COUNTING RATIONAL POINTS ON SMOOTH QUARTIC AND QUINTIC SURFACES

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ABSTRACT. Let X be a smooth projective surface of degree $d \geq 4$ defined over a number field K, and let $N_{X'}(B)$ be the number of rational points of X of height at most B that do not lie on lines contained in X. Assuming a suitable hypothesis on the size of the rank of abelian varieties, we show that $N_{X'}(B) \ll_X B^{4/3+\varepsilon}$ for any fixed $\varepsilon > 0$. This improves an unconditional and uniform bound from Salberger for $d = 4$ and $d = 5$. The proof, based on an argument of Heath-Brown, consists of cutting X by projective planes and using a uniform version of Faltings's Theorem due to Dimitrov, Gao, and Habegger, to bound the number of rational points on the plane sections of X.

1. INTRODUCTION

Let K be a number field, and let H_K be the height function in $\mathbb{P}^n$ given by:

$$H_K([x_0\colon \cdots : x_n]) = \prod_v \max\{\|x_1\|_v, \ldots \|x_n\|_v\},$$

where v runs through the places of K and $\|x\|_v$ is the normalized absolute value associated with the place v (this is the height function defined in [7, Section B.2]).

We consider a smooth hypersurface $X \subseteq \mathbb{P}^n_K$ of degree d, and we want to bound the number of rational points of bounded height on X. For this, define the counting function

$$N_X(B) := \#\{x \in X(K)\colon H_K(x) \leq B\}.$$

For $K = \mathbb{Q}$, Verzobio shows in [19] that for $n \geq 4$ and $d \geq 50$ we have

$$N_X(B) \ll_{n,d,\epsilon} B^{n-2+\varepsilon},$$

and gives other bounds for $n \geq 4$ and $d \geq 6$. It is conjectured that this bound holds for all $n \geq 4$ and $d \geq 3$.

For $n = 3$, it is clear that this bound does not hold if X contains a line defined over K, since in this case we would have $N_X(B) \gg B^2$. Therefore, when $X \subseteq \mathbb{P}^3$ is a smooth surface, we are interested in counting the rational points on the set $X' \subseteq X$, where X' is the complement of the union of all lines on X. By Lemma 2.2, for $d \geq 4$, X' is obtained from X by removing a finite number of lines. We define the counting function

$$N_{X'}(B) := \#\{x \in X'(K)\colon H_K(x) \leq B\}.$$

For $n = 3$ and $K = \mathbb{Q}$, Salberger shows in [16, Theorem 0.5] the bound

$$N_{X'}(B) \ll_d B^{3/\sqrt{d}}(\log B)^4 + B. \tag{1}$$

We aim to improve this bound (although not uniformly in X) for $d = 4$ and 5, using a technique developed by Heath-Brown in [4] and assuming a conjectural bound for the rank of abelian varieties. In his paper, Heath-Brown shows that for $d = 3$ and assuming such a hypothesis, we have, for any fixed $\varepsilon > 0$, a bound

$$N_{X'}(B) \ll_{X,\varepsilon} B^{4/3+\varepsilon}.$$

We will cover the points of bounded height with projective planes of bounded height in the dual projective space $(\mathbb{P}^3)^\vee$. This can be done using a result from Schmidt ([17]). We then apply another result from the same paper to bound the number of such planes asymptotically, and proceed by estimating the number of points of bounded height in the intersection between X and each of these planes. This intersection will be generically a smooth curve. In this case, we use a uniform version of Faltings's Theorem, due to Dimitrov, Gao and Habegger:

Lemma 1.1 (Uniform Faltings's Theorem ([2]))**.** *For each integer $g \geq 2$, there exists a constant $c(g) > 0$ depending on $[K\colon \mathbb{Q}]$ such that, for every curve C over K of (geometric) genus g, we have*

$$\#C(K) \leq c(g)^{1+\rho(C)},$$

where $\rho(C)$ denotes the rank of $\mathrm{Jac}(C)(K)$*.*

This is [2, Theorem 1]. Compare it also with [10, Theorem 4], which eliminates the dependency on the degree $[K : \mathbb{Q}]$.

We bound the number of rational points on curves of genus at least 2 that appear as irreducible components of the intersections between X and rational planes by applying the above result to their normalizations.

We can proceed similarly to bound the number of rational points on the irreducible components of genus 1, by using a uniform result due to Heath-Brown and Testa ([5]). Finally, we show that the curves of genus 0 that appear do not alter our estimation.

