{\nu}\Sigma$ denotes the interior of $\nu\Sigma$.

The *unknot U* in S^n is the (isotopy class of the) embedding $S^{n-2} \hookrightarrow S^n$ which extends to an embedding $D^{n-1} \hookrightarrow S^n$. By the disc theorem [Pal60], equivalently we could define the unknot to be the (isotopy class of the) embedding given by the composition of standard inclusions $S^{n-2} \hookrightarrow \mathbb{R}^{n-1} \hookrightarrow S^{n-1} \hookrightarrow S^n$.

Remark 2.1. Note that there is a subtle difference between the notions of knotted spheres as we defined them, i.e. smooth embeddings $\Sigma\colon S^k \hookrightarrow M^n$, and oriented embedded spheres $\Sigma(S^k) \subseteq M^n$. This difference arises due to the fact that the mapping class group $\mathrm{MCG}(S^k)$ is not trivial in general [Mil56], so we can have different embeddings with the same image that are not a priori isotopic. For $k \in \{1,2,3\}$, since the mapping class group of S^k is trivial [Sma59, Cer68], this distinction does not arise. More precisely, for $k \in \{1,2,3\}$, studying parametrised embeddings of S^k is equivalent to studying oriented embedded k-spheres. When $k \geq 4$, the two theories coincide exactly when S^{k+1} has a unique smooth structure. Indeed, by the h-cobordism theorem [Sma62], it is known that every exotic smooth structure on S^{k+1}, for $k+1 \geq 5$, arises from a non-trivial element in $\mathrm{MCG}(S^k)$, corresponding to the gluing map of the two hemispheres of S^{k+1} along the equator. See for example [Kos93, Chapter VIII] for details. It is known that S^{k+1} has a unique smooth structure for $k \in \{0,1,2,4,5,11,55,60\}$ and it is conjectured that these are the only values of k for which this is true, see for example [BHHM20] and [KM63].

As a consequence, we define the unknot as above, and not as an embedding of a sphere whose *image* bounds an $(n-1)$-ball as a submanifold. We will call the latter notion an *unknotted* sphere, see Definition 7.1. The unknot defines an unknotted embedding, but the converse is generally not true.

Of course, many invariants associated to a knotted sphere, such as its exterior or invariants derived from it, do not depend on the parametrisation. On the other hand, we will need to work with our definition using concrete embeddings in order for the notions of surgery and trace (see below) along a knotted sphere Σ to be well-defined. For the construction in Sections 3.3 and 3.4, we will first define embedded spheres and later parametrise them.

[3] Recall that the isotopy extension theorem implies that two knotted spheres are ambient isotopic if and only if they are isotopic.

2.2. **Framings, surgeries and traces.** For $n \geq 3$ and $1 \leq k \leq n$, let M^n be an n-manifold as above and let $\Sigma\colon S^k \hookrightarrow M^n$ be a knotted sphere. A *framing* of Σ is a trivialisation of the normal bundle of $\Sigma(S^k)$, i.e. an isomorphism f of oriented vector bundles from $TM/T\Sigma(S^k)$ to $\Sigma(S^k) \times \mathbb{R}^{n-k}$. The pair (Σ, f) is called a *framed knotted sphere*, or, in case of codimension $n-k=2$, a *framed knot*.

A framed knotted sphere $(\Sigma\colon S^k \hookrightarrow M^n, f)$ determines the isotopy class of an embedding $\hat{\Sigma}\colon S^k \times D^{n-k} \hookrightarrow M$. The result of *surgery on M along* (Σ, f) is the oriented, connected, compact, smooth n-manifold

$$M_f(\Sigma) := (M \setminus \operatorname{int} \hat{\Sigma}(S^k \times D^{n-k})) \cup_\partial (D^{k+1} \times S^{n-k-1})$$

with gluing map $\hat{\Sigma}|_{S^k \times S^{n-k-1}}$. This manifold is well-defined since, up to orientation-preserving diffeomorphism, it is independent of the choices involved. In the above description of $M_f(\Sigma)$, there is a preferred embedding of $D^{k+1} \times S^{n-k-1}$, which we can view as knotted sphere $\{0\} \times S^{n-k-1}$ with a preferred framing. If we surger on that framed knotted sphere, we recover M; see e.g. [GS99, Section 5.2]. This procedure is often called *reversing the surgery*. For $k = n-2$, we will also call it *loop surgery* and $\gamma_\Sigma := \{0\} \times S^1 \subseteq M_f(\Sigma)$ the *dual curve* of (Σ, f).

We will also use the following notation. Let W be an $(n+1)$-manifold and let (Σ, f) be a framed knotted k-sphere in ∂W. Then we denote by

$$W \cup h_{k+1}(\Sigma)$$

the $(n+1)$-manifold that is obtained from W by attaching an $(n+1)$-dimensional $(k+1)$-handle $h_{k+1} = D^{k+1} \times D^{n-k}$ along the framed k-sphere Σ. We denote the belt sphere of $h_{k+1}(\Sigma)$ by $\mathcal{B}_\Sigma$. Note that $W \cup h_{k+1}(\Sigma)$ inherits a canonical orientation from the orientation of W and (after smoothing corners) a preferred smooth structure up to diffeomorphism; see [GS99, Sections 1.3 and 4.1] for details. We can and we will usually assume that handles are attached in (weakly) increasing order of index; see [GS99, Proposition 4.2.7].

With this notation at hand, the *trace* of a framed knotted k-sphere (Σ, f) in S^n is the oriented, connected, compact, smooth $(n+1)$-manifold (again well-defined up to orientation-preserving diffeomorphism)

$$X_f(\Sigma) := D^{n+1} \cup h_{k+1}(\Sigma).$$

Note that $\partial X_f(\Sigma) = S^n_f(\Sigma)$. As the notation suggests, the oriented diffeomorphism type of the surgery or trace depends on the framed knotted sphere (Σ, f), but it is independent of the orientation of Σ.

Remark 2.2. The existence of a trivial normal bundle of a knotted sphere $\Sigma\colon S^k \hookrightarrow M^n$ and its possible framings are determined by algebraic topology data as follows. Orientable vector bundles of rank $n-k$ over S^k, and thus the possible normal bundles of Σ, are classified by $\pi_{k-1}(\mathrm{SO}(n-k))$; meanwhile, the possible framings of the trivial bundle of rank $n-k$ over S^k are classified by $\pi_k(\mathrm{SO}(n-k))$ (see e.g. [GS99, Chapter 4]).

We will mostly be interested in knotted spheres of codimension $n-k=2$, i.e. knots. As a consequence of the above classification, knots always have a trivial normal bundle if the ambient manifold has dimension $n \neq 4$, since $\pi_{n-3}(\mathrm{SO}(2))$ is trivial if and only if $n \neq 4$. For $n = 4$, an embedded sphere $\Sigma(S^2)$ in M has a trivial normal bundle if and only if its homological self intersection number $\Sigma \cdot \Sigma$ vanishes. The latter is automatically fulfilled if $H_2(M)$ vanishes; for example, if $M = S^4$. Since $\pi_{n-2}(\mathrm{SO}(2))$ is trivial if and only if $n \neq 3$, a knot $K\colon S^{n-2} \hookrightarrow M^n$ with trivial normal bundle has a *unique* framing (up to isotopy) if and only if $n \neq 3$. For $n = 3$, the framings are in bijection with the integers; see [GS99, Section 4.5] for further discussion.

Convention: Whenever there is a unique choice of framing for a knotted sphere Σ, for example if it is a knot in an ambient manifold M^n of dimension $n \neq 3$ as in Remark 2.2, we will just omit the framing f from the notation of (Σ, f), $M_f(\Sigma)$ and $X_f(\Sigma)$.

2.3. **Tubing.** For $n \geq 4$, let K_1 and K_2 be knots in an n-manifold M. Suppose that $\beta\colon D^1 \hookrightarrow M$ is an arc whose endpoints lie on $K_1(S^{n-2})$ and $K_2(S^{n-2})$, and whose image is otherwise disjoint from the image of K_1 and K_2. Thicken β to an embedding $h_\beta\colon D^1 \times D^{n-2} \hookrightarrow M$ such that the image of $\partial D^1 \times D^{n-2}$ lies on $K_1(S^{n-2})$ and $K_2(S^{n-2})$. Now, modify K_1 and K_2 by removing the pair of balls $h_\beta(\partial D^1 \times D^{n-2})$ from their images and gluing in the annulus $h_\beta(D^1 \times \partial D^{n-2})$. This operation is called *tubing K_1 and K_2 along β*. We denote the resulting submanifold of M by $K_1 \#_\beta K_2$. We will always choose the embedding h_β such that $K_1 \#_\beta K_2$ has an orientation that restricts to the one of $K_1(S^{n-2})$ and $K_2(S^{n-2})$. Notice that such thickenings h_β, as submanifolds, are parametrised by isotopy classes of paths with fixed endpoints in $\mathrm{Gr}^+_{n-2}(\mathbb{R}^{n-1})$, the Grassmannian of oriented $(n-2)$-planes in $\mathbb{R}^{n-1}$, using uniqueness of tubular neighbourhoods. Since $\mathrm{Gr}^+_{n-2}(\mathbb{R}^{n-1})$ is simply connected for $n \geq 4$, it follows that there is a unique choice of thickening h_β up to isotopy. So if we consider tubing as an operation on oriented submanifolds, and this is the case we will be interested in, then the resulting knot is unique up to ambient isotopy (as a submanifold) and only depends on the isotopy class of the arc β rel boundary.

If one considers tubing as an operation on parametrised knots, i.e. if one wants to define a new embedding $S^{n-2} \hookrightarrow S^n$, then there are a priori two ways to tube K_1 and K_2 along β. See e.g. [Gab20, Remark 5.3], where this operation of tubing is discussed for surfaces in 4-manifolds.

Note that the above described operation is usually called *band sum* in dimension $n = 3$. In this case, we have to be careful to specify the embedding h_β, for which we have infinitely many choices parametrised by $\mathbb{Z}$.

2.4. **Meridians and the knot group.** Let $K\colon S^{n-2} \hookrightarrow M^n$ be a knot in M with trivial normal bundle. The *meridian* of K is an oriented, simple, closed curve μ_K in M that is isotopic to the oriented boundary $\{\mathrm{pt}\} \times \partial D^2$ of a transverse disc $\{\mathrm{pt}\} \times D^2$ in the tubular neighbourhood $\nu K \cong S^{n-2} \times D^2$ in M. Sometimes we will also denote by μ_K the meridian seen as a curve in the knot exterior E_K. By abusing notation, we will also write μ_K for the conjugacy class of the push-forward of the meridian of K into the *knot group* $\pi_1(K) := \pi_1(E_K)$. Note that the knot group (but not its isomorphism type) depends on the choice of a base point $x_0 \in M$. In general, the meridian μ_K is not a curve based at x_0. So to see it as an element of $\pi_1(K)$, we choose a path w from x_0 to the base point of the curve μ_K. Different choices of w lead to a conjugation, so the conjugacy class of μ_K in the knot group is well-defined.

2.5. **Ribbon knots.** A knot $K\colon S^{n-2} \hookrightarrow S^n$ is *ribbon* if it bounds an immersed disc $D\colon D^{n-1} \looparrowright S^n$ with only ribbon singularities. This means that the components of the singular set of D are $(n-2)$-discs whose boundary $(n-3)$-spheres either lie on $D(\partial D^{n-1})$ or are disjoint from it. We call D a *ribbon disc*. Pushing the interior of $D(D^{n-1})$ appropriately into the ball D^{n+1} bounded by S^n produces an embedded disc $D^{n-1} \hookrightarrow D^{n+1}$ with boundary K, which we call an *embedded ribbon disc* and, by slight abuse of notation, also denote by D. Forming the *double* of this embedded ribbon disc $D\colon D^{n-1} \hookrightarrow D^{n+1}$, i.e. the union of two copies of D in D^{n+1} glued along their boundaries using the identity map, gives rise to a ribbon $(n-1)$-knot in S^{n+1}; see for example [CKS04, Section 2.2.1].

2.6. **Equivalence relations on knots.** For classical knots, i.e. knots in S^3, it is well known that (orientation-preserving) diffeomorphisms of pairs and (ambient) isotopies are equivalent notions. The harder direction of this equivalence follows from Cerf's result [Cer68] that the mapping class group of S^3 is trivial. However, the mapping class groups of n-spheres for $n \geq 6$ are generally non-trivial [Mil56], and it is an open question whether the mapping class group of S^4 is trivial. Nonetheless, the two notions of equivalence of knots still agree in higher dimensions, as we show next. We first define precisely which notions we consider.

Definition 2.3. Let $n \geq 3$ and $1 \leq k \leq n$. Two framed knotted k-spheres (Σ_1, f_1) and (Σ_2, f_2) in an n-manifold M^n are *equivalent* if there exists an orientation-preserving diffeomorphism $h\colon M \to M$ such that $h \circ \Sigma_1 = \Sigma_2$ and the following diagram commutes:

$$\begin{tikzcd} TM/T\Sigma_1(S^k) \arrow[r, "f_1"] \arrow[d, "\mathrm{T}h"] & \Sigma_1(S^k) \times \mathbb{R}^{n-k} \arrow[d, "h \times \mathrm{id}"] \\ TM/T\Sigma_2(S^k) \arrow[r, "f_2"] & \Sigma_2(S^k) \times \mathbb{R}^{n-k}. \end{tikzcd}$$

We will call such a map h an *equivalence* between (Σ_1, f_1) and (Σ_2, f_2).

Note that the term "equivalence" is also used slightly differently in the literature when knots are defined as submanifolds.

Definition 2.4. Let $n \geq 3$ and $1 \leq k \leq n$. Two framed knotted k-spheres (Σ_1, f_1) and (Σ_2, f_2) in an n-manifold M^n are *ambient isotopic* if there exists an equivalence between (Σ_1, f_1) and (Σ_2, f_2) which is isotopic to id_M.

The following lemma is a standard result for unframed spheres, which we found discussed at [Bud24]. We provide a proof in the setting of framed knotted spheres, as this is what we will need presently.

Lemma 2.5. *Let $n \geq 3$ and $1 \leq k \leq n$. Two framed knotted k-spheres (Σ_1, f_1) and (Σ_2, f_2) in S^n are equivalent if and only if they are ambient isotopic.*

Proof. Observe that there is only one non-trivial implication, so we suppose that the framed knotted spheres (Σ_1, f_1) and (Σ_2, f_2) are equivalent via a diffeomorphism $h\colon S^n \to S^n$. Think of S^n as the union of two n-balls D^n_N and D^n_S glued along their boundary. Up to ambient isotopy of framed knotted spheres, the images of both Σ_1 and Σ_2 can be assumed to be contained in D^n_S. The diffeomorphism h can be isotoped to restrict to the identity on D^n_N. Let $r_t\colon S^n \to S^n$, $t \in [0,1]$, be a 1-parameter family of rotations such that $r_0 = \mathrm{id}_{S^n}$ and r_1 swaps D^n_S and D^n_N. We now consider the family of framed knotted spheres $(\widetilde{\Sigma}_t, \widetilde{f}_t)$ where $\widetilde{\Sigma}_t = h \circ r_t \circ \Sigma_1$ and the framing $\widetilde{f}_t$ is induced by the framing f_1 via $h \circ r_t$. By construction, these framed knotted spheres are all ambient isotopic. Moreover, $(\widetilde{\Sigma}_0, \widetilde{f}_0) = (\Sigma_2, f_2)$ and $(\widetilde{\Sigma}_1, \widetilde{f}_1)$ is just $r_1 \circ \Sigma_1$ with the framing induced by r_1, since h is the identity on D^n_N. Clearly, (Σ_1, f_1) is ambient isotopic to $(\widetilde{\Sigma}_1, \widetilde{f}_1)$, thus to (Σ_2, f_2). □

Since there is a unique framing for knots with trivial normal bundle in M^n for $n \geq 4$ (see Remark 2.2), we will often not distinguish between knots and framed knots; in particular, we will sometimes speak about isotopy rather than ambient isotopy of framed knots.

2.7. **Dimensions and codimensions of knotted spheres.** The Schoenflies problem posits that any embedded $(n-1)$-sphere Σ in S^n (i.e. of codimension 1) manifests as the equator, i.e. there is a diffeomorphism of pairs (S^n, Σ) and (S^n, S^{n-1}). This holds for any $n \leq 3$ or $n \geq 5$, the latter as an application of the h-cobordism theorem in these dimensions due to Smale [Sma62]; see [Mil65, Section 9].[4] Isotopy classes of embeddings $S^k \hookrightarrow S^n$ of codimension $n-k=1$ are thus only interesting for $n=4$. We will not address this difficult remaining case of the Schoenflies problem here.

On the other hand, embeddings of high codimensions $n-k \geq 3$ are well-understood. In particular, isotopy classes of embeddings $S^k \hookrightarrow S^n$ together with the connected sum form a group in codimension at least 3. For example, any two smooth embeddings $S^1 \hookrightarrow S^n$ are isotopic; see e.g. [Hir76]. In the piecewise linear and topological categories, Zeeman [Zee63] and Stallings [Sta63] proved that embeddings $S^k \hookrightarrow S^n$ of codimension $n-k \geq 3$ are unknotted (cf. Definition 7.1). However, in the smooth category, knotting can happen even in codimension ≥ 3 as shown by Haefliger [Hae62]. Still, smooth embeddings $S^k \hookrightarrow S^n$ with $n-k \geq 3$ were classified by Haefliger [Hae66] and Levine [Lev65a]. Indeed, the former [Hae61] showed that the group of isotopy classes of smooth knotted k-spheres in S^n is trivial when $2n > 3(k+1)$, but non-trivial when $2n = 3(k+1)$.

