THE GROWTH OF RESIDUALLY SOLUBLE GROUPS

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Abstract. Building on work of Wilson, we show that if G is a finitely generated residually soluble group whose growth function γ satisfies $(\log \gamma(n))/n^{1/4} \to 0$ as $n \to \infty$ then G is virtually nilpotent. This shows that Grigorchuk's Gap Conjecture holds for all exponents $\beta < 1/4$ within the class of residually soluble groups (improving Wilson's exponent $1/6$). We also discuss stronger versions of the Gap Conjecture.

1. Introduction

If G is a group generated by a finite set X and $g \in G$, we denote by $\ell_X(g)$ the length of g with respect to X, i.e., the smallest integer n such that g can be written as a product of n elements in $X \cup X^{-1}$. For a finite subset $S \subseteq G$, we write $\ell_X(S) = \max\{\ell_X(g) : g \in S\}$. The *growth function* of X is

$$\gamma_X(n) = |\{g \in G : \ell_X(g) \le n\}|.$$

Usually, growth functions are only coarsely compared. Given two functions $f, g : \mathbf{N} \to [0, \infty)$, we write $f \preceq g$ if there is a constant $C > 0$ such that $f(n) \le Cg(Cn)$ for all $n > 0$. This relation defines a preorder on the set of functions. The equivalence class of the growth function $\gamma_X(n)$ is a group property (i.e., independent of the choice of generating set X), and in fact a quasi-isometry invariant.

In 1968, Milnor [Mil68b] asked some highly influential questions about the relation between the growth function $\gamma_X(n)$ and the group structure of G. One of his conjectures was that $\gamma_X(n) \preceq n^d$ for some d (if and) only if G is virtually nilpotent; this is now the celebrated theorem of Gromov [Gro81]. Milnor also asked whether the growth of a group is always either bounded by a polynomial or equivalent to e^n. This was proved for soluble groups by Milnor and Wolf [Mil68a, Wol68], for linear groups by Tits [Tit72], for elementary amenable groups by Chou [Cho80], but Grigorchuk [Gri84] famously discovered a counterexample, i.e., a group of *intermediate growth*. In fact, Grigorchuk [Gri85] showed that there are a continuum of groups with pairwise incomparable growth functions, which shows that the landscape of group growth functions is in truth completely dominated by the groups of intermediate growth.

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Notwithstanding the existence of groups of intermediate growth, there is undoubtedly a definite gap between polynomial growth and superpolynomial growth. This follows already from Gromov's theorem and some logical nonsense (see [Gro81, p. 71]), but without an explicit function. Pursuing this idea, Shalom and Tao [ST10] proved that if G is not virtually nilpotent then $\gamma_X(n) \succeq n^{(\log\log n)^c}$ for some $c > 0$. This is the greatest lower bound currently known for the growth of non-virtually-nilpotent groups in general.

On the other hand, Grigorchuk's groups all have growth $\succeq \exp(\sqrt{n})$. In particular, the growth of Grigorchuk's first group itself is of the form $\exp(n^{\alpha+o(1)})$, where $\alpha \approx 0.767$ (Erschler–Zheng [EZ20]), and this is still the slowest known superpolynomial growth function. For these reasons among others, Grigorchuk [Gri14] has advocated the following conjecture, known as the *Gap Conjecture*, which we formalize with a free parameter β.

Conjecture (Gap(β))**.** *If G is a group generated by a finite set X and $\gamma_X(n) \prec \exp(n^\beta)$ then G is virtually nilpotent.*

We have seen that Milnor's conjecture Gap(1) is false, but it does hold within certain large natural classes of groups including soluble groups and linear groups. "The" Gap Conjecture is Gap(1/2), and it was proved in [Gri89, LM91] that Gap(1/2) holds for residually nilpotent groups (such as all of Grigorchuk's groups). In [Wil05, Wil11], J. S. Wilson considered the wider class of residually soluble groups, and in the latter paper he proved that Gap(1/6) holds within this class.

Our main contribution in this paper is to show that Wilson's exponent can be improved to 1/4.

Theorem 1.1 (Gap($1/4-\epsilon$) for residually soluble groups)**.** *If $G = \langle X\rangle$ is a finitely generated residually soluble group such that*

$$\log \gamma_X(n)/n^{1/4} \to 0 \qquad \text{as } n \to \infty,$$

then G is virtually nilpotent. In particular, Gap(β) *holds for residually soluble groups for all $\beta < 1/4$.*

Our proof of Theorem 1.1 builds on Wilson's ideas, particularly the consideration of self-centralizing chief factors. We will review everything we need. The proof splits into two cases. If the self-centralizing chief factors of G have bounded rank then Gap(1/2) holds for G. Otherwise there are self-centralizing chief factors of arbitrarily large rank, and then we can use a generalization of a lemma of Milnor to produce many distinct elements of short length in the chief factor. To optimize the approach, we use an ad hoc notion of "modified derived length", which is a variant of derived length that allows class-2 nilpotent factors at a discount, and we bound the modified derived length of soluble linear groups.

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groups. We also wish to record that one of the key ideas in this paper, the generalized Milnor lemma (see Lemma 3.3), or a close variant anyway, is also critical in forthcoming joint work with Sabatini and Tracey on diameters of permutation groups, and really it originated in that project.

2. Self-centralizing chief rank

A key insight of Wilson [Wil05] was to consider self-centralizing chief factors. A chief factor M/N of a finite group G is a minimal normal subgroup of a quotient G/N. If M/N is abelian (e.g, if G is soluble) then $M/N \cong \mathbf{F}_p^n$ for some prime p and integer $n \geq 1$; the *rank* of M/N is n. A chief factor M/N of G is called *self-centralizing* if $M/N = C_{G/N}(M/N)$. If G/M is soluble, a Frattini argument shows that M/N is complemented in G/N, and it follows that G has a self-centralizing chief factor of order p^n if and only if G has a maximal subgroup of index p^n.