To reach our goal, we require the following hypothesis:

Hypothesis 1.2 (Rank Hypothesis)**.** *For any K-abelian variety A, let N_A denote its conductor (that is, the ideal norm of the conductor ideal of A) and let r_A denote its rank. Then we have $r_A = o(\log N_A)$ as $N_A \to \infty$.*

This hypothesis is formulated in [4] for elliptic curves over the rational numbers.

We remark that what we actually need is the a priori weaker bound $r_A = o(h_F(A))$, where $h_F(A)$ denotes the Faltings's height of A. However, the plausibility of the Rank Hypothesis is suggested by the following analytical argument:

Assuming modularity for the abelian variety A over the number field K, this hypothesis would hold for A by a combination of the Generalized Riemann Hypothesis and the Birch and Swinnerton-Dyer conjecture. Indeed, let $L(A, s)$ be the L-function associated with the abelian variety A. Then, if we define

$$\Lambda(A, s) \coloneqq N_A^{s/2}((2\pi)^{-s} \cdot \Gamma(s))^{\dim A} \cdot L(A, s),$$

by the modularity of A the function $\Lambda(A, s)$ extends to an entire function and satisfies the functional equation

$$\Lambda(A, 2 - s) = \pm\Lambda(A, s).$$

Assuming the Generalized Riemann Hypothesis and proceeding by classical analytic arguments, this functional equation would give us

$$\mathrm{ord}_{s=1}(L(A,s)) = O\left(\frac{\log N_A}{\log\log N_A}\right) = o(\log N_A).$$

Assuming the Birch and Swinnerton-Dyer conjecture, we obtain

$$\mathrm{ord}_{s=1}(L(A,s)) = r_A,$$

which gives the Rank Hypothesis. See [13, Proposition II.1] for a detailed proof of these implications in the case of the L-function of a modular form.

We now cite some known results that go in the direction of the Rank Hypothesis. By [14, Theorem 1], for an abelian variety A/K we have, unconditionally,

$$r_A = O_{K,\dim A}(N_A).$$

In the case of elliptic curves, more evidence is known. Notice that all elliptic curves over the rational number are modular by [1]. If we assume that an elliptic curve $E/\mathbb{Q}$ has at least one rational point of order 2, then it is known that (see [4, Section 6])

$$r_E = O\left(\frac{\log N_E}{\log\log N_E}\right) = o(\log N_E).$$

Our main result is the following:

Theorem 1.3. *Assume that the Rank Hypothesis holds. Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$, and let $\varepsilon > 0$. Then $N_{X'}(B) \ll_{X,K,\varepsilon} B^{4/3+\varepsilon}$.*

This exponent is smaller than the one in (1) for $d = 4$ and $d = 5$, thus improving the known bound in this case.

Another advantage of our method is that it works over any number field, so we obtain the same bound for every choice of K. In the proof of Theorem 1.3, we use a bound for the number of rational points of bounded height in an integral projective curve of a given degree. Over $\mathbb{Q}$, this bound is due to Salberger ([16, Theorem 0.11]). To obtain our result for any number field K, we use a generalization of Salberger's bound due to Paredes and Sasyk ([15, Theorem 1.8]). However, notice that we use the height H_K of the field K, while in [15] they use the absolute height $H = H_K^{1/[K:\,\mathbb{Q}]}$.

We remark that, assuming the stronger conjecture that the rank of all abelian varieties is bounded in terms of their dimensions and the field K, our proof will show that we can remove the dependence on ε and obtain

$$N_{X'}(B) \ll_{X,K} B^{4/3}.$$

Some of the references use a slightly different definition of H_K, taking the Archimedean norm in infinite places instead of the maximum norm. However, since both norms differ by a multiplicative constant, this distinction will not matter in what follows, so we shall ignore it.

By genus, we always mean geometric genus, unless explicitly stated. Besides, we will denote by x_0, x_1, x_2, x_3 the coordinates of $\mathbb{P}^3$ and by $F(x_0, x_1, x_2, x_3)$ the form of degree d that defines X.

2. Proof of Theorem 1.3

Let V be an n-dimensional K-vector space, and let $v \in V$. Let H_K be the height on V given by the identification $V = \mathbb{A}^n_K(K) \hookrightarrow \mathbb{P}^n_K(K)$. Let $1 \leq k \leq n-1$. We start by defining the height in the Grassmannian

$$\mathrm{Gr}_K(k,n) \coloneqq \{W \subseteq V \colon W \text{ is a } K\text{-linear subspace of dimension } k\}.$$

In our argument, we will only use $\mathrm{Gr}_K(3,4)$, which is the dual projective space $(\mathbb{P}^3)^\vee$. However, the more general concept of a Grassmanian will be useful if one wants to generalize this argument in higher dimensions.