This leaves the case of knotted spheres of codimension 2, for which no simple classification is known. In particular, the set of isotopy classes of codimension 2 embeddings does not form a group under connected sum. The rich theory of classical knots $S^1 \hookrightarrow S^3$ illustrates the complexity of this setting. In this article, we address some of the questions that arise when we broaden our focus to $n \geq 4$.

3. Constructing knotted spheres with the same trace

3.1. **The generalised RBG construction.** In this section, we describe a general method to construct pairs of knotted spheres with orientation-preservingly diffeomorphic traces. As mentioned in the introduction, this construction is motivated by similar 3-dimensional constructions and can be thought of as a generalisation of these. In particular, the RBG link constructions from [Pic19, Pic20] are relevant,

[4] The Schoenflies problem for topological, locally flat embeddings was proved by Brown [Bro60] in any dimension $n \geq 1$.

while our notation is inspired by [MP23, Tag24]. Other constructions for creating knots that share a surgery or trace in dimension 3 can be found for example in [Lic76, Akb77, Bra80, Liv82, Mot88, GL89a, Oso06, Ter07, AJOT13, AJLO15, Yas15, MP18, McC19, AT21].

Definition 3.1. We call an $(n+1)$-manifold W an *RBG manifold* if W comes equipped with a handle decomposition of the form

$$W \cong D^{n+1} \cup h_k(R) \cup h_{k+1}(B) \cup h_{k+1}(G),$$

for some $1 \leq k \leq n$, where R is a framed knotted $(k-1)$-sphere in ∂D^{n+1} and B, G are framed knotted k-spheres in $\partial(D^{n+1} \cup h_k(R))$ such that there exists

- (i) an isotopy of $B(S^k)$ in $\partial(D^{n+1} \cup h_k(R))$ after which it intersects the belt sphere $\mathcal{B}_R$ of $h_k(R)$ transversely in a single point;
- (ii) an isotopy of $G(S^k)$ in $\partial(D^{n+1} \cup h_k(R))$ after which it intersects $\mathcal{B}_R$ transversely in a single point.

Note that in Definition 3.1 we do not assume that $B(S^k)$ and $G(S^k)$ *simultaneously* intersect $\mathcal{B}_R$ transversely in a single point.

Any $(n+1)$-dimensional RBG manifold yields a pair of (not necessarily non-isotopic) knotted k-spheres in S^n that have orientation-preservingly diffeomorphic traces, as follows.

Proposition 3.2. *To any $(n+1)$-dimensional RBG manifold W as in Definition 3.1, we can assign the ambient isotopy classes of two framed k-spheres Σ_B and Σ_G in S^n whose traces are both orientation-preservingly diffeomorphic to W and thus to each other.*

Proof. By hypothesis, after an isotopy, the image of B intersects $\mathcal{B}_R$ transversely in a single point. Thus the handles $h_k(R)$ and $h_{k+1}(B)$ cancel. The same holds true if we replace B by G. We thus obtain diffeomorphisms

$$\begin{aligned} f_G &\colon D^{n+1} \cup h_k(R) \cup h_{k+1}(B) \longrightarrow D^{n+1}, \\ f_B &\colon D^{n+1} \cup h_k(R) \cup h_{k+1}(G) \longrightarrow D^{n+1}. \end{aligned} \tag{3.1}$$

Now, we define the framed knotted k-spheres Σ_B and Σ_G in $\partial D^{n+1} = S^n$ as the compositions of B and G with the diffeomorphisms f_B and f_G, i.e.

$$\Sigma_B := f_B \circ B \quad \text{and} \quad \Sigma_G := f_G \circ G$$

with framings induced by those of B and G. The diffeomorphisms f_B and f_G, and hence Σ_B and Σ_G, a priori depend on the choice of the isotopies of B and G and the exact cancellation of the handles. However, we now show that, up to ambient isotopy, Σ_B and Σ_G are independent of these choices. We describe the argument for Σ_G; the same works for Σ_B. Let

$$f_G,\, f_G' \colon D^{n+1} \cup h_k(R) \cup h_{k+1}(B) \longrightarrow D^{n+1}$$

be two diffeomorphisms as above yielding knotted spheres Σ_G and Σ_G'. Restricting the composition $f_G \circ (f_G')^{-1}$ to ∂D^{n+1}, we obtain a diffeomorphism $S^n \to S^n$ which gives rise to an equivalence between the (framed) knotted spheres $\Sigma_G' = f_G' \circ G$

and $\Sigma_G = f_G \circ G$ (see Definition 2.3). Thus Σ_G and Σ'_G are ambient isotopic by Lemma 2.5.

Finally, by construction, the traces of Σ_B and Σ_G are both orientation-preservingly diffeomorphic to W. Thus Σ_B and Σ_G share the same trace. $\square$

3.2. **Trivial cases.** In the construction from Section 3.1, it might very well happen that the framed knotted k-spheres Σ_B and Σ_G in S^n are actually ambient isotopic. Indeed, this will happen if the RBG manifold W is too simple or symmetric, or for certain ranges of k. For example, Σ_B and Σ_G are certainly isotopic when they arise as embeddings $S^k \hookrightarrow S^n$ for $2n > 3(k+1)$ [Hae61]. See also the discussion in Section 2.7. Another common situation that arises is as follows.

Lemma 3.3. *For $n \geq 4$, let W be an $(n+1)$-dimensional RBG manifold such that (possibly after an isotopy) the k-spheres $B(S^k)$ and $G(S^k)$ simultaneously intersect the belt sphere $\mathcal{B}_R$ transversely in a single point. Then Σ_B and Σ_G are ambient isotopic as unoriented submanifolds.*

Proof. If $B(S^k)$ and $G(S^k)$ simultaneously intersect $\mathcal{B}_R$ transversely in a single point, then we can choose an arc β connecting this pair of intersection points which lies entirely in $\mathcal{B}_R$. Take a tubular neighbourhood of β in ∂W of the form $D^k \times [0,1] \times D^{n-k-1}$, where $[0,1] \times D^{n-k-1} \subset \mathcal{B}_R$. This is possible since $\mathcal{B}_R$ is $(n-k)$-dimensional. Consider the "tube" $A = \partial D^k \times [0,1] \times \{0\}$. Take a push-off of the attaching sphere $G(S^k)$ and perform a handle slide over $h_{k+1}(B)$ guided by A. This yields a k-sphere S which we consider as an unoriented submanifold. The image of S under the diffeomorphism f_G which cancels $h_{k+1}(B)$ and $h_k(R)$ (see (3.1)) is isotopic, as an unoriented submanifold of S^n, to Σ_G.

We now reverse the roles of B and G and apply the same argument. The push-off of B that we take matches the one which implicitly appeared in our earlier handle slide; similarly, the handle slide of this sphere over $h_{k+1}(G)$ implicitly uses the same push-off of G that we started with. Since we also use the same arc β for the handle slide, the resulting k-sphere is again S. We deduce as before that the image of S under f_B is isotopic, as unoriented submanifold of S^n, to Σ_B.

In summary, the diffeomorphism $f_G \circ f_B^{-1}$, restricted to ∂D^{n+1}, sends $f_B(S)$ to $f_G(S)$ and thus can be used to define an equivalence of unoriented submanifolds between Σ_B and Σ_G, which are isotopic to $f_B(S)$ to $f_G(S)$, respectively. We can promote it to an ambient isotopy using Lemma 2.5. $\square$

However, there is no reason to expect that Σ_B and Σ_G are isotopic in general.

3.3. **Constructing RBG manifolds with $\boldsymbol{k = n-2}$.** The most interesting case in the construction of framed knotted k-spheres in S^n from $(n+1)$-dimensional RBG manifolds W is that of codimension 2, i.e. when $k = n-2$; see the discussion in Sections 2.7 and 3.2. In this case, W has a handle decomposition of the form

$$W = D^{n+1} \cup h_{n-2}(R) \cup h_{n-1}(B) \cup h_{n-1}(G). \tag{3.2}$$

The challenge here is to construct a sufficiently complicated RBG manifold W such that the associated framed knotted k-spheres, in this case (framed) knots $K_B := \Sigma_B$ and $K_G := \Sigma_G$ in S^n, are non-isotopic. At the same time, W should be simple enough to allow us to effectively compute invariants that distinguish K_B and K_G. In the following, we present a method to construct interesting RBG manifolds as in (3.2), where we expect K_B and K_G to usually be non-isotopic in S^n (see also Remark 3.6). In Section 3.4, we will use this general method to explicitly construct RBG manifolds with non-isotopic associated knots. Throughout this section, let $n \geq 4$.

3.3.1. *Construction of R.* Let R be the embedding induced by the standard inclusions $S^{n-3} \hookrightarrow \mathbb{R}^{n-2} \hookrightarrow \mathbb{R}^n \hookrightarrow S^n$. We choose the framing for R for which we obtain $D^{n+1} \cup h_{n-2}(R) \cong D^3 \times S^{n-2}$ with boundary $\partial(D^{n+1} \cup h_{n-2}(R)) \cong S^2 \times S^{n-2}$. Fix a point $z \in S^{n-2}$ and write $\mathcal{B}_R = S^2 \times \{z\} \subseteq S^2 \times S^{n-2}$ for the belt sphere of $h_{n-2}(R)$.

Next, we will construct knots B and G in $S^2 \times S^{n-2}$, so that after an isotopy, the image of each of B and G is *geometrically dual* to $\mathcal{B}_R$, i.e. after isotopies, $B(S^{n-2})$ and $\mathcal{B}_R$, or $G(S^{n-2})$ and $\mathcal{B}_R$, respectively, intersect transversely in a single point. We will first define B and G as embedded submanifolds of $\partial(D^{n+1} \cup h_{n-2}(R)) \cong S^2 \times S^{n-2}$, and then choose an embedding representing these submanifolds (see also Remark 2.1). Since there is a unique choice of framing for the spheres involved (see Remark 2.2), we will usually just omit the framing from the discussion.

For $\ell \geq 1$, choose $2\ell + 2$ distinct points $w, x_1, \ldots, x_{2\ell+1}$ in S^2 and consider the $(n-2)$-spheres $\{w\} \times S^{n-2}$ and $\{x_i\} \times S^{n-2}$, for $i \in \{1, \ldots, 2\ell+1\}$, in $S^2 \times S^{n-2}$. We will describe how to construct B and G by modifying these spheres.

3.3.2. *Construction of the knot B in $S^2 \times S^{n-2}$.* To construct B, we will add a local knot to $\{w\} \times S^{n-2}$; see Figure 1 for a schematic. More precisely, let K be a knot in S^n. The sphere B is obtained as the pairwise connected sum

$$(S^2 \times S^{n-2}, B) := (S^2 \times S^{n-2}, \{w\} \times S^{n-2}) \#(S^n, K).$$

Note that we perform the connected sum away from the spheres $\{x_i\} \times S^{n-2}$, $i \in \{1, \ldots, 2\ell+1\}$, and away from $\mathcal{B}_R = S^2 \times \{z\}$.

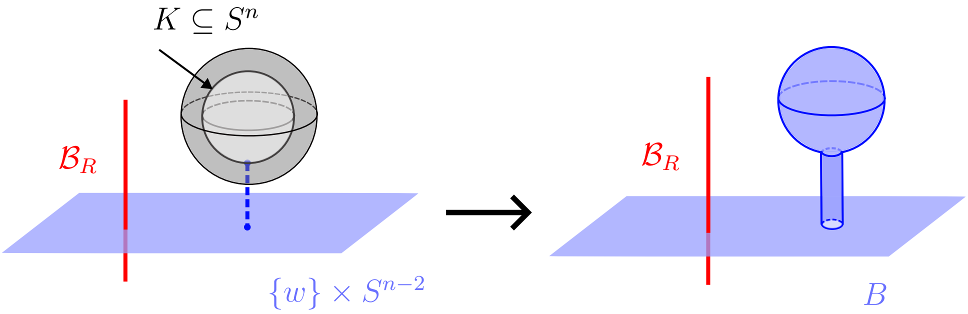

FIGURE 1. Construction of the knot B in $S^2 \times S^{n-2}$ as connected sum of a local knot K and $\{w\} \times S^{n-2}$. The belt sphere $\mathcal{B}_R = S^2 \times \{z\}$ is in red.

3.3.3. *Construction of the knot G in $S^2 \times S^{n-2}$.* To construct G, we begin by tubing together the spheres $\{x_i\} \times S^{n-2}$, $i \in \{1, \ldots, 2\ell+1\}$, along standard arcs. More precisely, choose orientations of these spheres such that all $\{x_i\} \times S^{n-2}$ for odd $i \in \{1, \ldots, 2\ell+1\}$ have the same orientation and such that the orientation of each $\{x_i\} \times S^{n-2}$ for even $i \in \{1, \ldots, 2\ell+1\}$ differs from that of $\{x_1\} \times S^{n-2}$.

Now, we take a collection of 2ℓ arcs $\beta_i\colon D^1 \hookrightarrow S^2 \times S^{n-2}$, $i \in \{1, \dots, 2\ell\}$, such that, for each i, the two endpoints of β_i lie on $\{x_i\} \times S^{n-2}$ and $\{x_{i+1}\} \times S^{n-2}$, respectively. These arcs can be chosen "straight" from $\{x_i\} \times S^{n-2}$ to $\{x_{i+1}\} \times S^{n-2}$, disjoint from each other and disjoint from B and $\mathcal{B}_R$. For example, we can choose 2ℓ arcs in S^2 that connect the points x_i and x_{i+1} in S^2 for $i \in \{1, \dots, 2\ell\}$, and use disjoint push-offs of these to define the arcs β_i in $S^2 \times S^{n-2}$ (see Figure 2). Tubing the spheres $\{x_i\} \times S^{n-2}$, $i \in \{1, \dots, 2\ell+1\}$, along the arcs β_i, $i \in \{1, \dots, 2\ell\}$, as explained in Section 2.3, we obtain an $(n-2)$-sphere $\widetilde{G}$ which is disjoint from B and intersects $\mathcal{B}_R = S^2 \times \{z\}$ transversely in $2\ell + 1$ points, but algebraically only once.

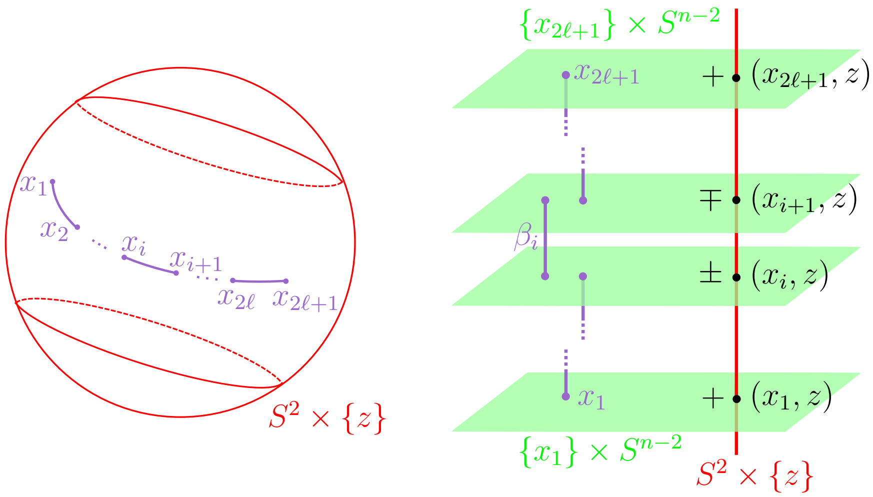

FIGURE 2. Choice of arcs in $S^2 \times \{z\}$ that give rise to "straight" arcs $\beta_1, \dots, \beta_{2\ell}$ in $S^2 \times S^{n-2}$. The knot $\widetilde{G}$ in $S^2 \times S^{n-2}$ is obtained by tubing $\bigcup \{x_i\} \times S^{n-2}$ along the arcs β_i.

Next, we will modify the collection of arcs $\{\beta_i\}$ in $S^2 \times S^{n-2}$ into a new collection of arcs $\{\beta_i'\colon D^1 \hookrightarrow S^2 \times S^{n-2}\}$ by linking them with B. Explicitly, choose 2ℓ disjoint embeddings $\gamma_i\colon S^1 \hookrightarrow S^n \setminus K$, $i \in \{1, \dots, 2\ell\}$, where K is the local knot used to construct B (see Construction 3.3.2). The embeddings γ_i, $i \in \{1, \dots, 2\ell\}$, are all contained in S^n and as such can be assumed to be disjoint from B in $S^2 \times S^{n-2} = S^2 \times S^{n-2} \# S^n$, and also disjoint from $\{x_i\} \times S^{n-2}$, $i \in \{1, \dots, 2\ell+1\}$. For each $i \in \{1, \dots, 2\ell\}$, modify the arc β_i by performing a band sum with γ_i along any arc δ_i disjoint from the $\{x_i\} \times S^{n-2}$ and B to obtain an arc $\beta_i' := \beta_i \#_{\delta_i} \gamma_i$ (see Figure 3). We define G to be the sphere obtained by tubing the spheres $\{x_i\} \times S^{n-2}$ along the arcs β_i'. Note that G a priori depends on the choices of the loops γ_i and the arcs δ_i. However, the next lemma shows that, up to isotopy in $S^2 \times S^{n-2}$ (i.e. ignoring B), it does not.