Definition 2.1. Let G be a finite soluble group. The *self-centralizing chief rank* (a.k.a. *non-Frattini chief rank*) of G is the maximum rank of its self-centralizing chief factors. We write $\mathfrak{S}(n)$ for the class of finite soluble groups of self-centralizing chief rank at most n.

Remarks.

1. A chief factor M/N is called *Frattini* if $M/N \leq \Phi(G/N)$. Every self-centralizing chief factor is non-Frattini, and conversely every non-Frattini chief factor is equivalent to a self-centralizing chief factor. Refer to [DH92, A.15].
2. Every nontrivial soluble group has positive self-centralizing chief rank, and the class $\mathfrak{S}(1)$ coincides with the class of finite supersoluble groups: see [DH92, VII 2.2(a, c)].

If G is a soluble group we denote by $\delta(G)$ its *derived length*: the least integer k such that $G^{(k)} = 1$. We need the following well-known result of Zassenhaus and Newman (see [New72]). The weak form, due to Zassenhaus, is sufficient in this section, but in the next section we will use the precise bound found by Newman.

Theorem 2.2. *Let $\rho(n)$ be the supremum of the derived lengths $\delta(G)$ of the soluble linear groups of degree n.*

(a) (Zassenhaus) $\rho(n) < \infty$.
(b) (Newman) $\rho(n) < 5\log_9(n) + 6$.

The following results should be compared with [Wil05, Theorem 1.2]. Our result is slightly more general. We also avoid both ultrafilter analysis and the Tits alternative.

Theorem 2.3. *If $G \in \mathfrak{S}(n)$ then $G^{(\rho(n))}$ is nilpotent.*

Proof. Let $d = \rho(n)$ and let $N = G^{(d)}$. We will show that N is nilpotent.

Suppose $\overline{G}$ is a quotient of G with a minimal normal self-centralizing subgroup V of order p^m, where $m \leq n$. Since $\overline{G}/V$ embeds in $\mathrm{GL}_m(p)$, it has derived length at most $\rho(n)$, so $\overline{N} \leq V$. In particular $[V, \overline{N}] = 1$.

Since N acts trivially by conjugation on every self-centralizing chief factor of G, it follows from [Wil05, Lemma 2.3] that N acts trivially on every chief factor of G. Intersecting a chief series for G with N thus gives a central series for N, so N is nilpotent. □

Corollary 2.4. *If G is residually $\mathfrak{S}(n)$ then $G^{(\rho(n))}$ is residually nilpotent.*

The following basic fact about virtually soluble groups is well-known.

Lemma 2.5. *If G is a group and $N \trianglelefteq G$ such that both N and G/N are virtually soluble, then G itself is virtually soluble.*

Proof. By passing to a finite-index subgroup we may assume G/N is soluble. Let $M \trianglelefteq N$ be a soluble normal subgroup of N of minimal index. If $g \in G$ then $MM^g \leq N$ is again soluble, so we must have $M^g = M$. Therefore $M \trianglelefteq G$. Replacing G with G/M, we may thus assume that N is finite. Now $C_G(N)$ is a finite-index soluble subgroup of G. □

Theorem 2.6. Gap(1/2) *holds for residually $\mathfrak{S}(n)$ groups. In fact, if $G = \langle X \rangle$ is residually $\mathfrak{S}(n)$ and not virtually nilpotent then $\gamma_X(n) \succeq \exp(n^{1/2})$.*

Proof. Suppose G is residually $\mathfrak{S}(n)$ and of subexponential growth. By Corollary 2.4, G has a residually nilpotent normal subgroup N such that G/N is soluble. By Milnor's lemma (recalled in Lemma 3.1 below), N is finitely generated. From [Gri89, LM91] we conclude that either G has growth $\succeq \exp(n^{1/2})$ or N is virtually nilpotent. Since the class of virtually soluble groups is closed under extension, in the latter case G itself is virtually soluble. Finally, by Milnor–Wolf, G is virtually nilpotent. □

3. Variations on a lemma of Milnor

The following fundamental lemma is due to Milnor: see [Mil68a, Lemma 1] or [Man12, Theorem 5.1]. Compare also Gromov's lemma (b) in [Gro81, Section 4]. It follows easily from this lemma and induction on derived length that any soluble group of subexponential growth is polycyclic, which is roughly half of the Milnor–Wolf theorem.

Lemma 3.1 (Milnor's lemma)**.** *Let G be a finitely generated group of subexponential growth and assume that $G/N \cong \mathbf{Z}$. Then N is finitely generated.*

If we assume a stronger growth hypothesis then a more general property holds: every finitely generated subgroup has finitely generated normal closure. This stronger property does not hold in general for groups of subexponential growth: see for example [BE12, Theorem B].

Lemma 3.2 (Generalized Milnor lemma)**.** *Let $G = \langle X \rangle$ be a finitely generated group and assume that $H \le G$ is a finitely generated subgroup whose normal closure $N = \langle H^G \rangle$ is not finitely generated. Then G has growth*

$$\gamma_X(n) \succeq \exp(n^{1/2}).$$

Proof. Suppose $H = \langle Y \rangle$ where Y is finite. Let

$$Y_i = \{y^g : y \in Y,\ \ell_X(g) \le i\}, \qquad H_i = \langle Y_i \rangle \qquad (i \ge 0). \tag{3.1}$$

Then $H_0 \le H_1 \le \cdots$ is an increasing chain of subgroups of N. If $H_k = H_{k+1}$ for some k then for $x \in X \cup X^{-1}$ we have

$$H_k^x = \langle Y_k^x \rangle \le \langle Y_{k+1} \rangle = H_{k+1} = H_k,$$

so $H_k \trianglelefteq G$, so $H_k = N$. Since N is not finitely generated, there is no such k.