Let $W \subseteq V$ be a k-dimensional subspace. Let $w_1, \dots, w_k$ be any basis of W. Consider the exterior product $w_1 \wedge \cdots \wedge w_k \in \bigwedge^k V \cong K^{\binom{n}{k}}$. We define $H_K(W) \coloneqq H_K(w_1 \wedge \cdots \wedge w_k)$. It is possible to show that this height does not depend on the choice of basis, so $H_K(W)$ is well-defined. For more on the heights of subspaces, see [17].

Returning to our original problem, let $x \in X(K)$. We want to find a projective plane $\Pi \cong \mathbb{P}^2$ of small height containing x. This is equivalent to finding a 3-dimensional subspace of a 4-dimensional space that contains a given vector x. By [17, Theorem 2], there exists such a plane Π with

$$H_K(\Pi) \ll_{K,n,k} H_K(x)^{\frac{4-3}{4-1}} = H_K(x)^{\frac{1}{3}}.$$

This shows that, for a given $B > 0$, the set

$$\{x \in X(K) \colon H_K(x) \leq B\}.$$

can be covered by planes of height $O_K(B^{\frac{1}{3}})$. Using now [17, Theorem 3] (or the more precise version [18, Theorem 1]), we see that the number of such planes is $O_K\big((B^{\frac{1}{3}})^4\big) = O_K(B^{4/3})$.

We proceed by counting the number of rational points in the intersection of X with each such plane. Let Π be one of the planes of bounded height as above. We have to consider some cases. If $\Pi \subseteq X$, suppose without loss of generality that Π is the plane $x_3 = 0$. This inclusion allows us to conclude that $F(x_0, x_1, x_2, 0)$ is the zero polynomial, hence x_3 divides F, which is then reducible, a contradiction since X is smooth. So this cannot occur. We conclude that $X \cap \Pi$ is a plane curve of degree d.

Let C be an irreducible degree $e \leq d$ component of this curve, and let $\tilde{C}$ be its normalization. We can compute the genus of C as being

$$g(C) = \frac{(e-1)(e-2)}{2} - \sum_p \frac{m_p(m_p - 1)}{2}, \tag{2}$$

where the sum is taken over all singular points of C, including infinitely near singular points, and m_P denotes the multiplicity of C at p (see, for example, [3, Example 3.9.2]). In particular, as $g(C) \geq 0$, we see that C has a finite number of singular points, and this number is bounded in terms of d. Since $\tilde{C}$ coincides with C outside the finite set of singular points, we see that counting the rational points of $C(K)$ and $\tilde{C}(K)$ is the same up to $O_d(1)$.

If $g(C) \geq 2$, we can apply the Uniform Faltings's Theorem (Lemma 1.1) to $\tilde{C}$ to find

$$\#\tilde{C}(K) \leq c(g)^{1+\rho(\tilde{C})}.$$

Let $\varepsilon > 0$. We will show that, assuming the Rank Hypothesis, we have

$$\#\tilde{C}(K) \ll_{X,K,\varepsilon} B^{\varepsilon}.$$

Since $\tilde{C}$ is obtained by normalizing an irreducible component of the intersection $X \cap \Pi$ and $H_K(\Pi) \ll B^{1/3}$, the Jacobian J of $\tilde{C}$ is defined by equations with coefficients of size $O_X(B^{1/3})$, thus

$$h_F(J) \ll_X \log B,$$

where $h_F(J)$ is the Faltings's height. It is well-known that $\log N_J \ll h_F(J)$ (one can deduce this inequality for Abelian varieties with semistable reduction over K by using [8, Theorem A], and from this deduce the general case by using [6, Lemma 3.4]). Therefore, by the Rank Hypothesis, we have

$$\rho(\tilde{C}) = r_J = o(\log N_J) = o(h_F(J)) = o(\log B).$$

Hence, there exists a constant $B_0(\varepsilon) > 0$ such that, for all $B \geq B_0(\varepsilon)$,

$$\rho(\tilde{C}) \leq \varepsilon \log B.$$

For $B \leq B_0(\varepsilon)$, we have $h_F(J) \ll_X \log B_0(\varepsilon)$. Since the number of abelian varieties of bounded height is finite, there exists some constant $c_1 > 0$ such that

$$1 + \rho(\tilde{C}) \leq c_1 \varepsilon \log B$$

for all $\tilde{C}$, thus

$$\#\tilde{C}(K) \leq c(g)^{1+\rho(\tilde{C})} \leq c(g)^{c_1 \varepsilon \log B} = B^{\varepsilon c_1 \log c(g)}.$$

Since $g \leq \frac{(d-1)(d-2)}{2}$, we have finitely many possibilities for the constant $c(g)$. By adjusting the constant $B_0(\varepsilon)$ if needed, we obtain

$$\#\tilde{C}(K) \ll_{X,K,\varepsilon} B^{\varepsilon},$$

as we wanted. We conclude that the total contribution of these curves to the number of rational points of X is $O_{X,K,\varepsilon}(B^{4/3+\varepsilon})$.