Lemma 3.4. *The spheres G and $\widetilde{G}$ are both isotopic to $\{x_1\} \times S^{n-2}$ in $S^2 \times S^{n-2}$. In particular, up to isotopy in $S^2 \times S^{n-2}$, the sphere G is independent of the choices of the loops γ_i and arcs δ_i.*

Proof. First, we show that $\widetilde{G}$ is isotopic to $\{x_1\} \times S^{n-2}$ in $S^2 \times S^{n-2}$. To that end, consider $\beta_{2\ell} \times S^{n-2}$. Observe that its boundary $(\partial \beta_{2\ell}) \times S^{n-2}$ is given by the disjoint union of $\{x_{2\ell}\} \times S^{n-2}$ and $\{x_{2\ell+1}\} \times S^{n-2}$, and that the complement of an open tubular neighbourhood of $\beta_{2\ell}$ in $\beta_{2\ell} \times S^{n-2}$ is diffeomorphic to an $(n-1)$-disc D

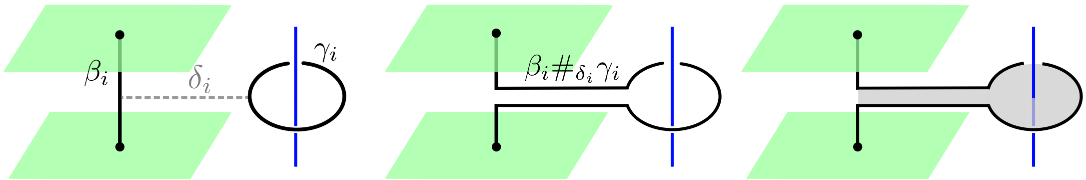

FIGURE 3. Band sum $\beta_i\#_{\delta_i}\gamma_i$ of the arc β_i and $\gamma_i\colon S^1 \hookrightarrow S^n \setminus K$ along an arc δ_i. On the right, the region giving an isotopy between $\beta_i\#_{\delta_i}\gamma_i$ and β_i in $S^2 \times S^{n-2}$ is shaded.

after smoothing corners. We can isotope $\widetilde{G}$ over D to the sphere obtained by tubing the $(2\ell-1)$ spheres $\{x_1\} \times S^{n-2}, \ldots, \{x_{2\ell-1}\} \times S^{n-2}$ along the arcs $\beta_1, \ldots, \beta_{2\ell-2}$. See Figure 4 for a schematic. By induction we get an isotopy from $\widetilde{G}$ to $\{x_1\} \times S^{n-2}$.

Now, note that the embeddings γ_i, $i \in \{1, \ldots, 2\ell\}$, extend to disjoint embeddings $(D^2, S^1) \hookrightarrow (S^n, S^n \setminus K)$ by general position when $n \geq 5$ and by using a small 2-sphere linking the boundary S^1 to tube away self intersections when $n = 4$, for details see [Nor69].[5] We can use these discs bounded by the γ_i and the arc δ_i to define, for each $i \in \{1, \ldots, 2\ell\}$, an isotopy rel boundary in the complement of the spheres $\{x_1\} \times S^{n-2}, \ldots, \{x_{2\ell+1}\} \times S^{n-2}$ in $S^2 \times S^{n-2}$ between the arcs β_i and $\beta_i\#_{\delta_i}\gamma_i$ (see Figure 3). Since isotopies of the arcs rel boundary induce an isotopy between the tubed surfaces, the sphere G is isotopic to $\widetilde{G}$ in $S^2 \times S^{n-2}$. $\square$

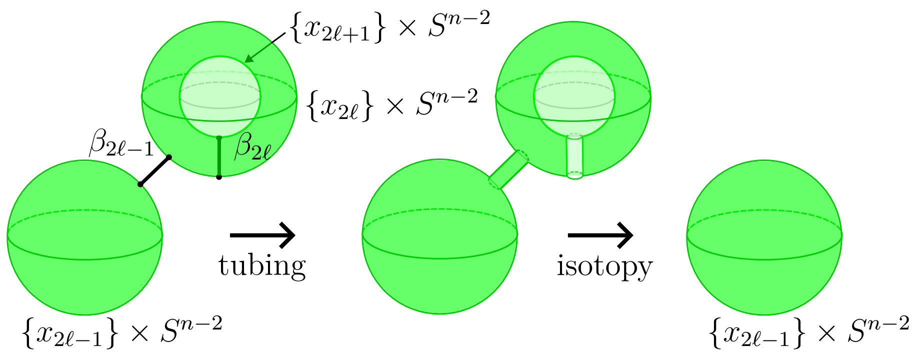

FIGURE 4. A schematic for an isotopy between $\widetilde{G}$ and $\{x_{2\ell-1}\} \times S^{n-2}$.

Note that, in general, G is not isotopic to $\widetilde{G}$ in the complement of B. See Remark 3.6 for further discussion.

We have so far defined B and G as embedded submanifolds of $S^2 \times S^{n-2}$, which is diffeomorphic to $\partial(D^{n+1} \cup h_{n-2}(R))$. To define the corresponding RBG manifold, we must choose a parametrisation and framing for each of B and G, so that we can attach $(n-1)$-handles to $D^{n+1} \cup h_{n-2}(R)$ along them. Both B and G have closed tubular neighbourhoods in $S^2 \times S^{n-2}$ that are diffeomorphic to $S^{n-2} \times D^2$, and we can use any such diffeomorphism for each of B and G to fix a parametrisation and framing. From now on, by slight abuse of notation, B and G will denote the corresponding framed knots in $\partial D^{n+1} \cup h_{n-2}(R)$.

[5] In fact, these extensions are unique up to isotopy rel S^1, which again uses general position for $n \geq 5$ and [Gab20] for $n = 4$.

3.3.4. *The construction gives an RBG manifold.*

Proposition 3.5. *The framed knots B and G together with the framed R from Constructions 3.3.1–3.3.3 always produce an RBG manifold W as in* (3.2).

Proof. We need to show that, for any B and G resulting from the above constructions, their images in $\partial(D^{n+1} \cup h_{n-2}(R))$ are geometrically dual to the belt sphere $\mathcal{B}_R$ up to isotopy. For any local knot K, the sphere $B(S^{n-2})$ is by construction geometrically dual to $\mathcal{B}_R$. Moreover, by Lemma 3.4, the embedded submanifold $G(S^{n-2})$ is isotopic to $\{x_1\} \times S^{n-2}$, which is geometrically dual to $\mathcal{B}_R$. □

Remark 3.6. In certain situations, $B(S^{n-2})$ and $G(S^{n-2})$ will not be simultaneously geometrically dual to $\mathcal{B}_R$. For example, this can happen if the embeddings $\gamma_1, \ldots, \gamma_{2\ell}$ in the above construction of G are chosen to represent distinct elements of $\pi_1(S^n \setminus K)$. We will make use of a particular instance of this situation in the next section.

Note that for the specific RBG manifolds constructed in Section 3.4, G will not be isotopic to $\widetilde{G}$ in the complement of B. Thus G, when considered as a knot in the complement of B, does indeed depend on the choices of γ_i and δ_i. So the isotopy class of G in $S^2 \times S^{n-2}$ is independent of the choices made, while the isotopy class of the 2-component link $B \cup G$ in $S^2 \times S^{n-2}$ depends on these choices.

3.4. **An explicit infinite family of RBG manifolds.** In this section, using the construction from Section 3.3, for every $n \geq 4$ and $m \geq 1$, we will build a pair of knots B_m and G_m in $\partial(D^{n+1} \cup h_{n-2}(R))$ that give rise to an RBG manifold W as in (3.2). By Proposition 3.2, we will obtain knots K_{B_m} and K_{G_m} in S^n with orientation-preservingly diffeomorphic traces. In Section 4, we will show that K_{B_m} and K_{G_m} are indeed non-isotopic.

3.4.1. *Construction of R.* Fix $n \geq 4$ and $m \geq 1$. As in Construction 3.3.1, let us denote by R an embedding $S^{n-3} \hookrightarrow S^n$ such that $D^{n+1} \cup h_{n-2}(R) \cong D^3 \times S^{n-2}$, and by $\mathcal{B}_R = S^2 \times \{z\} \subseteq S^2 \times S^{n-2}$ the belt sphere of $h_{n-2}(R)$ for some $z \in S^{n-2}$. We set $\ell = 1$, so $2\ell+1 = 3$. In the notation from Section 3.3, we will construct B_m and G_m from the $(n-2)$-spheres $\{w\} \times S^{n-2}$ and $\{x_i\} \times S^{n-2} \subseteq S^2 \times S^{n-2}$, $i \in \{1, 2, 3\}$, for four distinct points $w, x_1, x_2, x_3 \in S^2$. The construction of B_m is based on the choice of a local knot K_m in S^n.

3.4.2. *Construction of K_m and B_m.* We will build K_m by doubling an $(n-2)$-dimensional embedded ribbon disc D^b_m in D^n (see Section 2.5). First, we will explain the construction of the immersed ribbon disc D^b_m in S^{n-1}. The reader is invited to consult the left-hand side of Figure 5 throughout, which illustrates the situation for $n = 4$. For now, ignore the green and red parts of this figure and just view the blue part as a subset of $S^{n-1} = S^3$. We will explain the whole figure in Section 3.5.

Choose disjoint, oriented, smoothly embedded $(n-2)$-balls D^{n-2}_i in S^{n-1} for $i \in \{1, 2\}$, and denote the (oriented) meridians of $\partial D^{n-2}_i \cong S^{n-3}$ by μ_i. To construct D^b_m, choose an arc α_b whose interior lies in the complement of $\partial D^{n-2}_1 \cup \partial D^{n-2}_2$ as follows. If $m = 1$, α_b starts at ∂D^{n-2}_1, wraps once around ∂D^{n-2}_2 following μ_2, then wraps once around ∂D^{n-2}_1 following μ_1, and finally ends at ∂D^{n-2}_2. For general m, the arc α_b wraps in total m times first around ∂D^{n-2}_2 and then around ∂D^{n-2}_1 starting at ∂D^{n-2}_1

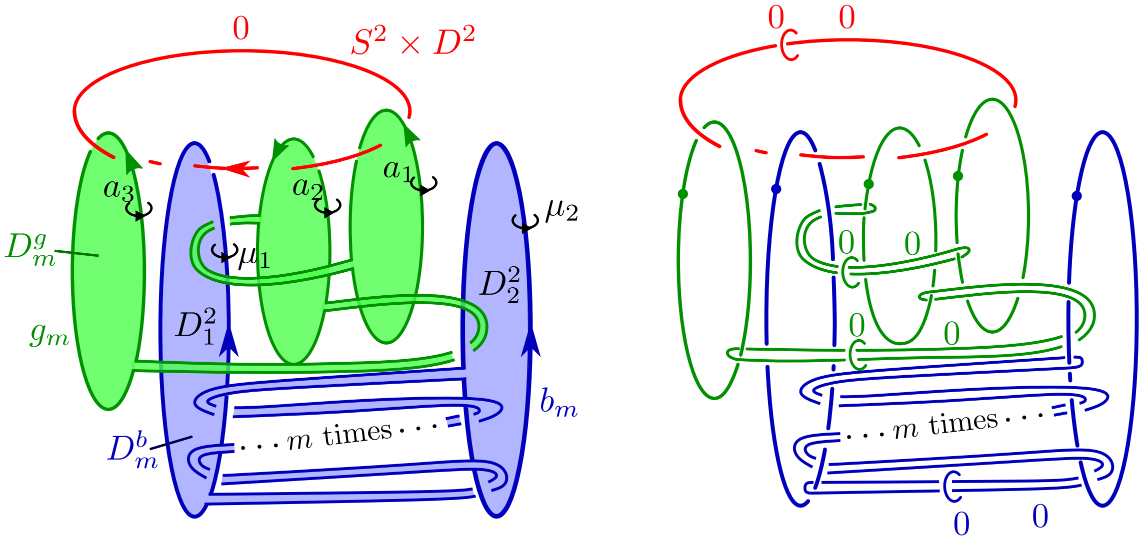

FIGURE 5. On the left: Discs D^g_m bounded by g_m (in green) and D^b_m bounded by b_m (in blue) in $S^2 \times D^2$ for $n = 4$. On the right: Kirby diagram for $S^2 \times S^2 \setminus (B_m \cup G_m)$.

and ending at ∂D^{n-2}_2. For example, for $m = 3$, the arc follows $\mu_2, \mu_1, \mu_2, \mu_1, \mu_2, \mu_1$ in this order. Note that the arc α_b is uniquely determined up to isotopy rel boundary, and it is chosen such that it transversely intersects D^{n-2}_1 and D^{n-2}_2 exactly m times each.

Now, tube ∂D^{n-2}_1 and ∂D^{n-2}_2 along α_b to obtain a sphere b_m (see Section 2.3). Here, for $n > 4$, in the tubing procedure there is a unique choice of thickening when α_b is thickened to $h_{\alpha_b} \colon D^1 \times D^{n-3} \hookrightarrow S^{n-1}$ and b_m is well-defined up to ambient isotopy. For $n = 4$, i.e. when α_b and $h_{\alpha_b} \colon D^1 \times D^1 \hookrightarrow S^3$ lie in S^3, we thicken by using the blackboard framing, as shown in Figure 5 (left) in blue. The two balls D^{n-2}_1 and D^{n-2}_2, together with $h_{\alpha_b}(D^1 \times D^{n-3})$ attached to them along $h_{\alpha_b}(S^0 \times D^{n-3})$, give rise to the desired immersed disc D^b_m in S^{n-1}, with $2m$ ribbon singularities and boundary b_m. The ribbon intersections disappear when the interior of D^b_m is appropriately pushed into D^n. We obtain an embedded disc in D^n with boundary b_m in S^{n-1}, which, by slight abuse of notation, we will also denote by D^b_m.

By doubling the pair (D^n, D^b_m), we obtain the ribbon $(n-2)$-knot K_m in S^n (see Section 2.5).[6] Finally, by Construction 3.3.2 we define

$$(S^2 \times S^{n-2}, B_m) := (S^2 \times S^{n-2}, \{w\} \times S^{n-2}) \# (S^n, K_m).$$

3.4.3. *Construction of G_m.* As in Construction 3.3.3, we start with $\widetilde{G}_m$, which is obtained by tubing the spheres $\{x_i\} \times S^{n-2}$, $i \in \{1, 2, 3\}$, along "straight" arcs β_1 and β_2. We now have to choose two disjoint embeddings $\gamma_i \colon S^1 \hookrightarrow S^n \setminus K_m$, $i \in \{1, 2\}$. By slightly abusing notation, we denote by μ_1 and μ_2 the images of the meridians μ_1 and μ_2 of ∂D^{n-2}_1 and ∂D^{n-2}_2 in $S^n \setminus K_m$. Then we choose γ_1 and γ_2 such that $\gamma_1(S^1)$ and $\gamma_2(S^1)$ represent μ_1^{-1} and μ_2^{-1} in $\pi_1(S^n \setminus K_m)$ up to conjugacy,

[6] For $n = 4$ and $m = 1$, we observe that $b_m = b_1 = \partial D^b_1$ is the knot $3_1 \# -3_1$, and the double K_1 of D^b_1 in S^4 is the spun trefoil, see e.g. [Suc85, Figure 2].

respectively (see also Section 2.4). We define G_m to be the sphere obtained by tubing the spheres $\bigcup_{i=1}^{3}\{x_i\} \times S^{n-2}$ along the arcs $\beta_i' = \beta_i \#_{\delta_i} \gamma_i$, $i \in \{1, 2\}$.

This finishes the construction of B_m and G_m.

3.4.4. *The construction gives an RBG manifold.* By Proposition 3.5, we obtain an RBG manifold $W_m = D^{n+1} \cup h_{n-2}(R) \cup h_{n-1}(B_m) \cup h_{n-1}(G_m)$. Therefore, by Proposition 3.2, we can summarise the work of this subsection as follows.

Proposition 3.7. *Let $n \geq 4$. For every $m \geq 1$, the pair of framed knots K_{B_m} and K_{G_m} in S^n have orientation-preservingly diffeomorphic traces.*

We will use these pairs in Section 4 to prove Theorem A as a direct application of Theorem 4.14.

3.5. **Kirby diagrams in dimension 4.** In dimension $n = 4$, we can describe B_m and G_m explicitly in Kirby diagrams as follows. Consider the Kirby diagram of $S^2 \times D^2$ given by the 0-framed red unknot r shown in Figure 5 on the left. The blue knot b_m and the green knot g_m in $\partial(S^2 \times D^2)$ bound the shaded ribbon discs. Here, the blue disc is the ribbon disc constructed in Construction 3.4.2. The three green discs in Figure 5 will produce by doubling the three green spheres $\{x_i\} \times S^2 \subseteq S^2 \times S^2$, $i \in \{1, 2, 3\}$, from Construction 3.4.3. We push the interiors of the blue and green discs into the interior of $S^2 \times D^2$, which gives disjointly and properly embedded 2-discs in $S^2 \times D^2$, which by slight abuse of notation we denote by D_m^b and D_m^g. Forming a relative double of $(S^2 \times D^2, D_m^b \cup D_m^g)$, we obtain two disjointly embedded 2-spheres in $S^2 \times S^2$, which are the 2-spheres B_m and G_m constructed above. In this perspective on our RBG manifold, the belt sphere of the red 2-handle in Figure 5 is the 2-sphere $S^2 \times \{z\}$ in $S^2 \times S^2$ which is given by gluing the core of this 2-handle to a pushed-in Seifert disc of the red unknot r.

On the right of Figure 5 we present a Kirby diagram of $(S^2 \times S^2) \setminus (B_m \cup G_m)$ which is obtained as follows. First, we double $S^2 \times D^2$, which corresponds to adding a 0-framed meridian to the red knot r, see for example [GS99, Example 4.6.3]. By construction, the green ribbon disc D_m^g admits a handle decomposition with three 0-handles and two 1-handles. Thus, its double, i.e. the green sphere G_m, admits a handle decomposition with three 0-handles, four 1-handles (two of which are duals of the 1-handles of D_m^g), and three 2-handles. Analogously, we get a handle decomposition for the blue sphere B_m consisting of two 0-handles, two 1-handles, and two 2-handles. Now, there is a standard argument for constructing a Kirby diagram of the complement of an embedded surface, see [GS99, Section 6.2]. For every 2-dimensional 0-handle of $B_m \cup G_m$ we obtain a 4-dimensional 1-handle of its complement. Likewise, for each 2-dimensional 1-handle we obtain a 4-dimensional 2-handle. The complement of the ribbon discs is constructed exactly as described in [GS99, p. 212]. In particular, the attaching regions of the 4-dimensional 2-handles that correspond to dual 1-handles in the handle decomposition of the ribbon spheres B_m and G_m are given by 0-framed meridians of the ribbon bands. Note that, as usual, by [LP72] the 3-handles of $(S^2 \times S^2) \setminus (B_m \cup G_m)$ corresponding to the 2-handles of $B_m \cup G_m$ can be omitted in our Kirby diagram (see also [GS99, Section 4.4]).