Therefore, for each $i > 0$ there is some $y_i \in Y_i \setminus H_{i-1}$. Now for all $k \ge 1$ we claim that the 2^k elements $y_1^{\epsilon_1} \cdots y_k^{\epsilon_k}$ with $\epsilon_i \in \{0, 1\}$ are distinct. Indeed, suppose

$$y_1^{\epsilon_1} \cdots y_k^{\epsilon_k} = y_1^{\delta_1} \cdots y_k^{\delta_k}$$

where $\epsilon, \delta \in \{0, 1\}^k$. Then

$$H_{k-1} y_k^{\epsilon_k} = H_{k-1} y_k^{\delta_k},$$

which implies $\epsilon_k = \delta_k$ since $y_k \notin H_{k-1}$. Cancelling $y_k^{\epsilon_k}$ and applying induction, it follows that $\epsilon = \delta$.

Now observe that

$$\ell_X(y_1^{\epsilon_1} \cdots y_k^{\epsilon_k}) \le \sum_{i=1}^{k} \ell_X(y_i) \le k(L + 2k),$$

where $L = \ell_X(Y)$. Thus

$$\gamma_X(kL + 2k^2) \ge 2^k$$

for all $k \ge 1$, which implies that $\gamma_X(n) \succeq \exp(n^{1/2})$. □

If we assume an even stronger growth hypothesis then we have a quantative form of the previous lemma. This is what we will use in the rest of the paper.

Lemma 3.3 (Generalized Milnor lemma, quantitative version)**.** *Let $G = \langle X \rangle$ be a finitely generated group whose growth function satisfies*

$$\gamma_X(n) \le \exp(Cn^\theta) \qquad (\textit{all } n > 0), \tag{3.2}$$

for some $\theta < 1/2$ and $C \ge 1$. Let $N = \langle Y^G \rangle$ be a normal subgroup of G, where Y is finite. Then $N = \langle Z \rangle$ where Z is finite and

$$\ell_X(Z) \le \ell_X(Y) + C_1 \ell_X(Y)^{\frac{\theta}{1-\theta}}.$$

Here $C_1 = 2(5C)^{1/(1-2\theta)}$ is a constant depending only on (C, θ).

Proof. Define Y_i and H_i as in (3.1). As before, $H_0 \leq H_1 \leq \cdots$ is an increasing chain of subgroups, and if $H_k = H_{k+1}$ then $H_k = N$. Moreover, $H_k = \langle Y_k \rangle$ where $\ell_X(Y_k) \leq \ell_X(Y) + 2k$. Thus it suffices to prove that there is such an integer k and to bound it.

Suppose $H_{i-1} < H_i$ for all $i = 1, \ldots, k$. For each such i pick $y_i \in Y_i \setminus H_{i-1}$. As before, the 2^k elements $y_1^{\epsilon_1} \ldots y_k^{\epsilon_k}$ with $\epsilon_i \in \{0, 1\}$ are distinct, and it follows that

$$\gamma_X(kL + 2k^2) \geq 2^k,$$

where $L = \ell_X(Y)$. Now from the growth hypothesis (3.2) we get

$$k \leq \log_2 \gamma_X(kL + 2k^2) \leq 2C(kL + 2k^2)^\theta.$$

If $L \leq k$, the above inequality gives $k^{1-2\theta} \leq 5C$. Otherwise, it gives $k^{1-\theta} \leq 5CL^\theta$. Thus, since $\theta < 1/2$, and since we may obviously assume $L \geq 1$, either way we get $k \leq (5C)^{1/(1-2\theta)} L^{\theta/(1-\theta)}$. This finishes the proof. ☐

Corollary 3.4. *Let $G = \langle X \rangle$ be a finitely generated group satisfying (3.2) for some $\theta < 1/2$. For each $k \geq 0$, $G^{(k)} = \langle X_k \rangle$ for some finite set X_k such that*

$$\ell_X(X_k) \leq C_2 4^k.$$

Here C_2 is another constant depending only on (C, θ).

Proof. We use induction on k. To start we may take $X_0 = X$. If $k > 0$, by induction we have $G^{(k-1)} = \langle X_{k-1} \rangle$ for a suitable set X_{k-1}. Let

$$Y_k = \{[x_1, x_2] : x_1, x_2 \in X_{k-1}\}.$$

Since $G^{(k-1)} / \left\langle Y_k^G \right\rangle$ is abelian, $G^{(k)} = \left\langle Y_k^G \right\rangle$. Therefore by the lemma we have $G^{(k)} = \langle X_k \rangle$ for some finite set X_k with

$$\ell_X(X_k) \leq 4\ell_X(X_{k-1}) + 4C_1 \ell_X(X_{k-1})^{\theta/(1-\theta)}.$$

This shows that $\ell_X(X_k) \leq L_k$, where the sequence (L_k) is defined by $L_0 = 1$ and

$$L_{k+1} = 4L_k + 4C_1 L_k^\delta \qquad (k \geq 1),$$

where $\delta = \theta/(1-\theta) < 1$. Clearly $L_k \geq 4^k$ for all k. Therefore also

$$\frac{L_{k+1}}{4L_k} \leq 1 + C_1 L_k^{-1+\delta} \leq 1 + C_1 4^{-(1-\delta)k} \leq \exp(C_1 4^{-(1-\delta)k}),$$

and it follows that

$$\frac{L_k}{4^k} = \prod_{j=0}^{k-1} \frac{L_{j+1}}{4L_j} \leq \exp\left(C_1 \sum_{j=0}^{\infty} 4^{-(1-\delta)j} \right),$$

and this proves the claim. ☐

By combining the previous corollary with the result of the previous section we can easily obtain Gap(1/4.16) for residually soluble groups, which is only slightly weaker than our main result Theorem 1.1. In the next section we will embellish the method to obtain Gap($1/4 - \epsilon$), as claimed in Theorem 1.1.

Proposition 3.5. Gap(1/4.16) *holds for residually soluble groups.*

Proof. Suppose that G is a non-virtually-nilpotent residually soluble group and X is a finite generating set for G such that $\gamma_X(n) \preceq \exp(n^\beta)$, where $\beta = 1/4.16$. Then in particular X satisfies (3.2) with $\theta = 1/3$ (say) and some constant C. By Corollary 3.4, $G^{(k)} = \langle X_k \rangle$ where X_k is a finite set such that

$$\ell_X(X_k) \le C_2 4^k.$$

In particular G is residually polycyclic. Since polycyclic groups are residually finite (see [Man12, Theorem 2.24]), G is residually finite-soluble.