If $g(C) = 1$, then either $\tilde{C}(K) = \emptyset$ or $\tilde{C}$ is an elliptic curve. In the first case, there is nothing to do. Suppose that we are in the second case. We will use the following result:

Lemma 2.1. *For any smooth plane cubic curve C, we have $N_C(B) \ll_K (\log B)^{1+\rho(\tilde{C})/2}$.*

Proof. It suffices to adapt the proof of [5, Proposition 2] to the case of number fields. The only important difference is that we use Merel's bound for the torsion of elliptic curves over number fields ([12]) instead of Mazur's bound for the torsion of elliptic curves over $\mathbb{Q}$ ([11]). □

Using again the Rank Hypothesis, we conclude that $N_C(B) = O_{X,K,\varepsilon}(B^{\varepsilon})$ for any $\varepsilon > 0$, so the total contribution of points coming from these curves is $O_{X,K,\varepsilon}(B^{4/3+\varepsilon})$.

All that is left is to bound the number of rational points in curves of genus 0 that lie inside X. If $C \subseteq X$ is a curve of genus 0, then either $C(K) = \emptyset$ or C is a rational curve. In the first case, there is nothing to do. In the second case, let e be the degree of C. Then $1 \leq e \leq d$. If $e = 1$, then C is a line, and by definition it does not contribute to the calculation of $N_{X'}(B)$. Suppose $e \geq 2$. By [15, Theorem 1.8],

$$N_C(B) \ll_{X,\varepsilon} B^{2/e+\varepsilon} \leq B^{1+\varepsilon}.$$

Since the contribution of a single rational curve of degree $e \geq 2$ is less than $O_{X,K,\varepsilon}(B^{4/3+\varepsilon})$, and since we have $O_d(1)$ possibilities for e, we conclude the proof of Theorem 1.3 if we show the following:

Lemma 2.2. *Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$. Fix a positive integer e. Then X contains finitely many rational plane curves of degree e.*

Proof. Assume by contradiction that X contains infinitely many rational plane curves of degree e. A plane curve of degree e has arithmetic genus $\frac{(e-1)(e-2)}{2}$, hence the Hilbert polynomial of any such curve is given by

$$P(m) = m \cdot e + 1 - \frac{(e-1)(e-2)}{2}.$$

Consider the Hilbert Scheme Hilb_X^P that parameterizes the subvarieties of X with Hilbert Polynomial P. It is a projective scheme (see [9, Section I.1]). Notice that the irreducible plane curves of degree e inside X form an open set $\mathrm{Irr}_X^e \subseteq \mathrm{Hilb}_X^P$. Let $\mathrm{Rat}_X^e \subseteq \mathrm{Irr}_X^e$ be the set of rational curves of degree e inside X. By (2), being a rational curve is a closed condition in Irr_X^e. Hence Rat_X^e is a closed subvariety of Irr_X^e, and therefore it is a quasi-projective scheme. In particular, it has finitely many irreducible components.

By our hypothesis, Rat_X^e has infinitely many points, hence there is an irreducible component $T \subseteq \mathrm{Rat}_X^e$ which contains infinitely many points. For any $t \in T$, let C_t be the curve associated with t. Let $U \subseteq X \times T$ be the universal family over T, that is,

$$U = \{(x,t) \colon x \in C_t\}.$$

Consider the projection in the first coordinate $\pi \colon U \to X$. Suppose that this map is not dominant. Then its image $\bigcup_{t \in T} C_t$ must lie inside a proper subvariety of X. As X is a surface, this means $\bigcup_{t \in T} C_t$ is contained in a curve. But this is impossible, since such a curve would have infinitely many irreducible components.

We conclude that $\pi \colon U \to X$ is dominant, that is, X is covered by a family of rational curves. This shows that X is uniruled, and thus it has Kodaira dimension $-\infty$.

However, as X is a smooth surface of degree $d \geq 4$, it is either a K3 surface (if $d = 4$) or a surface of general type (if $d \geq 5$). In the first case, X has Kodaira dimension 0, and in the second case it has Kodaira dimension 2. This contradiction proves the lemma. $\square$

This concludes the proof of Theorem 1.3.

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