Remark 3.8. Note that we could also construct B_m and G_m for general $n \geq 4$ by a relative double construction as described above for $n = 4$. For later use, we point out that this can be done by crossing the construction described in dimension 4 with D^{n-4}. More precisely, to get the spheres B_m and G_m in $S^2 \times S^{n-2}$, we can take the relative double of $((S^2 \times D^2) \times D^{n-4}, (D^b_m \cup D^g_m) \times D^{n-4})$. This description makes it evident that the latter pair deformation retracts onto $(S^2 \times D^2, D^b_m \cup D^g_m)$, and this retraction preserves the meridians of the boundaries of the ribbon discs. Note that one could depict the handle decomposition for $S^2 \times S^{n-2} \setminus (B_m \cup G_m)$ for every $n \geq 4$ in a diagram similar to Figure 8 in [Kim23].

3.6. **General remarks.** Let us end this section with two remarks on our construction.

Remark 3.9. A similar construction as in Definition 3.1 and Proposition 3.2, but more general, also works to construct knotted spheres with diffeomorphic surgeries, but a priori unrelated traces.[7] Let R be a framed s-sphere in S^n and let B and G be framed k-spheres in the manifold obtained by surgery on S^n along R such that

(i) surgery along B yields a manifold diffeomorphic to S^n via a diffeomorphism f_G;

(ii) surgery along G yields a manifold diffeomorphic to S^n via a diffeomorphism f_B.

Then the framed k-spheres $\Sigma_B = f_B(B)$ and $\Sigma_G = f_G(G)$ in S^n have orientation-preservingly diffeomorphic surgeries. This construction is more in line with the approach of [MP23].

Remark 3.10. In the original construction for $n = 3$, an RBG link consists of a 3-component link $R \cup B \cup G$ in S^3. This translates to an RBG 4-manifold W with $k = 1$ by viewing the link as a Kirby diagram in which the unknotted component R is a dotted 1-handle and the components B and G represent 0-framed 2-handles. There is an analogous construction in the higher-dimensional setting given by taking a 2-component link $B \cup G$ of $(n-2)$-spheres in S^n together with a circle r linking both components once. When $n \geq 4$, the circle always bounds a disc. We require that both B and G are individually isotopic in the complement of r to spheres that intersect the disc bounded by r geometrically once. Then $r \cup B \cup G$ can be understood as a *generalised RBG link* in S^n. The corresponding RBG manifold is obtained by pushing the disc bounded by r into D^{n+1}, removing a neighbourhood of it and then attaching $(n-1)$-handles along B and G as before.

4. Distinguishing knots with the same trace

In this section, we first provide a brief overview of relevant knot invariants. Then, in light of the examples constructed in Section 3, we define a new invariant in terms of the conjugacy classes of the meridians of knots. We will use this new invariant to finish the proof of Theorem A in Section 4.5.

[7] In Section 5, and in particular in Theorem C, we will study the relation between surgeries and traces of knotted $(n-2)$-spheres in dimensions $n \geq 4$.

4.1. **Knot invariants.** Let us first consider the exterior $E_K = S^n \setminus \mathring{\nu} K$ of a knot K in S^n for $n \geq 3$. By Alexander duality, it has the same homology as S^1. In particular, the homology groups of E_K are independent of the knot type K. However, useful homological invariants can be extracted from the *Alexander invariants* of K.

We briefly recall the definition of the most important of these invariants, the Alexander polynomial. Given a knot $K\colon S^{n-2} \hookrightarrow S^n$, the kernel of the Hurewicz homomorphism $\pi := \pi_1(E_K) \twoheadrightarrow H_1(E_K) \cong \mathbb{Z}$ determines the *infinite cyclic cover* $\widetilde{E_K} \to E_K$, whose fundamental group is the commutator subgroup $\pi' := [\pi, \pi]$. The deck transformation group of this cover is $\pi/\pi' \cong H_1(E_K) \cong \mathbb{Z}$, which induces an action of the group ring $\Lambda := \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[H_1(E_K)]$ on the homology groups $H_*(\widetilde{E_K})$. The *Alexander module* of K is defined as the (left) Λ-module $H_1(\widetilde{E_K})$. The order of this module is the *Alexander polynomial* $\Delta_K(t) \in \Lambda$ of K, which is well-defined up to multiplication by units in Λ (see e.g. [Rol76, Chapter 7] or [FNOP25, Section 15.1]). Note that, as groups, $H_1(\widetilde{E_K}) \cong \pi'/[\pi', \pi']$. A presentation matrix for $H_1(\widetilde{E_K})$, and thus $\Delta_K(t)$, can be computed from a presentation of π using the free differential calculus due to Fox [Fox53, CF63]. In Section 4.4, we will distinguish our infinitely many pairs of knots $(K_{B_m}, K_{G_m})_{m \geq 1}$ from each other using the Alexander polynomial (see Proposition 4.13).

The following proposition shows that, for $n \geq 4$, knots K_1, K_2 in M^n with homotopy equivalent surgeries $M(K_1) \simeq M(K_2)$, have isomorphic knot groups $\pi_1(K_1) \cong \pi_1(K_2)$. In particular, if two knots in S^n have orientation-preservingly diffeomorphic surgeries or orientation-preservingly diffeomorphic traces, then they cannot be distinguished by their Alexander polynomials or modules.

Proposition 4.1. *For $n \geq 4$, let K be a knot with trivial normal bundle in an n-manifold M, possibly with $\partial M \neq \emptyset$. Then the inclusion $E_K \hookrightarrow M(K)$ induces an isomorphism on the fundamental groups, which identifies the conjugacy class of the meridian μ_K with that of the dual curve of K in $M(K)$ for some orientation. In particular, if two such knots have homotopy equivalent surgeries, then their knot groups are isomorphic.*

Proof. This follows from the Seifert–van Kampen theorem applied to the decomposition $M(K) = E_K \cup_{S^{n-2} \times S^1} D^{n-1} \times S^1$, noting that the inclusion $S^{n-2} \times S^1 \hookrightarrow D^{n-1} \times S^1$ induces an isomorphism on the level of π_1 when $n \geq 4$. Via this inclusion, the dual curve $\gamma_K = \{0\} \times S^1$ of K is identified (for some orientation) with the element of $\pi_1(K)$ represented by μ_K, which is well-defined up to conjugacy (see also Section 2.4). $\square$

Remark 4.2. Although the fundamental group $\pi_1 := \pi_1(K_i) \cong \pi_1(S^n \setminus K_i)$, $i \in \{1, 2\}$, of the complement cannot distinguish between knots K_1 and K_2 with a shared surgery, the second homotopy groups π_2 of their complements could differ. We could compare these by considering each π_2 as a (left) $\mathbb{Z}[\pi_1]$-module. This method has been successfully employed to distinguish knots with isomorphic knot groups, e.g. in [Gor73, Suc85]. However, sometimes this is not enough. Plotnick–Suciu [PS85] gave examples of 2-knot complements with identical first and second homotopy groups (the latter as $\mathbb{Z}[\pi_1]$-module), whose homotopy types they distinguished using k-invariants. Many questions concerning the homotopy type of the complement of

2-knots remain open; see, for example, [Lom81, Section XII]. Note that, in contrast, classical 1-knots have aspherical complements [Pap57].

For an overview on other knot invariants that could potentially distinguish knots with the same surgery, we refer to the excellent survey article by Conway [Con22]. Many of these invariants are typically only computed for fibred knots, such as twist-spun knots, where the knot exterior has a simple structure. In our setting, our knot exteriors do not have a simple geometric structure, so these approaches are difficult to apply. Instead, we use a more elementary invariant: the conjugacy class of the meridians in the fundamental group. We will explain this next.

4.2. **Counting representations.** For each $m \geq 1$, the knots K_{B_m} and K_{G_m} constructed in Proposition 3.7 have orientation-preservingly diffeomorphic traces and thus surgeries. Proposition 4.1 tells us that their knot groups are isomorphic. As discussed above, it is therefore not possible to distinguish between K_{B_m} and K_{G_m} using abelian knot invariants derived from the knot groups for a fixed $m \geq 1$. In Section 4.3, we will compute a presentation of $\pi_1(K_{B_m}) \cong \pi_1(K_{G_m})$ in which two distinct generators represent the conjugacy classes of the meridians of K_{B_m} and K_{G_m}, respectively. This observation will be crucial in distinguishing these knots. Indeed, we will look at representations of the knot groups into a specific finite group A which map the elements representing the conjugacy classes of the meridians of K_{B_m} and K_{G_m} to specific elements of A. The number of such representations will differ for K_{B_m} and K_{G_m}.

Let us now explain this in detail. First, we introduce some notation.

Definition 4.3. For finitely presented groups H and A, and elements $h \in H$, $\sigma \in A$, we denote by $\operatorname{Rep}(H, A, h, \sigma) := \{\varphi \in \operatorname{Hom}(H, A) \mid \varphi(h) = \sigma\}$ the set of homomorphisms from H to A that map h to σ. Moreover, $\#\operatorname{Rep}(H, A, h, \sigma)$ will denote the number of such homomorphisms, which is possibly infinite.

Note that $\#\operatorname{Rep}(H, A, h, \sigma)$ is invariant under conjugation of $h \in H$ and $\sigma \in A$, i.e.

$$\#\operatorname{Rep}(H, A, h, \sigma) = \#\operatorname{Rep}(H, A, ghg^{-1}, \tau\sigma\tau^{-1}) \qquad \text{for any } g \in H, \tau \in A.$$

Definition 4.4. Let K be a knot in S^n with meridian μ_K. For a finitely presented group A and an element $\sigma \in A$, we define

$$\mathcal{N}_K(A, \sigma) := \#\operatorname{Rep}(\pi_1(K), A, \mu_K, \sigma).$$

Recall the following definition.

Definition 4.5. Let X, Y be n-manifolds with possibly non-empty boundary. We say that X and Y are *homotopy equivalent rel boundary* if the pairs $(X, \partial X)$ and $(Y, \partial Y)$ are homotopy equivalent, i.e. if there is a homotopy equivalence $f\colon X \to Y$ restricting to a homotopy equivalence $f|_{\partial X}\colon \partial X \to \partial Y$.

The following proposition shows that $\mathcal{N}_K(A, \sigma)$ defines a knot invariant.

Proposition 4.6. *For $n \geq 4$, suppose that two knots K_1 and K_2 in S^n have homotopy equivalent exteriors rel boundary (which happens e.g. if K_1 and K_2 are* *ambient isotopic). Let A be a finitely presented group and let $\sigma \in A$ be an element which is conjugate to its inverse σ^{-1} in A. Then $\mathcal{N}_{K_1}(A,\sigma) = \mathcal{N}_{K_2}(A,\sigma)$.*

Proof. Suppose that $(E_{K_1}, \partial\nu K_1)$ and $(E_{K_2}, \partial\nu K_2)$ are homotopy equivalent via a map $h\colon E_{K_1} \to E_{K_2}$. In particular, h induces a homotopy equivalence between the boundaries $\partial\nu K_1 \simeq \partial\nu K_2$. Since the fundamental groups of $\partial\nu K_1 \simeq \partial\nu K_2$ are infinite cyclic generated by the meridians, the induced isomorphism $h_*\colon \pi_1(K_1) \to \pi_1(K_2)$ maps the conjugacy class of μ_{K_1} either to the conjugacy class of μ_{K_2} or to the conjugacy class of $\mu_{K_2}^{-1}$. In the first case, h_* induces a bijection between

$$\operatorname{Rep}(\pi_1(K_1), A, \mu_{K_1}, \sigma) \qquad \text{and} \qquad \operatorname{Rep}(\pi_1(K_2), A, \mu_{K_2}, \sigma).$$

In the second case, there is a bijection between $\operatorname{Rep}(\pi_1(K_1), A, \mu_{K_1}, \sigma)$ and

$$\operatorname{Rep}(\pi_1(K_2), A, \mu_{K_2}^{-1}, \sigma) = \operatorname{Rep}(\pi_1(K_2), A, \mu_{K_2}, \sigma^{-1}).$$

The latter set is in bijection with $\operatorname{Rep}(\pi_1(K_2), A, \mu_{K_2}, \sigma)$, because σ and its inverse are conjugate in A. In both cases, we obtain $\mathcal{N}_{K_1}(A,\sigma) = \mathcal{N}_{K_2}(A,\sigma)$. ☐

Remark 4.7. The statement of Proposition 4.6 also holds for knots K_1, K_2 in S^3, although the proof is slightly different. According to [Wal68], if $(E_{K_1}, \partial\nu K_1)$ and $(E_{K_2}, \partial\nu K_2)$ are homotopy equivalent, then K_1 and K_2 have homeomorphic exteriors. By [GL89b], knots in S^3 with homeomorphic exteriors are ambient isotopic as unoriented knots up to taking mirror images. We thus find a homeomorphism of pairs $h\colon (S^3, K_1) \to (S^3, K_2)$ which induces an isomorphism $h_*\colon \pi_1(K_1) \to \pi_1(K_2)$ mapping the conjugacy class of μ_{K_1} either to μ_{K_2} or $\mu_{K_2}^{-1}$. We conclude that $\mathcal{N}_{K_1}(A,\sigma) = \mathcal{N}_{K_2}(A,\sigma)$ as above.

Remark 4.8. Suciu used the idea of distinguishing ribbon discs with the same exterior by their meridians in [Suc85], where he constructed infinitely many distinct (embedded) ribbon n-discs with the same exterior for $n \geq 3$.

Before we turn to an explicit calculation of the above defined invariant $\mathcal{N}_K(A,\sigma)$ for our knots $K \in \{K_{B_m}, K_{G_m}\}$, let us prove the following statement on knots K coming from fairly general RBG manifolds. Recall that the meridian μ_K is well-defined up to conjugacy in $\pi_1(K)$ and that we also denote this conjugacy class by μ_K.

Proposition 4.9. *Let W be an $(n+1)$-dimensional RBG manifold and suppose that $D^{n+1} \cup h_{n-2}(R) \cong D^3 \times S^{n-2}$ as in Construction 3.3.1, so B and G are $(n-2)$-spheres in $\partial(D^{n+1} \cup h_{n-2}(R)) \cong S^2 \times S^{n-2}$. Then, for the associated knots K_B and K_G from Proposition 3.2, there are isomorphisms of pairs*

$$\begin{aligned} (\pi_1(K_B),\ \mu_{K_B}) &\cong (\pi_1(S^2 \times S^{n-2} \setminus (\mathring{\nu} B \cup \mathring{\nu} G)),\ \mu_B), \\ (\pi_1(K_G),\ \mu_{K_G}) &\cong (\pi_1(S^2 \times S^{n-2} \setminus (\mathring{\nu} B \cup \mathring{\nu} G)),\ \mu_G). \end{aligned} \tag{4.1}$$

Proof. By definition, $K_B = f_B \circ B$ and $K_G = f_G \circ G$ for diffeomorphisms f_G and f_B as in (3.1). The surgery on $S^2 \times S^{n-2} \setminus \mathring{\nu} B$ along G is diffeomorphic, via the restriction of f_B, to $S^n \setminus \mathring{\nu} K_B$. Moreover this diffeomorphism maps μ_B to μ_{K_B}. Hence

$$\begin{aligned} (\pi_1(S^n \setminus \mathring{\nu} K_B),\ \mu_{K_B}) &\cong (\pi_1((S^2 \times S^{n-2} \setminus \mathring{\nu} B)(G)),\ \mu_B) \\ &\cong (\pi_1(S^2 \times S^{n-2} \setminus (\mathring{\nu} G \cup \mathring{\nu} B)),\ \mu_B), \end{aligned}$$

where the second isomorphism follows from Proposition 4.1. In the same way, one proves the result for K_G. □

4.3. **Explicit calculations for K_{B_m} and K_{G_m}.** We first determine a presentation for the knot groups $\pi_1(K_{B_m}) \cong \pi_1(K_{G_m})$.

Proposition 4.10. *Let $F_m := \pi_1(S^2 \times S^{n-2} \setminus (\mathring{\nu} B_m \cup \mathring{\nu} G_m))$. Then, for every $m \geq 1$, we have $F_m \cong \pi_1(K_{B_m}) \cong \pi_1(K_{G_m})$ and a presentation for F_m is given by*

$$F_m \cong \langle x, y, a \mid (yx)^m y (yx)^{-m} x^{-1} = 1,\, (x^{-1}ax)a^{-1}x^{-1}(yay^{-1}) = 1 \rangle,$$

where both x and y represent the conjugacy class of the meridian of K_{B_m}, while the generator a represents the conjugacy class of the meridian of K_{G_m}.

Proof. Throughout the proof, fix $m \geq 1$. The group isomorphisms $F_m \cong \pi_1(K_{B_m}) \cong \pi_1(K_{G_m})$ follow directly from Proposition 4.9. Note that under these isomorphisms, the meridians $\mu_{K_{B_m}}$ and $\mu_{K_{G_m}}$ are identified with meridians of the spheres B_m and G_m. We will now first compute a presentation for F_m in dimension $n = 4$ and then reduce the general case $n \geq 5$ to this computation.