On the other hand, Theorem 2.6 implies that G is not residually $\mathfrak{S}(n)$, so there are arbitrarily large integers n such that G has a finite soluble quotient $\overline{G} = G/N$ with a self-centralizing minimal normal subgroup $V = M/N$ of rank n, say $V \cong \mathbf{F}_p^n$. Since V is minimal normal and self-centralizing, V must coincide with the last nontrivial term of the derived series of $\overline{G}$. Thus

$$V = \overline{G}^{(k)}, \qquad \text{where } k = \delta(\overline{G}/V).$$

Since $\overline{G}/V$ embeds in $\mathrm{GL}_n(p)$, Theorem 2.2(b) gives

$$k \le \rho(n) < 5\log_9(n) + 6.$$

Since $G^{(k)} = \langle X_k \rangle$, we can find n elements $b_1, \ldots, b_n \in X_k$ whose images in V are linearly independent. It follows that the 2^n elements

$$b_1^{\epsilon_1} \cdots b_n^{\epsilon_n} \qquad (\epsilon_1, \ldots, \epsilon_n \in \{0, 1\})$$

are distinct, and

$$\ell_X(b_1^{\epsilon_1} \cdots b_n^{\epsilon_n}) \le C_2 4^k n \le C_2 4^{5\log_9(n)+6} n \le C_3 n^{5\log_9(4)+1} < C_3 n^{4.155}.$$

Therefore

$$\gamma_X(C_3 n^{4.155}) \ge 2^n.$$

Thus there is a constant $c > 0$ and infinitely many n such that

$$\gamma_X(n) \ge \exp(c n^{1/4.155}),$$

and this is a contradiction to the hypothesis $\gamma_X(n) \preceq \exp(n^\beta)$. □

4. Modified derived length

To improve the exponent 1/4.16 we will have to be creative, since Newman's bound on the derived length $\delta(G)$ of soluble linear groups of degree n is sharp. Our idea is to relax the notion of derived length. Rather than looking for elements of small length in the terms of the derived series, we will look for elements in the terms of another kind of subnormal series. We require that each factor in the series, if not abelian, is nilpotent of class 2.

The motivation for this idea is the observation that, instead of applying Lemma 3.3 to the derived subgroup N' of a given finitely generated normal subgroup $N = \langle X \rangle \trianglelefteq G$, which is normally generated by the set of commutators

$$Y = \{[x_1, x_2] : x_1, x_2 \in X\},$$

we can equally apply it to $\gamma_3(N)$, which is normally generated by the set of triple commutators

$$Y = \{[x_1, x_2, x_3] : x_1, x_2, x_3 \in X\}.$$

The word $[x_1, x_2, x_3]$ has length 10, and $10 < 4^2$, so there is a small saving available whenever there is a class-2 nilpotent section.

Let G be a soluble group. Consider a subnormal series

$$G = H_0 \geq H_1 \geq H_2 \geq \cdots \geq H_k = 1 \tag{$\mathcal{S}$}$$

such that $H_i \trianglelefteq H_{i-1}$ for each i and such that each factor H_{i-1}/H_i is either abelian or nilpotent of class 2. We say that $\mathcal{S}$ is a *modified soluble series* of G. Define

$$A(\mathcal{S}) = \{i \in \{1, \ldots, k\} : H_{i-1}/H_i \text{ is abelian}\}$$

and

$$\mu_S = \sum_{i \in A(\mathcal{S})} 1 + \sum_{i \in \{1,\ldots,k\} \setminus A(\mathcal{S})} \log_4(10).$$

We refer to the invariant

$$\mu(G) = \min\{\mu_{\mathcal{S}} : \mathcal{S} \text{ is a modified soluble series of } G\}$$

as the *modified derived length* of G. Trivially, $\mu(G) \leq \delta(G)$, but it can be smaller. This invariant μ behaves well with respect to subgroups, quotients, extensions, and direct powers. However, unlike the derived length δ, the modified derived length μ does *not* generally satisfy

$$\mu(G_1 \times G_2) = \max(\mu(G_1), \mu(G_2))$$

(e.g., this fails for $G_1 = \mathrm{SL}_2(3) \cong 2 \cdot A_4$ and $G_2 = \mathbf{F}_3^2 \rtimes Q_8$), and we will take some pains to avoid needing a bound of this kind.

Lemma 4.1. *Let G be a soluble group.*

(1) If $H \leq G$ then $\mu(H) \leq \mu(G)$.
(2) If $N \trianglelefteq G$ then $\mu(G/N) \leq \mu(G)$.
(3) If $N \trianglelefteq G$ then $\mu(G) \leq \mu(N) + \mu(G/N)$.
(4) If $n \geq 1$ then $\mu(G^n) = \mu(G)$.

Corollary 4.2. *Let A and B be soluble groups. Suppose B acts transitively on a set Ω and $W = A \wr_\Omega B = A^\Omega \rtimes B$. Then $\mu(W) \leq \mu(A) + \mu(B)$.*

In the proof of Proposition 3.5 we used Newman's bound Theorem 2.2(b) on the derived length of a soluble linear group. We want to prove a stronger bound for the modified derived length. This is plausible because Newman's bound is sharp only for certain groups (formed out of copies of $\mathrm{AGL}_2(3)$) that have many class-2 nilpotent sections in their derived series.

In order to prove such a bound, we will need Suprunenko's structure theorem for soluble primitive linear groups. This can be found in [Sup63, Sup76], and it is also stated in [Wil11, Lemma 2.7].