For the case $n = 4$, we will rely on the discussion from Section 3.5. Recall that $S^2 \times D^2$ has a handle decomposition that consists of one 0- and one 2-handle. We denote this 2-handle by h_2. Consider the ribbon discs D^b_m (in blue) and D^g_m (in green) in $S^2 \times D^2$ on the left-hand side of Figure 5 from Section 3.4. The green disc D^g_m has a handle decomposition with three 0-handles and two 1-handles; the blue disc D^b_m has a handle decomposition with two 0-handles D^2_1 and D^2_2 and one 1-handle. This provides us with a handle decomposition of the complement of $D^b_m \cup D^g_m$ in $S^2 \times D^2$, which consists of one 0-handle, five 1-handles and four 2-handles (see e.g. [GS99, Proposition 6.2.1]), where the fourth 2-handle corresponds to h_2. This handle decomposition can be described in a Kirby diagram by removing the small 0-framed unknots in the Kirby diagram on the right-hand side of Figure 5. From this handle decomposition, we can directly compute a presentation of $\pi_1(S^2 \times D^2 \setminus (D^b_m \cup D^g_m))$, as we do next.

Denote the meridians of the three 0-handles that make up the green disc D^g_m in Figure 5 (left) by a_3, a_2, a_1 from left to right. Recall that we denote the meridians of ∂D^2_1 and ∂D^2_2 from the construction of D^b_m by μ_1, μ_2. The push-forwards of all these five meridians represent generators of $\pi_1(S^2 \times D^2 \setminus (D^b_m \cup D^g_m))$. We can read off the following relations from the 2-handle attachments in $S^2 \times D^2 \setminus (D^b_m \cup D^g_m)$ corresponding to the 2-dimensional 1-handles of $D^b_m \cup D^g_m$, and the 2-handle h_2:

$$\begin{aligned}
\mu_2 a_2^{-1} \mu_2^{-1} a_3 &= 1 \quad (D^g_m) \quad \Rightarrow \quad a_3 = \mu_2 a_2 \mu_2^{-1}, \\
a_1^{-1} \mu_1^{-1} a_2 \mu_1 &= 1 \quad (D^g_m) \quad \Rightarrow \quad a_1 = \mu_1^{-1} a_2 \mu_1, \\
a_1 a_2^{-1} \mu_1^{-1} a_3 &= 1 \quad (h_2) \quad \Rightarrow \quad (\mu_1^{-1} a_2 \mu_1) a_2^{-1} \mu_1^{-1} (\mu_2 a_2 \mu_2^{-1}) = 1, \\
(\mu_2\mu_1)^m \mu_2 (\mu_2\mu_1)^{-m} \mu_1^{-1} &= 1 \quad (D^b_m).
\end{aligned}$$

Setting $x = \mu_1$, $y = \mu_2$ and $a = a_2$ for better readability, we obtain the presentation

$$\begin{aligned}
&\pi_1(S^2 \times D^2 \setminus (D^b_m \cup D^g_m)) \cong \\
&\qquad \langle x, y, a \mid (yx)^m y (yx)^{-m} x^{-1} = (x^{-1}ax)a^{-1}x^{-1}(yay^{-1}) = 1 \rangle.
\end{aligned}$$

Since the spheres B_m and G_m in $S^2 \times S^2$ are obtained as relative doubles of $D^b_m \cup D^g_m$ in $S^2 \times D^2$ (see Section 3.5), we have

$$\begin{aligned}\pi_1(S^2 \times S^2 \backslash (B_m \cup G_m)) &\cong \pi_1(S^2 \times D^2 \setminus (D^b_m \cup D^g_m)) \\ &\cong \langle x, y, a \mid (yx)^m y (yx)^{-m} x^{-1} = (x^{-1}ax)a^{-1}x^{-1}(yay^{-1}) = 1\rangle,\end{aligned}$$

where the first isomorphism can be obtained by a standard Seifert–van Kampen argument. Under these isomorphisms, x and y represent meridians of ∂D^b_m and thus B_m, while a represents a meridian of ∂D^g_m and thus G_m. By Proposition 4.9, this yields the claim for $n = 4$. Alternatively, we could have also read off the fundamental group presentation directly from the Kirby diagram for $S^2 \times S^2 \setminus (B_m \cup G_m)$ on the right-hand side of Figure 5. In that case, we have additional 2-handles coming from the relative doubling process described in Section 3.5, but the relations coming from these additional 2-handles are trivial.

For $n \geq 5$, as pointed out in Remark 3.8 we can obtain B_m and G_m also as doubles of $(n-2)$-dimensional (properly embedded) ribbon discs in $S^2 \times D^{n-2}$ by crossing the construction in dimension 4 with D^{n-4}. The deformation retraction obtained in this way induces, by restriction, a deformation retraction of the complement of these discs in $S^2 \times D^{n-2}$ onto $S^2 \times D^2 \setminus (D^b_m \cup D^g_m)$, preserving the meridians. By applying Seifert–van Kampen to the relative double, we get the desired result. $\square$

We will now compute the invariant $\mathcal{N}_K(A, \sigma)$ from Definition 4.4 for $K \in \{K_{B_m}, K_{G_m}\}$ using Proposition 4.10 and a specific finite group A. The alternating group of degree 5, which we denote by A_5, has order 60 and is the smallest non-abelian simple group.

Proposition 4.11. *Consider the 5-cycle $\sigma = (15432)$ in A_5. Let $m \geq 1$ be an integer with $m \equiv 1 \pmod{60}$. Then*

$$\mathcal{N}_{K_{B_m}}(A_5, \sigma) = 6 \qquad \textit{and} \qquad \mathcal{N}_{K_{G_m}}(A_5, \sigma) = 1.$$

Proof. By virtue of Proposition 4.10, we have to count the number of homomorphisms from F_m to A_5 that map x or a to $\sigma = (15432) \in A_5$, respectively. For $m = 1$, this is an easy task using the function *GroupHomomorphismByImages* in GAP via Sage [GAP25, Sag25] to check if a given assignment on the generators x, y, a of F_m extends to a group homomorphism. We simply apply this function to all possible maps $F_m \to A_5$ by assigning elements of A_5 to x, y, a. We obtain $\mathcal{N}_{K_{B_1}}(A_5, \sigma) = 6$ and $\mathcal{N}_{K_{G_1}}(A_5, \sigma) = 1$. See Appendix A for our Sage source code.

We claim that this determines the number of representations also for any other m congruent to 1 modulo 60. To see this, note that the second relation in the presentation of F_m given in Proposition 4.10 does not depend on m. Consider the first relation and suppose that $m = 60m' + 1$ for some $m' \in \mathbb{Z}$. Since the order of A_5 is 60, for every $\sigma_1, \sigma_2, \sigma_3 \in A_5$, the word $(\sigma_2\sigma_1)^m \sigma_2 (\sigma_2\sigma_1)^{-m} \sigma_1^{-1} = (\sigma_2\sigma_1)^{60m'+1} \sigma_2 (\sigma_2\sigma_1)^{-(60m'+1)} \sigma_1^{-1}$ is the trivial word in A_5 if and only if $(\sigma_2\sigma_1)^1 \sigma_2 (\sigma_2\sigma_1)^{-1} \sigma_1^{-1}$ is. Therefore the assignment $\varphi\colon x \mapsto \sigma_1, y \mapsto \sigma_2, a \mapsto \sigma_3$ defines a group homomorphism $F_m \to A_5$ if and only if it defines a group homomorphism $F_1 \to A_5$. We obtain

$$\begin{aligned}&\mathcal{N}_{K_{B_m}}(A_5, \sigma) = \mathcal{N}_{K_{B_1}}(A_5, \sigma) = 6 \qquad \text{and} \\ &\mathcal{N}_{K_{G_m}}(A_5, \sigma) = \mathcal{N}_{K_{G_1}}(A_5, \sigma) = 1 \qquad \text{for all } m \equiv 1 \pmod{60}.\end{aligned}$$ $\square$

Corollary 4.12. *Let $m \geq 1$ be an integer with $m \equiv 1 \pmod{60}$. Then the exteriors of K_{B_m} and K_{G_m} in S^n are not homotopy equivalent rel boundary.*

Proof. Notice that $\sigma = (15432) \in A_5$ is conjugate to its inverse. If the exteriors of K_{B_m} and K_{G_m} were homotopy equivalent rel boundary for some $m \equiv 1 \pmod{60}$, then by Proposition 4.6 we would have $\mathcal{N}_{K_{B_m}}(A_5, \sigma) = \mathcal{N}_{K_{G_m}}(A_5, \sigma)$. But Proposition 4.11 shows that these numbers differ, a contradiction. $\square$

4.4. **Fox calculus to distinguish the pairs.** In this subsection, using the Fox calculus [Fox53, CF63] mentioned in Section 4.1, we compute the Alexander polynomials of our knots K_{B_m} and K_{G_m} to distinguish the pairs $\{(K_{B_m}, K_{G_m})\}_{m\geq 1}$ from one another.

Proposition 4.13. *For every $m \geq 1$, the Alexander polynomials of K_{B_m} and K_{G_m}, up to multiplication by units in $\mathbb{Z}[t, t^{-1}]$, are*

$$\Delta_{K_{B_m}}(t) = \Delta_{K_{G_m}}(t) = \sum_{k=0}^{2m} (-1)^k t^{k-2}.$$

Proof. Fix $m \geq 1$. By Proposition 4.10, both knot groups $\pi_1(K_{B_m}) \cong \pi_1(K_{G_m})$ have a presentation F_m with generators x, y and a and relations

$$r_1 = x^{-1}(yx)^m y (yx)^{-m} \qquad \text{and} \qquad r_2 = (x^{-1}ax)a^{-1}x^{-1}(yay^{-1}).$$

Computing the Fox derivatives, we obtain

$$\begin{aligned}
\frac{\partial r_1}{\partial x} &= -x^{-1} + x^{-1}\left(\left(\sum_{k=0}^{m-1}(yx)^k\right) y + (yx)^m y \left(\sum_{k=0}^{m-1}(yx)^{-k}\right)\left(-x^{-1}\right)\right),\\
\frac{\partial r_1}{\partial y} &= x^{-1}\left(\sum_{k=0}^{m}(yx)^k - (yx)^m y (yx)^{-1}\sum_{k=0}^{m-1}(yx)^{-k}\right),\\
\frac{\partial r_1}{\partial a} &= 0,\\
\frac{\partial r_2}{\partial x} &= -x^{-1} + x^{-1}a - x^{-1}axa^{-1}x^{-1},\\
\frac{\partial r_2}{\partial y} &= x^{-1}axa^{-1}x^{-1}\left(1 - yay^{-1}\right),\\
\frac{\partial r_2}{\partial a} &= x^{-1}\left(1 + ax\left(-a^{-1} + a^{-1}x^{-1}y\right)\right).
\end{aligned}$$

Substituting t for x, y and a gives us

$$\left.\begin{pmatrix} \dfrac{\partial r_1}{\partial x} & \dfrac{\partial r_1}{\partial y} & \dfrac{\partial r_1}{\partial a} \\ \dfrac{\partial r_2}{\partial x} & \dfrac{\partial r_2}{\partial y} & \dfrac{\partial r_2}{\partial a} \end{pmatrix}\right|_{\{x=y=a=t\}} = \begin{pmatrix} \displaystyle\sum_{k=-1}^{2m-1}(-1)^k t^k & \displaystyle -\sum_{k=-1}^{2m-1}(-1)^k t^k & 0 \\ 1 - 2t^{-1} & t^{-1} - 1 & t^{-1} \end{pmatrix}.$$

Computing the determinant of a (2×2)-submatrix, we obtain the claim. $\square$

4.5. **Proof of Theorem A.** We now prove the following Theorem 4.14, which directly implies Theorem A.

Theorem 4.14. *For every $n \geq 4$, there exist families $(K_{B_m})_{m\geq 1}$ and $(K_{G_m})_{m\geq 1}$ of knots in S^n such that*

(i) the exteriors of K_{B_m} and $K_{B_{m'}}$ are homotopy equivalent if and only if $m = m'$,

(ii) the exteriors of K_{G_m} and $K_{G_{m'}}$ are homotopy equivalent if and only if $m = m'$,

(iii) for all m, the exteriors of K_{B_m} and K_{G_m} are not homotopy equivalent rel boundary (see Definition 4.5), but

(iv) for all m, the traces $X(K_{B_m})$ and $X(K_{G_m})$ are orientation-preservingly diffeomorphic.

In particular, for each $m \geq 1$, the two knots K_{B_m} and K_{G_m} are not ambient isotopic, but have orientation-preservingly diffeomorphic traces.

Proof. Fix $n \geq 4$. For every $m \geq 1$, the knots K_{B_m} and K_{G_m} in S^n have orientation-preservingly diffeomorphic traces (see Proposition 3.7), so we have (iv). By Proposition 4.13, the Alexander polynomials $\Delta_{K_{B_m}} = \Delta_{K_{G_m}}$ and $\Delta_{K_{B_{m'}}} = \Delta_{K_{G_{m'}}}$ have distinct breadths (i.e. differences between highest and lowest non-trivial powers) for $m \neq m'$. This shows (i) and (ii). Now suppose that $m \equiv 1 \pmod{60}$. By Corollary 4.12, the exteriors of K_{B_m} and K_{G_m} in S^n are not homotopy equivalent rel boundary. The pairs $\{(K_{B_m}, K_{G_m})\}$ for $m = 60m' + 1$, $m' \geq 1$, thus give rise to the desired infinite family with (i)–(iv). $\square$

4.6. **On a higher-dimensional light bulb theorem for links.** The 4-dimensional light bulb theorem proven by Gabai [Gab20] states that if S is a smoothly embedded 2-sphere in $S^2 \times S^2$ that is homologous (and therefore homotopic) to $\{\mathrm{pt}\} \times S^2$ and intersects $S^2 \times \{\mathrm{pt}\}$ transversely in one point, then S is isotopic to $\{\mathrm{pt}\} \times S^2$ via an isotopy fixing $S^2 \times \{\mathrm{pt}\}$ pointwise. An analogous result for knots in $S^2 \times S^{n-2}$ for $n \geq 5$, but under an additional condition, was proved by Litherland [Lit86]. Specifically, he showed that if S is a smoothly embedded $(n-2)$-sphere in $S^2 \times S^{n-2}$ that is homotopic to $\{\mathrm{pt}\} \times S^{n-2}$ and intersects $S^2 \times \{\mathrm{pt}\}$ transversely in one point, and $\iota(S) \subseteq S^{n+1}$ bounds a smoothly embedded $(n-1)$-ball in S^{n+1}, where

$$\iota\colon S^2 \times S^{n-2} \hookrightarrow \partial D^3 \times D^{n-1} \hookrightarrow \partial D^{n+2} = S^{n+1}$$

is given by the standard inclusions, then S is isotopic to $\{\mathrm{pt}\} \times S^{n-2}$. In contrast to the above results, the proof of Theorem 4.14 in the previous sections yields the following statement for links.

Corollary 4.15. *Let $n \geq 4$. There is a 2-component link L in $S^2 \times S^{n-2}$ such that*

(i) the image of each component of L is isotopic (as an oriented submanifold) in $S^2 \times S^{n-2}$ to $\{z\} \times S^{n-2}$ for $z \in S^2$, but

(ii) L is not isotopic to the link $\{z_1, z_2\} \times S^{n-2}$ for $z_1 \neq z_2 \in S^2$.

Proof. Consider an RBG manifold for which the associated knots K_B and K_G are non-isotopic. Examples of such manifolds are given in the proof of Theorem 4.14. This proof shows that $B(S^{n-2})$ and $G(S^{n-2})$ are both isotopic to $\{\mathrm{pt}\} \times S^{n-2}$ in $S^2 \times S^{n-2}$; see in particular Lemma 3.4 and Proposition 3.5. However, according to Lemma 3.3, we cannot simultaneously isotope $B(S^{n-2})$ and $G(S^{n-2})$ to $\{\mathrm{pt}\} \times S^{n-2}$. Therefore, the link $B(S^{n-2}) \cup G(S^{n-2})$ in $S^2 \times S^{n-2}$ fulfils the desired properties. □

The lightbulb theorem for links of spheres in a 4-manifold was shown in [Gab20] when each component has its own geometric dual sphere. Corollary 4.15 above demonstrates that this is a necessary condition for a higher-dimensional light bulb theorem for links to hold. In this setting with geometric duals, the generalization of Kosanovic–Teichner [KT24] can be applied to study the links.

Remark 4.16. For $n = 4$, Corollary 4.15 can also be obtained as follows. Take two standard spheres $\{x_1\} \times S^2$ and $\{x_2\} \times S^2$ in $S^2 \times S^2$. Adjust the second one by taking a connected sum with a non-trivial local knot K in $D^4 \subseteq S^4$ away from the first one. Then the link given by the disjoint union of these two knots in $S^2 \times S^2$ is not isotopic to the unlink of two $\{\mathrm{pt}\} \times S^2$ spheres, because the fundamental group of the link exterior is isomorphic to $\pi_1(K)$, which we can choose to be not isomorphic to $\mathbb{Z}$. However, by [Gab20], the link fulfils (i) in Corollary 4.15. A similar proof gives the analogous result for $n = 3$ (see also the construction described on [Lit86, p. 353]). However, by Lemma 3.3, these links are not interesting from the point of view of RBG manifolds, because they produce isotopic associated knots in S^n.

As noted above, for the RBG manifolds constructed in Section 3.4, the spheres $B_m(S^{n-2})$ and $G_m(S^{n-2})$ cannot be simultaneously geometrically dual to the belt sphere $\mathcal{B}_R$. Note that we can also recover this fact from the presentation of the knot groups F_m computed in Proposition 4.10. By Lemma 3.3, if $B(S^{n-2})$ and $G(S^{n-2})$ admitted a simultaneous geometric dual, then the elements x, a representing $\mu_{K_{B_m}}$ and $\mu_{K_{G_m}}$ would be conjugate in F_m. However, for every $m \equiv 1 \pmod{60}$, the map

$$\varphi\colon F_m \to A_5, \quad x \mapsto (15432), \quad y \mapsto (12453), \quad a \mapsto (245)$$

defines a homomorphism which does not map x and a to conjugate elements in A_5.

5. Surgeries, traces and Gluck twists

In this section, we will leave the realm of RBG manifolds and explore the relationship between surgeries, traces and Gluck twists. This will lay the groundwork for constructing in Section 6, for every $n \geq 5$, infinitely many non-isotopic knots in S^n that all have orientation-preservingly diffeomorphic traces.

Throughout this section we assume that $n \geq 4$.

5.1. Gluck twists.

We start by discussing Gluck twists.

Definition 5.1. Let $K\colon S^{n-2} \hookrightarrow S^n$ be a knot in S^n, and fix an identification of νK with $D^2 \times S^{n-2}$. The *Gluck twist of S^n along K* is the oriented, connected, closed, smooth n-manifold

$$S^n_K = (S^n \setminus \mathring{\nu} K) \cup_\psi (D^2 \times S^{n-2}), \qquad \text{where}$$

$$\psi\colon S^1 \times S^{n-2} \cong \partial(S^n \setminus \mathring{\nu} K) \to \partial D^2 \times S^{n-2} = S^1 \times S^{n-2},$$

$$(e^{it}, x) \mapsto (e^{it}, \rho_t(x)),$$

and ρ_t denotes the rotation of the $(n-2)$-sphere $\{e^{it}\} \times S^{n-2}$ about its standard diameter by angle t; see [Glu62] for the case $n = 4$. The embedding of $\{1\} \times S^{n-2}$ defines a knot $T(K)\colon S^{n-2} \hookrightarrow S^n_K$, which we call the *Gluck twist of* K.

In Definition 5.1, we use again that a knot in S^n has a unique framing. Thus the oriented diffeomorphism type of S^n_K is well-defined and does not depend on the choice of identification $\nu K \cong D^2 \times S^{n-2}$ (see also Section 2.2).

Notice that Gluck twisting S^n_K along $T(K)$, similarly to the operation in Definition 5.1, yields the original knot K, i.e. $T(T(K)) = K$. For $n = 4$, Gluck [Glu62] showed that there are at most two 2-knots with homeomorphic complements, and, if K and J are non-isotopic such 2-knots, then K has to be isotopic to the Gluck twist $T(J)$ of J. In higher dimensions, for $n \geq 5$, it has been shown in [Bro67, LS69] that there are at most two knots in S^n (up to reparametrisation) with diffeomorphic exteriors. Since K and $T(K)$ have diffeomorphic exteriors $S^n \setminus \mathring{\nu} K \cong S^n_K \setminus (D^2 \times S^{n-2})$ and the Gluck twist map ψ extends over $S^1 \times D^{n-1}$, it follows that K and $T(K)$ also share the same surgery, i.e. $S^n_K(T(K)) \cong S^n(K)$. On the other hand, their traces might differ, as we will discuss below in Corollary 5.13.

Remark 5.2. For any knot K in S^n, using the Seifert–van Kampen theorem and the Mayer–Vietoris sequence, one can easily see that the Gluck twist S^n_K along K is a simply connected homology sphere. By a standard argument using the Hurewicz and Whitehead theorems, it is a homotopy n-sphere. As a consequence of the topological Poincaré conjecture, S^n_K is hence homeomorphic to S^n [Fre82, Sma62]. Nevertheless, as a smooth n-manifold, S^n_K could possibly be a non-standard n-sphere.

Proposition 5.3. *For $n \geq 5$, let $K\colon S^{n-2} \hookrightarrow S^n$ be a knot in S^n. Suppose that K extends to an embedding $D^{n-1} \hookrightarrow D^{n+1}$. Then S^n_K is diffeomorphic to S^n.*

Remark 5.4. When S^{n-1} has a unique smooth structure (see Remark 2.1), the assumption on the knot K in Proposition 5.3 is equivalent to asking K to be *slice*, i.e. to asking $K(S^{n-2}) \subseteq S^n = \partial D^{n+1}$ to bound an $(n-1)$-dimensional smoothly and properly embedded disc in D^{n+1}. For even n, all knots in S^n are slice [Ker65, Ker71]. In particular, for each such knot, there is a reparametrisation of its embedding for which we can apply the proposition. For $n \in \{6, 62\}$, which are the even $n \geq 5$ such that S^{n-1} is known to have a unique smooth structure (see Remark 2.1), we obtain that S^n_K is diffeomorphic to S^n for *every* knot K in S^n.

Proof of Proposition 5.3. We will construct an h-cobordism from S^n to S^n_K. To that end, consider the $(n+1)$-manifold W obtained by first attaching an $(n-1)$-handle to D^{n+1} along K and then attaching a 2-handle along the dual curve γ_K in the surgery $S^n(K)$ of S^n along K. Notice that, since γ_K is the belt sphere of the $(n-1)$-handle, it naturally inherits a framing. We use the *opposite* framing for the latter handle attachment.[8] Standard arguments show that W is a simply connected $(n+1)$-manifold with $\partial W = S^n_K$, $H_k(W) \cong \mathbb{Z}$ for $k \in \{0, 2, n-1\}$ and $H_k(W) = 0$ otherwise. Notice that a generator of $H_{n-1}(W)$ is represented by a smoothly embedded $(n-1)$-sphere S given by the union of the core of the $(n-1)$-handle and a slice disc in D^{n+1}

[8] Here we are using $\pi_1(\mathrm{SO}(n-1)) \cong \mathbb{Z}/2\mathbb{Z}$.

for K. By Remark 2.2, S has a trivial normal bundle in W and its framing is unique. We now claim that the manifold

$$B := (W \setminus \nu S) \cup_\partial (D^n \times S^1)$$

obtained by surgery on W along S is a homotopy $(n+1)$-ball with boundary S^n_K. Indeed, a Mayer–Vietoris argument shows that $H_k(W \setminus \nu S) \cong \mathbb{Z}$ for $k \in \{0, n-1, n\}$ and all other homology groups are trivial, where generators for $H_{n-1}(W \setminus \nu S) \cong H_{n-1}(W)$ and $H_n(W \setminus \nu S) \cong H_n(\partial \nu S)$ can be represented by a parallel copy of S and by $\partial \nu S$, respectively. Since B is obtained by gluing $D^n \times S^1$ onto $\partial \nu S$, it is easy to check that all the homology groups $H_k(B)$ for $k > 0$ are trivial. Moreover, by the Seifert–van Kampen theorem, B is simply connected and hence a homotopy $(n+1)$-ball as claimed. By removing a small ball in the interior of B we obtain an h-cobordism between S^n and S^n_K, and the conclusion follows from the smooth h-cobordism theorem [Sma62]. $\square$

5.2. **Surgery diffeomorphisms and traces.** We now prove Theorem C, which we restate for the convenience of the reader.

Theorem C. *For $n \geq 5$, let K_1 and K_2 be knots in S^n and let $\varphi\colon S^n(K_1) \to S^n(K_2)$ be an orientation-preserving diffeomorphism between their surgeries. Then either there is an orientation-preserving diffeomorphism $X(K_1) \cong X(K_2)$ or the Gluck twist $T(K_1)$ is a knot in the standard smooth n-sphere with $X(T(K_1)) \cong X(K_2)$.*

Remark 5.5. Note that Theorem C does not hold in dimension $n = 3$ [MP23], compare also [Akb77, Boy86]. In fact, the proof of Theorem C is based on the same idea presented in [Akb77].

Proof of Theorem C. For $i = 1, 2$, we denote by $\gamma_i \subseteq S^n(K_i)$ the dual curve of K_i such that the knot K_i can be recovered by loop surgery on $S^n(K_i)$ along γ_i with some framing, which we denote by f_i. Since $X(K_2)$ is simply-connected and of dimension $n + 1 \geq 5$, there is a properly embedded 2-disc D in $X(K_2)$ bounded by $\varphi(\gamma_1)$. We can write

$$X(K_2) = (X(K_2) \setminus \mathring{\nu} D) \cup \nu D.$$

Using standard arguments as in Remark 5.2, one can then check that $V := X(K_2) \setminus \mathring{\nu} D$ is simply-connected and that its boundary $\partial V = \partial(X(K_2) \setminus \mathring{\nu} D)$ is a homotopy n-sphere. More precisely, $\partial V \cong \varphi(S^n(K_1) \setminus \mathring{\nu}\gamma_1) \cup_\partial S^{n-2} \times D^2$, is diffeomorphic either to S^n, when the push-forward via φ of f_1 extends over D, or to the Gluck twist $S^n_{K_1}$, when the push-forward of f_1 does not extend. Moreover, one can show that $H_i(V) \cong 0$ for $i > 0$. This follows from a Mayer-Vietoris argument on the pair $(V, \nu D)$, noticing for $i = n - 2$ that $V \cap \nu D \cong D^2 \times S^{n-2}$ is included in V via a map which factors through the homology sphere ∂V. Again, by a standard argument, it follows that V is contractible. Hence, by the smooth h-cobordism theorem [Sma62], it is a smooth $(n+1)$-ball.

Notice that this last step is where we are using the assumption $n \geq 5$. Now, note that νD is attached to $V = X(K_2) \setminus \mathring{\nu} D$ as an $(n-1)$-handle, with attaching locus an $(n-2)$-knot A in ∂V which is the core of the loop surgery on $\varphi(S^n(K_1))$

along $\varphi(\gamma_1)$. See Figure 6 for a pictorial description of our setting. We will now distinguish two cases.

First, suppose that the push-forward $\varphi_*(f_1)$ of the framing f_1 on $\varphi(\gamma_1)$ extends over D. In this case, A is equivalent to the core of the loop surgery on $S^n(K_1)$ along γ_1 with framing f_1. This means that the attaching locus of νD is equivalent to K_1, which yields an identification

$$X(K_2) = V \cup_A \nu D \cong D^{n+1} \cup_{K_1 = S^{n-2}\times\{0\}} (D^{n-1}\times D^2) = X(K_1). \tag{5.1}$$

On the other hand, it can also happen that the framing $\varphi_*(f_1)$ does not extend over D. Then we use the fact that surgery on $S^n(K_1)$ along γ_1 with framing not equivalent to f_1 yields the pair $(S^n_{K_1}, T(K_1))$. By the above argument, $S^n_{K_1}$ is diffeomorphic to the boundary of the $(n+1)$-ball V, i.e. $S^n_{K_1} \cong S^n$, and $A = T(K_1)$. This implies that

$$X(K_2) = V \cup_A \nu D \cong D^{n+1} \cup_{T(K_1) = S^{n-2}\times\{0\}} (D^{n-1}\times D^2) = X(T(K_1)). \qquad \square$$

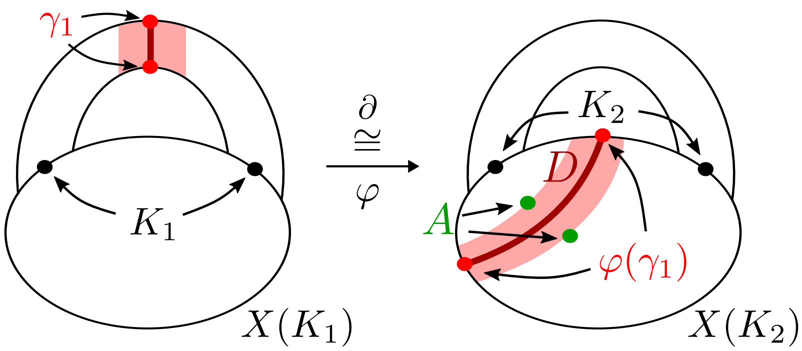

FIGURE 6. Extending a surgery diffeomorphism to the traces.

The same proof yields a similar result in the topological category for $n = 4$. The reason why we do not obtain a statement in the smooth setting for $n = 4$ is the failure of the smooth h-cobordism theorem in this dimension, which follows from the work of Donaldson [Don87].

Before stating our result in the topological category for $n = 4$, let us remark that the Gluck twist $T(K)$ of a knot K is a smooth 2-knot in the possibly exotic 4-sphere S^4_K (see Remark 5.2). For such knots we define the *trace* $X(T(K))$ of $T(K)$ by attaching a 3-handle along $T(K)\times\{1\}$ to $S^4_K \times [0,1]$ and then gluing D^4 to the resulting manifold via a homeomorphism between ∂D^4 and $S^4_K \times \{0\}$. It follows from the Alexander trick that this manifold is well-defined up to orientation-preserving homeomorphism.

Theorem 5.6. *Let K_1 and K_2 be 2-knots in S^4 and let $\varphi\colon S^4(K_1) \to S^4(K_2)$ be an orientation-preserving diffeomorphism between their surgeries. Then either there is an orientation-preserving diffeomorphism $X(K_1) \cong X(K_2)$ or there is an orientation-preserving homeomorphism $X(T(K_1)) \approx X(K_2)$. If the Gluck twist $S^4_{K_1}$ is a standard smooth 4-sphere, then we can conclude (as in the case $n \geq 5$) that either $X(K_1) \cong X(K_2)$ or $X(T(K_1)) \cong X(K_2)$.*

Proof. The proof is exactly the same as in the case $n \geq 5$. In particular, if the push-forward $\varphi_*(f_1)$ of the framing f_1 of γ_{K_1} extends over D, then we can still conclude that V is a smooth 5-disc since ∂V is a standard smooth 4-sphere and we can hence apply [Kos93, Chapter VIII, Corollary 4.7]. If $\varphi_*(f_1)$ does not extend over D, then we can only conclude that V is a topological 5-disc, and hence there is a homeomorphism $X(T(K_1)) \approx X(K_2)$. However, since ∂V is identified with the Gluck twist $S^4_{K_1}$, in the case in which $S^4_{K_1}$ is a standard S^4 we can again apply [Kos93, Chapter VIII, Corollary 4.7] to conclude that V is a standard smooth 5-disc and find a diffeomorphism $X(T(K_1)) \cong X(K_2)$. □

Remark 5.7. In the cases in which S^{n+1} has a unique smooth structure (see Remark 2.1), the diffeomorphisms $X(K_1) \cong X(K_2)$ and $X(T(K_1)) \cong X(K_2)$ can be constructed as smooth extensions of the map $\varphi\colon S^n(K_1) \to S^n(K_2)$ in Theorem C and Theorem 5.6. This follows from the fact that one can extend the identification induced by φ of (S^n, K_1) or $(S^n, T(K_1))$ with $(\partial V, A)$ by noticing that every self-diffeomorphism of the n-sphere extends to the $(n+1)$-ball in this case; see the next lemma. Note that any self-homeomorphism of the n-sphere extends to the $(n+1)$-ball by the Alexander trick. Therefore we can always extend φ to a homeomorphism $X(K_1) \approx X(K_2)$ or $X(T(K_1)) \approx X(K_2)$.

Lemma 5.8 (See e.g. [Kos93, Chapter VIII, Corollary 5.6].)**.** *If the $(n+1)$-dimensional smooth Poincaré conjecture is true, then any self-diffeomorphism of the n-sphere extends to a diffeomorphism of the $(n+1)$-disc.* □

5.3. **Extending surgery diffeomorphisms.** By Theorem 5.6 and Remark 5.7, any surgery diffeomorphism $\varphi\colon S^4(K_1) \to S^4(K_2)$ for 2-knots K_1 and K_2 in S^4 extends either to a trace diffeomorphism $X(K_1) \to X(K_2)$ or to a homeomorphism $X(T(K_1)) \to X(K_2)$. The following theorem provides the complete answer as to which of these two cases occurs, depending on the algebraic topology of the traces.

Theorem D. *Let K_1 and K_2 be 2-knots in S^4. An orientation-preserving surgery diffeomorphism $\varphi\colon S^4(K_1) \to S^4(K_2)$ extends to an orientation-preserving trace diffeomorphism $\Phi\colon X(K_1) \to X(K_2)$ if and only if the closed 5-manifold $X(K_1) \cup_\varphi -X(K_2)$ is spin.*

Both the trace and the surgery along a 2-knot K are spin, since

$$H^2(X(K); \mathbb{Z}/2) = 0 = H^2(S^4(K); \mathbb{Z}/2).$$

Recall that isomorphism classes of spin structures on a CW complex X are in bijection with $H^1(X; \mathbb{Z}/2)$ (see e.g. [GS99, Section 1.4]). Since $H^1(X(K_i); \mathbb{Z}/2) = 0$ for each $i \in \{1,2\}$, there is a unique spin structure $\mathfrak{s}_i$ on $S^4(K_i)$, up to isomorphism, which extends over $X(K_i)$. Therefore, Theorem D says that an orientation-preserving surgery diffeomorphism $\varphi\colon S^4(K_1) \to S^4(K_2)$ extends to an orientation-preserving trace diffeomorphism $\Phi\colon X(K_1) \to X(K_2)$ if and only if $\varphi^*(\mathfrak{s}_1) = \mathfrak{s}_2$.

Remark 5.9. Theorem D is analogous to the corresponding result for 1-knots referred to in [MP23, Theorem 3.7], which is deduced as a special case of Boyer's classification of simply-connected 4-manifolds with boundary [Boy86]. The aforementioned result concerns the extension of homeomorphisms of the 0-surgeries of 1-knots to homeomorphisms of the 0-framed traces. In the case of 2-knots, however, we prove a statement in the smooth category. On the other hand, we note that Theorems C and D also hold true in the topological category, i.e. for locally flat knots in S^n.

More care must be taken in this category when defining and working with all the relevant terms. We will not do this here, but just refer the reader to the excellent survey by Friedl–Nagel–Orson–Powell [FNOP25].

Proof of Theorem D. We follow the proof indicated by [MP23, Theorem 3.7], adapting it to one dimension up, and using Barden's classification of simply-connected 5-manifolds [Bar65].