Theorem 4.3 (Suprunenko)**.** *Let G be a maximal primitive soluble subgroup of $\mathrm{GL}_n(p)$. Then G has a unique maximal abelian normal subgroup A, which is cyclic of order $p^l - 1$ for some divisor l of n. Moreover, $C = C_G(A)$ embeds in $\mathrm{GL}_r(p^l)$ where $r = n/l$ and G/C is cyclic of order l_1 for some divisor l_1 of l. If $A = C$, then $l_1 = l$. Otherwise, $r > 1$ and there is an integer $u > 1$ that divides r and all of whose prime divisors divide $p^l - 1$, such that C has the following structure:*

(i) C/A has a unique maximal abelian normal subgroup B/A; the group B/A has elementary abelian Sylow subgroups and order u^2;

(ii) C/B embeds in $\prod_{i=1}^{s} \mathrm{Sp}_{2k_i}(q_i)$, where $u = \prod_{i=1}^{s} {q_i}^{k_i}$ is the prime factorization of u, and the projection of C/B in each symplectic group is completely reducible.

Remark. $A = Z(C)$ and B is nilpotent of class 2.

Essentially by applying Suprunenko's theorem, Newman obtained the following precise bound for the derived length of a completely reducible linear group.

Theorem 4.4 (Newman)**.** *Let $G \leq \mathrm{GL}_n(p)$ be soluble and completely reducible. Then $\delta(G) \leq \sigma(n)$, where*

$$\sigma(n) = \begin{cases} 1 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ 5 & \text{if } n \in [3,5], \\ 6 & \text{if } n \in [6,7], \end{cases}$$

and in general

$$\sigma(n) \leq 5\log_9(n) + C$$

where $C = 8 - 15\log_9(2) \approx 3.268$.

Proof. See [New72, Theorem C_S and Appendix]. □

We will prove a stronger bound for $\mu(G)$ when G is soluble and irreducible, using a similar approach. To begin, we need to bound $\mu(G)$ and $\delta(G)$ in several special cases.

Lemma 4.5. *Let $G \leq \mathrm{GL}_n(p)$ be soluble and irreducible.*

(1) If $n = 2$ then $\mu(G) \leq 2 + \log_4(10)$.
(2) If $n = 3$ then $\mu(G) \leq 1 + 2\log_4(10)$.
(3) If $n = 4$ then $\mu(G) \leq 3 + \log_4(10)$.

Furthermore, if $p = 2$ then the following bounds hold.

(4) If n is prime then $\delta(G) \leq 2$.
(5) If $n = 4$ then $\delta(G) \leq 3$.
(6) If $n = 6$ then $\delta(G) \leq 6$ and $\mu(G) \leq 2 + 2\log_4(10)$.
(7) If $n = 8$ then $\delta(G) \leq 5$.
(8) If $n = 9$ then $\delta(G) \leq 5$.
(9) If $n = 10$ then $\delta(G) \leq 4$.

Proof. Without loss of generality G is a maximal soluble irreducible subgroup of $\mathrm{GL}_n(p)$. If G is primitive then Theorem 4.3 applies and we use the notation from that theorem. In this case we will use the following observation several times: if there does not exist a factorization $n = rl$ such that $\gcd(r, p^l-1) > 1$, for example if $\gcd(n, p^n - 1) = 1$, then necessarily $A = C$ and $\delta(G) \leq 2$.

(1) Assume $n = 2$.

Imprimitive case. $G = \mathrm{GL}_1(p) \wr S_2$, so $\delta(G) \leq 2$.

Primitive case. If $A = C$ then $\delta(G) \leq 2$. If $A \neq C$, then $G = C$, $u = 2$, and $C/B \leq \mathrm{GL}_2(2) \cong S_3$. Since B is nilpotent of class 2 we have

$$\mu(G) \leq \mu(C/B) + \mu(B) \leq 2 + \log_4(10).$$

(2) Assume $n = 3$.

Imprimitive case. $G = \mathrm{GL}_1(p) \wr S_3$, $\delta(G) \leq 3$.

Primitive case. If $A = C$ then $\delta(G) \leq 2$. If $A \neq C$ then $l = 1$, $u = 3$, $G = C$, and $C/B \leq \mathrm{SL}_2(3) \cong 2 \cdot A_4$, so $\mu(C/B) \leq 1 + \log_4(10)$ and $\delta(C/B) \leq 3$, which gives $\mu(G) \leq 1 + 2\log_4(10)$.

We defer the proof of (3) until after the proof of (4) and (5).

(4) Assume n is prime and $p = 2$.

Imprimitive case. G is isomorphic to a soluble transitive subgroup of S_n, and hence $G \cong \mathrm{AGL}_1(n)$, so $\delta(G) \leq 2$.

Primitive case. $A \cong C_{2^l-1}$ for some l, and since A cannot be trivial we must have $l > 1$, so $l = n$. Thus $A = C$ and $\delta(G) \leq 2$.

(5) Assume $(n, p) = (4, 2)$.

Imprimitive case. Either $G \cong S_4$ or $G \cong \mathrm{GL}_2(2) \wr S_2 \cong S_3 \wr S_2$, so $\delta(G) \leq 3$.

Primitive case. Since $\gcd(4, 2^4 - 1) = 1$ we must have $A = C$, so $\delta(G) \leq 2$.

(3) Assume $n = 4$.

Imprimitive case. $G \leq \mathrm{GL}_d(p) \wr \mathrm{Sym}(4/d)$ for $d \in \{1, 2\}$. If $d = 1$ then $G = \mathrm{GL}_1(p) \wr S_4$ and $\delta(G) \leq 4$. If $d = 2$ then $G = L \wr S_2$ for some soluble irreducible $L \leq \mathrm{GL}_2(p)$, so $\mu(G) \leq 3 + \log_4(10)$ by (1).

Primitive case. If $A = C$ then $\delta(G) \leq 2$. Otherwise, $r > 1$ and $u \in \{2, 4\}$. If $u = 2$ then $C/B \leq \mathrm{Sp}_2(2) \cong S_3$. Since G/C is cyclic and B is nilpotent of class 2 it follows that $\mu(G) \leq 3 + \log_4(10)$. If $u = 4$ then $l = 1$, $G = C$, and $C/B \leq \mathrm{Sp}_4(2)$. If C/B is reducible then $\delta(C/B) \leq 2$ by (4). If C/B is irreducible then $\delta(C/B) \leq 3$ by (5). Thus $\mu(G) \leq 3 + \log_4(10)$ in all cases.

(6) Assume $(n, p) = (6, 2)$. The bound on $\delta(G)$ follows from Theorem 4.4, so we just need to prove the bound on $\mu(G)$.

Imprimitive case. $G \leq \mathrm{GL}_d(2) \wr \mathrm{Sym}(6/d)$ for some $d \in \{1, 2, 3\}$. If $d = 1$ then $G \leq S_6$ and $G \cong S_2 \wr S_3$ or $G \cong S_3 \wr S_2$, so $\delta(G) = 3$. If $d = 2$ then $G = \mathrm{GL}_2(2) \wr S_3 \cong S_3 \wr S_3$, so $\delta(G) = 4$. If $d = 3$ then $G = L \wr S_2$ for some soluble irreducible subgroup $L \leq \mathrm{GL}_3(2)$, and by (4) we have $\delta(G) \leq 3$.

Primitive case. If $A = C$ then $\delta(G) \leq 2$. Otherwise, $u \mid 6$ and the prime divisors of u divide $2^l - 1$, so $u = 3$ and $C/B \leq \mathrm{SL}_2(3) \cong 2 \cdot A_4$, so $\mu(C/B) \leq 1 + \log_4(10)$. Since B is nilpotent of class 2 we get

$$\mu(G) \leq \mu(G/C) + \mu(C/B) + \mu(B) \leq 2 + 2\log_4(10).$$

(7) Assume $(n, p) = (8, 2)$.

Imprimitive case. $G \le \mathrm{GL}_d(2) \wr \mathrm{Sym}(8/d)$ for some $d \in \{1, 2, 4\}$. If $d = 1$ then $G \le S_8$ and G is either $S_4 \wr S_2$, $S_2 \wr S_4$, or else $G \le \mathrm{AGL}_3(2)$. In the last case $G \cong \mathbf{F}_2^3 \rtimes L$ for an irreducible soluble subgroup $L \le \mathrm{GL}_3(2)$, and $\delta(L) \le 2$ by (4). Thus $\delta(G) \le 4$ in these cases. If $d = 2$ then $G = \mathrm{GL}_2(2) \wr S_4 \cong S_3 \wr S_4$ and $\delta(G) = 5$. If $d = 4$ then $G = L \wr S_2$ for some soluble irreducible $L \le \mathrm{GL}_4(2)$, and $\delta(L) \le 3$ by (5), so $\delta(G) \le 4$.

Primitive case. Since $\gcd(8, 2^8 - 1) = 1$ we must have $A = C$ and $\delta(G) \le 2$.

(8) Assume $(n, p) = (9, 2)$.

Imprimitive case. $G \le \mathrm{GL}_d(2) \wr \mathrm{Sym}(9/d)$ for some $d \in \{1, 3\}$. If $d = 1$ then $G \le S_9$ and G is either $S_3 \wr S_3$ or $\mathrm{AGL}_2(3)$, so $\delta(G) \le 5$.

Primitive case. Since $\gcd(9, 2^9 - 1) = 1$ we must have $A = C$ and $\delta(G) \le 2$.

(9) Assume $(n, p) = (10, 2)$.

Imprimitive case. $G \le \mathrm{GL}_d(2) \wr \mathrm{Sym}(10/d)$ for some $d \in \{1, 2, 5\}$. If $d = 1$ then $G \le S_{10}$ and $G \cong \mathrm{AGL}_1(5) \wr S_2$ or $G \cong S_2 \wr \mathrm{AGL}_1(5)$, so $\delta(G) = 3$. If $d = 2$ then $G = \mathrm{GL}_2(2) \wr \mathrm{AGL}_1(5) \cong S_3 \wr \mathrm{AGL}_1(5)$ and $\delta(G) = 4$. If $d = 5$ then $G = L \wr S_2$ for some irreducible $L \le \mathrm{GL}_5(2)$ and $\delta(L) \le 2$ by (4), so $\delta(G) \le 3$.

Primitive case. Since $\gcd(10, 2^{10} - 1) = 1$ we must have $A = C$ and $\delta(G) \le 2$. ☐

Theorem 4.6. *Let $n \ge 1$.*

(i) If G is a transitive soluble subgroup of S_n then

$$\mu(G) \le 3 \log_4(n).$$

(ii) If G is an irreducible soluble subgroup of $\mathrm{GL}_n(p)$ then

$$\mu(G) \le 3 \log_4(n) + K,$$

where $K = -5/2 + 3 \log_4(10) \approx 2.48$.

Proof. We prove both statements together by induction on n. More precisely, we will assume that both *(i)* and *(ii)* hold for all positive integers $m < n$, and additionally in the proof of *(ii)* we will assume that *(i)* holds for $m = n$.

(i) Let $G \le S_n$ be soluble and transitive. If $1 \le n \le 4$ then

$$\mu(G) \le \mu(S_n) \le n - 1 \le 3 \log_4(n),$$

so we may assume $n \ge 5$. If G is imprimitive then $G \le L \wr M$ for some soluble transitive subgroups $L \le \mathrm{Sym}(n/t)$ and $M \le \mathrm{Sym}(t)$ with $t \mid n$ and $1 < t < n$, so by induction

$$\mu(G) \le \mu(L) + \mu(M) \le 3 \log_4(n/t) + 3 \log_4(t) = 3 \log_4(n).$$

If G is primitive then $n = p^d$ and $G \cong V \rtimes H$ with $V = \mathbf{F}_p^d$ and $H \le \mathrm{GL}_d(p)$ irreducible. Since $d \le \log_2(n) < n$, by induction

$$\mu(G) \le 1 + 3 \log_4(d) + K,$$

and this gives $\mu(G) < 3\log_4(n)$ unless $n \in \{8, 9, 16\}$. In these cases we refer to Lemma 4.5. If $n \in \{8, 16\}$ then $H \leq \mathrm{GL}_d(2)$ with $d \in \{3, 4\}$, and Lemma 4.5(4)(5) give $\delta(H) \leq 3$, so

$$\mu(G) \leq \delta(G) \leq 4 < 3\log_4(n).$$

If $n = 9$ then $H \leq \mathrm{GL}_2(3)$, so

$$\mu(G) \leq 1 + \mu(\mathrm{GL}_2(3)) = 3 + \log_4(10) < 3\log_4(n).$$

(ii) Let p be a prime and let $G \leq \mathrm{GL}_n(p)$ be soluble and irreducible. If G is imprimitive then $G \leq L \wr M$ for some soluble irreducible subgroup L of $\mathrm{GL}_{n/t}(p)$ and some soluble transitive subgroup M of $\mathrm{Sym}(t)$, where $t \mid n$ and $1 < t \leq n$, so by induction we have

$$\mu(G) \leq \mu(L) + \mu(M) \leq 3\log_4(n/t) + K + 3\log_4(t) = 3\log_4(n) + K.$$