The "only if" argument is the same as in the classical case. Assume that φ extends to a diffeomorphism Φ. From Φ we can build a diffeomorphism

$$F\colon X(K_1) \cup_{\mathrm{id}} -X(K_1) \to X(K_1) \cup_{\varphi} -X(K_2).$$

Since $X(K_1)$ is spin, also $X(K_1) \cup_{\mathrm{id}} -X(K_1)$ is spin, and thus $X(K_1) \cup_{\varphi} -X(K_2)$ must be spin as well.

For the "if" direction, we wish to use the assumption that $Z := X(K_1) \cup_{\varphi} -X(K_2)$ is spin to construct an h-cobordism between the traces. First, notice that Z is a closed, simply-connected 5-manifold. Furthermore, Z is spin and has $H_2(Z) \cong \mathbb{Z}$, so by the classification of simply-connected 5-manifolds [Bar65, Theorem 2.3] we obtain a diffeomorphism $Z \cong S^2 \times S^3$. Thus Z bounds a 6-manifold W diffeomorphic to $D^3 \times S^3$, which provides us with a cobordism between $X(K_1)$ and $X(K_2)$. We will show that it is an h-cobordism. Indeed, by the Hurewicz and Whitehead theorems, it suffices to show that the inclusions $X(K_i) \to W$ induce isomorphisms on each homology group. Observe that the only homology groups we need to consider are $H_3(X(K_i))$, $i \in \{1, 2\}$, and $H_3(W)$, which are all isomorphic to $\mathbb{Z}$. Indeed, the inclusions $X(K_i) \hookrightarrow Z$ and $Z \hookrightarrow W$ induce isomorphisms on the level of H_3. Hence W gives rise to an h-cobordism. By the 6-dimensional h-cobordism theorem [Sma62], we obtain a diffeomorphism $X(K_1) \to X(K_2)$ extending φ. $\square$

Remark 5.10. For the proof of Theorem D, we used Barden's classification of simply-connected 5-manifolds [Bar65]. We record the following alternative direct approach, which could potentially be used to generalise Theorem D to all dimensions $n \geq 4$. First, use Whitehead's theorem and the fact that $Z = X(K_1) \cup_{\varphi} -X(K_2)$ is spin to construct a homotopy equivalence of Z to $S^2 \times S^{n-1}$. Then try to use surgery theory of simply-connected manifolds to upgrade this homotopy equivalence to a diffeomorphism. The rest of the proof would proceed exactly as the proof of Theorem D.

The obstruction to this technique lies in the surgery step. The structure set of $S^2 \times S^{n-1}$ for $n \geq 5$ has been determined in both the smooth [Cro10] and topological category [LM24, Theorem 18.11]. These structure sets depend on n, and in the smooth category can be complicated to compute. Topologically, the situation is slightly more manageable. In particular, when n is even, $\mathcal{S}^{\mathrm{TOP}}(S^2 \times S^{n-1}) \cong \mathbb{Z}_2$, and one can modify the homotopy equivalence $Z \to S^2 \times S^{n-1}$ with a Novikov pinch (see for example [CH90]) to obtain a homeomorphism. The topological structure set is more complicated for other values of n, where this method of altering the homotopy equivalence is ineffective. In the smooth category, it is also difficult to apply this pinching, since one has to factor in the existence of exotic spheres.

We observe that Theorem D implies the following corollary, which is in sharp contrast to the situation in dimension $n = 3$ (see for example [Akb91b, Akb91a, AM97]).

Corollary 5.11. *Let K_1 and K_2 be 2-knots in S^4. If there exists an orientation-preserving trace homeomorphism $X(K_1) \to X(K_2)$ that restricts to a diffeomorphism of the surgeries, then the traces $X(K_1)$ and $X(K_2)$ are orientation-preservingly diffeomorphic.*

Proof. Let $\Phi\colon X(K_1) \to X(K_2)$ be a homeomorphism such that its restriction to the surgeries $\varphi\colon S^4(K_1) \to S^4(K_2)$ is a diffeomorphism. Then we can build the smooth 5-manifold $X(K_1) \cup_\varphi -X(K_2)$. By the same argument as in the proof of Theorem D, this manifold is spin and thus φ extends to a diffeomorphism of the traces by Theorem D. Note that this does not imply that the original map Φ is a diffeomorphism. $\square$

Remark 5.12. The "only if" direction of Theorem D is true more generally for any $n \geq 3$, i.e. given knots K_1, K_2 in S^n, if a surgery diffeomorphism $\varphi\colon S^n(K_1) \to S^n(K_2)$ extends to a trace diffeomorphism $X(K_1) \to X(K_2)$, then the closed $(n+1)$-manifold $X(K_1) \cup_\varphi -X(K_2)$ is spin. For $n = 3$, this was proven in [MP23], whereas for $n \geq 5$ the proof works exactly the same as the one for $n = 4$ given in the proof of Theorem D. We believe that the "if" direction of Theorem D is true in all dimensions, too, and thus also Corollary 5.11; see Question 8.5 and the discussion there.

The following criterion is a direct consequence of Theorem 5.6 and Theorem D and provides a potential tool for distinguishing the isotopy class of a 2-knot K in S^4 from that of its Gluck twist $T(K)$.

Corollary 5.13. *Let K be a 2-knot in S^4 such that the Gluck twist S^4_K is a standard smooth S^4. Then $X(K) \ncong X(T(K))$ if and only if for every orientation-preserving diffeomorphism $\varphi\colon S^4(K) \to S^4(T(K))$ the manifold $X(K) \cup_\varphi -X(T(K))$ is not spin.* $\square$

6. Infinitely many knots with the same trace in dimensions ≥ 5

In this section, we prove Theorem B, restated below, which demonstrates for $n \geq 5$ the existence of infinitely many non-isotopic knots in S^n with orientation-preservingly diffeomorphic traces. This will be deduced by combining Plotnick's work in [Plo83b] with Theorem C.

Theorem B. *Let $n \geq 5$. There is an infinite collection $\{X_i\}_{i\geq 1}$ of pairwise non-diffeomorphic smooth $(n+1)$-manifolds such that, for each integer $i \geq 1$, there exists an infinite collection $\{K^i_j\}_{j\geq 1}$ of pairwise non-isotopic knots in S^n whose trace is orientation-preservingly diffeomorphic to X_i, i.e.*

$$X(K^i_j) \cong X_i \qquad \textit{for all integers} \quad i, j \geq 1.$$

In particular, for each $i \geq 1$, this results in an infinite family of pairwise non-isotopic knots in S^n with orientation-preservingly diffeomorphic traces.

Proof. Let $n \geq 5$. In [Plo83b, Theorem 1], Plotnick shows, for $n \geq 4$, the existence of infinitely many distinct smoothly embedded discs D^{n-1} in D^{n+1} with the same exterior. Theorem B will follow from looking at the boundaries of Plotnick's construction, paying particular attention to framings. There are infinitely many Brieskorn spheres $\Sigma(p,q,r)$ which bound a contractible smooth 4-manifold; see e.g. [CH81]. For each such Brieskorn sphere, Plotnick produces an $(n+1)$-manifold $X(p,q,r)$ and an infinite family of pairwise inequivalent (and thus non-isotopic) knots in S^n having $X(p,q,r)$ as their trace. The details are as follows.

Let $\Sigma(p,q,r)$ be a Brieskorn sphere which bounds a contractible smooth 4-manifold W. For $n \geq 5$, we define the n-manifold

$$V := (W \times S^{n-4}) \cup h_{n-3}(\{\mathrm{pt}\} \times S^{n-4}),$$

where the $(n-3)$-handle h_{n-3} is attached to $W \times S^{n-4}$ along $\{\mathrm{pt}\} \times S^{n-4}$. For $n = 4$, we define $V := W$. We obtain, for every $n \geq 4$, a contractible n-manifold V, whose boundary $\Sigma^{n-1} := \partial V$ is a homology $(n-1)$-sphere with fundamental group $\pi_1(\Sigma^{n-1}) \cong \pi_1(\partial W) \cong \pi_1(\Sigma(p,q,r))$.

In [Plo83b], Plotnick shows that there exists an infinite collection $A^{p,q,r}$ of normal generators of $\pi_1(\Sigma^{n-1})$ that are *algebraically distinct modulo the center* $Z(\pi_1(\Sigma^{n-1}))$. This means that, for any pair of distinct elements $\alpha, \beta \in A^{p,q,r}$, there is no automorphism of $\pi_1(\Sigma^{n-1})$ sending α to $(\beta c)^{\pm 1}$, for any $c \in Z(\pi_1(\Sigma^{n-1}))$. Set $t := [\{\mathrm{pt}\} \times S^1] \in \pi_1(\Sigma^{n-1} \times S^1)$ and perform loop surgeries on $\Sigma^{n-1} \times S^1$, with either of the two possible framings, along simple loops representing the elements

$$\{\alpha t \in \pi_1(\Sigma^{n-1} \times S^1) \mid \alpha \in A^{p,q,r}\}.$$

Note that the results of these loop surgeries on $\Sigma^{n-1} \times S^1$ correspond to the boundaries of the manifolds obtained by gluing a 2-handle $h_2^\alpha = D^2 \times D^{n-1}$ to $V \times S^1$ along the simple loops representing αt. Using the smooth h-cobordism theorem, Plotnick shows that the manifolds obtained by loop surgery are always smooth n-spheres. The cores $\{\mathrm{pt}\} \times S^{n-2} \subseteq D^2 \times \partial D^{n-1}$ of the surgeries thus give an infinite family of knots

$$\mathcal{K}_{p,q,r} = \{K_\alpha \colon S^{n-2} \hookrightarrow S^n\}_{\alpha \in A^{p,q,r}}$$

which are pairwise inequivalent, and in particular non-isotopic by Lemma 2.5. In fact, any equivalence between K_α and K_β would induce an automorphism of the group $\pi_1(\Sigma^{n-1} \times S^1)$ sending αt to βt (up to conjugacy). Such an automorphism produces, by restriction, an automorphism of $\pi_1(\Sigma^{n-1})$ sending α to βc, for some central element c, thus contradicting the assumptions on $A^{p,q,r}$. Here, we are using that $\pi_1(\Sigma^{n-1})$ is preserved by any automorphism of $\pi_1(\Sigma^{n-1} \times S^1)$, the former being the commutator subgroup of the latter[9], and that any automorphism of $\pi_1(\Sigma^{n-1} \times S^1)$ that induces the identity on the abelianisation must send t to ct for some central element $c \in \pi_1(\Sigma^{n-1})$. By construction, the surgery of any of these knots is orientation-preservingly diffeomorphic to $\Sigma^{n-1} \times S^1$. Since performing surgery on a loop with distinct framings yields knots that are related by a Gluck twist, Theorem C implies that the collection $\mathcal{K}_{p,q,r}$ can be built in such a way that all of its elements have diffeomorphic traces, by choosing in the above construction the correct framing of the simple loop that represents αt. We denote this shared trace by $X(p,q,r)$.

[9] Recall that Σ^{n-1} is a homology sphere.

Finally, note that the trace $X(p,q,r)$ of the knots in $\mathcal{K}_{p,q,r}$ is not diffeomorphic to the trace $X(p',q',r')$ of the knots in the family $\mathcal{K}_{p',q',r'}$ which is constructed analogously but using another Brieskorn sphere $\Sigma(p',q',r')$ as a starting point. Indeed, whenever we have $\{p,q,r\} \neq \{p',q',r'\}$, the traces $X(p,q,r)$ and $X(p',q',r')$ can be distinguished by the fundamental groups of their boundaries, namely $\pi_1(\Sigma(p,q,r))\times\mathbb{Z}$ and $\pi_1(\Sigma(p',q',r')) \times \mathbb{Z}$, respectively. In fact, the commutators of these groups are $\pi_1(\Sigma(p,q,r))$ and $\pi_1(\Sigma(p',q',r'))$, respectively, and they differ by the discussion at the end of Section 3 in [Mil75], where the fundamental group of the Brieskorn sphere is the group denoted by $[\Gamma,\Gamma]$. $\square$

Remark 6.1. Plotnick's construction [Plo83b] produces infinite families of *slice* knots in S^n which are pairwise inequivalent. Indeed, gluing the 2-handle h_2^α to $V \times S^1$ along a representative of αt yields a contractible $(n+1)$-manifold bounded by a homotopy n-sphere. By the smooth h-cobordism theorem, this is a smooth $(n+1)$-disc. A slice disc for the knot K_α is hence given by the cocore of h_α^2. In particular, such slice discs correspond to the 'knotted ball pairs' constructed by Plotnick in [Plo83b, Theorem 1].

A similar argument yields Theorem B in the topological category for $n = 4$, again following Plotnick's work.

Theorem 6.2. *There is an infinite collection $\{X_i\}_{i\geq 1}$ of pairwise non-homeomorphic topological 5-manifolds such that, for each integer $i \geq 1$, there exists an infinite collection $\{K_j^i\}_{j\geq 1}$ of pairwise non-isotopic locally flat 2-knots in S^4 whose trace is orientation-preservingly homeomorphic to X_i, i.e.*

$$X(K_j^i) \approx X_i \qquad \text{for all integers} \quad i,j \geq 1.$$

In particular, for each $i \geq 1$, this results in an infinite family of pairwise non-isotopic locally flat 2-knots in S^4 with orientation-preservingly homeomorphic traces.

Proof. This follows from noticing that the construction used in the proof of Theorem B works in the topological category for $n = 4$. In particular, construct the 4-manifold $\Sigma^3 \times S^1$ as in the proof of Theorem B starting from a Brieskorn sphere $\Sigma(p,q,r)$ which bounds a contractible smooth 4-manifold. Performing surgery on $\Sigma^3 \times S^1$ along any curve representing αt for $\alpha \in A^{p,q,r}$ as above yields a smooth knot K_α in a smooth homotopy 4-sphere S_α^4 which bounds a contractible 5-manifold. The topological h-cobordism theorem [Fre82] then implies that each S_α^4 is homeomorphic to S^4.[10] Hence in this way we produce an infinite family of locally flat knots in S^4, which are pairwise non-isotopic for the same reasons as described in the proof of Theorem B. Exactly as we explained for Gluck twists of knots before Theorem 5.6, we can define the (topological) trace of K_α. By construction, for any pair $\alpha, \beta \in A^{p,q,r}$, the manifolds $S_\alpha^4(K_\alpha)$ and $S_\beta^4(K_\beta)$ are orientation-preservingly diffeomorphic. This implies, by a slight variation of the proofs of Theorem C and Theorem 5.6, that $X(K_\beta)$ is orientation-preservingly homeomorphic either to $X(K_\alpha)$ or to $X(T(K_\alpha))$.[11] We conclude as in the proof of Theorem B. $\square$

[10] However, we are not able to prove the result in the smooth category for $n = 4$ since, as implied by the work of Donaldson in [Don87], the smooth h-cobordism theorem is false in this dimension and such a homotopy 4-sphere could potentially be an exotic S^4.

[11] Notice that, even if the trace is now only a topological manifold, $S_\beta^4 \times [0,1]$ union the 3-handle contained in the trace has a smooth structure and is simply connected, so we can use the theory of

Theorem 6.3. *For every $N > 0$, there exists a family of more than N pairwise non-isotopic 2-knots in S^4 with orientation-preservingly diffeomorphic traces.*

Proof. The construction of these families is essentially given in [Plo83a, Section 3], to which we refer the reader for more details. Fix p, q, r positive, pairwise coprime integers such that $\Sigma(p,q,r) \not\cong \Sigma(2,3,5)$. Recall that $\Sigma(p,q,r)$ is the r-fold branched cover of S^3 branched along the torus knot $T(p,q)$, and that the branched covering automorphisms form a group which is canonically identified with $\mathbb{Z}/r\mathbb{Z}$ (see e.g. the beginning of [Plo83a, Section 1]). Let σ denote the canonical generator of this group corresponding to $1 \in \mathbb{Z}/r\mathbb{Z}$. It is proved in [Zee65] that the r-twist spun knot of $T(p,q)$ is a fibred knot, with fibre $\Sigma(p,q,r) \setminus \operatorname{int} B^3$, and monodromy given by the restriction of σ. For $k > 0$ and $\gcd(k,r) = 1$, the k-branched cover of S^4 along this knot defines a knot $K^{r,k}_{p,q}$ in S^4 whose exterior fibres over S^1 with fibre $\Sigma(p,q,r) \setminus \operatorname{int} B^3$ and monodromy given by the restriction of σ^k [Pao78, Plo83a]. It is shown in [Plo83a, Proposition 3.5] that if $k \not\equiv \pm k' (\operatorname{mod} r)$, then the knots $K^{r,k}_{p,q}$ and $K^{r,k'}_{p,q}$ are not isotopic. Moreover, σ is isotopic to the identity (see for example the proof of Lemma 1.1 in [Mil75] and [Orl72, p. 143]), hence $S^4(K^{r,k}_{p,q}) \cong \Sigma(p,q,r) \times S^1$. By permuting p, q, r and varying k accordingly, one obtains a finite family of knots with the same surgery, which are not isotopic by [Plo83a, Theorem 3.7]. Up to taking Gluck twists, which in this case are again smooth spheres in S^4 by [Pao78, Theorem 5], these knots have orientation-preservingly diffeomorphic traces by Theorem 5.6. Taking p, q, r arbitrarily large, this gives arbitrarily large families of knots with the same trace. □

Note that Theorem B (for $n \geq 5$) and Theorem 6.3 (for $n = 4$) together provide an alternative proof of Theorem A.

7. Detecting the unknot by its surgery in higher dimensions

In the previous sections, we have seen that it is possible to construct many non-isotopic knots in S^n with the same trace, and therefore the same surgery. In this section, we will show that, in contrast to these results, the unknot in S^n is detected by its surgery, and consequently by its trace. Throughout this section, we will assume that $n \geq 4$.