Thus we may assume G is primitive, and without loss of generality we may take G to be maximal primitive soluble and apply Suprunenko's Theorem 4.3. If $A = C$ then $\mu(G) \leq \delta(G) \leq 2$, which is enough to conclude since $K \geq 2$, so we may assume $A \neq C$. Since B is nilpotent of class 2, $\mu(B) \leq \log_4(10)$. Now $C/B \leq \prod_{i=1}^s \mathrm{Sp}_{2k_i}(q_i)$ where $u = \prod_{i=1}^s q_i^{k_i} \mid n/l$, and the projection of C/B to each factor is completely reducible, so $\delta(C/B) \leq \sigma(2k)$ where $k = \max\{k_1, \dots, k_s\}$. Thus

$$\mu(G) \leq 1 + \delta(C/B) + \mu(B) \leq \sigma(2k) + 1 + \log_4(10).$$

We can check that this gives $\mu(G) \leq 3\log_4(n) + K$ unless $n \leq 6$ or $n \in \{8, 9, 16, 32\}$. We need to treat these cases specially, referring to Lemma 4.5.

Case $n = 1$. Trivial.

Cases $n = 2, 3$. See Lemma 4.5(1)(2).

Cases $n = 4, 5, 6$. The bound $\mu(G) \leq \delta(G) \leq \sigma(n)$ from Newman's Theorem 4.4 is sufficient.

Case $n = 8 = 2^3$. If $l > 1$ then $C/B \leq \mathrm{Sp}_4(2)$ and by Lemma 4.5(4)(5) we have $\delta(C/B) \leq 3$, so

$$\mu(G) \leq 1 + \delta(C/B) + \mu(B) \leq 4 + \log_4(10) < 3\log_4(n) + K.$$

If $l = 1$ then $G = C$ and $C/B \leq \mathrm{Sp}_6(2)$. If C/B is reducible then by Lemma 4.5(4)(5) we have $\mu(C/B) \leq \delta(C/B) \leq 3$. If C/B is irreducible then by Lemma 4.5(6) we have $\mu(C/B) \leq 2 + 2\log_4(10)$. Thus in all cases

$$\mu(G) \leq \mu(C/B) + \mu(B) \leq 2 + 3\log_4(10) = 3\log_4(n) + K.$$

Remark. This case is the source of the constant K.

Case $n = 9 = 3^2$. If $l > 1$ then $C/B \leq \mathrm{Sp}_2(3) \cong 2 \cdot A_4$, so $\mu(C/B) \leq 1 + \log_4(10)$ and

$$\mu(G) \leq 1 + \mu(C/B) + \mu(B) \leq 2 + 2\log_4(10) < 3\log_4(n) + K.$$

If $l = 1$ then $C/B \leq \mathrm{Sp}_4(3)$, so by Theorem 4.4 we have $\mu(C/B) \leq \sigma(4) = 5$ and

$$\mu(G) \leq \mu(C/B) + \mu(B) \leq 5 + \log_4(10) < 3\log_4(n) + K.$$

Case $n = 16 = 2^4$. If $l = 1$ then $G = C$ and $C/B \leq \mathrm{Sp}_8(2)$. By Lemma 4.5(4)–(7) we have $\delta(C/B) \leq 6$, so

$$\mu(G) \leq \delta(C/B) + \mu(B) \leq 6 + \log_4(10) < 3\log_4(n) + K.$$

If $l > 1$ then $C/B \leq \mathrm{Sp}_6(2)$. If C/B is reducible then similarly $\delta(C/B) \leq 3$ and

$$\mu(G) \leq 1 + \delta(C/B) + \mu(B) \leq 4 + \log_4(10) < 3\log_4(n) + K.$$

Finally, if C/B is an irreducible subgroup of $\mathrm{Sp}_6(2)$ then by Lemma 4.5(6) we have

$$\mu(G) \leq 1 + \mu(C/B) + \mu(B) \leq 3 + 3\log_4(10) < 3\log_4(n) + K.$$

Case $n = 32 = 2^5$. In this final case C/B is a completely reducible subgroup of $\mathrm{Sp}_{10}(2)$, so $\delta(C/B) \leq 6$ by Lemma 4.5(4)–(9) and

$$\mu(G) \leq 1 + \delta(C/B) + \mu(B) \leq 7 + \log_4(10) < 3\log_4(n) + K.$$

This completes the proof. □

Finally we can prove Theorem 1.1.

Proof of Theorem 1.1. The proof is mostly the same as the proof of Proposition 3.5. Suppose that G is a non-virtually-nilpotent residually soluble group and X is a finite generating set for G such that $\log \gamma_X(n)/n^{1/4} \to 0$ as $n \to \infty$. Then in particular G satisfies (3.2) with $\theta = 1/3$. Exactly as in the proof of Proposition 3.5, there are arbitrarily large integers n such that G has a finite soluble quotient $\overline{G} = G/N$ with a self-centralizing minimal normal subgroup $V = M/N \cong \mathbf{F}_p^n$.

Let

$$\overline{G} = H_0 \geq H_1 \geq H_2 \geq \cdots \geq H_k = 1 \qquad (\mathcal{S})$$

be a modified soluble series of $\overline{G}$ such that $\mu_{\mathcal{S}} = \mu(\overline{G})$. Since $\overline{G}/V$ embeds in $\mathrm{GL}_n(p)$, Theorem 4.6 gives $\mu(\overline{G}) \leq 3\log_4(n) + 3$. If $A(\mathcal{S}) = \{i \in \{1, \ldots, k\} : H_{i-1}/H_i$ is abelian$\}$, we may assume that

$$H_i = \begin{cases} (H_{i-1})' & \text{if } i \in A(\mathcal{S}), \\ \gamma_3(H_{i-1}) & \text{otherwise.} \end{cases}$$

Then H_{k-1} is a nontrivial normal subgroup of $\overline{G}$ that is either abelian or nilpotent of class 2. Since V is minimal normal and self-centralizing, we must have $V = H_{k-1}$. Now if H_{i-1} is generated by a set X_{i-1} then H_i is normally generated by a set Y_i with $\ell_X(Y_i) \leq a_i \ell(X_{i-1})$, where

$$a_i = \begin{cases} 4 & \text{if } i \in A(\mathcal{S}) \\ 10 & \text{otherwise.} \end{cases}$$

Thus Lemma 3.3 implies that each H_i is generated by a set of length $\leq L_i$, where $L_0 = 1$ and

$$L_i = a_i L_{i-1} + C L_{i-1}^{1/2} \qquad (1 \leq i \leq k),$$

where C is a constant. Clearly $L_i \geq 4^i$ for all i, so

$$\frac{L_i}{a_i L_{i-1}} \leq 1 + C2^{1-i},$$

and it follows easily that

$$L_i \leq C' a_1 \cdots a_{i+1}$$

for some constant C'. In particular

$$L_{k-1} \leq C' a_1 \cdots a_k = C' 4^{\mu(\overline{G})} \leq C'' n^3$$

We have established that some n elements $b_1, \ldots, b_n$ of length at most $C'' n^3$ project to linearly independent elements of V. The products

$$b_1^{\epsilon_1} \cdots b_n^{\epsilon_n} \qquad (\epsilon_1, \ldots, \epsilon_n \in \{0,1\})$$

are all distinct. Thus

$$\gamma_X(C'' n^4) \geq 2^n.$$

Since there are arbitrarily large such integers n, there is a constant $c > 0$ and infinitely many integers $n > 0$ such that

$$\gamma_X(n) \geq \exp(cn^{1/4}),$$

and this is a contradiction to the hypothesis $\log \gamma_X(n)/n^{1/4} \to 0$. □

Remark. As to whether the exponent $1/4$ can be improved, an important example is the pro-finite-soluble group

$$\mathcal{G} = \varprojlim S_4 \wr \cdots \wr S_4,$$

which is the automorphism group of an infinite rooted 4-regular tree. Although this group is not topologically finitely generated, its derived subgroup $\mathcal{G}'$ is. Let $G = \langle X \rangle$ be a finitely generated dense subgroup of $\mathcal{G}'$. Then Theorem 1.1 applies to G, so we have established that $\gamma_X(n) > \exp(cn^{1/4})$ infinitely often for some constant $c > 0$. Can the exponent $1/4$ be improved in this case? Can G even have subexponential growth?

5. Stronger gap conjectures

In [Gri14], Grigorchuk formulated the following stronger version of the gap conjecture.

Conjecture (Gap$^*(\beta)$)**.** *Let G be a group generated by a finite set X and assume G is not virtually nilpotent. Then $\gamma_X(n) \succeq \exp(n^\beta)$.*

Gap$^*(1/2)$ holds for residually nilpotent groups by the proof of [Gri89, LM91]. Theorem 2.6 extends Gap$^*(1/2)$ to residually $\mathfrak{S}(n)$ groups. In particular Gap$^*(1/2)$ holds for residually supersoluble groups, because supersoluble implies residually finite-supersoluble. This was already asserted in [Gri14, Theorem 5.6] without full explanation.

The generalized Milnor lemma (Lemma 3.2) ought to be useful for $\mathrm{Gap}^*(\beta)$, as it shows that any group having a finitely generated subgroup with non-finitely-generated normal closure has growth $\succeq \exp(n^{1/2})$.

Wilson's results do not give $\mathrm{Gap}^*(\beta)$ for residually soluble groups for any $\beta > 0$ (contrary to the claim in [Gri14, Theorem 5.7]), and neither have we succeeded in proving it, though our methods come close. A positive answer to the following question would help (compare Corollary 3.4).

Question. *Let $G = \langle X \rangle$ be a finitely generated group of subexponential growth. Do there exist generating sets X_k for $G^{(k)}$ for all $k \geq 0$ such that $\ell_X(X_k)$ grows at most exponentially as a function of k?*

Modifying the order of quantifiers one more time, we can formulate an even stronger version of the gap conjecture, which we label $\mathrm{Gap}^{**}(\beta)$.

Conjecture ($\mathrm{Gap}^{**}(\beta)$)**.** *There is a function $f(n) \succeq \exp(n^\beta)$ such that the following holds. Let G be a group generated by a finite set X and assume G is not virtually nilpotent. Then $\gamma_X(n) \geq f(n)$ for all n.*

This "uniform" version of the gap conjecture has not yet been established even for residually nilpotent groups, though remarkably the method of [Gri89, LM91] does give it for *nonlinear* residually nilpotent groups, and one may take $f(n)$ to be the partition function $p(n)$, which famously satisfies $\log p(n) \sim \pi\sqrt{2n/3}$ (Hardy–Ramanujan).

The theorem of Shalom–Tao (see [ST10, Corollary 1.10]) has this kind of strong uniformity: their result is that there is some universal constant $c > 0$ such that every non-virtually-nilpotent finitely generated group $G = \langle X \rangle$ satisfies

$$\gamma_X(n) \geq n^{c(\log\log n)^c} \qquad (n > 1/c).$$

For another result with this remarkable kind of uniformity, see the recent paper of Lyons–Mann–Tointon–Tessera [LMTT23].

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