The same reasons that make it more difficult to distinguish non-isotopic knots with the same trace in higher dimensions simplify the proof that, in higher dimensions, the unknot is detected by its surgery (and hence by its trace), compared to analogous results in dimension 3. Our statement will not only involve the unknot, but also any *unknotted* embedding of S^{n-2} in S^n (see also Remark 2.1).

Definition 7.1. A knot $K \colon S^{n-2} \hookrightarrow S^n$ is *unknotted* if its image bounds a ball.

We restate Theorem E for the convenience of the reader.

Theorem E. *For $n \geq 4$, let K be a knot in S^n. Suppose that its surgery $S^n(K)$ is (possibly orientation-reversingly) diffeomorphic to the surgery $S^n(U) \cong S^{n-1} \times S^1$ of*

smooth manifolds to find the embedded disc with tubular neighbourhood needed to start the proof of Theorem C (see Section 5.2). Then we proceed as in the proof of Theorem 5.6.

the unknot U. Then K is isotopic to U. In particular, if K and U have diffeomorphic traces, then K is isotopic to U.

We will prove Theorem E as an application of the following generalisation.

Theorem 7.2. *Let K_1 and K_2 be knots in S^n and suppose that K_2 is unknotted. If there exists a (possibly orientation-reversing) diffeomorphism between the surgeries $S^n(K_1)$ and $S^n(K_2)$, then K_1 is isotopic to K_2. In particular, if K_1 and K_2 have diffeomorphic traces, then K_1 is isotopic to K_2.*

Proof of Theorem E. By specialising the statement of Theorem 7.2 to the case where K_2 is the unknot U, we obtain the statement of Theorem E. □

In order to prove Theorem 7.2, we start by proving Lemma 7.3 below. Note that the diffeomorphism type of the exterior of a knot K does not depend on the choice of the parametrisation of K, but only on its image. In particular, the exterior of any unknotted knot in S^n is diffeomorphic to $D^{n-1} \times S^1$.

Lemma 7.3. *Let K_1 be a knot in S^n such that $S^n(K_1) \cong S^n(K_2)$, where K_2 is unknotted. Then the exterior of K_1 is diffeomorphic to $D^{n-1} \times S^1$.*

Proof. Let γ_{K_1} and γ_{K_2} denote the dual curves of K_1 and K_2 in the respective surgeries. Since $E_{K_2} \cong D^{n-1} \times S^1$, the fundamental group of $S^n(K_2)$ is isomorphic to $\mathbb{Z}$, generated by γ_{K_2}, by Proposition 4.1. If there is a diffeomorphism $\varphi\colon S^n(K_1) \to S^n(K_2)$, then $S^n(K_1)$ also has fundamental group isomorphic to $\mathbb{Z}$, and again by Proposition 4.1, the same holds for the exterior E_{K_1}. This implies that $\pi_1(E_{K_1}) \cong \mathbb{Z}$ is generated by μ_{K_1} and, by Proposition 4.1 once again, we deduce that $\pi_1(S^n(K_1))$ is generated by a conjugate of the dual curve γ_{K_1} for some orientation. The diffeomorphism φ thus maps the curve γ_{K_1} to a curve that (up to orientation) is freely homotopic to γ_{K_2}, since both curves represent the unique generator of $\pi_1(S^n(K_1)) \cong \pi_1(S^n(K_2)) \cong \mathbb{Z}$ up to sign. Since $n \geq 4$, we obtain that $\varphi(\gamma_{K_1})$ and γ_{K_2} are isotopic up to orientation. By the isotopy extension theorem, there is an ambient isotopy of $S^n(K_2)$ taking $\varphi(\gamma_{K_1})$ to γ_2, still up to orientation. This yields a diffeomorphism $\psi\colon S^n(K_2) \to S^n(K_2)$ such that $\psi(\varphi(\gamma_1)) = \gamma_2$, up to orientation. Now the composition $\psi \circ \varphi$, restricted to the exterior of γ_1, gives a diffeomorphism from the exterior E_{K_1} of γ_{K_1} to the exterior E_{K_2} of γ_{K_2}, where $E_{K_2} \cong D^{n-1} \times S^1$. □

Next, we prove that, under the same hypotheses, K_1 also defines an unknotted embedding.

Proposition 7.4. *Let K_1 be a knot in S^n such that $S^n(K_1) \cong S^n(K_2)$, where K_2 is unknotted. Then K_1 is also unknotted.*

Proof. We show that K_1 and K_2 are isotopic up to reparametrisation of the embedding, which implies the desired result since unknottedness only depends on the image of the embedding. By Lemma 7.3, the exterior of K_1 is diffeomorphic to $D^{n-1} \times S^1$. For $n \geq 5$, it was shown by [Sha68, Lev65b, Lev70, Tro73] that if the exterior of a knot in S^n is homotopy equivalent to a circle, then it is unknotted. This finishes the proof for $n \geq 5$, so let $n = 4$ for the rest of the proof. In [Glu62] it was proven that if the

2-knots K_1 and K_2 in S^4 have diffeomorphic exteriors, then, up to reparametrisation, either they are isotopic, or K_1 is isotopic to $T(K_2)$, the Gluck twist of K_2. So the image of K_1 can be identified with that of either K_2 or $T(K_2)$. Since K_2 is unknotted and the map used to define the Gluck twist (see Definition 5.1) extends to a diffeomorphism of $D^3 \times S^1$, we have that $T(K_2)$ is unknotted as well. Thus, in either case, K_1 is unknotted. □

Proof of Theorem 7.2. Suppose that $S^n(K_1)$ is diffeomorphic to $S^n(K_2)$, where K_2 is unknotted. If the diffeomorphism is not orientation-preserving, then we can compose it with an orientation-reversing diffeomorphism of $S^n(K_2)$, which is the product of a homotopy sphere and S^1, to obtain an orientation-preserving diffeomorphism between $S^n(K_1)$ and $S^n(K_2)$. By Proposition 7.4, both K_1 and K_2 are unknotted and the result follows from the fact that two unknotted embeddings have diffeomorphic surgeries if and only if they are isotopic [Sha68, Theorem 1.3]. □

The above proof of Proposition 7.4, which shows that being unknotted is detected by the surgery along a knot, mainly uses the fact that the knot group of an unknotted sphere is $\mathbb{Z}$ and thus has exactly two normal generators. In fact, we can give a partial generalisation of this detection result as follows. Recall from Section 2.2 that the dual curve γ_K in the surgery of S^n along a knot K comes equipped with a preferred framing.

Lemma 7.5. *Let K_1 and K_2 be knots in S^n with dual curves γ_{K_1} and γ_{K_2} in the respective surgeries. Suppose that $\varphi\colon S^n(K_1) \to S^n(K_2)$ is an orientation-preserving diffeomorphism between their surgeries such that $\varphi_*\gamma_{K_1}$ and γ_{K_2} are conjugate in $\pi_1(S^n(K_2))$. Then either K_2 is isotopic to K_1 or the Gluck twist $T(K_2)$ is a knot in the standard smooth S^n which is isotopic to K_1. The former happens if and only if $\varphi(\gamma_{K_1})$ is isotopic to γ_{K_2} as a framed loop.*

Proof. Since $\varphi_*\gamma_{K_1}$ and γ_{K_2} are conjugate in $\pi_1(S^n(K_2))$, we can assume, after isotopy, that $\varphi(\gamma_{K_1}) = \gamma_{K_2}$. Then, as explained in the proofs of Theorem C and Theorem 5.6, the two possible conclusions depend on whether $\varphi(\gamma_{K_1})$ and γ_{K_2} agree as framed loops or not. In the first case, by doing loop surgery, we get an equivalence, and hence an isotopy by Lemma 2.5, between K_1 and K_2; in the second case, we get an orientation-preserving diffeomorphism between S^n and the Gluck twist of S^n along K_2, mapping K_1 to $T(K_2)$, and we conclude again by Lemma 2.5. □

Proposition 7.6. *Let K_1 and K_2 be knots in S^n with dual curves γ_{K_1} and γ_{K_2} in the respective surgeries. Let $\Phi\colon X(K_1) \to X(K_2)$ be an orientation-preserving diffeomorphism between their traces such that $\Phi_*\gamma_{K_1}$ and γ_{K_2} are conjugate in $\pi_1(S^n(K_2))$. Then K_1 is isotopic to K_2.*

Proof. pply Lemma 7.5 to $(\Phi|_{\partial X(K_1)})^{-1}$. Then either K_1 is isotopic to K_2, in which case we are done, or the Gluck twist $T(K_1)$ is a knot in the standard smooth S^n which is isotopic to K_2. Thus, we assume that Φ is an orientation-preserving diffeomorphism between $X(K_1)$ and $X \coloneqq X(T(K_1))$. Since the dual curves $\Phi_*\gamma_{K_1}$ and $\gamma \coloneqq \gamma_{T(K_1)}$ are conjugate in $\pi_1(\partial X)$, we can suppose that $\Phi_*\gamma_{K_1} = \gamma_{T(K_1)} = \gamma$. Notice that γ inherits two framings f and $T(f)$, from γ_{K_1} and $\gamma_{T(K_1)}$ respectively. We will obtain the desired equivalence between K_1 and $T(K_1)$ if we are able to show that f and $T(f)$ coincide.

Suppose for a contradiction that the two framings are distinct. Using the identification of X with $X(K_1)$ and $X(T(K_1))$, we can decompose X in two different ways by gluing an $(n+1)$-dimensional 2-handle to $\partial X \times [0,1]$ along $\gamma \times \{1\}$ with framing f and $T(f)$ respectively, and a single $(n+1)$-handle. Let D_1 and D_2 be the cores of such 2-handles, and consider the 2-sphere $S \coloneqq D_1 \cup D_2 \subset DX$, where $DX \coloneqq X \cup_{\mathrm{id}_{\partial X}} -X$ is the double of X, and D_1 and D_2 lie in different copies of X. In particular, since the framings f and $T(f)$ are distinct, the tubular neighbourhood νS is diffeomorphic to the total space of the vector bundle $\epsilon^{n-1} \oplus \eta$. Here ϵ denotes a trivial line bundle on S^2 and η is a rank two vector bundle on S^2 with clutching function a map $c\colon S^1 \to \mathrm{SO}(2)$ corresponding to an odd class in $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$. Since $H^2(X; \mathbb{Z}/2\mathbb{Z}) = 0$, we know that X is spin. In particular, DX is also spin. This gives a contradiction, since it would imply that

$$0 = w_2(DX)|_S = w_2(\nu S) + w_1(\nu S) \cup w_1(TS) + w_2(TS) = w_2(\nu S),$$

which is false since $\langle w_2(\nu S), [S] \rangle = 1 \pmod 2$. Therefore, the two framings f and $T(f)$ of γ must coincide, which implies that the two knots K_1 and $T(K_1)$ are isotopic in S^n. $\square$

8. Further questions

We end this article by posing some questions that naturally arise from the results presented in this article.

8.1. **Detecting knots by their surgeries or traces.** In Theorem E, we showed that the surgery along a knot detects the unknot for $n \geq 4$. One might ask whether this property holds for other knot types.

Question 8.1. *For $n \geq 4$, are there knots in S^n (other than unknotted ones) that are detected by their surgeries or traces? Is there an infinite family?*

Note that in dimension $n = 3$, it is well known that Question 8.1 can be answered affirmatively [Gab87, Ghi08, BS24a, BS24b]. Indeed, every knot is detected by infinitely many of its surgeries [Lac19] (see also [KS25]). We think that the weaker version for the traces would be interesting as well; see also e.g. [BS25, BKM25] for work on the case when $n = 3$.

8.2. **Traces vs surgeries.** In general, it would be interesting to understand the difference between equivalent traces and equivalent surgeries. In Theorems A and B, we constructed families of pairwise non-isotopic knots with diffeomorphic traces. The following question remains unsolved.

Question 8.2. *For $n \geq 4$, are there any pairs of knots in S^n with diffeomorphic surgeries, but whose traces are not diffeomorphic?*

Maybe a construction as described in Remark 3.9 could help to answer Question 8.2. Again, Question 8.2 is known to have a positive answer in dimension $n = 3$, see for example [Akb91b, Akb91a, AM97, Yas15, MP23]. A related question is the following.

Question 8.3. *For $n \geq 4$, do there exist knots K_1 and K_2 in S^n with a surgery diffeomorphism $\varphi\colon S^n(K_1) \to S^n(K_2)$ that does not extend to a diffeomorphism of the traces?*

According to Theorem D and Remark 5.12, it would be enough to construct a pair of knots such that $X(K_1) \cup_\varphi -X(K_2)$ is not spin in order to answer Question 8.3 positively. If, in addition, the mapping class group of the surgery $S^n(K_1) \cong S^n(K_2)$ is trivial, then we would also get a positive answer to Question 8.2.

8.3. **Homeomorphisms vs diffeomorphisms.** In dimension $n = 3$, there exist knots that have homeomorphic but non-diffeomorphic traces, see for example [Akb91b, Akb91a, AM97, Yas15]. In higher dimensions, this remains unclear.

Question 8.4. *Let $n \geq 4$. Do there exist knots in S^n that have homeomorphic, but not diffeomorphic, traces, up to any reparametrisation of the knot?*

When K_1 is an $(n-2)$-knot, where $n \geq 8$ is such that $\mathrm{MCG}(S^{n-2}) \neq 1$, we can reparametrise K_1 using a non-trivial element of $\mathrm{MCG}(S^{n-2})$ to obtain a new knot K_2 (see the discussion in Remark 2.1). The traces $X(K_1)$ and $X(K_2)$ will always be homeomorphic by the Alexander trick, but they will in general not be diffeomorphic. For example, if K_1 is the unknot and K_2 is a reparametrisation of K_1, then we have $X(K_1) \cong S^{n-1} \times D^2$, whereas $X(K_2) \cong \widetilde{S} \times D^2$, where $\widetilde{S}$ is an exotic S^{n-1}. However, the two manifolds are not diffeomorphic, as can be seen by using the Product Structure Theorem (see Theorem 5.1 and Remark 2 thereafter in [KS77, Essay I]). The question remains open whether there are knots K_1 and K_2 such that for every possible reparametrisation of the knots, their traces are homeomorphic but not diffeomorphic.

For $n = 4$, Corollary 5.11 shows that if there exists a trace homeomorphism that restricts to a diffeomorphism of the surgeries, then the traces are already diffeomorphic. So to answer Question 8.4 positively, one needs to construct homeomorphic traces where no trace homeomorphism restricts to a surgery diffeomorphism. The proof of Corollary 5.11 depends on Theorem D, which we only proved in dimension $n = 4$. See Remark 5.10 for comments on dimensions $n \geq 5$.

Question 8.5. *For $n \geq 5$, let K_1 and K_2 be knots in S^n and let $\varphi\colon S^n(K_1) \to S^n(K_2)$ be a surgery diffeomorphism such that $X(K_1) \cup_\varphi -X(K_2)$ is spin. Does φ extend to a trace diffeomorphism $\Phi\colon X(K_1) \to X(K_2)$?*

As mentioned above, this holds for $n = 4$ by Theorem D. The converse direction is true in all dimensions, as discussed in Remark 5.12.

8.4. **Gluck twists.** Gluck twisting a knot in S^4 yields a smooth 4-manifold S^4_K which is always homeomorphic to S^4. For many knots K it is known that S^4_K is even diffeomorphic to S^4, but in general it remains unclear if S^4_K is always diffeomorphic to the standard 4-sphere (see Remark 5.2). In higher dimensions, we could not find such statements in the literature. In Proposition 5.3 we showed that for knots K in S^n with $n \geq 5$, the Gluck twist manifold S^n_K is diffeomorphic to S^n if K extends to an embedding $D^{n-1} \hookrightarrow D^{n+1}$. In general, however, this seems to be open (see also Remark 5.4).

Question 8.6. *For $n \geq 4$, given a knot K in S^n, is the Gluck twist S^n_K of S^n along K always diffeomorphic to the standard smooth n-sphere?*

8.5. **Other codimensions.** In this article, we have concentrated on knots in codimension 2. The RBG construction works in any codimension and we can thus ask many of the above questions for knotted spheres in any codimension.

Question 8.7. *Do there exist non-isotopic knotted k-spheres in S^n that have diffeomorphic traces or surgeries for $k \neq n-2$? Can the Haefliger spheres* [Hae62] *be realised by the RBG construction?*

Appendix A. Sage source code

Here we provide the Sage source code [GAP25, Sag25] used in Section 4.3.

A.1. **Counting representations.** The following function *repr* takes as input a finitely presented group H, a finite group A, an element $q \in A$ and an integer $m \geq 0$, and computes the number of homomorphisms from H to A which map the m-th generator of H (with labels starting from 0) to A.

```
from itertools import product
from sage.all import libgap, ConjugacyClass

def repr(H,A,q,m):
    R = 0                              # R = counter of reps
    L = A.list()                       # L = list of elements of A
    # T encodes the set of maps from the generators of H
    # (minus the m-th one) to elements of A
    T = product(L, repeat=len(H.gens())-1)
    F = list(T)                        # convert T into a list F
    for f in F:
        I = list(f)
        # add q such that the m-th generator is mapped to q
        I.insert(m,q)
        # check if assignement defined by I defines a homom.
        if (libgap.fail!=libgap.GroupHomomorphismByImages
            (H,A,H.gens(),I)):
            R = R+1                    # increase the counter R
    return R
```

A.2. **Example.** Here is an example of the usage of the function *repr* to count representations of the group F_1 into A_5 as needed in the proof of Proposition 4.11.

```
F.<x,y,a> = FreeGroup()
H = F/[x^-1*y*x*y*x^-1*y^-1, x^-1*a*x*a^-1*x^-1*y*a*y^-1]
A = AlternatingGroup(5)
q = A[1]                   # -> q = (1,5,4,3,2)
R = repr(H,A,q,0)          # -> R = 6
S = repr(H,A,q,2)          # -> S = 1
```

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